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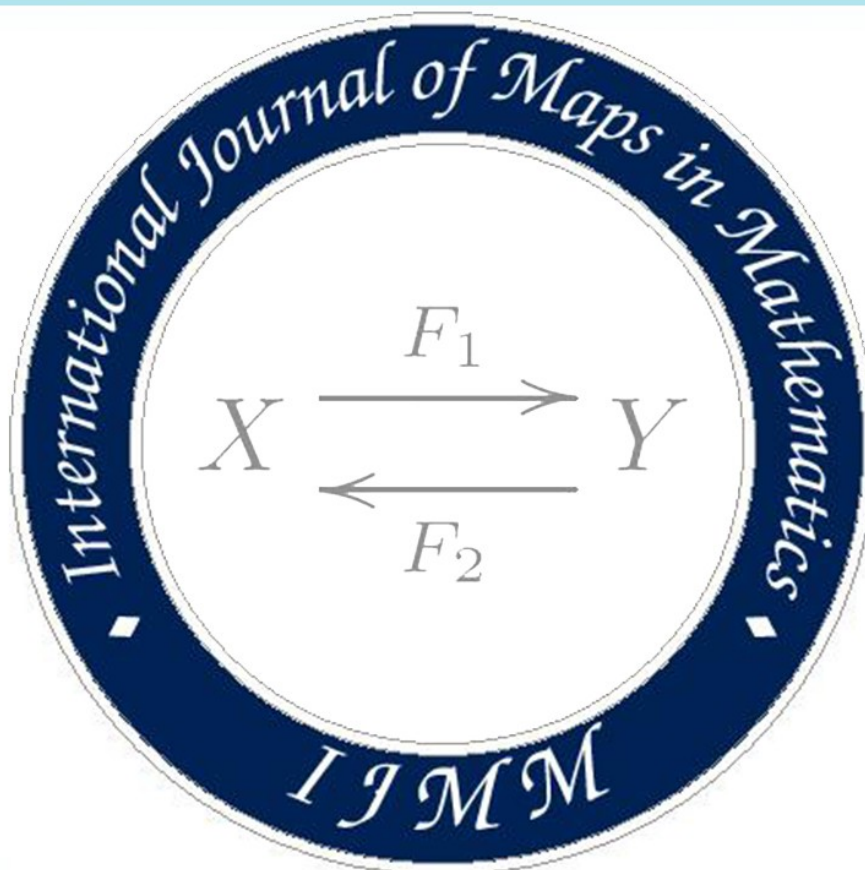
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Bayram SAHIN

Ege University, Izmir, TURKEY



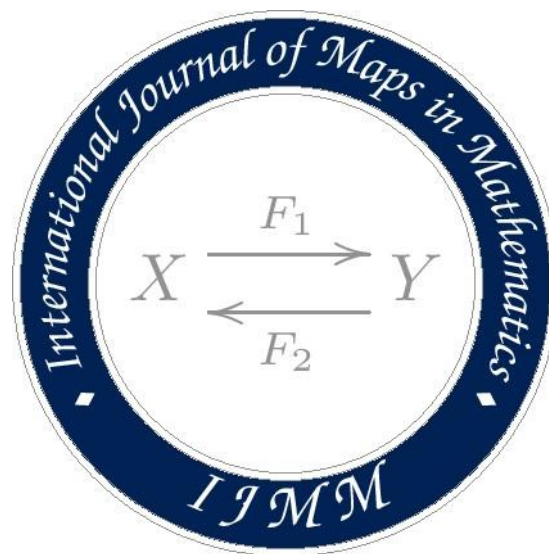
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Bayram Sahin
Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey
journalofmapsinmathematics@gmail.com

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arif.gursoy@ege.edu.tr

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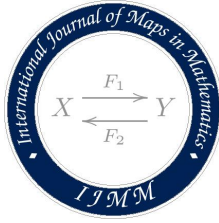
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Department of Mathematics, Faculty of Science, Ege
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SOME RESULTS AND EXAMPLES OF THE f -BIHARMONIC MAPS ON WARPED PRODUCT MANIFOLDS

ABDELHAMID BOULAL  AND SEDDIK OUAKKAS  *

ABSTRACT. In this paper, we present some constructions of f -biharmonic functions on the warped product. We studied particular cases and we give some examples of f -biharmonic maps.

1. INTRODUCTION

The smooth map $\phi : (M^m, g) \rightarrow (N^n, h)$ between two Riemannian manifolds is said to be harmonic if it is a critical point of the energy functional :

$$E(\phi) = \frac{1}{2} \int_M |d\phi|^2 dv_g.$$

ϕ is harmonic if it satisfies the Euler-Lagrange equation for the energy functional :

$$\tau(\phi) = \text{Tr}_g \nabla d\phi = 0,$$

$\tau(\phi)$ is called the tension field of ϕ . As a generalization, we define the bi-energy functional of ϕ :

$$E_2(\phi) = \frac{1}{2} \int_M |\tau(\phi)|^2 dv_g.$$

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* Corresponding author

ϕ is said to be biharmonic if and only if

$$\tau_2(\phi) = -Tr_g \left(\nabla^\phi \right)^2 \tau(\phi) - Tr_g R^N(\tau(\phi), d\phi) d\phi = 0.$$

$\tau_2(\phi)$ is called the bi-tension field of ϕ .

Let $f \in C^\infty(M)$ be a positive function, We respectively define the f -energy and the f -bienergy functional of ϕ by

$$E_f(\phi) = \frac{1}{2} \int_M f |d\phi|^2 dv_g$$

and

$$E_{2,f}(\phi) = \frac{1}{2} \int_M f |\tau(\phi)|^2 dv_g.$$

ϕ is said to be f -harmonic if and only if

$$\tau_f(\phi) = Tr_g \nabla f \tau(\phi) = 0, \quad (1.1)$$

and it is said to be f -biharmonic if and only if

$$\tau_{2,f}(\phi) = -Tr_g \left(\nabla^\phi \right)^2 f \tau(\phi) - Tr_g R^N(f \tau(\phi), d\phi) d\phi = 0. \quad (1.2)$$

$\tau_f(\phi)$ and $\tau_{2,f}(\phi)$ are called respectively the f -tension and f -bitension field of ϕ . Contrary to the fact that any harmonic map is biharmonic, an f -harmonic map is not necessarily f -biharmonic. By considering $(M^m, g), (N^n, h)$ two Riemannian manifolds and α a positive function on M , we recall that the warped product of M and N noted by $(M \times_\alpha N, G_\alpha)$ is the Riemannian manifold, where the Riemannian metric G_α is defined by

$$G_\alpha(X, Y) = g(d\pi(X), d\pi(Y)) + (\alpha \circ \pi)^2 h(d\eta(X), d\eta(Y)), \quad (1.3)$$

for all $X = (X_1, X_2), Y = (Y_1, Y_2) \in \Gamma(T(M \times N))$, $\pi : M \times N \rightarrow M$ and $\eta : M \times N \rightarrow N$ are respectively the first and the second projection. The Levi-Civita connection on $(M \times N, G)$ and $(M \times_\alpha N, G_\alpha)$ are noted respectively by ∇ and $\tilde{\nabla}$, the relation between $\tilde{\nabla}$ and ∇ is given by

$$\tilde{\nabla}_X Y = \nabla_X Y + X_1(\ln \alpha)(0, Y_2) + Y_1(\ln \alpha)(0, X_2) - \alpha^2 h(X_2, Y_2)(grad \ln \alpha, 0). \quad (1.4)$$

Similarly, the relation between the curvature tensor fields of $(M \times_\alpha N, G_\alpha)$ and $(M \times N, G)$ is given by

$$\begin{aligned} \tilde{R}(X, Y) &= R(X, Y) + (\nabla_{Y_1} grad \ln \alpha + Y_1(\ln \alpha) grad \ln \alpha, 0) \wedge_{G_\alpha} (0, X_2) \\ &\quad - (\nabla_{X_1} grad \ln \alpha + X_1(\ln \alpha) grad \ln \alpha, 0) \wedge_{G_\alpha} (0, Y_2) \\ &\quad - |grad \ln \alpha|^2 (0, X_2) \wedge_{G_\alpha} (0, Y_2), \end{aligned}$$

where

$$(X \wedge_{G_\alpha} Y) Z = G_\alpha(Z, Y) X - G_\alpha(Z, X) Y$$

for all $X, Y, Z \in \Gamma(T(M \times N))$, where $X = (X_1, X_2)$ and $Y = (Y_1, Y_2)$. In [12], The author studied the f -harmonicity on the doubly warped product manifold in order to construct a non-trivial f -harmonic map, he deals in particular with the case of projection. The author in [13] describes a new method for constructing f -biharmonic maps, this construction allowed him to give some examples of f -biharmonic map. In [5], the authors studied f -harmonic morphisms which are a subclass of f -harmonic maps. In [4], the authors studied biharmonic maps between warped products, in particular they gave the condition for the biharmonicity of the inclusion and of the projection. Our objective in this paper is to present the condition of f -biharmonic using the warped product of two Riemannian manifolds. In the first part of this paper, we give the conditions for the f -biharmonic of the maps $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ and $\Psi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ and $\Psi(x, y) = \psi(y)$ (Theorem 2.1 and Theorem 2.2), with this classification, we give some special cases and we construct an examples of f -biharmonic map. The study of the f -biharmonic of the identity maps $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$ and $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$ is presented in the second part of this paper (Theorem 2.3 and Theorem 2.4), where we give some particular results and we construct some examples of f -biharmonic maps.

2. THE MAIN RESULTS

In this section, we consider $\{e_i\}_{1 \leq i \leq m}$ an orthonormal frame on M and $\{f_j\}_{1 \leq j \leq n}$ an orthonormal frame on N . Then an orthonormal frame on $M \times_\alpha N$ is given by $\{(e_i, 0), \frac{1}{\alpha}(0, f_j)\}$. As a first result, we will study the f -biharmonic of the map $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$. We start by calculating the f -tension field of Φ .

Proposition 2.1. *The f -tension field of the map $\Phi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ is given by*

$$\tau_f(\Phi) = f(\tau(\phi) + d\phi(\text{grad} \ln f) + nd\phi(\text{grad} \ln \alpha)), \quad (2.5)$$

where $\phi : (M^m, g) \longrightarrow (P^p, k)$ is a smooth map.

Proof. By definition, we have

$$\begin{aligned}\tau_f(\Phi) &= Tr_{G_\alpha} \nabla f d\tilde{\phi} \\ &= \nabla_{(e_i,0)}^\Phi f d\Phi(e_i,0) + \frac{1}{\alpha^2} \nabla_{(0,f_j)}^\Phi f d\Phi(0,f_j) \\ &\quad - f d\Phi(\Phi_{(e_i,0)}(e_i,0)) - \frac{f}{\alpha^2} \Phi(\tilde{\nabla}_{(0,f_j)}(0,f_j)).\end{aligned}$$

Using the fact that $d\Phi(e_i,0) = d\phi(e_i)$ and $d\Phi(0,f_j) = 0$, a simple calculation gives

$$\nabla_{(e_i,0)}^\Phi f d\Phi(e_i,0) = f \nabla_{e_i}^\phi d\phi(e_i) + f d\phi(\text{grad} \ln f)$$

and

$$\nabla_{(0,f_j)}^\Phi d\Phi(0,f_j) = 0.$$

By using the equation (1.4), we deduce that

$$\tilde{\nabla}_{(e_i,0)}(e_i,0) = (\nabla_{e_i} e_i, 0)$$

and

$$\tilde{\nabla}_{(0,f_j)}(0,f_j) = (0, \nabla_{f_j} f_j) - n\alpha^2(\text{grad} \ln \alpha, 0).$$

It follows that

$$\tau_f(\Phi) = f \nabla_{e_i}^\phi d\phi(e_i) - f d\phi(\nabla_{e_i}^M e_i) + f d\phi(\text{grad} \ln f) + n f d\phi(\text{grad} \ln \alpha)$$

then, we obtain

$$\tau_f(\Phi) = f(\tau(\phi) + d\phi(\text{grad} \ln f) + n d\phi(\text{grad} \ln \alpha)).$$

Remark 2.1. In the case where $\phi = Id_M$, we conclude that the first projection $P_1 : (M^m \times_\alpha N^n, G_\alpha) \rightarrow (M^m, g)$ is f -harmonic if and only if the function $f\alpha^n$ is constant.

The expression of the f -bitension field of the map Φ is given by the following Theorem.

Theorem 2.1. The f -bitension field of the map $\Phi : (M^m \times_\alpha N^n, G_\alpha) \rightarrow (P^p, k)$ defined by $\Phi(x, y) = \phi(x)$ is given by the following formula

$$\begin{aligned}\tau_{2,f}(\Phi) &= f\tau_2(\phi) - nf \left(Tr_g \left(\nabla^\phi \right)^2 d\phi(\text{grad} \ln \alpha) + Tr_g R^P(d\phi(\text{grad} \ln \alpha), d\phi) d\phi \right) \\ &\quad - f \left(2\nabla_{\text{grad} \ln f}^\phi \tau(\phi) + n\nabla_{\text{grad} \ln \alpha}^\phi \tau(\phi) \right) \\ &\quad - nf \left(2\nabla_{\text{grad} \ln f}^\phi d\phi(\text{grad} \ln \alpha) + n\nabla_{\text{grad} \ln \alpha}^\phi d\phi(\text{grad} \ln \alpha) \right) \\ &\quad - f \left(|\text{grad} \ln f|^2 + \Delta \ln f + nd \ln f(\text{grad} \ln \alpha) \right) \tau(\phi) \\ &\quad - nf \left(|\text{grad} \ln f|^2 + \Delta \ln f + nd \ln f(\text{grad} \ln \alpha) \right) d\phi(\text{grad} \ln \alpha).\end{aligned}\tag{2.6}$$

Proof. By definition of the f -bitension field, we have

$$\tau_{2,f}(\Phi) = -Tr_{G_\alpha}(\nabla^\Phi)^2 f\tau(\Phi) - fTr_{G_\alpha}R^P(\tau(\Phi), d\Phi)d\Phi, \quad (2.7)$$

where

$$\tau(\Phi) = \tau(\phi) + nd\phi(grad \ln \alpha).$$

Looking at the first term $Tr_{G_\alpha}(\nabla^\Phi)^2 f\tau(\Phi)$, we have

$$\begin{aligned} Tr_{G_\alpha}(\nabla^\Phi)^2 f\tau(\Phi) &= \nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi f\tau(\Phi) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi f\tau(\Phi) \\ &\quad + \frac{1}{\alpha^2} \nabla_{(0,f_j)}^\Phi \nabla_{(0,f_j)}^\Phi f\tau(\Phi) - \frac{1}{\alpha^2} \nabla_{\tilde{\nabla}_{(0,f_j)}^\Phi(0,f_j)}^\Phi f\tau(\Phi) \end{aligned} \quad (2.8)$$

We will give a detailed calculation of this last equation. For the term $\nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi f\tau(\Phi) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi f\tau(\Phi)$, we have

$$\begin{aligned} &\nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi f\tau(\Phi) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi f\tau(\Phi) \\ &= \nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi f\tau(\phi) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi f\tau(\phi) \\ &\quad + n\nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi fd\phi(grad \ln \alpha) - n\nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi fd\phi(grad \ln \alpha). \end{aligned}$$

A simple calculation gives

$$\begin{aligned} &\nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi f\tau(\phi) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi f\tau(\phi) \\ &= fTr_g(\nabla^\phi)^2 \tau(\phi) + 2f\nabla_{grad \ln f}^\phi \tau(\phi) \\ &\quad + f(|grad \ln f|^2 + \Delta \ln f) \tau(\phi) \end{aligned}$$

and

$$\begin{aligned} &\nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi fd\phi(grad \ln \alpha) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi fd\phi(grad \ln \alpha) \\ &= fTr_g(\nabla^\phi)^2 d\phi(grad \ln \alpha) + 2f\nabla_{grad \ln f}^\phi d\phi(grad \ln \alpha) \\ &\quad + f(|grad \ln f|^2 + \Delta \ln f) d\phi(grad \ln \alpha). \end{aligned}$$

Then

$$\begin{aligned} &\nabla_{(e_i,0)}^\Phi \nabla_{(e_i,0)}^\Phi f\tau(\tilde{\phi}) - \nabla_{\tilde{\nabla}_{(e_i,0)}^\Phi(e_i,0)}^\Phi f\tau(\Phi) \\ &= fTr_g(\nabla^\phi)^2 \tau(\phi) + nfTr_g(\nabla^\phi)^2 d\phi(grad \ln \alpha) \\ &\quad + 2f\nabla_{grad \ln f}^\phi \tau(\phi) + 2nf\nabla_{grad \ln f}^\phi d\phi(grad \ln \alpha) \\ &\quad + f(|grad \ln f|^2 + \Delta \ln f) \tau(\phi) \\ &\quad + nf(|grad \ln f|^2 + \Delta \ln f) d\phi(grad \ln \alpha). \end{aligned} \quad (2.9)$$

In the same way, we obtain

$$\nabla_{(0,f_j)}^\Phi \nabla_{(0,f_j)}^\Phi f \tau(\Phi) = 0$$

and

$$\begin{aligned} \nabla_{\nabla_{(0,f_j)}^\Phi}^\Phi f \tau(\Phi) &= -n\alpha^2 \nabla_{grad \ln \alpha}^\phi f \tau(\phi) - n^2 \alpha^2 \nabla_{grad \ln \alpha}^\phi f d\phi(grad \ln \alpha) \\ &= -nf\alpha^2 \nabla_{grad \ln \alpha}^\phi \tau(\phi) - n^2 f \alpha^2 \nabla_{grad \ln \alpha}^\phi d\phi(grad \ln \alpha) \\ &\quad - nf\alpha^2 d \ln f(grad \ln \alpha) \tau(\phi) \\ &\quad - n^2 f \alpha^2 d \ln f(grad \ln \alpha) d\phi(grad \ln \alpha). \end{aligned} \quad (2.10)$$

By replacing (2.9) and (2.10) in (2.8), we deduce that

$$\begin{aligned} Tr_{G_\alpha} (\nabla^\Phi)^2 f \tau(\Phi) &= f Tr_g (\nabla^\phi)^2 \tau(\phi) + n f Tr_g (\nabla^\phi)^2 d\phi(grad \ln \alpha) \\ &\quad + 2f \nabla_{grad \ln f}^\phi \tau(\phi) + n f \nabla_{grad \ln \alpha}^\phi \tau(\phi) \\ &\quad + 2n f \nabla_{grad \ln f}^\phi d\phi(grad \ln \alpha) + n^2 f \nabla_{grad \ln \alpha}^\phi d\phi(grad \ln \alpha) \\ &\quad + f (|grad \ln f|^2 + \Delta \ln f + n d \ln f(grad \ln \alpha)) \tau(\phi) \\ &\quad + n f (|grad \ln f|^2 + \Delta \ln f + n d \ln f(grad \ln \alpha)) d\phi(grad \ln \alpha). \end{aligned} \quad (2.11)$$

Finally, the calculation of term $Tr_{G_\alpha} R^P(\tau(\Phi), d\Phi) d\Phi$ is simple, we have

$$\begin{aligned} Tr_{G_\alpha} R^P(\tau(\Phi), d\Phi) d\Phi &= R^P(\tau(\Phi), d\Phi(e_i, 0)) d\Phi(e_i, 0) \\ &\quad + \frac{1}{\alpha^2} R^P(\tau(\Phi), d\Phi(0, f_j)) d\Phi(0, f_j) \\ &= R^P(\tau(\Phi), d\Phi(e_i, 0)) d\Phi(e_i, 0) \\ &= R^P(\tau(\phi), d\phi(e_i)) d\phi(e_i) \\ &\quad + n R^P(d\phi(grad \ln \alpha), d\phi(e_i)) d\phi(e_i). \end{aligned}$$

It follows that

$$Tr_{G_\alpha} R^P(\tau(\Phi), d\Phi) d\Phi = Tr_g R^P(\tau(\phi), d\phi) d\phi + n Tr_g R^P(d\phi(grad \ln \alpha), d\phi) d\phi. \quad (2.12)$$

By substituting (2.11) and (2.12) in (2.7), we arrive at the following formula

$$\begin{aligned}
\tau_{2,f}(\Phi) &= f\tau_2(\phi) - nf \left(\text{Tr}_g \left(\nabla^\phi \right)^2 d\phi(\text{grad } \ln \alpha) + \text{Tr}_g R^P(d\phi(\text{grad } \ln \alpha), d\phi) d\phi \right) \\
&\quad - f \left(2\nabla_{\text{grad } \ln f}^\phi \tau(\phi) + n\nabla_{\text{grad } \ln \alpha}^\phi \tau(\phi) \right) \\
&\quad - nf \left(2\nabla_{\text{grad } \ln f}^\phi d\phi(\text{grad } \ln \alpha) + n\nabla_{\text{grad } \ln \alpha}^\phi d\phi(\text{grad } \ln \alpha) \right) \\
&\quad - f \left(|\text{grad } \ln f|^2 + \Delta \ln f + n d \ln f(\text{grad } \ln \alpha) \right) \tau(\phi) \\
&\quad - nf \left(|\text{grad } \ln f|^2 + \Delta \ln f + n d \ln f(\text{grad } \ln \alpha) \right) d\phi(\text{grad } \ln \alpha).
\end{aligned}$$

Theorem 2.1 allows us to establish the f -biharmonicity condition of Φ .

