ISSN:2636-7467(Online)

Volume 4
Issue 2
2021

international journal of MAPS IN MATHEMATICS

Editor in chief

Bayram SAHIN
Ege University, Izmir, TURKEY

International Journal of Maps in Mathematics is a fully refereed journal devoted to publishing recent results obtained in the research areas of maps in mathematics

www.journalmim.com

VOLUME 4 ISSUE 2
ISSN 2636-7467

September 2021
https://www.journalmim.com

International Journal of Maps in Mathematics

Editor-in-Chief

Bayram Sahin
Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey
journalofmapsinmathematics@gmail.com

This page intentionally left blank

International Journal of Maps in Mathematics

*Volume 4, Issue 2, 2021, Pages:**67-81***

ISSN: 2636-7467 (Online)

www.journalmim.com

SOME RESULTS IN THE THEORY OF QUASILINEAR SPACES

HACER BOZKURT * AND YILMAZ YILMAZ

Abstract. In this study, we present some new consequences and exercises of homogenized quasilinear spaces. We also research on the some characteristics of the homogenized quasilinear spaces. Then, we introduce the concept of equivalent norm on a quasilinear space. As in the linear functional analysis, we obtained some results with equivalent norms defined in normed quasilinear spaces.

Keywords: Quasilinear space, Normed quasilinear space, Inner product quasilinear space, Homogenized quasilinear space, Equivalent norms.

2010 Mathematics Subject Classification: 46C05, 46C07, 46C15, 46C50, 97H50.

1. Introduction

In the 1986, Aseev [1] presented the quasilinear spaces and normed quasilinear spaces which are generalization of linear spaces and normed linear spaces, respectively. The biggest difference between quasilinear space and linear space is that it has a partial order relation. He gave some properties and some results which are quasilinear provisions of some conclusions in classical linear functional analysis. Later, in [1], he presented the some new concepts in normed quasilinear spaces. Further, in ([7], [10], [11], [12], [2], [9], [8] etc.), they have proposed a series of new concepts and new results of quasilinear spaces. In [7], they introduced the concept of proper quasilinear space which is a new notion of quasilinear functional analysis.

Received: 2020.09.27 *Revised: 2020.11.18* *Accepted: 2020.11.24*

* Corresponding author

Hacer Bozkurt; hacer.bozkurt@batman.edu.tr; https://orcid.org/0000-0002-2216-2516

Yılmaz Yılmaz; yyilmaz44@gmail.com; https://orcid.org/0000-0003-1484-782X

.

In the same study, they presented concept of dimension of a quasilinear space which are very meaningful to improvement of quasilinear algebra.

In the light of all these studies, in [6], we extended the notion of inner product spaces to the quasilinear conditions. After giving this new definition, we obtained some new concepts on inner product quasilinear spaces such as Hilbert quasilinear spaces and some orthogonality concepts. Further, in [6], we examined the sample of quasilinear spaces "$I\mathbb{R}^n$" interval space and we presented the quasilinear spaces Is, Ic_0, Il_∞ and Il_2. Also, we have studied to clarify geometric properties of inner product quasilinear spaces in [13]. Furthermore, we tried to enlarge the results in quasilinear functional analysis in [3], [4] and [5].

In this paper, we present some new conclusions of homogenized quasilinear space. Also, we obtain some results with considerable advantages about features of homogenized quasilinear spaces. Furthermore, we obtain some results with equivalent norms in a normed quasilinear space.

2. Preliminaries

In this section, we give some definitions and results on quasilinear spaces given by Aseev [1].

Definition 2.1. [1] *A quasilinear space over a field $\mathbb{R}$ is a set Q with a partial order relation "$\preceq$" with the operations of addition $Q \times Q \to Q$ and scalar multiplication $\mathbb{R} \times Q \to Q$ satisfying the following conditions:*

(Q1) $q \preceq q$,

(Q2) $q \preceq z$, *if* $q \preceq w$ *and* $w \preceq z$,

(Q3) $q = w$, *if* $q \preceq w$ *and* $w \preceq q$,

(Q4) $q + w = w + q$,

(Q5) $q + (w + z) = (q + w) + z$,

(Q6) there exists an element $\theta \in Q$ *such that* $q + \theta = q$,

(Q7) $\alpha \cdot (\beta \cdot q) = (\alpha \cdot \beta) \cdot q$,

(Q8) $\alpha \cdot (q + w) = \alpha \cdot q + \alpha \cdot w$,

(Q9) $1 \cdot q = q$,

(Q10) $0 \cdot q = \theta$,

(Q11) $(\alpha + \beta) \cdot q \preceq \alpha \cdot q + \beta \cdot q$,

(Q12) $q + z \preceq w + v$, *if* $q \preceq w$ *and* $z \preceq v$,

(Q13) $\alpha \cdot q \preceq \alpha \cdot w$, *if* $q \preceq w$

for every $q, w, z, v \in Q$ *and every* $\alpha, \beta \in \mathbb{R}$.

The considerable instance which is a quasilinear space is the set of all closed intervals of $\mathbb{R}$ with the relation " $\subseteq$ ", algebraic sum operation $M + N = \{m + n : m \in M,\ n \in N\}$ and the real-scalar multiplication $\lambda \cdot M = \{\lambda m : m \in M\}$. We indicate this set by $\Omega_C(\mathbb{R})$. Also, the set of all compact subsets of $\mathbb{R}$ is $\Omega(\mathbb{R})$.

Let Q be a quasilinear space and $W \subseteq Q$. Then W is called a *subspace of* Q, whenever W is a quasilinear space with the same partial order relation and the restriction of the operations on Q to W. An element $q \in Q$ is said to be symmetric if $-q = q$, where $-q = (-1) \cdot q$, and Q_d denotes the set of all symmetric elements of Q.

Theorem 2.1. *W is a subspace of a quasilinear space Q if and only if, for every, $q, w \in W$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \cdot q + \beta \cdot w \in W$* [12].

Definition 2.2. [1] *Let Q be a quasilinear space. A function $\|.\|_Q : Q \longrightarrow \mathbb{R}$ is named a norm if the following circumstances hold:*

$(NQ1)$ $\|q\|_Q > 0$ *if* $q \neq 0$,

$(NQ2)$ $\|q + w\|_Q \leq \|q\|_Q + \|w\|_Q$,

$(NQ3)$ $\|\alpha \cdot q\|_Q = |\alpha| \cdot \|q\|_Q$,

$(NQ4)$ *if* $q \preceq w$, *then* $\|q\|_Q \leq \|w\|_Q$,

$(NQ5)$ *if for any $\varepsilon > 0$ there exists an element $q_\varepsilon \in Q$ such that, $q \preceq w + q_\varepsilon$ and $\|q_\varepsilon\|_Q \leq \varepsilon$ then $q \preceq w$ for any elements $q, w \in Q$ and any real number $\alpha \in \mathbb{R}$.*

Let Q be a normed quasilinear space. Hausdorff metric on Q is defined by the equality

$$h_Q(q, w) = \inf\left\{r \geq 0 : q \preceq w + z_1^r,\ w \preceq q + z_2^r, \|z_i^r\| \leq r\right\}.$$

Since $q \preceq w + (q - w)$ and $w \preceq q + (w - q)$, the quantity $h_Q(q, w)$ is well-defined for any elements $q, w \in Q$, and

$$h_Q(q, w) \leq \|q - w\|_Q .$$

Example 2.1. *Let X be a Banach space. A norm on $\Omega(X)$ is defined by*

$$\|A\|_{\Omega(X)} = \sup_{a \in A} \|a\|_X .$$

Then $\Omega(X)$ and $\Omega_C(X)$ are normed quasilinear spaces. The Hausdorff metric is described as ordinary:

$$h_{\Omega_C(X)}(A, B) = \inf\{r \geq 0 : A \subset B + S_r(\theta),\ B \subset A + S_r(\theta)\},$$

where $S_r(\theta)$ *demonstrates a closed ball of radius* r *about* $\theta \in X$ [1].

Definition 2.3. *Let* Q *be a quasilinear space,* $M \subseteq Q$ *and* $m \in M$*. The set*

$$F_m^M = \{z \in M_r : z \preceq m\}$$

is called floor in M of m*. If* $M = Q$*, then it is called* floor of m *and written* F_m *in place of* F_m^M [7].

Definition 2.4. *Let* Q *be a quasilinear space and* $M \subseteq Q$*. Then the set*

$$\mathcal{F}_M^Q = \bigcup_{m \in M} F_m^Q$$

is called floor in Q of M *and is indicated by* $\mathcal{F}_M^Q$ [7].

Definition 2.5. *Let* Q *be a quasilinear space.* Q *is called solid-floored quasilinear space whenever*

$$y = \sup\{m \in Q_r : m \preceq y\}$$

for all $y \in Q$*. Other than,* Q *is called non solid-floored quasilinear space* [7].

Example 2.2. $\Omega(\mathbb{R})$ *and* $\Omega_C(\mathbb{R})$ *are solid-floored quasilinear space. However, singular subspace of* $\Omega_C(\mathbb{R})$ *is non-solid floored quasilinear space. For example,*

$$\sup\left\{m : m \in \left((\Omega_C(\mathbb{R}))_s \cup \{0\}\right)_r,\ m \subseteq y\right\} = \{0\} \neq y$$

for element $y = [-2,2] \in (\Omega_C(\mathbb{R}))_s \cup \{0\}$*. Also, we can not find any element* $m \in \left((\Omega_C(\mathbb{R}))_s \cup \{0\}\right)_r$ *such that* $m \subseteq z$ *for* $z = [1,3] \in (\Omega_C(\mathbb{R}))_s \cup \{0\}$.

Definition 2.6. *Let* Q *be a quasilinear space.* Consolidation of floor of Q *is the smallest solid-floored quasilinear space* $\widehat{Q}$ *containing* Q*, that is, if there exists another solid-floored quasilinear space* W *containing* Q *then* $\widehat{Q} \subseteq W$ [13].

$\widehat{Q} = Q$ for some solid-floored quasilinear space Q. Besides, $\widehat{\Omega_C(\mathbb{R}^n)_s} = \Omega_C(\mathbb{R}^n)$. For a quasilinear space Q, the set

$$F_y^{\widehat{Q}} = \left\{z \in \left(\widehat{Q}\right)_r : z \preceq y\right\}.$$

is the *floor of* Q *in* $\widehat{Q}$.

Definition 2.7. *Let Q be a quasilinear space. A mapping $\langle \ , \ \rangle : Q \times Q \to \Omega(\mathbb{R})$ is called an inner product on Q if for any $q, w, z \in Q$ and $\alpha \in \mathbb{R}$ the following conditions hold:*

$(IPQ1)$ *If $q, w \in Q_r$ then $\langle q, w \rangle \in \Omega_C(\mathbb{R})_r \equiv \mathbb{R}$,*

$(IPQ2)$ $\langle q + w, z \rangle \subseteq \langle q, z \rangle + \langle w, z \rangle$,

$(IPQ3)$ $\langle \alpha \cdot q, w \rangle = \alpha \cdot \langle q, w \rangle$,

$(IPQ4)$ $\langle q, w \rangle = \langle w, q \rangle$,

$(IPQ5)$ $\langle q, q \rangle \geq 0$ *for* $q \in X_r$ *and* $\langle q, q \rangle = 0 \Leftrightarrow q = 0$,

$(IPQ6)$ $\|\langle q, w \rangle\|_{\Omega(\mathbb{R})} = \sup \left\{ \|\langle a, b \rangle\|_{\Omega(\mathbb{R})} : a \in F_q^{\widehat{Q}}, b \in F_w^{\widehat{Q}} \right\}$,

$(IPQ7)$ *if $q \preceq w$ and $u \preceq v$ then $\langle q, u \rangle \subseteq \langle w, v \rangle$,*

$(IPQ8)$ *if for any $\varepsilon > 0$ there exists an element $q_\varepsilon \in Q$ such that $q \preceq w + q_\varepsilon$ and $\langle q_\varepsilon, q_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$ then $q \preceq w$.*

A quasilinear space with an inner product is called an inner product quasilinear space [6].

Example 2.3. $\Omega_C(\mathbb{R})$, *is an example of inner product quasilinear space with*

$$\langle A, B \rangle = \{ab : a \in A, b \in B\}.$$

For any two elements q, w of an inner product quasilinear space Q, we have

$$\|\langle q, w \rangle\|_{\Omega(\mathbb{R})} \leq \|q\|_Q \|w\|_Q .$$

Every inner product quasilinear space Q is a normed quasilinear space with the norm described by

$$\|q\| = \sqrt{\|\langle q, q \rangle\|_{\Omega(\mathbb{R})}}$$

for every $q \in Q$.

Definition 2.8. *An element q of the inner product quasilinear space Q is said to be orthogonal to an element $w \in Q$ if*

$$\|\langle q, w \rangle\|_{\Omega(\mathbb{R})} = 0.$$

From here, we can call that q and w are orthogonal and we show $q \perp w$ [6].

An orthonormal set $M \subset Q$ is an orthogonal set in Q whose elements have norm 1, that is, for every q, $w \in M$

$$\| < q, w > \|_{\Omega(\mathbb{R})} = \begin{cases} 0, & q \neq w \\ 1, & q = w \end{cases} .$$

Definition 2.9. $A^{\perp}$, *is called the orthogonal complement of* A *and is showed by*

$$A^{\perp} = \{q \in Q : \|\langle q, w\rangle\|_{\Omega(\mathbb{R})} = 0,\ w \in A\}.$$

For any subset A of an inner product quasilinear space Q, $A^{\perp}$ is a closed subspace of Q [6].

Example 2.4. *Let* $X = (X_1, X_2, \ldots, X_n) \in I\mathbb{R}^n$ *and* $Y = (Y_1, Y_2, \ldots, Y_n) \in I\mathbb{R}^n$*. The algebraic sum operation on* $I\mathbb{R}^n$ *is defined by*

$$X + Y = (X_1 + Y_1, X_2 + Y_2, \ldots, X_n + Y_n)$$

and multiplication by a real number $\alpha \in \mathbb{R}$ *is defined by*

$$\alpha \cdot X = (\alpha \cdot X_1, \alpha \cdot X_2, \ldots, \alpha \cdot X_n).$$

If we will be assumed that the partial order on $I\mathbb{R}^n$ *is given by*

$$X \leq Y \Leftrightarrow X_i \preceq Y_i \qquad 1 \leq i \leq n$$

then $I\mathbb{R}^n$ *is quasilinear space according to the above processes. Furthermore, different two norm on* $I\mathbb{R}^n$ *are defined by*

$$\|X\| = \|(X_1, X_2, \ldots, X_n)\| = \sup_{1 \leq i \leq n} \|X_i\|_{I\mathbb{R}}$$

and

$$\|X\|_2 = \left(\sum_{i=1}^{n} \|X_i\|_{I\mathbb{R}}^2\right)^{\frac{1}{2}}.$$

The quasilinear space $I\mathbb{R}^n$*, with the inner product*

$$\langle X, Y\rangle = \sum_{i=1}^{n} \langle X_i, Y_i\rangle_{I\mathbb{R}}$$

is an inner product quasilinear space.

The quasilinear spaces $I\mathbb{R}^n$ and $\Omega_C(\mathbb{R}^n)$ are different from each other. For instance; while the set $A = \left\{(q, w) : q^2 + w^2 \leq 1\right\}$ is element of $\Omega_C(\mathbb{R}^2)$, it is not element of $I\mathbb{R}^2$. Further, $B = ([1, 3], \{4\}) \in I\mathbb{R}^2$ but $B \notin \Omega_C(\mathbb{R}^2)$. Thus, $I\mathbb{R}^n$ and $\Omega_C(\mathbb{R}^n)$ are two distinct instances of quasilinear spaces.

3. Main Results

In this section, we give the concept of homogenized quasilinear space by [5]. Then, we give new findings about this concept.

Definition 3.1. *Let Q be a quasilinear space. Q is called* ***homogenized quasilinear space*** *if for all $q \in Q$ and $\alpha\beta \geq 0$ the following circumstance is satisfied:*

$$(\alpha+\beta)\cdot q = \alpha \cdot q + \beta \cdot q.$$

Obviously, every vector space is a homogenized quasilinear space. However, the inverse is false.

Theorem 3.1. $\Omega_C(Q)$ *is a homogenized quasilinear space for every normed quasilinear space Q. However, $\Omega(Q)$ is not homogenized quasilinear space.*

Proof. Since $\Omega_C(Q)$ is a quasilinear space, we have $(\alpha+\beta)\cdot A \subseteq \alpha \cdot A + \beta \cdot A$ from (Q11) for every $A \in \Omega_C(Q)$. Now, we only prove the converse. Let $a \in \alpha \cdot A + \beta \cdot A$ for every $A \in \Omega_C(Q)$. Then, we obtain

$$a = \alpha \cdot q + \beta \cdot w$$

for a $q, w \in A$. From here, we can write

$$a = (\alpha+\beta)\left[\frac{\alpha}{\alpha+\beta}\cdot q + \frac{\beta}{\alpha+\beta}\cdot w\right].$$

If $t = \frac{\alpha}{\alpha+\beta}$ and $k = \frac{\beta}{\alpha+\beta}$, there is two different cases since $\alpha\beta \geq 0$:

i) If $\alpha \leq \alpha+\beta$ for $\alpha, \beta \in \mathbb{R}^+$, then we get $\frac{\alpha}{\alpha+\beta} \leq 1$ and $0 \leq \frac{\alpha}{\alpha+\beta}$.

ii) If $\alpha+\beta \leq \alpha$ for $\alpha, \beta \in \mathbb{R}^-$, then we get $1 \geq \frac{\alpha}{\alpha+\beta}$ and $0 \leq \frac{\alpha}{\alpha+\beta}$.

From i) and ii), we obtain $0 \leq t \leq 1$. Further, clearly $t+k=1$. According to the definition of convexity on quasilinear spaces, we get $\frac{\alpha}{\alpha+\beta}\cdot q + \frac{\beta}{\alpha+\beta}\cdot w \in A$. So, we show that

$$a = (\alpha+\beta)\cdot z \in A$$

for a $z \in A$.

Example 3.1. $\Omega(\mathbb{R})$ *is a non-homogenized quasilinear space. Consider the element $A = \{1,2,3\} \in \Omega(\mathbb{R})$. Clearly, $2\cdot A = \{2,4,6\}$. But $A+A = \{2,3,4,5,6\}$. Therefore $2\cdot A \neq A+A$ for $\alpha = 1$ and $\beta = 1$. This shows us that $\Omega(\mathbb{R})$ is not a homogenized quasilinear space.*

Theorem 3.2. *Let Q be a homogenized inner product quasilinear space and $q \in Q_d$. Then there exists at least one $w \in X$ such that $q = w - w$.*

Proof. We know that $(\alpha+\beta)\cdot w = \alpha\cdot w+\beta\cdot w$ for every $w \in Q$ and $\alpha, \beta \in \mathbb{R}^{+}$. Further $q = -q$ and $q = q$ since q is a symmetric element of Q. Same time, we get $q+q = q-q$. From here, we obtain $q = \frac{q}{2} - \frac{q}{2}$ since $2\cdot q = q-q$. This complete the proof.

Proposition 3.1. *Let Q be a homogenized quasilinear space and $q \in Q$. Then F_q is convex subset of Q.*

Proof. Let Q be a homogenized quasilinear space. From Definition 2.3, we have

$$F_q = \{a \in Q_r : a \preceq q\}$$

for a $q \in Q$. Thus, we obtain

$$a \preceq q \text{ and } b \preceq q$$

for every $a, b \in F_q$. From (Q13), we have

$$\gamma \cdot a \preceq \gamma \cdot q \text{ and } (1-\gamma)\cdot b \preceq (1-\gamma)\cdot q$$

for every $0 \preceq \gamma \preceq 1$. Hence,

$$\gamma \cdot a + (1-\gamma)\cdot b \preceq \gamma \cdot q + (1-\gamma)\cdot q.$$

Since, Q is a homogenized quasilinear space,

$$\gamma \cdot q + (1-\gamma)\cdot q = (\gamma + 1 - \gamma)\cdot q = q$$

for every $0 \preceq \gamma \preceq 1$. Therefore, we get

$$\gamma \cdot a + (1-\gamma)\cdot b \preceq q.$$

Thus, $\gamma \cdot a + (1-\gamma)\cdot b \in F_q$.

Remark 3.1. *Floor of an element of an inner product quasilinear space Q is convex if and only if this inner product quasilinear space Q is homogenized. If Q is not homogenized in the Proposition 3.1, then F_q is not convex since $(\alpha+\beta)\cdot q \neq \alpha \cdot q + \beta \cdot q$.*

Example 3.2. *$I\mathbb{R}^n$ is a homogenized inner product quasilinear space. In* [6]*, we showed that $I\mathbb{R}^n$ is an inner product quasilinear space with*

$$\langle X, Y\rangle = \sum_{i=1}^{n} \langle X_i, Y_i\rangle_{I\mathbb{R}}.$$

For every $X \in I\mathbb{R}^n$ and $\alpha\beta \geq 0$, we can write

$$\begin{aligned}(\alpha+\beta)\cdot X &= (\alpha+\beta)\cdot(X_1, X_2, \ldots, X_n)\\ &= ((\alpha+\beta)\cdot X_1, (\alpha+\beta)\cdot X_2, \ldots, (\alpha+\beta)\cdot X_n).\end{aligned}$$

Then, we obtain

$$\begin{aligned}(\alpha+\beta)\cdot X &= (\alpha\cdot X_1+\beta\cdot X_1, \alpha\cdot X_2+\beta\cdot X_2, \ldots, \alpha\cdot X_n+\beta\cdot X_n)\\ &= (\alpha\cdot X_1, \alpha\cdot X_2, \ldots, \alpha\cdot X_n)+(\beta\cdot X_1, \beta\cdot X_2, \ldots, \beta\cdot X_n)\\ &= \alpha\cdot X+\beta\cdot X\end{aligned}$$

since $I\mathbb{R}$ is a homogenized quasilinear space.

Example 3.3. *All interval sequence spaces Is, all bounded interval sequence spaces $Il_\infty = \{X = (X_n) \in I\mathbb{R}^\infty : |(X_n)| \leq \infty\}$ and all convergent interval sequence spaces*

$$Ic_0 = \{X = (X_n) \in I\mathbb{R}^\infty : (X_n) \to 0\}$$

are further example of homogenized quasilinear spaces.

