

ISSN:2636-7467(Online)

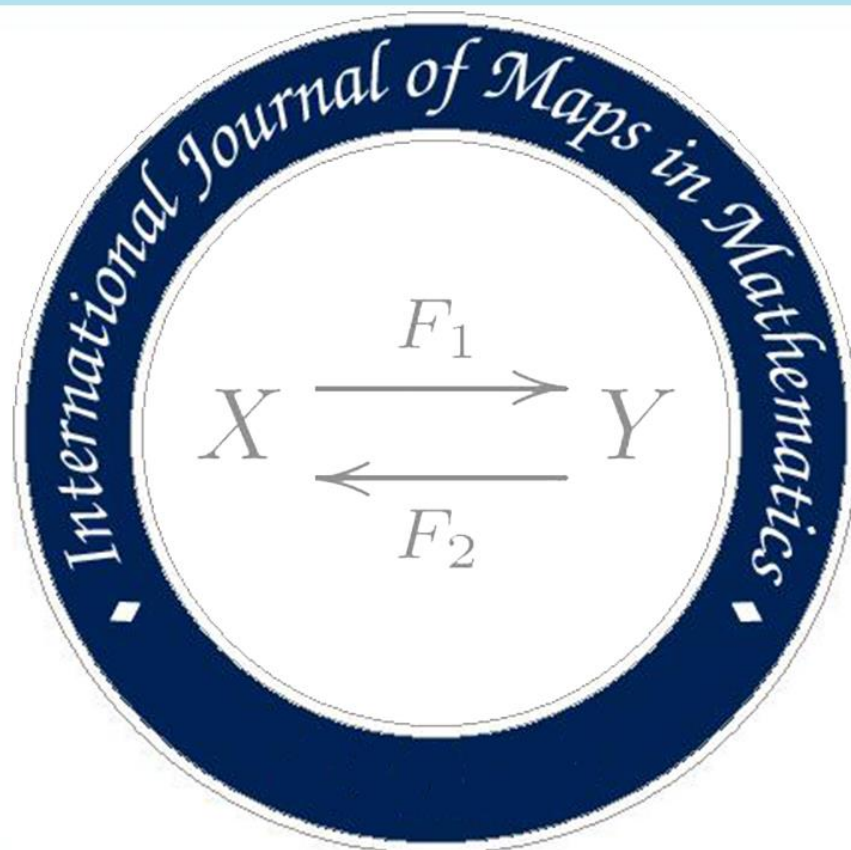
international journal of
MAPS IN MATHEMATICS

Volume 5
Issue 1
2022

Editor in chief

Bayram SAHIN

Ege University, Izmir, TURKEY



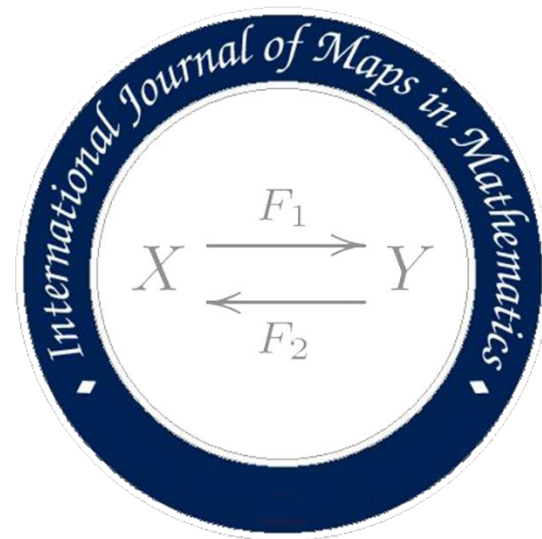
International Journal of Maps in Mathematics is a fully refereed journal devoted to publishing recent results obtained in the research areas of maps in mathematics

www.journalmim.com

VOLUME 5 ISSUE 1
ISSN 2636-7467

March 2022
<https://www.journalmim.com>

International Journal of Maps in Mathematics



Editor-in-Chief

Bayram Sahin
Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey
journalofmapsinmathematics@gmail.com

Managing Editor

Arif Gursoy
Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey
arif.gursoy@ege.edu.tr

Editorial Board

Syed Ejaz Ahmed
Brock University, Canada

Kamil Rajab Ayda-zade
Azerbaijan National Academy of Sciences, Azerbaijan

Erdal Ekici
Canakkale Onsekiz Mart University, Turkey

Arif Gursoy
Ege University, Turkey

Zulfiqar Habib
COMSATS Institute of Information Technology, Pakistan

Vatan Karakaya
Yildiz Technical University, Turkey

Andrey L. Karchevsky
Sobolev Mathematical Institute, Russia

Selcuk Kutluay
Inonu University, Turkey

Jae Won Lee
Gyeongsang National University, Republic of Korea

Jung Wook Lim
Kyungpook National University, Republic of Korea

Takashi Noiri
Yatsushiro College of Technology, Japan

Aldo Figallo Orellano
Universidad Nacional del Sur, Argentina

Bayram Sahin
Ege University, Turkey

Ali Taghavi
University of Mazandaran, Iran

Adnan Tercan
Hacettepe University, Turkey

Gabriel Eduard Vilcu
Petroleum-Gas University of Ploiesti, Romania

Technical Assistants

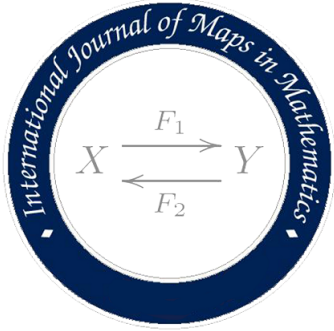
Ibrahim Senturk

Deniz Poyraz

Department of Mathematics, Faculty of Science, Ege
University, Izmir, Turkey

Department of Mathematics, Faculty of Science, Ege
University, Izmir, Turkey

This page intentionally left blank



International Journal of Maps in Mathematics

Volume 5, Issue 1, 2022, Pages:1

ISSN: 2636-7467 (Online)

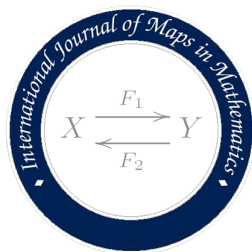
www.journalmim.com

EDITORIAL

BAYRAM ŞAHİN 

Dear Readers,

With this new issue, the **International Journal of Maps in Mathematics** has published its first issue in its 5th year. It is important for a scientific journal to complete 5 years. It proves that this journal is recognized by the scientific community and shows that it is an accepted platform for researchers working on the scope of the journal. We would like to thank our readers, editors, referees, technical assistants and you, our readers, who have contributed significantly to our journal's fifth year. The **International Journal of Maps in Mathematics** will continue to be a qualified platform for researchers in the research areas of the journal.



THE APPROXIMATION OF BIVARIATE GENERALIZED BERNSTEIN-DURRMEYER TYPE GBS OPERATORS

ECEM ACAR  * AND AYDIN İZGİ 

ABSTRACT. In the present paper, we introduce the generalized Bernstein-Durrmeyer type operators and obtain some approximation properties of these operators studied in the space of continuous functions of two variables on a compact set. The rate of convergence of these operators are given by using the modulus of continuity. The order of approximation using Lipschitz function and Peetre's K - functional are given. Further, we introduce Bernstein-Durrmeyer type GBS (Generalized Boolean Sum) operator by means of Bögel continuous functions which is more extensive than the space of continuous functions. We obtain the degree of approximation for these operators by using the mixed modulus of smoothness and mixed K -functional. Finally, we show comparisons by some illustrative graphics in Maple for the convergence of the operators to some functions.

Keywords: Bernstein-Durrmeyer operators, Modulus of continuity, Peetre's K - functional, GBS operators, B-continuous function, B-differentiable function, Mixed modulus of smoothness, Mixed K -functional.

2010 Mathematics Subject Classification: 41A10, 41A25, 41A36, 41A63.

1. INTRODUCTION

Let $f(x)$ be a function defined on the closed interval $[0, 1]$ the expression

$$B_n f(x) = B_n(f; x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad (1.1)$$

Received:2020.12.11

Revised:2021.06.01

Accepted:2021.06.03

* Corresponding author

Ecem Acar; karakusecem@harran.edu.tr; <https://orcid.org/0000-0002-2517-5849>

Aydın İzgi; aydinizgi@yahoo.com; <https://orcid.org/0000-0003-3715-8621>

is called Bernstein polynomial of order n of the function $f(x)$. The polynomials $B_n f(x)$ were introduced by S. Bernstein (see [5]) to give an especially simple proof of Weierstrass approximation theorem. The generalizations of Bernstein polynomials (1.1) were investigated in [15]- [12]. In 1988, [15] the function of two real variables function f be given over the unit square

$$s : [0, 1] \times [0, 1]$$

then the bivariate Bernstein polynomial of degree (n, m) , corresponding to the function f , is defined by means of the formula

$$B_{n,m}(x) = B_{n,m}(f; x, y) = \sum_{k=0}^n \sum_{j=0}^m f\left(\frac{k}{n}, \frac{j}{m}\right) \binom{n}{k} \binom{m}{j} x^k (1-x)^{n-k} y^j (1-y)^{m-j}. \quad (1.2)$$

There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two-dimensional Bernstein polynomials and their generalizations.

In 1967, Durrmeyer [11] introduced the following positive linear operators of the classical Bernstein operators, which modify with each function f integrable on the interval $[0, 1]$ the polynomial

$$M_n(f(x)) = (n + 1) \sum_{k=0}^n p_{n,k}(x) \int_0^1 p_{n,k}(t) f(t) dt,$$

which $p_{n,k}(x) = \binom{k}{n} \binom{n}{k} x^k (1-x)^{n-k}$. D. C. Morales and V. Gupta [9] studied two families of Bernstein-Durrmeyer type operators. The Baskakov Durrmeyer operators were introduced in 1985 and many properties of such operators were studied comprehensively. Gupta [13] presented the approximation properties of these operators. In 2007 [1] local approximation properties of a variant of the Bernstein-Durrmeyer operators were given.

In this paper, firstly we introduce bivariate generalized Bernstein-Durrmeyer operators. We investigate the properties of approximation of generalized Bernstein-Durrmeyer polynomials and the order of approximation using Lipschitz function and Peetre’s K - functional. Then, we define the Generalized Boolean Sum (GBS) operators of generalized Bernstein-Durrmeyer type and study the degree of approximation in terms of the mixed modulus of smoothness.

2. CONSTRUCTION OF THE BIVARIATE GENERALIZED BERNSTEIN-DURRMEYER TYPE
OPERATORS

Let $\mathbb{D} = [-1, 1] \times [-1, 1]$, $(x, y) \in \mathbb{D}$, $n, m \in \mathbb{N}$ and f defined on the interval $C(\mathbb{D})$. We define the linear positive operators $D_{n,m}(f; x, y)$ in the following way:

$$D_{n,m}(f; x, y) = \frac{n+1}{2} \frac{m+1}{2} \sum_{k=0}^n \sum_{j=0}^m \phi_{n,m}^{k,j}(x, y) \int_{-1}^1 \int_{-1}^1 \phi_{n,m}^{k,j}(t, u) f(t, u) dt du \quad (2.3)$$

where

$$\phi_{n,m}^{k,j}(x, y) = \varphi_n^k(x) \varphi_m^j(y)$$

and

$$\varphi_n^k(x) = \frac{1}{2^n} \binom{n}{k} (1+x)^k (1-x)^{n-k}.$$

Lemma 2.1. *For $\forall (x, y) \in \mathbb{D}$ and $\forall n, m \in \mathbb{N}$, Bernstein-Durrmeyer operators (2.3) satisfy the following equalities:*

$$D_{n,m}(1; x, y) = 1 \quad (2.4)$$

$$D_{n,m}(t; x, y) = x - \frac{2x}{n+2}$$

$$D_{n,m}(u; x, y) = y - \frac{2y}{m+2}$$

$$\begin{aligned} D_{n,m}(t^2 + u^2; x, y) &= x^2 - \frac{(6n+6)x^2 - 4nx}{(n+2)(n+3)} + \frac{2-2n}{(n+2)(n+3)} \\ &+ y^2 - \frac{(6m+6)y^2 - 4my}{(m+2)(m+3)} + \frac{2-2m}{(m+2)(m+3)} \end{aligned} \quad (2.5)$$

$$\begin{aligned} D_{n,m}(t^3 + u^3; x, y) &= x^3 - \frac{12n^2 + 24n + 24}{(n+2)(n+3)(n+4)} x^3 + \frac{6n^2 + 6n}{(n+2)(n+3)(n+4)} x \\ &+ \frac{12n + 48}{(n+2)(n+3)(n+4)} + y^3 - \frac{12m^2 + 24m + 24}{(m+2)(m+3)(m+4)} y^3 \\ &+ \frac{6m^2 + 6m}{(m+2)(m+3)(m+4)} y + \frac{12m + 48}{(m+2)(m+3)(m+4)} \end{aligned}$$

$$\begin{aligned} D_{n,m}(t^4 + u^4; x, y) &= x^4 - \frac{20n^3 + 60n^2 + 160n + 120}{(n+2)(n+3)(n+4)(n+5)} x^4 + \frac{12n^3 - 16n^2 + 4n}{(n+2)(n+3)(n+4)(n+5)} x^2 \\ &+ \frac{-4n^3 - 16n^2 + 32n}{(n+2)(n+3)(n+4)(n+5)} x + y^4 - \frac{20m^3 + 60m^2 + 160m + 120}{(m+2)(m+3)(m+4)(m+5)} y^4 \\ &+ \frac{12m^3 - 16m^2 + 4m}{(m+2)(m+3)(m+4)(m+5)} y^2 + \frac{-4m^3 - 16m^2 + 32m}{(m+2)(m+3)(m+4)(m+5)} y. \end{aligned}$$

From Lemma 2.1, we obtained the following lemma.

Lemma 2.2. *If the operator $D_{n,m}$ is defined by (2.3), then for $\forall(x, y) \in \mathbb{D}$ and $n, m \in \mathbb{N}$*

$$D_{n,m}((t-x)^2; x, y) = \frac{(-2n+6)x^2 + 4nx + 2 - 2n}{(n+2)(n+3)} \tag{2.6}$$

$$D_{n,m}((u-y)^2; x, y) = \frac{(-2m+6)y^2 + 4my + 2 - 2m}{(m+2)(m+3)} \tag{2.7}$$

$$D_{n,m}((t-x)^4; x, y) = \frac{72n^3 + 852n^2 + 1916n + 1680}{(n+2)(n+3)(n+4)(n+5)}x^4 + \frac{24n}{(n+2)(n+3)}x^3 + \frac{-24n^3 - 272n^2 - 830n + 840}{(n+2)(n+3)(n+4)(n+5)}x^2 + \frac{-4n^3 - 64n^2 - 464n - 960}{(n+2)(n+3)(n+4)(n+5)}x$$

$$D_{n,m}((u-y)^4; x, y) = \frac{72m^3 + 852m^2 + 1916m + 1680}{(m+2)(m+3)(m+4)(m+5)}y^4 + \frac{24m}{(m+2)(m+3)}y^3 + \frac{-24m^3 - 272m^2 - 830m + 840}{(m+2)(m+3)(m+4)(m+5)}y^2 + \frac{-4m^3 - 64m^2 - 464m - 960}{(m+2)(m+3)(m+4)(m+5)}y.$$

Let $C(\mathbb{D})$ is a continuous functions space on the $\mathbb{D} = [-1, 1] \times [-1, 1]$. $C(\mathbb{D})$ is a linear normed space with the norm

$$\|f\|_{C(\mathbb{D})} = \max_{x \in [-1,1] \times [-1,1]} |f(x, y)|.$$

If $f_{n,m}$ is a sequence on the space $C(\mathbb{D})$, for $f \in C(\mathbb{D})$

$$\lim_{n,m \rightarrow \infty} \|f_{n,m} - f\| = 0,$$

then it is called uniformly convergence to the function f .

Lemma 2.3. *Let $n \in \mathbb{N}$, for every fixed $x_0 \in [-1, 1]$, there exists a positive constant $M_1(x_0)$ such that $D_{n,n}((t-x_0)^4; x_0, y) \leq M_1(x_0)n^{-1}$.*

Theorem 2.1. *If $T_{n,m}$ is a sequence of linear positive operators satisfying the conditions*

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}(1; x, y) - 1\|_{C(\mathbb{X})} = 0,$$

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}((t-x); x, y) - x\|_{C(\mathbb{X})} = 0,$$

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}((u-y); x, y) - y\|_{C(\mathbb{X})} = 0,$$

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}(t^2 + u^2; x, y) - (x^2 + y^2)\|_{C(\mathbb{X})} = 0,$$

then for any function $f \in C(\mathbb{X})$, which is bounded in \mathbb{R}^2 and \mathbb{X} is a compact set,

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}(f; x, y) - f(x, y)\|_{C(\mathbb{X})} = 0.$$

In the following theorem we show that the linear positive operator $D_{n,m}$ converges to f uniformly with the help of Theorem 2.1 given by Volkov [18].

Theorem 2.2. *Let $f \in C(\mathbb{D})$, the operators $D_{n,m}$ defined by (2.3) converge uniformly to f on $\mathbb{D} \subset \mathbb{R}^2$ as $n, m \rightarrow \infty$.*

Proof. From (2.4)-(2.5), we obtain

$$\begin{aligned}\lim_{n,m \rightarrow \infty} \|D_{n,m}(1; x, y) - 1\|_{C(\mathbb{D})} &= 0, \\ \lim_{n,m \rightarrow \infty} \|D_{n,m}((t-x); x, y) - x\|_{C(\mathbb{D})} &= 0, \\ \lim_{n,m \rightarrow \infty} \|D_{n,m}((u-y); x, y) - y\|_{C(\mathbb{D})} &= 0, \\ \lim_{n,m \rightarrow \infty} \|D_{n,m}(t^2 + u^2; x, y) - (x^2 + y^2)\|_{C(\mathbb{D})} &= 0.\end{aligned}$$

The proof is obvious from Volkov's Theorem.

2.1. Degree of Approximation by $D_{n,m}$.

Definition 2.1. *Let $f \in C(\mathbb{D})$ be a continuous function and δ a positive number. For $x, y \in \mathbb{D}$, the full continuity modulus of the function $f(x, y)$ is*

$$\omega(f; \delta) = \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2} \leq \delta} |f(x_1, y_1) - f(x_2, y_2)|$$

and its partial continuity moduli with respect to x and y are defined by

$$\begin{aligned}\omega^{(1)}(f; \delta) &= \max_{-1 \leq y \leq 1} \max_{|x_1-x_2| \leq \delta} |f(x_1, y) - f(x_2, y)| \\ \omega^{(2)}(f; \delta) &= \max_{-1 \leq x \leq 1} \max_{|y_1-y_2| \leq \delta} |f(x, y_1) - f(x, y_2)|.\end{aligned}$$

It is also known that $\lim_{\delta \rightarrow 0} \omega(f; \delta) = 0$ and $\omega(f; \lambda\delta) \leq (\lambda + 1)\omega(f; \delta)$ for any $\lambda \geq 0$. The same properties are satisfied by partial continuity moduli.

Theorem 2.3. *Let $f \in C(\mathbb{D})$, the following inequalities hold:*

$$\|D_{n,m}(f; x, y) - f\|_{C(\mathbb{D})} \leq 3 \left(\omega^{(1)} \left(f; \frac{1}{\sqrt{n}} \right) + \omega^{(2)} \left(f; \frac{1}{\sqrt{n}} \right) \right) \quad (2.8)$$

$$\|D_{n,m}(f; x, y) - f\|_{C(\mathbb{D})} \leq 3\omega \left(f; \sqrt{\frac{1}{n} + \frac{1}{m}} \right). \quad (2.9)$$

Proof. From (2.3)-(2.4) and using the properties of the modulus of continuity we obtain

$$\begin{aligned} |D_{n,m}(f; x, y) - f(x, y)| &\leq |D_{n,m}(f(t, u) - f(t, y); x, y)| + |D_{n,m}(f(t, y) - f(x, y); x, y)| \\ &\leq D_{n,m}(|f(t, u) - f(t, y)|) + D_{n,m}(|f(t, y) - f(x, y)|) \\ &\leq \omega^{(1)}(f; \delta_n) \left\{ 1 + \frac{1}{\delta_n} \frac{n+1}{2} \sum_{k=0}^n \varphi_n^k(x) \int_{-1}^1 |t-x| \varphi_n^k(t) dt \right\} \\ &\quad + \omega^{(2)}(f; \delta_m) \left\{ 1 + \frac{1}{\delta_m} \frac{m+1}{2} \sum_{j=0}^m \varphi_m^j(y) \int_{-1}^1 |u-y| \varphi_m^j(u) du \right\} \end{aligned}$$

where δ_n, δ_m are the sequences which tend to zero as $n, m \rightarrow \infty$. Applying the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} |D_{n,m}(f; x, y) - f(x, y)| &\leq \omega^{(1)}(f; \delta_n) \left\{ 1 + \frac{1}{\delta_n} \frac{n+1}{2} \sum_{k=0}^n \varphi_n^k(x) \left(\int_{-1}^1 (t-x)^2 \varphi_n^k(t) dt \right)^{1/2} \left(\int_{-1}^1 \varphi_n^k(t) dt \right)^{1/2} \right\} \\ &\quad + \omega^{(2)}(f; \delta_m) \left\{ 1 + \frac{1}{\delta_m} \frac{m+1}{2} \sum_{j=0}^m \varphi_m^j(y) \left(\int_{-1}^1 (u-y)^2 \varphi_m^j(u) du \right)^{1/2} \left(\int_{-1}^1 \varphi_m^j(u) du \right)^{1/2} \right\}. \end{aligned}$$

Hence we get

$$\begin{aligned} |D_{n,m}(f; x, y) - f(x, y)| &\leq \omega^{(1)}(f; \delta_n) \left\{ 1 + \frac{1}{\delta_n} \frac{n+1}{2} \left(\sum_{k=0}^n \varphi_n^k(x) \right)^{1/2} \left(\int_{-1}^1 (t-x)^2 \varphi_n^k(t) dt \right)^{1/2} \right\} \\ &\quad + \omega^{(2)}(f; \delta_m) \left\{ 1 + \frac{1}{\delta_m} \frac{m+1}{2} \left(\sum_{j=0}^m \varphi_m^j(y) \right)^{1/2} \left(\int_{-1}^1 (u-y)^2 \varphi_m^j(u) du \right)^{1/2} \right\} \\ &= \omega^{(1)}(f; \delta_n) \left\{ 1 + \frac{1}{\delta_n} (D_{n,m}((t-x)^2; x, y))^{1/2} \right\} + \omega^{(2)}(f; \delta_m) \left\{ 1 + \frac{1}{\delta_m} (D_{n,m}((u-y)^2; x, y))^{1/2} \right\}. \end{aligned}$$

From (2.6) and (2.7), we obtain (2.8). Using (2.3), (2.4) and letting

$$\delta = \sqrt{(t-x)^2 + (u-y)^2}$$

we have

$$|f(t, u) - f(x, y)| \leq \omega(f; \delta_{nm}) \left(\frac{\sqrt{(t-x)^2 + (u-y)^2}}{\delta_{nm}} + 1 \right).$$

Hence, we obtain

$$\begin{aligned}
|D_{n,m}(f; x, y) - f(x, y)| &\leq D_{n,m}(|f(t, u) - f(x, y)|; x, y) \\
&\leq \omega(f; \delta_{nm}) \left\{ 1 + \frac{1}{\delta_{nm}} D_{n,m} \left(\sqrt{(t-x)^2 + (u-y)^2}; x, y \right) \right\} \\
&\leq \omega(f; \delta_{nm}) \left\{ 1 + \frac{1}{\delta_{nm}} \frac{n+1}{2} \frac{m+1}{2} \sum_{k=0}^n \sum_{j=0}^m \phi_{n,m}^{k,j}(x, y) \right. \\
&\quad \left. \int_{-1}^1 \int_{-1}^1 \left(\sqrt{(t-x)^2 + (u-y)^2} \right) \phi_{n,m}^{k,j}(t, u) dt du \right\}
\end{aligned}$$

applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
|D_{n,m}(f; x, y) - f(x, y)| &\leq \omega(f; \delta_{nm}) \left\{ 1 + \frac{1}{\delta_{nm}} \left(\frac{n+1}{2} \frac{m+1}{2} \sum_{k=0}^n \sum_{j=0}^m \phi_{n,m}^{k,j}(x, y) \right. \right. \\
&\quad \left. \left. \int_{-1}^1 \int_{-1}^1 \left((t-x)^2 + (u-y)^2 \right)^2 \phi_{n,m}^{k,j}(t, u) dt du \right)^{1/2} \right\} \\
&\leq \omega(f; \delta_{nm}) \left\{ 1 + \frac{1}{\delta_{nm}} \left(D_{n,m} \left((t-x)^2 + (u-y)^2; x, y \right) \right)^{1/2} \right\}.
\end{aligned}$$

With (2.6) and (2.7) we get desired result (2.9).

Now, we give the order of approximation using Lipschitz function and Peetre's K- functional.

Corollary 2.1. *If f additionally satisfies a Lipschitz condition*

$$|f(x_1, y_1) - f(x_2, y_2)| \leq K \left((x_1 - x_2)^2 + (y_1 - y_2)^2 \right)^{\alpha/2}, 0 < \alpha \leq 1$$

then the inequality

$$|D_{n,n}(f; x, y) - f(x, y)| \leq K' \left(\frac{1}{n} + \frac{1}{m} \right)^{\alpha/2},$$

where $K' = 3K$.

Corollary 2.2. *If f additionally satisfies a Lipschitz condition*

$$|f(x_1, y) - f(x_2, y)| \leq K_1 |x_1 - x_2|^{\alpha/2}$$

and

$$|f(x, y_1) - f(x, y_2)| \leq K_2 |y_1 - y_2|^{\gamma/2}$$

then the inequality

$$|D_{n,n}(f; x, y) - f(x, y)| \leq K'_1 \left(\frac{1}{n} \right)^{\alpha/2} + K'_2 \left(\frac{1}{m} \right)^{\alpha/2},$$

where $K'_1 = 3K_1$, $K'_2 = 3K_2$ holds.

Let $C^2(\mathbb{D})$ be the space of all functions $f \in C(\mathbb{D})$ such that $\frac{\partial^i f}{\partial x^i}, \frac{\partial^i f}{\partial y^i} \in C(\mathbb{D})$ for $i = 1, 2$. The norm on the space $C^2(\mathbb{D})$ is defined as

$$\|f\|_{C^2(\mathbb{D})} = \|f\|_{C(\mathbb{D})} + \sum_{i=1}^2 \left(\left\| \frac{\partial^i f}{\partial x^i} \right\|_{C(\mathbb{D})} + \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C(\mathbb{D})} \right).$$

Definition 2.2. Let $f \in C(\mathbb{D})$. The Peetre’s K -functional is defined by

$$\mathcal{K}(f; \delta) = \inf_{g \in C^2(\mathbb{D})} \left\{ \|f - g\|_{C(\mathbb{D})} + \delta \|g\|_{C^2(\mathbb{D})}, \delta > 0 \right\}. \tag{2.10}$$

Theorem 2.4. For the function $f \in C(\mathbb{D})$, we get

$$|D_{n,m}(f; x, y) - f(x, y)| \leq 2\mathcal{K}(f; \delta_{n,m}(x, y)),$$

where $\delta_{n,m}(x, y) = \max\left(\frac{2}{n+2}, \frac{2}{m+2}\right)$.

Proof. Let $g \in C^2(\mathbb{D})$ and $t, s \in [-1, 1]$. If we use Taylor’s theorem at point (x, y) for the function $g(t, s)$, we get

$$\begin{aligned} g(t, s) - g(x, y) &= \frac{\partial g(x, y)}{\partial x}(t - x) + \int_x^t (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du + \frac{\partial g(x, y)}{\partial y}(s - y) \\ &\quad + \int_y^s (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv. \end{aligned}$$

From Lemma 2.1, we have $D_{n,m}(t - x; x, y) = -\frac{2x}{n+2}$ ve $D_{n,m}(u - y; x, y) = -\frac{2y}{m+2}$. Applying the operator $D_{n,m}$ on the above equation, we obtain

$$\begin{aligned} D_{n,m}(g; x, y) - g(x, y) &= -\frac{2x}{n+2}g_x + D_{n,m}\left(\int_x^t (t - u) \frac{\partial^2 g(u, y)}{\partial u^2} du; x, y\right) \\ &\quad - \frac{2y}{m+2}g_y + D_{n,m}\left(\int_y^s (s - v) \frac{\partial^2 g(x, v)}{\partial v^2} dv; x, y\right). \end{aligned}$$

Hence,

$$\begin{aligned} &|D_{n,m}(g; x, y) - g(x, y)| \\ &\leq \left| \frac{2x}{n+2}g_x + \frac{2y}{m+2}g_y \right| + D_{n,m}\left(\left|\int_x^t |t - u| \left| \frac{\partial^2 g(u, y)}{\partial u^2} \right| du\right); x, y\right) \\ &\quad + D_{n,m}\left(\left|\int_y^s |s - v| \left| \frac{\partial^2 g(x, v)}{\partial v^2} \right| dv\right); x, y\right) \\ &\leq \left| \frac{2x}{n+2}g_x + \frac{2y}{m+2}g_y \right| + \frac{1}{2} \left| \frac{\partial^2 g}{\partial x^2} \right| |D_{n,m}((t - x)^2; x, y)| \\ &\quad + \frac{1}{2} \left| \frac{\partial^2 g(x, v)}{\partial v^2} \right| |D_{n,m}((u - y)^2; x, y)|. \end{aligned}$$

Using norm for $\forall x, y \in (D)$, we get

$$\begin{aligned}
\|D_{n,m}(g; x, y) - g(x, y)\|_{C(\mathbb{D})} &\leq \frac{2}{n+2} \|g_x\|_{C(\mathbb{D})} + \frac{2}{m+2} \|g_y\|_{C(\mathbb{D})} \\
&\quad + \frac{1}{n+2} \left\| \frac{\partial^2 g}{\partial x^2} \right\|_{C(\mathbb{D})} + \frac{1}{m+2} \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{C(\mathbb{D})} \\
&\leq \max\left(\frac{1}{n+2}, \frac{1}{m+2}\right) \left(\|g_x\|_{C(\mathbb{D})} + \|g_y\|_{C(\mathbb{D})} \right. \\
&\quad \left. + \left\| \frac{\partial^2 g}{\partial x^2} \right\|_{C(\mathbb{D})} + \left\| \frac{\partial^2 g}{\partial y^2} \right\|_{C(\mathbb{D})} \right) \\
&\leq \delta_{n,m} \|g\|_{C^2(\mathbb{D})},
\end{aligned}$$

where $\delta_{n,m} = \max\left(\frac{2}{n+2}, \frac{2}{m+2}\right)$. Since $D_{n,m}$ is a linear operator and for $\forall f \in C(\mathbb{D})$, $g \in C^2(\mathbb{D})$, we have

$$\begin{aligned}
\|D_{n,m}(f; x, y) - f(x, y)\|_{C(\mathbb{D})} &\leq \|D_{n,m}(f - g; x, y)\|_{C(\mathbb{D})} \\
&\quad + \|D_{n,m}(g; x, y) - g(x, y)\|_{C(\mathbb{D})} + \|f - g\|_{C(\mathbb{D})} \\
&\leq \|f - g\|_{C(\mathbb{D})} |D_{n,m}(1; x, y)| \\
&\quad + \|D_{n,m}(g; x, y) - g(x, y)\|_{C(\mathbb{D})} + \|f - g\|_{C(\mathbb{D})}.
\end{aligned}$$

Hence

$$\|D_{n,m}(f; x, y) - f(x, y)\|_{C(\mathbb{D})} \leq 2 \left(\|f - g\|_{C(\mathbb{D})} + \delta_{n,m} \|g\|_{C^2(\mathbb{D})} \right)$$

Taking the infimum on the right hand side, we get

$$|D_{n,m}(f; x, y) - f(x, y)| \leq 2\mathcal{K}(f; \delta_{n,m}(x, y)).$$

3. CONSTRUCTION OF GBS OPERATOR OF GENERALIZED BERNSTEIN-DURRMEYER TYPE

In 1934, Bögel introduced the term B -continuous and B -differentiable function and established important result for these functions [6]-[7]. In 1966, Dobrescu and Matei [10] gave some approximation properties for bivariate Bernstein polynomials using a generalized boolean sum. The Test function theorem is given by Badea et al. [4] for Bögel continuous functions. Sidharth et al. introduced GBS operators of Bernstein-Schurer-Kantorovich type and studied the degree of approximation by means of the mixed modulus of smoothness and the mixed Peetre's K -functional in [17].

In this section, we introduce Bernstein-Durrmeyer type GBS (Generalized Boolean Sum) operator by means of Bögel continuous functions which is more extensive than the space

of continuous functions. The degree of approximation for Bernstein-Durrmeyer type GBS operators are obtained by using the mixed modulus of smoothness and mixed K -functional.