Remark 2.2. *The map Φ is f -biharmonic if and only if*

$$\begin{aligned}
&\tau_2(\phi) - n \left(\text{Tr}_g \left(\nabla^\phi \right)^2 d\phi(\text{grad } \ln \alpha) + \text{Tr}_g R^P(d\phi(\text{grad } \ln \alpha), d\phi) d\phi \right) \\
&\quad - \left(|\text{grad } \ln f|^2 + \Delta \ln f + n d \ln f(\text{grad } \ln \alpha) \right) \tau(\phi) - \left(2\nabla_{\text{grad } \ln f}^\phi \tau(\phi) + n\nabla_{\text{grad } \ln \alpha}^\phi \tau(\phi) \right) \\
&\quad - n \left(|\text{grad } \ln f|^2 + \Delta \ln f + n d \ln f(\text{grad } \ln \alpha) \right) d\phi(\text{grad } \ln \alpha) \\
&\quad - n \left(2\nabla_{\text{grad } \ln f}^\phi d\phi(\text{grad } \ln \alpha) + n\nabla_{\text{grad } \ln \alpha}^\phi d\phi(\text{grad } \ln \alpha) \right) = 0.
\end{aligned}$$

And in the case where ϕ is harmonic, we obtain

Corollary 2.1. *If the map $\phi : (M^m, g) \rightarrow (P^p, k)$ is harmonic, we deduce that Φ is f -biharmonic if and only if*

$$\begin{aligned}
&\text{Tr}_g \left(\nabla^\phi \right)^2 d\phi(\text{grad } \ln \alpha) + \text{Tr}_g R^P(d\phi(\text{grad } \ln \alpha), d\phi) d\phi \\
&\quad + 2\nabla_{\text{grad } \ln f}^\phi d\phi(\text{grad } \ln \alpha) + n\nabla_{\text{grad } \ln \alpha}^\phi d\phi(\text{grad } \ln \alpha) \\
&\quad + \left(|\text{grad } \ln f|^2 + \Delta \ln f + n d \ln f(\text{grad } \ln \alpha) \right) d\phi(\text{grad } \ln \alpha) = 0.
\end{aligned}$$

In the particular case where $f = \alpha$, the map Φ is f -biharmonic if and only if

$$\begin{aligned}
&\text{Tr}_g \left(\nabla^\phi \right)^2 d\phi(\text{grad } \ln f) + \text{Tr}_g R^P(d\phi(\text{grad } \ln f), d\phi) d\phi \\
&\quad + \left((n+1) |\text{grad } \ln f|^2 + \Delta \ln f \right) d\phi(\text{grad } \ln f) \\
&\quad + (n+2) \nabla_{\text{grad } \ln f}^\phi d\phi(\text{grad } \ln f) = 0.
\end{aligned}$$

The first projection corresponds to the case where $\phi = Id_M$, its f -biharmonicity is given by

Corollary 2.2. *The first projection $P_1 : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m, g)$ defined by $P_1(x, y) = x$ is f -biharmonic if and only if*

$$\begin{aligned} & \text{grad} \Delta \ln \alpha + \left(|\text{grad} \ln f|^2 + \Delta \ln f + n d \ln f (\text{grad} \ln \alpha) \right) \text{grad} \ln \alpha \\ & + 2 \nabla_{\text{grad} \ln f}^\phi \text{grad} \ln \alpha + \frac{n}{2} \text{grad} \left(|\text{grad} \ln \alpha|^2 \right) + 2 \text{Ricci} (\text{grad} \ln \alpha) = 0. \end{aligned}$$

If $f = \alpha$, the f -biharmonicity condition of the first projection $P_1 : (M^m \times_f N^n, G_f) \longrightarrow (M^m, g)$ is given by the following equation

$$\begin{aligned} & \text{grad} \Delta \ln f + \left((n+1) |\text{grad} \ln f|^2 + \Delta \ln f \right) \text{grad} \ln f \\ & + \frac{(n+2)}{2} \text{grad} \left(|\text{grad} \ln f|^2 \right) + 2 \text{Ricci} (\text{grad} \ln f) = 0. \end{aligned}$$

Corollary 2.2 allows us to give an example of a f -biharmonic map.

Example 2.1. *Let $P_1 : \mathbf{R}_+^* \times_\alpha N^n \longrightarrow \mathbf{R}_+^*$ the first projection. By Corollary 2.2, P_1 is f -biharmonic if and only if the functions $f_1(t) = (\ln f(t))'$ and $\alpha_1(t) = (\ln \alpha(t))'$ satisfy the following differential equation*

$$f_1' \alpha_1 + f_1^2 \alpha_1 + n f_1 \alpha_1^2 + \alpha_1'' + n \alpha_1 \alpha_1' + 2 f_1 \alpha_1' = 0.$$

We will look for solutions of type $f_1(t) = \frac{a}{t}$ and $\alpha_1(t) = \frac{b}{t}$ where $a, b \in \mathbf{R}^*$, then the first projection P_1 is f -biharmonic if and only if

$$(a-1)(a+nb-2) = 0.$$

We distinguish two cases :

- (1) If $a = 1$, P_1 is f -biharmonic if and only if $f(t) = C_1 t$ and $\alpha(t) = C_2 t^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.
- (2) If $a = 2 - nb$, P_1 is f -biharmonic if and only if $f(t) = C_1 t^{2-nb}$ and $\alpha(t) = C_2 t^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.

Now we will determine the f -biharmonicity condition of the map $\Psi : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (P^p, k)$ defined by $\Psi(x, y) = \psi(y)$ where $\psi : (N^n, g) \longrightarrow (P^p, k)$ is a smooth map.

Theorem 2.2. *The f -tension field and the f -bitension field of Ψ are given by*

$$\tau_f(\Psi) = \left(\frac{f}{\alpha^2} \circ \pi \right) \tau(\psi) \tag{2.13}$$

and

$$\begin{aligned}
\tau_{2,f}(\Psi) &= \left(\frac{f}{\alpha^4} \circ \pi \right) \tau_2(\psi) \\
&\quad - \left(\frac{f}{\alpha^2} \left(\Delta \ln f + |\text{grad} \ln f|^2 \right) \circ \pi \right) \tau(\psi) \\
&\quad + \left(\frac{f}{\alpha^2} \left(2\Delta \ln \alpha + (2n-4) |\text{grad} \ln \alpha|^2 \right) \circ \pi \right) \tau(\psi) \\
&\quad - (n-4) \left(\frac{f}{\alpha^2} (d \ln f (\text{grad} \ln \alpha)) \circ \pi \right) \tau(\psi).
\end{aligned} \tag{2.14}$$

Proof. Let's start with the calculation of the f -tension field of Ψ , we have

$$\begin{aligned}
\tau_f(\Psi) &= \text{Tr}_{G_\alpha} \nabla f d\Psi \\
&= \nabla_{(e_i,0)}^\Psi f d\Psi(e_i, 0) - f d\Psi \left(\tilde{\nabla}_{(e_i,0)}(e_i, 0) \right) \\
&\quad + \left(\frac{f}{\alpha^2} \circ \pi \right) \nabla_{(0,f_j)}^\Psi d\Psi(0, f_j) - \left(\frac{f}{\alpha^2} \circ \pi \right) d\Psi \left(\tilde{\nabla}_{(0,f_j)}(0, f_j) \right).
\end{aligned}$$

By equation (1.4), we deduce that

$$\tau_f(\Psi) = \left(\frac{f}{\alpha^2} \circ \pi \right) \nabla_{f_j}^\psi d\psi(f_j) - \left(\frac{f}{\alpha^2} \circ \pi \right) d\psi(\nabla_{f_j} f_j),$$

then

$$\tau_f(\Psi) = \left(\frac{f}{\alpha^2} \circ \pi \right) \tau(\psi).$$

It follows that Ψ is f -harmonic if and only if ψ is harmonic. Let's go to the calculation of $\tau_{2,f}(\Psi)$, we have

$$\tau_{2,f}(\Psi) = -\text{Tr}_{G_\alpha} (\nabla^\Psi)^2 f \tau(\Psi) - \text{Tr}_{G_\alpha} R^P(f \tau(\Psi), d\Psi) d\Psi, \tag{2.15}$$

where

$$\tau(\Psi) = \left(\frac{1}{\alpha^2} \circ \pi \right) \tau(\psi).$$

By definition of $\text{Tr}_{G_\alpha} (\nabla^\Psi)^2 f \tau(\Psi)$, we have

$$\begin{aligned}
\text{Tr}_{G_\alpha} (\nabla^\Psi)^2 f \tau(\Psi) &= \nabla_{(e_i,0)}^\Psi \nabla_{(e_i,0)}^\Psi f \tau(\Psi) - \nabla_{\tilde{\nabla}_{(e_i,0)}(e_i,0)}^\Psi f \tau(\Psi) \\
&\quad + \left(\frac{1}{\alpha^2} \circ \pi \right) \nabla_{(0,f_j)}^\Psi \nabla_{(0,f_j)}^\Psi f \tau(\Psi) - \left(\frac{1}{\alpha^2} \circ \pi \right) \nabla_{\tilde{\nabla}_{(0,f_j)}(0,f_j)}^\Psi f \tau(\Psi).
\end{aligned}$$

The calculation of the terms of this equation gives us

$$\begin{aligned}
&\nabla_{(e_i,0)}^\Psi \nabla_{(e_i,0)}^\Psi f \tau(\Psi) - \nabla_{\tilde{\nabla}_{(e_i,0)}(e_i,0)}^\Psi f \tau(\Psi) \\
&= \left(\frac{f}{\alpha^2} \left(\Delta \ln f + |\text{grad} \ln f|^2 - 2\Delta \ln \alpha + 4 |\text{grad} \ln \alpha|^2 - 4d \ln f (\text{grad} \ln \alpha) \right) \circ \pi \right) \tau(\psi), \\
&\left(\frac{1}{\alpha^2} \circ \pi \right) \nabla_{(0,f_j)}^\Psi \nabla_{(0,f_j)}^\Psi f \tau(\Psi) = \left(\left(\frac{f}{\alpha^4} \right) \circ \pi \right) \nabla_{f_j}^\psi \nabla_{f_j}^\psi \tau(\psi)
\end{aligned}$$

and

$$\begin{aligned} \left(\frac{1}{\alpha^2} \circ \pi\right) \nabla_{\nabla_{(0,f_j)}^{\Psi}}^{(0,f_j)} f \tau(\Psi) &= \left(\frac{f}{\alpha^4} \circ \pi\right) \nabla_{\nabla_{f_j}^{\psi} f_j}^{\psi} \tau(\psi) \\ &+ n \left(\frac{f}{\alpha^2} \left(2 |\text{grad} \ln \alpha|^2 - d \ln f (\text{grad} \ln \alpha)\right) \circ \pi\right) \tau(\psi). \end{aligned}$$

Which gives us

$$\begin{aligned} \text{Tr}_{G_\alpha} \left(\nabla^{\tilde{\Psi}}\right)^2 f \tau(\tilde{\psi}) &= \left(\frac{f}{\alpha^4} \circ \pi\right) \text{Tr}_h \nabla^2 \tau(\psi) \\ &+ \left(\frac{f}{\alpha^2} \left(\Delta \ln f + |\text{grad} \ln f|^2\right) \circ \pi\right) \tau(\psi) \\ &- \left(\frac{f}{\alpha^2} \left(2 \Delta \ln \alpha + (2n - 4) |\text{grad} \ln \alpha|^2\right) \circ \pi\right) \tau(\psi) \\ &+ (n - 4) \left(\frac{f}{\alpha^2} (d \ln f (\text{grad} \ln \alpha)) \circ \pi\right) \tau(\psi). \end{aligned} \tag{2.16}$$

Finally for the first term $\text{Tr}_{G_\alpha} R^P (f \tau(\Psi), d\Psi) d\Psi$, it is easy to verify that

$$\text{Tr}_{G_\alpha} R^P (f \tau(\Psi), d\Psi) d\Psi = \left(\frac{f}{\alpha^4} \circ \pi\right) \text{Tr}_h R^P (\tau(\psi), d\psi) d\psi. \tag{2.17}$$

If we substitute (2.16) and (2.17) in (2.15), we obtain

$$\begin{aligned} \tau_{2,f}(\Psi) &= \left(\frac{f}{\alpha^4} \circ \pi\right) \tau_2(\psi) \\ &- \left(\frac{f}{\alpha^2} \left(\Delta \ln f + |\text{grad} \ln f|^2\right) \circ \pi\right) \tau(\psi) \\ &+ \left(\frac{f}{\alpha^2} \left(2 \Delta \ln \alpha + (2n - 4) |\text{grad} \ln \alpha|^2\right) \circ \pi\right) \tau(\psi) \\ &- (n - 4) \left(\frac{f}{\alpha^2} (d \ln f (\text{grad} \ln \alpha)) \circ \pi\right) \tau(\psi). \end{aligned}$$

If the map ψ a biharmonic non-harmonic, we obtain :

Corollary 2.3. *If ψ is a biharmonic non-harmonic map, then Ψ is f -biharmonic if and only if the functions f and α satisfy the following equation*

$$\Delta \ln f + |\text{grad} \ln f|^2 - 2 \Delta \ln \alpha + (4 - 2n) |\text{grad} \ln \alpha|^2 + (n - 4) d \ln f (\text{grad} \ln \alpha) = 0.$$

And if $f = \alpha$, the last equation becomes

$$\Delta \ln f + (n - 1) |\text{grad} \ln f|^2 = 0.$$

We will have two cases

- (1) If $n \neq 1$, by calculating the Laplacian of the function f^{n-1} , we deduce that the map $\Psi : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ is f -biharmonic if and only if the function f^{n-1} is harmonic.
- (2) If $n = 1$, $\Psi : (M^m \times_f N^n, G_f) \longrightarrow (P^p, k)$ is f -biharmonic if and only if the function $\ln f$ is harmonic.

In the same construction context, let's look at the identity map $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$.

Theorem 2.3. *The identity map $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$ is f -biharmonic if and only*

$$\begin{aligned} & grad \Delta \ln \alpha + 2 \nabla_{grad \ln f}^M grad \ln \alpha + \frac{n}{2} grad \left(|grad \ln \alpha|^2 \right) \\ & + \left(\Delta \ln f + |grad \ln f|^2 + n d \ln f (grad \ln \alpha) \right) grad \ln \alpha \\ & + 2 Ricci^M (grad \ln \alpha) = 0. \end{aligned} \quad (2.18)$$

Proof. By definition of the f -tension field of Id , we have

$$\begin{aligned} \tau_f (Id) &= \nabla_{(e_i, 0)} f (e_i, 0) - f \left(\tilde{\nabla}_{(e_i, 0)} (e_i, 0) \right) \\ &+ \frac{f}{\alpha^2} \nabla_{(0, f_j)} (0, f_j) - \frac{f}{\alpha^2} \left(\tilde{\nabla}_{(0, f_j)} (0, f_j) \right). \end{aligned}$$

It is simple to see that

$$\nabla_{(e_i, 0)} f (e_i, 0) = \nabla_{(e_i, 0)} (e_i, 0) + f (grad \ln f, 0) = (\nabla_{e_i} e_i, 0) + f (grad \ln \alpha, 0),$$

$$\nabla_{(e_i, 0)} (e_i, 0) = (\nabla_{e_i} e_i, 0),$$

$$\tilde{\nabla}_{(0, f_j)} (0, f_j) = (0, \nabla_{f_j} f_j),$$

and

$$\tilde{\nabla}_{(0, f_j)} (0, f_j) = (0, \nabla_{f_j} f_j) - n \alpha^2 (grad \ln \alpha, 0).$$

Then

$$\tau_f (Id) = f (grad \ln f, 0) + n f (grad \ln \alpha, 0) = f (grad \ln (f \alpha^n), 0).$$

From the expression of $\tau_f (Id)$, we deduce that Id is f -harmonic if and only if the function $f \alpha^n$ is constant. The biharmonicity condition of the identity map Id is given by the equation

$$Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0) + f^2 Tr_{G_\alpha} R^{M \times N} ((grad \ln f, 0), d\phi) d\phi = 0. \quad (2.19)$$

For the first term A $Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0)$ of equation (2.19)

$$\begin{aligned} Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0) &= \nabla_{(e_i,0)} \nabla_{(e_i,0)} f (grad \ln \alpha, 0) - \nabla_{\tilde{\nabla}_{(e_i,0)}(e_i,0)} f (grad \ln \alpha, 0) \\ &+ \frac{1}{\alpha^2} \left(\nabla_{(0,f_j)} \nabla_{(0,f_j)} f (grad \ln \alpha, 0) - \nabla_{\tilde{\nabla}_{(0,f_j)}(0,f_j)} f (grad \ln \alpha, 0) \right). \end{aligned} \quad (2.20)$$

The separate calculation of these terms gives us

$$\begin{aligned} \nabla_{(e_i,0)} \nabla_{(e_i,0)} f (grad \ln \alpha, 0) - \nabla_{\tilde{\nabla}_{(e_i,0)}(e_i,0)} f (grad \ln \alpha, 0) \\ = f (Tr_g \nabla^2 grad \ln \alpha, 0) + 2f (\nabla_{grad \ln f}^M grad \ln \alpha, 0) \\ + f (\Delta \ln f + |grad \ln f|^2) (grad \ln \alpha, 0), \end{aligned} \quad (2.21)$$

and

$$\begin{aligned} \nabla_{(0,f_j)} \nabla_{(0,f_j)} f (grad \ln \alpha, 0) - \nabla_{\tilde{\nabla}_{(0,f_j)}(0,f_j)} f (grad \ln \alpha, 0) \\ = nf\alpha^2 \left(\frac{1}{2} \left(grad \left(|grad \ln \alpha|^2 \right), 0 \right) + d \ln f (grad \ln \alpha) (grad \ln \alpha, 0) \right). \end{aligned} \quad (2.22)$$

From the equations (2.20), (2.21) and (2.22), we obtain

$$\begin{aligned} Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0) &= f (Tr_g \nabla^2 grad \ln \alpha, 0) + 2f (\nabla_{grad \ln f}^M grad \ln \alpha, 0) \\ &+ f (\Delta \ln f + |grad \ln f|^2 + nd \ln f (grad \ln \alpha)) (grad \ln \alpha, 0) \\ &+ \frac{n}{2} f \left(grad \left(|grad \ln \alpha|^2 \right), 0 \right). \end{aligned}$$

By using the fact that (see [17])

$$Tr_g \nabla^2 grad f = grad \Delta f + Ricci (grad f),$$

we conclude that

$$\begin{aligned} Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0) &= f (grad \Delta \ln \alpha, 0) + 2f (\nabla_{grad \ln f}^M grad \ln \alpha, 0) \\ &+ f (\Delta \ln f + |grad \ln f|^2 + nd \ln f (grad \ln \alpha)) (grad \ln \alpha, 0) \\ &+ \frac{n}{2} f \left(grad \left(|grad \ln \alpha|^2 \right), 0 \right) + f (Ricci^M (grad \ln \alpha), 0). \end{aligned} \quad (2.23)$$

Finally, it is clear that

$$Tr_{G_\alpha} R((grad \ln \alpha, 0), d\phi) d\phi = (Ricci (grad \ln \alpha), 0). \quad (2.24)$$

The equations (2.23) and (2.24) give us

$$\begin{aligned}
& Tr_{G_\alpha} \nabla^2 f (grad \ln \alpha, 0) + f Tr_{G_\alpha} R^{M \times N} ((grad \ln \alpha, 0), d\phi) d\phi \\
&= f (grad \Delta \ln \alpha, 0) + 2f (\nabla_{grad \ln f}^M grad \ln \alpha, 0) \\
&+ f \left(\Delta \ln f + |grad \ln f|^2 + nd \ln f (grad \ln \alpha) \right) (grad \ln \alpha, 0) \\
&+ \frac{n}{2} f \left(grad \left(|grad \ln \alpha|^2 \right), 0 \right) + 2f (Ricci^M (grad \ln \alpha), 0).
\end{aligned}$$

Then the identity map $Id : (M^m \times_\alpha N^n, G_\alpha) \longrightarrow (M^m \times N^n, G)$ is f -biharmonic if and only if

$$\begin{aligned}
& grad \Delta \ln \alpha + 2\nabla_{grad \ln f}^M grad \ln \alpha + \frac{n}{2} grad \left(|grad \ln \alpha|^2 \right) \\
&+ \left(\Delta \ln f + |grad \ln f|^2 + nd \ln f (grad \ln \alpha) \right) grad \ln \alpha \\
&+ 2Ricci^M (grad \ln \alpha) = 0.
\end{aligned}$$

The following corollary results from the case where $f = \alpha$.