Before giving the equivalent norms on the qasilinear spaces, we will give an example to cartesian product of quasilinear spaces.

Example 3.4. *Let Q be the Cartesian product of quasilinear spaces $Q_1, Q_2, ..., Q_n$, that is, $Q = Q_1 \times Q_2 \times ... \times Q_n$. The space Q is a quasilinear space with the algebraic sum operation*

$$(q_1, q_2, ..., q_n) + (w_1, w_2, ..., w_n) = (q_1 + w_1, q_2 + w_2 + ... + q_n + w_n),$$

real scalar multiplication

$$\alpha\cdot(q_1, q_2, ..., q_n) = (\alpha\cdot q_1, \alpha\cdot q_2, ..., \alpha\cdot q_n)$$

and order relation

$$(q_1, q_2, ..., q_n) \preceq (w_1, w_2, ..., w_n) \Leftrightarrow q_1 \preceq w_1, q_2 \preceq w_2, ..., q_n \preceq w_n$$

for every $(q_1, q_2, ...q_n), (w_1, w_2, ...w_n) \in Q_1 \times Q_2 \times ... \times Q_n = Q$.

Example 3.5. *Let Q and W be the normed quasilinear spaces with $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Define $Q \times W = \{z = (q, w) : q \in Q \text{ and } w \in W\}$. The functions*

$$\|z\| = \max(\|q\|_1, \|w\|_2) \tag{3.1}$$

$$\|z\|_0 = \|q\|_1 + \|w\|_2 \tag{3.2}$$

defines norms on $Q \times W$. Then $Q \times W$ is normed quasilinear space.

Proposition 3.2. *Let $\|\cdot\|_1$ be a norm on quasilinear space Q and $\|\cdot\|_2$ be a norm on quasilinear space W. From Example 3.5, we have $Z = Q \times W$ is normed quasilinear space with norms (3.1) and (3.2). Let $\{(q_n, w_n)\}$ be sequence in $Q \times W$. The following conditions are satisfied:*

i) The sequence $\{(q_n, w_n)\}$ is convergent to $\{(q, w)\}$ in Z if and only if $\{q_n\}$ is convergent to q in Q and $\{w_n\}$ is convergent to w in W.

ii) The sequence $\{(q_n, w_n)\}$ is Cauchy sequence in Z if and only if $\{q_n\}$ is Cauchy sequence in Q and $\{w_n\}$ is Cauchy sequence in W.

Proof. Suppose that $(q_n, w_n) \to (q, w) \in Z$. Then corresponding to each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that the following inequalities hold for $n > n_0$:

$$(q_n, w_n) \preceq (q, w) + a^{\epsilon}_{1,n}, \;\; (q, w) \preceq (q_n, w_n) + a^{\epsilon}_{2,n}, \;\; \left\|a^{\epsilon}_{i,n}\right\| \le \epsilon.$$

Here, $a^{\epsilon}_{1,n} = \left(b^{\epsilon}_{1,n}, c^{\epsilon}_{1,n}\right)$ and $a^{\epsilon}_{2,n} = \left(b^{\epsilon}_{2,n}, c^{\epsilon}_{2,n}\right)$. Since Z is quasilinear space, we get

$$q_n \preceq q + b^{\epsilon}_{1,n}, \;\; q \preceq q_n + b^{\epsilon}_{2,n}$$

and

$$w_n \preceq w + c^{\epsilon}_{1,n}, \;\; w \preceq w_n + c^{\epsilon}_{2,n}.$$

Also, since $\left\|a^{\epsilon}_{i,n}\right\| = \max\left(\left\|b^{\epsilon}_{i,n}\right\|_1, \left\|c^{\epsilon}_{i,n}\right\|_2\right) \le \epsilon$ or $\left\|a^{\epsilon}_{i,n}\right\|_0 = \left\|b^{\epsilon}_{i,n}\right\|_1 + \left\|c^{\epsilon}_{i,n}\right\|_2 \le \epsilon$, we obtain $\left\|b^{\epsilon}_{i,n}\right\|_1 \le \epsilon$ and $\left\|c^{\epsilon}_{i,n}\right\|_2 \le \epsilon$ according to (3.1) and (3.2). This proves that the sequence $\{q_n\}$ is convergent to q in Q and the sequence $\{w_n\}$ is convergent to w in W. The opposite can be shown in a similar way.

Let $\{(q_n, w_n)\}$ be a Cauchy sequence in Z. For an arbitrary $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$(q_n, w_n) \preceq (q_m, w_m) + a^{\epsilon}_{1,n}, \;\; (q_m, w_m) \preceq (q_n, w_n) + a^{\epsilon}_{2,n}, \quad \left\|a^{\epsilon}_{i,n}\right\| \le \epsilon$$

for all $m, n > n_0$, and thus also

$$q_n \preceq q_m + b^{\epsilon}_{1,n}, \;\; q_m \preceq q_n + b^{\epsilon}_{2,n}$$

and

$$w_n \preceq w_m + c^{\epsilon}_{1,n}, \;\; w_m \preceq w_n + c^{\epsilon}_{2,n}.$$

Further, we obtain $\left\|b_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$ and $\left\|c_{i,n}^{\epsilon}\right\|_2 \leq \epsilon$ for two norms defined in (3.1) and (3.2) since $\left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$. Now, let $\{q_n\}$ is Cauchy sequence in Q and $\{w_n\}$ is Cauchy sequence in W. Then for any $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$q_n \preceq q_m + b_{1,n}^{\epsilon},\ q_m \preceq q_n + b_{2,n}^{\epsilon}, \quad \left\|b_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$$

and

$$q_n \preceq q_m + c_{1,n}^{\epsilon},\ q_m \preceq q_n + c_{2,n}^{\epsilon}, \quad \left\|c_{i,n}^{\epsilon}\right\|_2 \leq \epsilon$$

for all $n, m > n_0$. Since Q and W are quasilinear space, we get

$$\begin{aligned} (q_n, w_n) &\preceq \left(q_m + b_{1,n}^{\epsilon}, w_m + c_{1,n}^{\epsilon}\right) = (q_m, w_m) + \left(b_{1,n}^{\epsilon}, c_{1,n}^{\epsilon}\right), \\ (q_m, w_m) &\preceq \left(q_n + b_{2,n}^{\epsilon}, w_n + c_{2,n}^{\epsilon}\right) = (q_n, w_n) + \left(b_{2,n}^{\epsilon}, c_{2,n}^{\epsilon}\right). \end{aligned}$$

Consequently, we obtain $\left\|\left(b_{i,n}^{\epsilon}, c_{i,n}^{\epsilon}\right)\right\| \leq \epsilon$ because $\left\|b_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$ and $\left\|c_{i,n}^{\epsilon}\right\|_2 \leq \epsilon$. This completes the proof.

Theorem 3.3. *Let $Q_1, Q_2, ..., Q_n$ be Banach quasilinear spaces over the same scalar field $\mathbb{R}$ with norm $\|\cdot\|_i$ $(1 \leq i \leq n)$, respectively. Then the product space $Q = Q_1 \times Q_2 \times ... \times Q_n$ is Banach quasilinear space with norm*

$$\|q\| = \max_{1 \preceq k \preceq n} \left(\|q_k\|_k\right).$$

Proof. Let $q^k = \left(\left(q_1^1, q_2^1, ..., q_n^1\right), \left(q_1^2, q_2^2, ..., q_n^2\right), ..., \left(q_1^k, q_2^k, ..., q_n^k\right), ...\right)$ be a Cauchy sequence in Q. For $\epsilon > 0$, there exists a number n_0 such that for $k, m > n_0$ there are elements $a_{1,n}^{\epsilon}, b_{2,n}^{\epsilon} \in Q$ for which

$$\begin{aligned} \left(q_1^k, q_2^k, ..., q_n^k\right) &\preceq \left(q_1^m, q_2^m, ..., q_n^m\right) + (a_i)_{1,k,m}^{\epsilon}, \\ \left(q_1^m, q_2^m, ..., q_n^m\right) &\preceq \left(q_1^k, q_2^k, ..., q_n^k\right) + (a_i)_{2,k,m}^{\epsilon}, \\ \left\|(a_i)_{j,k,m}^{\epsilon}\right\| &\leq \epsilon. \end{aligned}$$

From here, we get

$$\left\|\left(q_1^k, q_2^k, ..., q_n^k\right) - \left(q_1^m, q_2^m, ..., q_n^m\right)\right\| = \max_{1 \preceq i \preceq n} \left\|q_i^k - q_i^m\right\|_i \to 0$$

$(k, m \to \infty)$. Hence, $\left\|q_i^k - q_i^m\right\|_i \to 0$ for every $1 \leq i \leq n$ when $k, m \to \infty$. This proves that the $\left(q_i^k\right)$ is a Cauchy sequence in Q_i for every $1 \leq i \leq n$. Since Q_i is Banach, $\left(q_i^k\right)$ converges to a q_i in Q_i, $(k \to \infty)$. Note that this implies that for $\epsilon > 0$ there exists a n_0 such that for $k > n_0$:

$$q_i^k \preceq q_i + (a_i)_{1,k}^{\epsilon}, \quad q_i \preceq q_i^k + (a_i)_{2,k}^{\epsilon}, \qquad \left\|(a_i)_{j,k}^{\epsilon}\right\|_i \leq \epsilon$$

for every $1 \preceq i \preceq n$. Since

$$\begin{aligned}\left\|q^k - q\right\| &= \left\|\left(q_1^k, q_2^k, ..., q_n^k\right) - (q_1, q_2, ..., q_n)\right\| \\ &= \max_{1 \preceq i \preceq n}\left(\left\|q_i^k - q_i\right\|_i\right) \\ &\leq \epsilon,\end{aligned}$$

we have $q^k \to q \in Q, (k \to \infty)$. Consequently, Q is Banach quasilinear space.

Proposition 3.3. *If $Q_1, Q_2, ..., Q_n$ are solid-floored quasilinear space then $Q = Q_1 \times Q_2 \times ... \times Q_n$ is solid-floored quasilinear space.*

Proof. Let Q_i is solid-floored quasilinear space for every $1 \leq i \leq n$. From the Definition 2.5, we have

$$q_i = \sup\left\{w_i \in (Q_i)_r : w_i \preceq q_i\right\}$$

for every $q_i \in Q_i$. Since Q is a quasilinear space, we obtain

$$(w_1, w_2, ..., w_n) \preceq (q_1, q_2, ..., q_n)$$

such that $(w_1, w_2, ..., w_n) = w \in Q_r$ and $(q_1, q_2, ..., q_n) = q \in Q$. From here, we have

$$q = \sup\left\{(w_1, w_2, ..., w_n) \in Q_r : (w_1, w_2, ..., w_n) \preceq (q_1, q_2, ..., q_n)\right\}.$$

Now, we introduce the concept of equivalent norms on the same quasilinear space. Also, we concentrate on the Hausdorff metric properties for two equivalent norms that are defined on a quasilinear space.

Definition 3.2. *A norm $\|\cdot\|$ on a normed quasilinear space Q is said to be equivalent to a norm $\|\cdot\|_0$ on Q if there are positive real numbers a and b such that for all $q \in Q$ we have*

$$a\|q\|_0 \leq \|q\| \leq b\|q\|_0.$$

Example 3.6. *The following norms on $I\mathbb{R}^2 = \{(X_1, X_2) : X_1, X_2 \in \Omega_C(\mathbb{R})\}$ are equivalent:*

$$\begin{aligned}\|(x, y)\| &= \|x\| + \|y\| \\ \|(x, y)\|_1 &= \max\{\|x\|, \|y\|\}.\end{aligned}$$

Theorem 3.4. *Let Q be a quasilinear space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. The sequence $\{q_n\}$ is convergent to q in normed quasilinear space $(Q, \|\cdot\|)$ if and only if $\{q_n\}$ is convergent to q in $(Q, \|\cdot\|_1)$.*

Proof. Suppose that $\{q_n\} \to q$ in normed quasilinear space $(Q, \|\cdot\|)$. Then for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that:

$$q_n \preceq q + q_{1,n}^{\epsilon},\ q \preceq q_n + q_{2,n}^{\epsilon}, \quad \left\|q_{i,n}^{\epsilon}\right\| \leq \frac{\epsilon}{M}$$

$\forall n \geq N$ and $M \in \mathbb{N}^{+}$. Since the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, we have

$$\left\|q_{i,n}^{\epsilon}\right\|_1 \leq M \left\|q_{i,n}^{\epsilon}\right\| \leq \epsilon.$$

Hence $\{q_n\} \to q$ in $(Q, \|\cdot\|_1)$.

Conversely, let $\{q_n\} \to q$ in $(Q, \|\cdot\|_1)$. Then for every $\epsilon > 0$ there exists an index N such that

$$q_n \preceq q + q_{1,n}^{\epsilon},\ q \preceq q_n + q_{2,n}^{\epsilon}, \quad \left\|q_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$$

$\forall n \geq N$. Since the norms are equivalent, we get

$$m \|q\| \leq \|q\|_1 \leq \epsilon.$$

Hence, $\{q_n\}$ is convergent to q in $(Q, \|\cdot\|)$.

Theorem 3.5. *Let Q be a quasilinear space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. The sequence $\{q_n\}$ is Cauchy sequence in normed quasilinear space $(Q, \|\cdot\|)$ if and only if $\{q_n\}$ is Cauchy sequence in $(Q, \|\cdot\|_1)$.*

Proof. Let $\{q_n\}$ be a Cauchy sequence in $(Q, \|\cdot\|)$. For an arbitrary $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$q_n \preceq q_m + a_{1,n}^{\epsilon},\ q_m \preceq q_n + a_{2,n}^{\epsilon}, \quad \left\|a_{i,n}^{\epsilon}\right\| \leq \frac{\epsilon}{M}$$

for all $n, m > n_0$. Similar way to the above theorem, we obtain $\left\|a_{i,n}^{\epsilon}\right\|_1 \leq M \left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$. This proves that the sequence $\{q_n\}$ is Cauchy sequence in $(Q, \|\cdot\|_1)$. The proof of opposite can be proved by similar way.

Theorem 3.6. *Let Q be a quasilinear space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. $(Q, \|\cdot\|)$ is complete if and only if $(Q, \|\cdot\|_1)$ is complete.*

Proof. Let $(Q, \|\cdot\|)$ be a complete and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. If $\{q_n\}$ is a Cauchy sequence in $(Q, \|\cdot\|_1)$, then for an arbitrary $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$q_n \preceq q_m + a_{1,n}^{\epsilon},\ q_m \preceq q_n + a_{2,n}^{\epsilon}, \quad \left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$$

for all $n, m > n_0$. From Theorem 3.5, we have $\{q_n\}$ is a Cauchy sequence in $(Q, \|\cdot\|_1)$. We obtain $q_n \to q \in Q$ from the completeness of $(Q, \|\cdot\|)$. From Theorem 3.4, we get $\{q_n,\ n \in \mathbb{N}\}$ is convergent to q in $(Q, \|\cdot\|_1)$ which proves completeness of $(Q, \|\cdot\|_1)$. The converse can be proved similarly.

Corollary 3.1. *If two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a quasilinear space Q are equivalent, then $\|q_n - q\| \to 0$ if and only if $\|q_n - q\|_0 \to 0$ for any sequence (q_n) in Q and any $q \in Q$.*

If Q is finite dimensional normed quasilinear space, then any two norms on Q_r are equivalent since Q_r is a normed linear subspace of Q.

4. Conclusion

In this paper, we define the notion of homogenized quasilinear space as a new concept in quasilinear spaces. We also research on the some properties of the homogenized quasilinear spaces. Then, we introduce the concept of equivalent norm on a quasilinear space. As in the linear functional analysis, we obtained some results related to equivalent norms defined in normed quasilinear spaces.

References

[1] Aseev, S. M. (1986). Quasilinear operators and their application in the theory of multivalued mappings. Proceeding of the Steklov Instiute of Mathematics, 2, 23-52.

[2] Banazılı, H. K. (2014). On Quasilinear Operators Between Quasilinear Spaces. M. Sc. Thesis, Malatya, Turkey.

[3] Bozkurt, H. and Yılmaz, Y. (2016). On Inner Product Quasilinear Spaces and Hilbert Quasilinear Spaces. Information Sciences and Computing, vol.2016, Article ID ISC671116, 12 pp.

[4] Bozkurt, H. and Yılmaz, Y. (2016). Some New Properties of Inner Product Quasilinear Spaces. Bulletin of Mathematical Analysis and Applications, 8, No.1, 37-45.

[5] Bozkurt, H. and Yılmaz, Y. (2016). Some New Results on Inner Product Quasilinear Spaces. Cogent Mathematics, 3:1194801, 10 pp.

[6] Bozkurt, H. and Yılmaz, Y. (2016). New Inner Product Quasilinear Spaces on Interval Numbers. Journal of Function Spaces, vol. 2016, Article ID 2619271, 9 pp.

[7] Çakan, S. and Yılmaz, Y. (2015). Normed Proper Quasilinear Spaces. Journal of Nonlinear Sciences and Applications, 8, 816-836.

[8] Levent, H. and Yılmaz, Y. (2018). Translation, modulation and dilation systems set-valued signal processing. Carpathian Mathematical Publications, 10, No.1, 143-164.

[9] Levent, H. and Yılmaz, Y. (2018). Hahn- Banach extension theorem for interval-valued functions and existence of quasilinear functionals. New Trends in Mathematical Sciences. NTMSCI 6, No.2, 19-28.

[10] Markow, S. (2004). On the algebraic properties of the convex bodies and some applications. Journal of Convex Analysis, 162, No.1, 93-112.

[11] Rojas-Medar, M. A., Jiménez-Gamerob, M. D., Chalco-Canoa, Y. and Viera-Brandão, A. J. (2005) Fuzzy quasilinear spaces and applications. Fuzzy Sets and Systems, 152, 173-190.

[12] Yılmaz, Y., Çakan, S. and Aytekin, Ş. (2012). Topological quasilinear spaces. Abstract and Applied Analysis, vol.2012, 951374, 10 pp.

[13] Yılmaz, Y., Bozkurt, H. and Çakan, S. (2016). An orthonormal sets in inner product quasilinear spaces. Creative Mathematics and Informatics, 25, No.2, 237-247.

Batman University, Department of Mathematics, Batman, Turkey

İnönü University, Department of Mathematics, Malatya, Turkey

International Journal of Maps in Mathematics

Volume 4, Issue 2, 2021, Pages: ***82-92***

ISSN: 2636-7467 (Online)

www.journalmim.com

STABILITY OF CERTAIN NEUTRAL TYPE DIFFERENTIAL EQUATION AND NUMERICAL EXPERIMENT VIA DIFFERENTIAL TRANSFORM METHOD

YENER ALTUN

Abstract. In this study, we obtain both the asymptotically stability and the numerical solution of first order neutral type differential equation with multiple retarded arguments. We first obtain sufficient specific conditions expressed in terms of linear matrix inequality (LMI) using the Lyapunov method to establish the asymptotic stability of solutions. Secondly, we use the differential transform method (DTM) to show numerical solutions. Finally, two examples are presented to demonstrate the effectiveness and applicability of proposed methods by Matlab and an appropriate computer program.

Keywords: Stability, Lyapunov method, LMI, DTM.

2010 Mathematics Subject Classification: 34K20, 34K40, 65L10.

1. Introduction

The different particular cases of delay differential equations have been searched by many researchers for the past few decades. Recently, it can be seen from the related literature that qualitative properties of various neutral differential equations have been investigated by many authors and the researchers have obtained many interesting and important results on some qualitative properties such as stability, exponentially stability, asymptotically stability, oscillation, non-oscillations of solutions and etc.(see,[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

DTM, which is a semi-analytical-numerical technique, is based on the Taylor series expansion. The concept of method was first introduced by Pukhov [15] to solve linear and nonlinear problems in physical processes, and by Zhou [16] to study electrical circuits. This

Received:2020.12.24 *Revised:2021.01.12* *Accepted:2021.01.18*

Yener Altun; email.yeneraltun@yyu.edu.tr; https://orcid.org/0000-0003-1073-5513

.

method is advantageous in obtaining numerical, analytical and exact solutions of ordinary and partial differential equations it has been widely studied and applied in recent years (see,[17, 18, 19, 20, 21, 22, 23, 24, 25]). According to the current techniques in the literature, DTM is a reliable method that requires less work and does not require linearization.

In this study, we consider the following first order neutral type differential equation with multiple retarded arguments:

$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+a(t)f(x(t))+b(t)g(x(t-\sigma))+c(t)\int_{t-\delta}^{t}x(s)ds=0, \tag{1.1}$$

where $p(t), a(t),\ b(t), c(t) : [t_0,\infty) \to [0,\infty),\ t_0 \geq 0,$ and $f,\ g : \Re \to \Re$ with $f(0)=0, g(0)=0$ are continuous functions on their respective domains;τ, σ and δ are positive real constants. For each solution $x(t)$ of equation 1.1, we assume the existence following initial condition:

$$x(\theta)=\Phi(\theta), \quad \theta \in [t_0-H, t_0],$$

where $\Phi \in C([t_0-H,t_0],R), H=\max\{\tau,\sigma,\delta\}$.

Define

$$h_1(x)=\begin{cases} \frac{h(x)}{x}, & x \neq 0 \\ \\ \frac{dh(0)}{dt}, & x=0 \end{cases} \tag{1.2}$$

and

$$g_1(x)=\begin{cases} \frac{g(x)}{x}, x \neq 0 \\ \\ \frac{dg(0)}{dt}, x=0. \end{cases} \tag{1.3}$$

The main purpose and contribution of this work can be summarized as follows aspects:

i. This research on the stability of certain neutral type differential equation and their numerical solutions is still at the stage of developing. Therefore, we propose a novel stability criterion for further improvements.

ii. The proof technique for the asymptotically stability of the equation considered in this study includes the Lyapunov function method and the LMI technique. Also, DTM is used to obtain numerical solutions of the equation considered.

iii. The simulations showing the behaviors of the solutions of the equation addressed by applying the Lyapunov method and the numerical solutions of the equation addressed using DTM show that the proposed methods are useful and efficient.