Let X and Y be a compact real intervals and let $\Delta_{(x,y)}f [x_0, y_0; x, y]$ be mixed difference of f defined by

$$\Delta_{(x,y)}f [x_0, y_0; x, y] = f(x, y) - f(x, y_0) - f(x_0, y) + f(x_0, y_0)$$

for $(x, y), (x_0, y_0) \in X \times Y$. A function $f : X \times Y \rightarrow \mathbb{R}$ is called B -continuous (Bögel continuous) at $(x_0, y_0) \in X \times Y$, if

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \Delta_{(x,y)}f [x_0, y_0; x, y] = 0$$

for $(x, y) \in X \times Y$. Let the function $f : X \times Y \rightarrow \mathbb{R}$ if there exist $M > 0$ such that

$$|\Delta_{(x,y)}f [x_0, y_0; x, y]| \leq M$$

for every $(x, y), (x_0, y_0) \in X \times Y$, then the function f is defined by B -bounded (Bögel bounded) on $X \times Y$.

Throughout this paper $B_b(X \times Y)$ denotes all B -bounded functions on $X \times Y$ and $C_b(X \times Y)$ denotes B -continuous functions on $X \times Y$. As usual $B(X \times Y)$ and $C(X \times Y)$ predicate the space of all bounded functions and the space of all continuous functions on $X \times Y$.

The mixed modulus of smoothness of $f \in C_b(X \times Y)$ is defined by

$$\omega_{mixed}(f; \delta_1, \delta_2) := \sup \{ |\Delta_{(x,y)}f [x_0, y_0; x, y]| : |x - x_0| < \delta_1, |y - y_0| < \delta_2 \} \tag{3.11}$$

for $(x, y), (x_0, y_0) \in X \times Y$ and for any $(\delta_1, \delta_2) \in (0, \infty) \times (0, \infty)$ with $\omega_{mixed} : [0, \infty) \times [0, \infty) \rightarrow \mathbb{R}$.

In 1988-90's, Badea obtained the basic properties of the mixed modulus of smoothness ω_{mixed} and these properties are similar to usual modulus of continuity. Also; the mixed modulus of smoothness provide the next inequality for $\delta_1, \delta_2 > 0$

$$\omega_{mixed}(f; \lambda_1 \delta_1, \lambda_2 \delta_2) \leq (1 + \lambda_1)(1 + \lambda_2)\omega_{mixed}(f; \delta_1, \delta_2). \tag{3.12}$$

Let give the concept of Bögel differentiable function. A function $f : X \times Y \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ is called B -differentiable function at the point $(x_0, y_0) \in X \times Y$ if the limit

$$\lim_{(x,y) \rightarrow (x_0,y_0)} \frac{\Delta_{(x,y)}f [x_0, y_0; x, y]}{(x - x_0)(y - y_0)}$$

exists and is finite. The limit is call to be the B -differential of f at the point (x_0, y_0) and is denoted by $T_{xy}f(x_0, y_0) := T_B(f; x_0, y_0)$. The space of all B -differentiable functions is denoted by $T_B(X \times Y)$.

Let $f \in C_b(\mathbb{D})$, the mixed K -functional definition is given by

$$\begin{aligned} \mathcal{K}_{mixed}(f; t_1, t_2) = \inf_{g_1, g_2, h} \left\{ \|f - g_1 - g_2 - h\|_\infty + t_1 \left\| T_B^{2,0} g_1 \right\|_\infty + t_2 \left\| T_B^{0,2} g_2 \right\|_\infty \right. \\ \left. + t_1 t_2 \left\| T_B^{2,2} h \right\|_\infty \right\}, \end{aligned}$$

where $g_1 \in C_B^{2,0}$, $g_2 \in C_B^{0,2}$, $h \in C_B^{2,2}$ and for $0 \leq p, q \leq 2$ $C_B^{p,q}$ denotes the space of the functions $f \in C_b(\mathbb{D})$ with continuous mixed partial derivatives $T_B^{a,b} f$, $0 \leq a \leq p$, $0 \leq b \leq q$.

The partial derivatives are

$$T_x f(x_0, y_0) := T_B^{1,0}(f; x_0, y_0) = \lim_{x \rightarrow x_0} \frac{\Delta_x f([x_0, x]; y_0)}{(x - x_0)}$$

and

$$T_y f(x_0, y_0) := T_B^{0,1}(f; x_0, y_0) = \lim_{y \rightarrow y_0} \frac{\Delta_y f(x_0; [y_0, y])}{(y - y_0)}$$

where

$$\Delta_x f([x_0, x]; y_0) = f(x, y_0) - f(x_0, y_0)$$

and

$$\Delta_y f(x_0; [y_0, y]) = f(x_0, y) - f(x_0, y_0).$$

Definition 3.1. For $f \in C(\mathbb{D})$ and $m, n \in \mathbb{N}$, we define the Generalized Boolean Sum (GBS) operator of generalized Bernstein-Durrmeyer type operator $D_{n,m}$ as follows:

$$\begin{aligned} S_{n,m}(f(t, s); x, y) = \frac{n+1}{2} \frac{m+1}{2} \sum_{k=0}^n \sum_{j=0}^m \phi_{n,m}^{k,j}(x, y) \int_{-1}^1 \int_{-1}^1 \phi_{n,m}^{k,j}(t, u) \\ \times (f(t, y) + f(x, s) - f(t, s)) dt du, \end{aligned} \quad (3.13)$$

for $(x, y) \in \mathbb{D}$ where the operator $S_{n,m}$ is well defined on the space $C_b(\mathbb{D})$ and $f \in C_b(\mathbb{D})$.

3.1. Degree of Approximation by $S_{n,m}$.

Theorem 3.1. For every $f \in C_b(\mathbb{D})$, the operator (3.13) satisfy the following inequality at each point $(x, y) \in \mathbb{D}$

$$|S_{n,m}(f; x, y) - f(x, y)| \leq 9\omega_{mixed}\left(f; n^{-1/2}, m^{-1/2}\right).$$

Proof. Using the definition of $\omega_{mixed}(f; \delta_1, \delta_2)$ and for $\delta_1, \delta_2 > 0$ taking the inequality

$$\omega_{mixed}(f; \lambda_1 \delta_1, \lambda_2 \delta_2) \leq (1 + \lambda_1)(1 + \lambda_2)\omega_{mixed}(f; \delta_1, \delta_2)$$

we can write

$$\begin{aligned} |\Delta_{(x,y)} f [t, s; x, y]| &\leq \omega_{mixed} (f; |t - x|, |s - y|) \\ &\leq \left(1 + \frac{|t - x|}{\delta_1}\right) \left(1 + \frac{|s - y|}{\delta_2}\right) \omega_{mixed} (f; \delta_1, \delta_2) \end{aligned} \tag{3.14}$$

for every $(x, y), (t, s) \in \mathbb{D}$ and for any $(\delta_1, \delta_2) > 0$. From the definition of $\Delta_{(x,y)} f [t, s; x, y]$, we have

$$f(x, s) + f(t, y) - f(t, s) = f(x, y) - \Delta_{(x,y)} f [t, s; x, y]. \tag{3.15}$$

If we apply this equality the operator $D_{n,m}$ and take the definition $S_{n,m}$, we can write

$$S_{n,m} (f; x, y) = f(x, y)D_{n,m} (1; x, y) - D_{n,m} (\Delta_{(x,y)} f [t, s; x, y]; x, y).$$

From (2.4), we have $D_{n,m} (1; x, y) = 1$. Taking (3.14) into account and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |S_{n,m} (f; x, y) - f(x, y)| &\leq D_{n,m} (\Delta_{(x,y)} f [t, s; x, y]; x, y) \\ &\leq \left(D_{n,m} (1; x, y) + \delta_1^{-1} \sqrt{D_{n,m} ((t - x)^2; x, y)} \right. \\ &\quad \left. + \delta_2^{-1} \sqrt{D_{n,m} ((s - y)^2; x, y)} \right. \\ &\quad \left. + \delta_1^{-1} \delta_2^{-1} \sqrt{D_{n,m} ((t - x)^2; x, y) D_{n,m} ((s - y)^2; x, y)} \right) \\ &\quad \times \omega_{mixed} (f; \delta_1, \delta_2). \end{aligned}$$

From Lemma 2.2 and for every $(x, y) \in \mathbb{D}$, we have

$$D_{n,m} ((t - x)^2; x, y) \leq \frac{4}{n}$$

and

$$D_{n,m} ((u - y)^2; x, y) \leq \frac{4}{m}.$$

Therefore, choosing $\delta_1 = n^{-1/2}$ ve $\delta_2 = m^{-1/2}$ we get

$$|S_{n,m} (f; x, y) - f(x, y)| \leq 9\omega_{mixed} \left(f; n^{-1/2}, m^{-1/2} \right).$$

Theorem 3.2. *Let take $T_B f \in B(\mathbb{D})$ with the function $f \in T_b(\mathbb{D})$. Then, for every $(x, y) \in \mathbb{D}$, we get*

$$|S_{n,m} (f; x, y) - f(x, y)| \leq M. \left[\|T_B f\|_\infty + \omega_{mixed} \left(T_B f; n^{-1/2}, m^{-1/2} \right) \right] (nm)^{-1/2} \tag{3.16}$$

where M is any positive constant.

Proof. Let the function $f \in T_b(\mathbb{D})$. From [8], we have the identity

$$\Delta_{(x,y)}f [t, s; x, y] = (t - x)(s - y)T_Bf (\varsigma, \rho), \quad x < \varsigma < t, y < \rho < s. \quad (3.17)$$

From the definition $\Delta_{(x,y)}f [t, s; x, y]$ and applying T_Bf to each side of the equality (3.15), we get

$$T_Bf (\varsigma, \rho) = \Delta_{(x,y)}T_Bf (\varsigma, \rho) + T_Bf (\varsigma, y) + T_Bf (x, \rho) - T_Bf (x, y).$$

Taking $T_Bf \in B(\mathbb{D})$ and above equation into account, we can write

$$\begin{aligned} & |D_{n,m} (\Delta_{(x,y)}f [t, s; x, y]; x, y)| \\ &= |D_{n,m} ((t - x)(s - y)T_Bf (\varsigma, \rho); x, y)| \\ &\leq D_{n,m} (|t - x||s - y| |\Delta_{(x,y)}T_Bf (\varsigma, \rho)|; x, y) \\ &\quad + D_{n,m} (|t - x||s - y| (|T_Bf (\varsigma, y)| + |T_Bf (x, \rho)| + |T_Bf (x, y)|); x, y) \\ &\leq D_{n,m} (|t - x||s - y|\omega_{mixed}(T_Bf; |\varsigma - x|, |\rho - y|); x, y) \\ &\quad + 3 \|T_Bf\|_{\infty} D_{n,m} (|t - x||s - y|; x, y). \end{aligned}$$

Also, since the mixed modulus of smoothness ω_{mixed} is nondecreasing, we have

$$\begin{aligned} \omega_{mixed}(T_Bf; |\varsigma - x|, |\rho - y|) &\leq \omega_{mixed}(T_Bf; |t - x|, |s - y|) \\ &\leq (1 + \delta_1^{-1}|t - x|) (1 + \delta_2^{-1}|s - y|) \omega_{mixed}(f; \delta_1, \delta_2). \end{aligned}$$

Substituting in the above equality and applying the linearity of the operator $D_{n,m}$ and using the inequality of Cauchy-Schwarz, we get

$$\begin{aligned}
 |S_{n,m}(f; x, y) - f(x, y)| &= |D_{n,m}(\Delta_{(x,y)} f[t, s; x, y]; x, y)| \\
 &\leq 3 \|T_B f\|_\infty \sqrt{D_{n,m}((t-x)^2(s-y)^2; x, y)} \\
 &\quad + [D_{n,m}(|t-x||s-y|; x, y) \\
 &\quad + \delta_1^{-1} D_{n,m}((t-x)^2|s-y|; x, y) \\
 &\quad + \delta_2^{-1} D_{n,m}(|t-x|(s-y)^2; x, y) \\
 &\quad + \delta_1^{-1} \delta_2^{-1} D_{n,m}((t-x)^2(s-y)^2; x, y)] \omega_{mixed}(f; \delta_1, \delta_2) \\
 &\leq 3 \|T_B f\|_\infty \sqrt{D_{n,m}((t-x)^2(s-y)^2; x, y)} \\
 &\quad + \left[\sqrt{D_{n,m}((t-x)^2(s-y)^2; x, y)} \right. \\
 &\quad + \delta_1^{-1} \sqrt{D_{n,m}((t-x)^4(s-y)^2; x, y)} \\
 &\quad + \delta_2^{-1} \sqrt{D_{n,m}((t-x)^2(s-y)^4; x, y)} \\
 &\quad \left. + \delta_1^{-1} \delta_2^{-1} D_{n,m}((t-x)^2(s-y)^2; x, y) \right] \omega_{mixed}(f; \delta_1, \delta_2).
 \end{aligned}$$

From Lemma 2.2, we have

$$D_{n,m}((t-x)^2; x, y) \leq \frac{4}{n}$$

and

$$D_{n,m}((u-y)^2; x, y) \leq \frac{4}{m}.$$

For $(x, y), (t, s) \in \mathbb{D}$, $p, q \in 1, 2$ and taking

$$D_{n,m}((t-x)^{2p}(s-y)^{2q}; x, y) = D_{n,m}((t-x)^{2p}; x, y) D_{n,m}((s-y)^{2q}; x, y)$$

into account, choosing $\delta_1 = n^{-1/2}$ ve $\delta_2 = m^{-1/2}$, we get the desired result (3.16).

In the following theorem, we evaluate the order of approximation of the sequence $\{S_{n,m}(f)\}$ to the function $f \in C_b(\mathbb{D})$ in terms of mixed K -functional.

Theorem 3.3. *Let the operator $S_{n,m}$ given in (3.13). Then, for every $f \in C_b(\mathbb{D})$ we get*

$$|S_{n,m}(f; x, y) - f(x, y)| \leq 2\mathcal{K}_{mixed}\left(f; \frac{2}{n}, \frac{2}{m}\right). \tag{3.18}$$

Proof. For the function $g_1 \in C_B^{2,0}(\mathbb{D})$ using Taylor formula, we get

$$g_1(t, s) = g_1(x, y) + (t-x)T_B^{1,0}g_1(x, y) + \int_x^t (t-u)T_B^{2,0}g_1(u, y)du$$

([6]). Since the operator $S_{n,m}$ reproduces linear functions

$$S_{n,m}(g_1; x, y) = g_1(x, y) + S_{n,m} \left(\int_x^t (t-u) T_B^{2,0} g_1(u, y) du; x, y \right)$$

and the definition of $S_{n,m}$ operator for $g_1 \in C_B^{2,0}(\mathbb{D})$, we get

$$\begin{aligned} |S_{n,m}(g_1; x, y) - g_1(x, y)| &= \left| D_{n,m} \left(\int_x^t (t-u) \left[T_B^{2,0} g_1(u, y) - T_B^{2,0} g_1(u, s) \right] du; x, y \right) \right| \\ &\leq D_{n,m} \left(\left| \int_x^t |t-u| \left| T_B^{2,0} g_1(u, y) - T_B^{2,0} g_1(u, s) \right| du; x, y \right) \right) \\ &\leq \left\| T_B^{2,0} g_1 \right\|_{\infty} D_{n,m}((t-x)^2; x, y) \\ &< \left\| T_B^{2,0} g_1 \right\|_{\infty} \cdot \frac{4}{n}. \end{aligned}$$

For $g_2 \in C_B^{0,2}(\mathbb{D})$,

$$\begin{aligned} |S_{n,m}(g_2; x, y) - g_2(x, y)| &= \left| D_{n,m} \left(\int_y^s (s-v) \left[T_B^{0,2} g_2(v, y) - T_B^{0,2} g_2(v, s) \right] dv; x, y \right) \right| \\ &\leq D_{n,m} \left(\left| \int_y^s |s-v| \left| T_B^{0,2} g_2(v, y) - T_B^{0,2} g_2(v, s) \right| dv; x, y \right) \right) \\ &\leq \left\| T_B^{0,2} g_2 \right\|_{\infty} D_{n,m}((s-y)^2; x, y) \\ &< \left\| T_B^{0,2} g_2 \right\|_{\infty} \cdot \frac{4}{m}. \end{aligned}$$

For $h \in C_B^{2,2}(\mathbb{D})$, we get

$$\begin{aligned} h(t, s) &= h(x, y) + (t-x) T_B^{1,0} h(x, y) + (s-y) T_B^{0,1} h(x, y) + (t-x)(s-y) T_B^{1,1} h(x, y) \\ &\quad + \int_x^t (t-u) T_B^{2,0} h(u, y) du + \int_y^s (s-v) T_B^{0,2} h(x, v) dv \\ &\quad + \int_x^t (s-y)(t-u) T_B^{2,1} h(u, y) du + \int_y^s (t-x)(s-v) T_B^{1,2} h(x, v) dv \\ &\quad + \int_x^t \int_y^s (t-u)(s-v) T_B^{2,2} h(u, v) dv du. \end{aligned}$$

Since $S_{n,m}((t-x); x, y) = 0$, $S_{n,m}((s-y); x, y) = 0$ and the definition of the operator $S_{n,m}$

$$\begin{aligned} |S_{n,m}(h; x, y) - h(x, y)| &\leq \left| D_{n,m} \left(\int_x^t \int_y^s (t-u)(s-v) T_B^{2,2} h(u, v) dv du; x, y \right) \right| \\ &\leq D_{n,m} \left(\left| \int_x^t \int_y^s (t-u)(s-v) T_B^{2,2} h(u, v) dv du \right|; x, y \right) \\ &\leq D_{n,m} \left(\int_x^t \int_y^s |t-u||s-v| \left| T_B^{2,2} h(u, v) \right| dv du; x, y \right) \\ &\leq \frac{1}{4} \left\| T_B^{2,2} h \right\|_{\infty} D_{n,m}((t-x)^2(s-y)^2; x, y) \\ &\leq 4 \left\| T_B^{2,2} h \right\|_{\infty} \frac{1}{n} \frac{1}{m}. \end{aligned}$$

Therefore, we get

$$\begin{aligned}
 |S_{n,m}(f; x, y) - f(x, y)| &\leq |(f - g_1 - g_2 - h)(x, y)| + |(g_1 - S_{n,m}g_1)(x, y)| \\
 &\quad + |(g_2 - S_{n,m}g_2)(x, y)| + |(h - S_{n,m}h)(x, y)| \\
 &\quad + |S_{n,m}((f - g_1 - g_2 - h); x, y)| \\
 &\leq 2\|f - g_1 - g_2 - h\|_\infty + 4\left\|T_B^{2,0}g_1\right\|_\infty \frac{1}{n} \\
 &\quad + 4\left\|T_B^{0,2}g_2\right\|_\infty \frac{1}{m} + 4\left\|T_B^{2,2}h\right\|_\infty \frac{1}{n} \frac{1}{m}
 \end{aligned}$$

for $f \in C_b(\mathbb{D})$. Since the definition of the mixed K -functional and taking the infimum over all $g_1 \in C_B^{2,0}(\mathbb{D})$, $g_2 \in C_B^{0,2}(\mathbb{D})$, $h \in C_B^{2,2}(\mathbb{D})$ we get the desired result (3.18).

3.2. Numerical Examples. The convergence of the operators by illustrative graphics in Maple to certain functions for two dimensional cases are given and some numerical values are calculated as follows. For $n, m = 1, 2, 5, 10$ and the function $f(x, y) = x^2y + y^2$, the convergence of the operators $D_{n,m}$ is shown in Fig 1. For $n, m = 1, 2, 5, 10$ and the function $f(x, y) = 1 - x^3 + y^3$, the convergence of the operators $D_{n,m}$ is shown in Fig 2. It is seen that if the values of n, m increase, the convergence of $D_{n,m}$ to the function f becomes better. Finally, one can see that the convergence of the GBS operator $S_{n,m}$ has better approach than the operator $D_{n,m}$ for the function $f(x, y) = (1 + x + y)\sin(x + y)$ in Fig 3.

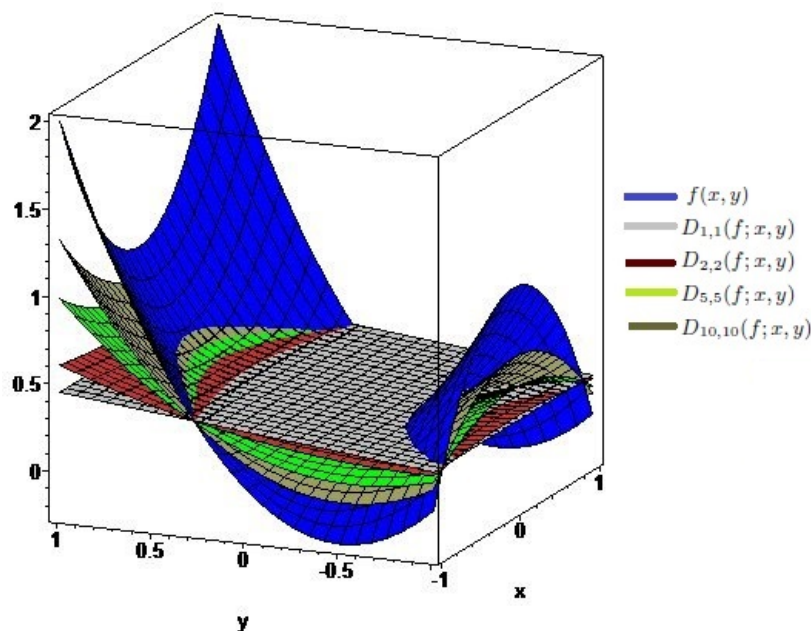


FIGURE 1. The convergence of the $D_{n,m}$ operators for $f(x, y) = x^2y + y^2$ and $n, m = 1, 2, 5, 10$.

TABLE 1. Mean errors of figure 1

(n,m)	$maximize D_{n,m}(x,y) - f(x,y) $
$n,m=5$	1,0204
$n,m=15$	0,5113
$n,m=25$	0,3390
$n,m=50$	0,1836
$n,m=100$	0,0957
$n,m=150$	0,0647

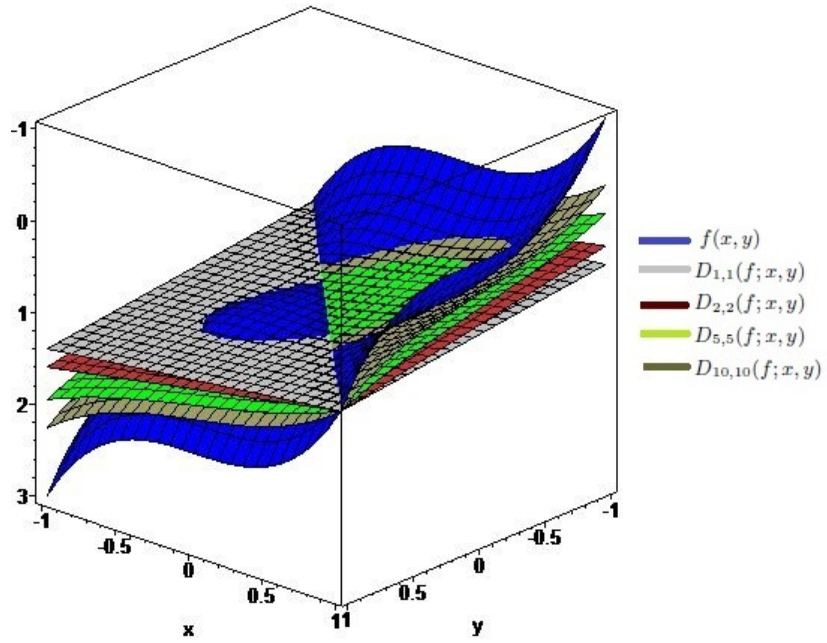


FIGURE 2. The convergence of the $D_{n,m}$ operators for $f(x,y) = 1 - x^3 + y^3$ and $n, m = 1, 2, 5, 10$.

TABLE 2. Mean errors of figure 2

(n,m)	maximize $ D_{n,m}(x, y) - f(x, y) $
n,m=5	1,0476
n,m=10	0,7362
n,m=50	0,2139
n,m=100	0,1131
n,m=500	0,0066

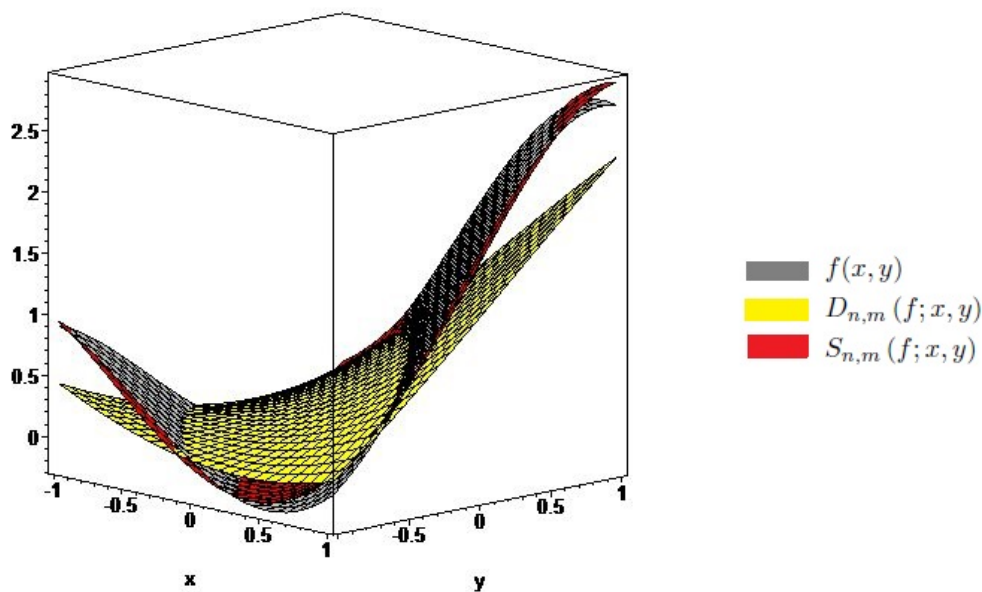


FIGURE 3. The convergence of the $D_{n,m}$ operators and the $S_{n,m}$ operators for $f(x, y) = (1 + x + y)\sin(x + y)$ and $n, m = 5$.

Acknowledgments. The authors would like to thank the referee for some useful comments and their helpful suggestions that have improved the quality of this paper.

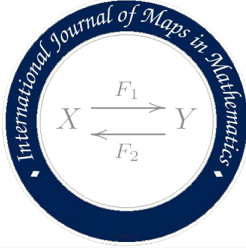
REFERENCES

[1] Abel, U., Gupta, V. & Mohapatra, R. N. (2008). Local approximation by a variant of Bernstein-Durrmeyer operators. *Nonlinear Analysis*, **68**, 3372-3381.


- [2] Acar, T. & Kajla, A. (2018). Degree of approximation for bivariate generalized Bernstein type operators. *Results Math.*, doi:10.1007/s00025-018-0838-1.
- [3] Aydın Arı, D., & Kırcı Serenbay, S. (2019). Approximation by a generalization of the Jakimovski -Leviatan operators. *Filomat*, 33(8).
- [4] Badea, C., Badea, I., & Gonska, H.H. (1986). A test function theorem and approximation by pseudopolynomials. *Bull. Aust. Math. Soc.* 34, 53–64.
- [5] Bernstein, S. (1912). Démonstration du théorème de Weierstrass, fondé sur le calcul des probabilités. *Commun. Soc. Math. Kharkov*, 13(2), 1-2.
- [6] Bögel, K. (1934). Mehrdimensionale differentiation von funktionen mehrerer veranderlicher. *J. Reine Angew. Math.*, 170, 197-217.
- [7] Bögel, K. (1935). Über die mehrdimensionale differentiation integration und beschränkte Variation. *J. Reine Angew. Math.*, 173, 5-29.
- [8] Bögel, K. (1962–1963). Über die mehrdimensionale differentiation. *Jber. Deutsch. Math-Verein.* 65, 45–71.
- [9] Cárdenas-Morales, D., & Gupta, V. (2014). Two families of Bernstein-Durrmeyer type operators. *Applied Mathematics and Computation*, 248, 342-353.
- [10] Dobrescu, E. & Matei, I. (1966). The approximation by Berntein type polynomials of bidimensionally continuous functions. *An. Univ. Timisoara Ser. Sti. Mat.-Fiz.*, 4, 85–90 (Romanian).
- [11] Durrmeyer, J. L. (1967). Une formule d'inversion de la transforméé de Laplace: applications á la théorie des moments, Thése de 3e cycle. Faculté des Sciences de l'Université de Paris.
- [12] Gurdek, M., Rempulska L., & Skorupa, M. (1999). The Baskakov operators for functions of two variables. *Colect. Math.* 50(3), 289-302.
- [13] Gupta, V. (2014). Different Durrmeyer Variants of Baskakov Operators. *Topics in Mathematical Analysis and Applications*, 94, 419-446.
- [14] İzgi, A. (2012). Approximation by a class of new type Bernstein polynomials of one two variables. *Global Journal of Pure and Applied Mathematics*, 8(5), 55-71.
- [15] Martinez, F. L. (1989). Some properties of two-dimensional Bernstein polynomials. *Journal of approximation theory*, 59(3), 300-306.
- [16] Pop, O.T. & Barbosu, D. (2008). GBS operators of Durrmeyer–Stancu type. *Miskolc Math. Notes*, 9, 53–60.
- [17] Sidharth, M., Ispir, N. & Agrawal, P.N. (2016). Approximation of B-continuous and B-differentiable functions by GBS operators of q-Bernstein–Schurer–Stancu type. *Turk. J. Math.*, 40(6), 1298–1315.
- [18] Volkov, V. I. (1957). On the convergence of sequences of linear positive operators in the space of continuous functions of two variables. *Dokl. Akad. Nauk SSSR*, 115, 17-19.

DEPARTMENT OF MATHEMATICS, ARTS AND SCIENCE FACULTY, HARRAN UNIVERSITY, SANLIURFA 63120,
TURKEY

DEPARTMENT OF MATHEMATICS, ARTS AND SCIENCE FACULTY, HARRAN UNIVERSITY, SANLIURFA 63120,
TURKEY



RULED SURFACES WITH THE BASE RECTIFYING CURVES IN EUCLIDEAN 3-SPACE

BEYHAN YILMAZ  * AND YUSUF YAYLI 

ABSTRACT. There are many studies about rectifying curves. In this present study, we examine the ruled surfaces that have rectifying curves as base curves. We say that co-centrode curves defined by Chen and Dillen are the parameter curves for the special case $u = 1$ on the ruled surface with a base rectifying curve. Also, we answer the question when does the parameter curves of the surface are geodesic.

Keywords: Rectifying curve, Ruled surface, Modified darbox vector

2010 Mathematics Subject Classification: Primary 53A10; Secondary 14Q05, 14H50, 14J26, 14Q10.

1. INTRODUCTION

The curves are the fundamental structure of differential geometry. In this study, we examine rectifying curves which are one of the subfamilies of the curves in Euclidean 3-space. A regular curve $\alpha(s)$ is called a rectifying curve, if its position vector always lies its rectifying plane. So, the position vector of a rectifying curve satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s)$$

*Received:*2021.01.18

*Revised:*2021.05.21

*Accepted:*2021.08.12

* Corresponding author

Beyhan Yılmaz; beyhanyilmaz@ksu.edu.tr; <https://orcid.org/0000-0002-5091-3487>

Yusuf Yaylı; yayli@science.ankara.edu.tr; <https://orcid.org/0000-0003-4398-3855>

for differentiable functions λ and μ according to arc length parameter s . The notion of rectifying curves is introduced by B.Y. Chen in [1]. Also B.Y. Chen and Dillen show that there exists a relationship between the rectifying curves and the centrodes [2].

In the differential geometry of a regular curve, the curvature functions κ and τ of a regular curve play an important role to determine what is the type of the curve. One of the most interesting characteristics of rectifying curves is that the ratio of their torsion and curvature is a non-constant linear function of the arc length parameter s .

There are many studies about rectifying curves. K. Ilarslan et.al in [4, 5] introduce the rectifying curves in the Minkowski 3–space. Also E. Özbey et.al study rectifying curves in dual Lorentzian space and they show that rectifying dual Lorentzian curves can be stated by the aid of dual unit spherical curves in [7]. In recent years, the rectifying curves from various viewpoints have been studied in Pseudo-Galilean space and three-dimensional sphere in [6, 8].

In this paper, we define the ruled surface whose the base curve is a rectifying curve by using modified Darboux vector field in Euclidean 3–space. So, we examine the relationship between rectifying curves and ruled surfaces. In [2], Chen and Dillen introduce co-centrode curves. Accordingly, we say that co-centrode curves are the parameter curve for the special case $u = 1$ on this ruled surface. Also, we give the hypothesis that the curve whose the base curve for the given surface is a rectifying curve. Finally, we investigate the connection between the rectifying curve and the parameter curves of the surface which are the geodesic. We study the whole theory for the any orthonormal frame and also examine for special cases.