Corollary 2.4. $Id : (M \times_f N, G_f) \longrightarrow (M \times N, G)$ is f -biharmonic if and only if

$$\begin{aligned}
& grad \Delta \ln f + \frac{n+2}{2} grad \left(|grad \ln f|^2 \right) + 2Ricci^M (grad \ln f) \\
&+ \left(\Delta \ln f + (n+1) |grad \ln f|^2 \right) grad \ln f = 0.
\end{aligned}$$

Theorem 2.3 gives us the following example.

Example 2.2. Let $Id : \mathbf{R}^m \setminus \{0\} \times_\alpha N^n \longrightarrow \mathbf{R}^m \setminus \{0\} \times N^n$ when we suppose that the positives functions f and α are radial. Then by Theorem 2.3, we deduce that the identity map Id is f -biharmonic if and only if the functions $f_1(r) = (\ln f(r))'$ and $\alpha_1(r) = (\ln \alpha(r))'$ are solutions of the following differential equation

$$f_1' \alpha_1 + f_1^2 \alpha_1 + n f_1 \alpha_1^2 + \alpha_1'' + n \alpha_1 \alpha_1' + 2 f_1 \alpha_1' + \frac{m-1}{r} \alpha_1' - \frac{m-1}{r^2} \alpha_1 = 0.$$

A method to solve this equation is to look at the solutions of the form $f_1(r) = \frac{a}{r}$ and $\alpha_1(r) = \frac{b}{r}$ ($a, b \in \mathbf{R}^*$), the f -biharmonicity of Id is expressed by the algebraic equation

$$a^2 + (nb - 3)a - (nb + 2m - 4) = 0.$$

For this equation, we can distinguish the following cases :

(1) If $m = 1$, we obtain two solutions $a = 1$ and $a = 2 - nb$.

- For $a = 1$, Id is f -biharmonic if and only if $f(r) = C_1 r$ and $\alpha(r) = C_2 r^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.

- For $a = 2 - nb$, Id is f -biharmonic if and only if $f(r) = C_1 r^{2-nb}$ and $\alpha(r) = C_2 r^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.

(2) For $m > 1$, the equation $a^2 + (nb - 3)a - (nb + 2m - 4) = 0$ has two real solutions

$$a = \frac{3 - nb + A}{2}$$

and

$$a = \frac{3 - nb - A}{2},$$

where

$$A = \sqrt{n^2 b^2 - 2nb + 8m - 7}.$$

- For $a = \frac{3-nb+A}{2}$, Id is f -biharmonic if and only if $f(r) = C_1 \sqrt{r^{3-nb+A}}$ and $\alpha(r) = C_2 r^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.
- For $a = \frac{3-nb-A}{2}$, Id is f -biharmonic if and only if $f(r) = C_1 \sqrt{r^{3-nb-A}}$ and $\alpha(r) = C_2 r^b$ for any $b \in \mathbf{R}^*$, where C_1 and C_2 are positive constants.

As a last result, we give a theorem analogous to Theorem 2.3 by considering the identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$.

Theorem 2.4. *The identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$ is biharmonic if and only if*

$$\begin{aligned} & grad \Delta \ln \alpha + 2 \nabla_{grad \ln f} grad \ln \alpha + \left(2 - \frac{n}{2} \alpha^2\right) grad \left(|grad \ln \alpha|^2\right) \\ & + \left(\Delta \ln f + |grad \ln f|^2 + 4d \ln f (grad \ln \alpha)\right) grad \ln \alpha \\ & + \left(2\Delta \ln \alpha + (4 - 2n\alpha^2) |grad \ln \alpha|^2\right) grad \ln \alpha + 2Ricci (grad \ln \alpha) = 0. \end{aligned}$$

Proof. By definition, we have

$$\begin{aligned} \tau_f (Id) &= f \tilde{\nabla}_{(e_i, 0)} (e_i, 0) + e_i (f) (e_i, 0) - fd\phi (\nabla_{e_i} e_i, 0) \\ &+ f \tilde{\nabla}_{(0, f_j)} (0, f_j) - fd\phi (0, \nabla_{f_j} f_j) \\ &= fd\phi (\nabla_{e_i} e_i, 0) + f (grad \ln f, 0) - fd\phi (\nabla_{e_i} e_i, 0) \\ &+ fd\phi (0, \nabla_{f_j} f_j) - n\alpha^2 (grad \ln \alpha, 0) - fd\phi (0, \nabla_{f_j} f_j), \end{aligned}$$

it follows that

$$\tau_f (Id) = f ((grad \ln f, 0) - n\alpha^2 (grad \ln \alpha, 0)).$$

It is simple to see that in this case Id is f -harmonic if and only if $f = Ce^{\frac{n}{2}\alpha^2}$. The identity map Id is f -biharmonic if and only if

$$Tr_G \tilde{\nabla}^2 f \alpha^2 (grad \ln \alpha, 0) + f \alpha^2 Tr_G \tilde{R} ((grad \ln \alpha, 0), \cdot) \cdot = 0. \quad (2.25)$$

For the first term $Tr_G \tilde{\nabla}^2 f \alpha^2 (grad \ln \alpha, 0)$, we have

$$\begin{aligned}
Tr_G \tilde{\nabla}^2 f \alpha^2 (grad \ln \alpha, 0) &= \tilde{\nabla}_{(e_i, 0)} \tilde{\nabla}_{(e_i, 0)} f \alpha^2 (grad \ln \alpha, 0) \\
&\quad - \tilde{\nabla}_{(\nabla_{e_i} e_i, 0)} f \alpha^2 (grad \ln \alpha, 0) \\
&\quad + f \alpha^2 \tilde{\nabla}_{(0, f_j)} \tilde{\nabla}_{(0, f_j)} (grad \ln \alpha, 0) \\
&\quad - f \alpha^2 \tilde{\nabla}_{(0, \nabla_{f_j} f_j)} (grad \ln \alpha, 0).
\end{aligned} \tag{2.26}$$

The terms of equation (2.26) are calculated from the same method used in Theorem 2.4, we find

$$\begin{aligned}
&\tilde{\nabla}_{(e_i, 0)} \tilde{\nabla}_{(e_i, 0)} f \alpha^2 (grad \ln \alpha, 0) - \tilde{\nabla}_{(\nabla_{e_i} e_i, 0)} f \alpha^2 (grad \ln \alpha, 0) \\
&= f \alpha^2 (grad \Delta \ln \alpha, 0) + 2f \alpha^2 (\nabla_{grad \ln f} grad \ln \alpha, 0) \\
&\quad + f \alpha^2 (\Delta \ln f + |grad \ln f|^2) (grad \ln \alpha, 0) \\
&\quad + 2f \alpha^2 (\Delta \ln \alpha + 2|grad \ln \alpha|^2) (grad \ln \alpha, 0) \\
&\quad + 4f \alpha^2 d \ln f (grad \ln \alpha) (grad \ln \alpha, 0) \\
&\quad + 2f \alpha^2 (grad (|grad \ln \alpha|^2), 0) \\
&\quad + f \alpha^2 (Ricci (grad \ln \alpha), 0),
\end{aligned} \tag{2.27}$$

and

$$\begin{aligned}
&\tilde{\nabla}_{(0, f_j)} \tilde{\nabla}_{(0, f_j)} (grad \ln \alpha, 0) - \tilde{\nabla}_{(0, \nabla_{f_j} f_j)} (grad \ln \alpha, 0) \\
&= -n \alpha^2 |grad \ln \alpha|^2 (grad \ln \alpha, 0).
\end{aligned} \tag{2.28}$$

If we replace (2.27) and (2.28) in (2.26), we deduce that

$$\begin{aligned}
Tr_G \tilde{\nabla}^2 f \alpha^2 (grad \ln \alpha, 0) &= f \alpha^2 (grad \Delta \ln \alpha, 0) + 2f \alpha^2 (\nabla_{grad \ln f} grad \ln \alpha, 0) \\
&\quad + 2f \alpha^2 (grad (|grad \ln \alpha|^2), 0) + f \alpha^2 \Delta \ln f (grad \ln \alpha, 0) \\
&\quad + 2f \alpha^2 \Delta \ln \alpha (grad \ln \alpha, 0) + f \alpha^2 |grad \ln f|^2 (grad \ln \alpha, 0) \\
&\quad + 4f \alpha^2 d \ln f (grad \ln \alpha) (grad \ln \alpha, 0) \\
&\quad + f \alpha^2 (4 - n \alpha^2) |grad \ln \alpha|^2 (grad \ln \alpha, 0) \\
&\quad + f \alpha^2 (Ricci (grad \ln \alpha), 0).
\end{aligned} \tag{2.29}$$

To calculate $Tr_G \tilde{R}((grad \ln \alpha, 0), \cdot) \cdot$, we use the relation between the curvature tensor fields of G_α and G , we obtain

$$\tilde{R}((grad \ln \alpha, 0), (e_i, 0)) (e_i, 0) = (Ricci (grad \ln \alpha), 0)$$

and

$$\tilde{R}((grad \ln \alpha, 0), (0, f_j)) (0, f_j) = -n\alpha^2 |grad \ln \alpha|^2 (grad \ln \alpha, 0) - \frac{n}{2}\alpha^2 \left(grad \left(|grad \ln \alpha|^2 \right), 0 \right).$$

It follows that

$$\begin{aligned} Tr_G \tilde{R}((grad \ln \alpha, 0), \cdot) \cdot &= \tilde{R}((grad \ln \alpha, 0), (e_i, 0)) (e_i, 0) \\ &+ \tilde{R}((grad \ln \alpha, 0), (0, f_j)) \\ &= -n\alpha^2 |grad \ln \alpha|^2 (grad \ln \alpha, 0) \\ &- \frac{n}{2}\alpha^2 \left(grad \left(|grad \ln \alpha|^2 \right), 0 \right) \\ &+ (Ricci (grad \ln \alpha), 0). \end{aligned} \quad (2.30)$$

By replacing (2.29) and (2.30) in (2.25), we conclude that the identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_\alpha N^n, G_\alpha)$ is f -biharmonic if and only if

$$\begin{aligned} &grad \Delta \ln \alpha + 2\nabla_{grad \ln f} grad \ln \alpha + \left(2 - \frac{n}{2}\alpha^2 \right) grad \left(|grad \ln \alpha|^2 \right) \\ &+ \left(\Delta \ln f + |grad \ln f|^2 + 4d \ln f (grad \ln \alpha) \right) grad \ln \alpha \\ &+ \left(2\Delta \ln \alpha + (4 - 2n\alpha^2) |grad \ln \alpha|^2 \right) grad \ln \alpha + 2Ricci (grad \ln \alpha) = 0. \end{aligned}$$

If $f = \alpha$, we obtain

Corollary 2.5. *The identity map $Id : (M^m \times N^n, G) \longrightarrow (M^m \times_f N^n, G_f)$ is f -biharmonic if and only if*

$$\begin{aligned} &grad \Delta \ln f + \left(3\Delta \ln f + (9 - 2nf^2) |grad \ln f|^2 \right) grad \ln f \\ &+ \left(3 - \frac{n}{2}f^2 \right) grad \left(|grad \ln f|^2 \right) + 2Ricci (grad \ln f) = 0. \end{aligned}$$

As an application of the Theorem 2.4, we give an example of a f -biharmonic map.

Example 2.3. *Let $Id : \mathbf{R}_+^* \times N^n \longrightarrow \mathbf{R}_+^* \times_\alpha N^n$ the identity map and let f and α a positive functions on \mathbf{R}_+^* . By Theorem 2.4, Id is f -biharmonic if and only*

$$f_1' \alpha_1 + f_1^2 \alpha_1 + 4f_1 \alpha_1^2 + \alpha_1'' + 6\alpha_1 \alpha_1' + 2f_1 \alpha_1' - n\alpha^2 \alpha_1 \alpha_1' - 2n\alpha^2 \alpha_1^3 + 4\alpha_1^3 = 0,$$

where $f_1(t) = (\ln f(t))'$ and $\alpha_1(t) = (\ln \alpha(t))'$. In solving this equation, we found particular solutions given by $f(t) = C_1 t$ and $\alpha(t) = C_2 \sqrt{t}$, where C_1 and C_2 are positive constants, which implies that the identity map $Id : \mathbf{R}_+^* \times N^n \longrightarrow \mathbf{R}_+^* \times_\alpha N^n$ is f -biharmonic.

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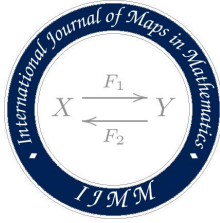
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DEPARTMENT OF MATHEMATICS - UNIVERSITY OF TIARET, ALGERIA.

Email address: hamidboulalsr@gmail.com

LABORATORY OF GEOMETRY, ANALYSIS, CONTROL AND APPLICATIONS, UNIVERSITY OF SAIDA, ALGERIA.

Email address: seddik.ouakkas@gmail.com



**CERTAIN CURVATURE CONDITIONS IN LORENTZIAN
PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE
SEMI-SYMMETRIC METRIC CONNECTION**

ABDUL HASEEB * AND RAJENDRA PRASAD 

ABSTRACT. The object of the present paper is to characterize Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying certain curvature conditions.

1. INTRODUCTION

In 1989, K. Matsumoto [12] introduced the notion of Lorentzian para-Sasakian manifolds. Again the same notion was studied by I. Mihai and R. Rosca [13] and obtained many results on this manifold. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [11], U. C. De et al. [2] and many others such as ([14], [16], [18]).

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a semi-symmetric connection [4] if the torsion tensor T of the connection $\bar{\nabla}$ defined by

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

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* Corresponding author

satisfies

$$T(X, Y) = \eta(Y)X - \eta(X)Y, \quad (1.1)$$

where η is a 1-form. If moreover, the connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y, Z) = 0 \quad (1.2)$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M , then $\bar{\nabla}$ is said to be a semi-symmetric metric connection, otherwise it is said to be a semi-symmetric non-metric connection. In 1932, H. A. Hayden [7] defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by K. Yano [21]. A semi-symmetric metric connection have been studied by many authors ([1], [5], [6], [17], [20]) in several ways to a different extent.

A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in Lorentzian para-Sasakian manifold M is given by [17, 21]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \quad (1.3)$$

The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by $R(X, Y) \cdot R = 0$, where $R(X, Y)$ acts on R as a derivation of the tensor algebra at each point of the manifold for tangent vector fields X, Y . A complete intrinsic classification of these manifolds was given by Z. I. Szabó in [19]. Also in [9], O. Kowalski classified 3-dimensional Riemannian spaces satisfying $R(X, Y) \cdot R = 0$. A Riemannian manifold is said to be Ricci semisymmetric if $R(X, Y) \cdot S = 0$, where S denotes the Ricci tensor of type $(0, 2)$. A general classification of these manifolds has been worked out by V. A. Mirzoyan [15].

We define endomorphisms $R(X, Y)$ and $X \wedge_A Y$ for an arbitrary vector field Z by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z, \quad (1.4)$$

and

$$(X \wedge_A Y)Z = A(Y, Z)X - A(X, Z)Y, \quad (1.5)$$

respectively, where $X, Y, Z \in \chi(M)$ and A is the symmetric $(0, 2)$ -tensor, R is the Riemannian curvature tensor of type $(1, 3)$.

Furthermore, the tensors $R \cdot R$ and $R \cdot S$ on (M, g) are defined by

$$\begin{aligned} (R(X, Y) \cdot R)(U, V)W &= R(X, Y)R(U, V)W - R(R(X, Y)U, V)W \\ &\quad - R(U, R(X, Y)V)W - R(U, V)R(X, Y)W, \end{aligned} \quad (1.6)$$

and

$$(R(X, Y) \cdot S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V), \quad (1.7)$$

respectively.

Recently, D. Kowalczyk [8] studied semi-Riemannian manifolds satisfying $Q(Q, R) = 0$ and $Q(S, g) = 0$, where S, R are the Ricci tensor and curvature tensor, respectively. For detailed study of semisymmetric manifolds we refer the readers to see ([3], [10]).

The paper is organized as follows: Section 2 is concerned with preliminaries. In Section 3, we obtain the expressions of the curvature tensor \bar{R} and the Ricci tensor \bar{S} with respect to the semi-symmetric metric connection. In Section 4, we prove that $R \cdot \bar{S} = 0$ if and only if the manifold is an Einstein manifold with respect to $\bar{\nabla}$. Next in Section 5 (resp., 6), we prove that if the manifold satisfies the curvature condition $\bar{S} \cdot R = 0$ (resp., $R \cdot \bar{R} = 0$), then it is an η -Einstein (resp., Einstein) manifold with respect to $\bar{\nabla}$. Section 7, deals with the study of Ricci semisymmetric Lorentzian para-Sasakian manifolds and prove that Ricci semisymmetries with respect to ∇ and $\bar{\nabla}$ are equivalent if the manifold is a generalized η -Einstein manifold. In Section 8, we prove that if $C(\xi, X) \cdot \bar{S} = 0$, then either the scalar curvature is constant or the manifold is an Einstein manifold with respect to $\bar{\nabla}$. In the last Section, it is shown that if $\bar{Q} \cdot C = 0$ (where C is the concircular curvature tensor with respect to ∇ and \bar{Q} is the Ricci operator with respect to $\bar{\nabla}$), then either the scalar curvature is constant or the manifold is a special type of η -Einstein manifold with respect to $\bar{\nabla}$. Finally, we construct an example of 5-dimensional Lorentzian para-Sasakian manifold.

2. PRELIMINARIES

A differentiable manifold M of dimension n is called a Lorentzian para-Sasakian manifold, if it admits a $(1, 1)$ -tensor field ϕ , a contravariant vector field ξ , a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad (2.1)$$

$$g(X, \xi) = \eta(X), \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad (2.3)$$

$$(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \quad (2.4)$$

$$\nabla_X \xi = \phi X, \quad (2.5)$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g . If we put

$$\Phi(X, Y) = g(\phi X, Y) \quad (2.6)$$

for all vector fields X and Y , then $\Phi(X, Y)$ is a symmetric $(0, 2)$ tensor field. Also since the 1-form η is closed in a Lorentzian para-Sasakian manifold, so we have

$$(\nabla_X \eta)(Y) = \Phi(X, Y), \quad \Phi(X, \xi) = 0 \quad (2.7)$$

for all vector fields $X, Y \in \chi(M)$.

Moreover, the curvature tensor R , the Ricci tensor S and the Ricci operator Q in a Lorentzian para-Sasakian manifold with respect to the Levi-Civita connection satisfy the following equations [2, 11]:

$$\eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \quad (2.8)$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.9)$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \quad (2.10)$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X + \eta(X)\xi, \quad (2.11)$$

$$S(X, \xi) = (n-1)\eta(X), \quad Q\xi = (n-1)\xi, \quad (2.12)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \quad (2.13)$$

for all $X, Y, Z \in \chi(M)$, where S and Q are related by $g(QX, Y) = S(X, Y)$.