2. Preliminaries and stability results

We suppose that there exist nonnegative constants $a_i, b_i, c_i, m_i, n_i \quad (i = 1, 2)$ and p_1 such that for $t \geq 0$,

$$a_1 \leq a(t) \leq a_2,\ b_1 \leq b(t) \leq b_2,\ c_1 \leq c(t) \leq c_2, \tag{2.4}$$

$$|p(t)| \leq p_1 < 1,\ m_1 \leq f_1(x) \leq m_2,\ n_1 \leq g_1(x) \leq n_2. \tag{2.5}$$

For convenience, define the operator $D : \Re \to \Re$ as

$$D(x_t) = x(t) + p(t)x(t-\tau) - \alpha \int_{t-\tau}^{t} x(s)ds - \beta \int_{t-\sigma}^{t} x(s)ds,$$

where α ve β are positive scalars to be chosen later. From 1.2 and 1.3, equation 1.1 can be readily rewritten as follows for $t \geq 0$,

$$\frac{d}{dt}[x(t) + p(t)x(t-\tau) - \alpha \int_{t-\tau}^{t} x(s)ds - \beta \int_{t-\sigma}^{t} x(s)ds] = -(f_1(x)a(t) + \alpha + \beta)x(t)$$
$$+\alpha x(t-\tau) + \beta x(t-\sigma) - g_1(x(t-\sigma))b(t)x(t-\sigma) - c(t) \int_{t-\delta}^{t} x(s)ds. \tag{2.6}$$

Theorem 2.1. *Let $a_i,\ b_i,\ c_i,\ m_i$ and n_i $(i = 1,\ 2)$ be nonnegative constants. Then trivial solution of neutral type differential equation 2.6 is asymptotically stability if the operator D is stable and there exist positive constants $\tau, \sigma, \delta, \alpha,\ \beta$ and λ_j $(j = 1,\ 2, \ldots, 5)$ such that*

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} & \beta - n_1 b_1 & \Pi_{14} & \Pi_{15} & -c_1 \\ * & \Pi_{22} & \Pi_{23} & -\alpha^2 & -\alpha\beta & -p_1 c_1 \\ * & * & -\lambda_2 & \Pi_{34} & \Pi_{35} & 0 \\ * & * & * & -\lambda_3 & 0 & \alpha c_2 \\ * & * & * & * & -\lambda_4 & \beta c_2 \\ * & * & * & * & * & -\lambda_5 \end{bmatrix} < 0, \tag{2.7}$$

where $\Pi_{11} = -2(m_1 a_1 + \alpha + \beta) + \lambda_1 + \lambda_2 + \lambda_3 \tau^2 + \lambda_4 \sigma^2 + \lambda_5 \delta^2, \Pi_{12} = \alpha - (m_1 a_1 + \alpha + \beta)p_1,$ $\Pi_{14} = m_2 a_2 \alpha + \alpha^2 + \alpha\beta, \Pi_{15} = m_2 a_2 \beta + \alpha\beta + \beta^2, \Pi_{22} = 2\alpha p_1 - \lambda_1, \Pi_{23} = \beta p_1 - n_1 b_1 p_1,$ $\Pi_{34} = -\alpha\beta + \alpha n_2 b_2, \Pi_{35} = -\beta^2 + \beta n_2 b_2$ *and the symbols "$*$" shows the elements below the main diagonal of the symmetric matrix* Π .

Proof. Consider the appropriate Lyapunov functional as

$$\begin{aligned}V(t) =&[D(x_t)]^2 + \lambda_1 \int_{t-\tau}^{t} x^2(s)ds + \lambda_2 \int_{t-\sigma}^{t} x^2(s)ds + \lambda_3\tau \int_{t-\tau}^{t} (\tau - t + s)x^2(s)ds \\ &+ \lambda_4\sigma \int_{t-\sigma}^{t} (\sigma - t + s)x^2(s)ds + \lambda_5\delta \int_{t-\delta}^{t} (\delta - t + s)x^2(s)ds,\end{aligned}$$

where $D(x_t) = x(t) + p(t)x(t-\tau) - \alpha \int\limits_{t-\tau}^{t} x(s)ds - \beta \int\limits_{t-\sigma}^{t} x(s)ds.$

When the time derivative of $V(t)$ along the trajectory of equation 2.6 are calculate, we obtain

$$\begin{aligned}\frac{dV}{dt} =&2[x(t) + p(t)x(t-\tau) - \alpha \int_{t-\tau}^{t} x(s)ds - \beta \int_{t-\sigma}^{t} x(s)ds] \\ &\times [-(f_1(x)a(t) + \alpha + \beta)x(t) + \alpha x(t-\tau) + \beta x(t-\sigma) \\ &- g_1(x(t-\sigma))b(t)x(t-\sigma) - c(t) \int_{t-\delta}^{t} x(s)ds] + \lambda_1[x^2(t) - x^2(t-\tau)] \\ &+ \lambda_2[x^2(t) - x^2(t-\sigma)] + \lambda_3\tau^2x^2(t) - \lambda_3\tau \int_{t-\tau}^{t} x^2(s)ds \\ &+ \lambda_4\sigma^2x^2(t) - \lambda_4\sigma \int_{t-\sigma}^{t} x^2(s)ds + \lambda_5\delta^2x^2(t) - \lambda_5\delta \int_{t-\delta}^{t} x^2(s)ds \\ =&(-2f_1(x)a(t) - 2\alpha - 2\beta + \lambda_1 + \lambda_2 + \lambda_3\tau^2 + \lambda_4\sigma^2 + \lambda_5\delta^2)x^2(t) \\ &+ 2\alpha x(t)x(t-\tau) + 2\beta x(t)x(t-\sigma) - 2g_1(x(t-\sigma))b(t)x(t)x(t-\sigma) \\ &- 2c(t)x(t) \int_{t-\delta}^{t} x(s)ds - 2(f_1(x)a(t) + \alpha + \beta)p(t)x(t)x(t-\tau) \\ &+ 2\alpha p(t)x^2(t-\tau) + 2\beta p(t)x(t-\tau)x(t-\sigma) \\ &- 2g_1(x(t-\sigma))b(t)p(t)x(t-\tau)x(t-\sigma) - 2p(t)c(t)x(t-\tau) \int_{t-\delta}^{t} x(s)ds \\ &+ 2(f_1(x)a(t) + \alpha + \beta)\alpha x(t) \int_{t-\tau}^{t} x(s)ds - 2\alpha^2x(t-\tau) \int_{t-\tau}^{t} x(s)ds \\ &- 2\alpha\beta x(t-\sigma) \int_{t-\tau}^{t} x(s)ds + 2\alpha g_1(x(t-\sigma))b(t)x(t-\sigma) \int_{t-\tau}^{t} x(s)ds \\ &+ 2\alpha c(t) \int_{t-\tau}^{t} x(s)ds \int_{t-\delta}^{t} x(s)ds + 2(f_1(x)a(t) + \alpha + \beta)\beta x(t) \int_{t-\sigma}^{t} x(s)ds \\ &- 2\alpha\beta x(t-\tau) \int_{t-\sigma}^{t} x(s)ds - 2\beta^2x(t-\sigma) \int_{t-\sigma}^{t} x(s)ds \\ &+ 2\beta g_1(x(t-\sigma))b(t)x(t-\sigma) \int_{t-\sigma}^{t} x(s)ds + 2\beta c(t) \int\limits_{t-\sigma}^{t} x(s)ds \int_{t-\delta}^{t} x(s)ds\end{aligned}$$

$$- \lambda_1 x^2(t-\tau) - \lambda_2 x^2(t-\sigma) - \lambda_3 \tau \int\limits_{t-\tau}^{t} x^2(s)ds - \lambda_4 \sigma \int\limits_{t-\sigma}^{t} x^2(s)ds$$
$$- \lambda_5 \delta \int_{t-\delta}^{t} x^2(s)ds.$$

By using hölder inequality we can easily see that

$$\begin{aligned} \tau \int_{t-\tau}^{t} x^2(s)ds &\geq \left(\int_{t-\tau}^{t} x(s)ds \right)^2, \\ \sigma \int_{t-\sigma}^{t} x^2(s)ds &\geq \left(\int_{t-\sigma}^{t} x(s)ds \right)^2, \\ \delta \int_{t-\delta}^{t} x^2(s)ds &\geq \left(\int_{t-\delta}^{t} x(s)ds \right)^2. \end{aligned}$$

Taking into account conditions 2.4 and 2.5, we have

$$\begin{aligned} \frac{dV}{dt} \leq & (-2m_1a_1 - 2\alpha - 2\beta + \lambda_1 + \lambda_2 + \lambda_3\tau^2 + \lambda_4\sigma^2 + \lambda_5\delta^2)x^2(t) \\ & + [2\alpha - 2(m_1a_1 + \alpha + \beta)p_1]x(t)x(t-\tau) + (2\beta - 2n_1b_1)x(t)x(t-\sigma) \\ & - 2c_1 x(t) \int\limits_{t-\delta}^{t} x(s)ds + (2\alpha p_1 - \lambda_1)x^2(t-\tau) \\ & + (2\beta p_1 - 2n_1b_1p_1)x(t-\tau)x(t-\sigma) - 2p_1c_1x(t-\tau) \int_{t-\delta}^{t} x(s)ds \\ & + 2(m_2a_2\alpha + \alpha^2 + \alpha\beta)x(t) \int_{t-\tau}^{t} x(s)ds - 2\alpha^2 x(t-\tau) \int_{t-\tau}^{t} x(s)ds \\ & - (2\alpha\beta - 2\alpha n_2 b_2)x(t-\sigma) \int_{t-\tau}^{t} x(s)ds + 2\alpha c_2 \int_{t-\tau}^{t} x(s)ds \int_{t-\delta}^{t} x(s)ds \\ & + 2\beta c_2 \int_{t-\sigma}^{t} x(s)ds \int_{t-\delta}^{t} x(s)ds + 2(m_2a_2\beta + \alpha\beta + \beta^2)x(t) \int_{t-\sigma}^{t} x(s)ds \\ & - 2\alpha\beta x(t-\tau) \int_{t-\sigma}^{t} x(s)ds - \lambda_2 x^2(t-\sigma) - (2\beta^2 - 2\beta n_2 b_2)x(t-\sigma) \int_{t-\sigma}^{t} x(s)ds \\ & - \lambda_3 \left(\int_{t-\tau}^{t} x(s)ds \right)^2 - \lambda_4 \left(\int_{t-\sigma}^{t} x(s)ds \right)^2 - \lambda_5 \left(\int_{t-\delta}^{t} x(s)ds \right)^2. \end{aligned}$$

The last estimate implies that

$$\frac{dV}{dt} \leq \xi^T(t)\Pi\xi(t),$$

where $\xi^T(t) = \left[x(t) \quad x(t-\tau) \quad x(t-\sigma) \quad \int\limits_{t-\tau}^{t} x(s)ds \quad \int\limits_{t-\sigma}^{t} x(s)ds \quad \int\limits_{t-\delta}^{t} x(s)ds \right]$ and Π is defined in 2.7. Thus, 2.7 implied that there exists a positive constant $\mu > 0$ such that $\frac{dV}{dt} \leq -\mu \left\| D(x_t) \right\|$. Therefore, equation 2.6 is asymptotically stable according to [[8],Theorem 8.1, pp. 292–293]. This completes the proof.

Example 2.1. *Consider neutral differential equation 2.6 with*

$$a_1 = a_2 = 1, b_1 = b_2 = 0.5, c_1 = c_2 = 0, m_1 = m_2 = 2, n_1 = n_2 = 0.4, |p(t)| \leq p_1 = 0.25 < 1, \tag{2.8}$$

$$\tau = 0.2, \sigma = 0.4., \delta = 0.3, \alpha = 0.1, \beta = 0.3, \lambda_1 = 1.6, \lambda_2 = \lambda_3 = 1.2, , \lambda_4 = 0.8, \lambda_5 = 1.5. \tag{2.9}$$

Under the above assumptions, by solving matrix inequality 2.7 using Matlab, we found that the all eigenvalues of this matrix are -0.3125, -1.1539, -1.1931, -1.4085, -1.5000 and -2.3669. As a result, it is clear that all the conditions of Theorem 2.1 hold. This discussion implies that the zero solution of equation 2.6 is asymptotically stable.

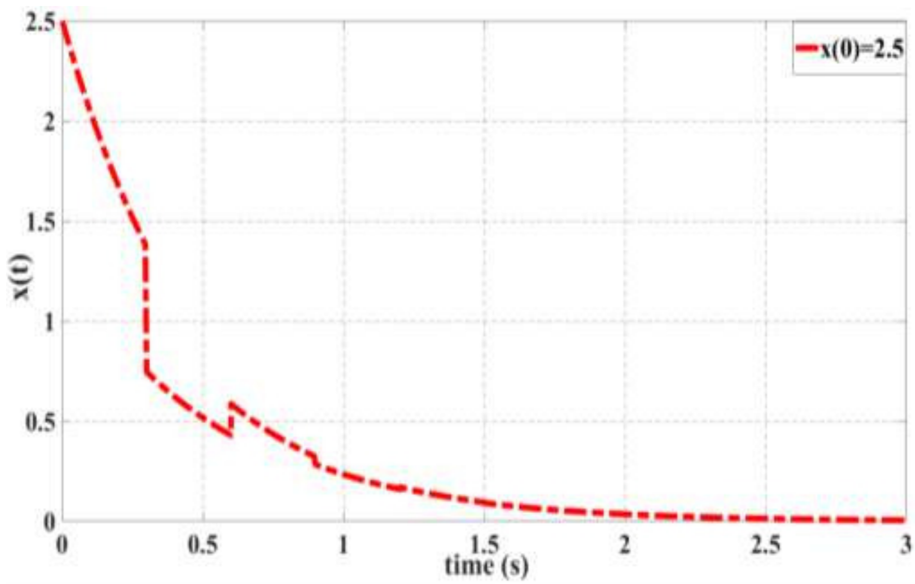

FIGURE 1. The simulation of the Example 2.1.

3. DTM and Numerical Experiment

The theory of DT can be found in [15, 16]. In this research paper, we will explain briefly. The DT of function $x(t)$ is defined as

$$X(k) = \frac{1}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}, \tag{3.10}$$

where $x(t)$is the original function and $X(k)$ is the transformed function.

Differential inverse transform of $X(k)$ is defined as

$$x(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\frac{d^k x(t)}{dt^k}\right]_{t=0}. \tag{3.11}$$

From 3.10 and 3.11, if the function $x(t)$ can be expressed in a finite series as follows

$$x(t) = \sum_{k=0}^{\infty} X(k)\, t^k = X(0) + X(1)\,t + X(2)\,t^2 + \ldots, \tag{3.12}$$

then it is called series solution of the DTM.

The following fundamental theorems can be easily deduced from equations 3.10 and 3.11 (also see,[17],[20]).

Theorem 3.1. *If* $x(t) = \frac{dx(t)}{dt}$*,then* $X(k) = \frac{(k+1)!}{k!} X(k+1) = (k+1)\,X(k+1)$.

Theorem 3.2. *If* $x(t) = \alpha x(t)$*, then* $X(k) = \alpha X(k)$*,where* α *is a constant.*

Theorem 3.3. *If* $x(t) = x(t-a)$*,* $a > 0$ *and reel constant, then*

$$X(k) = \sum_{i=k}^{N} (-1)^{i-k} \begin{pmatrix} i \\ k \end{pmatrix} a^{i-k} X(i).$$

Theorem 3.4. *If* $\frac{d}{dt}x(t-a)$ *, then* $X(k) = (k+1) \sum_{i=k+1}^{N} (-1)^{i-k-1} \begin{pmatrix} i \\ k+1 \end{pmatrix} a^{i-k-1} X(i)$.

Theorem 3.5. *If* $x(t) = \int_{t_0}^{t} x(s) ds$*, then* $X(k) = \frac{X(k-1)}{k}$*,* $k \geq 1$*,* $X(0) = 0$.

Now, we demonstrate potentiality, advantages and effectiveness of our method on an example.

Example 3.1. *Under initial condition* $x(0) = 2.5$*, we consider the first order neutral differential equation 2.6 with 2.8 and 2.9. Taking into account Theorem 3.1 - 3.5, applying DTM on both sides of equation 3.10 and condition 3.11, we obtain the following recurrence relation*

$$\begin{aligned} X(0) =& 2.5, \\ (k+1)\,X(k+1) =& [-0.25\,(k+1) \sum_{i=k+1}^{N} (-1)^{i-k-1} \begin{pmatrix} i \\ k+1 \end{pmatrix} 0.2^{i-k-1} X(i) - 2X(k) \\ & - 0.2 \sum_{i=k}^{N} (-1)^{i-k} \begin{pmatrix} i \\ k \end{pmatrix} 0.4^{i-k} X(i)],\ \ k = 0, 1, ..., 6. \end{aligned}$$

Using this recurrence relation, the following series coefficients $X(k)$ *can be obtained. For* $N = 4$,

$X(1)$ =*-4.256423713,* $X(2)$=*4.173891756,* $X(3)$=*-3.190591724,* $X(4)$=*2.211301195,*

$X(5)$ =*-1.326780717,* $X(6)$=*0.4422602390,* $X(7)$=*-0.1263600683,* $k = 0, 1, ..., 6$.

For $N = 6$,

$X(1)$ =*-4.256931168,* $X(2)$=*4.169113047,* $X(3)$=*-3.134489650,* $X(4)$=*2.052537892,*

$X(5)$ =*-1.272263766,* $X(6)$=*0.7624052530,* $X(7)$=*-0.3703111229,* $k = 0, 1, ..., 6.$

For $N = 8$,

$X(1)$ =*-4.256957370,* $X(2)$=*4.169240023,* $X(3)$=*-3.133772360,* $X(4)$=*2.045844921,*

$X(5)$ =*-1.257197863,* $X(6)$=*0.7759998430,* $X(7)$=*-0.4899948359,* $k = 0, 1, ..., 6.$

Finally, using above mentioned relations, taking $N = 4, \quad 6, \quad 8$ *and using equation 3.12, we reach approximate solutions of equation 2.6 with 7 iterations as follows:*

$N = 4$,

$$\begin{aligned} x_{DTM}(t) =& 2.5 - 4.256423713t + 4.173891756t^2 - 3.190591724t^3 + 2.211301195t^4 \\ & - 1.326780717t^5 + 4.422602390t^6 - 1.263600683t^7, \end{aligned}$$

$N = 6$,

$$\begin{aligned} x_{DTM}(t) =& 2.5 - 4.256931168t + 4.169113047t^2 - 3.134489650t^3 + 2.052537892t^4 \\ & - 1.272263766t^5 + 7.624052530t^6 - 3.703111229t^7, \end{aligned}$$

$N = 8$,

$$\begin{aligned} x_{DTM}(t) =& 2.5 - 4.256957370t + 4169240023t^2 - 3.133772360t^3 + 2.045844921t^4 \\ & - 1.257197863t^5 + 7.759998430t^6 - 4.899948359t^7. \end{aligned}$$

As a result, it is seen that in the cases of $N = 4, \quad N = 6$ *and* $N = 8$, *our numerical results are almost the same.*

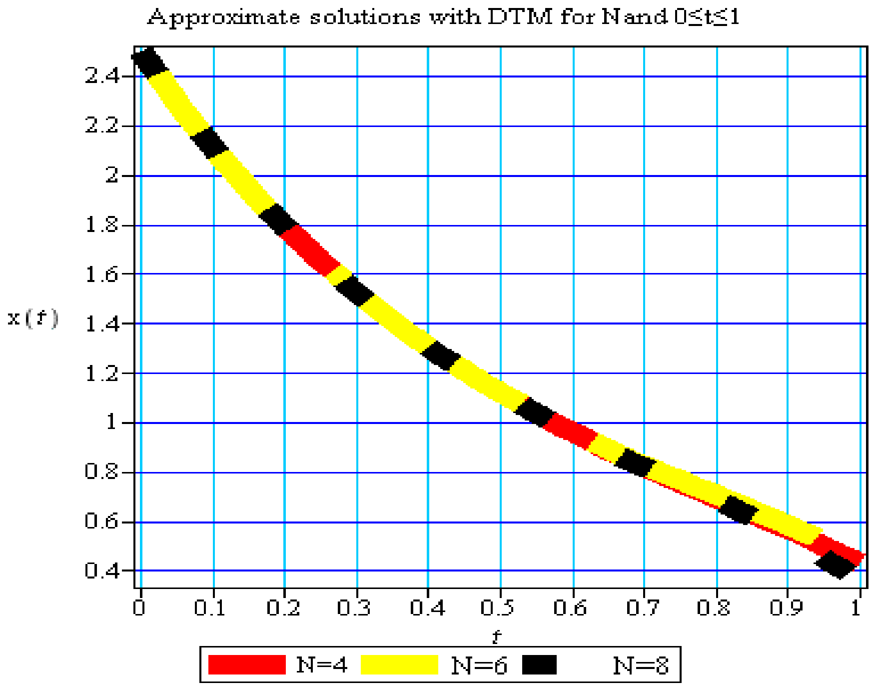

FIGURE 2. Comparison between approximate solutions using DTM.

TABLE 1. Comparison of numerical results obtained with DTM.

t	$N=4$	$N=6$	$N=8$
0.0	2.5	2.5	2.5
0.1	2.113114246	2.113056779	2.113055630
0.2	1.793286393	1.793223362	1.793222389
0.3	1.527559403	1.527518365	1.527507432
0.4	1.305682872	1.305711365	1.305609636
0.5	1.119104674	1.119546373	1.118984564
0.6	0.960089977	0.961954349	0.959727281
0.7	0.820903961	0.826068104	0.819026077
0.8	0.692994495	0.703852936	0.684940092
0.9	0.566111176	0.584166389	0.539253944
1.0	0.427296967	0.450060485	0.353162358

4. CONCLUSIONS

In this study, we first derived some novel sufficient conditions to prove the asymptotic stability of solutions the first order neutral type differential equation. Subsequently, using

DTM, we obtained numerical approximations for different N ve t by an appropriate computer program. We constructed the Table 1 to make a comparison between the numerical results for $N = 4$, $N = 6$ and $N = 8$. By Matlab and an appropriate computer program, we provided two examples to show the effectiveness of proposed method. When the simulations of Example 2.1 and Example 3.1 are examined, the obtained results shows that the proposed methods are useful and applicable. As a suggestion, the techniques and methods presented for equation 1.1 can be improved with different situational or time-dependent delays.