2. PRELIMINARIES

Let $\alpha : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be an arbitrary curve in three dimensional Euclidean space. A moving orthonormal frame is defined as $\{N_1, N_2, N_3\}$ in the \mathbb{E}^3 along to curve α . Derivative of the frame is given by

$$\begin{bmatrix} N_1'(s) \\ N_2'(s) \\ N_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & \kappa_3(s) \\ -\kappa_2(s) & -\kappa_3(s) & 0 \end{bmatrix} \begin{bmatrix} N_1(s) \\ N_2(s) \\ N_3(s) \end{bmatrix} \quad (2.1)$$

where $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the curvatures of the curve α . This any orthonormal frame encompasses some other frames. So, this frame is substantially important in terms of generality. For example, if we take $N_1 = T, N_2 = N, N_3 = B, \kappa_1 = \kappa, \kappa_2 = 0$ and $\kappa_3 = \tau$, above orthonormal frame coincides with the Serret Frenet frame. Also, if we take

$N_1 = T, N_2 = N_1, N_3 = N_2, \kappa_1 = k_1, \kappa_2 = 0$ and $\kappa_3 = k_3$, we have Bishop frame. Similarly, if we take $N_1 = T, N_2 = Y, N_3 = Z, \kappa_1 = k_g, \kappa_2 = k_n$ and $\kappa_3 = \tau_r$, orthonormal frame coincides with the Darboux frame on a curve. Using the equations $N_1 = N, N_2 = C, N_3 = W, \kappa_1 = f, \kappa_2 = 0$ and $\kappa_3 = g$, we get the alternative moving frame defined by Uzunoglu et.al in [9].

In the Euclidean space, the Darboux vector may be interpreted kinematically as the direction of the instantaneous axis of rotation in the moving trihedron. The direction of the Darboux vector is the instantaneous axis of rotation. In terms of the moving frame apparatus, the general Darboux vector field D can be expressed as

$$D = \kappa_3(s) N_1(s) - \kappa_2(s) N_2(s) + \kappa_1(s) N_3(s) \tag{2.2}$$

and it provides the following symmetrical properties

$$D \times N_1(s) = N_1'(s) \tag{2.3}$$

$$D \times N_2(s) = N_2'(s)$$

$$D \times N_3(s) = N_3'(s)$$

where \times is the wedge product in Euclidean space \mathbb{E}^3 .

Izumiya and Takeuchi define the modified Darboux vector field as follows

$$\bar{D} = \left(\frac{\tau}{\kappa}\right)(s) T(s) + B(s)$$

with $\kappa(s) \neq 0$ and another modified Darboux vector field is defined as $\tilde{D} = T(s) + \left(\frac{\kappa}{\tau}\right)(s) B(s)$ with $\tau(s) \neq 0$ [3].

In [1], Chen proves that the curve $\alpha(s)$ is congruent to a rectifying curve if and only if the ratio $\frac{\tau}{\kappa}$ with $\kappa > 0$ is a non-constant linear according to arc length parameter s in \mathbb{E}^3 .

3. RULED SURFACES WITH THE BASE RECTIFYING CURVES IN EUCLIDEAN 3-SPACE

In this section, we examine the relationship between rectifying curves and ruled surfaces according to any orthonormal frame $\{N_1, N_2, N_3\}$. We consider this any orthonormal frame with $\kappa_2 = 0$ but note that the frame different from Frenet frame. Also, we give the hypothesis the parameter curves of the ruled surfaces with the base rectifying curve are geodesic. We can define the rectifying curve with this orthonormal frame. So, if the rate of the curvatures $\frac{\kappa_3}{\kappa_1}$ is a non-constant linear function according to arc length function s , then we can say the curve is a rectifying curve.

Theorem 3.1. *Let $\alpha(s) = \int N_1(s) ds$ be a unit speed curve with any orthonormal frame $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. The curve α is a rectifying curve if and only s -parameter curves of the surface $\phi(s, u) = \alpha(s) + u\bar{D}(s)$ are rectifying curve where $\bar{D}(s) = \left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3$ is modified Darboux vector field and $u \neq -\frac{1}{a}$.*

Proof. Let $\alpha(s) = \int N_1(s) ds$ be a unit speed and rectifying curve with the frame apparatus $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. If the parameter u is a constant, we obtain the s -parameter curves of the surface as $\beta(s) = \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1}\right) (s) N_1(s) + N_3(s) \right)$. If we take the derivative of β according to its arc length parameter, then we have

$$\begin{aligned} \frac{d\beta}{d\bar{s}} &= \frac{d\beta}{ds} \frac{ds}{d\bar{s}}, \\ \bar{N}_1 &= \left(1 + u \left(\frac{\kappa_3}{\kappa_1}\right)' \right) N_1 \frac{ds}{d\bar{s}} \end{aligned}$$

where $\{\bar{N}_1, \bar{N}_2, \bar{N}_3, \bar{\kappa}_1, \bar{\kappa}_3\}$ is the any orthonormal frame apparatus of β . If we take the norm of both sides of above equation, we have

$$d\bar{s} = \left(1 + u \left(\frac{\kappa_3}{\kappa_1}\right)' \right) ds.$$

If we integrate the last equation, we obtain

$$\bar{s} = s + u \left(\frac{\kappa_3}{\kappa_1}\right) + c, \quad c \text{ constant}, \quad (3.4)$$

and we can easily see that

$$\bar{N}_1 = N_1.$$

Similarly, if a derivative of this equation is taken with respect to s , we obtain

$$\begin{aligned} \frac{d\bar{N}_1}{d\bar{s}} \frac{d\bar{s}}{ds} &= \kappa_1 N_2, \\ \bar{\kappa}_1 \bar{N}_2 &= \kappa_1 N_2 \frac{1}{1 + u \left(\frac{\kappa_3}{\kappa_1}\right)'}, \end{aligned}$$

where $\left(\frac{\kappa_3}{\kappa_1}\right)' \neq -\frac{1}{u}$. If we take the norm of last equation, we get

$$\bar{\kappa}_1 = \frac{\kappa_1}{1 + u \left(\frac{\kappa_3}{\kappa_1}\right)'}. \quad (3.5)$$

So, we can easily see that

$$\bar{N}_2 = N_2.$$

Hence, we know that $\bar{N}_3 = N_3$. If we take the derivative of this equation according to s , we have

$$\bar{\kappa}_3 = \frac{\kappa_3}{1 + u \left(\frac{\kappa_3}{\kappa_1} \right)}. \tag{3.6}$$

If we look at the ratio of the Eq. (3.5) and Eq. (3.6), we can say that

$$\frac{\kappa_3}{\kappa_1} = \frac{\bar{\kappa}_3}{\bar{\kappa}_1}. \tag{3.7}$$

Since α is a rectifying curve, we know that $\frac{\kappa_3}{\kappa_1} = as + b$ non-constant linear function for some constants a and b with $a \neq 0$ and $a \neq -\frac{1}{u}$. Let us write this equality in equation (3.4).

$$\begin{aligned} \bar{s} &= s + u(as + b) + c, \\ \bar{s} &= (1 + au)s + buc, \\ \bar{s} &= es + f, \end{aligned}$$

where e, f are some constants with $e \neq 0$. So, we obtain the arc length parameter of the curve α as follows

$$s = \frac{\bar{s} - f}{e}.$$

From equation (3.7), we get

$$\frac{\kappa_3}{\kappa_1} = \frac{\bar{\kappa}_3}{\bar{\kappa}_1} = a \left(\frac{\bar{s} - f}{e} \right) + b.$$

Hence, we can easily see that

$$\frac{\bar{\kappa}_3}{\bar{\kappa}_1} = \lambda \bar{s} + \mu$$

where λ and μ are some constants with $\lambda \neq 0$.

Finally, if the curve α is a rectifying curve, then s -parameter curves of the surface $\phi(s, u) = \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1} \right) N_1 + N_3 \right)$ are rectifying curve.

Conversely, let s -parameter curves of the surface $\beta(s) = \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1} \right) N_1 + N_3 \right)$ are rectifying curve. The ratio of the curvatures of the curve β is the non-constant linear function according to \bar{s} for some constants λ and μ with $\lambda \neq 0$ as

$$\frac{\bar{\kappa}_3}{\bar{\kappa}_1} = \lambda \bar{s} + \mu.$$

From the equations (3.4) and (3.7), we can easily see that

$$\frac{\bar{\kappa}_3}{\bar{\kappa}_1} = \frac{\kappa_3}{\kappa_1} = \lambda \left(s + u \left(\frac{\kappa_3}{\kappa_1} \right) + c \right) + \mu.$$

If the necessary arrangements are made, we get

$$\frac{\kappa_3}{\kappa_1} = as + b$$

where a, b are some constants with $a \neq 0$. This means that α is a rectifying curve.

Corollary 3.1. *Let $\beta(s) = \int N_3(s) ds$ be a unit speed curve with $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. The curve β is a rectifying curve if and only if s -parameter curves of the surface $\phi(s, u) = \beta(s) + v\tilde{D}(s)$ are rectifying curve where $\tilde{D}(s) = N_1 + \left(\frac{\kappa_1}{\kappa_3}\right)N_3$ is modified Darboux vector field.*

Corollary 3.2. *Let $\gamma(s) = \int T(s) ds$ be a unit speed curve with $\{T, N, B, \kappa, \tau\}$. Then the curve γ is a rectifying curve if and only if s -parameter curves of the surface $\phi(s, u) = \gamma(s) + u\bar{D}(s)$ are rectifying curve where $\bar{D}(s) = \left(\frac{\tau}{\kappa}\right)T + B$ is modified Darboux vector field.*

Remark 3.1. *For a regular curve γ in \mathbb{E}^3 with $\kappa \neq 0$, the curve given by the Darboux vector $D = \tau T + \kappa B$ is called the centrode of γ and the curves $C_{\pm} = \gamma \pm D$ are called the co-centrodes of γ . Chen and Dillen show that a curve γ with non-zero constant curvature and non-constant torsion is a rectifying curve if and only if one of its co-centrodes is a rectifying curve [2]. If we select $u = 1$ for u constant parameter curves, then we define the u constant parameter curves correspond to the co-centrodes.*

Corollary 3.3. *Let $\sigma(s) = \int N(s)$ be a unit speed curve with $\{N, C, W, f, g\}$ defined by Uzunoğlu [9]. The curve σ is a rectifying curve if and only if s -parameter curves of the surface $\phi(s, u) = \sigma(s) + u\bar{D}(s)$ are rectifying curve where $\bar{D}(s) = \left(\frac{g}{f}\right)N + W$ is modified Darboux vector field.*

Theorem 3.2. *Let $\alpha(s) = \int N_1(s) ds$ be a unit speed curve with any orthonormal frame apparatus $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. If α is a rectifying curve, the parameter curves of the surface $\phi(s, u) = \alpha(s) + u\bar{D}(s)$ are geodesic curve where $\bar{D}(s) = \left(\frac{\kappa_3}{\kappa_1}\right)N_1 + N_3$ is modified Darboux vector field and $u \neq -\frac{1}{a}$.*

Proof. The curve α has been always geodesic on the surface, but the parameter curves of the surface are geodesic if α is a rectifying curve. The normal vector of the surface

is as follows

$$\begin{aligned} \phi_s &= \left(1 + u \left(\frac{\kappa_3}{\kappa_1}\right)'\right) N_1 \text{ and } \phi_u = \left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3, \\ N_\phi &= -\left(1 + u \left(\frac{\kappa_3}{\kappa_1}\right)'\right) N_2. \end{aligned}$$

Let α be a unit speed rectifying curve. Let's examine s -parameter curves of the surface

$$\phi(s, u) = \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1}\right)(s) N_1(s) + N_3(s) \right),$$

$$\beta(s) = \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3 \right),$$

$$\begin{aligned} \frac{d\beta}{d\bar{s}} &= \frac{d\beta}{ds} \frac{ds}{d\bar{s}} = a N_1 \frac{ds}{d\bar{s}}, \\ \frac{d^2\beta}{d\bar{s}^2} &= \frac{d^2\beta}{ds^2} \frac{ds^2}{d\bar{s}^2} = b \kappa_1 N_2, \end{aligned}$$

where a and b are some constants.

Similar to the above thought, if we examine u -parameter curves of the surface $\phi(s, u) = \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3 \right)$, then we have

$$\begin{aligned} \beta(s) &= \int N_1(s) ds + u \left(\left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3 \right), \\ \frac{d^2\beta}{d\bar{u}^2} &= 0. \end{aligned}$$

So, if the curve α is a rectifying curve, then the parameter curves of the surface $\phi(s, u) = \alpha(s) + u \left(\left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3 \right)$ are geodesic curve.

Corollary 3.4. *Let $\gamma = \int T(s) ds$ be a unit speed curve with $\{T, N, B, \kappa, \tau\}$. If γ is a rectifying curve, the parameter curves of the surface $\phi(s, u) = \gamma(s) + u\bar{D}(s)$ are geodesic curve where $\bar{D}(s) = \left(\frac{\tau}{\kappa}\right) T + B$ is modified Darboux vector field.*

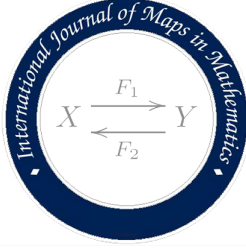
Corollary 3.5. *Let $\sigma(s) = \int N(s)$ be a unit speed curve with $\{N, C, W, f, g\}$. If σ is a rectifying curve, the parameter curves of the surface $\phi(s, u) = \sigma(s) + u\bar{D}(s)$ are geodesic curve where $\bar{D}(s) = \left(\frac{g}{f}\right) N + W$ is modified Darboux vector field.*

REFERENCES

- [1] Chen, B. Y. (2003). When does the position vector of a space curve always lie in its rectifying plane?. *The American mathematical monthly*, 110(2), 147-152.
- [2] Chen, B. Y. & Dillen, F. (2005). Rectifying curves as centrodes and extremal curves. *Bull. Inst. Math. Acad. Sinica*, 33(2), 77-90.
- [3] Izumiya, S. & Takeuchi, N. (2004). New special curves and developable surfaces. *Turk. J. Math.*, 28(2), 153-163.
- [4] İlarıslan, K., Nesovic, E. & Petrovic-Torgasev, M. (2003). Some characterizations of rectifying curves in the Minkowski 3-space. *Novi. Sad. J. Math.*, 33(2), 23-32.
- [5] İlarıslan, K. & Nesovic, E. (2007). On rectifying curves as centrodes and extremal curves in the Minkowski in the Minkowski 3-space. *Novi. Sad. J. Math.*, 37(1), 53-64.
- [6] Lucas, P. & Ortega-Yagües, J.A. (2015). Rectifying curves in the three-dimensional sphere. *J. Math. Anal. Appl.*, 421, 1855-1868.
- [7] Özbey, E. & Oral, M. (2009). A study on rectifying curves in the dual lorentzian space. *Bull. Korean Math. Soc.*, 46(5), 967-978
- [8] Şahin, T. & Yılmaz, M. (2015). The rectifying developable and rectifying gaussian surface of curve in pseudo-galilean geometry. *Analele Ştiintifice ale Universităţii "Al.I.Cuza" din Iaşi, Matematica*, 61, 217-234.
- [9] Uzunoğlu, B., Gök, İ. & Yaylı, Y. (2016). A new approach on curves of constant precession. *Applied Mathematics and Computation*, 275, 317-323.

DEPARTMENT OF MATHEMATICS, KAHRAMANMARAŞ SÜTÇÜ İMAM UNIVERSITY, KAHRAMANMARAŞ, ,
TURKEY

DEPARTMENT OF MATHEMATICS, ANKARA UNIVERSITY, ANKARA, TURKEY



FRENET CURVES IN 3-DIMENSIONAL LORENTZIAN CONCIRCULAR STRUCTURE MANIFOLDS

MÜSLÜM AYKUT AKGÜN *

ABSTRACT. In this paper, we give some characterizations of Frenet curves in 3-dimensional Lorentzian concircular structure manifolds($(LCS)_3$ manifolds). We define Frenet equations and the Frenet elements of these curves. We also obtain the curvatures of non-geodesic Frenet curves on $(LCS)_3$ manifolds. Finally we give some theorems, corollaries and examples for these curves.

Keywords: Lorentzian manifold, Concircular structure, Frenet curve

2010 Mathematics Subject Classification: 53D10, 53A04.

1. INTRODUCTION

The differential geometry of curves in manifolds investigated by several authors. Especially the curves in contact and para-contact manifolds drew attention and studied by the authors. B. Olszak[17], derived the conditions for an a.c.m structure on M to be normal and point out some of their consequences. B. Olszak completely characterized the local nature of normal a.c.m. structures on M by giving suitable examples. Moreover B. Olszak gave some restrictions on the scalar curvature in contact metric manifolds which are conformally flat or of constant ϕ -sectional curvature in[16].

J. Welyczko[22], generalized some of the results for Legendre curves in three dimensional normal a.c.m. manifolds, especially, quasi-Sasakian manifolds. J. Welyczko [23], studied

Received:2021.04.12

Revised:2021.08.12

Accepted:2021.08.29

* Corresponding author

Müslüm Aykut Akgün; muslumakgun@adiyaman.edu.tr; <https://orcid.org/0000-0002-8414-5228>

the curvatures of slant Frenet curves in three-dimensional normal almost paracontact metric manifolds.

B. E. Acet and S. Y. Perktaş [1] obtained the curvatures of Legendre curves in 3-dimensional (ε, δ) trans-Sasakian manifolds. Ji-Eun Lee, defined Lorentzian cross product in a three-dimensional almost contact Lorentzian manifold and proved that $\frac{\kappa}{\tau-1} = \text{cons.}$ along a Frenet slant curve in a Sasakian Lorentzian three-manifold. Furthermore, Ji-Eun Lee proved that γ is a slant curve if and only if M is Sasakian for a contact magnetic curve γ in contact Lorentzian 3-manifold M in[12]. Ji-Eun Lee, also gave some characterizations for the generalized Tanaka-Webster connection in a contact Lorentzian manifold in[13].

A. Yıldırım[25] obtained the Frenet apparatus for Frenet curves on three dimensional normal almost contact manifolds and characterized some results for these curves.

U.C.De and K.De[10] studied Lorentzian Trans-Sasakian and conformally flat Lorentzian Trans-Sasakian manifolds.

The LCS manifolds was introduced by [19] with an example. A. A. Shaikh[20] studied various types of $(LCS)_n$ -manifolds and proved that in such a manifold the Ricci operator commutes with the structure tensor φ .

In this framework, the paper is organized in the following way. Section 2 with two subsections, we give basic definitions of a $(LCS)_n$ -manifolds manifold. In the second subsection we give the Frenet-Serret equations of a curve in $(LCS)_3$ manifold. We give finally the Frenet elements of a Frenet curve in $(LCS)_3$ manifold and give theorems, corollaries and examples for these curves in the third and fourth sections.

2. PRELIMINARIES

2.1. Lorentzian Conircular Structure Manifolds. A Lorentzian manifold of dimension n is a doublet (\bar{N}, \bar{g}) , where \bar{N} is a smooth connected para-compact Hausdorff manifold of dimension n and \bar{g} is a Lorentzian metric, that is, \bar{N} admits a smooth symmetric tensor field g of type $(0, 2)$ such that for each point $p \in \bar{N}$ the tensor $\bar{g}_p : T_p\bar{N} \times T_p\bar{N} \rightarrow R$ is a non degenerate inner product of signature $(-, +, \dots, +)$, where $T_p\bar{N}$ denotes the tangent space of \bar{N} at p and R is the real number space. A non zero vector field $V \in T_p\bar{N}$ is called spacelike (resp. non-spacelike, null and timelike) if it satisfies $\bar{g}_p(V, V) > 0$ (resp., $\leq 0, =, < 0$).[15]

Definition 2.1. In a Lorentzian manifold (\bar{N}, \bar{g}) a vector field w is defined by

$$\bar{g}(U, \rho) = A(U) \tag{2.1}$$

for any $U \in \chi(\bar{N})$ is said to be a concircular vector field if

$$(\nabla_U A)(V) = \alpha \{ \bar{g}(U, V) + w(U)w(V) \}, \tag{2.2}$$

where α is a non-zero scalar and w is a closed 1-form.[24]

If a Lorentzian manifold \bar{N} admits a unit timelike concircular vector field ξ , called generator of the manifold, then we have

$$\bar{g}(\xi, \xi) = -1. \tag{2.3}$$

Since ξ is the unit concircular vector field on \bar{N} , there exists a non-zero 1-form η such that

$$\bar{g}(U, \xi) = \eta(U), \tag{2.4}$$

which satisfies the following equation

$$(\nabla_U \eta)(V) = \alpha \{ \bar{g}(U, V) + \eta(U)\eta(V) \}, \quad (\alpha \neq 0) \tag{2.5}$$

for all vector fields U and V , where ∇ gives the covariant differentiation with respect to the Lorentzian metric \bar{g} and α is a non-zero scalar function satisfies

$$(\nabla_U \alpha) = U\alpha = d\alpha(U) = \rho\eta(U), \tag{2.6}$$

where ρ is a certain scalar function defined by $\rho = -(\xi\alpha)$. If we take

$$\varphi U = \frac{1}{\alpha} \nabla_U \xi, \tag{2.7}$$

then with the help of (2.3), (2.4) and (2.6), we can find

$$\varphi U = U + \eta(U)\xi, \tag{2.8}$$

which shows that φ is a tensor field of type (1,1), called the structure tensor of the manifold \bar{N} . Hence the Lorentzian manifold \bar{N} of class C^∞ equipped with a unit timelike concircular vector field ξ , its associated 1-form η and (1,1) tensor field φ is said to be a Lorentzian concircular structure manifold (i.e. $(LCS)_n$ manifold)[19]. Moreover, if $\alpha = 1$, then we have the LP-Sasakian structure of Matsumoto[14]. So we can say the generalization of LP-Sasakian manifold gives us the $(LCS)_n$ manifold. It is noteworthy to mention that LCS-manifold is invariant under a conformal change whereas LP-Sasakian structure is not so[18]. In $(LCS)_3$ manifolds, the following relations hold[19]

$$\varphi^2 U = U + \eta(U)\xi, \quad \eta(\xi) = -1, \tag{2.9}$$

$$\varphi(\xi) = 0, \quad \eta(\varphi U) = 0,$$

and

$$\bar{g}(\varphi U, \varphi V) = \bar{g}(U, V) + \eta(U)\eta(V). \quad (2.10)$$

2.2. Frenet Curves. Let $\zeta : I \rightarrow \bar{N}$ be a unit speed curve in $(LCS)_3$ manifold \bar{N} such that ζ' satisfies $\bar{g}(\zeta', \zeta') = \varepsilon_1 = \mp 1$. The constant ε_1 is called the casual character of ζ . The constants ε_2 and ε_3 defined by $\bar{g}(n, n) = \varepsilon_2$ and $\bar{g}(b, b) = \varepsilon_3$ and called the second casual character and third casual character of ζ , respectively. Thus we $\varepsilon_1\varepsilon_2 = -\varepsilon_3$.

A unit speed curve ζ is said to be a spacelike or timelike if its casual character is 1 or -1, respectively. A unit speed curve ζ is said to be a Frenet curve if $\bar{g}(\zeta', \zeta') \neq 0$. A Frenet curve ζ admits an orthonormal frame field $\{t = \zeta', n, b\}$ along ζ . Then the Frenet-Serret equations given as follows:

$$\begin{aligned} \nabla_{\zeta'} t &= \varepsilon_2 \kappa n \\ \nabla_{\zeta'} n &= -\varepsilon_1 \kappa t - \varepsilon_3 \tau b \\ \nabla_{\zeta'} b &= \varepsilon_2 \tau n \end{aligned} \quad (2.11)$$

where $\kappa = |\nabla_{\zeta'} \zeta'|$ is the geodesic curvature of ζ and τ is geodesic torsion [12]. The vector fields t , n and b are called the tangent vector field, the principal normal vector field and the binormal vector field of ζ , respectively.

If the geodesic curvature of the curve ζ vanishes, then the curve is called a geodesic curve. If $\kappa = \text{cons.}$ and $\tau = 0$, then the curve is called a pseudo-circle and pseudo-helix if the geodesic curvature and torsion are constant.

A curve in a three dimensional Lorentzian manifold is a slant curve if the tangent vector field of the curve has constant angle with the Reeb vector field, i.e. $\eta(\zeta') = -\bar{g}(\zeta', \xi) = \cos\theta = \text{constant}$. If $\eta(\zeta') = -\bar{g}(\zeta', \xi) = 0$, then the curve ζ is called a Legendre curve [12].

3. MAIN RESULTS

In this section we consider a $(LCS)_3$ manifold \bar{N} . Let $\zeta : I \rightarrow \bar{N}$ be a Frenet curve with the geodesic curvature $\kappa \neq 0$, given with the arc-parameter s and $\bar{\nabla}$ be the Levi-Civita connection on \bar{N} . From the basis $(\zeta', \varphi\zeta', \xi)$ we obtain an orthonormal basis $\{e_1, e_2, e_3\}$ which

satisfy the equations

$$\begin{aligned} e_1 &= \zeta', \\ e_2 &= \frac{\varepsilon_2 \varphi \zeta'}{\sqrt{\varepsilon_1 + \rho^2}}, \\ e_3 &= \varepsilon_2 \frac{\varepsilon_1 \xi - \rho \zeta'}{\sqrt{\varepsilon_1 + \rho^2}} \end{aligned} \tag{3.12}$$

where

$$\eta(\zeta') = \bar{g}(\zeta', \xi) = \rho. \tag{3.13}$$

Then if we write the covariant differentiation of ζ' as

$$\bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3 \tag{3.14}$$

such that

$$\nu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_2) \tag{3.15}$$

is a certain function. Moreover we obtain ν by

$$\mu = \bar{g}(\bar{\nabla}_{\zeta'} e_1, e_3) = \varepsilon_2 \left(\frac{\rho'}{\sqrt{\varepsilon_1 + \rho^2}} - \varepsilon_1 \alpha \sqrt{\varepsilon_1 + \rho^2} \right), \tag{3.16}$$

where $\rho'(s) = \frac{d\rho(\zeta(s))}{ds}$. Then we find

$$\bar{\nabla}_{\zeta'} e_2 = -\nu e_1 + \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) e_3 \tag{3.17}$$

and

$$\bar{\nabla}_{\zeta'} e_3 = -\mu e_1 - \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) e_2. \tag{3.18}$$

The fundamental forms of the tangent vector ζ' on the basis of the equation (3.12) is

$$[\omega_{ij}(\zeta')] = \begin{pmatrix} 0 & \nu & \mu \\ -\nu & 0 & \varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \\ -\mu & -\varepsilon_3 \alpha - \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} & 0 \end{pmatrix} \tag{3.19}$$

and the Darboux vector connected to the vector ζ' is

$$\omega(\zeta') = \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) e_1 - \mu e_2 + \nu e_3. \tag{3.20}$$

So we can write

$$\bar{\nabla}_{\zeta'} e_i = \omega(\zeta') \wedge \varepsilon_i e_i \quad (1 \leq i \leq 3). \tag{3.21}$$

Thus, for any vector field $Z = \sum_{i=1}^3 \theta^i e_i \in \chi(\bar{N})$ strictly dependent on the curve ζ on \bar{N} and we have the following equation

$$\bar{\nabla}_{\zeta'} Z = \omega(\zeta') \wedge Z + \sum_{i=1}^3 \varepsilon_i e_i [\theta^i] e_i. \quad (3.22)$$

3.1. Frenet Elements of ζ . Let a curve $\zeta : I \rightarrow \bar{N}$ be a Frenet curve with the geodesic curvature $\kappa \neq 0$, given with the arc parameter s and the elements $\{t, n, b, \kappa, \tau\}$. The Frenet elements of the curve ζ can be calculated as above:

If we consider the equation (3.14), then we get

$$\varepsilon_2 \kappa n = \bar{\nabla}_{\zeta'} e_1 = \nu e_2 + \mu e_3. \quad (3.23)$$

If we consider (3.16) and (3.23) we find

$$\kappa = \sqrt{\nu^2 + \left(\frac{\rho'}{\sqrt{\varepsilon_1 + \rho^2}} - \varepsilon_1 \alpha \sqrt{\varepsilon_1 + \rho^2} \right)^2}. \quad (3.24)$$

On the other hand

$$\begin{aligned} \bar{\nabla}_{\zeta'} n &= \left(\frac{\nu}{\varepsilon_2 \kappa} \right)' e_2 + \frac{\nu}{\varepsilon_2 \kappa} \nabla_{\zeta'} e_2 + \left(\frac{\mu}{\varepsilon_2 \kappa} \right)' e_3 + \frac{\mu}{\varepsilon_2 \kappa} \nabla_{\zeta'} e_3 \\ &= -\varepsilon_1 \kappa t - \varepsilon_3 \tau B. \end{aligned} \quad (3.25)$$

By means of the equation (3.17) and (3.18) we find

$$\begin{aligned} -\varepsilon_3 \tau B &= \left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' - \frac{\mu}{\varepsilon_2 \kappa} \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] e_2 \\ &+ \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' + \frac{\nu}{\varepsilon_2 \kappa} \left(\varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] e_3. \end{aligned} \quad (3.26)$$

By a direct computation we find following

$$\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \left[\left(\frac{\mu}{\varepsilon_2 \kappa} \right)' \right]^2 = \left[- \left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \frac{\mu}{\varepsilon_2 \kappa} + \frac{\nu}{\varepsilon_2 \kappa} \left(\frac{\mu}{\varepsilon_2 \kappa} \right)' \right]^2. \quad (3.27)$$

Taking the norm of the last equation by using (3.26) and if we consider the equations (3.16) and (3.27) in (3.26) we obtain

$$\tau = \left| \varepsilon_3 \alpha + \frac{\varepsilon_1 \rho \nu}{\sqrt{\varepsilon_1 + \rho^2}} - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \left[\left(\frac{\varepsilon_2 \left(\frac{\rho'}{\sqrt{\varepsilon_1 + \rho^2}} - \varepsilon_1 \alpha \sqrt{\varepsilon_1 + \rho^2} \right)}{\kappa} \right)' \right]^2} \right|. \quad (3.28)$$

Moreover we can write the Frenet vector fields of ζ as in the following theorem

Theorem 3.1. *Let \bar{N} be a $(LCS)_3$ manifold and ζ be a Frenet curve on \bar{N} . The Frenet vector fields t , n and b are in the form of*

$$\begin{aligned} t &= \zeta' = e_1, \\ n &= \frac{\nu}{\varepsilon_2\kappa}e_2 + \frac{\mu}{\varepsilon_2\kappa}e_3, \\ b &= -\frac{1}{\varepsilon_3\tau} \left[\left(\frac{\nu}{\varepsilon_2\kappa} \right)' - \frac{\mu}{\varepsilon_2\kappa} \left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] e_2 \\ &\quad - \frac{1}{\varepsilon_3\tau} \left[\left(\frac{\mu}{\varepsilon_2\kappa} \right)' + \frac{\nu}{\varepsilon_2\kappa} \left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] e_3. \end{aligned} \tag{3.29}$$

Note that

$$\begin{aligned} \xi &= \varepsilon_1\rho t - \frac{\mu\sqrt{\varepsilon_1 + \rho^2}}{\kappa}n \\ &\quad - \frac{\sqrt{\varepsilon_1 + \rho^2}}{\varepsilon_3\tau} \left[\left(\frac{\mu}{\varepsilon_2\kappa} \right)' + \frac{\nu}{\varepsilon_2\kappa} \left(\varepsilon_3\alpha + \frac{\varepsilon_1\rho\nu}{\sqrt{\varepsilon_1 + \rho^2}} \right) \right] b. \end{aligned} \tag{3.30}$$

Let ζ be a non-geodesic Frenet curve given with the arc-parameter s in $(LCS)_3$ manifold \bar{N} . So one can state the above theorems.

Theorem 3.2. *Let \bar{N} be a $(LCS)_3$ manifold and ζ be a Frenet curve on \bar{N} . ζ is a slant curve ($\rho = \eta(\zeta') = \cos\theta = \text{cons.}$) on \bar{N} if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of ζ are as follows*

$$\begin{aligned} t &= e_1 = \zeta', \\ n &= e_2 = \frac{\varepsilon_2\varphi\zeta'}{\sqrt{\varepsilon_1 + \cos^2\theta}}, \\ b &= e_3 = \varepsilon_2 \frac{\varepsilon_1\xi - \cos\theta\zeta'}{\sqrt{\varepsilon_1 + \cos^2\theta}}, \\ \kappa &= \sqrt{\nu^2 + \alpha^2(\varepsilon_1 + \cos^2\theta)}, \\ \tau &= \left| \varepsilon_3\alpha + \frac{\varepsilon_1\cos\theta\nu}{\sqrt{\varepsilon_1 + \cos^2\theta}} - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2\kappa} \right) \right]^2 + \left[\left(\frac{\alpha\sqrt{\varepsilon_1 + \cos^2\theta}}{\kappa} \right) \right]^2} \right|. \end{aligned} \tag{3.31}$$

Proof. Let the curve ζ be a slant curve in $(LCS)_3$ manifold \bar{N} . If we take account the condition $\rho = \eta(\zeta') = \cos\theta = \text{constant}$ in the equations (3.12), (3.24) and (3.28) we find (3.31). If the equations in (3.31) hold, from the definition of slant curves it is obvious that the curve ζ is a slant curve.

