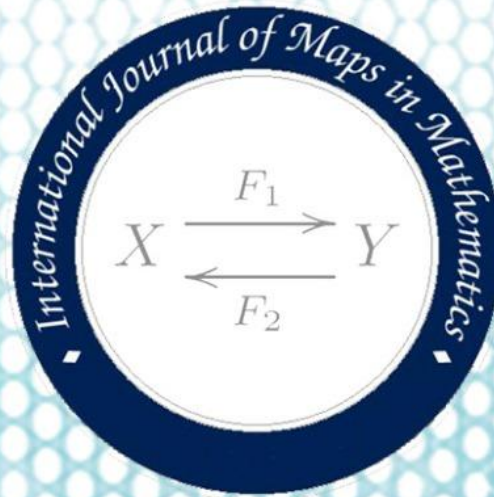


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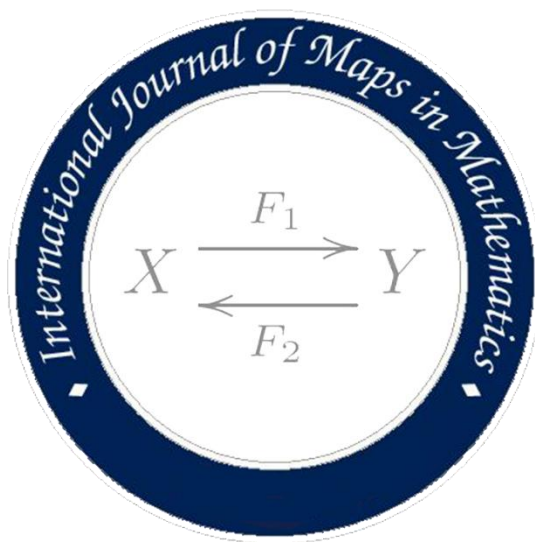
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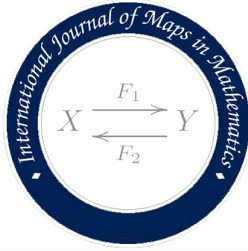
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η -RICCI-YAMABE SOLITONS ON HYPERBOLIC KENMOTSU MANIFOLDS

GAZALA , ABDUL HASEEB *, MOBIN AHMAD , AND OĞUZHAN BAHADIR 

Abstract. In this article, we characterize hyperbolic Kenmotsu manifolds endowed with η -Ricci-Yamabe solitons. Finally, the existence of η -Ricci-Yamabe solitons in a hyperbolic Kenmotsu manifold has been proved by constructing a non-trivial example.

Keywords: Ricci-Yamabe solitons, hyperbolic Kenmotsu manifolds, conharmonic curvature tensor, Einstein manifolds.

MSC (2020): 53C25, 53C44, 53D15.

1. Introduction

During 1988-1989, Hamilton proposed the notion of Ricci and Yamabe flows [9, 10]. These flows on a smooth Riemannian (or semi-Riemannian) manifold are respectively defined by

$$\frac{\partial}{\partial t}g(t) + 2S(g(t)) = 0, \quad \text{and} \quad \frac{\partial}{\partial t}g(t) + r(t)g(t) = 0, \quad g(0) = g_0, \quad (1.1)$$

where S and $r(t)$ are the Ricci tensor and scalar curvature of the metric $g(t)$, respectively.

In two dimensional case, the Yamabe and Ricci flows are equivalent. However, for more than two dimension, there is not such an equivalence, since the Yamabe flow preserves the conformal class of the metric but the Ricci flow does not, in general.

On a Riemannian manifold M admitting a vector field $\mathcal{V}$, the Ricci and Yamabe solitons are respectively defined by [2, 3]

$$\mathcal{L}_{\mathcal{V}}g + 2S + 2\lambda g = 0, \quad (1.2)$$

$$\mathcal{L}_{\mathcal{V}}g + 2(r - \lambda)g = 0, \quad (1.3)$$

here $\mathcal{L}_{\mathcal{V}}$ is the Lie derivative operator along the vector field $\mathcal{V}$ and λ is a real number.

A new class of geometric flows named "Ricci-Yamabe flow" of the type (p, q) was introduced in [8], and is defined by

$$\frac{\partial}{\partial t}g(t) = -2pS(g(t)) + qr(t)g(t), \quad g(0) = g_0, \quad (1.4)$$

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where p and q are scalars.