References

[1] Agarwal, R. P., Grace, S. R. (2000). Asymptotic stability of certain neutral differential equations. Math. Comput. Modelling, 31(8-9), 9–15.

[2] Altun, Y. (2019). A new result on the global exponential stability of nonlinear neutral volterra integro-differential equation with variable lags. Math. Nat. Sci., 5, 29–43.

[3] Altun Y. (2019). Further results on the asymptotic stability of Riemann–Liouville fractional neutral systems with variable delays. Adv. Difference Equ., 437, 1-13.

[4] Altun, Y. (2020). Improved results on the stability analysis of linear neutral systems with delay decay approach. Math Meth Appl Sci., 43, 1467–1483.

[5] Altun, Y., Tunç, C. (2017). On the global stability of a neutral differential equation with variable time-lags. Bull. Math. Anal. Appl., 9(4), 31-41.

[6] El-Morshedy, H. A., Gopalsamy, K. (2000). Nonoscillation, oscillation and convergence of a class of neutral equations. Nonlinear Anal., 40(1-8), Ser. A: Theory Methods, 173-183.

[7] Fridman, E. (2002). Stability of linear descriptor systems with delays a Lyapunov-based approach. J. Math. Anal. Appl., 273(1), 24-44.

[8] Hale, J., Verduyn Lunel, S.M. (1993). Introduction to functional-differential equations, Springer Verlag, New York.

[9] Keadnarmol, P., Rojsiraphisal, T. (2014). Globally exponential stability of a certain neutral differential equation with time-varying delays. Adv. Difference Equ., 32, 1-10.

[10] Kolmanovskii, V., Myshkis, A. (1992). Applied Theory of Functional Differential Equations. Kluwer Academic Publisher Group, Dordrecht.

[11] Kulenovic, M., Ladas, G., Meimaridou, A. (1987). Necessary and sufficient conditions for oscillations of neutral differential equations. J. Aust. Math. Soc. Ser. B 28, 362-375.

[12] Park, J. H. (2005). LMI optimization approach to asymptotic stability of certain neutral delay differential equation with time-varying coefficients. Appl. Math. Comput., 160, 355-361.

[13] Park, J. H., Kwon, O. M. (2008). Stability analysis of certain nonlinear differential equation. Chaos Solitons Fractals, 37, 450-453.

[14] Tunç, C., Altun, Y. (2016). Asymptotic stability in neutral differential equations with multiple delays. J. Math. Anal., 7(5), 40-53.

[15] Pukhov, G.E. (1986). Differential Transformations and Mathematical Modelling of Physical Processes. Naukova Dumka, Kiev.

[16] Zhou, J.K. (1986). Differential transformation and its application for electrical circuits. Huazhong University Press, Wuhan.

[17] Arikoglu, A., Ozkol, I. (2005). Solution of boundary value problems for integro-differential equations by using differential transform method. Applied Mathematics and Computation, 168, 1145–1158.

[18] Arslan D. (2019). A novel hybrid method for singularly perturbed delay differential equations. Gazi University Journal of Sciences, 32 (1), 217-223.

[19] Ayaz, F. (2004). Applications of differential transform method to differential-algebraic equations. Applied Mathematics and Computation, 152, 649-657.

[20] Cakir, M., Arslan, D. (2015). The adomian decomposition method and the differential transform method for numerical solution of multi-pantograph delay differential equations. Applied Mathematics, 6, 1332-1343.

[21] Rebenda, J., Smarda, Z., Khan, Y. (2017). A new semi-analytical approach for numerical solving of Cauchy problem for differential equations with delay. Filomat, 31(15), 4725–4733.

[22] Karakoç, F., Bereketoğlu, H. (2009). Solutions of delay differential equations by using differential transform method. International Journal of Computer Mathematics, 86, 914-923.

[23] Liu, B., Zhou, X., Du, Q. (2015). Differential transform method for some delay differential equations. Applied Mathematics, 6, 585-593.

[24] Düşünceli, F., and Çelik, E. (2018). Numerical solution for high-order linear complex differential equations with variable coefficients. Numerical Methods for Partial Differential Equations, 34(5), 1645-1658.

[25] Düşünceli, F., and Çelik, E. (2017). Numerical solution for high-order linear complex differential equations by hermite polynomials. Igdır University Journal of the Institute of Science and Technology, 7(4), 189-201.

Department of Statistics, Van Yuzuncu Yil University, 65080 Van, Turkey

International Journal of Maps in Mathematics

*Volume 4, Issue 2, 2021, Pages:**93-106***

ISSN: 2636-7467 (Online)

www.journalmim.com

A CONSTRUCTION OF VERY TRUE OPERATOR ON SHEFFER STROKE MTL-ALGEBRAS

IBRAHIM SENTURK AND TAHSIN ONER *

Abstract. In this paper, we introduce Sheffer stroke very true operator on MTL-algebras. We handle some fundamental properties of this operator. We obtain some equalities and inequalities which are used in our construction. Moreover, we give some relations among very true operator, supremum and infimum relations. Finally, we construct bridges among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them.

Keywords: Sheffer stroke MTL-algebra, Very True Operator, Reduction, BL-algebra, MV-algebra, Gödel algebra.

2010 Mathematics Subject Classification: 03F50, 06F99.

1. Introduction

When a structure is established as a mathematical model, we must firstly throw off redundant statements. For this aim, we venture to give equivalent statements as possible as with the least number of axioms or the least number of operations and so on. For instance, Tarski achieved to explain Abelian groups with the least number of axioms from the point of divisor operator. [19]

The concept of monoidal t-norm-based logic (shortly, MTL) is given by Godo and Esteva [8]. Montogna and Jenei show that MTL corresponds to the logic of all left continuous t-norms and their residua [11]. In accordance wtih these studies, MTL-algebras are defined as a counterpart of this logical system [8]. In recent times, the structure of MTL-algebras

Received:2021.01.20 *Revised:2021.03.09* *Accepted:2021.03.19*

* Corresponding author

Tahsin Oner; tahsin.oner@ege.edu.tr; https://orcid.org/0000-0002-6514-4027

Ibrahim Senturk; ibrahim.senturk@ege.edu.tr; https://orcid.org/0000-0001-8296-2796

.

has been supported with important structural works [13, 20]. These works get a constructive effect on its algebraic structure. For instance, Vetterlein demonstrate that MTL-algebras correspond to the positive cone of a partially ordered group [20]. Moreover, he confirm that this algebra is a commutative, bounded, integral and pre-linear residuated lattice [13]. And, MTL-algebras are the basis residuated structures having all algebras induced by their residua and continuous t-norms. So, MTL-algebras have an important position in different structures which are related with fuzzy logic [21].

Oner and Senturk introduced Sheffer stroke basic algebras [14]. Sheffer stroke basic algebras play an important role in great numbers of logics as many-valued Łukasiewicz logics, non-classical logics, fuzzy logics and etc. This reduction topic is studied in recent times such as [15]. In harmony with these logical roles, Senturk gives a reduction of MTL-algebras by means of only Sheffer stroke operation which is called Sheffer stroke MTL-algebras [18].

The notion of "very true" was firstly established by Hájek giving an answer to the question "whether any natural axiomatization is possible and how far can even this sort of fuzzy logic be captured by standard methods of mathematical logic?" [10]. To put in a different way, very true operator is used to reduce the number of possible logical values in many-valued logic. After this operator was effectively used in particular tasks in various fields of mathematics [9, 5, 1, 23], this operator has been implemented to other logical algebras such as effect algebras [6], commutative basic algebras [3], equality algebras [22], Rℓ-monoids [16], MV-algebras [12] and so on.

In this paper, we give some fundamental concepts which are needed for our construction in Section 2. In Section 3, we introduce Sheffer stroke very true operator on Sheffer stroke MTL-algebras. We handle some fundamental properties of this operator. We obtain some equalities and inequalities. We give some relations among very true operator, supremum and infimum. Then, we engaging links among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them. In Section 4, we briefly mention what we do during this work.

2. Preliminaries

The basic definitions, lemmas, theorems and etc. which are used throughout the paper are given in this section.

The fundamental concepts in this chapter are taken from [17] and [2].

Definition 2.1. *If the binary operations $\vee$ and $\wedge$ satisfy the following conditions on the non-empty set L:*

(L_1) $k \wedge l = l \wedge k$ *and* $k \vee l = l \vee k$,

(L_2) $k \wedge (l \wedge m) = (k \wedge l) \wedge m$ *and* $k \vee (l \vee m) = (k \vee l) \vee m$,

(L_3) $k \wedge k = k$ *and* $k \vee k = k$,

(L_4) $k \wedge (k \vee l) = k$ *and* $k \vee (k \wedge l) = k$

then $\mathfrak{L} = (L; \wedge, \vee)$ *is called a lattice.*

Definition 2.2. *An algebraic structure $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ is called bounded lattice if it satisfies the following properties:*

(i) *for each* $k \in L$, $k \wedge 1 = k$ *and* $k \vee 1 = 1$,

(ii) *for each* $k \in L$, $k \wedge 0 = 0$ *and* $k \vee 0 = k$.

The elements 1 is called the greatest element and 0 is called called the least element of the lattice.

Definition 2.3. *Let the structure $\mathcal{L} = (L; \vee, \wedge)$ be a lattice. A mapping $k \mapsto k^{\perp}$ is said to be an antitone involution if it verifies the following conditions:*

(i) $k^{\perp\perp} = k$ *(involution),*

(ii) $k \leq l$ *implies* $l^{\perp} \leq k^{\perp}$ *(antitone).*

Definition 2.4. *Let $\mathcal{L}$ be a bounded lattice with an antitone involution. If the below conditions*

$$k \vee k^{\perp} = 1 \quad \textit{and} \quad k \wedge k^{\perp} = 0,$$

are satisfied then $k^{\perp}$ is called the complement of k and the lattice $\mathcal{L} = (L; \vee, \wedge, ^{\perp}, 0, 1)$ is also an ortholattice.

Lemma 2.1. *Let $\mathcal{L} = (L; \vee, \wedge, ^{\perp})$ be a lattice which verifies the antitone involution condition. Then the De Morgan laws*

$$k^{\perp} \wedge l^{\perp} = (k \vee l)^{\perp} \textit{ and } k^{\perp} \vee l^{\perp} = (k \wedge l)^{\perp}$$

are satisfied.

Definition 2.5. [4] *Let $\mathcal{G} = (G, |)$ be a groupoid. If the following conditions are satisfied, then the operation $| : G \times G \to G$ is called a Sheffer stroke operation.*

$(S1)$ $g_1|g_2 = g_2|g_1$,

$(S2)$ $(g_1|g_1)|(g_1|g_2) = g_1$,

$(S3)$ $g_1|((g_2|g_3)|(g_2|g_3)) = ((g_1|g_2)|(g_1|g_2))|g_3$,

$(S4)$ $(g_1|((g_1|g_1)|(g_1|g_1)))|(g_1|((g_1|g_1)|(g_2|g_2))) = g_1$.

If also the following identity

$(S5)$ $g_2|(g_1|(g_1|g_1)) = g_2|g_2$,

is satisfied, then it is said to be an ortho-Sheffer stroke operation.

Lemma 2.2. [4] *Let $\mathcal{G} = (G, |)$ be a groupoid with Sheffer stroke operation. Then the following equalities are verified for each $g_1, g_2, g_3 \in G$:*

(i) $(g_1|g_2)|(g_1|(g_2|g_3)) = g_1$,

(ii) $(g_1|g_1)|g_2 = g_2|(g_1|g_2)$,

(iii) $g_1|((g_2|g_2)|g_1) = g_1|g_2$.

Lemma 2.3. [4] *Let $\mathcal{G} = (G, |)$ be a groupoid. The binary relation $\leq$ defined on G as below*

$$g_1 \leq g_2 \text{ if and only if } g_1|g_2 = g_1|g_1$$

is a partial order on G.

Lemma 2.4. [4] *Let $|$ be a Sheffer stroke operation on G and $\leq$ order relation of $\mathcal{G}$. Then, the following equalities:*

(i) $g_1 \leq g_2$ *if and only if* $g_2|g_2 \leq g_1|g_1$,

(ii) $g_1|(g_2|(g_1|g_1)) = g_1|g_1$ *is the identity of* $\mathcal{G}$,

(iii) $g_1 \leq g_2$ *implies* $g_2|g_3 \leq g_1|g_3$, *for all* $g_3 \in G$,

(iv) $g_3 \leq g_1$ *and* $g_3 \leq g_2$ *imply* $g_1|g_2 \leq g_3|g_3$

are verified.

Lemma 2.5. [14] *Let $\mathfrak{G} = (G; |)$ be a Sheffer stroke basic algebra with the constant element 1. Then, the following identities:*

(i) $g_1|(g_1|g_1) = 1$,

(ii) $g_1|(1|1) = 1$,

(iii) $1|(g_1|g_1) = g_1$,

(iv) $((g_1|(g_2|g_2))|(g_2|g_2))|(g_2|g_2) = g_1|(g_2|g_2)$,

(v) $(g_2|(g_1|(g_2|g_2)))|(g_1|(g_2|g_2)) = 1$

are verified.

Definition 2.6. [21] *Let X be a non-empty set, the operations $\vee$, $\wedge$, $\rightarrow$ and $\circledast$ be binary operations on X and the elements 0 and 1 be algebraic constant of X. If the following* *conditions:*

(MTL_1) $(X; \wedge, \vee, 0, 1)$ *is a bounded lattice,*

(MTL_2) $(X; \circledast, 0, 1)$ *is a commutative monoid,*

(MTL_3) $x \leq y \to z$ *if and only if* $x \circledast y \leq z$,

(MTL_4) $(x \to y) \vee (y \to x) = 1$

are satisfied for each $x, y, z \in X$, *then the algebraic structure* $\mathcal{X} = (X; \vee, \wedge, \to, \circledast, 0, 1)$ *is called an MTL-algebra.*

Definition 2.7. [21] *Let* $\mathcal{X} = (X; \vee, \wedge, \to, \circledast, 0, 1)$ *be an MTL-algebra. Then* $\mathcal{X}$ *is called*

(i) *a BL-algebra if* $x \wedge y = x \circledast (x \to y)$ *for each* $x, y \in X$,

(ii) *an MV-algebra if* $(x \to y) \to y = (y \to x) \to x$ *for each* $x, y \in X$,

(iii) *a Gödel algebra if* $x \circledast x = x$ *for each* $x \in X$.

Theorem 2.1. [18] *Let* $\mathcal{X} = (X; \vee, \wedge, \to, \circledast, 0, 1)$ *an* MTL*-algebra. If the operations are defined as:*

$x_1 \wedge x_2 := (((x_2|x_2)|x_1)|x_1)|(((x_2|x_2)|x_1)|x_1)$

$x_1 \vee x_2 := (x_1|(x_2|x_2))|(x_2|x_2)$

$x_1 \circledast x_2 := (x_1|x_2)|(x_1|x_2)$

$x_1 \to x_2 := x_1|(x_2|x_2)$

for each $x_1, x_2 \in X$, *then* $\mathcal{X} = (X; |)$ *is a Sheffer stroke reduction of* MTL*-algebra.*

Corollary 2.1. [18] *Let* $\mathcal{X} = (X; |)$ *is a Sheffer stroke reduction of MTL-algebra. Then, it is also a Sheffer stroke basic algebra.*

During this paper, Sheffer stroke reduction of MTL-algebras are shortly called Sheffer stroke MTL-algebras.

3. A Construction of Very True Operator On Sheffer Stroke MTL-Algebras

In this part of the paper, we construct Sheffer stroke very true operator on Sheffer stroke MTL-algebras. We examine some fundamental properties of this operator. We attain some equalities and inequalities. Moreover, we give some relations among very true operator, supremum and infimum. On the other hand, we build links among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them.

Definition 3.1. *Let* $\mathcal{M} = (M; |)$ *be a Sheffer stroke* MTL*-algebra. If the following conditions:*

$(SV_{SM}1)$ $\vartheta(1) = 1$

$(SV_{SM}2)$ $\vartheta(m) \leq m$

$(SV_{SM}3)$ $\vartheta(m|(n|n)) \leq \vartheta(m)|(\vartheta(n)|\vartheta(n))$

$(SV_{SM}4)$ $\vartheta(m) \leq \vartheta(\vartheta(m))$

$(SV_{SM}5)$ $(\vartheta(m|(n|n))|(\vartheta(n|(m|m))|\vartheta(n|(m|m))))|(\vartheta(n|(m|m))|\vartheta(n|(m|m))) = 1$

are satisfied for each $m, n \in M$, then the mapping $\vartheta : M \to M$ is called a Sheffer stroke very true operator.

Example 3.1. *Let $M = \{0, k, l, m, n, 1\}$. The relations of elements in M are given as Figure 1 and the operation $|$ on this structure is defined as the Table 1.*

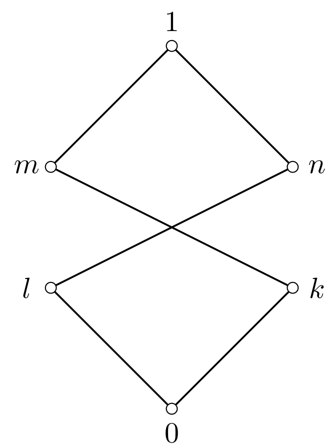

Figure 1. Hasse Diagram of M

$\vert$	0	l	k	m	n	1
0	1	1	1	1	1	1
l	1	m	1	1	m	m
k	1	1	n	n	1	n
m	1	1	n	l	1	l
n	1	m	1	1	k	k
1	1	m	n	l	k	0

Table 1. $|-$operation on M

If the binary operations $\wedge, \vee, \circledast$ and $\to$ are defined as Theorem 2.1, then we have the following Cayley tables for these operations.

$\wedge$	0	l	k	m	n	1
0	0	0	0	0	0	0
l	0	l	0	0	l	l
k	0	0	k	k	0	k
m	0	0	k	m	0	m
n	0	l	0	0	n	n
1	0	l	k	m	n	1

Table 2. $\wedge-$operation on M

and

$\vee$	0	l	k	m	n	1
0	0	l	k	m	n	1
l	l	l	1	1	n	1
k	k	1	k	m	1	1
m	m	1	m	m	1	1
n	n	n	1	1	n	1
1	1	1	1	1	1	1

Table 3. $\vee-$operation on M

$\circledast$	0	l	k	m	n	1
0	0	0	0	0	0	0
l	0	l	0	0	l	l
k	0	0	k	k	0	k
m	0	0	n	m	0	m
n	0	l	0	m	n	n
1	0	l	k	m	n	1

Table 4. $\circledast-$operation on M

and

$\rightarrow$	0	l	k	m	n	1
0	1	1	1	1	1	1
l	m	1	m	m	1	1
k	n	m	1	1	n	1
m	l	l	1	1	n	1
n	k	1	k	m	1	1
1	0	l	k	m	n	1

Table 5. $\rightarrow -$operation on M

So, the algebraic structure $\mathcal{M} = (M; |)$ is a Sheffer stroke MTL-algebra. If the operation $\vartheta : M \rightarrow M$ is defined by

$$\vartheta(u) := \begin{cases} 0, & u = 0, \\ 1, & u = 1, \\ n, & u \in \{l, n\}, \\ m, & x \in \{k, m\} \end{cases}$$

then, this mapping is a Sheffer stroke very true operator on M.

Proposition 3.1. *Assume that the mapping $\vartheta : M \rightarrow M$ be a Sheffer stroke very true operator. Then, the following statements*

(i) $\vartheta(0) = 0$,

(ii) $m = 1$ *if and only if* $\vartheta(m) = 1$,

(iii) ϑ *is increasing,*

(iv) $\vartheta(m|m) \leq \vartheta(m)|\vartheta(m)$

hold for each $m, n, k \in L$.

Proof. *(i)* By $(SV_{SM}2)$, we get $\vartheta(0) \leq 0$. Moreover, we have $m \leq \vartheta(m)$ for each $m \in M$. So, we obtain $\vartheta(0) = 0$.

(ii) ($\Rightarrow$:) It is clear from $(SV_{SM}1)$.

($\Leftarrow$:) Assume that $\vartheta(m) = 1$. Since $\vartheta(m) = 1 \leq m \leq 1$, we get $m = 1$.

(iii) Assume that $m \leq n$. Then, we have $m|(n|n) = 1$. By the help of $(SV_{SM}1)$ and $(SV_{SM}3)$, we get $\vartheta(m|(n|n)) = \vartheta(1) = 1 \leq \vartheta(m)|(\vartheta(n)|\vartheta(n)) \leq 1$. We obtain $\vartheta(m)|(\vartheta(n)|\vartheta(n)) = 1$. So, we conclude that $\vartheta(m) \leq \vartheta(n)$, i.e., the mapping ϑ is increasing.

(iv) Let m be any element of M. Then, we have

$$\begin{aligned}\vartheta(m|m) &= \vartheta(m|1)\\ &= \vartheta(m|(0|0))\\ &\leq \vartheta(m)|(\vartheta(0)|\vartheta(0))\\ &= \vartheta(m)|(0|0)\\ &= \vartheta(m)|1\\ &= \vartheta(m)|\vartheta(m).\end{aligned}$$

So, the inequality $\vartheta(m|m) \leq \vartheta(m)|\vartheta(m)$ is verified for each $m \in M$.

Lemma 3.1. *Let $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then, the equality $\vartheta(m) = \vartheta^2(m)$ is verified for each $m \in M$.*

Proof. Let m be any element of M. By using Proposition 3.1 (iii) and $(SV_{SM}1)$, we obtain $\vartheta(\vartheta(m)) \leq \vartheta(m)$. From $(SV_{SM}4)$, we have $\vartheta(m) \leq \vartheta(\vartheta(m))$. Hence, we obtain $\vartheta(m) = \vartheta(\vartheta(m))$ for each $m \in M$.

Lemma 3.2. *The following inequalities*

$$(\vartheta(m)|\vartheta(n))|(\vartheta(m)|\vartheta(n)) \leq (m|n)|(m|n) \leq \vartheta(m|n)|\vartheta(m|n)$$

hold for each $m, n \in L$.