Definition 2.1. *A Lorentzian para-Sasakian manifold M is said to be a generalized η -Einstein manifold if its Ricci tensor S is of the form [23]*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + c\Omega(X, Y),$$

where a, b, c are smooth functions on M and $\Omega(X, Y) = g(\phi X, Y)$. If $c = 0$ (resp., $b = c = 0$), then the manifold reduces to an η -Einstein (resp., an Einstein) manifold.

Definition 2.2. *The concircular curvature tensor C in an n -dimensional Lorentzian para-Sasakian manifold M is defined by [22]*

$$C(X, Y)Z = R(X, Y)Z - \frac{r}{n(n-1)}[g(Y, Z)X - g(X, Z)Y] \quad (2.14)$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

3. CURVATURE TENSOR OF A LORENTZIAN PARA-SASAKIAN MANIFOLD WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

Let M be an n -dimensional Lorentzian para-Sasakian manifold. The curvature tensor \bar{R} with respect to $\bar{\nabla}$ is defined by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z. \quad (3.1)$$

By using (1.2), (1.3), (2.1), (2.2), (2.5) and (2.7) in (3.1), we get

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + g(X, \phi Z)Y - g(Y, \phi Z)X - g(Y, Z)\phi X + g(X, Z)\phi Y \\ &\quad + (g(Y, Z)\eta(X) - g(X, Z)\eta(Y))\xi + g(Y, Z)X - g(X, Z)Y \\ &\quad + (\eta(Y)X - \eta(X)Y)\eta(Z), \end{aligned} \quad (3.2)$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z$$

is the Riemannian curvature tensor with respect to ∇ . By contracting (3.2) over X , we obtain

$$\bar{S}(Y, Z) = S(Y, Z) - (n-2)g(Y, \phi Z) + (n-2-\psi)g(Y, Z) + (n-2)\eta(Y)\eta(Z), \quad (3.3)$$

where \bar{S} and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇ , respectively and $\psi = \text{trace}\phi$.

The equation (3.3) yields

$$\bar{Q}Y = QY - (n-2)\phi Y + (n-2-\psi)Y + (n-2)\eta(Y)\xi, \quad (3.4)$$

where \bar{Q} and Q are the Ricci operators of the connections $\bar{\nabla}$ and ∇ , respectively.

Contracting again Y and Z in (3.3), it follows that

$$\bar{r} = r + (n-1)(n-2-2\psi), \quad (3.5)$$

where \bar{r} and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇ , respectively.

Lemma 3.1. *Let M be an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection. Then*

$$\bar{R}(X, Y)\xi = \eta(Y)(X - \phi X) - \eta(X)(Y - \phi Y), \quad (3.6)$$

$$\bar{R}(\xi, X)Y = (g(X, Y) - g(X, \phi Y))\xi - \eta(Y)(X - \phi X), \quad (3.7)$$

$$\bar{R}(\xi, X)\xi = X - \phi X + \eta(X)\xi, \quad (3.8)$$

$$\bar{S}(X, \xi) = (n-1-\psi)\eta(X), \quad \bar{Q}\xi = (n-1-\psi)\xi, \quad (3.9)$$

$$\bar{S}(\phi X, \phi Y) = \bar{S}(X, Y) + (n - 1 - \psi)\eta(X)\eta(Y). \quad (3.10)$$

Proof. By taking $Z = \xi$ in (3.2) and using (2.1), (2.2), (2.10), we get (3.6). (3.7) follows from (2.1), (2.2), (2.9) and (3.6). By taking $Y = \xi$ in (3.7) and using (2.1), (2.2) we obtain (3.8). From (3.3), (2.1), (2.2) and (2.12) we find (3.9). By replacing $Y = \phi X$ and $Z = \phi Y$ in (3.3) and then using (2.1)-(2.3) and (2.13) we get (3.10).

4. LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION SATISFYING $R(X, Y) \cdot \bar{S} = 0$

Suppose that a Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection $\bar{\nabla}$ satisfies the condition

$$R(X, Y) \cdot \bar{S} = 0. \quad (4.1)$$

Then in view of (1.7), it follows that

$$\bar{S}(R(X, Y)U, V) + \bar{S}(U, R(X, Y)V) = 0$$

which by putting $X = \xi$ and using (2.9) takes the form

$$g(Y, U)\bar{S}(\xi, V) - \eta(U)\bar{S}(Y, V) + g(Y, V)\bar{S}(U, \xi) - \eta(V)\bar{S}(U, Y) = 0. \quad (4.2)$$

By taking $U = \xi$ in (4.2) and using (2.1), (2.2) and (3.9), we obtain

$$\bar{S}(Y, V) = (n - 1 - \psi)g(Y, V). \quad (4.3)$$

From which we have

$$\bar{Q}V = (n - 1 - \psi)V. \quad (4.4)$$

Conversely, if (4.3) satisfies, then by using (4.4) in the expression $(R(X, Y) \cdot \bar{S})(U, V) = -\bar{S}(R(X, Y)U, V) - \bar{S}(U, R(X, Y)V) = -g(R(X, Y)U, \bar{Q}V) - g(\bar{Q}U, R(X, Y)V)$, we find

$$(R(X, Y) \cdot \bar{S})(U, V) = -(n - 1 - \psi)(g(R(X, Y)U, V) + g(U, R(X, Y)V)) \quad (4.5)$$

which by using the fact that $g(R(X, Y)U, V) + g(U, R(X, Y)V) = 0$ reduces to $(R(X, Y) \cdot \bar{S})(U, V) = 0$. Thus we can state the following theorem:

Theorem 4.1. *If an n -dimensional Lorentzian para-Sasakian manifold with respect to semi-symmetric metric connection satisfies the condition $R \cdot \bar{S} = 0$, then the manifold is an Einstein manifold of the form (4.3) and the converse is also true.*

5. LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION SATISFYING $\bar{S} \cdot R = 0$

Suppose that a Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $(\bar{S}(X, Y) \cdot R)(U, V)Z = 0$. Then we have [8]

$$\begin{aligned} (X \wedge_{\bar{S}} Y)R(U, V)Z + R((X \wedge_{\bar{S}} Y)U, V)Z + R(U, (X \wedge_{\bar{S}} Y)V)Z \\ + R(U, V)(X \wedge_{\bar{S}} Y)Z = 0 \end{aligned} \quad (5.1)$$

for any $X, Y, Z, U, V \in \chi(M)$. Taking $Y = \xi$ in (5.1), we have

$$\begin{aligned} (X \wedge_{\bar{S}} \xi)R(U, V)Z + R((X \wedge_{\bar{S}} \xi)U, V)Z + R(U, (X \wedge_{\bar{S}} \xi)V)Z \\ + R(U, V)(X \wedge_{\bar{S}} \xi)Z = 0 \end{aligned} \quad (5.2)$$

which in view of (1.5) takes the form

$$\begin{aligned} \bar{S}(\xi, R(U, V)Z)X - \bar{S}(X, R(U, V)Z)\xi + R(\bar{S}(\xi, U)X - \bar{S}(X, U)\xi, V)Z \\ + R(U, \bar{S}(\xi, V)X - \bar{S}(X, V)\xi)Z + R(U, V)(\bar{S}(\xi, Z)X - \bar{S}(X, Z)\xi) = 0. \end{aligned} \quad (5.3)$$

By using (3.9) in (5.3), we find

$$\begin{aligned} (n - 1 - \psi)[\eta(R(U, V)Z)X + \eta(U)R(X, V)Z + \eta(V)R(U, X)Z + \eta(Z)R(U, V)X] \\ - \bar{S}(X, R(U, V)Z)\xi - \bar{S}(X, U)R(\xi, V)Z - \bar{S}(X, V)R(U, \xi)Z - \bar{S}(X, Z)R(U, V)\xi = 0. \end{aligned} \quad (5.4)$$

Now taking inner product of (5.4) with ξ , we get

$$\begin{aligned} (n - 1 - \psi)[\eta(R(U, V)Z)\eta(X) + \eta(U)\eta(R(X, V)Z) + \eta(V)\eta(R(U, X)Z) \\ + \eta(Z)\eta(R(U, V)X)] + \bar{S}(X, R(U, V)Z) - \bar{S}(X, U)\eta(R(\xi, V)Z) \\ - \bar{S}(X, V)\eta(R(U, \xi)Z) - \bar{S}(X, Z)\eta(R(U, V)\xi) = 0 \end{aligned}$$

which by putting $U = Z = \xi$ and using (3.6)-(3.8) reduces to

$$(n - 1 - \psi)(g(X, V) + \eta(X)\eta(V)) + \bar{S}(X, V + \eta(V)\xi) = 0$$

from which it follows that

$$\bar{S}(X, V) = -(n - 1 - \psi)g(X, V) - 2(n - 1 - \psi)\eta(X)\eta(V). \quad (5.5)$$

Thus we can state the following theorem:

Theorem 5.1. *If an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $\bar{S} \cdot R = 0$, then the manifold is an η -Einstein manifold of the form (5.5).*

6. LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION SATISFYING $R \cdot \bar{R} = 0$

Let M be an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $(R(X, Y) \cdot \bar{R})(U, V)W = 0$. Then in view of (1.6), it follows that

$$\begin{aligned} R(X, Y)\bar{R}(U, V)W - \bar{R}(R(X, Y)U, V)W - \bar{R}(U, R(X, Y)V)W \\ - \bar{R}(U, V)R(X, Y)W = 0. \end{aligned} \quad (6.1)$$

By substituting $X = U = \xi$ in (6.1) and using (2.2), (2.9), (2.11) and (3.7), we find

$$\begin{aligned} g(V, W)Y - g(V, \phi W)Y - \bar{R}(Y, V)W - \eta(V)g(Y, \phi W)\xi \\ + \eta(V)\eta(W)\phi Y - g(Y, W)V + g(Y, W)\phi V = 0. \end{aligned} \quad (6.2)$$

Taking inner product of (6.2) with Z , we have

$$\begin{aligned} g(V, W)g(Y, Z) - g(V, \phi W)g(Y, Z) - g(\bar{R}(Y, V)W, Z) - \eta(V)\eta(Z)g(Y, \phi W) \\ + \eta(V)\eta(W)g(\phi Y, Z) - g(Y, W)g(V, Z) + g(Y, W)g(\phi V, Z) = 0. \end{aligned} \quad (6.3)$$

Let $\{e_1, e_2, e_3, \dots, e_{n-1}, e_n = \xi\}$ be a frame of orthonormal basis of the tangent space at any point of the manifold. If we put $V = W = e_i$ in (6.3) and summing up with respect to $i(1 \leq i \leq n)$, then we obtain

$$\bar{S}(Y, Z) = (n - 1 - \psi)g(Y, Z). \quad (6.4)$$

Thus we can state the following theorem:

Theorem 6.1. *If an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $R \cdot \bar{R} = 0$, then the manifold is an Einstein manifold of the form (6.4).*

7. RICCI SEMISYMMETRIES IN LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT
TO THE CONNECTIONS $\bar{\nabla}$ AND ∇

Assuming that the manifold is Ricci symmetric with respect to the semi-symmetric metric connection $\bar{\nabla}$, therefore we have

$$(\bar{R}(X, Y) \cdot \bar{S})(U, V) = -\bar{S}(\bar{R}(X, Y)U, V) - \bar{S}(U, \bar{R}(X, Y)V) \quad (7.1)$$

for all $X, Y, U, V \in \chi(M)$. In view of (3.2) and (3.3), (7.1) takes the form

$$\begin{aligned} (\bar{R}(X, Y) \cdot \bar{S})(U, V) &= (R(X, Y) \cdot S)(U, V) - (n - 2 - \psi)[R(X, Y, U, V) \\ &+ R(X, Y, V, U)] + (n - 2)[g(R(X, Y)U, \phi V) + g(R(X, Y)V, \phi U)] \\ &- (n - 2)[\eta(R(X, Y)U)\eta(V) + \eta(R(X, Y)V)\eta(U)] \\ &- g(X, \phi U)\bar{S}(Y, V) - g(X, \phi V)\bar{S}(U, Y) + g(Y, \phi U)\bar{S}(X, V) \\ &+ g(Y, \phi V)\bar{S}(X, U) + g(Y, U)\bar{S}(\phi X, V) + g(Y, V)\bar{S}(U, \phi X) \\ &- g(X, U)\bar{S}(\phi Y, V) - g(X, V)\bar{S}(U, \phi Y) - g(Y, U)\eta(X)\bar{S}(\xi, V) \\ &- g(Y, V)\eta(X)\bar{S}(U, \xi) + g(X, U)\eta(Y)\bar{S}(\xi, V) + g(X, V)\eta(Y)\bar{S}(\xi, U) \\ &- g(Y, U)\bar{S}(X, V) - g(Y, V)\bar{S}(X, U) + g(X, U)\bar{S}(Y, V) \\ &+ g(X, V)\bar{S}(U, Y) - \eta(Y)\eta(U)\bar{S}(X, V) - \eta(Y)\eta(V)\bar{S}(X, U) \\ &+ \eta(X)\eta(U)\bar{S}(Y, V) + \eta(X)\eta(V)\bar{S}(Y, U) \end{aligned}$$

which by using (2.8) and the fact that $R(X, Y, U, V) + R(X, Y, V, U) = 0$ turns to

$$\begin{aligned} (\bar{R}(X, Y) \cdot \bar{S})(U, V) &= (R(X, Y) \cdot S)(U, V) + (n - 2)[g(R(X, Y)U, \phi V) \\ &+ g(R(X, Y)V, \phi U)] - (2n - 3 - \psi)[g(Y, U)\eta(X)\eta(V) - g(X, U)\eta(Y)\eta(V) \\ &+ g(Y, V)\eta(X)\eta(U) - g(X, V)\eta(Y)\eta(U)] - g(X, \phi U)\bar{S}(Y, V) \\ &- g(X, \phi V)\bar{S}(U, Y) + g(Y, \phi U)\bar{S}(X, V) + g(Y, \phi V)\bar{S}(X, U) \\ &+ g(Y, U)\bar{S}(\phi X, V) + g(Y, V)\bar{S}(U, \phi X) - g(X, U)\bar{S}(\phi Y, V) \\ &- g(X, V)\bar{S}(U, \phi Y) - g(Y, U)\bar{S}(X, V) - g(Y, V)\bar{S}(X, U) \\ &+ g(X, U)\bar{S}(Y, V) + g(X, V)\bar{S}(U, Y) - \eta(Y)\eta(U)\bar{S}(X, V) \\ &- \eta(Y)\eta(V)\bar{S}(X, U) + \eta(X)\eta(U)\bar{S}(Y, V) + \eta(X)\eta(V)\bar{S}(Y, U). \end{aligned} \quad (7.2)$$

Suppose that $(\bar{R}(X, Y) \cdot \bar{S})(U, V) = (R(X, Y) \cdot S)(U, V)$, then from (7.2), it follows that

$$(n - 2)[g(R(X, Y)U, \phi V) + g(R(X, Y)V, \phi U)]$$

$$\begin{aligned}
& -(2n - 3 - \psi)[g(Y, U)\eta(X)\eta(V) - g(X, U)\eta(Y)\eta(V) \\
& + g(Y, V)\eta(X)\eta(U) - g(X, V)\eta(Y)\eta(U)] - g(X, \phi U)\bar{S}(Y, V) \\
& - g(X, \phi V)\bar{S}(U, Y) + g(Y, \phi U)\bar{S}(X, V) + g(Y, \phi V)\bar{S}(X, U) \\
& + g(Y, U)\bar{S}(\phi X, V) + g(Y, V)\bar{S}(U, \phi X) - g(X, U)\bar{S}(\phi Y, V) \\
& - g(X, V)\bar{S}(U, \phi Y) - g(Y, U)\bar{S}(X, V) - g(Y, V)\bar{S}(X, U) \\
& + g(X, U)\bar{S}(Y, V) + g(X, V)\bar{S}(U, Y) - \eta(Y)\eta(U)\bar{S}(X, V) \\
& - \eta(Y)\eta(V)\bar{S}(X, U) + \eta(X)\eta(U)\bar{S}(Y, V) + \eta(X)\eta(V)\bar{S}(Y, U) = 0
\end{aligned}$$

which by taking $X = U = \xi$ and then using (2.1), (2.2) and (2.8) reduces to

$$\bar{S}(\phi Y, V) = (n - 2)g(Y, V) + (n - 2)\eta(Y)\eta(V) - (\psi - 1)g(Y, \phi V). \quad (7.3)$$

Now replacing V by ϕV in (7.3) and using (2.1), (2.2) and (3.10), we obtain

$$\bar{S}(Y, V) = (1 - \psi)g(Y, V) + (n - 2)g(Y, \phi V) - (n - 2)\eta(Y)\eta(V). \quad (7.4)$$

Thus we can state the following theorem:

Theorem 7.1. *Ricci semisymmetries with respect to $\bar{\nabla}$ and ∇ are equivalent if the manifold is a generalized η -Einstein manifold with respect to the semi-symmetric metric connection.*

8. LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION SATISFYING $C(\xi, X) \cdot \bar{S} = 0$

We consider that an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $C(\xi, X) \cdot \bar{S} = 0$. Then we have

$$\bar{S}(C(\xi, X)Y, Z) + \bar{S}(Y, C(\xi, X)Z) = 0. \quad (8.1)$$

From (2.14), we find

$$C(\xi, X)Y = [1 - \frac{r}{n(n-1)}](g(X, Y)\xi - \eta(Y)X). \quad (8.2)$$

By virtue of (8.2), (8.1) takes the form

$$[1 - \frac{r}{n(n-1)}](g(X, Y)\bar{S}(\xi, Z) - \eta(Y)\bar{S}(X, Z) + g(X, Z)\bar{S}(Y, \xi) - \eta(Z)\bar{S}(X, Y)) = 0$$

which by taking $Z = \xi$ and using (2.1), (2.2) and (3.9) gives

$$[1 - \frac{r}{n(n-1)}](\bar{S}(X, Y) - (n - 1 - \psi)g(X, Y)) = 0. \quad (8.3)$$

Thus we have either $r = n(n - 1)$, or

$$\bar{S}(X, Y) = (n - 1 - \psi)g(X, Y). \quad (8.4)$$

Thus we can state the following theorem:

Theorem 8.1. *If an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $C(\xi, X) \cdot \bar{S} = 0$, then either the scalar curvature is constant or the manifold is an Einstein manifold of the form (8.4).*

9. LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION SATISFYING $\bar{Q} \cdot C = 0$

In this section we suppose that an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $\bar{Q} \cdot C = 0$. Then we have

$$\bar{Q}(C(X, Y)Z) - C(\bar{Q}X, Y)Z - C(X, \bar{Q}Y)Z - C(X, Y)\bar{Q}Z = 0 \quad (9.1)$$

for all $X, Y, Z \in \chi(M)$. In view of (2.14), it follows from (9.1) that

$$\begin{aligned} &\bar{Q}(R(X, Y)Z) - R(\bar{Q}X, Y)Z - R(X, \bar{Q}Y)Z - R(X, Y)\bar{Q}Z \\ &\quad + \frac{2r}{n(n-1)}(\bar{S}(Y, Z)X - \bar{S}(X, Z)Y) = 0 \end{aligned}$$

which by taking inner product with ξ yields

$$\begin{aligned} &\eta(\bar{Q}(R(X, Y)Z)) - \eta(R(\bar{Q}X, Y)Z) - \eta(R(X, \bar{Q}Y)Z) - \eta(R(X, Y)\bar{Q}Z) \\ &\quad + \frac{2r}{n(n-1)}(S(Y, Z)\eta(X) - S(X, Z)\eta(Y)) = 0. \end{aligned} \quad (9.2)$$

Putting $Y = \xi$ in (9.2), we have

$$\begin{aligned} &\eta(\bar{Q}(R(X, \xi)Z)) - \eta(R(\bar{Q}X, \xi)Z) - \eta(R(X, \bar{Q}\xi)Z) - \eta(R(X, \xi)\bar{Q}Z) \\ &\quad + \frac{2r}{n(n-1)}(S(\xi, Z)\eta(X) - S(X, Z)\eta(\xi)) = 0. \end{aligned} \quad (9.3)$$