A solution of (1.4) is called a Ricci-Yamabe soliton and satisfies the following equation [6]:

$$\mathcal{L}_V g + 2pS + (2\lambda - qr)g = 0. \quad (1.5)$$

Ricci-Yamabe soliton of types $(p, 0)$ and $(0, q)$ are called p -Ricci and q -Yamabe solitons, respectively. The Ricci-Yamabe solitons has been studied by many others in several ways to a different extent in [1, 11, 12, 13, 15, 19]. Recently, the authors Traore et al. [21] studied almost η -Ricci-Bourguignon solitons and obtain certain interesting results.

As a generalization of Ricci-Yamabe soliton, the η -Ricci Yamabe soliton of type (p, q) on $\mathbb{M}$ is defined by [23]

$$\mathcal{L}_V g + 2pS + (2\lambda - qr)g + 2\mu\eta \otimes \eta = 0, \quad (1.6)$$

here μ is a constant. The soliton is called expanding, steady or shrinking according to $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. An η -Ricci-Yamabe soliton of type $(p, 0)$ and $(0, q)$ is called p - η -Ricci soliton and q - η -Yamabe soliton, respectively.

The outline of this article is as follows: In section 2, we present some basic results and definitions on hyperbolic Kenmotsu manifolds. In Section 3, we explore hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons. Section 4 deals with the study of Ricci symmetric hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons. In Section 5, hyperbolic Kenmotsu manifolds with the η -Recurrent Ricci tensor admitting η -Ricci-Yamabe solitons have been studied. Ricci semi-symmetric and Einstein semi-symmetric hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons have been studied in Section 6 and Section 7, respectively. In Section 8, we study conharmonic curvature tensor satisfying certain curvature conditions on hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons. Finally, we construct a 3-dimensional non-trivial example of a hyperbolic Kenmotsu manifold to prove the existence of an η -Ricci-Yamabe soliton on such manifold.

2. Preliminaries

A $(2n + 1)$ -dimensional smooth manifold $\mathbb{M}$ is named an almost hyperbolic contact metric manifold, if $\mathbb{M}$ admits a $(1, 1)$ type tensor field ϕ , a timelike vector field ξ , a 1-form η and a semi-Riemannian metric g satisfying [22]

$$\phi^2 K = K + \eta(K)\xi, \quad (2.7)$$

$$\phi\xi = 0, \quad \eta(\xi) = -1, \quad (2.8)$$

$$\eta(\phi K) = 0, \quad rank(\phi) = 2n, \quad (2.9)$$

$$g(\phi K, \phi L) = -g(K, L) - \eta(K)\eta(L), \quad (2.10)$$

$$\eta(K) = g(K, \xi), \quad g(\phi K, L) = -g(K, \phi L), \quad (2.11)$$

for all smooth vector fields K, L on $\mathbb{M}$. Then, the structure (ϕ, ξ, η, g) on $\mathbb{M}$ is known as almost hyperbolic contact metric structure.

If an almost hyperbolic contact metric manifold $\mathbb{M}$ fulfills the following condition:

$$(\nabla_K \phi)L = g(\phi K, L)\xi - \eta(K)\phi L, \quad (2.12)$$

here ∇ indicates the Levi-Civita connection of g . Then, $\mathbb{M}$ is named a hyperbolic Kenmotsu manifold [4, 5, 17].

From the above equation, it is clear that

$$\nabla_K \xi = -K - \eta(K)\xi, \quad (2.13)$$

$$(\nabla_K \eta)L = g(\phi K, \phi L) = -g(K, L) - \eta(K)\eta(L). \quad (2.14)$$

Also, the following relations hold on $\mathbb{M}$:

$$\eta(R(K, L)U) = g(L, U)\eta(K) - g(K, U)\eta(L), \quad (2.15)$$

$$R(K, L)\xi = \eta(L)K - \eta(K)L, \quad (2.16)$$

$$R(K, \xi)\xi = -K - \eta(K)\xi, \quad (2.17)$$

$$R(\xi, K)L = g(K, L)\xi - \eta(L)K, \quad (2.18)$$

$$S(K, \xi) = 2n\eta(K), \quad (2.19)$$

$$S(\xi, \xi) = -2n \iff Q\xi = 2n\xi, \quad (2.20)$$

$$(\mathcal{L}_\xi g)(K, L) = -2[g(K, L) + \eta(K)\eta(L)], \quad (2.21)$$

here S and R are the Ricci tensor and the Riemannian curvature tensor of $\mathbb{M}$, respectively; and the Ricci operator Q is defined by $S(K, L) = g(QK, L)$.