Proof. Let m and n be any elements of M. By using $(SV_{SM}2)$, we get $\vartheta(m) \leq m$ and $\vartheta(n) \leq n$. From Lemma 2.4 (i), we have $m|n \leq \vartheta(m)|\vartheta(n)$. If we use again the same step for the last equation, we get the following inequality:

$$(\vartheta(m)|\vartheta(n))|(\vartheta(m)|\vartheta(n)) \leq (m|n)|(m|n). \tag{3.1}$$

By $(SV_{SM}2)$, we have $\vartheta(m|n) \leq m|n$. Similarly, we obtain

$$(m|n)|(m|n) \leq \vartheta(m|n)|\vartheta(m|n). \tag{3.2}$$

From Inequalities (3.1) and (3.2), we attain our assumption.

Lemma 3.3. *The following inequalities*

(i) $\vartheta(m|m) \leq \vartheta(m|n)$,

(ii) $\vartheta(m|n)|\vartheta(m|n) \leq \vartheta(m)$, *(iii)* $\vartheta(m) \leq \vartheta((m|n)|n)$

hold for each $m, n \in L$.

Proof. (i) Let m and n be any two elements of M. We have $m \leq 1$ and $n \leq 1$. Then,

$$\begin{aligned} n \leq 1 \quad &\Rightarrow \quad 1|m \leq n|m, && \text{(By Lemma 2.4 } (iii)) \\ &\Rightarrow \quad m|m \leq n|m, && \text{(By Lemma 2.5 and Corollary 2.1)} \\ &\Rightarrow \quad \vartheta(m|m) \leq \vartheta(m|n). && \text{(By Proposition 3.1 } (iii)) \end{aligned}$$

(ii) We have the inequality $m|m \leq n|m$ from Lemma 3.3 (i). By the help of Lemma 2.4 (i) and Definition 2.5 $(S2)$, we get $(n|m)|(n|m) \leq m$. By increasing property of ϑ mapping, we conclude that $\vartheta((n|m)|(n|m)) \leq \vartheta(m)$ for each $m, n \in M$.
(iii) We have $n \leq 1$ for each $n \in M$. We obtain $m \leq (m|n)|n$ by using Lemma 2.4 (iii), Lemma 2.5 (iii) and Lemma 2.2, respectively. Since ϑ is an increasing mapping, we obtain $\vartheta(m) \leq \vartheta((m|n)|n)$ for each $m, n \in M$.

Theorem 3.1. *Let $\vartheta : M \to M$ be a Sheffer stroke very true operator. Let* sup *and* inf *be the least upper bound and greatest lower bound functions, respectively. Then the following equalities*

$$\sup\{\vartheta(m), \vartheta(n)\} = \vartheta(\sup\{m, n\}) \qquad \text{and} \qquad \inf\{\vartheta(m), \vartheta(n)\} = \vartheta(\inf\{m, n\})$$

are satisfied for each $m, n \in M$.

Proof. Let $m, n \in M$ and the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. We have $m \leq \sup\{m, n\}$ and $n \leq \sup\{m, n\}$. Since ϑ is an increasing mapping, we get $\vartheta(n) \leq \vartheta(\sup\{m, n\})$ and $\vartheta(m) \leq \vartheta(\sup\{m, n\})$. Then, we obtain the following inequality

$$\sup\{\vartheta(m), \vartheta(n)\} \leq \vartheta(\sup\{m, n\}) \tag{3.3}$$

for each $m, n \in M$.
Let $\sup\{\vartheta(m), \vartheta(n)\} = k$ for $k \in M$. So, we have $\vartheta(m) \leq k$ and $\vartheta(n) \leq k$. By the help of Lemma 3.1 and Proposition 3.1 (iii), we get $\vartheta(m) \leq \vartheta(k)$ and $\vartheta(n) \leq \vartheta(k)$. Using again Proposition 3.1 (iii), we get $m \leq k$ and $n \leq k$. Then, we attain $\sup\{m, n\} \leq k$. From Definition 3.1 $(SV_{SM}2)$ and Proposition 3.1 (iii), we obtain following the inequalities

$$\vartheta(\sup\{m, n\}) \leq \vartheta(k) \leq k = \sup\{\vartheta(m), \vartheta(n)\}. \tag{3.4}$$

From Inequalities (3.3) and (3.4), we prove that $\sup\{\vartheta(m), \vartheta(n)\} = \vartheta(\sup\{m, n\})$ for each $m, n \in M$.

For the infimum part of the proof, we have $\inf\{m, n\} \leq m$ and $\inf\{m, n\} \leq n$ for each $m, n \in M$. Since ϑ is an increasing mapping, we get $\vartheta(\inf\{m, n\}) \leq \vartheta(m)$ and $\vartheta(\inf\{m, n\}) \leq \vartheta(n)$. So, we obtain the following inequality

$$\vartheta(\inf\{m, n\}) \leq \inf\{\vartheta(m), \vartheta(n)\}. \tag{3.5}$$

By Definition 3.1 ($SV_{SM}2$), we have $\vartheta(m) \leq m$ and $\vartheta(n) \leq n$. Then, we get $\inf\{\vartheta(m), \vartheta(n)\} \leq \inf\{m, n\}$. From Proposition 3.1 (*iii*) and Lemma 3.1, we handle $\vartheta(\inf\{\vartheta(m), \vartheta(n)\}) \leq \vartheta(\vartheta(\inf\{m, n\}))$, i.e.,

$$\inf\{\vartheta(m), \vartheta(n)\} \leq \vartheta(\inf\{m, n\}). \tag{3.6}$$

From Inequalities (3.5) and (3.6), we show that $\inf\{\vartheta(m), \vartheta(n)\} = \vartheta(\inf\{m, n\})$ for each $m, n \in M$.

Example 3.2. *Let $M = \{0, k, l, m, n, 1\}$ and $\vartheta : M \to M$ be defined as Example 3.1. Then we show that Theorem 3.1 is satisfied for each $a, b \in M$. If one of $\{a, b\}$ equals 0 or 1, the equalities $\sup\{\vartheta(a), \vartheta(b)\} = \vartheta(\sup\{a, b\})$ and $\inf\{\vartheta(a), \vartheta(b)\} = \vartheta(\inf\{a, b\})$ are obtained clearly. We examine $a \in \{k, l, m, n\}$ and $b \in \{k, l, m, n\}$. So, we need to examine the sets such as $\{k, l\}$, $\{k, m\}$, $\{k, n\}$, $\{l, m\}$, $\{l, n\}$ and $\{m, n\}$.*

- *We analyze for $\{k, l\}$:*

$$\sup\{\vartheta(k), \vartheta(l)\} = \sup\{m, n\} = 1 = \vartheta(1) = \vartheta(\sup\{k, l\}).$$

$$\inf\{\vartheta(k), \vartheta(l)\} = \inf\{m, n\} = 0 = \vartheta(0) = \vartheta(\inf\{k, l\}).$$

- *We analyze for $\{k, m\}$:*

$$\sup\{\vartheta(k), \vartheta(m)\} = \sup\{m, m\} = m = \vartheta(m) = \vartheta(\sup\{k, m\}).$$

$$\inf\{\vartheta(k), \vartheta(m)\} = \inf\{m, m\} = m = \vartheta(k) = \vartheta(\inf\{k, m\}).$$

- *We analyze for $\{k, n\}$:*

$$\sup\{\vartheta(k), \vartheta(n)\} = \sup\{m, n\} = 1 = \vartheta(1) = \vartheta(\sup\{k, n\}).$$

$$\inf\{\vartheta(k), \vartheta(n)\} = \inf\{m, n\} = 0 = \vartheta(0) = \vartheta(\inf\{k, n\}).$$

- *We analyze for $\{l, m\}$:*

$$\sup\{\vartheta(l), \vartheta(m)\} = \sup\{n, m\} = 1 = \vartheta(1) = \vartheta(\sup\{l, m\}).$$

$$\inf\{\vartheta(l), \vartheta(m)\} = \inf\{n, m\} = 0 = \vartheta(0) = \vartheta(\inf\{l, m\}).$$

- *We analyze for* $\{l, n\}$*:*

$$\sup\{\vartheta(l), \vartheta(n)\} = \sup\{n, n\} = n = \vartheta(n) = \vartheta(\sup\{l, n\}).$$

$$\inf\{\vartheta(l), \vartheta(n)\} = \inf\{n, n\} = n = \vartheta(l) = \vartheta(\inf\{l, n\}).$$

- *We analyze for* $\{m, n\}$*:*

$$\sup\{\vartheta(m), \vartheta(n)\} = \sup\{m, n\} = 1 = \vartheta(1) = \vartheta(\sup\{m, n\}).$$

$$\inf\{\vartheta(m), \vartheta(n)\} = \inf\{m, n\} = 0 = \vartheta(o) = \vartheta(\inf\{m, n\}).$$

Corollary 3.1. *Let $m, n \in M$ and the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then the following equalities*

$$\sup\{\vartheta(m), \vartheta(n)\} = \vartheta(\sup\{\vartheta(m), \vartheta(n)\}) \quad \textit{and} \quad \inf\{\vartheta(m), \vartheta(n)\} = \vartheta(\inf\{\vartheta(m), \vartheta(n)\})$$

are verified for each $m, n \in M$.

Proof. It is straightforward from Theorem 3.1 and Proposition 3.1 (iii).

Theorem 3.2. *Let $Fix_\vartheta(M)$ be the set of the points of M such that $\vartheta(m) = m$. Then, the equality $Fix_\vartheta(M) = \vartheta(M)$ is satisfied.*

Proof. Assume that $n \in \vartheta(M)$. Then, we have any element m of M such that $\vartheta(m) = n$. Using Lemma 3.1, we obtain $\vartheta(n) = \vartheta(\vartheta(m)) = \vartheta(m) = n$. So, we get $n \in Fix_\vartheta(M)$. Hence, we handle the following relation

$$\vartheta(M) \subseteq Fix_\vartheta(M). \tag{3.7}$$

Let $n \in Fix_\vartheta(M)$. This means that $\vartheta(n) = n$. Since $n \in M$, $n = \vartheta(n) \in \vartheta(M)$. Therefore, we get the following relation

$$Fix_\vartheta(M) \subseteq \vartheta(M). \tag{3.8}$$

From the relations (3.7) and (3.8), we prove that $Fix_\vartheta(M) = \vartheta(M)$.

Example 3.3. *Let $M = \{0, k, l, m, n, 1\}$ and $\vartheta : M \to M$ be defined as Example 3.1. Then, we have $Fix_\vartheta(M) = \{0, n, m, 1\}$ and also $\vartheta(M) = \{0, n, m, 1\}$. So, we verify $Fix_\vartheta(M) = \vartheta(M)$ for Example 3.1.*

Now, when we consider on Theorem 3.1 and Theorem 3.2, we can reach the following corollary.

Corollary 3.2. *Let the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then the following equalities*

$$\sup\{Fix_\vartheta(M)\} = \vartheta(\sup(M)) \qquad \textit{and} \qquad \inf\{Fix_\vartheta(M)\} = \vartheta(\inf(M))$$

are verified.

Lemma 3.4. *Let $id : M \to M$ be defined as $Id(m) = m$ for each $m \in M$. Then, the mapping Id is a Sheffer stroke very true operator on M.*

Proof. It is clear from Definition 3.1, Definition 2.6 and Theorem 2.1.

Theorem 3.3. *Let $\mathcal{M} = (M; |)$ be a Sheffer stroke MTL-algebra and the mapping $\vartheta : M \to M$ be a Sheffer stroke very true operator. Then,*

(i) $\mathcal{M} = (M; \vee, \wedge, \to, \circledast, 0, 1)$ is a BL-algebra if and only if $\vartheta(\inf\{m,n\}) = \vartheta((((m|m)|n)|n) |(((m|m)|n)|n))$ for each very true operator ϑ on M and for each $m, n \in M$,

(ii) $\mathcal{M} = (M; \vee, \wedge, \to, \circledast, 0, 1)$ is a MV-algebra if and only if $\vartheta(\sup\{m,n\}) = \vartheta((m|(n|n)|(n|n)))$ for each very true operator ϑ on M and for each $m, n \in M$,

(iii) $\mathcal{M} = (M; \vee, \wedge, \to, \circledast, 0, 1)$ is a Gödel algebra if and only if $\vartheta(\inf\{m,n\}) = (\vartheta(m)|\vartheta(n))| (\vartheta(m)|\vartheta(n))$ for each very true operator ϑ on M and for each $m, n \in M$,

Proof. The proof is clear from Lemma 3.4 and Theorem (3.7) in [18].

4. Conclusion

In this paper, we define Sheffer stroke very true operator on MTL-algebras. We get some fundamental properties of this operator. We give some equalities and inequalities which are used in our construction. Then, we attain some relations among very true operator, supremum and infimum relations. Finally, we construct paths among Sheffer stroke MTL-algebras, BL-algebras, MV-algebras and Gödel algebras by using them. After this work, we will use this operator other algebraic structures. By this means, we want to obtain new paths among new algebraic structures.

References

[1] Bělohlávek, R., and Vychodil, V. (2005, May). Reducing the size of fuzzy concept lattices by hedges. In The 14th IEEE International Conference on Fuzzy Systems, 2005. FUZZ'05. (pp. 663-668). IEEE.

[2] Birkhoff, G. (1940). Lattice theory (Vol. 25). American Mathematical Soc.

[3] Botur, M., and Švrček, F. (2010). Very true on CBA fuzzy logic. Mathematica Slovaca, 60(4), 435-446.

[4] Chajda, I. (2005). Sheffer operation in ortholattices. Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, 44(1), 19-23.

[5] Chajda, I. (2011). Hedges and successors in basic algebras. Soft Computing, 15(3), 613-618.

[6] Chajda, I., and Kolařík, M. (2012). Very true operators in effect algebras. Soft Computing, 16(7), 1213-1218.

[7] Chajda, I. (2015). Basic algebras, logics, trends and applications. Asian-European Journal of Mathematics, 8(03), 1550040.

[8] Esteva, F., and Godo, L. (2001). Monoidal t-norm based logic: towards a logic for left-continuous t-norms. Fuzzy sets and systems, 124(3), 271-288.

[9] Hájek, P., and Harmancová, D. (2000). A hedge for Gödel fuzzy logic. International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems, 8(04), 495-498.

[10] Hájek, P. (2001). On very true. Fuzzy sets and systems, 124(3), 329-333.

[11] Jenei, S., and Montagna, F. (2002). A proof of standard completeness for Esteva and Godo's logic MTL. Studia Logica, 70(2), 183-192.

[12] Leuştean, I. (2006). Non-commutative Łukasiewicz propositional logic. Archive for Mathematical Logic, 45(2), 191-213.

[13] Noguera, C., Esteva, F., and Gispert, J. (2005). On some varieties of MTL-algebras. Logic Journal of the IGPL, 13(4), 443-466.

[14] Oner, T., and Senturk, I. (2017). The Sheffer stroke operation reducts of basic algebras. Open Mathematics, 15(1), 926-935.

[15] Oner, T., Kalkan, T., and Gursoy, N. K. (2021). Sheffer stroke BG-algebras. International Journal of Maps in Mathematics-IJMM, 4(1), 27-39.

[16] Rachůnek, J., and Šalounová, D. (2006). Truth values on generalizations of some commutative fuzzy structures. Fuzzy sets and systems, 157(24), 3159-3168.

[17] Sankappanavar, H. P., and Burris, S. (1981). A course in universal algebra. Graduate Texts Math, 78.

[18] Senturk, I. (2020). A bridge construction from Sheffer stroke basic algebras to MTL-algebras. Balıkesir Üniversitesi Fen Bilimleri Enstitüsü Dergisi, 22(1), 193-203.

[19] Tarski, A. (1938). Ein beitrag zur axiomatik der abelschen gruppen. Fundamenta Mathematicae, 1(30), 253-256.

[20] Vetterlein, T. (2010). MTL-algebras arising from partially ordered groups. Fuzzy sets and systems, 161(3), 433-443.

[21] Wang, J. T., Xin, X. L., and Saeid, A. B. (2016). Very true operators on MTL-algebras. Open Mathematics, 14(1), 955-969.

[22] Wang, J. T., Xin, X. L., and Jun, Y. B. (2018). Very true operators on equality algebras. J. Comput. Anal. Appl, 24(3).

[23] Zadeh, L. A. (1975). Fuzzy logic and approximate reasoning. Synthese, 30(3-4), 407-428.

Ege University, Faculty of Science, Department of Mathematics, Izmir, Turkey

Ege University, Faculty of Science, Department of Mathematics, Izmir, Turkey

International Journal of Maps in Mathematics

*Volume 4, Issue 2, 2021, Pages:**107-120***

ISSN: 2636-7467 (Online)

www.journalmim.com

ON TRANS-SASAKIAN 3-MANIFOLDS WITH RESPECT TO THE SCHOUTEN-VAN KAMPEN CONNECTION

SEMRA ZEREN AND AHMET YILDIZ *

Abstract. The object of the present paper is to characterize trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection. Also, we consider Ricci solitons, η-Ricci solitons and Yamabe solitons of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. Then we give an example of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection.

Keywords: Schouten-van Kampen connection, trans-Sasakian manifolds, semi-symmetry, Ricci semi-symmetry, almost Ricci soliton, almost η-Ricci soliton, almost Yamabe soliton.

2010 Mathematics Subject Classification: 53C15, 53C25, 53A30.

1. Introduction

In [19], Oubina defined a new class of almost contact metric structure, which is said to be trans-Sasakian structure of type (α, β). In [7], Chinea and Gonzales introduced two subclasses of trans-Sasakian structures which contain the Kenmotsu and Sasakian structures. Trans-Sasakian structures of type $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α-Sasakian, β-Kenmotsu and cosymplectic, respectively [3, 14].

The Schouten-van Kampen connection defined as adapted to a linear connection for studying non holonomic manifolds and it is one of the most natural connections on a differentiable manifold [2, 13, 23]. Solov'ev studied hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [24, 25, 26, 27]. Then Olszak studied the Schouten-van

Received:2021.01.16 *Revised:2021.03.05* *Accepted:2021.03.18*

* Corresponding author

Semra Zeren; zerensemra@hotmail.com; https://orcid.org/0000-0003-0083-3507

Ahmet Yıldız; a.yildiz@inonu.edu.tr; https://orcid.org/0000-0002-9799-1781

.

Kampen connection to almost (para) contact metric structures [18]. In recent times, Perktaş and Yildiz studied some symmetry conditions and some soliton types of quasi-Sasakian manifolds and f-Kenmotsu manifolds with respect to the Schouten-van Kampen connection [21, 22].

Let (M, g) be a Riemannian manifold. Then the metric g is called a Ricci soliton if [12]

$$L_X g + 2Ric + 2\delta g = 0, \tag{1.1}$$

where L is the Lie derivative, Ric is the Ricci tensor, X is a complete vector field and δ is a constant on M. In [8], Cho and Kimura given the notion of η-Ricci solitons. The manifold (M, g) is called an η-Ricci soliton if there exist a smooth vector field X such that the Ricci tensor satisfies

$$L_X g + 2Ric + 2\delta g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where and μ is also constant on M. Note that Ricci solitons and η-Ricci solitons are said to be shrinking, steady and expanding according as δ is negative, zero and positive, respectively.

In [12], Hamilton defined Yamabe flow to solve the Yamabe problem. The Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively [1, 5, 6, 10, 17].

A Yamabe soliton on a Riemannian manifold (M, g) satisfying [1]

$$\frac{1}{2}(L_X g) = (\tau - \delta)g, \tag{1.3}$$

where τ is the scalar curvature of M. Moreover, if (M, g) is of constant scalar curvature τ, then the Riemannian metric g is called a Yamabe metric.Yamabe solitons are said to be shrinking, steady and expanding according as δ is positive, zero and negative, respectively.

This paper is organized as follows: After preliminaries, we give some basic information about the Schouten-van Kampen connection and trans-Sasakian manifolds. Then we adapte the Schouten-van Kampen connection on trans-Sasakian 3-manifolds. In section 4, we consider Ricci semisymmetric trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection. In the last section, firstly we study Ricci solitons, η-Ricci solitons and Yamabe solitons of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. Then we give an example of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(U) = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.4}$$

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.5}$$

$$g(U, \phi V) = -g(\phi U, V), \quad g(U, \xi) = \eta(U), \tag{2.6}$$

for all $U, V \in TM$ [3]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(U, V) = g(U, \phi V). \tag{2.7}$$

This may be expressed by the condition [4]

$$(\nabla_U \phi)V = \alpha(g(U, V)\xi - \eta(V)U) + \beta(g(\phi U, V)\xi - \eta(V)\phi U), \tag{2.8}$$

for smooth functions α and β on M. Here we say that the trans-Sasakian structure is of type (α, β). From the formula (2.8) it follows that

$$\nabla_U \xi = -\alpha\phi U + \beta(U - \eta(U)\xi), \tag{2.9}$$

$$(\nabla_U \eta)V = -\alpha g(\phi U, V) + \beta g(\phi U, \phi V). \tag{2.10}$$

An explicit example of trans-Sasakian 3-manifolds was constructed in [15]. In [9], the Ricci tensor and curvature tensor for trans-Sasakian 3-manifolds were studied and their explicit formulae were given.

From [9] we know that for a trans-Sasakian 3-manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{2.11}$$

$$Ric(U, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(U) - U\beta - (\phi U)\alpha, \tag{2.12}$$

$$\begin{aligned} Ric(U, V) \;\; = \;\; & (\frac{\tau}{2} + \xi\beta - (\alpha^2 - \beta^2))g(U, V) - (\frac{\tau}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(U)\eta(V) \\ & -(V\beta + (\phi V)\alpha)\eta(U) - (U\beta + (\phi U)\alpha)\eta(V), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} R(U,V)W &= (\frac{\tau}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(V,W)U-g(U,W)V) \\ &-g(V,W)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(U)\xi \\ &-\eta(U)(\phi grad\alpha-grad\beta)+(U\beta+(\phi U)\alpha)\xi] \\ &+g(U,W)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(V)\xi \\ &-\eta(V)(\phi grad\alpha-grad\beta)+(V\beta+(\phi V)\alpha)\xi] \qquad (2.14) \\ &-[(W\beta+(\phi W)\alpha)\eta(V)+(V\beta+(\phi V)\alpha)\eta(W) \\ &+(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(V)\eta(W)]U \\ &+[(W\beta+(\phi W)\alpha)\eta(U)+(U\beta+(\phi U)\alpha)\eta(W) \\ &+(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(U)\eta(W)]V, \end{aligned}$$

where Ric is the Ricci tensor, R is the curvature tensor and τ is the scalar curvature of the manifold M, respectively.