From (2.9), we can easily find

$$\begin{aligned} \eta(\bar{Q}(R(X, \xi)Z)) &= \eta(R(X, \bar{Q}\xi)Z) = (n - 1 - \psi)(g(X, Z) + \eta(X)\eta(Z)), \\ \eta(R(\bar{Q}X, \xi)Z) &= \eta(R(X, \xi)\bar{Q}Z) = (n - 1 - \psi)\eta(X)\eta(Z) + \bar{S}(X, Z). \end{aligned} \quad (9.4)$$

By making use of (2.1), (3.9) and (9.4), (9.3) reduces to

$$\left[\frac{r}{n(n-1)} - 1\right](\bar{S}(X, Z) + (n - 1 - \psi)\eta(X)\eta(Z)) = 0.$$

Thus we have either $r = n(n - 1)$, or

$$\bar{S}(X, Z) = -(n - 1 - \psi)\eta(X)\eta(Z). \quad (9.5)$$

Thus we can state the following theorem:

Theorem 9.1. *If an n -dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $\bar{Q} \cdot C = 0$, then either the scalar curvature is constant or the manifold is a special type of η -Einstein manifold of the form (9.5).*

Example. We consider the 5-dimensional manifold $M = \{(x_1, x_2, x_3, x_4, x_5) \in R^5\}$, where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinates in R^5 . Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$\begin{aligned} e_1 &= \cosh x_5 \frac{\partial}{\partial x_1} + \sinh x_5 \frac{\partial}{\partial x_2}, & e_2 &= \sinh x_5 \frac{\partial}{\partial x_1} + \cosh x_5 \frac{\partial}{\partial x_2}, \\ e_3 &= \cosh x_5 \frac{\partial}{\partial x_3} + \sinh x_5 \frac{\partial}{\partial x_4}, & e_4 &= \sinh x_5 \frac{\partial}{\partial x_3} + \cosh x_5 \frac{\partial}{\partial x_4}, & e_5 &= \frac{\partial}{\partial x_5} = \xi, \end{aligned}$$

which are linearly independent at each point of M and hence form a basis of $T_p M$. Let g be the Lorentzian metric on M defined by

$$g(e_i, e_i) = 1, \text{ for } 1 \leq i \leq 4 \text{ and } g(e_5, e_5) = -1,$$

$$g(e_i, e_j) = 0, \text{ for } i \neq j, 1 \leq i \leq 5 \text{ and } 1 \leq j \leq 5.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1, 1)$ -tensor field defined by

$$\phi e_1 = -e_2, \phi e_2 = -e_1, \phi e_3 = -e_4, \phi e_4 = -e_3, \phi e_5 = 0.$$

By applying linearity of ϕ and g , we have

$$\eta(\xi) = g(\xi, \xi) = -1, \phi^2 X = X + \eta(X)\xi \text{ and } g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Thus for $e_5 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M . Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,$$

$$[e_1, e_5] = -e_2, [e_2, e_5] = -e_1, [e_3, e_5] = -e_4, [e_4, e_5] = -e_3.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) + g(Y, [Z, X]) + g(Z, [X, Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we find

$$\begin{aligned}\nabla_{e_1}e_1 &= 0, \nabla_{e_1}e_2 = -e_5, \nabla_{e_1}e_3 = 0, \nabla_{e_1}e_4 = 0, \nabla_{e_1}e_5 = -e_2, \\ \nabla_{e_2}e_1 &= -e_5, \nabla_{e_2}e_2 = 0, \nabla_{e_2}e_3 = 0, \nabla_{e_2}e_4 = 0, \nabla_{e_2}e_5 = -e_1, \\ \nabla_{e_3}e_1 &= 0, \nabla_{e_3}e_2 = 0, \nabla_{e_3}e_3 = 0, \nabla_{e_3}e_4 = -e_5, \nabla_{e_3}e_5 = -e_4, \\ \nabla_{e_4}e_1 &= 0, \nabla_{e_4}e_2 = 0, \nabla_{e_4}e_3 = -e_5, \nabla_{e_4}e_4 = 0, \nabla_{e_4}e_5 = -e_3, \\ \nabla_{e_5}e_1 &= 0, \nabla_{e_5}e_2 = 0, \nabla_{e_5}e_3 = 0, \nabla_{e_5}e_4 = 0, \nabla_{e_5}e_5 = 0.\end{aligned}$$

Also one can easily verify that

$$\nabla_X\xi = \phi X \quad \text{and} \quad (\nabla_X\phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Therefore, the manifold is a Lorentzian para-Sasakian manifold. By using (1.3), we find

$$\begin{aligned}\bar{\nabla}_{e_1}e_1 &= -e_5, \bar{\nabla}_{e_1}e_2 = -e_5, \bar{\nabla}_{e_1}e_3 = 0, \bar{\nabla}_{e_1}e_4 = 0, \bar{\nabla}_{e_1}e_5 = -e_1 - e_2, \\ \bar{\nabla}_{e_2}e_1 &= -e_5, \bar{\nabla}_{e_2}e_2 = -e_5, \bar{\nabla}_{e_2}e_3 = 0, \bar{\nabla}_{e_2}e_4 = 0, \bar{\nabla}_{e_2}e_5 = -e_1 - e_2, \\ \bar{\nabla}_{e_3}e_1 &= 0, \bar{\nabla}_{e_3}e_2 = 0, \bar{\nabla}_{e_3}e_3 = -e_5, \bar{\nabla}_{e_3}e_4 = -e_5, \bar{\nabla}_{e_3}e_5 = -e_3 - e_4, \\ \bar{\nabla}_{e_4}e_1 &= 0, \bar{\nabla}_{e_4}e_2 = 0, \bar{\nabla}_{e_4}e_3 = -e_5, \bar{\nabla}_{e_4}e_4 = -e_5, \bar{\nabla}_{e_4}e_5 = -e_3 - e_4, \\ \bar{\nabla}_{e_5}e_1 &= 0, \bar{\nabla}_{e_5}e_2 = 0, \bar{\nabla}_{e_5}e_3 = 0, \bar{\nabla}_{e_5}e_4 = 0, \bar{\nabla}_{e_5}e_5 = 0.\end{aligned}$$

From the above results, we can easily obtain the components of the curvature tensor as follows:

$$\begin{aligned}R(e_1, e_2)e_1 &= e_2, R(e_1, e_2)e_2 = -e_1, R(e_1, e_3)e_1 = 0, R(e_1, e_3)e_3 = 0, \\ R(e_1, e_4)e_1 &= 0, R(e_1, e_4)e_4 = 0, R(e_1, e_5)e_1 = -e_5, R(e_1, e_5)e_5 = -e_1, \\ R(e_2, e_3)e_2 &= 0, R(e_2, e_3)e_3 = 0, R(e_2, e_4)e_2 = 0, R(e_2, e_4)e_4 = 0, \\ R(e_2, e_5)e_2 &= -e_5, R(e_2, e_5)e_5 = -e_2, R(e_3, e_4)e_3 = e_4, R(e_3, e_4)e_4 = -e_3, \\ R(e_3, e_5)e_3 &= -e_5, R(e_3, e_5)e_5 = -e_3, R(e_4, e_5)e_4 = -e_5, R(e_4, e_5)e_5 = -e_4,\end{aligned}$$

and

$$\begin{aligned}\bar{R}(e_1, e_2)e_1 &= 0, \bar{R}(e_1, e_2)e_2 = 0, \bar{R}(e_1, e_3)e_1 = -e_3 - e_4, \bar{R}(e_1, e_3)e_3 = e_1 + e_2, \\ \bar{R}(e_1, e_4)e_1 &= -e_3 - e_4, \bar{R}(e_1, e_4)e_4 = e_1 + e_2, \bar{R}(e_1, e_5)e_1 = -e_5, \bar{R}(e_1, e_5)e_5 = -e_1 - e_2, \\ \bar{R}(e_2, e_3)e_2 &= -e_3 - e_4, \bar{R}(e_2, e_3)e_3 = -e_1 - e_2, \bar{R}(e_2, e_4)e_2 = -e_3 - e_4, \bar{R}(e_2, e_4)e_4 = e_1 + e_2, \\ \bar{R}(e_2, e_5)e_2 &= -e_5, \bar{R}(e_2, e_5)e_5 = -e_1 - e_2, \bar{R}(e_3, e_4)e_3 = 0, \bar{R}(e_3, e_4)e_4 = 0, \\ \bar{R}(e_3, e_5)e_3 &= -e_5, \bar{R}(e_3, e_5)e_5 = -e_3 - e_4, \bar{R}(e_4, e_5)e_4 = -e_5, \bar{R}(e_4, e_5)e_5 = -e_3 - e_4.\end{aligned}$$

From these curvature tensors, we calculate

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = 0, \quad S(e_5, e_5) = -4, \quad (9.6)$$

$$\bar{S}(e_1, e_1) = \bar{S}(e_2, e_2) = \bar{S}(e_3, e_3) = \bar{S}(e_4, e_4) = 3, \quad \bar{S}(e_5, e_5) = -4. \quad (9.7)$$

Therefore, from (9.6) and (9.7) we obtain $r = 4$ and $\bar{r} = 16$, respectively. Thus it can be seen that the equation (3.5) is satisfied, where $\psi = \sum_{i=1}^5 \epsilon_i g(\phi e_i, e_i) = 0$.

From (1.1), we calculate the components of torsion tensor as follows:

$$T(e_i, e_j) = 0, \text{ for } 1 \leq i, j \leq 5, \quad T(e_i, e_5) = -e_i, \text{ for } i = 1, 2, 3, 4. \quad (9.8)$$

From (1.2), it can be easily seen that

$$(\bar{\nabla}_{e_i} g)(e_j, e_k) = 0, \text{ for any } 1 \leq i, j, k \leq 5. \quad (9.9)$$

Thus by virtue of (9.8) and (9.9), we say that the linear connection $\bar{\nabla}$ defined by (1.3) on the manifold M is a semi-symmetric metric connection.

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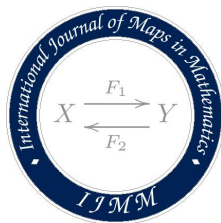
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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, JAZAN UNIVERSITY, JAZAN-2097, KINGDOM OF SAUDI ARABIA.

Email address: malikhaseeb80@gmail.com, haseeb@jazanu.edu.sa

DEPARTMENT OF MATHEMATICS AND ASTRONOMY, UNIVERSITY OF LUCKNOW, LUCKNOW-226007, INDIA.

Email address: rp.manpur@rediffmail.com



SEMI-SYMMETRIC METRIC CONNECTION ON COSYMPLECTIC MANIFOLDS

BABAK HASSANZADEH  *

ABSTRACT. In this paper we studied almost contact manifolds with semi-symmetric connection, especially Sasakian manifolds. Curvature, sectional curvature and ϕ -sectional curvature are calculated by semi-symmetric connection. Furthermore; geometric properties of integral submanifold of Sasakian manifolds are investigated.

1. INTRODUCTION

The idea of a semi symmetric connection on a smooth manifolds was first introduce by Friedmann and Schouten in 1924, [3]. The Sasakian manifolds were introduced in the 1960's by S. Sasaki as an odd-dimensional analogous of Kaehler manifolds. Kaehler manifolds are a classical object of differential geometry and well studied in literature. Compared to that Sasakian manifolds have only recently become subject of deeper research in mathematics and physics. Semi-symmetric connection studied by many authors from 1924 so far. In 1993, Benjancu and Duggal [2] introduced the concept of (ε) -Sasakian manifolds. Afterwards, in 2014, Ram Nawal Singh, Shravan Kumar Pandey, Giteshwari Pandey and Kiran Tiwari examined semi-symmetric connection in an (ε) -Kenmotsu manifold.

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* Corresponding author

In the present paper, in the first section, Sasakian manifold are examined, then in next section cosymplectic manifolds are studied using semi symmetric metric connection.

2. PRELIMINARIES

Let M be an odd dimensional smooth manifold with a Riemannian metric g and Riemannian connection ∇ . Denote by TM the Lie algebra of vector fields on M . Then M is said to be an almost contact metric manifold if there exist on M a tensor ϕ of type $(1, 1)$, a vector field ξ called structure vector field and η , the dual 1-form of ξ satisfying the following

$$\phi^2 X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X) \quad (2.1)$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \quad (2.2)$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

for any $X, Y \in TM$. In this case

$$g(\phi X, Y) = -g(X, \phi Y). \quad (2.4)$$

If $d\eta(X, Y) = g(X, \phi Y)$, for every $X, Y \in TM$, then we say that M is a contact metric manifold. If ξ is a killing vector field with respect to g , the contact metric structure is called a K-contact structure. It is easy to prove that a contact metric manifold is K-contact if and only if $\nabla_X \xi = -\phi X$, for any $X \in TM$, where ∇ denotes the Levi-Civita connection on M . We are thus led to define four tensors N^1, N^2, N^3, N^4 by

$$N^{(1)}(X, Y) = [\phi, \phi](X, Y) + 2d\eta(X, Y)\xi,$$

$$N^{(2)}(X, Y) = (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X),$$

$$N^{(3)} = (\mathcal{L}_{\xi}\phi)X,$$

$$N^{(4)} = (\mathcal{L}_{\xi}\eta)X.$$

An almost contact structure (ϕ, ξ, η) is normal if and only if these four tensors are equal to zero. Now we give some useful theorems.

Theorem 2.1. [2] *An almost contact metric structure (ϕ, ξ, η, g) is Sasakian if and only if*

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X.$$

A Sasakian manifold is K-contact then ξ is Killing vector field and $\nabla_X \xi = -\phi X$.

Proposition 2.1. [2] *On a Sasakian manifold,*

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y.$$

Theorem 2.2. [2] *A contact metric manifold is K-contact if and only if the sectional curvature of all plane sections containing ξ are equal to 1. Moreover, on a K-contact manifold,*

$$R(X, \xi)\xi = X - \eta(X)\xi.$$

Let M be a submanifold of \tilde{M} and TM and $T^\perp M$ be the Lie algebras of vector fields tangential and normal to \tilde{M} , respectively. Suppose $\tilde{\nabla}$ is the induced Levi-Civita connection on \tilde{M} . The Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad (2.5)$$

$$\tilde{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad (2.6)$$

for all $X, Y \in TM$ and $V \in T^\perp M$, where ∇^\perp is the connection on the normal bundle $T^\perp M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X, Y), V), \quad (2.7)$$

for then using the standard formula namely Koszul formula for the Levi-Civita connection,

$$\begin{aligned} g(\nabla_X Y, Z) = & \frac{1}{2} \{ Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ & + g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X) \}, \end{aligned} \quad (2.8)$$

for all $X, Y \in TM$.

3. SEMI-SYMMETRIC METRIC CONNECTIONS

A linear connection $\bar{\nabla}$ defined on contact metric manifold M is said to be semi-symmetric connection[3], if its torsion tensor

$$\bar{T}(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y]$$

satisfies

$$\bar{T}(X, Y) = \eta(Y)X - \eta(X)Y.$$

Further, a connection is called a semi-symmetric metric connection[5] if

$$(\bar{\nabla}_X g)(Y, Z) = 0.$$

The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection is given by[4]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X, Y)\xi. \quad (3.9)$$

Let M be a Sasakian manifold and ∇ be a Levi-Civita connection defined on M. Using 3.9 we obtain

$$(\bar{\nabla}_X \eta)Y = \bar{\nabla}_X g(Y, \xi) - \eta(\bar{\nabla}_X Y) = -\eta(\nabla_X Y) - \eta(Y)\eta(X) + g(X, Y). \quad (3.10)$$

From the definition immediately we obtain the following useful facts.

- 1) $\bar{\nabla}_X \phi Y = \nabla_X \phi Y - g(X, \phi Y)\xi,$
- 2) $\bar{\nabla}_{\phi X} Y = \nabla_{\phi X} Y + \eta(Y)\phi X - g(\phi X, Y)\xi,$
- 3) $\bar{\nabla}_{\phi X} \phi Y = \nabla_{\phi X} \phi Y + \eta(X)\eta(Y)\xi - g(X, Y)\xi,$
- 4) $\bar{\nabla}_{\phi X} \xi = 0,$
- 5) $\bar{\nabla}_\xi X = \nabla_\xi X,$

for all $X, Y \in TM$.

Lemma 3.1. *On Sasakian manifold,*

$$(\bar{\nabla}_{\phi X} \phi)Y = g(\phi X, Y)\xi - g(X, Y)\xi - \eta(Y)\phi X + \eta(Y)X.$$

Proof.

$$\begin{aligned} (\bar{\nabla}_{\phi X} \phi)Y &= \bar{\nabla}_{\phi X} \phi Y - \phi \bar{\nabla}_{\phi X} Y = \nabla_{\phi X} \phi Y - g(\phi X, \phi Y)\xi - \phi \nabla_{\phi X} Y - \eta(Y)\phi^2 X, \\ &= \nabla_{\phi X} \phi Y - \phi \nabla_{\phi X} Y + \eta(Y)\phi X - g(X, Y)\xi, \\ &= g(\phi X, Y)\xi - g(X, Y)\xi - \eta(Y)\phi X + \eta(Y)X. \end{aligned}$$

The proof is completed.

Let the curvature tensor \bar{R} given by

$$\bar{R}(X, Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X, Y]} Z,$$

where $\bar{\nabla}$ is semi-symmetric connection. Using 3.9, we obtain routinely

$$\begin{aligned} \bar{R}(X, Y)Z &= R(X, Y)Z + \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y, \\ &\quad - g(Y, Z)X + g(X, Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi, \\ &\quad + g(\nabla_X \xi, Z)Y - g(Y, Z)\nabla_X \xi - g(Z, \nabla_Y \xi)X + g(X, Z)\nabla_Y \xi. \end{aligned} \quad (3.11)$$

For Sasakian manifolds the equation 3.9 reduces to

$$\begin{aligned}\bar{R}(X, Y)Z &= R(X, Y)Z + \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y, \\ &- g(Y, Z)X + g(X, Z)Y + g(Y, Z)\eta(X)\xi - g(X, Z)\eta(Y)\xi, \\ &- g(\phi X, Z)Y + g(Y, Z)\phi X + g(Z, \phi Y)X - g(X, Z)\phi Y.\end{aligned}\tag{3.12}$$

To calculate the sectional curvature, first we have

$$\begin{aligned}\bar{R}(X, Y, Y, X) &= R(X, Y, Y, X) + \eta(Y)\eta(Y)g(X, X), \\ &- g(Y, Y)g(X, X) + g(X, Y)g(X, Y) + \eta(X)\eta(X)g(Y, Y).\end{aligned}\tag{3.13}$$

Assume $\{X, Y\}$ are orthonormal, then

$$\bar{R}(X, Y, Y, X) = R(X, Y, Y, X) + \eta(Y)\eta(Y) + \eta(X)\eta(X) - 1,\tag{3.14}$$

therefore

$$\bar{K}(X, Y) = K(X, Y) + \eta(Y)\eta(Y) + \eta(X)\eta(X) - 1.\tag{3.15}$$

For Sasakian manifolds we have $R(X, Y)\xi = \eta(Y)X - \eta(X)Y$, then from 3.9, we obtain

$$\bar{R}(X, Y)\xi = \eta(Y)X - \eta(X)Y - \eta(X)\phi Y + \eta(Y)\phi X.$$

3.1. Integral submanifolds.

Definition 3.1. A submanifold N of M is an integral submanifold, if $\eta(X) = 0$ for every $X \in TN$. [1]

Lemma 3.2. Let M be a Sasakian manifold with a semi-symmetric metric connection. Assume N be an integral submanifold, then

$$(\bar{\nabla}_X \phi)Y = g(X, Y)\xi,$$

for any $X, Y \in TN$.

Proof. If N be an integral submanifold, then ξ is normal to N , hence

$$(\bar{\nabla}_X \phi)Y = \bar{\nabla}_X \phi Y - \phi \bar{\nabla}_X Y = \nabla_X \phi Y - g(X, \phi Y)\xi - \phi \nabla_X Y + g(Y, \xi)\phi X = (\nabla_X \phi)Y.$$

Using theorem 2.1 the proof is trivial.