Corollary 3.1. *Let \bar{N} be a $(LCS)_3$ manifold and ζ be a slant curve on \bar{N} . If the geodesic curvature κ of the curve ζ is non-zero constant, then the geodesic torsion of ζ is $\tau = \left| \left(\varepsilon_3 \alpha + \varepsilon_1 \frac{\cos \theta \nu}{\sqrt{\varepsilon_1 + \cos^2 \theta}} \right) \right|$ and ζ is a pseudo-helix on \bar{N} .*

Corollary 3.2. *Let \bar{N} be a $(LCS)_3$ manifold and ζ be a slant curve on \bar{N} . If the geodesic curvature κ of the curve ζ is not constant and the geodesic torsion of ζ is $\tau = 0$ then ζ is a plane curve on \bar{N} and function ν satisfies the equation*

$$\nu = \int (c_1 + c_2 \nu) \kappa^2 ds, \quad (3.32)$$

where $c_1 = \frac{\varepsilon_3}{\sqrt{\varepsilon_1 + \cos^2 \theta}}$ and $c_2 = \frac{\varepsilon_1 \cos \theta}{\alpha(\varepsilon_1 + \cos^2 \theta)}$.

Theorem 3.3. *Let \bar{N} be a $(LCS)_3$ manifold and ζ is a Frenet curve on \bar{N} . ζ is a spacelike Legendre curve ($\rho = \eta(\zeta') = 0$) in this manifold if and only if the Frenet elements $\{t, n, b, \kappa, \tau\}$ of ζ are as follows*

$$\begin{aligned} t &= e_1 = \zeta', \\ n &= e_2 = \varepsilon_2 \varphi \zeta', \\ b &= e_3 = -\varepsilon_3 \xi, \\ \kappa &= \sqrt{\nu^2 + \alpha^2}, \\ \tau &= \left| \varepsilon_3 \alpha - \sqrt{\left[\left(\frac{\nu}{\varepsilon_2 \kappa} \right)' \right]^2 + \alpha^2 \left[\frac{\kappa'}{\kappa^2} \right]^2} \right|. \end{aligned} \quad (3.33)$$

Proof. Let the curve ζ be a Legendre curve in $(LCS)_3$ manifold \bar{N} . If we take account the condition $\rho = \eta(\zeta') = 0$ in the equations (3.12), (3.24) and (3.28) we find (3.33). If the equations in (3.33) hold, from the definition of Legendre curves it is obvious that the curve ζ is a Legendre curve on \bar{N} .

Corollary 3.3. *Let the curve ζ is a Legendre curve in $(LCS)_3$ manifold \bar{N} . If the geodesic curvature κ of the curve ζ is non-zero constant, then the geodesic torsion of ζ is $\tau = 0$ and ζ is a plane curve on \bar{N} .*

4. EXAMPLES

Let \bar{N} be the 3-dimensional manifold given

$$\bar{N} = \{(x, y, z) \in \mathfrak{R}^3, z \neq 0\}, \quad (4.34)$$

where (x,y,z) denote the standart co-ordinates in \mathfrak{R}^3 . Then

$$E_1 = e^z \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right), \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z} \tag{4.35}$$

are linearly independent of each point of \bar{N} . Let g be the Lorentzian metric tensor defined by

$$\begin{aligned} \bar{g}(E_1, E_1) = \bar{g}(E_2, E_2) = 1, \quad \bar{g}(E_3, E_3) = -1, \\ \bar{g}(E_i, E_j) = 0, \quad i \neq j, \end{aligned} \tag{4.36}$$

for $i, j = 1, 2, 3$ [2]. Let η be the 1-form defined by $\eta(Z) = \bar{g}(Z, E_3)$ for any $Z \in \Gamma(T\bar{N})$. Let φ be the (1,1)-tensor field defined by

$$\varphi E_1 = E_1, \quad \varphi E_2 = E_2, \quad \varphi E_3 = 0. \tag{4.37}$$

Then using the condition of the linearity of φ and \bar{g} , we obtain $\eta(E_3) = -1$,

$$\begin{aligned} \varphi^2 Z = Z + \eta(Z)E_3, \\ \bar{g}(\varphi Z, \varphi W) = \bar{g}(Z, W) + \eta(Z)\eta(W), \end{aligned} \tag{4.38}$$

for all $Z, W \in \Gamma(T\bar{N})$. Thus for $\xi = E_3$, $(\varphi, \xi, \eta, \bar{g})$ defines a Lorentzian paracontact structure on \bar{N} .

Now, let ∇ be the Levi-Civita connection with respect to the Lorentzian metric \bar{g} . Then we obtain

$$[E_1, E_2] = -e^z E_2, \quad [E_1, E_3] = -E_1, \quad [E_2, E_3] = -E_2. \tag{4.39}$$

If we use the Koszul formulae for the Lorentzian metric tensor \bar{g} , we can easily calculate the covariant derivations as follows:

$$\begin{aligned} \nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_2} E_1 = e^z E_2, \quad \nabla_{E_1} E_3 = -E_1, \\ \nabla_{E_2} E_3 = -E_2, \quad \nabla_{E_2} E_2 = -e^z E_1 - E_3, \\ \nabla_{E_1} E_2 = \nabla_{E_3} E_1 = \nabla_{E_3} E_2 = \nabla_{E_3} E_3 = 0. \end{aligned} \tag{4.40}$$

From the about representatations, one can easily see that $(\varphi, \xi, \eta, \bar{g})$ is a $(LCS)_3$ structure on \bar{N} , that is, \bar{N} is an $(LCS)_3$ -manifold with $\alpha = -1$ and $\rho = 0$.

Example 4.1. Let β be a spacelike Legendre curve in the $(LCS)_3$ manifold \bar{N} and defined as

$$\begin{aligned}\beta: I &\rightarrow \bar{N} \\ s &\rightarrow \beta(s) = (s^2, s^2, \ln 2),\end{aligned}$$

where the curve β parametrized by the arc length parameter t . If we differentiate $\beta(t)$ and consider (3.12) we find

$$e_1 = \beta'(t), \quad (4.41)$$

$$e_2 = \frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_2, \quad (4.42)$$

$$e_3 = \varepsilon_2 E_3. \quad (4.43)$$

If we consider the equations (3.13), (3.14), (3.16), (3.24) and (3.28) we can write

$$\rho = 0, \quad \mu = -\varepsilon_2 \alpha, \quad \nu = -\frac{1}{\sqrt{2}}, \quad (4.44)$$

$$\kappa = \sqrt{\alpha^2 + \frac{1}{2}} = \sqrt{\frac{3}{2}}, \quad \tau = |\alpha| = 1.$$

From the above equations we see that the curve β is a Legendre helix curve in \bar{N} .

Example 4.2. Let v be a spacelike Legendre curve in the $(LCS)_3$ manifold \bar{N} and defined as

$$\begin{aligned}v: I &\rightarrow \bar{N} \\ s &\rightarrow v(s) = (\cos s, \sin s, 1),\end{aligned}$$

where the curve v parametrized by the arc length parameter t . If we differentiate $v(t)$ and consider (3.12) we find

$$e_1 = v'(t), \quad (4.45)$$

$$e_2 = \varepsilon_2 \left(-\sin\left(\frac{t}{e}\right)E_1 + \cos\left(\frac{t}{e}\right)E_2 \right), \quad (4.46)$$

$$e_3 = \varepsilon_2 \partial_3. \quad (4.47)$$

If we consider the equations (3.13), (3.14), (3.16), (3.24) and (3.28) we can write

$$\rho = 0, \quad \mu = -\varepsilon_2 \alpha, \quad \nu = 0, \quad (4.48)$$

$$\kappa = \tau = |\alpha|.$$

So, the curve $v(s)$ is a Legendre helix curve in \bar{N} .

REFERENCES

- [1] Acet B. E., Perktas S. Y. (2018). Curvature and torsion of a legendre curve in (ε, δ) Trans-Sasakian manifolds. *Malaya Journal of Matematik*, Vol. 6, No. 1, 140-144.
- [2] Atçeken M. (2013). Some Curvature Properties of $(LCS)_n$ -Manifolds. Hindawi Publishing Corporation Abstract and Applied Analysis, Volume 2013, <http://dx.doi.org/10.1155/2013/380657>.
- [3] Blair D.E. (2002). *Riemannian Geometry of Contact and Symplectic Manifolds*. Progr. Math., Birkhäuser, Boston.
- [4] Calvaruso G. (2007). Homogeneous structures on three-dimensional Lorentzian manifolds. *J. Geom. Phys.* 57, 1279–1291. G. Calvaruso, *J. Geom. Phys.* 58 (2008) 291–292, Addendum.
- [5] Calvaruso G. (2007). Einstein-like metrics on three-dimensional homogeneous Lorentzian manifolds. *Geometriae Dedicata*. 127, 99–119.
- [6] Calvaruso G., Perrone D. (2010). Contact pseudo-metric manifolds. *Differ. Geom. App.* 28, 615–634.
- [7] Calvaruso G. (2011). Contact Lorentzian manifolds. *Differ. Geom. App.* 29, S41–S51.
- [8] Camci, C. (2011). Extended cross product in a 3-dimensional almost contact metric manifold with applications to curve theory. *Turk. J. Math.* 35, 1–14.
- [9] Cho, J. T., Inoguchi, J. I., and Lee, J. E. (2006). On slant curves in Sasakian 3-manifolds. *Bulletin of the Australian Mathematical Society*, 74(3), 359-367.
- [10] De, U. C. and De, K. (2013). On Lorentzian Trans-Sasakian Manifolds. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.*, 62(2), 37-51.
- [11] Duggal, K.L. and Jin, D.H. (2007). *Null Curves and Hypersurfaces of Semi-Riemannian Manifolds*. World Scientific Publishing: Singapore.
- [12] Lee, J. E. (2019). Slant Curves and Contact Magnetic Curves in Sasakian Lorentzian 3-Manifolds. *Symmetry*. 11, 784; doi:10.3390/sym11060784.
- [13] Lee, J. E. (2020). Slant Curves in Contact Lorentzian Manifolds with CR Structures. *Mathematics*. 8, 46; doi:math8010046.
- [14] Matsumoto K. (1989). On Lorentzian paracontact manifolds. *Bull. Yamagata Univ. Natur. Sci.* 12, no.2, 151-156.
- [15] O'Neill B. (1983). *Semi-Riemannian Geometry*. Academic Press, New York.
- [16] Olszak B. (1979). On contact metric manifolds. *Tôhoku Math. J.* 31, 247–253.
- [17] Olszak B. (1986). Normal Almost Contact Metric Manifolds of Dimension Three. *Ann. Pol. Math.*, XLVII.
- [18] Shaikh, A. A., and De, U. C. (2000). On 3-dimensional LP-Sasakian manifolds. *Soochow Journal of Mathematics*, 26(4), 359-368.
- [19] Shaikh, A. A. (2003). On Lorentzian almost paracontact manifolds with a structure of the concircular type. *Kyungpook Math. J.* 43(2), 305-314.
- [20] Shaikh A.A. (2009). Some Results on $(LCS)_n$ -manifolds. *J. Korean Math. Soc.* 46(3), 449-461.
- [21] Şahin, B. (2012). *Manifoldların diferensiyel geometrisi*. Nobel Yayın, 310.
- [22] Welyczko J. (2007). On Legendre curves in 3-dimensional normal almost contact metric manifolds. *Soochow Journal of Mathematics*, 33(4), 929-937.

- [23] Welyczko J. (2014). Slant Curves in 3-Dimensional Normal Almost Paracontact Metric Manifolds. *Mediterr. J. Math.* 11, 965–978, DOI 10.1007/s00009-013-0361-2 0378-620X/14/030965-14.
- [24] Yano K. (1940). Concircular Geometry.I. Concircular transformations. *proc. Imp. Acad. Tokyo.* 16, 195-200.
- [25] Yıldırım A. (2020). On curves in 3-dimensional normal almost contact metric manifolds. *Int. J. Geom. Methods M. Physics*, 1-18.

ADIYAMAN UNIVERSITY, TECHNICAL SCIENCES VOCATIONAL SCHOOL, TURKEY



REMARKS ON COMBINATORIAL SUMS ASSOCIATED WITH SPECIAL NUMBERS AND POLYNOMIALS WITH THEIR GENERATING FUNCTIONS

NESLİHAN KILAR  AND YILMAZ ŞİMŞEK  *

ABSTRACT. The purpose of this article is to give some novel identities and inequalities associated with combinatorial sums involving special numbers and polynomials. In particular, by using the method of generating functions and their functional equations, we derive not only some inequalities, but also many formulas, identities, and relations for the parametrically generalized polynomials, special numbers and special polynomials. Our identities, relations, inequalities and combinatorial sums are related to the Bernoulli numbers and polynomials of negative order, the Euler numbers and polynomials of negative order, the Stirling numbers, the Daehee numbers, the Changhee numbers, the Bernoulli polynomials, the Euler polynomials, the parametrically generalized polynomials, and other well-known special numbers and polynomials. Moreover, using Mathematica with the help of the Wolfram programming language, we illustrate some plots of the parametrically generalized polynomials under some of their randomly selected special conditions. Finally, we give some remarks and observations on our results.

Keywords: Bernoulli numbers and polynomials, Euler numbers and polynomials, Stirling numbers, Daehee numbers, Changhee numbers, Parametrically generalized polynomials, Generating functions, Special functions, Special numbers and polynomials.

2010 Mathematics Subject Classification: 05A15, 11B68, 11B73, 26C05, 33B10.

*Received:*2021.05.24

*Accepted:*2021.10.01

* Corresponding author

Neslihan Kılar; neslihankilar@gmail.com; <https://orcid.org/0000-0001-5797-6301>

Yılmaz Şimşek; ysimsek@akdeniz.edu.tr; <https://orcid.org/0000-0002-0611-7141>

1. INTRODUCTION AND PRELIMINARIES

Combinatorial sums and combinatorial numbers and polynomials have many applications in mathematics and other applied sciences. These numbers are related to the special functions and also some classes of special numbers and polynomials. The motivation of this paper is to use not only generating functions, but also their functional equations, we give many new formulas and combinatorial sums involving the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers, and also combinatorial numbers and polynomials such as the Daehee numbers, the Changhee numbers, and the parametrically generalized polynomials. By using these formulas and combinatorial sums, we provide some inequalities applications. In order to illustrate graph and plots of special polynomials, here we use Mathematica with the help of the Wolfram programming language.

Throughout of this paper, we use the following notations and definitions. Let

$$\mathbb{N} = \{1, 2, 3, \dots\} \quad \text{and} \quad \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

As usual, \mathbb{Z} , \mathbb{R} and \mathbb{C} denote the set of integer numbers, the set of real numbers, the set of complex numbers, respectively. We assume that:

$$0^n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{N}. \end{cases}$$

Furthermore,

$$\binom{\lambda}{0} = 1 \quad \text{and} \quad \binom{\lambda}{n} = \frac{(\lambda)_n}{n!} \quad (n \in \mathbb{N}; \lambda \in \mathbb{C}),$$

where $(\lambda)_n$ is the falling factorial defined by

$$(\lambda)_n = \lambda(\lambda - 1)(\lambda - 2) \dots (\lambda - n + 1),$$

with $(\lambda)_0 = 1$ (*cf.* [1–34]; and references therein).

The Stirling numbers of the second kind are defined by means of the following generating function:

$$F_S(t, k) = \frac{(e^t - 1)^k}{k!} = \sum_{n=0}^{\infty} S_2(n, k) \frac{t^n}{n!}, \quad (1.1)$$

(*cf.* [1–34]; and references therein).

The Stirling numbers of the second kind are also given by the falling factorial polynomials:

$$x^n = \sum_{j=0}^n S_2(n, j) (x)_j, \quad (1.2)$$

(*cf.* [1–34]; and references therein).

By using (1.1), an explicit formula for the numbers $S_2(n, k)$ is given as follows:

$$S_2(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} j^n, \tag{1.3}$$

where $n, k \in \mathbb{N}_0$ and for $k > n$,

$$S_2(n, k) = 0,$$

(cf. [1–34]; and references therein).

Let $v \in \mathbb{Z}$. The Bernoulli numbers and polynomials of higher order are defined by means of the following generating functions:

$$F_B(t, v) = \left(\frac{t}{e^t - 1} \right)^v = \sum_{n=0}^{\infty} B_n^{(v)} \frac{t^n}{n!}, \tag{1.4}$$

and

$$G_B(t, x, v) = F_B(t, v) e^{xt} = \sum_{n=0}^{\infty} B_n^{(v)}(x) \frac{t^n}{n!}, \tag{1.5}$$

such that $v = 0$,

$$B_n^{(0)}(x) = x^n,$$

(cf. [13, 23, 29, 30, 34]; and references therein).

Substituting $v = 1$ and $x = 0$ into (1.4) and (1.5), the Bernoulli numbers and polynomials are derived, respectively,

$$B_n = B_n^{(1)},$$

and

$$B_n(x) = B_n^{(1)}(x),$$

(cf. [1–34]; and references therein).

By using (1.5), an explicit formula for the polynomials $B_n^{(-k)}(x)$ is given as follows:

$$B_n^{(-k)}(x) = \frac{1}{\binom{n+k}{k} k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (x+j)^{n+k}, \tag{1.6}$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ (cf. [5, Equation (3.20)]).

Putting $n = x = k$ in (1.6), we have the following presumably known result:

$$B_n^{(-n)}(n) = \frac{n!}{(2n)!} \sum_{j=0}^n (-1)^{n-j} \binom{n}{j} (n+j)^{2n}.$$

Substituting $x = 0$ into the above equation, and using (1.3), we have the following well-known identity:

$$B_n^{(-k)} = \frac{1}{\binom{n+k}{k}} S_2(n+k, k) \tag{1.7}$$

(cf. [33, Equation (7.17)]).

Let $v \in \mathbb{Z}$. The Euler numbers and polynomials of higher order are defined by means of the following generating functions:

$$F_E(t, v) = \left(\frac{2}{e^t + 1} \right)^v = \sum_{n=0}^{\infty} E_n^{(v)} \frac{t^n}{n!} \quad (1.8)$$

and

$$G_E(t, x, v) = F_E(t, v) e^{xt} = \sum_{n=0}^{\infty} E_n^{(v)}(x) \frac{t^n}{n!}, \quad (1.9)$$

such that $v = 0$,

$$E_n^{(0)}(x) = x^n$$

(*cf.* [13, 23, 28, 29, 34]; and references therein).

Substituting $v = 1$ and $x = 0$ into (1.8) and (1.9), the Euler numbers and polynomials are derived, respectively,

$$E_n = E_n^{(1)}(0)$$

and

$$E_n(x) = E_n^{(1)}(x)$$

(*cf.* [1–34]; and references therein).

By using (1.9), we have

$$E_n^{(-k)}(x) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \sum_{d=0}^j \binom{d-k-1}{d} \frac{d!(-1)^d}{2^d} S_2(j, d), \quad (1.10)$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ (*cf.* [23, 28, 29, 34]).

Putting $n = x = k$ in (1.10), we have the following presumably known result:

$$E_n^{(-n)}(n) = \sum_{j=0}^n \binom{n}{j} n^{n-j} \sum_{d=0}^j \binom{d-n-1}{d} \frac{d!(-1)^d}{2^d} S_2(j, d).$$

By using (1.4) and (1.8), a relation between the numbers $E_n^{(-k)}$ and the numbers $B_n^{(-k)}$ is given as follows:

$$B_n^{(-k)} = \frac{1}{2^n} \sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)}, \quad (1.11)$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$ (*cf.* [13, Equation (3.1)]).

By using (1.1) and (1.8), a relation between the numbers $E_n^{(-k)}$ and the numbers $S_2(n, k)$ is given as follows:

$$S_2(n, k) = \frac{2^{k-n}}{k!} \sum_{m=0}^n \sum_{j=0}^k (-1)^{k-j} \binom{n}{m} \binom{k}{j} j^m E_{n-m}^{(-k)}, \quad (1.12)$$

(*cf.* [13, Theorem 2.14]).

The Euler numbers of the second kind, E_n^* , are defined by means of the following generating function:

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{\infty} E_n^* \frac{t^n}{n!} \tag{1.13}$$

(cf. [19, 26, 28, 30]; and references therein).

By using (1.9) and (1.13), a relation between the Euler numbers E_n^* and the Euler polynomials is given as follows:

$$E_n^* = 2^n E_n \left(\frac{1}{2} \right)$$

(cf. [19, 21, 28, 30]).

Kilar and Simsek [13, Corollary 3.5] gave the following identity for the numbers $S_2(n, k)$:

$$S_2(n + k, k) = \sum_{j=0}^n \frac{\binom{n}{j} \binom{n+k}{k}}{2^{k+n} \binom{j+k}{k}} S_2(j + k, k) B(n - j, k), \tag{1.14}$$

where $n, k \in \mathbb{N}_0$ and

$$\begin{aligned} B(n, k) &= \sum_{j=0}^n \binom{k}{j} j! 2^{k-j} S_2(n, j) \\ &= \sum_{j=0}^k \binom{k}{j} j^n \end{aligned}$$

(cf. [32, Identity 12.]; see also [7, 29]).

Substituting $n = k$ into (1.14), we have

$$S_2(2n, n) = \binom{2n}{n} \sum_{j=0}^n \frac{\binom{n}{j}}{4^n \binom{j+n}{n}} S_2(j + n, n) B(n - j, n).$$

The Daehee numbers, D_n , are defined by means of the following generating function:

$$\frac{\log(1 + t)}{t} = \sum_{n=0}^{\infty} D_n \frac{t^n}{n!} \tag{1.15}$$

(cf. [17, 25, 30]).

By using (1.15), an explicit formula for the Daehee numbers is given by

$$D_n = (-1)^n \frac{n!}{n + 1} \tag{1.16}$$

(cf. [17, 25, 30]).

The Changhee numbers, Ch_n , are defined by means of the following generating function:

$$\frac{2}{2 + t} = \sum_{n=0}^{\infty} Ch_n \frac{t^n}{n!} \tag{1.17}$$

(cf. [18, 30]).

By using (1.17), an explicit formula for the Changhee numbers is given by

$$Ch_n = (-1)^n \frac{n!}{2^n} \quad (1.18)$$

(*cf.* [18, 30]).

Kucukoglu and Simsek [22] defined the numbers $\beta_n(k)$ by means of the following generating function:

$$\left(1 - \frac{z}{2}\right)^k = \sum_{n=0}^{\infty} \beta_n(k) \frac{z^n}{n!}, \quad (1.19)$$

where $k \in \mathbb{N}_0$, $z \in \mathbb{C}$ with $|z| < 2$.

By using (1.19), we have

$$\beta_n(k) = \frac{(-1)^n n!}{2^n} \binom{k}{n} = \binom{k}{n} Ch_n \quad (1.20)$$

where $n, k \in \mathbb{N}_0$ (*cf.* [22, Equations (4.9) and (4.10)]).

The polynomials $C_n(x, y)$ and $S_n(x, y)$ are defined by means of the following generating functions:

$$G_C(t, x, y) = e^{xt} \cos(yt) = \sum_{n=0}^{\infty} C_n(x, y) \frac{t^n}{n!} \quad (1.21)$$

and

$$G_S(t, x, y) = e^{xt} \sin(yt) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!}, \quad (1.22)$$

(*cf.* [9–12, 14–16, 20, 24]).

By using (1.21) and (1.22), the polynomials $C_n(x, y)$ and $S_n(x, y)$ are computed by the following formulas:

$$C_n(x, y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} x^{n-2j} y^{2j}$$

and

$$S_n(x, y) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^j \binom{n}{2j+1} x^{n-2j-1} y^{2j+1},$$

respectively (*cf.* [9–12, 14–16, 20, 24]).

By using (1.21) and (1.22), the polynomials $C_n(x, y)$ and $S_n(x, y)$ are also computed by the following formulas:

$$C_n(x, y) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{d=0}^{n-2j} (-1)^j \binom{n}{2j} S_2(n-2j, d) y^{2j} (x)_d \quad (1.23)$$

and

$$S_n(x, y) = \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{d=0}^{n-2j-1} (-1)^j \binom{n}{2j+1} S_2(n-2j-1, d) y^{2j+1} (x)_d \quad (1.24)$$

(cf. [2]).

Simsek [31] defined new classes of special numbers and polynomials by means of the following generating functions:

$$F_{\mathcal{Y}}(t, k, a) = \frac{at}{4 \sinh\left(\frac{(k+2)t}{2}\right) \cosh\left(\frac{kt}{2}\right)} = \sum_{n=0}^{\infty} \mathcal{Y}_n(k, a) \frac{t^n}{n!} \tag{1.25}$$

and

$$G_{\mathcal{Y}}(t, x, k, a) = e^{xt} F_{\mathcal{Y}}(t, k, a) = \sum_{n=0}^{\infty} Q_n(x, k, a) \frac{w^n}{n!}, \tag{1.26}$$

where $k \in \mathbb{Z}$ and $a \in \mathbb{R}$ (or \mathbb{C}).

Substituting $x = 0$ into (1.26), we have

$$\mathcal{Y}_n(k, a) = Q_n(0, k, a).$$

Simsek also gave the representation of equation (1.25) as follows:

$$F_{\mathcal{Y}}(t, k, a) = \frac{tae^{(k+1)t}}{(e^{(k+2)t} - 1)(e^{kt} + 1)}.$$

(cf. [31]).

By using (1.25) and (1.26), a relation between the polynomials $Q_n(x, k, a)$ and the numbers $\mathcal{Y}_n(k, a)$ is given as follows:

$$Q_n(x, k, a) = \sum_{j=0}^n \binom{n}{j} x^{n-j} \mathcal{Y}_j(k, a)$$

(cf. [31]).

By using (1.5), (1.8) and (1.25), we have the following identity:

$$\mathcal{Y}_n(k, a) = \frac{a}{2(k+2)} \sum_{s=0}^n \binom{n}{s} k^{n-s} (k+2)^s E_{n-s} B_s \left(\frac{k+1}{k+2}\right), \tag{1.27}$$

where $n \in \mathbb{N}_0$ (cf. [31, Equation (15)]).

Recently, Bayad and Simsek [2] defined new classes of the parametrically generalized polynomials, the polynomials $Q_n^{(C)}(x, y, k, a)$ and $Q_n^{(S)}(x, y, k, a)$, by means of the following generating functions, respectively:

$$H_C(t, x, y, a, k) = \frac{e^{xt} \cos(yt) at}{4 \sinh\left(\frac{(k+2)t}{2}\right) \cosh\left(\frac{kt}{2}\right)} = \sum_{n=0}^{\infty} Q_n^{(C)}(x, y, k, a) \frac{t^n}{n!} \tag{1.28}$$

and

$$H_S(t, x, y, a, k) = \frac{e^{xt} \sin(yt) at}{4 \sinh\left(\frac{(k+2)t}{2}\right) \cosh\left(\frac{kt}{2}\right)} = \sum_{n=0}^{\infty} Q_n^{(S)}(x, y, k, a) \frac{t^n}{n!}, \tag{1.29}$$

where $k \in \mathbb{Z}$ and $a \in \mathbb{R}$ (or \mathbb{C}).

By using (1.28) and (1.29), the polynomials $Q_n^{(C)}(x, y, k, a)$ and $Q_n^{(S)}(x, y, k, a)$ are computed by the following formulas:

$$Q_n^{(C)}(x, y, k, a) = \sum_{j=0}^n \binom{n}{j} \mathcal{Y}_j(k, a) C_{n-j}(x, y) \quad (1.30)$$

and

$$Q_n^{(S)}(x, y, k, a) = \sum_{j=0}^n \binom{n}{j} \mathcal{Y}_j(k, a) S_{n-j}(x, y) \quad (1.31)$$

(cf. [2]).

The rest of this article is briefly summarized as follows:

In Section 2, by using generating functions and functional equations techniques, we derive some formulas, combinatorial sums and relations including the parametrically generalized polynomials, the Bernoulli numbers and polynomials of higher order, the Euler numbers and polynomials of higher order, the Euler numbers of the second kind, the polynomials $C_n(x, y)$, and the polynomials $S_n(x, y)$.

In Section 3, we give many inequalities for combinatorial sums including the Bernoulli numbers of negative order, the Euler numbers of negative order, the Bernoulli polynomials, the Changhee numbers, the Daehee numbers, the Stirling numbers, the numbers $B(n, k)$ and the numbers $\beta_n(k)$.

In Section 4, using Mathematica with the help of the Wolfram programming language, we present some plots of the parametrically generalized polynomials under some of their randomly selected special cases.

Finally, in Section 5, we give remarks and observations on our results.

2. COMBINATORIAL SUMS AND IDENTITIES FOR THE PARAMETRICALLY GENERALIZED POLYNOMIALS, AND SPECIAL NUMBERS AND POLYNOMIALS

In this section, using generating functions and functional equations, we give some interesting identities and combinatorial sums related to the parametrically generalized polynomials, the polynomials $C_n(x, y)$, the polynomials $S_n(x, y)$, the Bernoulli numbers and polynomials of higher order, the Euler numbers and polynomials of higher order and the Euler numbers of the second kind.

Theorem 2.1. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$C_n(x + k + 1, y) = \sum_{d=0}^n \sum_{j=0}^d \binom{d}{j} \binom{n}{d} \frac{2(k+2)^{j+1} k^{d-j}}{a(j+1)} E_{d-j}^{(-1)} Q_{n-d}^{(C)}(x, y, k, a).$$

Proof. Combining (1.28) with (1.4), (1.8) and (1.21), we get the following functional equation:

$$\frac{a}{2(k+2)} G_C(t, x+k+1, y) = F_B((k+2)t, -1) F_E(kt, -1) H_C(t, x, y, a, k).$$

From the above equation, we obtain

$$\begin{aligned} & \frac{a}{2(k+2)} \sum_{n=0}^{\infty} C_n(x+k+1, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (k+2)^n B_n^{(-1)} \frac{t^n}{n!} \sum_{n=0}^{\infty} k^n E_n^{(-1)} \frac{t^n}{n!} \sum_{n=0}^{\infty} Q_n^{(C)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{a}{2(k+2)} \sum_{n=0}^{\infty} C_n(x+k+1, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{d=0}^n \sum_{j=0}^d \binom{d}{j} \binom{n}{d} (k+2)^j k^{d-j} B_j^{(-1)} E_{d-j}^{(-1)} Q_{n-d}^{(C)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, and combining with following well-known formula

$$B_n^{(-1)} = \frac{1}{n+1},$$

we arrive at the desired result.