If α and β are constants, then equations (2.11)-(2.14) become

$$\begin{aligned} R(U,V)W &= (\frac{\tau}{2}-2(\alpha^2-\beta^2))(g(V,W)U-g(U,W)V) \\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))(g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi \qquad (2.15) \\ &\quad +\eta(V)\eta(W)U-\eta(U)\eta(W)V), \end{aligned}$$

$$\begin{aligned} Ric(U,V) &= (\frac{\tau}{2}-(\alpha^2-\beta^2))g(U,V) \qquad (2.16) \\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(U)\eta(V), \end{aligned}$$

$$Ric(U,\xi)=2(\alpha^2-\beta^2)\eta(U), \qquad (2.17)$$

$$R(U,V)\xi=(\alpha^2-\beta^2)(\eta(V)U-\eta(U)V), \qquad (2.18)$$

$$R(\xi,U)V=(\alpha^2-\beta^2)(g(U,V)\xi-\eta(V)U), \qquad (2.19)$$

$$\begin{aligned} QU &= (\frac{\tau}{2}-(\alpha^2-\beta^2))U \qquad (2.20) \\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(U)\xi. \end{aligned}$$

From (2.11) it follows that if α and β are constants, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic, respectively.

On the other hand we have two naturally defined distributions in the tangent bundle TM of M as follows:

$$H = \ker \eta, \qquad V = \mathrm{span}\{\xi\}. \tag{2.21}$$

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition allows one to define the Schouten-van Kampen connection $\tilde{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\tilde{\nabla}$ on an almost contact metric manifold with respect to Levi-Civita connection ∇ is defined by [24]

$$\tilde{\nabla}_U V = \nabla_U V - \eta(V)\nabla_U \xi + (\nabla_U \eta)(V)\xi. \tag{2.22}$$

Thus with the help of the Schouten-van Kampen connection given by (2.22), many properties of some geometric objects connected with the distributions H, V can be characterized [24, 25, 26]. For example g, ξ and η are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0$, $\tilde{\nabla} g = 0$,$\tilde{\nabla}\eta = 0$. Also the torsion $\tilde{T}$ of $\tilde{\nabla}$ is defined by

$$\tilde{T}(U,V) = \eta(U)\nabla_V \xi - \eta(V)\nabla_U \xi + 2d\eta(U,V)\xi.$$

3. Trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection

Let M be a trans-Sasakian 3-manifold with α and β are constants with respect to the Schouten-van Kampen connection. Then using (2.9) and (2.10) in (2.22), we get

$$\tilde{\nabla}_U V = \nabla_U V + \alpha\{\eta(V)\phi U - g(\phi U, V)\xi\} + \beta\{g(U,V)\xi - \eta(V)U\}. \tag{3.23}$$

Let R and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\tilde{\nabla}$ are given by

$$R(U,V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}, \qquad \tilde{R}(U,V) = [\tilde{\nabla}_U, \tilde{\nabla}_V] - \tilde{\nabla}_{[U,V]}.$$

Using (3.23), by direct calculations, we obtain the following formula connecting R and $\tilde{R}$ on a trans-Sasakian 3-manifold

$$\begin{aligned} \tilde{R}(U,V)W = {} & R(U,V)W \\ & + \alpha^2\{g(\phi V, W)\phi U - g(\phi U, W)\phi V + \eta(U)\eta(W)V \\ & - \eta(V)\eta(W)U - g(V,W)\eta(U)\xi + g(U,W)\eta(V)\xi\} \\ & + \beta^2\{g(V,W)U - g(U,W)V\}. \end{aligned} \tag{3.24}$$

We will also consider the Riemann curvature $(0,4)$-tensors $\tilde{R}, R$, the Ricci tensors $\tilde{Ric}, Ric$, the Ricci operators $\tilde{Q}, Q$ and the scalar curvatures $\tilde{\tau}, \tau$ of the connections $\tilde{\nabla}$ and ∇ are given by

$$\begin{aligned}\tilde{R}(U,V,W,Z) &= R(U,V,W,Z)\\ &+\alpha^2\{g(\phi V,W)g(\phi U,Z)-g(\phi U,W)g(\phi V,Z)\\ &+g(V,Z)\eta(U)\eta(W)-g(U,Z)\eta(V)\eta(W)\\ &-g(V,W)\eta(U)\eta(Z)+g(U,W)\eta(V)\eta(Z)\}\\ &+\beta^2\{g(V,W)g(U,Z)-g(U,W)g(V,Z)\},\end{aligned} \tag{3.25}$$

$$\begin{aligned}\tilde{Ric}(V,W) &= Ric(V,W)\\ &+2\beta^2 g(V,W)-2\alpha^2\eta(V)\eta(W),\end{aligned} \tag{3.26}$$

$$\tilde{Q}U = QU + 2\beta^2 U - 2\alpha^2\eta(U)\xi, \tag{3.27}$$

$$\tilde{\tau} = \tau - 2\alpha^2 + 6\beta^2, \tag{3.28}$$

respectively, where $\tilde{R}(U,V,W,Z) = g(\tilde{R}(U,V)W,Z)$ and $R(U,V,W,Z) = g(R(U,V)W,Z)$.

4. Ricci semisymetric trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection

In this section, we study Ricci semisymetric trans-Sasakian 3-manifolds with α and β are constants with respect to the Schouten-van Kampen connection.

If a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection is *Ricci semisymmetric* then we can write

$$(\tilde{R}(U,V)\cdot\tilde{Ric})(W,Y) = 0, \tag{4.29}$$

which turns to

$$\tilde{Ric}(\tilde{R}(U,V)W,Y) + \tilde{Ric}(W,\tilde{R}(U,V)Y) = 0. \tag{4.30}$$

Using (3.26) in (4.30), we obtain

$$\begin{aligned}& Ric(\tilde{R}(U,V)W,Y) - 2\alpha^2\eta(\tilde{R}(U,V)W)\eta(Y) + 2\beta^2 g(\tilde{R}(U,V)W,Y)\\ & +Ric(W,\tilde{R}(U,V)Y) - 2\alpha^2\eta(\tilde{R}(U,V)Y)\eta(W) + 2\beta^2 g(W,\tilde{R}(U,V)Y)\\ = \ & Ric(\tilde{R}(U,V)W,Y) + Ric(W,\tilde{R}(U,V)Y) = 0.\end{aligned} \tag{4.31}$$

Now using (3.24) in (4.31), we get

$$
\begin{aligned}
&Ric(R(U,V)W,Y)+Ric(W,R(U,V)Y)+\alpha^2\{g(\phi V,W)Ric(\phi U,Y)\\
&-g(\phi U,W)Ric(\phi V,Y)+Ric(V,Y)\eta(U)\eta(W)-Ric(U,Y)\eta(V)\eta(W)\\
&+g(U,W)\eta(V)Ric(Y,\xi)-g(V,W)\eta(U)Ric(Y,\xi)+g(\phi V,Y)Ric(\phi U,W)\\
&-g(\phi U,Y)Ric(\phi V,W)+Ric(V,W)\eta(U)\eta(Y)-Ric(U,W)\eta(V)\eta(Y) \qquad (4.32)\\
&+g(U,Y)\eta(V)Ric(W,\xi)-g(V,Y)\eta(U)Ric(W,\xi)\}\\
&+\beta^2\{g(V,W)Ric(Y,U)-g(U,W)Ric(Y,V)\\
&+g(V,Y)Ric(W,U)-g(U,Y)Ric(W,V)\}=0.
\end{aligned}
$$

Let $\{e_i\}$, $(1\leq i\leq 3)$, be an orthonormal basis of the tangent space at any point of M. Then the sum for $1\leq i\leq 3$ of the relation (4.32) for $U=Y=e_i$ gives

$$
\begin{aligned}
&Ric(R(e_i,V)W,e_i)+Ric(W,R(e_i,V)e_i)\\
&+\alpha^2\left\{Ric(V,W)-\tau\eta(V)\eta(W)\right\}\\
&+2\alpha^2\left(\alpha^2-\beta^2\right)\{3\eta(V)\eta(W)-g\left(V,W\right)\} \qquad (4.33)\\
&+\beta^2\{\tau g(V,W)-3Ric(V,W)\}=0,
\end{aligned}
$$

which is equal to

$$
\begin{aligned}
&\lambda\left\{\tau g(V,W)-3Ric\left(V,W\right)\right\}+2\mu(\alpha^2-\beta^2)\eta(V)\eta(W)\\
&+\mu Ric(V,W)-2\mu(\alpha^2-\beta^2)g(V,W)+4\mu(\alpha^2-\beta^2)\eta(V)\eta(W)\\
&-\mu\tau\eta(V)\eta(W) \qquad (4.34)\\
&+\alpha^2\left\{Ric(V,W)-\tau\eta(V)\eta(W)\right\}\\
&+2\alpha^2(\alpha^2-\beta^2)\left\{3\eta(V)\eta(W)-g(V,W)\right\}\\
&+\beta^2\{\tau g(V,W)-3Ric(V,W)\}=0,
\end{aligned}
$$

where $\lambda=\frac{\tau}{2}-2(\alpha^2-\beta^2)$ and $\mu=\frac{\tau}{2}-3(\alpha^2-\beta^2)$. After some calculations we have

$$
\begin{aligned}
&[-3\left(\lambda+\beta^2\right)+\left(\mu+\alpha^2\right)]Ric(V,W)\\
&+[(\lambda+\beta^2)\tau-2\left(\mu+\alpha^2\right)(\alpha^2-\beta^2)]g(V,W)\\
&+[6(\mu+\alpha^2)(\alpha^2-\beta^2)-\left(\lambda+\beta^2\right)\tau]\eta(V)\eta(W)=0,
\end{aligned}
$$

i.e.,

$$Ric(V,W) = [\frac{\tau}{2} - \left(\alpha^2 - \beta^2\right)]g\left(V,W\right) + [3\left(\alpha^2 - \beta^2\right) - \frac{\tau}{2}]\eta(V)\eta(W). \tag{4.35}$$

Hence M is an η-Einstein manifold with respect to the Levi-Civita connection. Now using (4.35) in (3.26), we have

$$\tilde{R}ic(V,W) = [\frac{\tau}{2} - \alpha^2 + 3\beta^2]g(V,W) - [\frac{\tau}{2} - \alpha^2 + 3\beta^2]\eta(V)\eta(W).$$

Thus M is also an η-Einstein manifold with respect to the Schouten-van Kampen connection. Therefore we have the following:

Theorem 4.1. *Let M be a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. If M is Ricci semisymmetric with respect to the Schouten-van Kampen connection then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection and Levi-Civita connection.*

5. Soliton types on trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection

In this section we study Ricci solitons, η-Ricci solitons and Yamabe solitons on a trans-Sasakian 3-manifold with α and β are constants with respect to the Schouten-van Kampen connection.

In a trans-Sasakian 3-manifold M endowed with respect to the Schouten-van Kampen connection bearing an Ricci soliton, we can write

$$(\tilde{L}_X g + 2\tilde{R}ic + 2\delta g)(U,V) = 0. \tag{5.36}$$

Using (3.23) in (5.36), since $\tilde{\nabla} g = 0$ and $\tilde{T} \neq 0$, we have

$$(\tilde{L}_X g)(U,V) = g(\nabla_U X, V) + g(U, \nabla_V X) = (L_X g)(U,V),$$

that is,

$$g(\nabla_U X, V) + g(U, \nabla_V X) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) = 0. \tag{5.37}$$

Putting $X = \xi$ in (5.37), we obtain

$$g(\nabla_U \xi, V) + g(U, \nabla_V \xi) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) = 0. \tag{5.38}$$

Now using (2.9) in (5.38), we get

$$g(-\alpha\phi U + \beta(U - \eta(U)\xi), V) + g(U, -\alpha\phi V + \beta(V - \eta(V)\xi) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) = 0,$$

i.e.,

$$\tilde{R}ic(U,V) = -(\beta+\delta)g(U,V) + \beta\eta(U)\eta(V). \tag{5.39}$$

Thus M is an η-Einstein manifold with respect to the Schouten-van Kampen connection. Also using (3.26) in (5.39), we get

$$Ric(U,V) = -(2\beta^2+\beta+\delta)g(U,V) + (\beta+2\alpha^2)\eta(U)\eta(V).$$

Hence M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 5.1. *Let M be a trans-Sasakian 3-manifold bearing a Ricci soliton (ξ,δ,g) with respect to the Schouten-van Kampen connection. Then M is an η-Einstein manifold both with respect to the Schouten-van Kampen connection and Levi-Civita connection.*

Putting $V=\xi$ and using (3.26) in (5.39), we give the following:

Corollary 5.1. *A Ricci soliton (ξ,δ,g) on a trans-Sasakian 3-manifold M with respect to the Schouten-van Kampen connection is always steady.*

On the other hand, from (2.16) and (3.26), it is easy to see that a trans-Sasakian 3-manifold M is always η-Einstein with respect to the Schouten-van Kampen connection of the form $\tilde{R}ic = \gamma g + \sigma\eta\otimes\eta$, where $\gamma = -\sigma = \frac{\tau}{2} - \alpha^2 + 3\beta^2$. Then, we write

$$(\tilde{L}_\xi g + 2\tilde{R}ic + 2\delta g)(U,V) = ((2\gamma+2\delta)g - 2\sigma\eta\otimes\eta)(U,V), \tag{5.40}$$

for all $U,V\in\chi(M)$, which implies that the manifold M admits a Ricci soliton (ξ,δ,g) if $\gamma+\delta=0$ and $\sigma=0$.

Using (5.39), we can also state the following:

Corollary 5.2. *The scalar curvature of a trans-Sasakian 3-manifold M bearing a Ricci soliton (ξ,δ,g) with respect to the Schouten-van Kampen connection is $\tilde{\tau} = -3\delta - 2\beta$.*

Now we consider an η-Ricci soliton on a trans-Sasakian 3-manifold M with respect to the Schouten-van Kampen connection. Then

$$(\tilde{L}_X g + 2\tilde{R}ic + 2\delta g + 2\mu\eta\otimes\eta)(U,V) = 0, \tag{5.41}$$

that is,

$$g(\nabla_U X, V) + g(U,\nabla_V X) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) + 2\mu\eta(U)\eta(V) = 0. \tag{5.42}$$

Putting $X = \xi$ in (5.42), we obtain

$$\tilde{R}ic(U,V) = -\delta g(U,V) - \mu\eta(U)\eta(V). \tag{5.43}$$

Hence M is an η-Einstein manifold with respect to the Schouten-van Kampen connection. Taking $V = \xi$ in (5.43), we get $\delta + \mu = 0$. Using (3.26) in (5.43), we have

$$Ric(U,V) = [-2\beta^2 - \delta]g(U,V) + [2\alpha^2 - \mu]\eta(U)\eta(V).$$

Thus M is an η-Einstein manifold with respect to the Levi-Civita connection. Now we have the following:

Theorem 5.2. *Let M be a trans-Sasakian 3-manifold bearing an η-Ricci soliton (ξ, δ, μ, g) with respect to the Schouten-van Kampen connection. Then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection and the Levi-Civita connection.*

Again let us consider equations (5.36) and (5.37). Using (3.26), we obtain

$$g(\nabla_U X, V) + g(U, \nabla_V X) + 2Ric(U,V) + 2(2\beta^2 + \delta)g(U,V) - 2\alpha^2\eta(U)\eta(V) = 0.$$

Thus we write

$$(L_X g)(U,V) + 2Ric(U,V) + 2(2\beta^2 + \delta)g(U,V) - 2\alpha^2\eta(U)\eta(V) = 0.$$

This last equation shows that if (X, δ, g) is a Ricci soliton on a trans-Sasakian 3-manifold M with respect to the Schouten-van Kampen connection, then the manifold admits an η-Ricci soliton $(X, 2\beta^2 + \delta, \alpha^2, g)$ with respect to the Levi-Civita connection. If $\alpha = 0$, then

$$(L_X g)(U,V) + 2Ric(U,V) + 2(2\beta^2 + \delta)g(U,V) = 0.$$

So we have the following:

Corollary 5.3. *Let M be a trans-Sasakian 3-manifold bearing a Ricci soliton (X, δ, g) with respect to the Schouten-van Kampen connection. Then we have: (i) If $\alpha = 0$, then M admits a Ricci soliton $(X, 2\beta^2 + \delta, g)$ with respect to the Levi-Civita connection. (ii) If $\alpha \neq 0$, then M admits an η-Ricci soliton $(X, 2\beta^2 + \delta, \alpha^2, g)$ with respect to the Levi-Civita connection.*

Example 5.1. *We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ y \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields*

$$e_1 = e^y \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = e^y \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1,e_3) &= g(e_2,e_3)=g(e_1,e_2)=0, \\ g(e_1,e_1) &= g(e_2,e_2)=g(e_3,e_3)=1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z)=g(Z,e_2)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi(e_1)=e_3$, $\phi(e_2)=0$, $\phi(e_3)=-e_1$. Then using linearity of ϕ and g we have

$$\eta(e_2)=1, \quad \phi^2 W = -W+\eta(W)e_3,$$

$$g(\phi W,\phi Z)=g(W,Z)-\eta(W)\eta(Z),$$

for any $W,Z \in \chi(M)$. Thus for $e_2=\xi$, (ϕ,ξ,η,g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1,e_2]=-e_1, \quad [e_2,e_3]=e_3, \quad [e_1,e_3]=0.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_U V,W) &= Ug(V,W)+Vg(W,U)-Wg(U,V) \\ &\quad -g(U,[V,W])-g(V,[U,W])+g(W,[U,V]). \end{aligned} \tag{5.44}$$

Using (5.44), we obtain

$$\begin{aligned} \nabla_{e_1}e_1 &= e_2, \quad \nabla_{e_1}e_2=-e_1, \quad \nabla_{e_1}e_3=0, \\ \nabla_{e_2}e_1 &= 0, \quad \nabla_{e_2}e_2=0, \quad \nabla_{e_2}e_3=0, \\ \nabla_{e_3}e_1 &= 0, \quad \nabla_{e_3}e_2=-e_3, \quad \nabla_{e_3}e_3=e_2. \end{aligned} \tag{5.45}$$

By (5.45), we see that the manifold satisfies (2.8) for $U=e_1$, $\alpha=0$, $\beta=-1$, and $e_2=\xi$. Similarly, it can be shown that for $U=e_2$ and $U=e_3$ the manifold also satisfies (2.8) for $\alpha=0$, $\beta=-1$, and $e_2=\xi$. Hence the manifold is a trans-Sasakian manifold of type $(0,-1)$ [20]. *Now we consider the Schouten-van Kampen connection to this example. From (5.45), we have*

$$\begin{aligned} R(e_1,e_2)e_1 &= e_2, \quad R(e_1,e_2)e_2=-e_1, \quad R(e_1,e_2)e_3=0, \\ R(e_1,e_3)e_1 &= e_3, \quad R(e_1,e_3)e_2=0, \quad R(e_1,e_3)e_3=-e_1, \\ R(e_2,e_3)e_1 &= 0, \quad R(e_2,e_3)e_2=e_3, \quad R(e_2,e_3)e_3=-e_2. \end{aligned} \tag{5.46}$$

Again using (3.23) and (5.45), we obtain

$$\begin{aligned}
\tilde{\nabla}_{e_1}e_1 &= (\beta+1)e_2, \qquad \tilde{\nabla}_{e_1}e_2 = -(\beta+1)e_1+\alpha e_3,\\
\tilde{\nabla}_{e_1}e_3 &= -\alpha e_2, \quad \tilde{\nabla}_{e_2}e_1 = 0, \quad \tilde{\nabla}_{e_2}e_2 = 0,\\
\tilde{\nabla}_{e_2}e_3 &= 0, \quad \tilde{\nabla}_{e_3}e_1 = \alpha e_2, \qquad (5.47)\\
\tilde{\nabla}_{e_3}e_2 &= -(\beta+1)e_3-\alpha e_1, \quad \tilde{\nabla}_{e_3}e_3 = (\beta+1)e_2.
\end{aligned}$$

Considering (5.47), we can see that $\tilde{\nabla}_{e_i}\xi = 0$, $(1 \le i \le 3)$, *for* $\xi = e_2$ *and* $\alpha = 0$, $\beta = -1$. *Hence* M *is a trans-Sasakian* 3*-manifold of type* $(0,-1)$ *with respect to the Schouten-van Kampen connection. Thus from (5.47), we get*

$$\begin{aligned}
\tilde{R}(e_1,e_2)e_1 &= (1+\alpha^2-\beta^2)e_2, \qquad \tilde{R}(e_1,e_2)e_2 = -(1+\alpha^2-\beta^2)e_1,\\
\tilde{R}(e_1,e_2)e_3 &= 0, \qquad \tilde{R}(e_1,e_3)e_1 = (1-\alpha^2-\beta^2)e_3,\\
\tilde{R}(e_1,e_3)e_2 &= 0, \qquad \tilde{R}(e_1,e_3)e_3 = (-1-\alpha^2+\beta^2)e_1, \qquad (5.48)\\
\tilde{R}(e_2,e_3)e_1 &= 0, \qquad \tilde{R}(e_2,e_3)e_2 = (1+\alpha^2-\beta^2)e_3,\\
\tilde{R}(e_2,e_3)e_3 &= (-1+\alpha^2+\beta^2)e_2.
\end{aligned}$$

Now using (5.48), we see that the non-zero components of the Ricci tensor $\tilde{R}ic$ *with respect to the Schouten-van Kampen connection as follows:*

$$\tilde{R}ic(e_1,e_1) = -2+2\beta^2, \quad \tilde{R}ic(e_2,e_2) = -2-2\alpha^2+2\beta^2, \quad \tilde{R}ic(e_3,e_3) = -2+2\beta^2.$$

For any $U, V \in \chi(M)$, *we write*

$$U = a_1e_1 + a_2e_2 + a_3e_3,$$

$$V = b_1e_1 + b_2e_2 + b_3e_3.$$

Thus we have

$$\begin{aligned}
(\tilde{L}_\xi g)(X,Y) + 2\tilde{S}(X,Y) + 2\delta g(X,Y) + 2\mu\eta(X)\eta(Y) &= (-2+2\beta^2+\delta)a_1b_1\\
&\quad +(-2-2\alpha^2+2\beta^2+\delta+\mu)a_2b_2\\
&\quad +(-2+2\beta^2+\delta)a_3b_3.
\end{aligned}$$

If $\delta = 2-2\beta^2$ *and* $\mu = 2\alpha^2$, *then* M *admits an* η*-Ricci soliton* (ξ,δ,μ,g) *with respect to the Schouten-van Kampen connection.*

Finally we study Yamabe solitons on a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. Assume that (M, X, δ, g) is a Yamabe soliton on a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. From (1.3), we can write

$$\frac{1}{2}(\tilde{L}_X g)(U,V) = (\tilde{\tau} - \delta)g(U,V), \tag{5.49}$$

that is,

$$\frac{1}{2}\{g(\tilde{\nabla}_U X, V) + g(U, \tilde{\nabla}_V X)\} = (\tilde{\tau} - \delta)g(U,V). \tag{5.50}$$

Putting $X = \xi$ in (5.50), we obtain $\tilde{\tau} = \delta$, which implies that the following:

Theorem 5.3. *The scalar curvature $\tilde{\tau}$ of a trans-Sasakian 3-manifold bearing a Yamabe soliton (M, ξ, δ, g) with respect to the Schouten-van Kampen connection is equal to δ.*

So we give the followings:

Corollary 5.4. *A trans-Sasakian 3-manifold bearing a Yamabe soliton (M, ξ, δ, g) with respect to the Schouten-van Kampen connection is of constant scalar curvature with respect to the Schouten-van Kampen connection.*

Corollary 5.5. *If a trans-Sasakian 3-manifold bearing a Yamabe soliton (M, ξ, δ, g) with respect to the Schouten-van Kampen connection, then the Riemannian metric g is a Yamabe metric.*

References

[1] Barbosa E., Riberio E. (2013). On conformal solutions of the Yamabe flow, Arch. Math., 101, 79-89.