For integral submanifolds the equation 3.9 become to

$$\bar{R}(X, Y)Z = R(X, Y)Z + g(X, Z)Y - g(Y, Z)X - g(X, Z)\phi Y + g(Y, Z)\phi X, \quad (3.16)$$

which present the relation between curvature tensors of connections $\bar{\nabla}$ and ∇ in integral submanifolds of Sasakian manifolds. From 3.16, we get

$$g(\bar{R}(X, Y)Z, V) = g(R(X, Y)Z, V) + g(X, Z)g(Y, V) - g(Y, Z)g(X, V). \quad (3.17)$$

Suppose $\bar{R}(X, Y)Z = 0$, which by virtue of the equation 3.17 yields

$$g(R(X, Y)Z, V) = g(Y, Z)g(X, V) - g(X, Z)g(Y, V). \quad (3.18)$$

We know $R(X, \xi)\xi = X$, and we can caculate easily $R(\xi, X)\xi = X$, hence

$$\bar{R}(\xi, X)\xi = 2X - \phi X,$$

it's trivial $\bar{R}(X, Y)\xi = R(X, Y)\xi = 0$ and $\bar{R}(X, \xi)\xi = X$. Also, ϕ -sectional curvature is defined by

$$K(\square) = K(X, \phi X) = R(X, \phi X; \phi X, X).$$

Assume $X \in N$ be an unit vector field, then

$$\begin{aligned} \bar{R}(X, \phi X; \phi X, X) &= R(X, \phi X; \phi X, X) - g(X, X)g(\phi X, \phi X) \\ &= R(X, \phi X; \phi X, X) - 1, \end{aligned}$$

and we conclude $\bar{K} = K - 1$.

Lemma 3.3. *Let N be an integral submanifold of Sasakian maniold M , then*

$$\nabla_{\xi} Y = [\xi, Y],$$

for all $X, Y \in N$.

Proof. By 2.8, following equations are obtained

$$2g(\nabla_{\xi} X, Y) = \xi g(X, Y) + g([\xi, X], Y) + g([Y, \xi], X),$$

using $\xi g(X, Y) = g(\nabla_{\xi} X, Y) + g(X, \nabla_{\xi} Y)$ leads to

$$g(\nabla_{\xi} X, Y) = g(X, \nabla_{\xi} Y) + g([\xi, X], Y) + g([Y, \xi], X). \quad (3.19)$$

Also,

$$2g(\nabla_X \xi, Y) = \xi g(X, Y) + g([X, \xi], Y) + g([Y, \xi], X) = 0. \quad (3.20)$$

Comparing 3.19 and 3.20 complete the proof.

4. Cosymplectic manifolds

A normal almost contact metric manifold M is called a cosymplectic manifold if

$$(\nabla_X \phi)Y = 0, \quad \nabla_X \xi = 0, \quad (4.21)$$

where ∇ denotes Levi-Civita connection. From we have

$$(\bar{\nabla}_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi. \quad (4.22)$$

Following facts easily can be obtained

- (1) $(\bar{\nabla}_X \phi Y) = \phi \nabla_X Y - g(X, \phi Y)\xi,$
- (2) $\bar{\nabla}_\xi \phi X = \nabla_\xi \phi X,$
- (3) $\bar{\nabla}_\xi Y = \nabla_\xi Y.$

Using obtained facts we obtain

$$g(\bar{\nabla}_X \phi Y, \xi) = g(\phi X, Y), \quad (4.23)$$

$$g(\nabla_\xi \phi X, Y) = g(\nabla_\xi \phi Y, X). \quad (4.24)$$

From 2.4 and 3.9 we get

$$\bar{\nabla}_X \xi = -\phi^2 X, \quad (\bar{\nabla}_X \phi)\xi = -\phi X. \quad (4.25)$$

Lemma 4.1. *Let M be a cosymplectic manifold, then*

$$\eta((\bar{\nabla}_X \phi)Y) = \eta(\bar{\nabla}_X \phi Y),$$

for all $X, Y \in TM$.

Proof. Using 4.22 and other obtained facts for Cosymplectic manifolds we get

$$\eta((\bar{\nabla}_X \phi)Y) = \eta(g(\phi X, Y)\xi) = g(\phi X, Y) = \eta(\bar{\nabla}_X \phi Y),$$

the proof is complete.

Based on theorem 6.8 [1] it can be seen $d\eta = 0$, then

$$2d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]) = 0. \quad (4.26)$$

Assume $X, Y \in TM$ are orthogonal elements. Using 2.8, 4.25 and 4.26 we find out

$$2g(\nabla_X \xi, Y) = X\eta(Y) - Y\eta(X) + g([X, \xi], Y) + \eta([X, Y]) + g([Y, \xi], X).$$

Therefore $g([X, \xi], Y) + g([Y, \xi], X) = 0$. Using 2.8 we have

$$\eta(\nabla_X Y) = X\eta(Y), \quad \eta(\nabla_Y X) = Y\eta(X). \quad (4.27)$$

Since M is an almost cosymplectic manifold, from 4.26 following statement is valid

$$(\nabla_X \eta)(Y) - (\nabla_Y \eta)(X) = 0.$$

Also, we have

$$(\bar{\nabla}_X \eta)(Y) = \eta(\bar{\nabla}_X \xi, Y) = g(\phi X, \phi Y).$$

Thus

$$(\bar{\nabla}_X \eta)Y + (\bar{\nabla}_Y \eta)X,$$

for all $X, Y \in TM$. Assume $\bar{\nabla}_X \phi Y = 0$, for all $X, Y \in TM$, from 3.9 we obtain

$$\nabla_X \phi Y = g(X, \phi Y)\xi. \quad (4.28)$$

For cosymplectic manifold we have

$$g(\nabla_X \phi Y, \xi) = X\eta(\phi Y) - g(\phi Y, \nabla_X \xi) = 0.$$

On the other side we know $g(\nabla_X \phi Y, \xi) = g(X, \phi Y)$ we realized that X is orthogonal to $Im\phi$. From 4.28 we have $\phi^2 \nabla_X Y = 0$, using 2.1 leads to

$$\nabla_X Y = \eta(\nabla_X Y)\xi.$$

Furthermore we have $\nabla_Y X = \eta(\nabla_Y X)\xi$, comparing last two equations we have

$$[X, Y] = (\nabla_X Y - \nabla_Y X)\xi. \quad (4.29)$$

Now we have proved

Theorem 4.1. *Let M be a cosymplectic manifold with semi symmetric metric connection $\bar{\nabla}$. If there is vector fields $X, Y \in TM$, such that $\bar{\nabla}_X Y = 0$, then*

$$\phi([X, Y]) = 0.$$

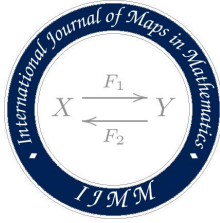
Proof. From 4.29 the proof is trivial.

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INSTITUTE OF MATHEMATICS AND STATISTICS, MASARYK UNIVERSITY, BUILDING 08, KOTLARSKA 2,
BRNO, 611 37, CZECH REPUBLIC

Email address: babakmath777@gmail.com- bhassanzadeh@math.muni.cz



CHARACTERISTICS OF LIGHTLIKE HYPERSURFACES OF TRANS-PARA SASAKIAN MANIFOLDS

MOHD DANISH SIDDIQI *

ABSTRACT. In this research article, we study three lightlike hypersurfaces of trans-para Sasakian manifolds with a quarter-symmetric metric connection: (1) re-current, (2) Lie re-current and (3) Hopf-lightlike hypersurface. Also, we discuss some properties of a screen semi-invariant lightlike hypersurface of trans-para Sasakian manifolds with a quarter-symmetric metric connection. Furthermore, we show that a conformal hypersurface is screen totally geodesic lightlike hypersurfaces. Finally we prove the integrability conditions for the distributions of screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold with a quarter-symmetric metric connection.

1. INTRODUCTION

In 1975, S. Golab [11] initiated the study of a quarter-symmetric connection in a differentiable manifold.

A linear connection $\bar{\nabla}$ on an n -dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection if its torsion tensor T of the connection $\bar{\nabla}$ [11]

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \quad (1.1)$$

satisfies

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* Corresponding author

$$T(X, Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \quad (1.2)$$

where η is a 1-form and φ is a $(1, 1)$ tensor field.

In particular, if $\varphi(X) = X$, then the quarter-symmetric connection is reduces to the semi-symmetric connection [9]. Thus the notion of a quarter-symmetric connection is the extension of the semi-symmetric connection. Moreover if, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y, Z) = 0 \quad (1.3)$$

for all $X, Y, Z \in T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M , then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection. Otherwise it is said to be a quarter-symmetric non-metric connection.

After S. Golab [11], numerous geometers (see [16], [24], [25], [2]) continued the systematic and specific study of a quarter-symmetric metric connection with various structures in several ways to a different extent .

The differential geometry of lightlike hypersurfaces is one of the most specific topic in the theory of lightlike submanifolds. Lightlike hypersurfaces have several significant applications in mathematical physics [4], electromagnetic [5], black hole theory [3], string theory and general relativity [10]. A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric is degenerate. In 1996, K. L. Duggal, A. Bejancu established the conception of lightlike submanifolds of almost contact metric manifolds [5].

Also K. L. Duggal with B. Sahin and others geometers have further developed this concept and studied many new classes of lightlike submanifolds (for more details see [1], [6], [7], [8], [13]).

Furthermore, K. L. Duggal and R. Sharma [9] also studied some properties of semi-Riemannian manifold with a semi-symmetric metric connection. They proved that these geometric results have many physical applications in real world.

Inspired by the above studies others geometers like D. H. Jin have been exclusively studied on lightlike hypersurfaces with respect to the different connections such as semi symmetric metric and quarter-symmetric metric connection (cf. [17], [18], [19], [20], [21], [22]).

On the other hand, in 1985, S. Kaneyuki and M. Konzai [23] initiated the study of a para-complex structure and almost para-contact structure on a semi-Riemannian manifold.

S. Zamkovoy [26] has extensively studied para contact metric manifolds after that there are many papers discussed the contribution of para-contact geometry of a semi-Riemannian manifolds ([27, 28]. In 2019, S. Zamkovoy [27] also introduced the geometry of trans-para-Sasakian manifolds. An almost contact structure on a manifold M is called a trans-Sasakian structure if the product manifolds $M \times \mathbb{R}$ belongs to the class W_4 [12]. In ([14], [15]), J. C. Marrero and D. Chinea are completely characterized trans-Sasakian structures of types (α, β) . We note that the trans-Sasakian structures of type $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α -Sasakian [12], β -Kenmotsu [14], and cosymplectic [14], respectively. In [27], S. Zamkovoy consider the trans-para-Sasakian manifolds as an analogue of the trans-Sasakian manifolds. A trans-para-Sasakian manifolds is a trans-para-Sasakian structure of type (α, β) , where α and β are smooth functions. The trans-para-Sasakian manifolds of type (α, β) , are called para-Sasakian manifolds ($\alpha = 1$), para-Kenmotsu manifolds ($\beta = 1$) [14] and para-cosymplectic manifolds ($\alpha = \beta = 0$).

Motivated by the above research articles, we consider the three types of lightlike hypersurfaces of trans-para-Sasakian manifolds with respect to the quarter-symmetric metric connection in the present framework.

2. PRELIMINARIES

A $(2n + 1)$ -dimensional smooth manifold M has an almost paracontact structure (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying the following compatibility conditions

$$\varphi^2 X = X - \eta(X)\xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1. \quad (2.4)$$

The distribution $\mathbb{D} : p \in M \longrightarrow \mathbb{D}_p \subset T_p M : \mathbb{D}_p = \text{Ker}\eta = \{X \in T_p M : \eta(X) = 0\}$ is called a paracontact distribution generated by η .

By the definition of an almost paracontact structure the endomorphism φ has rank $2n$.

If a $(2n + 1)$ -dimensional manifold M with (φ, ξ, η) structure admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad (2.5)$$

where $X, Y \in T(M)$ then we say that M has an almost paracontact metric structure with compatible metric g . Any compatible metric g with a given almost paracontact structure with signature $(n + 1, n)$. Note that setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$. Further, any

almost paracontact structure admits a compatible metric.

Definition 2.1. *An almost paracontact metric manifold $(M, \varphi, \eta, \xi, g)$ is said to be a paracontact metric manifold if $(g(X, \varphi Y) = d\eta(X, Y)$, where $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y]))$ and η is a paracontact form.*

A paracontact structure on M naturally gives rise to an almost paracomplex structure on the product $M \times \mathbb{R}$. If this almost paracomplex structure is integrable, then the given paracontact metric manifold is said to be a para-Sasakian (see [20]). A paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X. \quad (2.6)$$

The manifold $(M, \varphi, \xi, \eta, g)$ of dimension $(2n + 1)$ is said to be trans-para-Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = \alpha(-g(X, Y)\xi + \eta(Y)X) + \beta(g(X, \varphi Y)\xi + \eta(Y)\varphi X). \quad (2.7)$$

From (2.7), we also have

$$\nabla_X \xi = -\alpha\varphi X - \beta(X - \eta(X)\xi). \quad (2.8)$$

Now, we have the following lemma [17]

Lemma 2.1. [17] *Let $(M, \varphi, \eta, \xi, g)$ be a trans-para-Sasakian manifold. Then we have*

$$R(X, Y)\xi = -(\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y], \quad (2.9)$$

$$R(\xi, Y)Z = -(\alpha^2 + \beta^2)[g(Y, Z)\xi - \eta(Z)X], \quad (2.10)$$

$$S(X, \xi) = -2n(\alpha^2 + \beta^2)\eta(X), \quad (2.11)$$

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) - \beta(g(X, Y) - \eta(X)\eta(Y)), \quad (2.12)$$

for all $X, Y, Z \in T(M)$, where R is a Riemannian curvature tensor and S is a Ricci curvature tensor.

3. QUARTER-SYMMETRIC METRIC CONNECTION

In this section, we express few tensorial relations for a trans-para Sasakian manifold with quarter-symmetric metric connection.

Let $\bar{\nabla}$ be a linear connection and ∇ be the Levi-Civita connection of a almost paracontact metric manifold M such that

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{3.13}$$

where H is a $(1, 1)$ -tensor type. For $\bar{\nabla}$ to be a quarter-symmetric metric connection in M , we have [11]

$$H(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \tag{3.14}$$

where

$$g(T'(X, Y), Z) = g(T(Z, X)Y). \tag{3.15}$$

From (1.1) and (3.15), we find

$$T'(X, Y) = \eta(X)\varphi Y - g(\varphi X, Y)\xi. \tag{3.16}$$

Using (1.1) and (3.16) in (3.14), we arrive at

$$H(X, Y) = \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{3.17}$$

Therefore, a quarter-symmetric metric connection $\bar{\nabla}$ in a trans-para Sasakian manifold is given by [16]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{3.18}$$

Now, using (3.18), (2.7) and (2.8), we obtain the following results:

Theorem 3.1. *Let M be a trans-para Sasakian manifold with a quarter-symmetric metric connection. Then*

$$(\bar{\nabla}_X \varphi)Y = (1 - \alpha) \{g(X, Y)\xi - \eta(Y)X\} + \beta \{g(X, \varphi Y)\xi + \eta(Y)\varphi X\} \tag{3.19}$$

$$\bar{\nabla}_X \xi = (1 - \alpha)\varphi X - \beta(X - \eta(X)\xi). \tag{3.20}$$

4. LIGHTLIKE HYPERSURFACES

Let \bar{M} be a semi-Riemannian manifold with index r , $0 < r < 2n + 1$ and M be a hypersurface of \bar{M} , with induced metric $g = \bar{g}$. M is a null hypersurface of \bar{M} if the metric g is of rank $2n - 1$. The orthogonal complement TM^\perp of the tangent space TM , given as

$$TM^\perp = \left\{ X_p \in T_p M^\perp : g_p(X_p, Y_p) = 0, \forall Y_p \in \Gamma(T_p M) \right\}$$

is a distribution of rank 1 on M . If $TM^\perp \subset TM$ and then coincides with the *radical distribution* $Rad(TM)$ such that

$$Rad(TM) = TM \cap TM^\perp. \quad (4.21)$$

A complementary bundle of TM^\perp in TM is a non-degenerate distribution of constant rank $2n - 1$ over M . It is known as a *screen distribution* and denoted by $S(TM)$.

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold \bar{M} . Then there exists a unique rank over subbundle $tr(TM)$ called the lightlike *transversal vector* bundle of M with respect to $S(TM)$, such that for any null section ξ of $Rad(TM)$ on coordinate neighborhood U of M , there exists a unique section N of $tr(TM)$ on U satisfying

$$g(N, X) = 0, \quad g(N, N) = 0, \quad g(N, \xi) = 1, \forall X \in \Gamma(S(TM)) \quad (4.22)$$

Then, we have the decomposition on the tangent bundle

$$TM = S(TM) \perp Rad(TM), \quad (4.23)$$

$$T\bar{M} = TM \oplus tr(TM) = S(TM) \perp \{Rad(TM) \oplus tr(TM)\}. \quad (4.24)$$

Let $P : TM \rightarrow S(TM)$ be the projection morphism. Then, we have the local Gauss-Weingarten formulas of M and $S(TM)$ as follows

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \quad (4.25)$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{tr} N, \quad (4.26)$$

$$\nabla_X P T = \nabla_X^* P Y + C(X, P Y)\xi, \quad (4.27)$$

$$\bar{\nabla}_X \xi = -A_\xi^* X - \tau(X)\xi \quad (4.28)$$

for any $X, Y \in \Gamma(TM)$, where ∇ is a linear connection on M and ∇^* is a linear connection on $S(TM)$ and B , A_N and τ are called the local second fundamental form on $T(M)$ respectively.

It is well know that the induced connection ∇ is quarter-symmetric non-metric connection and we get

$$(\nabla_X g) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{4.29}$$

$$T(X, Y) = \eta(X)Y - \eta(Y)X. \tag{4.30}$$

where T is the torsion tensor with respect to the induced connection ∇ on M , B is symmetric on $T(M)$ and $\eta(X) = g(X, N)$ is a differential 1-form on TM .

For the second fundamental form B , we have

$$B(X, \xi) = 0. \tag{4.31}$$

The local second fundamental forms are related to their shape operators by

$$B(X, PY) = g(A_\xi^* X, PY), \quad g(A_\xi^* X, N) = 0, \tag{4.32}$$

$$C(X, PY) = g(A_N X, PY), \quad g(A_N X, N) = 0. \tag{4.33}$$

From (4.32), A_ξ^* is a $S(TM)$ -valued real self-adjoint operator and satisfies

$$A_\xi^* \xi = 0. \tag{4.34}$$

5. SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACES

This segment deal with screen semi-invariant lightlike hypersurfaces of a trans-para Sasakian manifold equipped with a quarter-symmetric metric connection.

Let M be a lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with $\xi \in \Gamma(TM)$. If ξ is a local section of $\Gamma Rad(TM)$, then

$$g(\varphi\xi, \xi) = 0, \tag{5.35}$$

and $\varphi\xi$ is tangent to M . Therefore, we obtain a distribution $\varphi(Rad(TM))$ of dimension 1 on M .

If

$$\varphi((tr(TM)) = (tr(TM), \quad \text{and} \quad \varphi(Rad(TM)) = Rad(TM), \tag{5.36}$$

then lightlike hypersurface M is called a screen semi-invariant lightlike hypersurface of \bar{M} [1].

Since M is a screen semi-invariant lightlike hypersurface

$$g(\varphi N, N) = 0 \quad (5.37)$$

$$g(\varphi N, \xi) = -g(N, \varphi \xi) = 0. \quad (5.38)$$

$$g(N, \xi) = 1 \quad (5.39)$$

from (2.5), we obtain

$$g(\varphi \xi, \varphi N) = -1. \quad (5.40)$$

Therefore, $\varphi(Rad(TM)) \oplus \varphi(tr(TM))$ is a non-degenerate vector subbundle of screen distributions $S(TM)$.