Theorem 2.2. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$S_n(x+k+1, y) = \sum_{d=0}^n \sum_{j=0}^d \binom{d}{j} \binom{n}{d} \frac{2(k+2)^{j+1} k^{d-j}}{a(j+1)} E_{d-j}^{(-1)} Q_{n-d}^{(S)}(x, y, k, a).$$

Proof. Combining (1.29) with (1.4), (1.8) and (1.22), we have

$$\frac{a}{2(k+2)} G_S(t, x+k+1, y) = F_B((k+2)t, -1) F_E(kt, -1) H_S(t, x, y, a, k).$$

From the above functional equation, we obtain

$$\begin{aligned} & \frac{a}{2(k+2)} \sum_{n=0}^{\infty} S_n(x+k+1, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (k+2)^n B_n^{(-1)} \frac{t^n}{n!} \sum_{n=0}^{\infty} k^n E_n^{(-1)} \frac{t^n}{n!} \sum_{n=0}^{\infty} Q_n^{(S)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} & \frac{a}{2(k+2)} \sum_{n=0}^{\infty} S_n(x+k+1, y) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{d=0}^n \sum_{j=0}^d \binom{d}{j} \binom{n}{d} (k+2)^j k^{d-j} B_j^{(-1)} E_{d-j}^{(-1)} Q_{n-d}^{(S)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Theorem 2.3. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$B_n = \frac{1}{a(n+1)(k+2)^{n-1}} \sum_{d=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{n+1-2d} (-1)^d \binom{n+1-2d}{j} \binom{n+1}{2d} 2^{2d+1} y^{2d-1} k^j \quad (2.32)$$

$$\times E_j^{(-1)} \left(\frac{-x-k-1}{k} \right) B_{2d} \left(\frac{1}{2} \right) Q_{n+1-2d-j}^{(S)}(x, y, k, a).$$

Proof. By using (1.4), (1.8) and (1.29), we get the following functional equation:

$$\frac{a}{(k+2)} F_B((k+2)t, 1) = \frac{2}{\sin(yt)} G_E \left(kt, \frac{-x-k-1}{k}, -1 \right) H_S(t, x, y, a, k).$$

Combining above equation with the following well-known identity:

$$\frac{t}{\sin(t)} = \sum_{n=0}^{\infty} (-1)^n 2^{2n} B_{2n} \left(\frac{1}{2} \right) \frac{t^{2n}}{(2n)!} \quad (2.33)$$

(cf. [19, Equation (2.24)]), we have

$$\frac{ay}{2(k+2)} \sum_{n=0}^{\infty} (k+2)^n B_n \frac{t^{n+1}}{n!} = \sum_{n=0}^{\infty} (-1)^n (2y)^{2n} B_{2n} \left(\frac{1}{2} \right) \frac{t^{2n}}{(2n)!}$$

$$\times \sum_{n=0}^{\infty} E_n^{(-1)} \left(\frac{-x-k-1}{k} \right) \frac{(kt)^n}{n!} \sum_{n=0}^{\infty} Q_n^{(S)}(x, y, k, a) \frac{t^n}{n!}.$$

Therefore

$$\frac{ay}{2} \sum_{n=0}^{\infty} n (k+2)^{n-2} B_{n-1} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \sum_{d=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2d} \sum_{j=0}^{n-2d} \binom{n-2d}{j} k^j E_j^{(-1)} \left(\frac{-x-k-1}{k} \right)$$

$$\times Q_{n-2d-j}^{(S)}(x, y, k, a) (-1)^d (2y)^{2d} B_{2d} \left(\frac{1}{2} \right) \frac{t^n}{n!}.$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Theorem 2.4. *Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have*

$$E_n = \frac{1}{ak^n(n+1)} \sum_{d=0}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{j=0}^{n+1-2d} (-1)^d \binom{n+1-2d}{j} \binom{n+1}{2d} 2^{2d+1} y^{2d-1} (k+2)^{j+1}$$

$$\times B_j^{(-1)} \left(\frac{-x-k-1}{k+2} \right) B_{2d} \left(\frac{1}{2} \right) Q_{n+1-2d-j}^{(S)}(x, y, k, a).$$

Proof. By using (1.5), (1.8) and (1.29), we get the following functional equation:

$$\frac{a}{2} F_E(kt, 1) = \frac{k+2}{\sin(yt)} G_B\left((k+2)t, \frac{-x-k-1}{k+2}, -1\right) H_S(t, x, y, a, k).$$

Combining above equation with (2.33), we have

$$\begin{aligned} \frac{ayt}{2} \sum_{n=0}^{\infty} k^n E_n \frac{t^n}{n!} &= \sum_{n=0}^{\infty} (-1)^n (2y)^{2n} B_{2n} \left(\frac{1}{2}\right) \frac{t^{2n}}{(2n)!} \\ &\times \sum_{n=0}^{\infty} (k+2)^n B_n^{(-1)} \left(\frac{-x-k-1}{k+2}\right) \frac{t^n}{n!} \sum_{n=0}^{\infty} Q_n^{(S)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{ay}{2} \sum_{n=0}^{\infty} n k^{n-1} E_{n-1} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} \sum_{d=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^d \binom{n}{2d} \sum_{j=0}^{n-2d} \binom{n-2d}{j} (k+2)^j \\ &\times B_j^{(-1)} \left(\frac{-x-k-1}{k+2}\right) Q_{n-2d-j}^{(S)}(x, y, k, a) (2y)^{2d} B_{2d} \left(\frac{1}{2}\right) \frac{t^n}{n!}. \end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Theorem 2.5. Let $n \in \mathbb{N}_0$ and $a \neq 0$. Then we have

$$\frac{a}{2} \sum_{j=0}^n \binom{n}{j} \frac{(k+2)^{j-1}}{k^{j-n}} B_j \left(\frac{x}{k+2}\right) E_{n-j} \left(\frac{k+1}{k}\right) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n}{2j} y^{2j} E_{2j}^* Q_{n-2j}^{(C)}(x, y, k, a).$$

Proof. By using (1.5), (1.9) and (1.28), we get the following functional equation:

$$\frac{a}{2(k+2)} G_B\left((k+2)t, \frac{x}{k+2}, 1\right) G_E\left(kt, \frac{k+1}{k}, 1\right) = \sec(yt) H_C(t, x, y, a, k).$$

Combining above equation with the following well-known identity:

$$\sec(t) = \sum_{n=0}^{\infty} (-1)^n E_{2n}^* \frac{t^{2n}}{(2n)!} \tag{2.34}$$

(cf. [19, Equation (2.40)]), we have

$$\begin{aligned} &\frac{a}{2(k+2)} \sum_{n=0}^{\infty} (k+2)^n B_n \left(\frac{x}{k+2}\right) \frac{t^n}{n!} \sum_{n=0}^{\infty} k^n E_n \left(\frac{k+1}{k}\right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} (-1)^n E_{2n}^* \frac{t^{2n}}{(2n)!} \sum_{n=0}^{\infty} Q_n^{(C)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} &\frac{a}{2(k+2)} \sum_{n=0}^{\infty} \sum_{j=0}^n \binom{n}{j} (k+2)^j k^{n-j} B_j \left(\frac{x}{k+2}\right) E_{n-j} \left(\frac{k+1}{k}\right) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2j} (-1)^j E_{2j}^* Q_{n-2j}^{(C)}(x, y, k, a) \frac{t^n}{n!}. \end{aligned}$$

Comparing coefficient of $\frac{t^n}{n!}$ on both sides of the above equation, after some elementary calculations, we arrive at the desired result.

Combining (1.24) with (1.12), after some elementary calculations, we obtain the following theorem:

Theorem 2.6. *Let $n \in \mathbb{N}$. Then we have*

$$\begin{aligned} S_n(x, y) &= \sum_{j=0}^{\lfloor \frac{n-1}{2} \rfloor} \sum_{d=0}^{n-2j-1} (-1)^j \binom{n}{2j+1} \frac{2^{d-n+2j+1} y^{2j+1} (x)_d}{d!} \\ &\quad \times \sum_{m=0}^{n-2j-1} \sum_{v=0}^d (-1)^{d-v} \binom{n-2j-1}{m} \binom{d}{v} v^m E_{n-2j-1-m}^{(-d)}. \end{aligned}$$

3. INEQUALITIES APPLICATIONS FOR COMBINATORIAL SUMS INVOLVING SPECIAL NUMBERS

In this section, we give the upper bound and the lower bound for the special numbers and polynomials, and combinatorial sums involving the Bernoulli numbers of negative order, the Euler numbers of negative order, the Changhee numbers, the Daehee numbers, the Stirling numbers of the second kind, the numbers $B(n, k)$ and the numbers $\beta_n(k)$.

In order to give our results, we need the following inequalities for the special numbers.

Gun and Simsek [8] gave the lower bound and the upper bound for the Bernoulli numbers of negative order $B_n^{(-k)}$ as follows:

$$B_n^{(-k)} \geq \frac{k^n}{\binom{n+k}{k}} \quad (3.35)$$

and

$$B_n^{(-k)} \leq \frac{\binom{n+k-1}{k-1} k^n}{\binom{n+k}{k}}, \quad (3.36)$$

where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$.

Comtet [6] gave the lower bound and the upper bound for the Stirling numbers of the second kind $S_2(n, k)$ as follows:

$$S_2(n, k) \geq k^{n-k} \quad (3.37)$$

and

$$S_2(n, k) \leq \binom{n-1}{k-1} k^{n-k}. \quad (3.38)$$

Abramowitz and Stegun [1, p. 805] gave the following inequality for the Bernoulli numbers:

$$\frac{2(2n)!}{(2\pi)^{2n}} < (-1)^{n+1} B_{2n} < \frac{2(2n)!}{(2\pi)^{2n} (1 - 2^{1-2n})}, \quad (3.39)$$

where $n \in \mathbb{N}$.

Combining (1.11) with (3.35), we get the following theorem for the Euler numbers of negative order and the Bernoulli numbers of negative order:

Theorem 3.1. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \geq \frac{2^n k^n}{\binom{k+n}{k}}. \tag{3.40}$$

By using (1.6), (1.10), (1.18) and (3.40), we derive the following corollary:

Corollary 3.1. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \sum_{d=0}^k \sum_{m=0}^{n-j} \frac{(-1)^{k-d}}{\binom{j+k}{k} k!} \binom{n}{j} \binom{k}{d} \binom{m-k-1}{m} d^{j+k} Ch_m S_2(n-j, m) \geq \frac{2^n k^n}{\binom{k+n}{k}}.$$

By using (1.18), (1.20) and (3.40), we get the following corollary:

Corollary 3.2. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \geq \frac{(2k)^n Ch_k}{\beta_k(n+k)}.$$

Combining (1.11) with (3.36), we obtain the following theorem:

Theorem 3.2. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \leq \frac{2^n \binom{n+k-1}{k-1} k^n}{\binom{n+k}{k}}. \tag{3.41}$$

Substituting $n = k$ into (3.41), we arrive at the following result:

Corollary 3.3. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} B_j^{(-n)} E_{n-j}^{(-n)} \leq \frac{2^n \binom{2n-1}{n-1} n^n}{\binom{2n}{n}}.$$

By using (1.18), (1.20) and (3.41), we obtain the following corollary:

Corollary 3.4. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \binom{n}{j} B_j^{(-k)} E_{n-j}^{(-k)} \leq (2k)^n \frac{\beta_{k-1}(n+k-1) Ch_k}{\beta_k(n+k) Ch_{k-1}}.$$

Combining (1.14) with (3.37), we get the following theorem:

Theorem 3.3. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \frac{\binom{n}{j} \binom{n+k}{k}}{2^{k+n} \binom{j+k}{k}} S_2(j+k, k) B(n-j, k) \geq k^n. \tag{3.42}$$

By using (1.14), (1.16), (1.20) and (3.37), we have the following corollary:

Corollary 3.5. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\sum_{j=0}^n \frac{\binom{n}{j} \beta_k(n+k)}{2^n \binom{j+k}{k} D_k} S_2(j+k, k) B(n-j, k) \geq (k+1) k^n.$$

Combining (1.12) with (3.37), we arrive at the following theorem:

Theorem 3.4. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\frac{2^{k-n}}{k!} \sum_{m=0}^n \sum_{j=0}^k (-1)^{k-j} \binom{n}{m} \binom{k}{j} j^m E_{n-m}^{(-k)} \geq k^{n-k}.$$

Combining (1.12) with (1.20) and (3.38), we get the following theorem for the Euler numbers of negative order:

Theorem 3.5. *Let $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$. Then we have*

$$\frac{2^{k-n}}{k!} \sum_{m=0}^n \sum_{j=0}^k (-1)^{k-j} \binom{n}{m} \binom{k}{j} j^m E_{n-m}^{(-k)} \leq \frac{k^{n-k} \beta_{k-1}(n-1)}{Ch_{k-1}}.$$

4. SOME PLOTS OF THE PARAMETRICALLY GENERALIZED POLYNOMIALS

In this section, with the help of Wolfram programming language in Mathematica 35, we illustrated the plots of the parametrically generalized polynomials by applying the formulas given by (1.30) and (1.31).

Figure 1 is obtained by $y = 2$, $k = -10$, $a = 2$, and $n \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-50, 50]$.

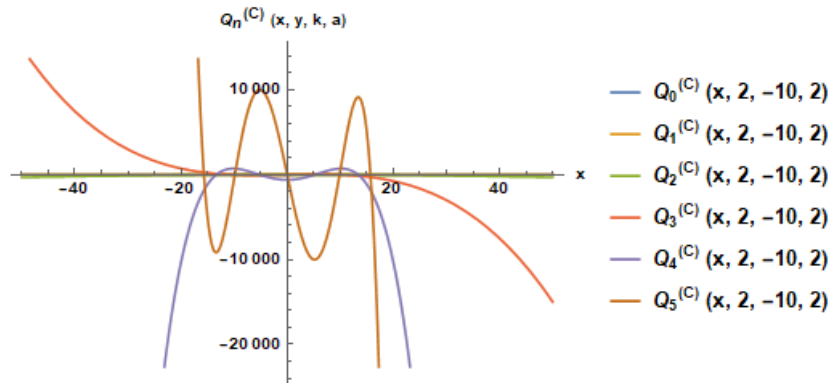


FIGURE 1. Plots of the polynomials $Q_n^{(C)}(x, 2, -10, 2)$ for randomly selected special cases when $n \in \{0, 1, 2, 3, 4, 5\}$ and $x \in [-50, 50]$.

Figure 2 is obtained by $n = 4$, $y = 2$, $a = 2$, and $k \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-5, 5]$.

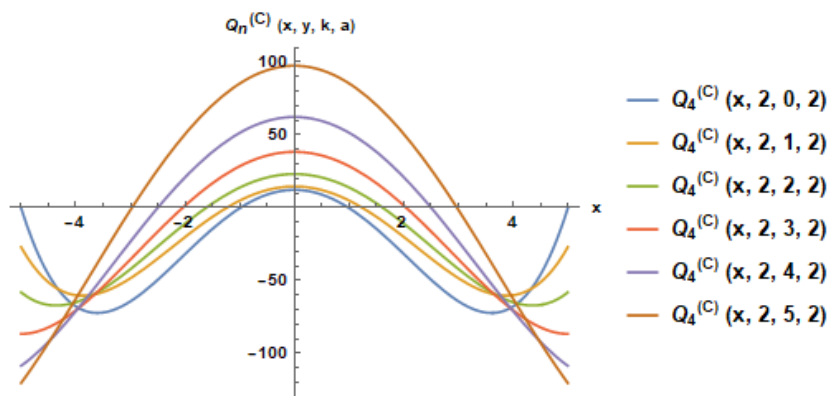


FIGURE 2. Plots of the polynomials $Q_n^{(C)}(x, 2, k, 2)$ for randomly selected special cases when $k \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-5, 5]$.

Figure 3 is obtained by $n = 4$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-6, 6]$.

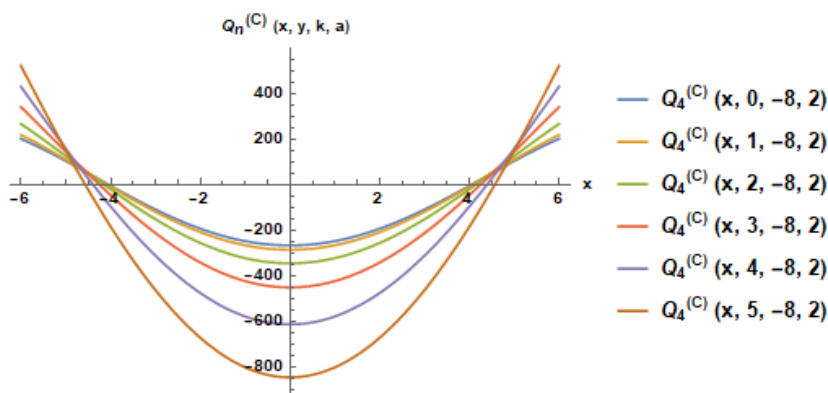


FIGURE 3. Plots of the polynomials $Q_n^{(C)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-6, 6]$.

Figure 4 is obtained by $n = 15$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.30) for $x \in [-6, 6]$.

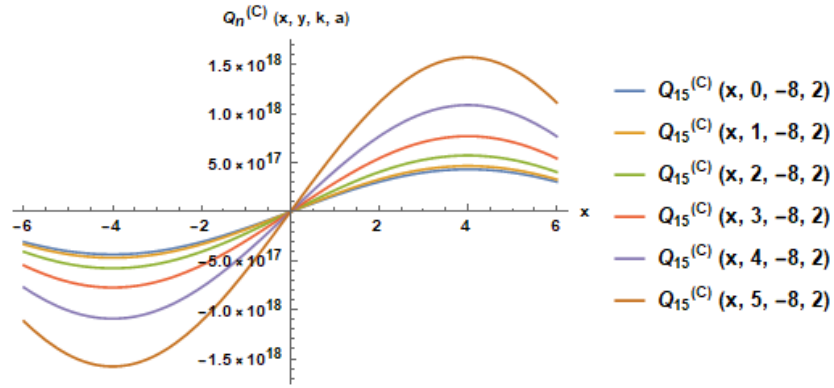


FIGURE 4. Plots of the polynomials $Q_n^{(C)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 15$ and $x \in [-6, 6]$.

Figure 5 is obtained by $y = 2$, $k = -10$, $a = 2$, and $n \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-50, 50]$.

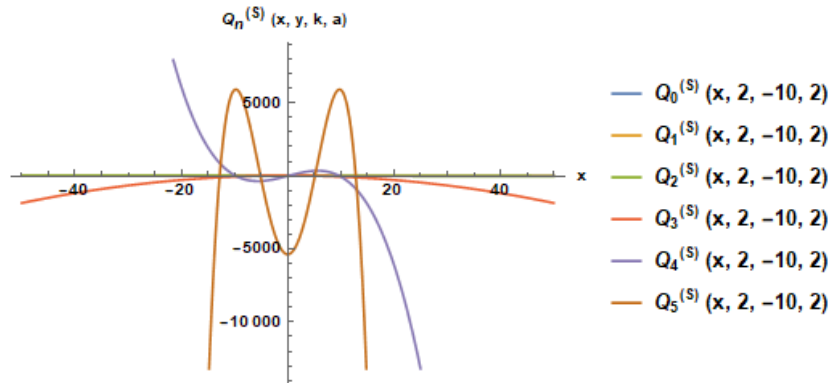


FIGURE 5. Plots of the polynomials $Q_n^{(S)}(x, 2, -10, 2)$ for randomly selected special cases when $n \in \{0, 1, 2, 3, 4, 5\}$ and $x \in [-50, 50]$.

Figure 6 is obtained by $n = 4$, $y = 2$, $a = 2$, and $k \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-5, 5]$.

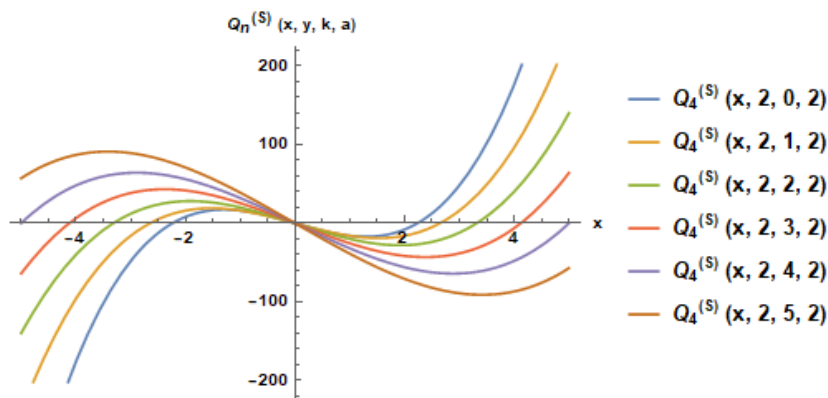


FIGURE 6. Plots of the polynomials $Q_n^{(S)}(x, 2, k, 2)$ for randomly selected special cases when $k \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-5, 5]$.

Figure 7 is obtained by $n = 4$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-5, 5]$.

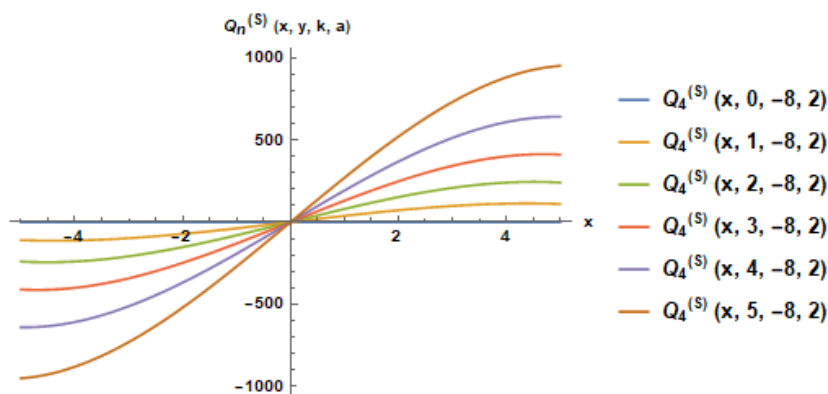


FIGURE 7. Plots of the polynomials $Q_n^{(S)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 4$ and $x \in [-5, 5]$.

Figure 8 is obtained by $n = 15$, $k = -8$, $a = 2$, and $y \in \{0, 1, 2, 3, 4, 5\}$ using (1.31) for $x \in [-6, 6]$.

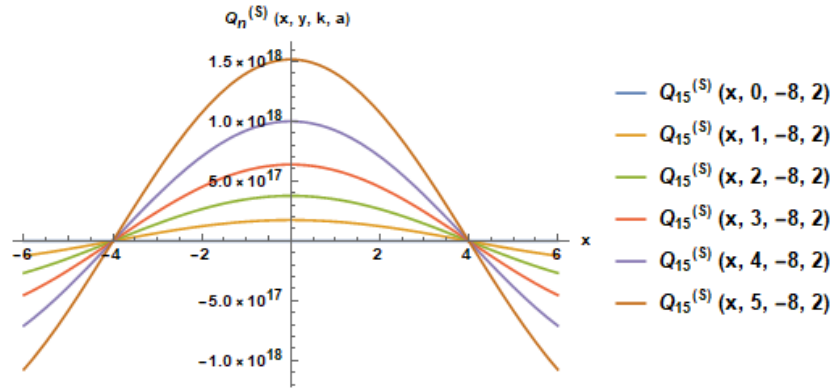


FIGURE 8. Plots of the polynomials $Q_n^{(S)}(x, y, -8, 2)$ for randomly selected special cases when $y \in \{0, 1, 2, 3, 4, 5\}$ with $n = 15$ and $x \in [-6, 6]$.

5. CONCLUSION

Special numbers, special polynomials and trigonometric functions are among remarkably wide used in applied mathematics, combinatorial analysis, mathematical analysis, analytic number theory, mathematical physics, and engineering. Recently using different techniques and methods, many properties of parametrically polynomials involving trigonometric functions have been studied by many researchers. Using both the generating functions and their functional equations techniques and some known results, we obtained many interesting identities, combinatorial sums and inequalities including the Euler numbers and polynomials of higher order, the Bernoulli numbers of higher order, the Changhee numbers, the Daehee numbers, the parametrically generalized polynomials, the Stirling numbers and also well-known special polynomials. By using Mathematica with the help of the Wolfram programming language, we gave some plots of the parametrically generalized polynomials under the special cases. Consequently, the results of this article have the potential to be used both pure and applied mathematics, physics, engineering and other related areas, and to attract the attention of researchers working in this areas.

Acknowledgments. The second-named author was supported by the Scientific Research Project Administration of the University of Akdeniz.

REFERENCES

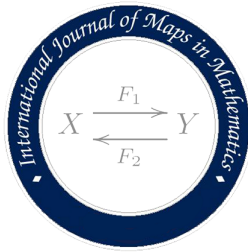
- [1] Abramowitz, M. & Stegun, I. A. (1970). Handbook of mathematical functions with formulas, graphs and mathematical tables. New York: Dover Publication.

- [2] Bayad, A. & Simsek, Y. (2021). On generating functions for parametrically generalized polynomials involving combinatorial, Bernoulli and Euler polynomials and numbers. Preprint.
- [3] Carlitz, L. (1952). Some theorems on Bernoulli numbers of higher order. *Pac. J. Math.*, 2(2), 127-139.
- [4] Carlitz, L. (1963). Some formulas for the BERNOULLI and EULER polynomials. *Math. Nachr.*, 25, 223-231.
- [5] Carlitz, L. (1965). Recurrences for the Bernoulli and Euler numbers. II. *Math. Nachr.*, 29, 151-160.
- [6] Comtet, L. (1974). *Advanced combinatorics*. Dordrecht-Holland/ Boston-U.S.A: D. Reidel Publication Company.
- [7] Golombek, R. (1994). Aufgabe 1088. *El Math*, 49, 126.
- [8] Gun, D. & Simsek, Y. (2020). Some new identities and inequalities for Bernoulli polynomials and numbers of higher order related to the Stirling and Catalan numbers. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM*, 114, <https://doi.org/10.1007/s13398-020-00899-z>.
- [9] Kilar, N. & Simsek, Y. (2019). Relations on Bernoulli and Euler polynomials related to trigonometric functions. *Adv. Stud. Contemp. Math.*, 29, 191-198.
- [10] Kilar, N. & Simsek, Y. (2019). Two parametric kinds of Eulerian-type polynomials associated with Euler's formula. *Symmetry*, 11, 1097.
- [11] Kilar, N. & Simsek, Y. (2020). A special approach to derive new formulas for some special numbers and polynomials. *Turk. J. Math.*, 44, 2217-2240.
- [12] Kilar, N. & Simsek, Y. (2020). Y. Formulas involving sums of powers, special numbers and polynomials arising from p -adic integrals, trigonometric and generating functions. *Publ. Inst. Math.*, 108, 103-120.
- [13] Kilar, N. & Simsek, Y. (2021). Formulas and relations of special numbers and polynomials arising from functional equations of generating functions. *Montes Taurus J. Pure Appl. Math.*, 3(1), 106-123.
- [14] Kilar, N. & Simsek, Y. (2021). Identities and relations for Hermite-based Milne–Thomson polynomials associated with Fibonacci and Chebyshev polynomials. *Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM*, 115, <https://doi.org/10.1007/s13398-020-00968-3>.
- [15] Kilar, N. & Simsek, Y. (2021). Computational formulas and identities for new classes of Hermite-based Milne–Thomson type polynomials: Analysis of generating functions with Euler's formula. *Math Meth Appl Sci.*, 44, 6731-6762.
- [16] Kilar, N. & Simsek, Y. (2021). New computational formulas for special numbers and polynomials derived from applying trigonometric functions to generating functions. *Milan J. Math.*, 89, 217-239.
- [17] Kim, D. S. & Kim, T. (2013). Daehee numbers and polynomials. *Appl. Math. Sci. (Ruse)*, 7(120), 5969-5976.
- [18] Kim, D. S., Kim, T. & Seo, J. (2013). A note on Changhee numbers and polynomials. *Adv. Stud. Theor. Phys.*, 7, 993-1003.
- [19] Kim, D. S. & Kim, T. (2018). Some p -adic integrals on \mathbb{Z}_p associated with trigonometric functions. *Russ. J. Math. Phys.*, 25(3), 300-308.
- [20] Kim, T. & Ryoo, C. S. (2018). Some identities for Euler and Bernoulli polynomials and their zeros. *Axioms*, 7, 56.

- [21] Knuth, D. E. & Buckholtz, T. J. (1967). Computation of tangent, Euler, and Bernoulli numbers. *Math. Comput.*, 21, 663-688.
- [22] Kucukoglu, I. & Simsek, Y. (2021). New formulas and numbers arising from analyzing combinatorial numbers and polynomials. *Montes Taurus J. Pure Appl. Math.*, 3(3), 238-259.
- [23] Luo, Q-M. (2006). Apostol-Euler polynomials of higher order and Gaussian hypergeometric functions. *Taiwanese J. Math.*, 10, 917-925.
- [24] Masjed-Jamei, M. & Koepf, W. (2017). Symbolic computation of some power-trigonometric series. *J. Symb. Comput.*, 80, 273-284.
- [25] Riordan, J. (1958). *An introduction to combinatorial analysis*. New York: John Wiley Sons, Inc.
- [26] Simsek, Y. (2010). Special functions related to Dedekind-type DC-sums and their applications. *Russ. J. Math. Phys.*, 17(4), 495-508.
- [27] Simsek, Y. (2013). Generating functions for generalized Stirling type numbers, array type polynomials, Eulerian type polynomials and their applications. *Fixed Point Theory Appl.*, 87, 1-28.
- [28] Simsek, Y. (2017). Computation methods for combinatorial sums and Euler-type numbers related to new families of numbers. *Math. Meth. Appl. Sci.*, 40(7), 2347-2361.
- [29] Simsek, Y. (2018). New families of special numbers for computing negative order Euler numbers and related numbers and polynomials. *Appl. Anal. Discrete Math.*, 12, 1-35.
- [30] Simsek, Y. (2019). Explicit formulas for p -adic integrals: Approach to p -adic distributions and some families of special numbers and polynomials. *Montes Taurus J. Pure Appl. Math.*, 1(1), 1-76.
- [31] Simsek, Y. (2021). New classes of recurrence relations involving hyperbolic functions, special numbers and polynomials. *Appl. Anal. Discrete Math.*, <https://doi.org/10.2298/AADM201020015S>.
- [32] Spivey, M. Z. (2007). Combinatorial sums and finite differences. *Discrete Math.*, 307(24), 3130-3146.
- [33] Srivastava, H. M. (2011). Some generalizations and basic (or q -) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inf. Sci.*, 5(3), 390-444.
- [34] Srivastava, H. M. & Choi, J. (2012). *Zeta and q -zeta functions and associated series and integrals*. Amsterdam: Elsevier Science Publishers.
- [35] Wolfram Research Inc. (2020). *Mathematica online (Wolfram Cloud)*, Champaign, IL, <https://www.wolframcloud.com>

DEPARTMENT OF COMPUTER TECHNOLOGIES, BOR VOCATIONAL SCHOOL, NIGDE OMER HALISDEMIR UNIVERSITY, TR-51700, NIGDE-TURKEY

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE UNIVERSITY OF AKDENIZ TR-07058 ANTALYA-TURKEY



HARMONICITY OF MUS-GRADIENT METRIC

NOUR EL HOUDA DJAA  * AND ABDERRAHIM ZAGANE 

ABSTRACT. Let (M^m, g) be an m -dimensional Riemannian manifold. In this paper, we introduce an other class of metric on (M^m, g) called Mus-gradient metric. First we investigate the Levi-Civita connection of this metric. Secondly we study some properties of harmonicity with respect to the Mus-gradient metric. In the last section, we investigate the harmonicity of Mus-gradient metric on product manifolds. Also, we construct some examples of harmonic maps.

Keywords: Levi-Civita connexion, Mus-Gradient metric, Harmonic maps

2010 Mathematics Subject Classification: 53A45, 53C20, 58E20.

1. INTRODUCTION

The theory of harmonic maps studies the mapping between different metric manifolds from the energy-minimization point of view (solutions to a natural geometrical variational problem). This concept has several applications such as geodesics, minimal surfaces and harmonic functions. Harmonic maps are also closely related to holomorphic maps in several complex variables, to the theory of stochastic processes, to nonlinear field theory in theoretical physics, and to the theory of liquid crystals in materials science. The last years this subject has been developed extensively by several authors (for example see [1], [3], [4], [5], [7], [8], [12], [10], [11], [12] etc...).

The main idea in this note consists in the modification of the metric of the Riemannian

Received: 2021.06.12

Revised: 2021.09.26

Accepted: 2021.10.01

* Corresponding author

Nour El Houda Djaa; Djaanor@hotmail.fr; <https://orcid.org/0000-0002-0568-0629>

Abderrahim Zagane; Zaganeabr2018@gmail.com; <https://orcid.org/0000-0001-9339-3787>

manifold (M^m, g) . Firstly we introduce the Mus-gradient metric on M noted by \tilde{g} and we investigate the Levi-Civita connection of this metric (Theorem 2.1). Secondly we study the harmonicity with respect to the Mus-gradient metric, then we establish necessary and sufficient conditions under which the Identity Map is harmonic with respect to this metric (Theorem 3.2 and Theorem 3.4). Next we study the harmonicity of the map $\sigma : (M, \tilde{g}) \rightarrow (N, h)$ (Theorem 3.6) and the map $\sigma : (M, g) \rightarrow (N, \tilde{h})$ (Theorem 3.8). In the last section, we investigate the harmonicity of Mus-gradient metric on product manifolds (Theorem 4.1 to Theorem 4.7). We also construct some examples of harmonic maps.

2. MUS-GRADIENT METRIC

Definition 2.1. *Let (M^m, g) be a Riemannian manifold and $f : M \rightarrow]0, +\infty[$ be a strictly positive smooth function. We define the Mus-gradient metric on M noted \tilde{g} by*

$$\tilde{g}(X, Y)_x = f(x)g(X, Y)_x + X_x(f)Y_x(f), \quad (2.1)$$

where $x \in M$ and $X, Y \in \mathfrak{S}_0^1(M)$, f is called twisting function.

In the following, we consider $\|\text{grad } f\| = 1$, where $\|\cdot\|$ denote the norm with respect to (M^m, g) .

Lemma 2.1. *Let $\text{grad } f$ (resp. $\widetilde{\text{grad}} f$) denote the gradient of f with respect to g (resp. \tilde{g}), then we have*

$$\widetilde{\text{grad}} f = \frac{1}{f+1} \text{grad } f. \quad (2.2)$$

Proof. We have

$$\begin{aligned} X(f) &= g(X, \text{grad } f) \\ &= \frac{1}{f} (\tilde{g}(X, \text{grad } f) - X(f)(\text{grad } f)(f)) \\ &= \frac{1}{f} (\tilde{g}(X, \text{grad } f) - X(f)) \end{aligned}$$

on the other hand, we have $X(f) = \tilde{g}(X, \widetilde{\text{grad}} f)$, then

$$\begin{aligned} \tilde{g}(X, \widetilde{\text{grad}} f) &= \frac{1}{f} (\tilde{g}(X, \text{grad } f) - \tilde{g}(X, \widetilde{\text{grad}} f)) \\ &= \frac{1}{f+1} \tilde{g}(X, \text{grad } f) \end{aligned}$$

so, thus $\widetilde{\text{grad}} f = \frac{1}{f+1} \text{grad } f$.

We shall calculate the Levi-Civita connection $\tilde{\nabla}$ of (M^m, \tilde{g}) , as follows.

Theorem 2.1. *Let (M^m, g) be a Riemannian manifold, the Levi-Civita connection $\tilde{\nabla}$ of (M^m, \tilde{g}) , is given by*

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \frac{X(f)}{2f} Y + \frac{Y(f)}{2f} X \\ &\quad + \left(\frac{\text{Hess}_f(X, Y)}{f+1} - \frac{X(f)Y(f)}{f(f+1)} - \frac{g(X, Y)}{2(f+1)} \right) \text{grad } f \end{aligned} \tag{2.3}$$

for all vector fields $X, Y \in \mathfrak{S}_0^1(M)$, where ∇ denote the Levi-Civita connection of (M^m, g) and $\text{Hess}_f(X, Y) = g(\nabla_X \text{grad } f, Y)$ is the Hessian of f with respect to g .