[2] Bejancu A., Faran H. (2006). Foliations and geometric structures, Math. and its Appl. 580. Springer, Dordrecht.

[3] Blair D. E. (1976). Contact manifolds in Riemannian geometry, Lecture Notes in Mathematics Vol 509. Springer-Verlag, Berlin-New York.

[4] Blair D. E., Oubina J. A. (1990). Conformal and related changes of metric on the product of two almost contact metric manifolds, Publications Matematiques, 34, 199-207.

[5] Cao H. D., Sun X., Zhang Y. (2012). On the structure of gradient Yamabe solitons, Mathematical Research Letters, 19, 767-774.

[6] Chen B. Y., Deshmukh S. (2018). Yamabe and quasi Yamabe solitons on Euclidean submanifolds, Mediterr. J. Math., 15(5), 194, 1-9.

[7] Chinea D., Gonzales J. C. (1990). A classification of almost contact metric manifolds, Ann. Mat. Pura Appl., 156(4), 15-30.

[8] Cho J. C., Kimura M. (2009). Ricci solitons and real hypersurfaces in a complex space form, Tohoku Math. J., 61(2), 205-212.

[9] De U.C., Tripathi M. M. (2003). Ricci Tensor in 3-dimensional Trans-Sasakian Manifolds. Kyungpook Math. J., 43, 247-255.

[10] Deshmukh S., Chen B. Y. (2018). A note on Yamabe solitons, Balkan J. Geom. Appl., 23(1), 37-43.

[11] Gray A., Hervella L. M. (1980). The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura Appl., 123(4), 35-58.

[12] Hamilton R. S. (1988). The Ricci flow on surfaces, Mathematics and general relativity. Contemp. Math., 71, 237-262.

[13] Ianuş S. (1971). Some almost product structures on manifolds with linear connection, Kodai Math. Sem. Rep., 23, 305-310.

[14] Janssens D., Vanhecke L. (1981). Almost contact structures and curvature tensors, Kodai Math. J., 4(1), 1-27.

[15] Marrero J. C. (1992). The local structure of trans-Sasakian manifolds, Ann. Mat. Pura Appl., 162(4), 77-86.

[16] Marrero J. C., Chinea D. (1989). On trans-Sasakian manifolds in Proceedings of the XIVth Spanish-Portuguese Conference on Mathematics. Vol. I-III. (Spanish). Puerto de la Cruz, Yniv. La Laguna, La Laguna, 655-659.

[17] Neto, B. L. (2017). A note on (anti-)self dual quasi Yamabe gradient solitons, Results in Math., 71, 527-533.

[18] Olszak Z. (2013). The Schouten-van Kampen affine connection adapted an almost (para) contact metric structure, Publ. De L'inst. Math., 94, 31-42.

[19] Oubina J. A. (1985). New classes of almost contact metric structures, Publ. Mat. Debrecen., 32(3-4), 187-193.

[20] Pahan A.S., Bhattacharyya B. A. (2016). Some properties of three-dimensional trans-Sasakian manifolds with a semi-symmetric metric connection, Lobachevskii J. of Math., 37(2), 177-184.

[21] Perktaş S. Y., Yıldız A. (2020). On quasi-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection, Int. Electronic J. of Geom., 13(2), 62-74.

[22] Perktaş S. Y., Yıldız A. (2021). On f-Kenmotsu 3-manifolds with respect to the Schouten-van Kampen connection, Turkish J. of Math., 45(1), 387-409.

[23] Schouten J., Van Kampen E. (1930). Zur Einbettungs-und Krümmungsthorie nichtholonomer Gebilde, Math. Ann., 103, 752-783.

[24] Solov'ev A. F. (1978). On the curvature of the connection induced on a hyperdistribution in a Riemannian space, Geom. Sb., 19, 12-23, (in Russian).

[25] Solov'ev A. F. (1979). The bending of hyperdistributions,Geom. Sb. 20, 101-112 (in Russian).

[26] Solov'ev A. F. (1982). Second fundamental form of a distribution, Mat. Zametki, 35, 139-146.

[27] Solov'ev A. F. (1984). Curvature of a distribution, Mat. Zametki, 35, 111-124.

Education Faculty, Department of Mathematics, Inonu University, Malatya, TURKEY

International Journal of Maps in Mathematics
*Volume 4, Issue 2, 2021, Pages:**121-135***
ISSN: 2636-7467 (Online)
www.journalmim.com

ON PARA-KAHLER-NORDEN PROPERTIES OF THE φ-SASAKI METRIC ON TANGENT BUNDLE

ABDERRAHIM ZAGANE

Abstract. In the present paper, we investigate para-Nordenian properties of the φ-Sasaki metric on the tangent bundle.

Keywords: Horizontal lift and vertical lift, tangent bundle, φ-Sasaki metric, almost para-complex structure, pure metric.

2010 Mathematics Subject Classification: Primary, 53C15,53C55, Secondary, 53C20, 53B35.

1. Introduction

In this field, the notion of almost para-complex structure on a smooth manifold has been studied, in the first papers by Libermann, P. [9], Patterson, E. M.[12] until now, from several different points of view. Moreover, the papers related to it have appeared many times in a rather disperse way, and a survey of further results on para-complex geometry (including para-Kähler geometry) can be found for instance in [2, 3, 5]. Also, other further signifiant developments are due in some recent surveys [1, 8, 13], where some aspects concerning the geometry of para-complex manifolds are presented on the tangent and cotangent bundles. See also [7, 6, 11, 15, 16].

The main idea in this note consists in the modification of the Sasaki metric. First we introduce a new metric called φ-Sasaki metric on the tangent bundle TM over a para-Kahler-Norden manifold (M^{2m}, φ, g). This new metric will lead us to interesting results. Afterward we construct almost para-complex Norden structures on tangent bundle equipped with the

Received:2020.09.27 *Revised:2021.01.30* *Accepted:2021.04.17*

Abderrahim Zagane; zaganeabr2018@gmail.com; https://orcid.org/0000-0001-9339-3787

.

φ-Sasaki metric and investigate necessary and sufficient conditions for these structures to become para-Kähler-Norden, quasi-para-Kähler-Norden. Finally we characterize some properties of almost para-complex Norden structures in context of almost product Riemannian manifolds.

2. Preliminaries

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=\overline{1,m}}$ on TM. Let $C^\infty(M)$ (resp. $C^\infty(TM)$) be the ring of real-valued C^∞ functions on M (resp. TM) and $\Im^r_s(M)$ (resp. $\Im^r_s(TM)$) be the module over $C^\infty(M)$ (resp. $C^\infty(TM)$) of C^∞ tensor fields of type (r, s).

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned}\mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)},\ a^i \in \mathbb{R}\},\\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}|_{(x,u)},\ a^i \in \mathbb{R}\},\end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i}, \tag{2.1}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma^k_{ij} \frac{\partial}{\partial y^k}\}. \tag{2.2}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=\overline{1,m}}$ is a local adapted frame on TTM.

Lemma 2.1. [18] *Let (M, g) be a Riemannian manifold and R its tensor curvature, then for all vector fields $X, Y \in \Im^1_0(M)$ we have:*

(1) $[X^H, Y^H]_p = [X, Y]^H_p - (R_x(X, Y)u)^V$,
(2) $[X^H, Y^V]_p = (\nabla_X Y)^V_p$,
(3) $[X^V, Y^V]_p = 0$,

where $p = (x, u) \in TM$.

An almost product structure φ on a manifold M is a $(1, 1)$ tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1, 1)$ on M). The pair (M, φ)

is called an almost product manifold.

A linear connection ∇ on (M, φ) such that $\nabla\varphi = 0$ is said to be an almost product connection. There exists an almost product connection on every almost product manifold[4].

An almost para-complex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost paracomplex manifold is necessarily even [3].

An almost para-complex Norden manifold (M^{2m}, φ, g) is a real $2m$-dimensional differentiable manifold M^{2m} with an almost para-complex structure φ and a Riemannian metric g such that

$$g(\varphi X, Y) \;=\; g(X, \varphi Y), \tag{2.3}$$

for all $X, Y \in \Im_0^1(M)$, in this case g is called a pure metric with respect to φ or para-Norden metric (B-metric)[13].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that φ is integrable i.e $\nabla\varphi = 0$ (B-manifold), where ∇ is the Levi-Civita connection of g [13, 16].

A Tachibana operator $\phi_\varphi : \Im_0^2(M) \to \Im_0^3(M)$ applied to the pure metric g is given by

$$\begin{aligned}(\phi_\varphi g)(X, Y, Z) \;=\;& (\varphi X)(g(Y, Z)) + X(g(\varphi Y, Z)) + g((L_Y\varphi)X, Z)\\ &+ g((L_Z\varphi)X, Y),\end{aligned} \tag{2.4}$$

for all $X, Y, Z \in \Im_0^1(M)$ [17], where L_Y denotes the Lie differentiation with respect to Y.

In an almost para-complex Norden manifold, a para-Norden metric g is called para-holomorphic if

$$(\phi_\varphi g)(X, Y, Z) \;=\; 0, \tag{2.5}$$

for all $X, Y, Z \in \Im_0^1(M)$[13].

A para-holomorphic Norden manifold is an almost para-complex Norden manifold (M^{2m}, φ, g) such that g is a para-holomorphic i.e $\phi_\varphi g = 0$.

In [13], Salimov and his collaborators showed that for an almost para-complex Norden manifold, the condition $\phi_\varphi g = 0$ is equivalent to $\nabla\varphi = 0$. By virtue of this point of view, para-holomorphic Norden manifolds are similar to para-Kähler-Norden manifolds (For complex version see [8]).

The purity conditions for a tensor field $\omega \in \Im_0^q(M)$ with respect to the para-complex structure φ given by

$$\omega(\varphi X_1, X_2, \cdots, X_q) = \omega(X_1, \varphi X_2, \cdots, X_q) = \cdots = \omega(X_1, X_2, \cdots, \varphi X_q),$$

for all $X_1, X_2, \cdots, X_q \in \Im_0^1(M)$ [13].

In [17], an operator $\phi_\varphi : \Im_0^q(M) \to \Im_0^{q+1}(M)$ joined with φ and applied to the pure tensor field ω, given by

$$\begin{aligned}(\phi_\varphi \omega)(Y, X_1, \cdots, X_q) &= (\varphi Y)(\omega(X_1, \cdots, X_q)) + Y(\omega(\varphi X_1, \cdots, X_q)) \\ &+\omega((L_{X_1}\varphi)Y, X_2, \cdots, X_q) + \cdots + \omega((X_1, \cdots, (L_{X_q}\varphi)Y),\end{aligned}$$

for all $Y, X_1, X_2, \cdots, X_q \in \Im_0^1(M)$. If $\phi_\varphi \omega$ vanishes, then ω is said to be almost para-holomorphic.

It is well known that if (M^{2m}, φ, g) is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [13], and

$$\begin{cases} \nabla_Y(\varphi Z) &= \varphi \nabla_Y Z, \\ R(\varphi Y, Z) &= R(Y, \varphi Z) = R(Y, Z)\varphi = \varphi R(Y, Z), \\ R(\varphi Y, \varphi Z) &= R(Y, Z), \end{cases} \tag{2.6}$$

for all $Y, Z \in \Im_0^1(M)$.

Let (M^{2m}, φ, g) be a non-integrable almost para-complex Norden manifold, if

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0.$$

for all $X, Y, Z \in \Im_0^1(M)$, where σ is the cyclic sum by three arguments, then the triple (M^{2m}, φ, g) is a quasi-para-Kähler-Norden manifold [5, 10]. It is well known that

$$\underset{X,Y,Z}{\sigma} g((\nabla_X \varphi)Y, Z) = 0 \Leftrightarrow \underset{X,Y,Z}{\sigma} (\phi_\varphi g)(X, Y, Z) = 0, \tag{2.7}$$

which was proven in [14].

3. φ-Sasaki metric

Definition 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold. On the tangent bundle TM, we define a φ-Sasaki metric noted g_φ by*

$$\begin{aligned} (1)\ g_\varphi(X^H, Y^H)_{(x,u)} &= g_x(X, Y), \\ (2)\ g_\varphi(X^H, Y^V)_{(x,u)} &= 0, \\ (3)\ g_\varphi(X^V, Y^V)_{(x,u)} &= g_x(X, \varphi Y), \end{aligned}$$

where $X, Y \in \Im_0^1(M)$ *and* $(x,u) \in TM$.

Lemma 3.1. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold, we have the following*

$$\begin{aligned} (1)\ X^H g_\varphi(Y^H, Z^H) &= Xg(Y,Z), \\ (2)\ X^V g_\varphi(Y^H, Z^H) &= 0, \\ (3)\ X^H g_\varphi(Y^V, Z^V) &= g_\varphi((\nabla_X Y)^V, Z^V) + g_\varphi(Y^V, (\nabla_X Z)^V), \\ (4)\ X^V g_\varphi(Y^H, Z^H) &= 0, \end{aligned}$$

for any $X, Y, Z \in \Im_0^1(M)$, *where* ∇ *denote the Levi-Civita connection of* (M^{2m}, φ, g).

Theorem 3.1. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold and* (TM, g_φ) *its tangent bundle equipped with the* φ*-Sasaki metric. If* ∇ *(resp* $\widetilde{\nabla}$*) denote the Levi-Civita connection of* (M,g) *(resp* (TM, g_φ) *), then we have:*

$$\begin{aligned} (1)\ \ (\widetilde{\nabla}_{X^H} Y^H)_{(x,u)} &= (\nabla_X Y)^H_{(x,u)} - \frac{1}{2}(R_x(X,Y)u)^V, \\ (2)\ \ (\widetilde{\nabla}_{X^H} Y^V)_{(x,u)} &= (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}(R_x(\varphi u, Y)X)^H, \\ (3)\ \ (\widetilde{\nabla}_{X^V} Y^H)_{(x,u)} &= \frac{1}{2}(R_x(\varphi u, X)Y)^H, \\ (4)\ \ (\widetilde{\nabla}_{X^V} Y^V)_{(x,u)} &= 0, \end{aligned}$$

for all vector fields $X, Y \in \Im_0^1(M)$ *and* $(x,u) \in TM$, *where* R *denote the curvature tensor of* (M^{2m}, φ, g).

The proof of Theorem 3.1 follows directly from Kozul formula, Lemma 2.1 and Lemma 3.1.

4. Some almost Para-complex Structures

4.1. We Consider the tensor field $J_\varphi \in \Im_1^1(TM)$ by

$$\begin{cases} J_\varphi X^H & = (\varphi X)^H \\ J_\varphi X^V & = (\varphi X)^V \end{cases} \tag{4.8}$$

for all $X \in \Im_0^1(M)$.

Lemma 4.1. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold and* (TM, g_φ) *its tangent bundle equipped with the* φ*-Sasaki metric. The couple* (TM, J_φ) *is an almost para-complex manifold .*

Proof. By virtue of (4.8), we have

$$\begin{cases} J_\varphi^2 X^H = J_\varphi(J_\varphi X^H) = J_\varphi((\varphi X)^H) = (\varphi(\varphi X))^H = (\varphi^2 X)^H = X^H, \\ J_\varphi^2 X^V = J_\varphi(J_\varphi X^V) = J_\varphi((\varphi X)^V) = (\varphi(\varphi X))^V = (\varphi^2 X)^V = X^V, \end{cases}$$

for any $X \in \Im_0^1(M)$, then $J_\varphi^2 = id_{TM}$.

Let $\{E_1, \cdots, E_m, E_{m+1}, \cdots, E_{2m}\}$ be local frame of eigenvectors on M such that $\varphi E_i = E_i$, $\varphi E_{m+i} = -E_{m+i}$, for all $i = \overline{1, m}$.

If $Z = Z_1^i E_i^H + Z_2^i E_i^V$, then

$J_\varphi Z = Z_1^i (\varphi E_i)^H + Z_2^i (\varphi E_i)^V = Z_1^i E_i^H + Z_2^i E_i^V = Z$,

i.e. $TTM^+ = Span(E_1^H, \cdots, E_m^H, E_1^V, \cdots, E_m^V)$,

If $Z = Z_1^{m+i} E_{m+i}^H + Z_2^{m+i} E_{m+i}^V$, then

$J_\varphi Z = Z_1^{m+i} (\varphi E_{m+i})^H + Z_2^{m+i} (\varphi E_{m+i})^V = -Z_{m+1}^i E_{m+i}^H - Z_2^{m+i} E_{m+i}^V = -Z$,

i.e. $TTM^- = Span(E_{m+1}^H, \cdots, E_{2m}^H, E_{m+1}^V, \cdots, E_{2m}^V)$.

Theorem 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8). *The triple $(TM, J_\varphi, g_\varphi)$ is an almost para-complex Norden manifold.*

Proof. For all $X, Y \in \Im_0^1(M)$, from (4.8) we have

$$\begin{aligned} (i)\ g_\varphi(J_\varphi X^H, Y^H) &= g_\varphi((\varphi X)^H, Y^H) = g(\varphi X, Y) = g(X, \varphi Y) \\ &= g_\varphi(X^H, (\varphi Y)^H) = g_\varphi(X^H, J_\varphi Y^H), \\ (ii)\ g_\varphi(J_\varphi X^H, Y^V) &= g_\varphi((\varphi X)^H, Y^V) = 0 = g_\varphi(X^H, Y^V) = g_\varphi(X^H, J_\varphi Y^V), \\ (iii)\ g_\varphi(J_\varphi X^V, Y^V) &= g_\varphi((\varphi X)^V, Y^V) = g(\varphi X, \varphi Y) = g(X, Y) \\ &= g(X, \varphi^2 Y) = g_\varphi(X^V, (\varphi Y)^V) = g_\varphi(X^V, J_\varphi Y^V). \end{aligned}$$

Since g is pure with respect to φ, then g_φ is pure with respect to J_φ.

Proposition 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8), *then we get*

1. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^H) = 0$,
2. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^H) = 0$,
3. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^H) = 0$,
4. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^V) = 0$,

5. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^V, Z^H) = 0,$
6. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^V) = 0,$
7. $(\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^V) = 0,$
8. $(\phi_{J_\varphi} g_\varphi)(X^V, Y^V, Z^V) = 0,$

for all $X, Y, Z \in \Im_0^1(M)$.