Now, since $S(TM)$ and $\varphi(Rad(TM)) \oplus \varphi(tr(TM))$ are non-degenerate distribution \bar{D}_0 such that

$$S(TM) = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\}. \quad (5.41)$$

Therefore, $\varphi(D_0) = D_0$ and $\xi \in D_0$. In view of (4.23), (4.25) and (5.41) we obtain the followings

$$TM = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\} \perp Rad(TM) \quad (5.42)$$

$$T\bar{M} = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\} \perp \{Rad(TM) \oplus tr(TM)\}. \quad (5.43)$$

Now, we take $D_1 = Rad(TM) \perp \varphi(Rad(TM)) \perp D_0$ and $D_2 = \varphi(tr(TM))$ on M , we get

$$TM = D_1 \oplus D_2. \quad (5.44)$$

Let the local null vector fields $V = \varphi \xi$ and $U = \varphi N$ and denote the projection morphism of TM into D_1 and D_2 by P_1 and P_2 , respectively. Therefore, for $X \in \Gamma(TM)$, we have

$$X = P_1 X + P_2 X, \quad P_2 X = u(X) \bar{U}, \quad (5.45)$$

where u is a differential 1-form locally defined by

$$u(X) = -g(\varphi \xi, X), \quad \text{and} \quad v(X) = -g(\varphi N, X). \quad (5.46)$$

Operating φ on X , we get

$$\varphi X = \varphi(P_1 X) + u(X) N. \quad (5.47)$$

If we put $\varphi X = \varphi(P_1 X)$ in above relation, we obtain the following:

$$\varphi X = \omega X + u(X)N, \quad (5.48)$$

where ω is a tensor field defined as $\omega = \varphi \circ P_1$ of type $(1, 1)$.

Again operating ω to (5.48), we get

$$\omega^2 X = X - \eta(X)\xi - u(X)(U), \quad u(U) = 1. \quad (5.49)$$

Replacing Y by ξ in (4.25) with (3.19) and (5.48), we have

$$\nabla_X \xi = (1 - \alpha)\omega X + \beta(X - \eta(X)\xi), \quad (5.50)$$

$$B(X, \xi) = (1 - \alpha)u(X). \quad (5.51)$$

From the covariant derivative of $g(\xi, N) = 0$ in terms of X with (3.19), (5.49) and (4.33), we obtained that

$$C(X, \xi) = (1 - \alpha)v(X) + \beta\eta(X). \quad (5.52)$$

Now, from (4.23) comparing the different components, we get

$$(\nabla_X \omega)Y = (1 - \alpha)[g(X, Y)\xi - \eta(Y)X] + \beta[g(X, \varphi Y)\xi + \eta(Y)\omega X] \quad (5.53)$$

$$+B(X, Y)\bar{U} + u(Y)A_N X,$$

$$(\nabla_X u)Y = u(Y)\tau(X) - B(X, \omega X) + \beta\eta(Y)u(X), \quad (5.54)$$

$$(\nabla_X v)Y = v(Y)\tau(X) + g(A_N X, \omega Y) + [(1 - \alpha)\eta(X) + \beta v(X)]\eta(Y), \quad (5.55)$$

$$\nabla_X \bar{U} = \omega(A_N X - \tau(X)\bar{U}) + [(1 - \alpha)\eta(X) + \beta v(X)]\xi, \quad (5.56)$$

$$\nabla_X \bar{V} = \omega(A_E^* X) - \tau(X)U + \beta u(X)\xi, \quad (5.57)$$

$$B(X, \bar{U}) = C(X, \bar{V}). \quad (5.58)$$

where \bar{U} and \bar{V} are the structure tensor fields on M .

6. RE-CURRENT SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACE

Now, we give the following definition

Definition 6.1. *Let M be a screen semi-invariant lightlike hypersurface of trans-para Sasakian manifold \bar{M} and μ be a 1-form on M . If M admits a re-current tensor field ω such that*

$$(\nabla_X \omega)Y = \mu(X)\omega Y \quad (6.59)$$

then said to be recurrent [9].

Theorem 6.1. *Let M is a re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection. Then*

- (1) $\alpha = 1, \beta = 0$ i.e., \bar{M} is a para-Sasakian manifold,
- (2) ω is parallel with respect to the induced connection ∇ on M ,
- (3) $A_N X = -\mu(X)\bar{U} - v(X)\xi$
- (4) $A_\xi^* X = -\mu(X)\bar{V} - u(X)\xi$.

for all $X, Y \in \Gamma T(M)$.

Proof. (1) From (5.53), we have

$$\begin{aligned} \mu(X)\omega Y &= (1 - \alpha)(g(X, Y)\xi - \eta(Y)X) + \beta(g(X, \varphi Y)\xi + \eta(Y)\omega X) \\ &\quad + B(X, Y)\bar{U} + u(Y)A_N X. \end{aligned} \quad (6.60)$$

Setting $Y = \xi$ in (6.60) and using (2.4), we obtained that

$$(1 - \alpha) \{X - \eta(X)\xi + u(X)U\} + \beta\omega X = 0. \quad (6.61)$$

Putting $X = \xi$ in (6.61) and using the fact that $\omega\xi = V$, we have

$$(1 - \alpha)\xi + \beta V = 0. \quad (6.62)$$

Taking the scalar product with N and \bar{U} to the above equation, we get

$$\alpha = 1, \quad \beta = 0. \quad (6.63)$$

Therefore, \bar{M} is a para-Sasakian manifold with a quarter-symmetric metric connection and we arrive at (1).

(2) Taking $Y = \xi$ to (6.60) and in view (4.32) and (5.46), we get

$$\mu(X)V = -g(X, \xi)\xi. \tag{6.64}$$

Taking inner product of \bar{U} it follows that $\mu = 0$. Thus, ω is parallel with respect to the connection ∇ and we arrive at (2).

(3) Now taking $Y = \bar{U}$ in (6.60) and using the fact that $\mu(X) = 0$, we obtain (3). Similarly taking inner product \bar{V} to (6.60), we get (4).

Theorem 6.2. *Let M be a re-current screen semi-invariant lightlike hypersurface of a transpara Sasakian manifold \bar{M} with a quarter-symmetric metric connection. Then D_1 and D_2 are parallel distributions on M .*

Proof. Taking inner product with \bar{V} to (5.53) and in view of (6.59), we can write as

$$B(X, Y) = u(Y)u(A_N X). \tag{6.65}$$

Putting $Y = \bar{V}$ and $Y = \omega Z$ in (6.65), we get

$$B(X, Y) = 0, \quad \text{and} \quad B(X, \omega Z) = 0. \tag{6.66}$$

Now, from (5.48) and (5.57), we find for all $Z \in \Gamma(D_0)$,

$$g(\nabla_X \xi, \bar{V}) = B(X, \bar{V}), \tag{6.67}$$

$$g(\nabla_X Z, \bar{V}) = B(X, \omega Z), \quad g(\nabla_X \bar{V}, \bar{V}) = 0. \tag{6.68}$$

From these equations and (6.66), we see that

$$\nabla_X Y \in \Gamma(D_1), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(D_1).$$

and hence D_1 is a parallel distribution on M .

On the other hand, setting $Y = \bar{U}$ in (6.60), we have

$$B(X, \bar{U})\bar{U} = A_N X. \tag{6.69}$$

Using $\omega\bar{U} = 0$ in (6.69), it is obtained that

$$\omega(A_N X) = 0. \tag{6.70}$$

Using this result and equation (5.56) reduced to

$$\nabla_X \bar{U} = \tau(X)\bar{U}. \quad (6.71)$$

It follows that

$$\nabla_X \bar{U} \in \Gamma(D_2), \quad \forall X \in \Gamma(TM),$$

and hence D_2 is a parallel distribution on M .

Therefore immediate consequence of the above theorem and from equation (5.44), we have the following theorem

Theorem 6.3. *Let M be a re-current screen semi-invariant lightlike hypersurface of a transpara Sasakian manifold \bar{M} with a quarter-symmetric metric connection . Then M is locally a product manifolds $C_{\bar{U}} \times M$ [23], where $C_{\bar{U}}$ is a null curve tangent to D_2 and M is a leaf of the distribution D_1 .*

Now, we have following

Definition 6.2. [9] *A lightlike hypersurface of semi-Riemannian manifold is said to be screen conformal if there exists a non-zero smooth function λ such that*

$$A_N X = \lambda A_N^* X \quad \text{or} \quad C(X, PY) = \lambda B(X, Y). \quad (6.72)$$

Theorem 6.4. *Let M be a re-current screen semi-invariant lightlike hypersurface of a transpara Sasakian manifold \bar{M} with a quarter-symmetric metric connection. Consider that M is a screen conformal lightlike hypersurface. Then M is either geodesic or screen totally geodesic if and only if $X \in \Gamma(D_0)$.*

Proof. Since M is screen conformal, from Theorem (6.1) using relations (3) and (4), we get

$$\mu(X)U + v(X)\xi = \lambda(\mu(X)\bar{V} + u(X)\xi). \quad (6.73)$$

Taking an inner product with \bar{V} to (6.73), we have

$$\mu(X) = 0. \quad (6.74)$$

So, by using relation (3), (4) and Theorem (6.1), we get the required assertion.

7. LIE RE-CURRENT SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACE

This section starts with the following definition:

Definition 7.1. [9] *Let M be a screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection and ρ be a 1-form on M . Then M is said to be Lie re-current if it admits a Lie re-current tensor field ω such that*

$$(\mathcal{L}_X\omega)Y = \rho(X)\omega Y, \tag{7.75}$$

where \mathcal{L}_X denotes the Lie derivative on M with respect to X that is

$$(\mathcal{L}_X\omega)Y = [X, \omega Y] - \omega[X, Y]. \tag{7.76}$$

If the structure tensor field ω satisfies the condition

$$\mathcal{L}_X\omega = 0, \tag{7.77}$$

then ω is said to be Lie parallel. A screen semi-invariant lightlike hypersurface M of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection is called Lie re-current if its structure tensor field ω is Lie re-current.

Theorem 7.1. *Let M be a Lie re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection . Then the structure tensor field ω is Lie parallel.*

Proof. In view of (7.76), (7.77) and (5.53), we get

$$\begin{aligned} \rho(X)\omega Y &= -\nabla_{\omega Y}X + \omega\nabla_YX + u(Y)A_NX - B(X, Y)\bar{U} \\ &+ (1 - \alpha)[g(X, Y)\xi - \eta(Y)X] + \beta g[X, \varphi Y]\xi + \beta\eta(Y)\omega X. \end{aligned} \tag{7.78}$$

Putting $Y = E$ in (7.78) and by the use of (4.31), we have

$$\rho(X)\bar{V} = -\nabla_{\bar{V}}X + \omega\nabla_EX - \beta u(X)\xi. \tag{7.79}$$

Taking inner product with \bar{V} to (7.79), we obtain

$$g(\nabla_{\bar{V}}X, \bar{V}) = u(\nabla_{\bar{V}}X) = 0, \quad \text{and} \quad \eta(\nabla_{\bar{V}}X) = \beta u(X). \tag{7.80}$$

Replacing Y by \bar{V} in (7.78) and using the fact that $\eta(Y) = 0$, we have

$$\rho(X)E = -\nabla_{\bar{E}}X + \omega\nabla_{\bar{V}}X + B(X, \bar{V}) + \bar{U} + (1 - \alpha)u(X)\xi. \tag{7.81}$$

Applying ω to the above equation, using (5.49) with (7.80), it is obtained that

$$\rho(X)E = -\nabla_{\bar{E}}X + \omega\nabla_{\bar{V}}X + \bar{U} + \beta u(X)\xi. \quad (7.82)$$

Comparing the above equation with (7.79), we get $\rho = 0$. Therefore we arrive at ω is Lie-parallel.

Theorem 7.2. *Let M be a Lie re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection. Then $\alpha = 1$, $\beta = 0$ and \bar{M} is a para Sasakian manifold.*

Proof. Replacing X by \bar{U} in (7.79) and using (4.32), (4.33), (5.46), (5.53)-(5.56) and $\omega\bar{U} = 0$ and $\omega\xi = 0$, it is obtained that

$$u(Y)A_N\bar{U} - \omega(A_N\omega Y) - A_NY - \tau(\omega Y)\bar{U} \quad (7.83)$$

$$-\alpha v(Y)\xi + \beta\eta(Y)\xi - \alpha\eta(Y)\bar{U} = 0.$$

Taking an inner product with ξ into (7.83) and using the fact that

$$C(X, \xi) = (1 - \alpha)v(X) + \beta\eta(X),$$

it is obtained that $(1 - \alpha)v(Y) = 0$ and $\beta\eta(Y) = 0$, and hence $\alpha = 1$, $\beta = 0$. That is, \bar{M} is a para Sasakian manifold.

Theorem 7.3. *Let M be a Lie re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection. Then the following statements are holds:*

- (1) $\tau = \beta\eta$ on TM , and
- (2) $A_\xi^*\bar{U} = 0$, and $A_\xi^*\bar{V} = 0$.

for all $X, Y \in T(M)$.

Proof. Taking inner product with N to (7.79) and using, (4.33), we have

$$-g(\nabla_Y X, N) + g(\nabla_Y X, \bar{U}) = \beta\eta(Y)u(X), \quad (7.84)$$

since $\alpha = 1$ in (7.84). Replacing X by ξ in (7.84) and using (4.28) and (4.32), we get

$$B(X, \bar{U}) = \tau(\omega X). \quad (7.85)$$

Taking $X = \bar{U}$ and using (5.58) and $\omega\bar{U} = 0$, we have

$$C(\bar{U}, \bar{V}) = B(\bar{U}, \bar{U}) = 0. \tag{7.86}$$

taking the inner product with \bar{V} in (7.82) and using (4.32), (5.58), (7.86), and $\alpha = 0$, it is obtained that

$$B(X, \bar{U}) = -\tau(\omega X). \tag{7.87}$$

Comparing the above equation with (7.81), it is obtained that $\tau(\omega X) = 0$.

Replacing X by \bar{V} in (7.83) and using (5.57), we have

$$B(\omega Y, \bar{U}) + \beta\eta(Y) = \tau(Y). \tag{7.88}$$

Taking $Y = \bar{U}$ and $Y = \xi$ and using $\omega\bar{U} = \omega\xi = 0$, it is obtained that

$$\tau(\bar{U}) = 0, \quad \tau(\xi) = -\beta. \tag{7.89}$$

Setting $X = \omega Y$ to $\tau\omega X = 0$ and using (5.49) and (7.89), we get $\tau(X) = -\beta\eta(X)$. Thus we have (1).

As $\tau(\omega X) = 0$, from (4.32) and (7.84), we have $g(A_\xi^*\bar{U}, X) = 0$. The non-degeneracy of $S(TM)$ implies that $A_E^*\bar{U} = 0$. Putting X by ξ to (7.80) and using (4.34) and $\tau(\omega X) = 0$, then we obtained $A_\xi^*\bar{V} = 0$, thus we arrive at (2).

8. SCREEN SEMI-INVARIANT HOPF LIGHTLIKE HYPERSURFACE

Definition 8.1. *Let M be a screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} and \bar{U} be a structure tensor field on M . The structure tensor field \bar{U} is called principal if there exists a smooth function σ and $X \in (TM)$ such that*

$$A_\xi^*X = \sigma\bar{U}. \tag{8.90}$$

A screen semi-invariant lightlike hypersurface M of a trans-para Sasakian manifold \bar{M} is called a Hopf lightlike hypersurface if it admits principal vector field $\bar{U} \in (M)$ [9].

If we consider (8.90), from (4.32) and (5.46), we obtain

$$B(X, \bar{U}) = -\sigma v(X), \quad \text{and} \quad C(X, \bar{V}) = -\sigma u(X). \tag{8.91}$$

Now, we have the following theorems

Theorem 8.1. *Let M be a screen semi-invariant Hopf lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection. If M is screen totally umbilical then $\kappa = 0$ and M is a screen totally geodesic null hypersurface for $X, Y \in \Gamma T(M)$.*

Proof. We know that M is screen totally umbilical lightlike hypersurface if there exists a smooth function f such that $A_N X = fg(X, Y)$ or

$$C(X, PY) = fg(X, Y), \quad (8.92)$$

and $f = 0$, we say that M is a screen totally geodesic lightlike hypersurface.

Therefore, in (8.92) replacing PY with \bar{V} and use of (5.46) and (8.91), we find

$$fv(X) = fu(X). \quad (8.93)$$

Putting $X = \bar{U}$ in (8.93) we obtain $f = 0$. So, we get $A_N = 0 = C$ and $\kappa = 0 = g(A_N X, \bar{V})$.

Therefore $\kappa = 0$ and M is a screen totally geodesic lightlike hypersurface.

Theorem 8.2. *Let M be a screen semi-invariant Hopf lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection. If \bar{V} is a parallel null vector field then M is a Hopf lightlike hypersurface such that $\kappa = 0$.*

Proof. Let us consider \bar{V} is parallel null vector field, from (5.47) and (5.57), we find

$$\varphi(A_E^* X) - \beta u(A_E^* X)N + \tau(X)\bar{V}. \quad (8.94)$$

Applying φ to (8.94) and in view of (2.4), we have

$$A_E^* X - \beta u(A_E^* X)\bar{U} + \tau(X)E = 0. \quad (8.95)$$

Taking inner product with N to (8.95), we get at $\tau = 0$, which yields

$$A_E^* X = \beta u(A_E^* X)\bar{U}. \quad (8.96)$$

Therefore, we can say that M is a Hopf lightlike hypersurface. If we take inner product with \bar{U} to (8.96), we find $\kappa(X) = 0 = B(X, \bar{U})$.

9. INTEGRABILITY OF SCREEN SEMI-INVARIANT LIGHTLIKE HYPERSURFACE

This section explores the integrability conditions for the distributions engage with the screen semi-invariant hypersurface of a trans-para Sasakian manifold with a quarter-symmetric metric connection :

We note that $X \in D_1$ if and only if $u(X) = 0$. Now from (5.54), we have for all $X, Y \in \Gamma(TM)$.

$$u(\nabla_Y X) = \nabla_X u(Y) + u(Y)\tau(X) - B(X, \omega Y) + \beta\eta(Y)u(X) \tag{9.97}$$

from which we get

$$u([X, Y]) = B(X, \omega Y) - B(\omega X, Y) + \nabla_X u(Y) - \nabla_Y u(X) \tag{9.98}$$

$$+u(Y)\tau(X) - u(X)\tau(Y) + \beta\eta(Y)u(X) - \beta\eta(X)u(Y).$$

Let $X, Y \in D_1$. Then $u(X) = 0 = u(Y)$, and from the equation (9.98) we get

$$u([X, Y]) = B(X, \omega Y) - B(\omega X, Y),$$

for all $X, Y \in D_1$. Thus we obtain a necessary and sufficient condition for the integrability of the distribution D_1 in the following:

Theorem 9.1. *Let M be a screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection . Then the distribution D_1 is integrable if and only if*

$$B(X, \omega Y) = B(\omega X, Y), \quad X, Y \in \Gamma(D_1). \tag{9.99}$$

As a consequence of the theorem (9.1), we obtain a results based on radical anti-invariant lightlike hypersurface of trans-para Sasakian manifolds:

Theorem 9.2. *Let M be a radical anti-invariant lightlike hypersurface of a trans-para Sasakian manifold \bar{M} with a quarter-symmetric metric connection. Then the screen distribution $S(TM)$ of M is an integrable distribution if and only if*

$$B(X, \omega Y) = B(\omega X, Y). \tag{9.100}$$

Now, we find a necessary and sufficient condition for the distribution D_2 to be integrable.

Theorem 9.3. *Let M be a screen semi-invariant lightlike hypersurface of a trans-a Sasakian manifold \bar{M} with a quarter-symmetric metric connection . Then the distribution D_2 is integrable if and only if*

$$A_N \xi + (1 - \alpha)\bar{U} + \beta\omega\bar{U} = 0 \tag{9.101}$$

Proof. It is Noted here that $X \in D_2$ if and only if $\varphi X = \omega X = 0$. Now for all $X, Y \in \Gamma(TM)$, in view of (5.53), we arrive at

$$\omega(\nabla_X Y) = \nabla_X \omega(Y) - u(Y)A_N X - B(X, Y)\bar{U} \quad (9.102)$$

$$(1 - \alpha)(g(X, Y)\xi - \eta(Y)X) + \beta(g(X, \varphi Y)\xi + \eta(Y)\omega X).$$

From (9.102), we get

$$\omega([X, Y]) = \nabla_X \omega(Y) - \nabla_Y \omega(X) + u(X)A_N Y - u(Y)A_N X \quad (9.103)$$

$$+(1 - \alpha)(\eta(Y)X - \eta(X)Y) + \beta(\eta(Y)\omega X - \eta(X)\omega Y).$$

In particular for $X, Y \in D_2$, we get

$$\omega([X, Y]) = +u(X)A_N Y - u(Y)A_N X + (1 - \alpha)(\eta(Y)X - \eta(X)Y) \quad (9.104)$$

$$+\beta(\eta(Y)\omega X - \eta(X)\omega Y).$$

Setting $X = \bar{U}$ and $Y = \xi$ and hence, D_2 is integrable if and only if

$$\omega[\bar{U}, \xi] = 0 \quad (9.105)$$

which, in view of (9.105), is equivalent to (9.101).