Proof. From Kozul formula and Lemma 2.1, we have

$$\begin{aligned} 2\tilde{g}(\tilde{\nabla}_X Y, Z) &= X\tilde{g}(Y, Z) + Y\tilde{g}(Z, X) - Z\tilde{g}(X, Y) + \tilde{g}(Z, [X, Y]) \\ &\quad + \tilde{g}(Y, [Z, X]) - \tilde{g}(X, [Y, Z]) \\ &= X(fg(Y, Z) + Y(f)Z(f)) + Y(fg(Z, X) + Z(f)X(f)) \\ &\quad - Z(fg(X, Y) + X(f)Y(f)) + fg(Z, [X, Y]) + Z(f)[X, Y](f) \\ &\quad + fg(Y, [Z, X]) + Y(f)[Z, X](f) - fg(X, [Y, Z]) \\ &\quad - X(f)[Y, Z](f) \\ &= X(f)g(Y, Z) + fXg(Y, Z) + X(Y(f))Z(f) + Y(f)X(Z(f)) \\ &\quad + Y(f)g(Z, X) + fYg(Z, X) + Y(Z(f))X(f) + Z(f)Y(X(f)) \\ &\quad - Z(f)g(X, Y) - fZg(X, Y) - Z(X(f))Y(f) - X(f)Z(Y(f)) \\ &\quad + fg(Z, [X, Y]) + Z(f)(X(Y(f)) - Y(X(f))) + fg(Y, [Z, X]) \\ &\quad + Y(f)(Z(X(f)) - X(Z(f))) - fg(X, [Y, Z]) \\ &\quad - X(f)(Y(Z(f)) - Z(Y(f))) \\ &= 2fg(\nabla_X Y, Z) + X(f)g(Y, Z) + Y(f)g(Z, X) - Z(f)g(X, Y) \\ &\quad + 2X(Y(f))Z(f) \\ &= 2\tilde{g}(\nabla_X Y, Z) - 2(\nabla_X Y)(f)Z(f) + 2X(Y(f))Z(f) \\ &\quad + \frac{X(f)}{f}(\tilde{g}(Y, Z) - Y(f)Z(f)) + \frac{Y(f)}{f}(\tilde{g}(Z, X) - Z(f)X(f)) \\ &\quad - Z(f)g(X, Y). \end{aligned}$$

From the definition of Hessian, we obtain

$$\begin{aligned}
2\tilde{g}(\tilde{\nabla}_X Y, Z) &= 2\tilde{g}(\nabla_X Y, Z) + \frac{X(f)}{f}\tilde{g}(Y, Z) + \frac{Y(f)}{f}\tilde{g}(Z, X) \\
&\quad + (2Hess_f(X, Y) - \frac{2X(f)Y(f)}{f} - g(X, Y))Z(f) \\
&= 2\tilde{g}(\nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X, Z) \\
&\quad + 2(Hess_f(X, Y) - \frac{X(f)Y(f)}{f} - \frac{1}{2}g(X, Y))\tilde{g}(\widetilde{grad f}, Z).
\end{aligned}$$

From the formula (2.2), we get

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + \frac{X(f)}{2f}Y + \frac{Y(f)}{2f}X \\
&\quad + \left(\frac{Hess_f(X, Y)}{f+1} - \frac{X(f)Y(f)}{f(f+1)} - \frac{g(X, Y)}{2(f+1)} \right) grad f.
\end{aligned}$$

Lemma 2.2. *Let (M^m, g) be a Riemannian manifold, then for all vector field $X \in \mathfrak{S}_0^1(M)$, we have*

$$\tilde{\nabla}_X grad f = \nabla_X grad f + \frac{1}{2f}X - \frac{X(f)}{2f(f+1)}grad f. \quad (2.4)$$

Proof. Using the theorem 2.1, we have

$$\begin{aligned}
\tilde{\nabla}_X grad f &= \nabla_X grad f + \frac{X(f)}{2f}grad f + \frac{(grad f)(f)}{2f}X \\
&\quad + \left(\frac{Hess_f(X, grad f)}{f+1} - \frac{X(f)(grad f)(f)}{f(f+1)} - \frac{g(X, grad f)}{2(f+1)} \right) grad f.
\end{aligned}$$

Since $\|grad f\| = 1$, we obtain $(grad f)(f) = 1$ and $Hess_f(X, grad f) = 0$. then we get

$$\tilde{\nabla}_X grad f = \nabla_X grad f + \frac{1}{2f}X - \frac{X(f)}{2f(f+1)}grad f.$$

3. HARMONICITY OF MUS-GRADIENT METRIC

Consider a smooth map $\phi : (M^m, g) \rightarrow (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X, Y) = \nabla_X^\phi d\phi(Y) - d\phi(\nabla_X Y). \quad (3.5)$$

Here ∇ is the Riemannian connection on M and ∇^ϕ is the pull-back connection on the pull-back bundle $\phi^{-1}TN$. The tension field of ϕ is defined by

$$\tau(\phi) = trace_g \nabla d\phi = \sum_{i=1}^m (\nabla_{E_i}^\phi d\phi(E_i) - d\phi(\nabla_{E_i} E_i)), \quad (3.6)$$

where $\{E_i\}_{i=1, \overline{m}}$ is an orthonormal frame on (M^m, g) . A map ϕ is called harmonic if and only if $\tau(\phi) = 0$.

Remark 3.1. Let (M^m, g) be a Riemannian manifold and \tilde{g} the Mus-gradient metric on M . If $\{E_i\}_{i=1, \overline{m}}$ be an orthonormal frame on (M^m, g) , such that $E_1 = \text{grad } f$, the set $\{\tilde{E}_i\}_{i=1, \overline{m}}$, which is defined as below, is an orthonormal frame on (M^m, \tilde{g}) , then

$$\tilde{E}_1 = \frac{1}{\sqrt{f+1}}E_1, \tilde{E}_i = \frac{1}{\sqrt{f}}E_i, \quad i = \overline{2, m}, \tag{3.7}$$

where $f : M \rightarrow]0, +\infty[$ be a strictly positive smooth function.

Theorem 3.1. The tension field of the Identity Map $I : (M^m, \tilde{g}) \rightarrow (M^m, g)$ is given by

$$\tau(I) = \frac{1}{f(f+1)} \left(\frac{(m-2)f+m-1}{2(f+1)} - \Delta(f) \right) \text{grad } f, \tag{3.8}$$

where $\Delta(f) = \text{trace}_g \text{Hess}_f = \sum_{i=1}^m g(\nabla_{E_i} \text{grad } f, E_i)$.

Proof. Let $\{\tilde{E}_i\}_{i=1, \overline{m}}$ be a locale orthonormal frame on (M^m, \tilde{g}) defined by (3.10), then

$$\begin{aligned} \tau(I) &= \sum_{i=1}^m (\nabla_{\tilde{E}_i}^I dI(\tilde{E}_i) - dI(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i)) \\ &= \sum_{i=1}^m (\nabla_{\tilde{E}_i} \tilde{E}_i - \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i) \\ &= \sum_{i=1}^m \left(-\frac{\tilde{E}_i(f)}{f} \tilde{E}_i - \left(\frac{\text{Hess}_f(\tilde{E}_i, \tilde{E}_i)}{f+1} - \frac{\tilde{E}_i(f)^2}{f(f+1)} - \frac{g(\tilde{E}_i, \tilde{E}_i)}{2(f+1)} \right) \text{grad } f \right) \\ &= \left(\frac{-1}{f(f+1)} - \frac{\Delta(f)}{f(f+1)} + \frac{1}{f(f+1)^2} + \frac{1}{2(f+1)^2} + \frac{m-1}{2f(f+1)} \right) \text{grad } f \\ &= \frac{1}{f(f+1)} \left(\frac{(m-2)f+m-1}{2(f+1)} - \Delta(f) \right) \text{grad } f. \end{aligned}$$

From the Theorem 3.1 we obtain

Theorem 3.2. The Identity Map $I : (M^m, \tilde{g}) \rightarrow (M^m, g)$ is harmonic if and only if $f = \text{const}$ or

$$\Delta(f) = \frac{(m-2)f+m-1}{2(f+1)}. \tag{3.9}$$

Example 3.1. Let $M =]0, +\infty[\times_F \mathbb{R}^{m-1}$ be the Riemannian twisted product manifold equipped with the Riemannian metric g defined by

$$g = dx_1^2 + F(x_1)g_{\mathbb{R}^{m-1}}$$

where $g_{\mathbb{R}^{m-1}}$ is the standard metric and

$$F(x_1) = e^{\frac{m-2}{m-1}x_1} (x_1 + 1)^{\frac{1}{m-1}}.$$

Let $f(x_1, \dots, x_m) = x_1$, it's clear that $\|\text{grad}f\| = 1$

as we have

$$\Delta(f) = \frac{(m-2)f + m - 1}{2(f+1)}.$$

So, thus the Identity Map $I : (M^m, \tilde{g}) \rightarrow (M^m, g)$ is harmonic.

Example 3.2. Let $m = 2$ and $f(x, y) = F_1(y - Ix) + F_2(y + Ix) + \frac{1}{2}x^2 + \frac{1}{2}y^2$, where $F_1, F_2 : \mathbb{C} \rightarrow \mathbb{R}_+^*$ and $I^2 = -1$. Then the Identity Map $I : (M^m, \tilde{g}) \rightarrow (M^m, g)$ is harmonic.

Theorem 3.3. The tension field of the Identity Map $I : (M^m, g) \rightarrow (M^m, \tilde{g})$ is given by

$$\tau(I) = \frac{1}{f+1}(\Delta(f) + \frac{2-m}{2})\text{grad}f. \quad (3.10)$$

Proof. Let $\{E_i\}_{i=1,2m}$ be a locale orthonormal frame on M , then

$$\begin{aligned} \tau(I) &= \sum_{i=1}^m (\nabla_{E_i}^I dI(E_i) - dI(\nabla_{E_i} E_i)) \\ &= \sum_{i=1}^m \tilde{\nabla}_{dI(E_i)} dI(E_i) - \nabla_{E_i} E_i \\ &= \sum_{i=1}^m \tilde{\nabla}_{E_i} E_i - \nabla_{E_i} E_i \\ &= \sum_{i=1}^m \left(\frac{E_i(f)}{f} E_i + \left(\frac{\text{Hess}_f(E_i, E_i)}{f+1} - \frac{E_i(f)^2}{f(f+1)} - \frac{g(E_i, E_i)}{2(f+1)} \right) \text{grad}f \right) \\ &= \frac{1}{f} \text{grad}f + \left(\frac{\Delta(f)}{f+1} - \frac{1}{f(f+1)} - \frac{m}{2(f+1)} \right) \text{grad}f \\ &= \frac{1}{f+1} \left(\frac{2-m}{2} + \Delta(f) \right) \text{grad}f. \end{aligned}$$

From the Theorem 3.3 we obtain

Theorem 3.4. The Identity Map $I : (M^m, g) \rightarrow (M^m, \tilde{g})$ is harmonic if and only if

$$\Delta(f) = \frac{m-2}{2}. \quad (3.11)$$

Example 3.3. The Identity Map $I : (IR^2, g = dx^2) \rightarrow (IR^2, \tilde{g})$ is harmonic if and only if

$$\Delta(f) = \frac{\partial^2 f}{(\partial x)^2} + \frac{\partial^2 f}{(\partial y)^2} = 0. \quad (3.12)$$

Example 3.4. Let $M =]0, +\infty[\times \frac{-\pi}{4}, \frac{3\pi}{4}[$ be endowed with the Riemannian metric g in polar coordinate defined by

$$g = dr^2 + r^2 d\theta^2.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}, \Gamma_{22}^1 = -r.$$

Relatively to the orthonormal frame

$$e_1 = \frac{\partial}{\partial r}, e_2 = \frac{1}{r} \frac{\partial}{\partial \theta},$$

we have

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \nabla_{e_2} e_1 = \frac{1}{r^2} \frac{\partial}{\partial \theta}, \nabla_{e_2} e_2 = \frac{-1}{r} \frac{\partial}{\partial r}.$$

Let $f(r, \theta) = r \sin(\theta + \frac{\pi}{4})$, for all $(r, \theta) \in M$.

By direct computations we obtain

$$\begin{aligned} \text{grad } f &= \sin(\theta + \frac{\pi}{4}) \frac{\partial}{\partial r} + \frac{1}{r} \cos(\theta + \frac{\pi}{4}) \frac{\partial}{\partial \theta}, \\ \|\text{grad } f\| &= 1, \\ \Delta(f) &= 0. \end{aligned}$$

By virtue of the Theorem 3.4 the identity map $I : (M^m, g) \rightarrow (M^m, \tilde{g})$ is harmonic, where

$$\tilde{g} = (r \sin(\theta + \frac{\pi}{4}) + \sin^2(\theta + \frac{\pi}{4})) dr^2 + r^2 (r \sin(\theta + \frac{\pi}{4}) + \cos^2(\theta + \frac{\pi}{4})) d\theta^2 + r \cos(2\theta) dr d\theta.$$

Theorem 3.5. The tension field of the map $\sigma : (M^m, \tilde{g}) \rightarrow (N^n, h)$ is given by

$$\begin{aligned} \tilde{\tau}(\sigma) &= \frac{1}{f} \tau(\sigma) + \frac{1}{f(f+1)} \left(\frac{(m-2)f + m - 1}{2(f+1)} - \Delta(f) \right) d\sigma(\text{grad } f) \\ &\quad - \frac{1}{f(f+1)} \nabla_{d\sigma(\text{grad } f)}^N d\sigma(\text{grad } f), \end{aligned} \tag{3.13}$$

where $f : M \rightarrow]0, +\infty[$ be a strictly positive smooth function and $\tau(\sigma)$ is the tension field of $\sigma : (M, g) \rightarrow (N, h)$.

Proof. Let $\{\tilde{E}_i\}_{i=1, \dots, m}$ be a locale orthonormal frame on (M^m, \tilde{g}) defined by (3.10), then

$$\begin{aligned} \tau(I) &= \sum_{i=1}^m (\nabla_{\tilde{E}_i}^\sigma d\sigma(\tilde{E}_i) - d\sigma(\tilde{\nabla}_{\tilde{E}_i}^M \tilde{E}_i)) \\ &= \sum_{i=1}^m \nabla_{\tilde{E}_i}^\sigma d\sigma(\tilde{E}_i) - \sum_{i=1}^m d\sigma(\tilde{\nabla}_{\tilde{E}_i}^M \tilde{E}_i). \end{aligned}$$

By direct computations we obtain

$$\begin{aligned}
\sum_{i=1}^m \nabla_{\tilde{E}_i}^\sigma d\sigma(\tilde{E}_i) &= \nabla_{\tilde{E}_1}^\sigma \tilde{E}_1 + \sum_{i=2}^m \nabla_{\tilde{E}_i}^\sigma \tilde{E}_i \\
&= \frac{1}{\sqrt{f+1}} \nabla_{E_1}^\sigma \frac{1}{\sqrt{f+1}} E_1 + \sum_{i=2}^m \frac{1}{\sqrt{f}} \nabla_{E_i}^\sigma \frac{1}{\sqrt{f}} E_i \\
&= \frac{-1}{2(f+1)^2} d\sigma(\text{grad } f) - \frac{1}{f(f+1)} \nabla_{d\sigma(\text{grad } f)}^N d\sigma(\text{grad } f) \\
&\quad + \frac{1}{f} \sum_{i=1}^m \nabla_{E_i}^\sigma d\sigma(E_i),
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^m d\sigma(\tilde{\nabla}_{\tilde{E}_i}^M \tilde{E}_i) &= d\sigma\left(\sum_{i=1}^m \tilde{\nabla}_{\tilde{E}_i}^M \tilde{E}_i\right) \\
&= d\sigma\left(\tilde{\nabla}_{\tilde{E}_1}^M \tilde{E}_1 + \sum_{i=2}^m \tilde{\nabla}_{\tilde{E}_i}^M \tilde{E}_i\right) \\
&= d\sigma\left(\frac{1}{\sqrt{f+1}} \tilde{\nabla}_{E_1}^M \frac{1}{\sqrt{f+1}} E_1 + \sum_{i=2}^m \frac{1}{\sqrt{f}} \tilde{\nabla}_{E_i}^M \frac{1}{\sqrt{f}} E_i\right) \\
&= \frac{1}{f} \sum_{i=1}^m d\sigma(\nabla_{E_i}^\sigma E_i) + \left(\frac{\Delta(f)}{f(f+1)} - \frac{m-1}{2f(f+1)}\right) d\sigma(\text{grad } f),
\end{aligned}$$

hence we get

$$\begin{aligned}
\tilde{\tau}(\sigma) &= \frac{1}{f} \tau(\sigma) + \frac{1}{f(f+1)} \left(\frac{(m-2)f+m-1}{2(f+1)} - \Delta(f)\right) d\sigma(\text{grad } f) \\
&\quad - \frac{1}{f(f+1)} \nabla_{d\sigma(\text{grad } f)}^N d\sigma(\text{grad } f).
\end{aligned}$$

From the Theorem 3.5 we obtain

Theorem 3.6. *Let $\sigma : (M^m, g) \rightarrow (N^n, h)$ be harmonic. Then the map $\sigma : (M^m, \tilde{g}) \rightarrow (N^n, h)$ is harmonic if and only if*

$$\begin{aligned}
\tau(\sigma) &= \frac{1}{f+1} \left(\Delta(f) - \frac{(m-2)f+m-1}{2(f+1)}\right) d\sigma(\text{grad } f) \\
&\quad + \frac{1}{f+1} \nabla_{d\sigma(\text{grad } f)}^N d\sigma(\text{grad } f).
\end{aligned} \tag{3.14}$$

Example 3.5. *If we set $\sigma = \text{Id}_M$ and $f = \text{const}$ then $\sigma : (M^m, \tilde{g}) \rightarrow (N^n, h)$ is harmonic.*

Lemma 3.1. [1] *Given a smooth map $\sigma : (M^m, g) \rightarrow (N^n, h)$ between two Riemannian manifolds and $f \in C^\infty(N)$, then we have*

$$\Delta(f \circ \sigma) = \text{trace}_g \text{Hess}_f(d\sigma, d\sigma) + df(\tau(\sigma)). \tag{3.15}$$

Proof. Let $X, Y \in \mathfrak{S}_0^1(M)$, we have $f \circ \sigma \in C^\infty(M)$ then

$$\begin{aligned} \nabla d(f \circ \sigma)(X, Y) &= \nabla_X^{f \circ \sigma} d(f \circ \sigma)(Y) - d(f \circ \sigma)(\nabla_X^M Y) \\ &= \nabla_{d\sigma(X)}^f df(d\sigma(Y)) - df(d\sigma(\nabla_X^M Y)) \\ &= \nabla df(d\sigma(X), d\sigma(Y)) + df(\nabla_{d\sigma(X)}^N d\sigma(Y)) - df(d\sigma(\nabla_X^M Y)) \\ &= \nabla df(d\sigma(X), d\sigma(Y)) + df(\nabla d\sigma(X, Y)). \end{aligned}$$

By passing to the trace in the last equation and using

$$\text{trace}_g \nabla df = \text{trace}_g \text{Hess}_f$$

we get

$$\Delta(f \circ \sigma) = \text{trace}_g \text{Hess}_f(d\sigma, d\sigma) + df(\tau(\sigma)).$$

Theorem 3.7. *The tension field of the map $\sigma : (M^m, g) \rightarrow (N^n, \tilde{h})$ is given by*

$$\begin{aligned} \tilde{\tau}(\sigma) &= \tau(\sigma) + \frac{1}{f} d\sigma(\text{grad}(f \circ \sigma)) \\ &\quad + \frac{1}{f+1} (\Delta(f \circ \sigma) - df(\tau(\sigma)) - \frac{\|\text{grad}(f \circ \sigma)\|^2}{f} - \frac{\|d\sigma\|^2}{2})(\text{grad } f) \circ \sigma, \end{aligned} \tag{3.16}$$

where $f : N \rightarrow]0, +\infty[$ be a strictly positive smooth function and $\tau(\sigma)$ is the tension field of $\sigma : (M, g) \rightarrow (N, h)$.

Proof. Let $\{E_i\}_{i=1, \overline{m}}$ be a locale orthonormal frame on (M^m, g) , then

$$\begin{aligned}
\tilde{\tau}(\sigma) &= \sum_{i=1}^m (\tilde{\nabla}_{E_i}^\sigma d\sigma(E_i) - d\sigma(\nabla_{E_i}^M E_i)) \\
&= \sum_{i=1}^m (\tilde{\nabla}_{d\sigma(E_i)}^N d\sigma(E_i) - d\sigma(\nabla_{E_i}^M E_i)) \\
&= \sum_{i=1}^m (\nabla_{d\sigma(E_i)}^N d\sigma(E_i) + \frac{d\sigma(E_i)(f)}{f} d\sigma(E_i) \\
&\quad + (\frac{Hess_f(d\sigma(E_i), d\sigma(E_i))}{f+1} - \frac{(d\sigma(E_i)(f))^2}{f(f+1)} \\
&\quad - \frac{h(d\sigma(E_i), d\sigma(E_i))}{2(f+1)})(grad f) \circ \sigma - d\sigma(\nabla_{E_i}^M E_i)) \\
&= \sum_{i=1}^m (\nabla_{E_i}^\sigma d\sigma(E_i) - d\sigma(\nabla_{E_i}^M E_i) + \frac{E_i(f \circ \sigma)}{f} d\sigma(E_i) \\
&\quad + (\frac{Hess_f(d\sigma(E_i), d\sigma(E_i))}{f+1} - \frac{(E_i(f \circ \sigma))^2}{f(f+1)} \\
&\quad - \frac{h(d\sigma(E_i), d\sigma(E_i))}{2(f+1)})(grad f) \circ \sigma) \\
&= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f \circ \sigma)) \\
&\quad + (\frac{trace Hess_f(d\sigma, d\sigma)}{f+1} - \frac{\|grad(f \circ \sigma)\|^2}{f(f+1)} - \frac{\|d\sigma\|^2}{2(f+1)})(grad f) \circ \sigma \\
&= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f \circ \sigma)) \\
&\quad + (\frac{\Delta(f \circ \sigma) - df(\tau(\sigma))}{f+1} - \frac{\|grad(f \circ \sigma)\|^2}{f(f+1)} - \frac{\|d\sigma\|^2}{2(f+1)})(grad f) \circ \sigma \\
&= \tau(\sigma) + \frac{1}{f} d\sigma(grad(f \circ \sigma)) \\
&\quad + \frac{1}{f+1} (\Delta(f \circ \sigma) - df(\tau(\sigma)) - \frac{\|grad(f \circ \sigma)\|^2}{f} - \frac{\|d\sigma\|^2}{2})(grad f) \circ \sigma.
\end{aligned}$$

From the Theorem 3.7 we obtain

Theorem 3.8. *The map $\sigma : (M^m, g) \longrightarrow (N^n, \tilde{h})$ is harmonic if and only if*

$$\begin{aligned}
\tau(\sigma) &= \frac{-1}{f+1} (\Delta(f \circ \sigma) - df(\tau(\sigma)) - \frac{\|grad(f \circ \sigma)\|^2}{f} - \frac{\|d\sigma\|^2}{2})(grad f) \circ \sigma \\
&\quad - \frac{1}{f} d\sigma(grad(f \circ \sigma)). \tag{3.17}
\end{aligned}$$

4. HARMONICITY ON PRODUCT MANIFOLD

Let (M, g) and (N, h) be a Riemannian manifolds.

Definition 4.1. *Let (M, g) and (N, h) be two Riemannian manifolds of dimension m and n respectively. We define the product metric on $M \times N$ by*

$$G = \pi^*g + \eta^*h$$

where $\pi : M \times N \rightarrow M$ and $\eta : M \times N \rightarrow N$ denote the first and the second canonical projection.

Proposition 4.1. *For all vector fields $X_1, X_2 \in \mathcal{H}(M)$ and $Y_1, Y_2 \in \mathcal{H}(N)$ we have*

$$\begin{aligned} G((X_1, Y_1), (X_2, Y_2)) &= g(X_1, X_2) + h(Y_1, Y_2) \\ G((X_1, 0), (X_2, 0)) &= g(X_1, X_2) \\ G((0, Y_1), (0, Y_2)) &= h(Y_1, Y_2) \\ G((X_1, 0), (0, Y_2)) &= 0. \end{aligned}$$

Subsequently, if $X \in \mathcal{H}(M)$ and $Y \in \mathcal{H}(N)$, then we denote (X, Y) by $X + Y$.

Remark 4.1. • *Any vector field of $\mathcal{H}(M)$ is orthogonal to all vector fields of $\mathcal{H}(N)$.*

• *Let (E_1, \dots, E_m) (resp $(E_{m+1}, \dots, E_{m+n})$) is an orthonormal basis of $\mathcal{H}(M)$ (resp $\mathcal{H}(N)$) then (E_1, \dots, E_{m+n}) is an orthonormal basis of $\mathcal{H}(M \times N)$.*

• *Let $f \in C^\infty(M)$, then $\Delta(f) = \sum_{i=1}^m \text{Hess}_f(E_i, E_i)$.*

Proposition 4.2. *Let (M, g) and (N, h) be two Riemannian manifolds. If ${}^M\nabla$ (resp ${}^N\nabla$) denote the connection of Levi-Civita on M (resp N), then the levi-civita connection ∇ on the manifold $M \times N$ associated with the product metric $G = \pi^*g + \eta^*h$ is verifies the following properties:*

$$\left\{ \begin{aligned} \nabla_{X_1} X_2 &= {}^M \nabla_{X_1} X_2 \\ \nabla_{Y_1} Y_2 &= {}^N \nabla_{Y_1} Y_2 \\ \nabla_{X_1} Y_1 &= \nabla_{Y_2} X_2 = 0 \\ \nabla_{(X_1+Y_1)} (X_2 + Y_2) &= {}^M \nabla_{X_1} X_2 + {}^N \nabla_{Y_1} Y_2 \end{aligned} \right.$$

for any $X_1, Y_1 \in \mathcal{H}(M)$ and $X_2, Y_2 \in \mathcal{H}(N)$.

Lemma 4.1. *Let (M^m, g) and (N^n, h) be two Riemannian manifolds and $f \in C^\infty(M)$. If $P : (x, y) \in M \times N \rightarrow y \in N$ (resp $P : (x, y) \in M \times N \rightarrow (0, y) \in M \times N$) is the second projection, then we have*

$$\begin{aligned} \text{grad}(f) &= \text{grad}_G(f) = \text{grad}_g(f), \\ dP(\text{grad}(f)) &= 0, \\ dP(\widetilde{\nabla}_X X) &= dP(\nabla_X X) + \frac{X(f)}{f} dP(X) \end{aligned} \quad (4.18)$$

where $X \in \mathcal{H}(M \times N)$.

Proof. The proof of the formula (4.18) is a direct consequence of Theorem 2.1.

Theorem 4.1. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the second projection*

$$\begin{aligned} P : (M \times N, \widetilde{G}) &\rightarrow (N, h) \\ (x, y) &\mapsto y \end{aligned}$$

is harmonic map. where $G = g + h$.

Proof. Let (E_1, \dots, E_m) be an orthonormal basis on (M^m, g) such as $E_1 = \text{grad}(f)$ and $(E_{m+1}, \dots, E_{m+n})$ be an orthonormal basis on (N^n, h) such as ${}^N \nabla_{E_i} E_j = 0$, $(i, j \geq m+1)$, then (E_1, \dots, E_{m+n}) is an orthonormal basis on $(M \times N, g + h)$.

From Lemma 4.1, we obtain:

$$\begin{aligned} {}^N \nabla_{dP(\widetilde{E}_i)} dP(\widetilde{E}_i) - dP(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i) &= -dP(\nabla_{\widetilde{E}_i} \widetilde{E}_i) \\ &= -dP({}^M \nabla_{\widetilde{E}_i} \widetilde{E}_i) \\ &= 0 \end{aligned}$$

for $1 \leq i \leq m$, and

$$\begin{aligned} {}^N \nabla_{dP(\widetilde{E}_i)} dP(\widetilde{E}_i) - dP(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i) &= {}^N \nabla_{\widetilde{E}_i} \widetilde{E}_i - dP(\nabla_{\widetilde{E}_i} \widetilde{E}_i) \\ &= 0 \end{aligned}$$

for $m+1 \leq i \leq m+n$. We therefore deduce $\tau(P) = 0$.

We find the same result for the following theorem

Theorem 4.2. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then*

$$P : (M \times N, \tilde{G}) \rightarrow (M \times N, G)$$

$$(x, y) \mapsto (0, y)$$

is harmonic map. where $G = g + h$.

Theorem 4.3. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$P : (M \times N, G) \rightarrow (M \times N, \tilde{G})$$

$$(x, y) \mapsto (0, y)$$

is given by

$$\tau(P) = \frac{-n}{2(f+1)} grad(f).$$

Proof. Similarly to the proof of Theorem 4.1, we obtain

$$\tilde{\nabla}_{dP(E_i)} dP(E_i) - dP(\nabla_{E_i} E_i) = 0, \quad (i \leq m).$$

$$\tilde{\nabla}_{dP(E_i)} dP(E_i) - dP(\nabla_{E_i} E_i) = -\frac{1}{2(f+1)} grad(f), \quad (i \geq m+1).$$

Theorem 4.4. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$P : (M \times N, \tilde{G}) \rightarrow (M \times N, \tilde{G})$$

$$(x, y) \mapsto (0, y)$$

is given by

$$\tau(P) = -\frac{n}{2f(f+1)} grad(f)$$

where $G = g + h$.

Proof. Let $i \in \{m + 1, \dots, n + m\}$, from Theorem 2.1 and Lemma 4.1 we obtain

$$\begin{aligned}
\widetilde{\nabla}_{dP(\widetilde{E}_i)} dP(\widetilde{E}_i) - dP(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i) &= \widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i - dP(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i) \\
&= \widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i - dP(\nabla_{\widetilde{E}_i} \widetilde{E}_i) \\
&= \widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i - \nabla_{\widetilde{E}_i} \widetilde{E}_i \\
&= -\frac{G(\widetilde{E}_i, \widetilde{E}_i)}{2(f+1)} \text{grad}(f) \\
&= -\frac{1}{2f(f+1)} \text{grad}(f).
\end{aligned}$$

Example 4.1. Let $(M, g) = (IR^m, dx^2)$, $(m \geq 3)$ and $f(x_1, x_2, x_3, \dots, x_m) = f(x_1, x_2)$ such that $(\frac{\partial f}{\partial x_1})^2 + (\frac{\partial f}{\partial x_2})^2 = 1$. If we put

$$\begin{aligned}
\widetilde{P} : (M, \widetilde{g}) &\rightarrow (M, g) \\
(x_1, x_2, x_3, \dots, x_m) &\mapsto (0, 0, x_3, \dots, x_m)
\end{aligned}$$

then we obtain

$$\begin{aligned}
E_1 &= \partial_1(f)\partial_1 + \partial_2(f)\partial_2 \\
E_2 &= \partial_2(f)\partial_1 - \partial_1(f)\partial_2 \\
E_i &= \partial_i, \quad (i \geq 3) \\
d\widetilde{P}(\widetilde{\nabla}_X X) &= d\widetilde{P}(\nabla_X X) + \frac{X(f)}{f} dP(X).
\end{aligned}$$

So

$$\begin{aligned}
\tau(\widetilde{P}) &= \sum_i \nabla_{d\widetilde{P}(\widetilde{E}_i)} d\widetilde{P}(\widetilde{E}_i) - \sum_i d\widetilde{P}(\widetilde{\nabla}_{\widetilde{E}_i} \widetilde{E}_i) \\
&= 0.
\end{aligned}$$

Then \widetilde{P} is harmonic.

On the other hand, the tension field of the projection

$$\begin{aligned}
P : (M, g) &\rightarrow (M, \widetilde{g}) \\
(x_1, x_2, x_3, \dots, x_m) &\mapsto (0, 0, x_3, \dots, x_m)
\end{aligned}$$

is given by the following formula

$$\tau(P) = \frac{2-m}{2(f+1)} \text{grad}(f).$$

Therefore, P is non-harmonic.

Theorem 4.5. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$Q : (M \times N, \tilde{G}) \rightarrow (M, g)$$

$$(x, y) \mapsto x$$

is given by

$$\tau(Q) = -\frac{1}{f(f+1)} \left[\Delta(f) + \frac{(n+m-2)(f+1)+1}{2(f+1)} \right] \text{grad}(f).$$

Proof. Let (E_1, \dots, E_m) be an orthonormal basis on (M^m, g) such as $E_1 = \text{grad}(f)$ and $(E_{m+1}, \dots, E_{m+n})$ be an orthonormal basis on (N^n, h) such as ${}^N\nabla_{E_i} E_j = 0, \quad (i, j \geq m+1)$, then (E_1, \dots, E_{m+n}) is an orthonormal basis on $(M \times N, g+h)$.