Definition 2.1. A hyperbolic Kenmotsu manifold $\mathbb{M}$ is named an η -Einstein manifold if its Ricci tensor $S(\neq 0)$ is of the form

$$S(K, L) = a_1 g(K, L) + a_2 \eta(K)\eta(L), \quad (2.22)$$

where a_1 and a_2 are smooth functions on $\mathbb{M}$. In particular, if $a_2 = 0$, then $\mathbb{M}$ is termed as an Einstein manifold.

Replacing K and U by ξ in (2.22) and using the identities (2.8), (2.11) and (2.20), we have

$$-2n = -a_1 + a_2. \quad (2.23)$$

Now contracting (2.22) over K and L , we have

$$r = (2n + 1)a_1 - a_2. \quad (2.24)$$

From (2.23) and (2.24), we find $a_1 = (\frac{r}{2n} - 1)$ and $a_2 = (\frac{r}{2n} - 2n - 1)$. Thus, (2.22) leads to

$$S(K, L) = (\frac{r}{2n} - 1)g(K, L) + (\frac{r}{2n} - 2n - 1)\eta(K)\eta(L). \quad (2.25)$$

This gives

$$QK = \left(\frac{r}{2n} - 1\right)K + \left(\frac{r}{2n} - 2n - 1\right)\eta(K)\xi. \quad (2.26)$$

Taking the covariant derivative of (2.26) along L and using (2.13), (2.14) and (2.26), we have

$$(\nabla_L Q)K = \frac{L(r)}{2n}(K + \eta(K)\xi) - \left(\frac{r}{2n} - 2n - 1\right)[g(L, K)\xi + 2\eta(K)\eta(L)\xi + \eta(K)L],$$

which by contracting over L and using the formula $(\text{div} Q)K = \frac{1}{2}K(r)$ gives

$$K(r) = \frac{\xi(r)}{n-1}\eta(K) - \frac{2n(r-2n(2n+1))}{n-1}\eta(K). \quad (2.27)$$

Putting $K = \xi$ in (2.27) and using (2.8), we infer

$$\xi(r) = 2(r - 2n(2n + 1)).$$

From the last two equations, it follows that

$$K(r) = -2(r - 2n(2n + 1))\eta(K).$$

Now, we have the following proposition:

Proposition 2.1. *In a hyperbolic Kenmotsu manifold $\mathbb{M}$, we have*

$$\xi(r) = 2(r - 2n(2n + 1)), \quad (2.28)$$

$$K(r) = -2(r - 2n(2n + 1))\eta(K), \quad (2.29)$$

for any K on $\mathbb{M}$.

Remark: If the scalar curvature r of a $(2n + 1)$ -dimensional hyperbolic Kenmotsu manifold is constant, then $r = 2n(2n + 1)$.

Definition 2.2. *The Ricci tensor S of a hyperbolic Kenmotsu manifold $\mathbb{M}$ is called η -recurrent if*

$$(\nabla_K S)(L, U) = \eta(K)S(L, U), \quad (2.30)$$

for all smooth vector fields K, L, U on $\mathbb{M}$.

Definition 2.3. *A hyperbolic Kenmotsu manifold $\mathbb{M}$ is said to be Ricci symmetric if*

$$(\nabla_K S)(L, U) = 0, \quad (2.31)$$

for all smooth vector fields K, L, U on $\mathbb{M}$.

Definition 2.4. *A hyperbolic Kenmotsu manifold $\mathbb{M}$ is called a manifold of quasi-constant curvature if its curvature tensor R is expressed as*

$$\begin{aligned} R(K, L, U, W) = & a[g(L, U)g(K, W) - g(K, U)g(L, W)] \\ & + b[g(K, W)T(L)T(U) - g(K, U)T(L)T(W) \\ & + g(L, U)T(K)T(W) - g(L, W)T(L)T(U)], \end{aligned} \quad (2.32)$$

here $R(K, L, U, W) = g(R(K, L)U, W)$; a, b are the scalar functions and ρ is a unit vector field defined by

$$g(K, \rho) = T(K), \quad (2.33)$$

for any K, L, U, W on $\mathbb{M}$.