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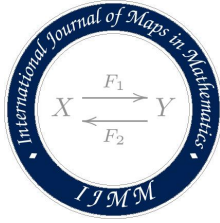
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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA

Email address: anallintegral@gmail.com, msiddiqi@jazanu.edu.sa



CONTRIBUTION TO NULL KILLING MAGNETIC TRAJECTORIES

GÖZDE ÖZKAN TÜKEL * AND TUNAHAN TURHAN 

ABSTRACT. We analyze null magnetic trajectories of a magnetic field on a timelike surface in Minkowski 3–space \mathbb{E}_1^3 . We show that the Lorentz force can be written into the Darboux frame field of a null trajectory on the surface. We give the necessary and sufficient condition for writing a null curve as the magnetic trajectory of the magnetic field. After creating a variation, we derive the Killing magnetic flow equations with regard to the geodesic curvature, geodesic torsion and normal curvature of the curve γ on the timelike surface. Finally we examine the geodesics of some timelike surfaces in \mathbb{E}_1^3 .

1. INTRODUCTION

Any magnetic vector field is known divergence zero vector field in three- dimensional spaces. A magnetic trajectory of a magnetic flow created by magnetic vector field is a curve called as magnetic. Although the problem of investigating magnetic trajectories appears to be physical problem, recent studies show that the characterization of magnetic flow in a magnetic field have brought variational perspective in more geometrical manner. In particular, magnetic curves have been developed by techniques of differential geometry and methods of

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* Corresponding author

calculus of variation from basic spaces to manifolds because the Lorentz force equation is a minimizer of the functional $\mathcal{L} : \Gamma \rightarrow \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) : \frac{1}{2} \int_{\gamma} \langle \gamma', \gamma \rangle' dt + \omega(\gamma') dt,$$

where Γ is a family of smooth curves that connect two fixed point of U , γ is a curve choosing from Γ and ω is a potential 1-form. The Euler-Lagrange equation of the functional \mathcal{L} is derived as

$$\phi(\gamma') = \nabla_{\gamma'} \gamma', \quad (1.1)$$

where ϕ is the skew-symmetric operator. The critical point of the functional \mathcal{L} corresponds to a solution of the Lorentz force equation. So the solutions of the equations could be interpreted with a more geometric point of view [1, 3, 4, 5, 7, 10, 13].

In this work we consider null Killing magnetic trajectories on a timelike surface S in Minkowski 3-space \mathbb{E}_1^3 . Also, we get equation of the Lorentz force by using the Darboux frame field of a null magnetic curve on the such surface and give equations of the Killing magnetic flow by means of the structures of a magnetic vector field in \mathbb{E}_1^3 . Then we apply this formulation to give results about magnetic curves on the pseudo-sphere and the pseudo-cylinder surfaces, so we show that geodesics of these surfaces are null magnetic curves.

2. PRELIMINARIES

We consider that \mathbb{E}_1^3 denotes Minkowski 3-space with the inner product

$$\langle u, w \rangle = -u_1 w_1 + u_2 w_2 + u_3 w_3$$

which is a non-degenerate, symmetric and bilinear form and the vector product

$$u \times w = (-u_2 w_3 + u_3 w_2, u_3 w_1 - u_1 w_3, u_1 w_2 - u_2 w_1),$$

where $u = (u_1, u_2, u_3)$, $w = (w_1, w_2, w_3) \in \mathbb{E}_1^3$. A vector u in \mathbb{E}_1^3 is called a spacelike vector if $\langle u, u \rangle > 0$ or $u = 0$, a timelike vector if $\langle u, u \rangle < 0$, or null (lightlike) vector if $\langle u, u \rangle = 0$ and $u \neq 0$. A regular curve in \mathbb{E}_1^3 is called spacelike, timelike or null, if its velocity vector is spacelike, timelike or null, respectively. A non-degenerate surface is named in terms of the induced metric. If the induced metric is indefinite, a non-degenerate surface is called timelike [9 12].

We can assign a frame to any point of a null curve since we investigate the geometry of the curve. This frame is known as Cartan frame field along a null curve in \mathbb{E}_1^3 . Let $\gamma = \gamma(s)$

be a null curve in \mathbb{E}_1^3 . Let T denote a null vector field along γ . So, there exists a null vector field B along γ satisfying $\langle T, B \rangle = 1$. If we write $N = B \times T$, then we can obtain a Cartan frame field $\mathcal{F} = \{T, N, B\}$ along γ . A Cartan framed null curve (γ, \mathcal{F}) is given by

$$T(s) = \gamma'(s), \quad N(s) = \gamma''(s), \quad B(s) = -\gamma'''(s) - \frac{1}{2} \langle \gamma'''(s), \gamma'''(s) \rangle \gamma'(s)$$

at a point $\gamma(s)$, where

$$\begin{aligned} \langle T, T \rangle = \langle B, B \rangle = \langle T, N \rangle = \langle N, B \rangle = 0, \\ \langle N, N \rangle = \langle T, B \rangle = 1. \end{aligned}$$

We have the following derivative equations of the Cartan frame (generally known as Frenet equations)

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa & 0 & -1 \\ 0 & \kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where

$$\kappa(s) = \frac{1}{2} \langle \gamma'''(s), \gamma'''(s) \rangle,$$

[2, 8, 12].

In order to study the geometry of a null curve on a timelike surface, we can construct a suitable frame, which is known the Darboux frame field, to any point of the curve. Let (γ, \mathcal{F}) be a null curve with frame $\mathcal{F} = \{T, N, B\}$ and S an oriented timelike surface in Minkowski 3-space. The Darboux frame at $\gamma(s)$ of γ is the orthonormal basis $\{T, Q, n\}$ of \mathbb{E}_1^3 , where Q is the unique vector obtained by

$$Q = \frac{1}{\langle V, T \rangle} \left\{ V - \frac{\langle V, V \rangle}{2\langle V, T \rangle} T \right\}, \quad V \in T_{\gamma(s)}M, \quad \langle V, T \rangle \neq 0,$$

and n is the spacelike unit normal of S which is defined by $n = T \times Q$. So, we have

$$\begin{aligned} \langle T, T \rangle = \langle Q, Q \rangle = \langle Q, n \rangle = \langle T, n \rangle = 0, \\ \langle n, n \rangle = \langle T, Q \rangle = 1. \end{aligned}$$

The first order variation of $\{T, Q, n\}$ is expressed as follow

$$\begin{bmatrix} T' \\ Q' \\ n' \end{bmatrix} = \begin{bmatrix} \kappa_g & 0 & \kappa_n \\ 0 & -\kappa_g & \tau_g \\ -\tau_g & -\kappa_n & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ n \end{bmatrix}, \quad (2.2)$$

where the functions κ_g , κ_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion of the curve γ , respectively. From the comparison of Cartan and Darboux frames, we have

$$\kappa_n = \pm 1 \quad (2.3)$$

[6, 12].

3. MAGNETIC VECTOR FIELDS

The Lorentz force ϕ corresponding the magnetic field V is given by

$$\phi(\gamma') = V \times \gamma'.$$

A curve γ in \mathbb{E}_1^3 is called magnetic curve of a magnetic field V if its tangent vector field satisfies

$$\nabla_{\gamma'} \gamma' = \phi(\gamma') = V \times \gamma'. \quad (3.4)$$

The Lorentz force ϕ of a magnetic field F in \mathbb{E}_1^3 is defined to be skew symmetric operator given by

$$\langle \phi(X), Y \rangle = F(X, Y)$$

for vector fields X and Y . The mixed product of the vector fields X , Y and Z is given by

$$\langle X \times Y, Z \rangle = \Omega(X, Y, Z),$$

where Ω a volume on \mathbb{E}_1^3 . So, the Lorentz force of the corresponding Killing magnetic force is given as $\phi(X) = V \times X$, where V is a Killing vector field [13].

Then we can give the following proposition for the Lorentz force.

Proposition 3.1. *Let γ be a null magnetic curve on a timelike surface $S \subset \mathbb{E}_1^3$ and $\{T, Q, n\}$ is the Darboux frame field along γ . Then the Lorentz force in the Darboux frame $\{T, Q, n\}$ is written as follows*

$$\phi(T) = \kappa_g T + \kappa_n n, \quad (3.5)$$

$$\phi(Q) = -\kappa_g Q + \omega n \quad (3.6)$$

and

$$\phi(n) = -\omega T - \kappa_n Q, \quad (3.7)$$

where the function $\omega(s) = \langle \phi(Q(s)), n(s) \rangle$ associated with each magnetic curve is quasi-slope measured with respect to the magnetic vector field V .

Proof. The unit tangent vector to γ at a point $\gamma(s)$ of γ is $T(s) = \gamma'(s)$. Then from (1.1), we have

$$\phi(T) = \nabla_T T = V \times T.$$

By using the Darboux formulas (2.2), we get

$$\phi(T) = \kappa_g T + \kappa_n n$$

and

$$\langle \phi(T), Q \rangle = \kappa_g \quad \text{and} \quad \langle \phi(T), n \rangle = \kappa_n.$$

Similarly, we can write the linear expansion of $\phi(Q), \phi(n) \in S$ as follows

$$\phi(Q) = \langle \phi(Q), Q \rangle T + \langle \phi(Q), T \rangle Q + \langle \phi(Q), n \rangle n$$

and

$$\phi(n) = \langle \phi(n), Q \rangle T + \langle \phi(n), T \rangle Q + \langle \phi(n), n \rangle n,$$

respectively. Taking into consideration Eqs. (3.4) and (3.5), we get

$$\langle \phi(Q), T \rangle = \langle V \times Q, T \rangle = - \langle V \times T, Q \rangle = - \langle \phi(T), Q \rangle = -\kappa_g$$

and

$$\langle \phi(n), T \rangle = \langle V \times n, T \rangle = - \langle V \times T, n \rangle = - \langle \phi(T), n \rangle = -\kappa_n.$$

Since ϕ is a skew-symmetric operator, we get $\langle \phi(Q), Q \rangle = \langle \phi(n), n \rangle = 0$.

Then by using Proposition 3.1 we can write the magnetic vector field according to Darboux frame on a timelike surface S in the following.

Proposition 3.2. *A null curve $\gamma : I \subset \mathbb{R} \rightarrow S$ is a magnetic trajectory of a magnetic field V if and only if V can be written along γ as*

$$V = \omega T - \kappa_n Q + \kappa_g n. \quad (3.8)$$

Proof. Suppose that γ is a null magnetic curve along a magnetic field V with the Darboux frame field $\{T, Q, n\}$. Then, V can be written as $V = \langle V, Q \rangle T + \langle V, T \rangle Q + \langle V, n \rangle n$. To find coefficient of V , we use the Lorentz force in Darboux frame equations (3.5–3.7):

$$\omega = \langle \phi(Q), n \rangle = \langle V, Q \times n \rangle = \langle V, Q \rangle,$$

$$\kappa_n = \langle \phi(T), n \rangle = - \langle V, n \times T \rangle = - \langle V, T \rangle$$

and

$$\kappa_g = \langle \phi(T), Q \rangle = \langle V, T \times Q \rangle = \langle V, n \rangle .$$

4. KILLING MAGNETIC FLOW EQUATION FOR NULL MAGNETIC TRAJECTORIES

Let $\gamma : I \rightarrow S$ be pseudo-parametrized null curve on a timelike surface in \mathbb{E}_1^3 and V a magnetic vector field along that curve. One can take a variation of γ in the direction of V , say a map

$$\begin{aligned} \Gamma : [0, 1] \times (-\varepsilon, \varepsilon) &\rightarrow S \\ (s, t) &\rightarrow \Gamma(s, t) \end{aligned}$$

which satisfies

$$\Gamma(s, 0) = \gamma(s), \quad \left(\frac{\partial \Gamma(s, t)}{\partial t} \right)_{t=0} = V(s) \quad \text{and} \quad \left(\frac{\partial \Gamma(s, t)}{\partial s} \right)_{t=0} = \gamma'(s).$$

We recall that a spacelike or timelike curve in \mathbb{E}_1^3 can be reparametrize by an arclength. However, there would be not sense reparametrize by the arclength for a null curve γ . However, it has pseudo arc-length parametrized $\alpha(s) = \gamma(\phi(s))$, such that $\|\alpha''(s)\| = 1$, where ϕ is the differential function in suitable interval. Thus, we have the following equations:

$$\begin{aligned} T(s, t) &= \left(\frac{\partial \Gamma(s, t)}{\partial s} \right)_{t=0} = \gamma'(s), \\ \beta(s, t) &= \left(\left\langle \left(\frac{\partial^2 \Gamma(s, t)}{\partial s^2} \right)_{t=0}, \left(\frac{\partial^2 \Gamma(s, t)}{\partial s^2} \right)_{t=0} \right\rangle \right)^{1/4}, \end{aligned}$$

(see [9, 12]).

By using above variational formulas, we have the following equalities (by similar method that of [3, 10]).

Lemma 4.1. *We consider that γ is a null curve on a timelike surface in \mathbb{E}_1^3 and a magnetic vector field V is a variational vector field along the variation Γ . So we can give the following expressions;*

$$V(\beta) = \frac{1}{2\beta^3} \langle \nabla_T \nabla_T V, \nabla_T T \rangle, \quad (4.9)$$

$$V(\kappa) = \frac{1}{2} V \langle \nabla_T \nabla_T T, \nabla_T \nabla_T T \rangle = \langle \nabla_T^3 V, \nabla_T^2 T \rangle. \quad (4.10)$$

Proposition 4.1. *(see [11]) . Let $V(s)$ be the restriction to $\gamma(s)$ of a Killing vector field, then*

$$V(\beta) = V(\kappa) = 0. \quad (4.11)$$

Thus, Killing magnetic flow equations can be given the following theorem.

Theorem 4.1. *Let γ be a null curve on S in \mathbb{E}_1^3 . Suppose that $V = \omega T - \kappa_n Q + \kappa_g n$ is a Killing vector field along γ . Then the magnetic trajectories are curves on S satisfying following differential equations*

$$b\kappa_g + c\kappa_n = 0 \quad (4.12)$$

and

$$\begin{aligned} -a' + 2c\tau_g + b'\kappa_g' - b\kappa_g\kappa_g' - c\kappa_n\kappa_g' + \kappa_g^2 b' - b\kappa_g^3 \\ - c\kappa_n\kappa_g^2 - \kappa_n\tau_g b' + 2b\kappa_g\kappa_n\tau_g + c'\kappa_g\kappa_n = 0, \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} a &= \omega'' + 2\omega'\kappa_g + \omega\kappa_g' - 2\kappa_g'\tau_g - \kappa_g\tau_g' + \omega\kappa_g^2 - \kappa_g^2\tau_g \\ &\quad - \omega\kappa_n\tau_g + \kappa_n\tau_g^2, \\ b &= -\omega + \tau_g - \kappa_g'\kappa_n, \\ c &= 2\omega'\kappa_n + \omega\kappa_g\kappa_n - \kappa_g\kappa_n\tau_g - \kappa_n\tau_g' + \kappa_g''. \end{aligned}$$

Proof. Assume that V is a Killing vector field along γ on S . Along any magnetic trajectory γ , we have $V = \omega T - \kappa_n Q + \kappa_g n$. Using (2.3), we get

$$\nabla_T V = (\omega' + \omega\kappa_g - \kappa_g\tau_g) T + (\omega\kappa_n - \kappa_n\tau_g + \kappa_g') n. \quad (4.14)$$

We calculate derivative of (4.14) as follows

$$\begin{aligned} \nabla_T^2 V &= (\omega'' + 2\omega'\kappa_g + \omega\kappa_g' - 2\kappa_g'\tau_g - \kappa_g\tau_g' + \omega\kappa_g^2 \\ &\quad - \kappa_g^2\tau_g - \omega\kappa_n\tau_g + \kappa_n\tau_g^2) T + (-\omega + \tau_g - \kappa_g'\kappa_n) Q \\ &\quad (2\omega'\kappa_n + \omega\kappa_g\kappa_n - \kappa_g\kappa_n\tau_g - \kappa_n\tau_g' + \kappa_g'') n \\ &= aT + bQ + cn. \end{aligned} \quad (4.15)$$

Substituting (4.15) into (4.9), we derive

$$V(\beta) = b\kappa_g + c\kappa_n = 0.$$

For variation of κ , taking derivative of (4.15), we have,

$$\begin{aligned} \nabla_T^3 V &= (a' + a\kappa_g - c\tau_g) T + (b' - b\kappa_g - c\kappa_n) Q \\ &\quad + (a\kappa_n + b\tau_g + c') n. \end{aligned} \quad (4.16)$$

Substituting (4.16), (2.2) and (2.3) into (4.10), we obtain

$$\begin{aligned} V(\kappa) &= -a' + 2c\tau_g + b'\kappa_g' - b\kappa_g\kappa_g' - c\kappa_n\kappa_g' + \kappa_g^2 b' - b\kappa_g^3 \\ &\quad - c\kappa_n\kappa_g^2 - \kappa_n\tau_g b' + 2b\kappa_g\kappa_n\tau_g + c'\kappa_g\kappa_n = 0. \end{aligned}$$

Definition 4.1. *Any null curve on a timelike surface S is called the null magnetic trajectory of a magnetic field V if it satisfies the differential equation system (4.12) and (4.13).*

5. APPLICATIONS

Magnetic trajectories on a timelike pseudo-sphere: We consider the timelike pseudo-sphere with radius r ,

$$\mathbb{S}_1^2(r) = \{(x_1, x_2, x_3) \in E_1^3 : x_1^2 + x_2^2 + x_3^2 = r^2\}.$$

The geodesic torsion τ_g vanishes for all curves on $\mathbb{S}_1^2(r)$ and the normal curvature $\kappa_n^2 = 1$ [12]. Then any null geodesic curve γ on $\mathbb{S}_1^2(r)$ is a magnetic trajectory of a magnetic field V if and only if V can be written along γ as

$$V = \omega T \pm Q,$$

where ω is a constant.

Magnetic trajectories on a pseudo-cylinder: The pseudo-cylinder

$$\mathbb{C}_1^2(1) = \{(x, y, z) \in \mathbb{E}_1^3 \mid -x^2 + y^2 = 1, z \in \mathbb{R}\}$$

is a timelike surface and parametrized by

$$X(u, v) = (\sinh s, \cosh s, s),$$

where r is radius of the circle. Then for a null geodesic

$$\gamma(s) = (\sinh s, \cosh s, s)$$

on $\mathbb{C}_1^2(1)$, we have

$$\kappa_g = 0, \kappa_n = 1 \text{ and } \tau_g = -\frac{1}{2},$$

(see [6, 12]). So, the null geodesic γ on a pseudo-cylinder are magnetic trajectories of the magnetic field

$$V = \omega T - Q$$

where ω is a constant (see Fig (5.1)).

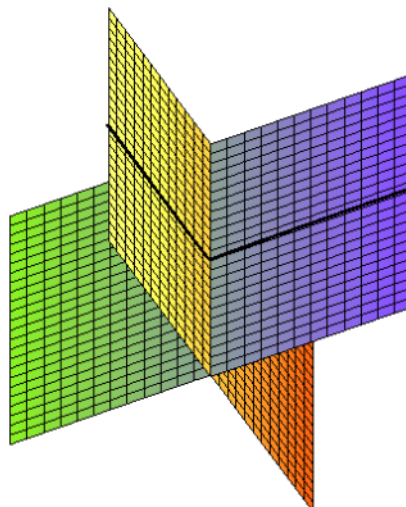


FIGURE 1. A null magnetic trajectory on the pseudo-cylinder

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ISPARTA UNIVERSITY OF APPLIED SCIENCES,, 32200, ISPARTA, TURKEY.

Email address: gozdetukel@isparta.edu.tr

ISPARTA UNIVERSITY OF APPLIED SCIENCES,, 32200, ISPARTA, TURKEY.

Email address: tunahanturhan@isparta.edu.tr

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