From Remark 3.1 and Theorem 2.1, we have:

$$\begin{aligned} \sum_{i=m+1}^{m+n} \left[{}^M\nabla_{dQ(\tilde{E}_i)} dQ(\tilde{E}_i) - dQ(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i) \right] &= - \sum_{i=m+1}^{m+n} dQ(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i) \\ &= \sum_{i=m+1}^{m+n} \frac{G(\tilde{E}_i, \tilde{E}_i)}{2(f+1)} \text{grad}(f) \\ &= \frac{n}{2f(f+1)} \text{grad}(f) \end{aligned}$$

$$\begin{aligned} {}^M\nabla_{dQ(\tilde{E}_1)} dQ(\tilde{E}_1) - dQ(\tilde{\nabla}_{\tilde{E}_1} \tilde{E}_1) &= {}^M\nabla_{\tilde{E}_1} \tilde{E}_1 - \tilde{\nabla}_{\tilde{E}_1} \tilde{E}_1 \\ &= -\frac{\tilde{E}_1(f)}{f} \tilde{E}_1 + \frac{(\tilde{E}_1(f))^2}{f(f+1)} \text{grad}(f) + \frac{G(\tilde{E}_1, \tilde{E}_1)}{2(f+1)} \text{grad}(f) \\ &= -\frac{1}{f(f+1)} \text{grad}(f) + \frac{1}{f(f+1)^2} \text{grad}(f) + \frac{1}{2(f+1)^2} \text{grad}(f) \\ &= \left[\frac{-1}{2(f+1)^2} - \frac{\text{Hess}_f(E_1, E_1)}{f(f+1)} \right] \text{grad}(f) \end{aligned}$$

$$\begin{aligned} \sum_{i=2}^m \left[{}^M\nabla_{dQ(\tilde{E}_i)} dQ(\tilde{E}_i) - dQ(\tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i) \right] &= \sum_{i=2}^m \left[{}^M\nabla_{\tilde{E}_i} \tilde{E}_i - \tilde{\nabla}_{\tilde{E}_i} \tilde{E}_i \right] \\ &= \sum_{i=2}^m \left[\frac{G(\tilde{E}_i, \tilde{E}_i)}{2(f+1)} - \frac{\text{Hess}_f(E_i, E_i)}{f(f+1)} \right] \text{grad}(f) \\ &= \left[\frac{m-1}{2f(f+1)} - \frac{\Delta(f)}{f(f+1)} \right] \text{grad}(f). \end{aligned}$$

Theorem 4.6. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} Q : (M \times N, \tilde{G}) &\rightarrow (M \times N, \tilde{G}) \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

is given by

$$\tau(Q) = -\frac{n}{2f(f+1)} \text{grad}(f).$$

The proof of Theorem 4.6 follows immediately from the Remark 3.1, Remark 4.1 and Theorem 2.1.

Theorem 4.7. *Let (M^m, g) be a Riemannian manifolds and (N^n, h) be an Euclidian manifold. If $f \in C^\infty(M)$ is a smooth positif function, then the tension field of*

$$\begin{aligned} Q : (M \times N, G) &\rightarrow (M \times N, \tilde{G}) \\ (x, y) &\mapsto (x, 0) \end{aligned}$$

is given by

$$\tau(Q) = \frac{1}{f(f+1)} \left[\Delta(f) - \frac{(2-m)f+1-m}{2(f+1)} \right] \text{grad}(f).$$

Proof. Let (E_1, \dots, E_m) be an orthonormal basis on (M^m, g) such as $E_1 = \text{grad}(f)$ and $(E_{m+1}, \dots, E_{m+n})$ be an orthonormal basis on (N^n, h) such as ${}^N\nabla_{E_i} E_j = 0$, $(i, j \geq m+1)$, then (E_1, \dots, E_{m+n}) is an orthonormal basis on $(M \times N, g+h)$.

From Remark 3.1, Remark 4.1 and Theorem 2.1, we obtain:

$$\tilde{\nabla}_{dQ(E_i)} dQ(E_i) - dQ(\nabla_{E_i} E_i) = 0, \quad (m+1 \leq i \leq m+n).$$

$$\begin{aligned} \tilde{\nabla}_{dQ(E_1)} dQ(E_1) - dQ(\nabla_{E_1} E_1) &= \tilde{\nabla}_{E_1} E_1 - \nabla_{E_1} E_1 \\ &= \left[\frac{\text{Hess}_f(E_1, E_1)}{f(f+1)} + \frac{1}{2(f+1)^2} \right] \text{grad}(f). \end{aligned}$$

$$\begin{aligned} \tilde{\nabla}_{dQ(E_i)} dQ(E_i) - dQ(\nabla_{E_i} E_i) &= \tilde{\nabla}_{E_i} E_i - \nabla_{E_i} E_i \quad (2 \leq i \leq m) \\ &= \left[\frac{\text{Hess}_f(E_i, E_i)}{f(f+1)} - \frac{1}{2f(f+1)} \right] \text{grad}(f). \end{aligned}$$

Acknowledgement

This work was supported by PRFU and LGACA Saida Laboratory.

The authors would like to thank and express their special gratitude to Prof. Mustafa Djaa for his suggestion of this work as well as for helping him with some accounts for his helpful suggestions and valuable comments which helped to improve the paper.

REFERENCES

- [1] Baird P., & Wood J.C. (2003). Harmonic morphisms between Riemannian manifolds. Oxford University Press.
- [2] Benkartab, A., & Cherif, A. M. (2019). New methods of construction for biharmonic maps. *Kyungpook Mathematical Journal*, 59(1), 135-147.
- [3] Boulal, A., & Ouakkas, S. (2020). Some results and examples of the f-biharmonic maps on warped product manifolds. *International Journal of Maps in Mathematics*, 3(2), 68-84.
- [4] Chen G., Liu Y. & Wei J. (2020). Nondegeneracy of harmonic maps from R^2 to S^2 , *Discrete & Continuous Dynamical Systems*, 40(6) : 3215-3233.
- [5] Deschamps G., Loubeau E., & Pantilie R. (2020). Harmonic maps and twistorial structures. *Mathematika*, Volume66, Issue1 January, 112-124.
- [6] Djaa N. E. H., Boulal A., & Zagane A. (2012). Generalized warped product manifolds And biharmonic maps. *Acta Math. Univ. Comenianae*, 81(2), 283-298.
- [7] Eells, J., & Lemaire, L. (1988). Another report on harmonic maps. *Bulletin of the London Mathematical Society*, 20(5), 385-524. <https://doi.org/10.1112/blms/20.5.385>.
- [8] Erdogan, F. E., Perktas, S. Y., & Polat, G. (2020). f-biharmonic normal section curves. *International Journal of Maps in Mathematics*, 3(1), 28-38.
- [9] Latti F., Djaa M., & Zagane A. (2018). Mus-Sasaki Metric and Harmonicity. *Mathematical Sciences and Applications E-Notes* , 6(1) 29-36.
- [10] Ouakkas, S., Nasri, R., & Djaa, M. (2010). On the f-harmonic and f-biharmonic maps. *JP J. Geom. Topol*, 10(1), 11-27.
- [11] Perktas, S. Y., Erdogan, F. E., & Acet, B. E. (2018). On f-Biharmonic Hypersurfaces. *International Journal of Maps in Mathematics*, 1(1), 116-136.
- [12] Zagane, A. (2021). On para-Kahler-Norden properties of the φ -Sasaki metric on tangent bundle. *International Journal of Maps in Mathematics*, 4(2), 121-135.

RELIZANE UNIVERSITY, FACULTY OF SCIENCES AND TECHNOLOGY, DEPARTMENT OF MATHEMATICS,
ALGERIA

RELIZANE UNIVERSITY, FACULTY OF SCIENCES AND TECHNOLOGY, DEPARTMENT OF MATHEMATICS,
ALGERIA



CONFORMAL SLANT RIEMANNIAN MAPS

ŞENER YANAN * AND BAYRAM ŞAHİN

ABSTRACT. Conformal slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds are introduced. We give a non-trivial example of proper conformal slant Riemannian maps, obtain conditions for certain distributions to be integrable and find totally geodesicity conditions for leaves of distributions. We adjust the notion of pluriharmonicity by considering distributions on the total manifold of a conformal slant Riemannian map, and get conditions for such maps to be horizontally homothetic maps.

Keywords: Kaehler manifold, Slant immersion, Slant submersion, Slant Riemannian map

2010 Mathematics Subject Classification: 53C43.

1. INTRODUCTION

The concept of Riemannian submersion was introduced by Gray [13] and O'Neill [19]. Then, this notion was widely studied [10] and new kinds of Riemannian submersions such as invariant, anti-invariant and slant submersion were introduced [26]. Let F be a Riemannian submersion (respectively, horizontally conformal submersion, $m > n$) from (M^m, g_M, J) an almost Hermitian manifold to (N^n, g_N) a Riemannian manifold. If the angle $\theta(U)$ between the space $(\ker F_{*p})$ and JU is a constant for any non-zero vector field $U \in \Gamma(\ker F_{*p})$; $p \in M$, i.e., it is independent from the choice of the tangent vector field U in $(\ker F_{*p})$ and choice of the point $p \in M$, then we say that F is a slant submersion (respectively, conformal slant submersion) [5, 14, 22].

Received: 2021.04.11

Revised: 2021.11.25

Accepted: 2021.12.05

* Corresponding author

Şener Yanan; syanan@adiyaman.edu.tr; <https://orcid.org/0000-0003-1600-6522>

Bayram Şahin; bayram.sahin@gmail.com; <https://orcid.org/0000-0002-9372-1151>

The notions of isometric immersions and Riemannian submersions are generalized by Riemannian maps between Riemannian manifolds [10, 11, 13, 19]. Let $F : (M_1, g_1) \rightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < \text{rank} F < \min\{\dim(M_1), \dim(M_2)\}$. So, the tangent bundle TM_1 of M_1 has the sequent decomposition:

$$TM_1 = \ker F_* \oplus (\ker F_*)^\perp.$$

Because of $\text{rank} F < \min\{\dim(M_1), \dim(M_2)\}$, we always have $(\text{range} F_*)^\perp$. Consequently, the tangent bundle TM_2 of M_2 has the sequent decomposition:

$$TM_2 = (\text{range} F_*) \oplus (\text{range} F_*)^\perp.$$

Hence, a smooth map $F : (M_1^m, g_1) \rightarrow (M_2^n, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F_{*p_1}^h : (\ker F_{*p_1})^\perp \rightarrow (\text{range} F_*)$ is a linear isometry. Therefore a Riemannian map provides the equation

$$g_1(E, G) = g_2(F_*(E), F_*(G)) \tag{1.1}$$

for $E, G \in \Gamma((\ker F_*)^\perp)$. Isometric immersions and Riemannian submersions are particular Riemannian maps with $\ker F_* = \{0\}$ and $(\text{range} F_*)^\perp = \{0\}$, respectively, [11]. As an another generalization of Riemannian submersions defined and studied independently horizontally conformal submersions [12, 15]. By following these studies and B. Şahin’s papers including anti-invariant Riemannian, semi-invariant, slant submersions (see also [20]) and conformal anti-invariant [3], conformal slant [7], conformal semi-invariant [4] and conformal semi-slant submersions [2] have appeared in the literature. At the same time, the notion of slant submanifolds was introduced by Chen [9]. Inspiring from this notion, as a general map of Hermitian, anti-invariant and slant submersions, slant Riemannian maps were given in [24, 25] as follows; let F be a Riemannian map from an almost Hermitian manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If the angle $\theta(U)$ is a constant between JU and the space $\ker F_*$ for any non-zero vector field $U \in \Gamma(\ker F_*)$; i.e., it is independent from the choice of the tangent U in $\ker F_*$ and choice of the point $p \in M$, then we say that F is a slant Riemannian map [24, 25]. On the other hand, we say that $F : (M^m, g_M) \rightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < \text{rank} F_{*p} \leq \min\{m, n\}$ and F_{*p} maps the horizontal space $(\ker F_{*p})^\perp$ conformally onto $\text{range}(F_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p}(E), F_{*p}(G)) = \lambda^2(p)g_M(E, G)$$

for $E, G \in \Gamma((\ker F_{*p})^\perp)$. Also F is said to be conformal Riemannian if F is conformal Riemannian at each $p \in M$ [21]. Conformal Riemannian maps have many application areas, some of them are computer vision [16], geometric modelling [29] and medical imaging [30]. In a previous paper, the second author and Akyol have studied conformal slant Riemannian maps from a Riemannian manifold to a Kaehler manifold and they have studied the geometry determined by the existence of these maps [5].

In this paper, we present conformal slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, investigate geometric properties of the base manifold and the total manifold by the existence of such maps and give examples. We also obtain certain geodesicity conditions for conformal slant Riemannian maps. Moreover, we obtain several conditions for conformal slant Riemannian maps to be horizontally homothetic maps by using the adapted version of the notion of pluri-harmonic maps.

2. PRELIMINARIES

In this section, some definitions and useful results which will be used at this paper for conformal slant Riemannian maps are given. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \rightarrow N$ is a smooth map between them. The second fundamental form of F is given by

$$(\nabla F_*)(X, Y) = \nabla_X^N F_*(Y) - F_*(\nabla_X^M Y) \quad (2.2)$$

for $X, Y \in \Gamma(TM)$. We know that (∇F_*) is symmetric [17]. Here, ∇^N is pull-back connection of ∇ on N along F .

Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) to a Riemannian manifold (N^n, g_N) . We characterize \mathcal{T} and \mathcal{A} as

$$\mathcal{A}_X Y = h \nabla_{hX}^M vY + v \nabla_{hX}^M hY, \quad (2.3)$$

$$\mathcal{T}_X Y = h \nabla_{vX}^M vY + v \nabla_{vX}^M hY, \quad (2.4)$$

for $X, Y \in \Gamma(TM)$, where ∇^M is the Levi-Civita connection of g_M . Actually, we could see that these are O'Neill's tensor fields for Riemannian submersions [19]. \mathcal{T}_X and \mathcal{A}_X are skew-symmetric operators and reversing the vertical and the horizontal distributions on $(\Gamma(TM), g)$ for any $X \in \Gamma(TM)$. Also, it can be seen easily that \mathcal{T} is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$, and \mathcal{A} is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. We should know that \mathcal{T} is symmetric on the vertical distribution

[10, 19]. Following these, from (2.3) and (2.4) we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \tag{2.5}$$

$$\overset{M}{\nabla}_U E = h\overset{M}{\nabla}_U E + \mathcal{T}_U E, \tag{2.6}$$

$$\overset{M}{\nabla}_E V = \mathcal{A}_E V + v\overset{M}{\nabla}_E V, \tag{2.7}$$

$$\overset{M}{\nabla}_E G = h\overset{M}{\nabla}_E G + \mathcal{A}_E G \tag{2.8}$$

for $E, G \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [10].

A vector field on M is called a projectable vector field if it is related to a vector field on N . Thus, we say a vector field is basic on M if it is both a horizontal and a projectable vector field. From now on, when we mention a horizontal vector field, we always consider a basic vector field [8].

On the other hand, let F be a conformal Riemannian map between Riemannian manifolds (M^m, g_M) and (N^n, g_N) . Then, we have

$$\begin{aligned} (\nabla F_*)(E, G) |_{\text{range } F_*} &= E(\ln \lambda)F_*(G) + G(\ln \lambda)F_*(E) \\ &- g_M(E, G)F_*(\text{grad}(\ln \lambda)), \end{aligned} \tag{2.9}$$

where $E, G \in \Gamma((\ker F_*)^\perp)$ [6, 21]. Therefore from (2.9), we obtain $\overset{N}{\nabla}_E^F F_*(G)$ as

$$\begin{aligned} \overset{N}{\nabla}_E^F F_*(G) &= F_*(h\overset{M}{\nabla}_E G) + E(\ln \lambda)F_*(G) + G(\ln \lambda)F_*(E) \\ &- g_M(E, G)F_*(\text{grad}(\ln \lambda)) + (\nabla F_*)^\perp(E, G) \end{aligned} \tag{2.10}$$

where $(\nabla F_*)^\perp(E, G)$ is the component of $(\nabla F_*)(E, G)$ on $(\text{range } F_*)^\perp$ for $E, G \in \Gamma((\ker F_*)^\perp)$ [27, 28].

Finally, we recall the following notion. A map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if F provides the following equation

$$(\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) = 0 \tag{2.11}$$

for $X, Y \in \Gamma(TM)$ [18].

3. CONFORMAL SLANT RIEMANNIAN MAPS

In this section we are going to introduce conformal slant Riemannian maps as a generalization of slant Riemannian maps and conformal slant submersions, present examples and examine the geometry of source manifolds, target manifolds and maps themselves. We present the sequent definition.

Definition 3.1. Let $F : (M, g_M, J_M) \longrightarrow (N, g_N)$ be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J_M) to a Riemannian manifold (N, g_N) . If for any non-zero vector $X \in \Gamma(\ker F_*)$ at a point $p \in M$; the angle $\theta(X)$ between the space $\ker F_*$ and $J_M X$ is a constant, i.e. it is independent of the choice of the tangent vector $X \in \Gamma(\ker F_*)$ and choice of the point $p \in M$, then we say that F is a conformal slant Riemannian map. In this situation, the angle θ is called the slant angle of the conformal slant Riemannian map.

We say that a conformal slant Riemannian map is proper if F is not a conformal invariant and a conformal anti-invariant Riemannian map. The sequent example is for a proper conformal slant Riemannian map.

Example 3.1. Let $F : (R^4, g_4, J) \longrightarrow (R^4, g_4)$ be a map from a Kaehlerian manifold (R^4, g_4, J) to a Riemannian manifold (R^4, g_4) defined by

$$(e^{x_2} \sin x_4, e^{x_2} \cos x_4, -e^{x_2} \sin x_4, -e^{x_2} \cos x_4).$$

Then, F is a conformal Riemannian map with $\lambda = e^{x_2} \sqrt{2}$ and $\text{rank} F = 2$. One can easily see that F is a proper conformal slant Riemannian map with the slant angle $\theta = \alpha$ via $J_\alpha = \cos \alpha(-c, -d, a, b) + \sin \alpha(-b, a, d, -c)$, $0 < \alpha \leq \frac{\pi}{2}$.

Let F be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then for $V \in \Gamma(\ker F_*)$, we write

$$JV = \phi V + \omega V, \tag{3.12}$$

where $\phi V \in \Gamma(\ker F_*)$ and $\omega V \in \Gamma((\ker F_*)^\perp)$. Also for $X \in \Gamma((\ker F_*)^\perp)$, we write

$$JX = BX + CX, \tag{3.13}$$

where $BX \in \Gamma(\ker F_*)$ and $CX \in \Gamma((\ker F_*)^\perp)$. We have covariant derivatives of ϕ and ω :

$$\overset{M}{(\nabla_U \omega)} V = h \overset{M}{\nabla_U} \omega V - \omega \hat{\nabla}_U V, \tag{3.14}$$

$$\overset{M}{(\nabla_U \phi)} V = \hat{\nabla}_U \phi V - \phi \hat{\nabla}_U V \tag{3.15}$$

for any $U, V \in \Gamma(\ker F_*)$.

We give the following result by using equations (2.5), (2.6), (3.12), (3.13) and covariant derivatives of ϕ and ω .

Lemma 3.1. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then F is a conformal slant Riemannian map, we get*

$$h \overset{M}{\nabla}_U \omega V - \omega \hat{\nabla}_U V = CT_U V - T_U \phi V,$$

$$\hat{\nabla}_U \phi V - \phi \hat{\nabla}_U V = BT_U V - T_U \omega V$$

for any $U, V \in \Gamma(\ker F_*)$.

Now, we present the following characterization for conformal slant Riemannian maps.

Theorem 3.1. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then F is a conformal slant Riemannian map if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$\phi^2 U = \lambda U$$

for $U \in \Gamma(\ker F_*)$. If F is a conformal slant Riemannian map, then $\lambda = -\cos^2 \theta$.

Proof. For $U \in \Gamma(\ker F_*)$ we have $\cos \theta = \frac{\|\phi U\|}{\|JU\|}$. Since M is a Kaehler manifold, we get

$$g_M(\phi^2 U, U) = -g_M(\phi U, \phi U) = -\cos^2 \theta g_M(U, U).$$

Hence, we have $\phi^2 U = \lambda U$. Conversely, suppose that $\phi^2 U = \lambda U$ for $\forall U \in \Gamma(\ker F_*)$ with $\lambda \in [-1, 0]$. Hence, we obtain

$$\cos \theta = \frac{g_M(JU, \phi U)}{\|JU\| \|\phi U\|} = -\lambda \frac{\|JU\|}{\|\phi U\|}. \tag{3.16}$$

Using $\cos \theta = \frac{\|\phi U\|}{\|JU\|}$ in (3.16) we get $\lambda = -\cos^2 \theta$.

From (3.12) and Theorem 3.1. we have the next result.

Theorem 3.2. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with the slant angle θ . Then, we have*

$$g_M(\phi U, \phi V) = \cos^2 \theta g_M(U, V) \tag{3.17}$$

$$g_M(\omega U, \omega V) = \sin^2 \theta g_M(U, V) \tag{3.18}$$

for any $U, V \in \Gamma(\ker F_*)$.

Let F be a conformal slant Riemannian map from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the slant angle θ ; then we say that ω is parallel with respect to $\overset{M}{\nabla}$ on $\ker F_*$ if its covariant derivative according to $\overset{M}{\nabla}$ vanishes, i.e.

$$(\overset{M}{\nabla}_U \omega)V = 0 \quad (3.19)$$

for $U, V \in \Gamma(\ker F_*)$.

Theorem 3.3. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If ω is parallel according to $\overset{M}{\nabla}$ on $\ker F_*$, then we have*

$$T_{\phi U} \phi U = -\cos^2 \theta T_U U \quad (3.20)$$

for $U \in \Gamma(\ker F_*)$.

Proof. If ω is parallel according to $\overset{M}{\nabla}$ on $\ker F_*$, we obtain using (3.14) and Lemma 3.1. for $U, V \in \Gamma(\ker F_*)$

$$CT_U V = T_U \phi V. \quad (3.21)$$

Now, changing roles of U and V in (3.21) we get

$$CT_V U = T_V \phi U. \quad (3.22)$$

Because vertical vector field T is symmetric, from (3.21) and (3.22) we get

$$T_U \phi V = T_V \phi U. \quad (3.23)$$

Since $\phi^2 V = \lambda V$ and for $V = \phi U$ in (3.23) we obtain

$$-\cos^2 \theta T_U U = T_{\phi U} \phi U,$$

which gives the assertion.

Theorem 3.4. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then, two of the below assertions imply the third assertion,*

- i- *The horizontal distribution $(\ker F_*)^\perp$ is integrable,*
- ii- $X(\ln \lambda)g_M(Y, \omega \phi U) = Y(\ln \lambda)g_M(X, \omega \phi U),$
- iii- $g_N(F_*(Ch \overset{M}{\nabla}_X \omega U + \omega A_X \omega U), F_*(Y)) + g_N(\overset{N}{\nabla}_X^F F_*(\omega \phi U), F_*(Y))$
 $= g_N(F_*(Ch \overset{M}{\nabla}_Y \omega U + \omega A_Y \omega U), F_*(X)) + g_N(\overset{N}{\nabla}_Y^F F_*(\omega \phi U), F_*(X))$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$.

Proof. Now, for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$, using (2.8) and (3.12), we obtain

$$\begin{aligned}
 g_M([X, Y], U) &= g_M(\overset{M}{\nabla}_X J\phi U, Y) + g_M(JA_X\omega U + Jh\overset{M}{\nabla}_X\omega U, Y) \\
 &\quad - g_M(\overset{M}{\nabla}_Y J\phi U, X) - g_M(JA_Y\omega U + Jh\overset{M}{\nabla}_Y\omega U, X).
 \end{aligned}$$

Since F is a conformal map, from Theorem 3.1., (2.8) and (3.13) we get

$$\begin{aligned}
 g_M([X, Y], U) &= \cos^2\theta g_M([X, Y], U) + \frac{1}{\lambda^2} \{g_N(F_*(h\overset{M}{\nabla}_X\omega\phi U), F_*(Y)) \\
 &\quad + g_N(F_*(\omega A_X\omega U), F_*(Y)) + g_N(F_*(Ch\overset{M}{\nabla}_X\omega U), F_*(Y)) \\
 &\quad - g_N(F_*(h\overset{M}{\nabla}_Y\omega\phi U), F_*(X)) - g_N(F_*(\omega A_Y\omega U), F_*(X)) \\
 &\quad - g_N(F_*(Ch\overset{M}{\nabla}_Y\omega U), F_*(X))\}.
 \end{aligned}$$

Now, from (2.2) and (2.9) we have

$$\begin{aligned}
 \sin^2\theta g_M([X, Y], U) &= \frac{1}{\lambda^2} \{g_N(F_*(Ch\overset{M}{\nabla}_X\omega U + \omega A_X\omega U), F_*(Y)) \\
 &\quad - g_N(F_*(Ch\overset{M}{\nabla}_Y\omega U + \omega A_Y\omega U), F_*(X)) \\
 &\quad + g_N(F_*(\overset{N}{\nabla}_X^F F_*(\omega\phi U), F_*(Y)) \\
 &\quad - g_N(F_*(\overset{N}{\nabla}_Y^F F_*(\omega\phi U), F_*(X)) \\
 &\quad - X(\ln \lambda)g_N(F_*(\omega\phi U), F_*(Y)) \\
 &\quad - \omega\phi U(\ln \lambda)g_N(F_*(X), F_*(Y)) \\
 &\quad + g_M(X, \omega\phi U)g_N(F_*(grad(\ln \lambda)), F_*(Y)) \\
 &\quad - g_N((\nabla F_*)^\perp(X, \omega\phi U), F_*(Y)) \\
 &\quad + Y(\ln \lambda)g_N(F_*(\omega\phi U), F_*(X)) \\
 &\quad + \omega\phi U(\ln \lambda)g_N(F_*(Y), F_*(X)) \\
 &\quad - g_M(Y, \omega\phi U)g_N(F_*(grad(\ln \lambda)), F_*(X)) \\
 &\quad + g_N((\nabla F_*)^\perp(Y, \omega\phi U), F_*(X))\}.
 \end{aligned}$$

Using conformality of F we obtain

$$\begin{aligned}
\sin^2\theta g_M([X, Y], U) &= \frac{1}{\lambda^2} \{g_N(F_*(Ch\overset{M}{\nabla}_X\omega U + \omega A_X\omega U), F_*(Y)) \\
&- g_N(F_*(Ch\overset{M}{\nabla}_Y\omega U + \omega A_Y\omega U), F_*(X)) \\
&+ g_N(F_*(\overset{N}{\nabla}_X^F F_*(\omega\phi U), F_*(Y)) \\
&- g_N(F_*(\overset{N}{\nabla}_Y^F F_*(\omega\phi U), F_*(X)))\} \\
&+ 2Y(\ln \lambda)g_M(X, \omega\phi U) - 2X(\ln \lambda)g_M(Y, \omega\phi U).
\end{aligned}$$

The proof is completed from the above equation.

Now we will examine the geometry of leaves of the vertical distribution.

Theorem 3.5. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then, the vertical distribution $\ker F_*$ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla F_*)(U, JX), F_*(\omega V)) = g_N((\nabla F_*)(U, X), F_*(\omega\phi V))$$

for $X \in \Gamma((\ker F_*)^\perp)$ and $U, V \in \Gamma(\ker F_*)$.

Proof. Because of M is a Kaehler manifold and from Theorem 3.1., (3.12) and (3.13), we have

$$\begin{aligned}
g_M(\overset{M}{\nabla}_U V, X) &= -\cos^2\theta g_M(\overset{M}{\nabla}_U X, V) - g_M(\overset{M}{\nabla}_U X, \omega\phi V) \\
&- g_M(\overset{M}{\nabla}_U BX, \omega V) - g_M(\overset{M}{\nabla}_U CX, \omega V).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\sin^2\theta g_M(\overset{M}{\nabla}_U V, X) &= -g_M(h\overset{M}{\nabla}_U X, \omega\phi V) - g_M(T_U BX, \omega V) \\
&- g_M(h\overset{M}{\nabla}_U CX, \omega V).
\end{aligned}$$

Now, from (2.2) we get

$$\begin{aligned}
\sin^2\theta g_M(\overset{M}{\nabla}_U V, X) &= \frac{1}{\lambda^2} \{-g_N(F_*(h\overset{M}{\nabla}_U X), F_*(\omega\phi V)) \\
&- g_N(F_*(T_U BX), F_*(\omega V)) \\
&- g_N(F_*(h\overset{M}{\nabla}_U CX), F_*(\omega V))\} \\
&= \frac{1}{\lambda^2} \{g_N((\nabla F_*)(U, JX), F_*(\omega V)) \\
&- g_N((\nabla F_*)(U, X), F_*(\omega\phi V))\}.
\end{aligned}$$

This completes the proof.

Now, we examine the geometry of the horizontal distribution.

Theorem 3.6. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then, two of the below assertions imply the third assertion,*

- i- *the horizontal distribution $(ker F_*)^\perp$ defines a totally geodesic foliation on M ,*
- ii- *F is a horizontally homothetic map,*
- iii- $g_M(A_X Y, U) = \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(Y), F_*(\omega\phi U + C\omega U))$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$.

Proof. Now, from (2.8), (3.17) and (3.18) we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= g_M(JA_X Y + Jh\overset{M}{\nabla}_X Y, \phi U) \\ &+ g_M(JA_X Y + Jh\overset{M}{\nabla}_X Y, \omega U) \\ &= \cos^2\theta g_M(A_X Y, U) - g_M(h\overset{M}{\nabla}_X Y, J\phi U) \\ &+ \sin^2\theta g_M(A_X Y, U) - g_M(h\overset{M}{\nabla}_X Y, J\omega U) \\ &= g_M(A_X Y, U) - g_M(h\overset{M}{\nabla}_X Y, \omega\phi U) - g_M(h\overset{M}{\nabla}_X Y, C\phi U) \end{aligned}$$

for $X, Y \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$. From (2.2) and (2.9), we obtain

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= g_M(A_X Y, U) - \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(Y), F_*(\omega\phi U + C\omega U)) \\ &+ X(\ln \lambda)g_M(Y, \omega\phi U) + Y(\ln \lambda)g_M(X, \omega\phi U) \\ &- \omega\phi U(\ln \lambda)g_M(X, Y) + X(\ln \lambda)g_M(Y, C\omega U) \\ &+ Y(\ln \lambda)g_M(X, C\omega U) - C\omega U(\ln \lambda)g_M(X, Y). \end{aligned} \tag{3.24}$$

If the horizontal distribution $(ker F_*)^\perp$ defines a totally geodesic foliation on M for $X, Y \in \Gamma((ker F_*)^\perp)$, $U \in \Gamma(ker F_*)$ and $g_M(A_X Y, U) = \frac{1}{\lambda^2} g_N(\nabla_X^F F_*(Y), F_*(\omega\phi U + C\omega U))$, we show that the map F is a horizontally homothetic map. If (i) and (iii) are satisfied, then we have

$$\begin{aligned} 0 &= X(\ln \lambda)g_M(Y, \omega\phi U) + Y(\ln \lambda)g_M(X, \omega\phi U) \\ &- \omega\phi U(\ln \lambda)g_M(X, Y) + X(\ln \lambda)g_M(Y, C\omega U) \\ &+ Y(\ln \lambda)g_M(X, C\omega U) - C\omega U(\ln \lambda)g_M(X, Y) \end{aligned} \tag{3.25}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $U \in \Gamma(\ker F_*)$. Suppose that $X = \omega\phi U$, $Y = C\omega U$ in equation (3.25), we have

$$C\omega U(\ln \lambda)g_M(\omega\phi U, \omega\phi U) + \omega\phi U(\ln \lambda)g_M(C\omega U, C\omega U) = 0. \quad (3.26)$$

If $C\omega U(\ln \lambda) = 0$ from (3.26) we get $\omega\phi U(\ln \lambda)g_M(C\omega U, C\omega U) = 0$ for $C\omega U \in \Gamma(C(\ker F_*)^\perp)$. Therefore λ is a constant on $\Gamma(\omega(\ker F_*))$. At the same time, if $\omega\phi U(\ln \lambda) = 0$ we derive $C\omega U(\ln \lambda)g_M(\omega\phi U, \omega\phi U) = 0$ from (3.26) for $\omega\phi U \in \Gamma(\omega(\ker F_*))$. Thus λ is a constant on $\Gamma(C(\ker F_*)^\perp)$. So, F is a horizontally homothetic map. The rest of the proof is clear.

Now we are going to slightly modify the notion of pluriharmonic map and use this new notion to obtain certain conditions for conformal slant Riemannian maps to be horizontally homothetic map. We say that a conformal slant Riemannian map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is $\ker F_*$ - (respectively, $(\ker F_*)^\perp$, $\omega(\ker F_*)$, μ) pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(U, V) + (\nabla F_*)(JU, JV) = 0$$

for $U, V \in \Gamma(\ker F_*)$ (respectively, $(\ker F_*)^\perp$, $\omega(\ker F_*)$, μ) [27, 28].