Proof. We calculate Tachibana operator ϕ_{J_φ} applied to the pure metric g_φ. This operator is characterized by (2.4), from Lemma 3.1 we have

$$\begin{aligned}
1.\, (\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^H) &= (J_\varphi X^H) g_\varphi(Y^H, Z^H) - X^H g_\varphi(J_\varphi Y^H, Z^H) \\
&\quad + g_\varphi\big((L_{Y^H} J_\varphi) X^H, Z^H\big) + g_\varphi\big(Y^H, (L_{Z^H} J_\varphi) X^H\big) \\
&= (\varphi X)^H g_\varphi(Y^H, Z^H) - X^H g_\varphi((\varphi Y)^H, Z^H) \\
&\quad + g_\varphi\big(L_{Y^H} J_\varphi X^H - J_\varphi(L_{Y^H} X^H), Z^H\big) \\
&\quad + g_\varphi\big(Y^H, L_{Z^H} J_\varphi X^H - J_\varphi(L_{Z^H} X^H)\big) \\
&= (\varphi X) g(Y, Z) - X g(\varphi Y, Z) \\
&\quad + g_\varphi\big([Y^H, (\varphi X)^H] - J_\varphi[Y^H, X^H], Z^H\big) \\
&\quad + g_\varphi\big(Y^H, [Z^H, (\varphi X)^H] - J_\varphi[Z^H, X^H]\big) \\
&= (\varphi X) g(Y, Z) - X g(\varphi Y, Z) + g\big([Y, \varphi X] - \varphi[Y, X], Z\big) \\
&\quad + g\big(Y, [Z, \varphi X] - \varphi[Z, X]\big) \\
&= (\varphi X) g(Y, Z) - X g(\varphi Y, Z) + g\big((L_Y \varphi) X, Z\big) \\
&\quad + g\big(Y, (L_Z \varphi) X\big) \\
&= (\phi_\varphi g)(X, Y, Z).
\end{aligned}$$

Since (M^{2m}, φ, g) is a para-Kähler-Norden manifold, then $(\phi_\varphi g)(X, Y, Z) = 0$.

$$\begin{aligned}
2.\, (\phi_{J_\varphi} g_\varphi)(X^V, Y^H, Z^H) &= (J_\varphi X^V) g_\varphi(Y^H, Z^H) - X^V g_\varphi(J_\varphi Y^H, Z^H)) \\
&\quad + g_\varphi\big((L_{Y^H} J_\varphi) X^V, Z^H\big) + g_\varphi\big(Y^H, (L_{Z^H} J_\varphi) X^V\big) \\
&= (\varphi X)^V g_\varphi(Y^H, Z^H) - X^V g_\varphi((\varphi Y)^H, Z^H) \\
&\quad + g_\varphi\big([Y^H, (\varphi X)^V] - J_\varphi[Y^H, X^V], Z^H\big) \\
&\quad + g_\varphi\big(Y^H, [Z^H, (\varphi X)^V] - J_\varphi[Z^H, X^V]\big) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
3.\ (\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^H) &= (J_\varphi X^H) g_\varphi(Y^V, Z^H) - X^H g_\varphi(J_\varphi Y^V, Z^H) \\
&\quad + g_\varphi\big((L_{Y^V} J_\varphi) X^H, Z^H\big) + g_\varphi\big(Y^V, (L_{Z^H} J_\varphi) X^H\big) \\
&= g_\varphi\big([Y^V, (\varphi X)^H] - J_\varphi [Y^V, X^H], Z^H\big) \\
&\quad + g_\varphi\big(Y^V, [Z^H, (\varphi X)^H] - J_\varphi [Z^H, X^H]\big) \\
&= g_\varphi\big(Y^V, (-R(Z, \varphi X)u)^V + (\varphi R(Z, X)u)^V\big) \\
&= -g\big(R(Z, \varphi X)u, \varphi Y) + g\big(\varphi R(Z, X)u, \varphi Y\big).
\end{aligned}$$

Since the Riemann curvature R of a para-Kähler-Norden manifold is pure, then

$$\begin{aligned}
(\phi_{J_\varphi} g_\varphi)(X^H, Y^V, Z^H) &= -g\big(R(Z, X)u, Y) + g\big(R(Z, X)u, Y) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
4.\ (\phi_{J_\varphi} g_\varphi)(X^H, Y^H, Z^V) &= (J_\varphi X^H) g_\varphi(Y^H, Z^V) - X^H g_\varphi(J_\varphi Y^H, Z^V) \\
&\quad + g_\varphi\big((L_{Y^H} J_\varphi) X^H, Z^V\big) + g_\varphi\big(Y^H, (L_{Z^V} J_\varphi) X^H\big) \\
&= g_\varphi\big([Y^H, (\varphi X)^H] - J_\varphi [Y^H, X^H], Z^V\big) \\
&\quad + g_\varphi\big(Y^H, [Z^V, (\varphi X)^H] - J_\varphi [Z^V, X^H]\big) \\
&= g_\varphi\big((-R(Y, \varphi X)u)^V + (\varphi R(Y, X)u)^V, Z^V\big) \\
&= -g(R(Y, \varphi X)u, \varphi Z) + g(\varphi R(Y, X)u, \varphi Z) \\
&= -g(R(Y, X)u, Z) + g(R(Y, X)u, Z) \\
&= 0.
\end{aligned}$$

The other formulas are obtained by a similar calculation.

Therefore, we have the following result.

Theorem 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8)*,then the triple $(TM, J_\varphi, g_\varphi)$ is a para-Kähler-Norden manifold.*

Corollary 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure J_φ defined by* (4.8)*, then the triple $(TM, J_\varphi, g_\varphi)$ is a quasi-para-Kähler-Norden manifold.*

4.2. We Consider the tensor field $P_\varphi \in \Im_1^1(TM)$ defined by:

$$\begin{cases} P_\varphi X^H &= -(\varphi X)^H \\ P_\varphi X^V &= -(\varphi X)^V \end{cases} \tag{4.9}$$

for all $X \in \Im_0^1(M)$, satisfies the following:

1. $P\varphi = -J\varphi$.
2. g_φ is pure with respect to P_φ.
3. $\phi_{P_\varphi} g_\varphi = \phi_{J_\varphi} g_\varphi$.

Therefore we have the following results.

Theorem 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure P_φ defined by* (4.9)*, then the triple $(TM, P_\varphi, g_\varphi)$ is a para-Kähler-Norden manifold.*

4.3. We Consider the tensor field $Q_\varphi \in \Im_1^1(TM)$ defined by:

$$\begin{cases} Q_\varphi X^H &= (\varphi X)^V \\ Q_\varphi X^V &= (\varphi X)^H \end{cases} \tag{4.10}$$

for all $X \in \Im_0^1(M)$.

Lemma 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric. The couple (TM, Q_φ) is an almost para-complex manifold .*

Proof. By virtue of (4.10), we have

$$\begin{cases} Q_\varphi^2 X^H = Q_\varphi(Q_\varphi X^H) = Q_\varphi((\varphi X)^V) = (\varphi(\varphi X))^H = (\varphi^2 X)^H = X^H, \\ Q_\varphi^2 X^V = Q_\varphi(Q_\varphi X^V) = Q_\varphi((\varphi X)^H) = (\varphi(\varphi X))^V = (\varphi^2 X)^V = X^V, \end{cases}$$

for any $X \in \Im_0^1(M)$, then $Q_\varphi^2 = id_{TM}$.
Let $\{E_1, \cdots, E_m, E_{m+1}, \cdots, E_{2m}\}$ be local frame of eigenvectors on M such that $\varphi E_i = E_i$, $\varphi E_{m+i} = -E_{m+i}$, for all $i = \overline{1,m}$, then

$$TTM^+ = Span(E_1^H + E_1^V, \cdots, E_m^H + E_m^V, E_{m+1}^H - E_{m+1}^V, \cdots, E_{2m}^H - E_{2m}^V),$$

$$TTM^- = Span(E_1^H - E_1^V, \cdots, E_m^H - E_m^V, E_{m+1}^H + E_{m+1}^V, \cdots, E_{2m}^H + E_{2m}^V).$$

Theorem 4.4. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure Q_φ defined by* (4.10)*. The φ-Sasaki metric is never pure with respect to Q_φ i.e The triple $(TM, Q_\varphi, g_\varphi)$ is never an almost para-complex Norden manifold.*

4.4. We Consider the tensor field $F_\varphi \in \Im_1^1(TM)$ by

$$\begin{cases} F_\varphi X^H &= -(\varphi X)^H \\ F_\varphi X^V &= (\varphi X)^V \end{cases} \tag{4.11}$$

for all $X \in \Im_0^1(M)$.

Lemma 4.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric. The couple (TM, F_φ) is an almost para-complex manifold .*

Theorem 4.5. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by* (4.11)*. The triple $(TM, F_\varphi, g_\varphi)$ is an almost para-complex Norden manifold.*

Proof. With the same steps in the proof of Theorem 4.1, we get the results.

Proposition 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by* (4.11)*, then we get*

1. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^H, Z^H) = 0,$
2. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^H, Z^H) = 0,$
3. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^V, Z^H) = 2g\big(R(X,Z)Y, u\big),$
4. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^H, Z^V) = 2g\big(R(X,Y)Z, u\big),$
5. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^V, Z^H) = 0,$
6. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^H, Z^V) = 0,$
7. $(\phi_{F_\varphi} g_\varphi)(X^H, Y^V, Z^V) = 0,$
8. $(\phi_{F_\varphi} g_\varphi)(X^V, Y^V, Z^V) = 0,$

for all $X, Y, Z \in \Im_0^1(M)$, where R denote the curvature tensor of (M, g).

Proof. We calculate Tachibana operator ϕ_{F_φ} applied to the pure metric g_φ. With the same steps in the proof of Proposition 4.1, we get the results.

Theorem 4.6. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by (4.11). The triple (TM,F_φ,g_φ) is a para-Kähler-Norden manifold if and only if M is flat.*

Proof. For all $X,Y,Z\in\Im_0^1(M)$ and $h,k,l\in\{H,V\}$

$$(\phi_{F_\varphi}g_\varphi))(X^h,Y^k,Z^l)=0 \quad\Leftrightarrow\quad \begin{cases} g(R(X,Z)Y,u) &= 0\\ g(R(X,Y)Z,u) &= 0\end{cases}$$

$$\Leftrightarrow \quad R \;= 0.$$

Theorem 4.7. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure F_φ defined by (4.11). The triple (TM,F_φ,g_φ) is a quasi-para-Kähler-Norden manifold.*

Proof. For all $\widetilde{X},\widetilde{Y},\widetilde{Z}\in\Im_0^1(TM)$,

$$\underset{\widetilde{X},\widetilde{Y},\widetilde{Z}}{\sigma}(\phi_{J_\varphi}g_\varphi)(\widetilde{X},\widetilde{Y},\widetilde{Z})=(\phi_{J_\varphi}g_\varphi)(\widetilde{X},\widetilde{Y},\widetilde{Z})+(\phi_{J_\varphi}g_\varphi)(\widetilde{Y},\widetilde{Z},\widetilde{X})+(\phi_{J_\varphi}g_\varphi)(\widetilde{Z},\widetilde{X},\widetilde{Y})$$

By virtue of Proposition 4.1 we have

1. $\underset{X^H,Y^H,Z^H}{\sigma}(\phi_{J_\varphi}g_\varphi)(X^H,Y^H,Z^H) \;=\; 0,$
2. $\underset{X^V,Y^H,Z^H}{\sigma}(\phi_{J_\varphi}g_\varphi)(X^V,Y^H,Z^H) \;=\; 2g(R(Y,Z)X,u)+2g(R(Z,Y)X,u)=0,$
3. $\underset{X^V,Y^V,Z^H}{\sigma}(\phi_{J_\varphi}g_\varphi)(X^V,Y^V,Z^H) \;=\; 0,$
4. $\underset{X^V,Y^V,Z^V}{\sigma}(\phi_{J_\varphi}g_\varphi)(X^V,Y^V,Z^V) \;=\; 0,$

then, (TM,J_φ,g_φ) is a quasi-para-Kähler-Norden manifold.

4.5. We Consider the tensor field $K_\varphi\in\Im_1^1(TM)$ defined by:

$$\begin{cases} K_\varphi X^H &= (\varphi X)^H\\ K_\varphi X^V &= -(\varphi X)^V\end{cases} \tag{4.12}$$

for all $X\in\Im_0^1(M)$, satisfies the following:

1. $K\varphi=-F\varphi.$
2. g_φ is pure with respect to K_φ.
3. $\phi_{K_\varphi}g_\varphi=-\phi_{F_\varphi}g_\varphi.$

Therefore we have the following results.

Theorem 4.8. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost para-complex structure K_φ defined by* (4.12), *then we have*

1. *The triple (TM,K_φ,g_φ) is is a quasi-para-Kähler-Norden manifold.*
2. *The triple (TM,K_φ,g_φ) is a para-Kähler-Norden manifold if and only if M is flat.*

4.6. Now consider the almost product structure F_φ defined by (4.11). We define a tensor field S of type $(1,2)$ and linear connection $\widehat{\nabla}$ on TM by,

$$S(\widetilde{X},\widetilde{Y}) = \frac{1}{2}\big[(\widetilde{\nabla}_{F_\varphi\widetilde{Y}}F_\varphi)\widetilde{X} + F_\varphi\big((\widetilde{\nabla}_{\widetilde{Y}}F_\varphi)\widetilde{X}\big) - F_\varphi\big((\widetilde{\nabla}_{\widetilde{X}}F_\varphi)\widetilde{Y}\big)\big]. \tag{4.13}$$

$$\widehat{\nabla}_{\widetilde{X}}\widetilde{Y} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - S(\widetilde{X},\widetilde{Y}). \tag{4.14}$$

for all $\widetilde{X},\widetilde{Y}\in\Im^1_0(TM)$, where $\widetilde{\nabla}$ is the Levi-Civita connection of (TM,g_φ) given by Theorem 3.1. $\widehat{\nabla}$ is an almost product connection on TM (see [4, p.151] for more details).

Lemma 4.4. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, (TM,g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost product structure F_φ defined by* (4.11). *Then tensor field S is as follows,*

(1) $S(X^H,Y^H) = -\frac{1}{2}(R(X,Y)u)^V$,
(2) $S(X^H,Y^V) = \frac{1}{2}(R(\varphi u,Y)X)^H$,
(3) $S(X^V,Y^H) = -(R(\varphi u,X)Y)^H$,
(4) $S(X^V,Y^V) = 0$,

for all $X,Y\in\Im^1_0(M)$.

Proof. (1) Using (4.11) and (4.13), we have

$$\begin{aligned}
S(X^H,Y^H) &= \frac{1}{2}\big[(\widetilde{\nabla}_{F_\varphi Y^H}F_\varphi)X^H + F_\varphi\big((\widetilde{\nabla}_{Y^H}F_\varphi)X^H\big) - F_\varphi\big((\widetilde{\nabla}_{X^H}F_\varphi)Y^H\big)\big]\\
&= \frac{1}{2}\big[\widetilde{\nabla}_{(\varphi Y)^H}(\varphi X)^H + F_\varphi(\widetilde{\nabla}_{(\varphi Y)^H}X^H) - F_\varphi\big(\widetilde{\nabla}_{Y^H}(\varphi X)^H\big)\\
&\quad -\widetilde{\nabla}_{Y^H}X^H + F_\varphi\big(\widetilde{\nabla}_{X^H}(\varphi Y)^H\big) + \widetilde{\nabla}_{X^H}Y^H\big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\big[(\nabla_{\varphi Y}\varphi X)^H - \frac{1}{2}(R(\varphi Y,\varphi X)u)^V - (\varphi\nabla_{\varphi Y}X)^H\\
&\quad -\frac{1}{2}(\varphi R(\varphi Y,X)u)^V + (\varphi\nabla_Y\varphi X)^H + \frac{1}{2}(\varphi R(Y,\varphi X)u)^V\\
&\quad -(\nabla_Y X)^H + \frac{1}{2}(R(Y,X)u)^V - (\varphi\nabla_X\varphi Y)^H\\
&\quad -\frac{1}{2}(\varphi R(X,\varphi Y)u)^V + (\nabla_X Y)^H - \frac{1}{2}(R(X,Y)u)^V\big].
\end{aligned}$$

Using (2.6) we have

$$S(X^H, Y^H) = -\frac{1}{2}(R(X,Y)u)^V.$$

(2) By a similar calculation to (1), we have

$$\begin{aligned} S(X^H, Y^V) &= \frac{1}{2}\big[(\widetilde{\nabla}_{F_\varphi Y^V} F_\varphi)X^H + F_\varphi\big((\widetilde{\nabla}_{Y^V} F_\varphi)X^H\big) - F_\varphi\big((\widetilde{\nabla}_{X^H} F_\varphi)Y^V\big)\big] \\ &= \frac{1}{2}\big[- \widetilde{\nabla}_{(\varphi Y)^V}(\varphi X)^H - F_\varphi(\widetilde{\nabla}_{(\varphi Y)^V} X^H) - F_\varphi\big(\widetilde{\nabla}_{Y^V}(\varphi X)^H\big) \\ &\quad -\widetilde{\nabla}_{Y^V} X^H - F_\varphi\big(\widetilde{\nabla}_{X^H}(\varphi Y)^V\big) + \widetilde{\nabla}_{X^H} Y^V\big] \\ &= \frac{1}{2}\big[- \frac{1}{2}(R(\varphi u, \varphi Y)\varphi X)^H + \frac{1}{2}(\varphi R(\varphi u, \varphi Y)X)^H \\ &\quad +\frac{1}{2}(\varphi R(\varphi u, Y)\varphi X)^H - \frac{1}{2}(R(\varphi u, Y)X)^H \\ &\quad -(\varphi \nabla_X \varphi Y)^V + \frac{1}{2}(\varphi R(\varphi u, \varphi Y)X)^H \\ &\quad +(\nabla_X Y)^V + \frac{1}{2}(R(\varphi u, Y)X)^H\big]. \end{aligned}$$

Using (2.6) we get

$$S(X^H, Y^V) = \frac{1}{2}(R(\varphi u, Y)X)^H.$$

The other formulas are obtained by a similar calculation.

Theorem 4.9. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost product structure F_φ defined by* (4.11). *Then the almost product connection $\widehat{\nabla}$ defined by* (4.14) *is as follows,*

$$\begin{aligned} (1)\ \widehat{\nabla}_{X^H} Y^H &= (\nabla_X Y)^H, \\ (2)\ \widehat{\nabla}_{X^H} Y^V &= (\nabla_X Y)^V, \\ (3)\ \widehat{\nabla}_{X^V} Y^H &= \frac{3}{2}(R(\varphi u, X)Y)^H, \\ (4)\ \widehat{\nabla}_{X^V} Y^V &= 0, \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$.

Proof. The proof of Theorem 4.9 follows directly from Theorem 3.1, Lemma 4.4 and formula (4.14).

Lemma 4.5. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, (TM, g_φ) be its tangent bundle equipped with the φ-Sasaki metric and the almost product structure F_φ defined by*

(4.11) *and* $\widehat{T}$ *denote the torsion tensor of* $\widehat{\nabla}$*, then we have:*

$$\begin{array}{rcl} (1)\ \widehat{T}(X^H,Y^H) & = & (R(X,Y)u)^V, \\ (2)\ \widehat{T}(X^H,Y^V) & = & -\frac{3}{2}(R(\varphi u,Y)X)^H, \\ (3)\ \widehat{T}(X^V,Y^H) & = & \frac{3}{2}(R(\varphi u,X)Y)^H, \\ (4)\ \widehat{T}(X^V,Y^V) & = & 0, \end{array}$$

for all $X,Y \in \Im^1_0(M)$*.*

Proof. The proof of Lemma 4.5 follows directly from Lemma 4.4 and formula

$$\begin{array}{rcl} \widehat{T}(\widetilde{X},\widetilde{Y}) & = & \widehat{\nabla}_{\widetilde{X}}\widetilde{Y} - \widehat{\nabla}_{\widetilde{Y}}\widetilde{X} - [\widetilde{X},\widetilde{Y}] \\ & = & S(\widetilde{Y},\widetilde{X}) - S(\widetilde{X},\widetilde{Y}) \end{array}$$

for all $\widetilde{X},\widetilde{Y} \in \Im^1_0(TM)$.

From Lemma 4.5 we obtain

Theorem 4.10. *Let* (M^{2m},φ,g) *be a para-Kähler-Norden manifold,* (TM,g_φ) *be its tangent bundle equipped with the* φ*-Sasaki metric and the almost product structure* F_φ *defined by* (4.11)*, then* $\widehat{\nabla}$ *is symmetric if and only if* M *is flat.*

Acknowledgments. The author would like to thank the referees for useful comments and their helpful suggestions that have improved the quality of this paper.

References

[1] Alekseevsky, D.V., Medori, C., & Tomassini, A. (2009). Para-Kähler Einstein metrics on homogeneous manifolds. C. R. Acad. Sci. Paris, Ser. I, 347, 69-72.

[2] Cruceanu, V., Gadea, P.M., & Munoz Masque, J. (1995). Para-Hermitian and para-Kähler manifolds. Quaderni Inst. Mat. Univ. Messina, 1 , 1-72.

[3] Cruceanu, V., Fortuny, P. & Gadea, P. M. (1996). A survey on paracomplex geometry. Rocky Mountain J. Math., 26, 83–115.

[4] De León, M., & Rodrigues, P. R. (1989). Methods of Differential Geometry in Analytical Mechanics. North-Holland Mathematics Studies.

[5] Ganchev, G. T., & Borisov, A. V. (1986). Note on the almost complex manifolds with a Norden metric. C. R. Acad. Bulgare Sci., 39(5), 31-34.

[6] Gezer, A., & Ozkan, M. (2014). Notes on the tangent bundle with deformed complete lift metric. Turkish J. Math., 38, 1038-1049.

[7] Gezer, A.,& Altunbas, M. (2016). On the Rescaled Riemannian Metric of Cheeger Gromoll Type on the Cotangent Bundle. Hacet. J. Math. Stat., 45(2), 355-365.

[8] Iscan, M., & Salimov, A.A. (2009). On Kähler–Norden manifolds. Proc. Indian Acad. Sci. Math. Sci., 119(1), 71-80.

[9] Libermann, P. (1952). Sur les structures presque paracomplexes. C. R. Acad. Sci. Paris, 234, 2517-2519.

[10] Manev, M., & Mekerov, D. (2008). On Lie groups as quasi-Kähler manifolds with Killing Norden metric. Adv. Geom., 8(3), 343-352.

[11] Ocak, F., & Salimov, A.A. (2014). Geometry of the cotangent bundle with Sasakian metrics and its applications. Proc. Proc. Indian Acad. Sci. Math. Sci., 124(3), 427-436.

[12] Patterson, E.M. (1954). Riemann extensions which have Kähler metrics. Proc. Roy. Soc. Edinburgh Sect. A, 64, 113-126.

[13] Salimov, A.A., Iscan, M., & Etayo, F. (2007) Para-holomorphic B-manifold and its properties. Topology Appl., 154(4), 925-933.

[14] Salimov, A.A., Iscan, M., & Akbulut, K. (2010). Notes on para-Norden-Walker 4-manifolds. Int. J. Geom. Methods Mod. Phys., 7(8), 1331–1347.

[15] Salimov, A.A., & Agca, F. (2011). Some Properties of Sasakian Metrics in Cotangent Bundles. Mediterr. J. Math., 8(2), 243-255.

[16] Salimov, A.A., Gezer, A., & Iscan, M. (2012). On para-Kähler-Norden structures on the tangent bundles. Ann. Polon. Math., 103(3), 247–261.

[17] Yano, K., & Ako, M. (1968). On certain operators associated with tensor field. Kodai Math. Sem. Rep, 20 , 414-436.

[18] Yano, K., & Ishihara, S. (1973). Tangent and Cotangent Bundles. Marcel Dekker. INC. New York.

University Center Ahmed Zabana-Relizane, Dept. of mathematics, 48000, Relizane-Algeria

VOLUME 4 ISSUE 2 September 2021
ISSN 2636-7467 https://www.journalmim.com

Contents