Theorem 3.7. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If F is a $\ker F_*$ -pluriharmonic map, then one of the below assertions imply the second assertion,*

- i- F is a horizontally homothetic map,
- ii- $F_*(A_{\omega U}\phi V + A_{\omega V}\phi U) = F_*(h\nabla_U^M \omega\phi V + \omega T_U \omega V + Ch\nabla_U^M \omega V)$
and $(\nabla F_*)^\perp(\omega U, \omega V) = 0$

for $U, V \in \Gamma(\ker F_*)$.

Proof. From the definition of $\ker F_*$ -pluriharmonic map, (2.2) and (2.10), we have

$$\begin{aligned} 0 &= F_*(\nabla_U^M J\phi V + J\nabla_U^M \omega V) - F_*(\nabla_{\phi U}^M \phi V) - F_*(\nabla_{\omega V}^M \phi U) \\ &\quad - F_*(\nabla_{\omega U}^M \phi V) + (\nabla F_*)^\perp(\omega U, \omega V) + \omega U(\ln \lambda)F_*(\omega V) \\ &\quad + \omega V(\ln \lambda)F_*(\omega U) - g_M(\omega U, \omega V)F_*(\text{grad}(\ln \lambda)). \end{aligned}$$

Now, using (2.6), (3.20) and Theorem 3.1., we get

$$\begin{aligned}
 0 &= F_*(h\nabla_U^M \omega\phi V + \omega T_U \omega V + Ch\nabla_U^M \omega V - A_{\omega U} \phi V - A_{\omega V} \phi U) \\
 &+ (\nabla F_*^\perp(\omega U, \omega V) + \omega U(\ln \lambda)F_*(\omega V) + \omega V(\ln \lambda)F_*(\omega U) \\
 &- g_M(\omega U, \omega V)F_*(grad(\ln \lambda))).
 \end{aligned}
 \tag{3.27}$$

If (i) is provided we have from (3.27)

$$\omega U(\ln \lambda)F_*(\omega V) + \omega V(\ln \lambda)F_*(\omega U) - g_M(\omega U, \omega V)F_*(grad(\ln \lambda)) = 0$$

for $U, V \in \Gamma(ker F_*)$. So one can see second assertion clearly. Now if (ii) is satisfied in (3.27) we have $F_*(A_{\omega U} \phi V + A_{\omega V} \phi U) = F_*(h\nabla_U^M \omega\phi V + \omega T_U \omega V + Ch\nabla_U^M \omega V)$ and $(\nabla F_*^\perp(\omega U, \omega V) = 0$ for $U, V \in \Gamma(ker F_*)$, respectively. Thus, by (3.27) we get

$$\begin{aligned}
 0 &= \omega U(\ln \lambda)F_*(\omega V) + \omega V(\ln \lambda)F_*(\omega U) \\
 &- g_M(\omega U, \omega V)F_*(grad(\ln \lambda)).
 \end{aligned}
 \tag{3.28}$$

For $\omega U \in \Gamma(\omega(ker F_*))$ from (3.28) we get $0 = \lambda^2 \omega V(\ln \lambda)g_M(\omega U, \omega U)$, which implies that $\omega(ker F_*)(grad(\ln \lambda)) = 0$. At the same time, from (3.28) if we take $\omega U = \omega V$ and for $X \in \Gamma(C(ker F_*^\perp))$ we get

$$0 = 2\lambda^2 \omega U(\ln \lambda)g_M(X, \omega U) - \lambda^2 X(\ln \lambda)g_M(\omega U, \omega U).
 \tag{3.29}$$

Because of λ is a constant on $\omega(ker F_*)$ we have $2\lambda^2 \omega U(\ln \lambda)g_M(X, \omega U) = 0$. Thus, by (3.29) we get $\lambda^2 X(\ln \lambda)g_M(\omega U, \omega U) = 0$, which implies that $(C(ker F_*^\perp))(grad(\ln \lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln \lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

Theorem 3.8. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If F is a $(ker F_*)^\perp$ -pluriharmonic map, then F is a horizontally homothetic map if and only if the following conditions*

$$(\nabla F_*^\perp(X, Y) + (\nabla F_*^\perp(CX, CY)) = 0$$

and

$$F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) = 0,$$

are satisfied for $X, Y \in \Gamma((ker F_*^\perp))$.

Proof. From the definition of a $(ker F_*)^\perp$ -pluriharmonic map, (2.2) and (2.9), we have

$$\begin{aligned}
0 &= (\nabla F_*)^\perp(X, Y) + X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\
&- g_M(X, Y)F_*(grad(\ln \lambda)) + (\nabla F_*)^\perp(CX, CY) + CX(\ln \lambda)F_*(CY) \\
&+ CY(\ln \lambda)F_*(CX) - g_M(CX, CY)F_*(grad(\ln \lambda)) \\
&- F_*\overset{M}{(\nabla_{BX}BY)} - F_*\overset{M}{(\nabla_{CY}BX)} - F_*\overset{M}{(\nabla_{CX}BY)}
\end{aligned}$$

or

$$\begin{aligned}
0 &= (\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(CX, CY) + X(\ln \lambda)F_*(Y) \\
&+ Y(\ln \lambda)F_*(X) - g_M(X, Y)F_*(grad(\ln \lambda)) + CX(\ln \lambda)F_*(CY) \\
&+ CY(\ln \lambda)F_*(CX) - g_M(CX, CY)F_*(grad(\ln \lambda)) \\
&- F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) \tag{3.30}
\end{aligned}$$

for $X, Y \in \Gamma((ker F_*)^\perp)$. If F is a horizontally homothetic map we have from equation (3.30)

$$\begin{aligned}
0 &= X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\
&- g_M(X, Y)F_*(grad(\ln \lambda)) + CX(\ln \lambda)F_*(CY) \\
&+ CY(\ln \lambda)F_*(CX) - g_M(CX, CY)F_*(grad(\ln \lambda))
\end{aligned}$$

for $X, Y \in \Gamma((ker F_*)^\perp)$. Since F is a horizontally homothetic map from (3.30) we obtain $(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(CX, CY) = 0$ and $F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) = 0$ for $X, Y \in \Gamma((ker F_*)^\perp)$. Now suppose that $(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(CX, CY) = 0$ and $F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) = 0$ in (3.30) for $X, Y \in \Gamma((ker F_*)^\perp)$, respectively. Thus, by (3.30) we get

$$\begin{aligned}
0 &= X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\
&- g_M(X, Y)F_*(grad(\ln \lambda)) + CX(\ln \lambda)F_*(CY) \\
&+ CY(\ln \lambda)F_*(CX) - g_M(CX, CY)F_*(grad(\ln \lambda)). \tag{3.31}
\end{aligned}$$

For $X = CX$, $Y = CY$ and $CY \in \Gamma(C(ker F_*)^\perp)$ in (3.31), we get $0 = 2\lambda^2 CX(\ln \lambda)g_M(CY, CY)$, which implies that $(C(ker F_*)^\perp)(grad(\ln \lambda)) = 0$. At the same time, from (3.31) if we take $X = Y = CX$ and $\omega U \in \Gamma(\omega(ker F_*))$, we get

$$0 = 4\lambda^2 CX(\ln \lambda)g_M(CX, \omega U) - 2\lambda^2 \omega U(\ln \lambda)g_M(CX, CX). \tag{3.32}$$

Since λ is a constant on $C(\ker F_*)^\perp$ we have $4\lambda^2 CX(\ln \lambda)g_M(CX, \omega U) = 0$. Thus, by (3.32) we get $-2\lambda^2\omega U(\ln \lambda)g_M(CX, CX) = 0$, which implies that $(\omega(\ker F_*))(grad(\ln \lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln \lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

We say that a conformal slant Riemannian map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is $\{(ker F_*)^\perp - (ker F_*)\}$ - pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X, V) + (\nabla F_*)(JX, JV) = 0$$

for $X \in \Gamma((ker F_*)^\perp)$ and $V \in \Gamma(ker F_*)$ [27, 28].

Theorem 3.9. *Let $F : (M, g_M, J) \rightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If F is a $\{(ker F_*)^\perp - (ker F_*)\}$ -pluriharmonic map, then two of the below assertions imply the third assertion,*

- i- F is a horizontally homothetic map,
- ii- $F_*(T_{BX}\omega U + A_{\omega U}BX + A_{CX}\phi U + h\nabla_X^M\omega\phi U) = F_*(\omega A_X\omega U + Ch\nabla_X^M\omega U)$
and $(\nabla F_*)^\perp(CX, \omega U) = 0$,
- iii- The vertical distribution $ker F_*$ is parallel along the horizontal distribution $(ker F_*)^\perp$ on M ,

for $X \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$.

Proof. From the definition of $\{(ker F_*)^\perp - (ker F_*)\}$ -pluriharmonic map we get

$$0 = (\nabla F_*)(X, U) + (\nabla F_*)(JX, JU)$$

for $X \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$. Using symmetry property of second fundamental form of a map by (2.2), (3.12) and (3.13) we get

$$\begin{aligned} 0 &= -F_*(\nabla_X^M U) + (\nabla F_*)(BX, \phi U) + (\nabla F_*)(\omega U, BX) \\ &+ (\nabla F_*)(CX, \phi U) + (\nabla F_*)(CX, \omega U). \end{aligned}$$

From (2.7), (2.8) and (2.10) we get

$$\begin{aligned} 0 &= F_*(\nabla_X^M J\phi U) + F_*(JA_X\omega U + Jh\nabla_X^M\omega U) - F_*(T_{BX}\phi U) \\ &- F_*(A_{\omega U}BX) - F_*(A_{CX}\phi U) + (\nabla F_*)^\perp(CX, \omega U) \\ &+ CX(\ln \lambda)F_*(\omega U) + \omega U(\ln \lambda)F_*(CX) \\ &- g_M(CX, \omega U)F_*(grad(\ln \lambda)). \end{aligned}$$

Now, from Theorem 3.1. , we have

$$\begin{aligned}
\cos^2\theta F_*^M(\nabla_X^M U) &= F_*(h\nabla_X^M \omega\phi U + \omega A_X \omega U + Ch\nabla_X^M \omega U) \\
&- F_*(T_{BX}\phi U + A_{\omega U}BX + A_{CX}\phi U) \\
&+ (\nabla F_*)^\perp(CX, \omega U) \\
&+ CX(\ln \lambda)F_*(\omega U) + \omega U(\ln \lambda)F_*(CX) \\
&- g_M(CX, \omega U)F_*(grad(\ln \lambda))
\end{aligned} \tag{3.33}$$

for $X \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$. If (i) and (ii) are satisfied in (3.33) we have

$$\begin{aligned}
0 &= CX(\ln \lambda)F_*(\omega U) + \omega U(\ln \lambda)F_*(CX) - g_M(CX, \omega U)F_*(grad(\ln \lambda)), \\
&(\nabla F_*)^\perp(CX, \omega U) = 0
\end{aligned}$$

and

$$F_*(T_{BX}\omega U + A_{\omega U}BX + A_{CX}\phi U + h\nabla_X^M \omega\phi U) = F_*(\omega A_X \omega U + Ch\nabla_X^M \omega U),$$

respectively. Then we get $F_*^M(\nabla_X^M U) = 0$. Therefore the vertical distribution $ker F_*$ is parallel along the horizontal distribution $(ker F_*)^\perp$ on M for $X \in \Gamma((ker F_*)^\perp)$ and $U \in \Gamma(ker F_*)$. Suppose that (i) and (iii) are satisfied in (3.33), one can see clearly that (ii) is satisfies. Assume that (ii) and (iii) are satisfied in (3.33) we get

$$\begin{aligned}
0 &= CX(\ln \lambda)F_*(\omega U) + \omega U(\ln \lambda)F_*(CX) \\
&- g_M(CX, \omega U)F_*(grad(\ln \lambda)).
\end{aligned} \tag{3.34}$$

For $CX \in \Gamma(C(ker F_*)^\perp)$ in (3.34) we get $0 = \lambda^2 \omega U(\ln \lambda)g_M(CX, CX)$, which implies that $(\omega(ker F_*))(grad(\ln \lambda)) = 0$. At the same time, from (3.34) for $\omega U \in \Gamma(\omega(ker F_*))$ we get $0 = \lambda^2 CX(\ln \lambda)g_M(\omega U, \omega U)$, which implies that $(C(ker F_*)^\perp)(grad(\ln \lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln \lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

Theorem 3.10. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If F is a $\omega(ker F_*)$ -pluriharmonic map, then F is a horizontally homothetic map if and only if the following conditions*

$$(\nabla F_*)^\perp(Z, Y) + (\nabla F_*)^\perp(CZ, CY) = 0$$

and

$$F_*(T_{BZ}BY + A_{CZ}BY + A_{CY}BZ) = 0$$

are satisfied for $Z, Y \in \Gamma(\omega(\ker F_*))$.

Proof. From the definition of $\omega(\ker F_*)$ - pluriharmonic map we have

$$0 = (\nabla F_*)(Z, Y) + (\nabla F_*)(JZ, JY)$$

for $Z, Y \in \Gamma(\omega(\ker F_*))$. From (2.2), (2.9) and (3.13) we get

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(Z, Y) + Z(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(Z) \\ &\quad - g_M(Z, Y)F_*(grad(\ln \lambda)) - F_*({}^M \nabla_{BZ} BY) - F_*({}^M \nabla_{CZ} BY) \\ &\quad - F_*({}^M \nabla_{CY} BZ) + (\nabla F_*)^\perp(CY, CZ) + CZ(\ln \lambda)F_*(CY) \\ &\quad + CY(\ln \lambda)F_*(CZ) - g_M(CZ, CY)F_*(grad(\ln \lambda)). \end{aligned}$$

Using (2.5) and (2.7) we get

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(Z, Y) + (\nabla F_*)^\perp(CY, CZ) + Z(\ln \lambda)F_*(Y) \\ &\quad + Y(\ln \lambda)F_*(Z) - g_M(Z, Y)F_*(grad(\ln \lambda)) + CZ(\ln \lambda)F_*(CY) \\ &\quad + CY(\ln \lambda)F_*(CZ) - g_M(CZ, CY)F_*(grad(\ln \lambda)) \\ &\quad - F_*(T_{BZ} BY) - F_*(A_{CZ} BY) - F_*(A_{CY} BZ). \end{aligned} \tag{3.35}$$

If F is a horizontally homothetic map we have from (3.35)

$$\begin{aligned} 0 &= Z(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(Z) - g_M(Z, Y)F_*(grad(\ln \lambda)) \\ &\quad + CZ(\ln \lambda)F_*(CY) + CY(\ln \lambda)F_*(CZ) - g_M(CZ, CY)F_*(grad(\ln \lambda)) \end{aligned}$$

for $Z, Y \in \Gamma(\omega(\ker F_*))$. Since F is a horizontally homothetic map from (3.35) we obtain $(\nabla F_*)^\perp(Z, Y) + (\nabla F_*)^\perp(CZ, CY) = 0$ and $F_*(T_{BZ} BY + A_{CZ} BY + A_{CY} BZ) = 0$ for $Z, Y \in \Gamma(\omega(\ker F_*))$. Suppose that

$$(\nabla F_*)^\perp(Z, Y) + (\nabla F_*)^\perp(CZ, CY) = 0$$

and $F_*(T_{BZ} BY + A_{CZ} BY + A_{CY} BZ) = 0$ in (3.35) for $Z, Y \in \Gamma(\omega(\ker F_*))$. Thus, by (3.35) we get

$$\begin{aligned} 0 &= Z(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(Z) - g_M(Z, Y)F_*(grad(\ln \lambda)) \\ &\quad + CZ(\ln \lambda)F_*(CY) + CY(\ln \lambda)F_*(CZ) \\ &\quad - g_M(CZ, CY)F_*(grad(\ln \lambda)). \end{aligned} \tag{3.36}$$

We know $g_M(Y, CY) = g_M(Y, JY - BY) = g_M(Y, JY) = 0$. For $Z = Y$ and $CY \in \Gamma(C(\ker F_*)^\perp)$ in (3.36) we get $0 = -\lambda^2 CY(\ln \lambda)\{g_M(Y, Y) - g_M(CY, CY)\}$ which implies that $(C(\ker F_*)^\perp)(\text{grad}(\ln \lambda)) = 0$. At the same time, from (3.36) if we take $Z = Y$ and $Y \in \Gamma(\omega(\ker F_*))$ we get $0 = \lambda^2 Y(\ln \lambda)\{g_M(Y, Y) - g_M(CY, CY)\}$ which implies that $(\omega(\ker F_*))(\text{grad}(\ln \lambda)) = 0$. Thus $\mathcal{H}(\text{grad}(\ln \lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

We say that a conformal slant Riemannian map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is $(\mu - \omega(\ker F_*))$ -pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) = 0$$

for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(\ker F_*))$.

Theorem 3.11. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . If F is a $(\mu - \omega(\ker F_*))$ -pluriharmonic map, then F is a horizontally homothetic map if and only if the following conditions*

$$(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(JX, CY) = 0$$

and

$$F_*(A_{JX}BY) = 0$$

are satisfied for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(\ker F_*))$.

Proof. From the definition of $(\mu - \omega(\ker F_*))$ -pluriharmonic map, (2.2), (2.10) and (3.13) we have

$$\begin{aligned} 0 &= (\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) \\ 0 &= (\nabla F_*)^\perp(X, Y) + X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\ &\quad - g_M(X, Y)F_*(\text{grad}(\ln \lambda)) + (\nabla F_*)(JX, BY) + (\nabla F_*)(JX, CY). \end{aligned}$$

Since the distributions μ and $\omega(\ker F_*)$ are orthogonal to each other, we have $g_M(X, Y) = 0$. So, we obtain

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X, Y) + X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\ &\quad - F_*(A_{JX}BY) + (\nabla F_*)^\perp(JX, CY) + JX(\ln \lambda)F_*(CY) \\ &\quad + CY(\ln \lambda)F_*(JX) - g_M(JX, CY)F_*(\text{grad}(\ln \lambda)) \end{aligned} \tag{3.37}$$

for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(\ker F_*))$. Suppose that F is a horizontally homothetic map. From (3.37) we have

$$\begin{aligned} 0 &= X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\ &+ JX(\ln \lambda)F_*(CY) + CY(\ln \lambda)F_*(JX) \\ &- g_M(JX, CY)F_*(grad(\ln \lambda)). \end{aligned} \tag{3.38}$$

Since F is a horizontally homothetic map from (3.37) we obtain $F_*(A_{JX}BY) = 0$ and $(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(JX, CY) = 0$ for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(\ker F_*))$. Suppose that $F_*(A_{JX}BY) = 0$ and $(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(JX, CY) = 0$ for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(\ker F_*))$ in (3.37). Using conformality of F for $X \in \Gamma(\mu)$ in (3.38) we get

$$\begin{aligned} 0 &= \lambda^2\{X(\ln \lambda)g_M(Y, X) + Y(\ln \lambda)g_M(X, X) \\ &+ JX(\ln \lambda)g_M(CY, X) + CY(\ln \lambda)g_M(JX, X) \\ &- X(\ln \lambda)g_M(JX, CY)\}. \end{aligned} \tag{3.39}$$

We know $g_M(CY, X) = g_M(JY, X) = -g_M(Y, JX) = 0$, $g_M(JX, CY) = 0$ for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(\ker F_*))$ from (3.13). Then we obtain from (3.39) $\lambda^2Y(\ln \lambda)g_M(X, X) = 0$, which implies that $\omega(\ker F_*)(grad(\ln \lambda)) = 0$. For $X \in \Gamma(\mu)$ and $JX = X$ in (3.38) we get

$$\begin{aligned} 0 &= \lambda^2\{X(\ln \lambda)g_M(Y, X) + Y(\ln \lambda)g_M(X, X) \\ &+ X(\ln \lambda)g_M(CY, X) + CY(\ln \lambda)g_M(X, X) \\ &- X(\ln \lambda)g_M(X, CY)\}. \end{aligned} \tag{3.40}$$

Since λ is a constant on $\omega(\ker F_*)$ we have $Y(\ln \lambda) = 0$. We get from (3.40) $0 = \lambda^2CY(\ln \lambda)g_M(X, X)$ that implies $(C(\ker F_*)^\perp)(grad(\ln \lambda)) = 0$. It means λ is a constant on $C(\ker F_*)^\perp$. Lastly for $Y \in \Gamma(\omega(\ker F_*))$ and $JX = X$ in (3.38) we get

$$\begin{aligned} 0 &= \lambda^2\{X(\ln \lambda)g_M(Y, Y) + Y(\ln \lambda)g_M(X, Y) \\ &+ X(\ln \lambda)g_M(CY, Y) + CY(\ln \lambda)g_M(X, Y) \\ &- Y(\ln \lambda)g_M(X, CY)\}. \end{aligned} \tag{3.41}$$

We know $g_M(CY, Y) = g_M(JY, Y) = 0$ from (3.13) for $Y \in \Gamma(\omega(\ker F_*))$. Then we obtain from (3.41) $0 = \lambda^2X(\ln \lambda)g_M(Y, Y)$, which implies that $\mu(grad(\ln \lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln \lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

Theorem 3.12. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . F is a μ -pluriharmonic map if and only if λ is a constant on $\omega(\ker F_*)$.*

Proof. From the definition of μ - pluriharmonic map and (2.10), we have

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X, Y) + X(\ln \lambda)F_*(Y) + Y(\ln \lambda)F_*(X) \\ &\quad - g_M(X, Y)F_*(\text{grad}(\ln \lambda)) + (\nabla F_*)^\perp(JX, JY) + JX(\ln \lambda)F_*(JY) \\ &\quad + JY(\ln \lambda)F_*(JX) - g_M(JX, JY)F_*(\text{grad}(\ln \lambda)) \end{aligned}$$

for $X, Y \in \Gamma(\mu)$. Since $g_M(X, Y) = g_M(JX, JY)$ we obtain

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(JX, JY) + X(\ln \lambda)F_*(Y) \\ &\quad + Y(\ln \lambda)F_*(X) + JX(\ln \lambda)F_*(JY) + JY(\ln \lambda)F_*(JX) \\ &\quad - 2g_M(X, Y)F_*(\text{grad}(\ln \lambda)). \end{aligned} \tag{3.42}$$

Now taking $X = Y$ in (3.42) we get

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X, X) + (\nabla F_*)^\perp(JX, JX) \\ &\quad + 2X(\ln \lambda)F_*(X) + 2JX(\ln \lambda)F_*(JX) \\ &\quad - 2g_M(X, X)F_*(\text{grad}(\ln \lambda)). \end{aligned} \tag{3.43}$$

For $Z \in \Gamma(\omega(\ker F_*))$ in (3.43) we get

$$\begin{aligned} 0 &= g_N((\nabla F_*)^\perp(X, X), F_*(Z)) + g_N((\nabla F_*)^\perp(JX, JX), F_*(Z)) \\ &\quad + 2X(\ln \lambda)g_N(F_*(X), F_*(Z)) + 2JX(\ln \lambda)g_N(F_*(JX), F_*(Z)) \\ &\quad - 2g_M(X, X)g_N(F_*(\text{grad}(\ln \lambda)), F_*(Z)). \end{aligned}$$

Because of F is a conformal map and μ is a invariant distribution we obtain

$$\begin{aligned} 0 &= 2\lambda^2\{X(\ln \lambda)g_M(X, Z) + JX(\ln \lambda)g_M(JX, Z)\} \\ &\quad - 2\lambda^2g_M(X, X)g_M(\text{grad}(\ln \lambda), Z) \\ 0 &= -2\lambda^2Z(\ln \lambda)g_M(X, X). \end{aligned} \tag{3.44}$$

From equation (3.44) we obtain $Z(\ln \lambda) = 0$, which implies that λ is a constant on $\omega(\ker F_*)$ for $Z \in \Gamma(\omega(\ker F_*))$. The converse is clear.

We now give necessary and sufficient conditions for a conformal slant Riemannian map to be totally geodesic map.

Theorem 3.13. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) . Then, F is a totally geodesic map if and only if the following conditions are satisfied for $X, Y, Z \in \Gamma((ker F_*)^\perp)$ and $U, V \in \Gamma(ker F_*)$;*

- i- $g_N(F_*(Ch \overset{M}{\nabla}_U \omega V) + F_*(\omega \hat{\nabla}_U \phi V + \omega T_U \omega V), F_*(X)) = 0$,
- ii- F is a horizontally homothetic map and $(\nabla F_*)^\perp(X, Y) = 0$.

Proof. Now, from (2.2), (2.5), (3.12) and (3.13) we have

$$\begin{aligned}
 (\nabla F_*)(U, V) &= F_*(JT_U \phi V + J \hat{\nabla}_U \phi V) \\
 &+ F_*(\omega T_U \omega V + Ch \overset{M}{\nabla}_U \omega V).
 \end{aligned}$$

Because T is symmetric, we get

$$\begin{aligned}
 (\nabla F_*)(U, V) &= \cos^2 \theta F_*(T_V U) + F_*(\omega \hat{\nabla}_U \phi V) \\
 &+ F_*(\omega T_U \omega V + Ch \overset{M}{\nabla}_U \omega V)
 \end{aligned}$$

which implies that

$$\sin^2 \theta (\nabla F_*)(U, V) = F_*(\omega \hat{\nabla}_U \phi V) + F_*(\omega T_U \omega V + Ch \overset{M}{\nabla}_U \omega V) \tag{3.45}$$

for $U, V \in \Gamma(ker F_*)$. Thus, we obtain from (3.45)

$$\begin{aligned}
 \sin^2 \theta g_N((\nabla F_*)(U, V), F_*(X)) &= g_N(F_*(\omega \hat{\nabla}_U \phi V + \omega T_U \omega V), F_*(X)) \\
 &+ g_N(F_*(Ch \overset{M}{\nabla}_U \omega V), F_*(X))
 \end{aligned} \tag{3.46}$$

for $X \in \Gamma((ker F_*)^\perp)$. (i) is satisfied in (3.46). Now, from (2.9) we get

$$\begin{aligned}
 (\nabla F_*)(X, Y) &= (\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\top(X, Y) \\
 &= (\nabla F_*)^\perp(X, Y) + X(\ln \lambda) F_*(Y) + Y(\ln \lambda) F_*(X) \\
 &- g_M(X, Y) F_*(grad(\ln \lambda))
 \end{aligned} \tag{3.47}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$. From (3.47) we have

$$\begin{aligned}
 g_N((\nabla F_*)(X, Y), F_*(X)) &= g_N((\nabla F_*)^\perp(X, Y), F_*(X)) \\
 &+ X(\ln \lambda)g_N(F_*(Y), F_*(X)) \\
 &+ Y(\ln \lambda)g_N(F_*(X), F_*(X)) \\
 &- g_M(X, Y)g_N(F_*(\text{grad}(\ln \lambda)), F_*(X)) \\
 &= Y(\ln \lambda)g_N(F_*(X), F_*(X)) \\
 &= \lambda^2 Y(\ln \lambda)g_M(X, X)
 \end{aligned}$$

for $X \in \Gamma((\ker F_*)^\perp)$. We have $0 = \lambda^2 Y(\ln \lambda)g_M(X, X)$ which implies $Y(\ln \lambda) = 0$. So, λ is a constant on $(\ker F_*)^\perp$. F is a horizontally homothetic map and from (3.47) we get $(\nabla F_*)^\perp(X, Y) = 0$. Therefore, (ii) is satisfied. We complete the proof.

Acknowledgments. The authors would like to thank the referee for useful comments and their helpful suggestions that have improved the quality of this paper. This paper was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) with number 114F339.

REFERENCES

- [1] Akyol, M. A., & Gündüzalp, Y. (2018). On the geometry of conformal anti-invariant ξ^\perp -submersions, *Int. J. Maps Math.* 1(1), 50-67.
- [2] Akyol, M.A. (2017). Conformal semi-slant submersions. *Int. J. Geom. Method in Modern Physic.*, 14(7), 1750114.
- [3] Akyol, M. A., & Şahin, B. (2016). Conformal anti-invariant submersions from almost Hermitian manifolds. *Turk. J. Math.*, 40, 43-70.
- [4] Akyol, M. A., & Şahin, B. (2017). Conformal semi-invariant submersions. *Commun. Contemp. Math.*, 19(2), 1650011.
- [5] Akyol, M. A., & Şahin, B. (2017). Conformal slant Riemannian maps to Kaehler manifolds. *Tokyo J. Math.*, 42(1), 225-237.
- [6] Akyol, M. A., & Şahin, B. (2018). Conformal anti-invariant Riemannian maps to Kaehler manifolds. *U.P.B. Sci Bull Series A*, 80(4), 187-198.
- [7] Akyol, M. A., & Şahin, B. (2019). Conformal slant submersions. *Hacet. J. Math. Stat.*, 48(1), 28-44.
- [8] Baird, P., & Wood, J. C. (2003). *Harmonic morphisms between Riemannian manifolds* (No. 29). Oxford University Press.
- [9] Chen, B. Y. (1990). Slant immersions. *Bull. Austral. Math. Soc.*, 41, 135-147.
- [10] Falcitelli, M., Ianus, S., & Pastore, A. M. (2004). *Riemannian submersions and related topics*, World Sci. Publishing, River Edge, NJ.

- [11] Fischer, A. E. (1992). Riemannian maps between Riemannian manifolds. *Contemp. Math.*, 132, 331-366.
- [12] Fuglede, B. (1978). Harmonic morphism between Riemannian manifolds, *Ann. de l'Institut Fourier*, 28(2), 107-144.
- [13] Gray, A. (1967). Pseudo-Riemannian almost product manifolds and submersions. *J. Math. Mech.*, 16, 715-738.
- [14] Gündüzalp, Y., & Akyol, M. A. (2018). Conformal slant submersions from cosymplectic manifolds. *Turk. J. Math.*, 42(5), 2672-2689.
- [15] Ishiara, T. (1979). A mapping of Riemannian manifolds which preserves harmonic functions. *J. Math. Kyoto Univ.*, 19, 215-229.
- [16] Miao, J., Wang, Y., Gu, X., & Yau, S. T. (2005). Optimal global conformal surface parametrization for visualization. *Commun. Inf. Syst.*, 4(2), 117-134.
- [17] Nore, T. (1987). Second fundamental form of a map. *Ann. Mat. Pur. and Appl.*, 146, 281-310.
- [18] Ohnita, Y. (1987). On pluriharmonicity of stable harmonic maps. *Jour. London Math. Soc.*, 35(2), 563-587.
- [19] O'Neill, B. (1966). The fundamental equations of a submersion. *Michigan Math. J.*, 13(4), 459-469.
- [20] Park, K.S., & Prasad, R. (2013). Semi-slant submersions. *Bull. Korean Math. Soc.*, 50(3), 951-962.
- [21] Şahin, B. (2010). Conformal Riemannian maps between Riemannian manifolds, their harmonicity and decomposition theorems. *Acta Appl. Math.*, 109, 829-847.
- [22] Şahin, B. (2011). Slant submersions from almost Hermitian manifolds. *Bull. Math. Soc. Sci. Math. Roumanie (N.S.)*, 54(102), 93-105.
- [23] Şahin, B. (2012). *Manifoldların Diferensiyel Geometrisi (Turkish)*. Nobel Academic Press.
- [24] Şahin, B. (2013). Slant Riemannian maps from almost Hermitian manifolds. *Quaest. Math.*, 3(36), 449-461.
- [25] Şahin, B. (2013). Slant Riemannian maps to Kaehler manifolds. *Int. J. Geom. Methods Mod. Phys.*, 10(2), 1250080.
- [26] Şahin, B. (2017). *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*. Elsevier.
- [27] Şahin, B., & Yanan, Ş. (2018). Conformal Riemannian maps from almost Hermitian manifolds. *Turk. J. Math.*, 42, 2436-2451.
- [28] Şahin, B., & Yanan, Ş. (2019). Conformal semi-invariant Riemannian maps from almost Hermitian manifolds. *Filomat*, 33(4), 1125-1134.
- [29] Wang, Y., Gu, X., & Yau, S. T. (2003). Volumetric harmonic map. *Commun. Inf. Syst.*, 3(3), 191-201.
- [30] Wang, Y., Gu, X., Chan, T. F., Thompson, P. M., & Yau, S. T. (2007). Brain surface conformal parametrization with the Ricci flow. In: *IEEE International Symposium on Biomedical Imaging-From nano to macro (ISBI)*; Washington D.C., 1312-1315.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND LETTERS, ADIYAMAN UNIVERSITY, ADIYAMAN-TURKEY

DEPARTMENT OF MATHEMATICS, SCIENCE FACULTY, EGE UNIVERSITY İZMİR-TURKEY

Contents

1. [The approximation of bivariate generalized Bernstein-Durrmeyer type GBS operators](#)
Ecem Acar, Aydın İzgi 2-20
2. [Ruled surfaces with the base rectifying curves in Euclidean 3-space](#)
Beyhan Yılmaz, Yusuf Yaylı 21-28
3. [Frenet curves in 3-dimensional Lorentzian concircular structure manifolds](#)
Müslüm Aykut Akgün 29-40
4. [Remarks on combinatorial sums associated with special numbers and polynomials with their generating functions](#)
Neslihan Kılar, Yılmaz Şimşek 41-60
5. [Harmonicity of deformed gradient metric](#)
Nour El Houda Djaa, Abderrahim Zagane 61-77
6. [Conformal slant Riemannian maps](#)
Şener Yanan, Bayram Şahin 78-100