3. η -Ricci-Yamabe solitons on hyperbolic Kenmotsu manifolds

Let a hyperbolic manifold $\mathbb{M}$ admit an η -Ricci-Yamabe soliton $(g, \xi, p, q, \lambda, \mu)$. Then, we have

$$(\mathcal{L}_\xi g)(K, L) + 2pS(K, L) + (2\lambda - qr)g(K, L) + 2\mu\eta(K)\eta(L) = 0. \quad (3.34)$$

Using (2.21) in (3.34), we find

$$S(K, L) = -\frac{1}{p}\left(\lambda - \frac{qr}{2} - 1\right)g(K, L) - \frac{(\mu - 1)}{p}\eta(K)\eta(L), \quad p \neq 0. \quad (3.35)$$

Putting $L = \xi$ in (3.35), then using (2.8) and (2.11), we have

$$S(K, \xi) = A\eta(K), \quad (3.36)$$

where $A = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)$. Now, from (2.19) and (3.36), it follows that

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (3.37)$$

Thus, we state the following theorem:

Theorem 3.1. *An $\mathbb{M}$ admitting η -Ricci-Yamabe solitons $(g, \xi, p, q, \lambda, \mu)$ is an η -Einstein manifold of the form (3.35). Moreover, the scalars λ and μ are connected by $\lambda - \mu = \frac{qr}{2} - 2np$.*

In particular, if $\mu = 0$, then (3.37) turns to $\lambda = \frac{qr}{2} - 2np$.

Corollary 3.1. *Let $\mathbb{M}$ be a hyperbolic Kenmotsu manifold with the constant scalar curvature and admit Ricci-Yamabe solitons. Then, the soliton on $\mathbb{M}$ is shrinking, steady and expanding if $q < \frac{2p}{2n+1}$, $q = \frac{2p}{2n+1}$ and $q > \frac{2p}{2n+1}$, respectively.*

4. Ricci symmetric hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons

Let a hyperbolic Kenmotsu manifold $\mathbb{M}$ admit η -Ricci-Yamabe solitons. Then, (3.35) holds, i.e.,

$$S(K, L) = -\frac{1}{p}(\lambda - \frac{qr}{2} - 1)g(K, L) - \frac{(\mu - 1)}{p}\eta(K)\eta(L), \quad p \neq 0. \quad (4.38)$$

Now, taking the covariant differentiation of (4.38) with respect to U , we have

$$(\nabla_U S)(K, L) = -\frac{(\mu - 1)}{p}\{\eta(L)(\nabla_U \eta)K + \eta(K)(\nabla_U \eta)L\},$$

which by using (2.14) takes the form

$$(\nabla_U S)(K, L) = -\frac{(\mu - 1)}{p}\{g(U, K)\eta(L) + g(U, L)\eta(K) + 2\eta(K)\eta(L)\eta(U)\}. \quad (4.39)$$

In view of (2.31), (4.39) turns to

$$\frac{(\mu - 1)}{p}\{g(U, K)\eta(L) + g(U, L)\eta(K) + 2\eta(K)\eta(L)\eta(U)\} = 0. \quad (4.40)$$

Now, putting $L = \xi$ in (4.40), then using (2.8), (2.10) and (2.11), we obtain

$$\frac{(\mu - 1)}{p}g(\phi U, \phi K) = 0. \quad (4.41)$$

Since, $g(\phi U, \phi K) \neq 0$ and $p \neq 0$, then it follows that $\mu = 1$. Thus, (4.38) turns to

$$S(K, L) = -\frac{1}{p}(\lambda - \frac{qr}{2} - 1)g(K, L). \quad (4.42)$$

Putting $L = \xi$ in (4.42) then using (2.11) and (2.19), we find

$$\lambda = \frac{qr}{2} - 2np + 1. \quad (4.43)$$

Thus, we have the following theorem:

Theorem 4.1. *A Ricci symmetric hyperbolic Kenmotsu manifold $\mathbb{M}$ admitting η -Ricci-Yamabe solitons is an Einstein manifold of the form (4.42). Moreover, the scalar λ is given by $\lambda = \frac{qr}{2} - 2np + 1$.*

Next, putting $U = \xi$ in (4.39), then using (2.8) and (2.11), it follows that

$$(\nabla_{\xi} S)(K, L) = 0. \quad (4.44)$$

This relation shows that the Ricci tensor S is parallel to the vector field ξ . Now, we state the following corollary:

Corollary 4.1. *The Ricci tensor S on $\mathbb{M}$ admitting η -Ricci-Yamabe solitons is parallel to the vector field ξ .*

5. η -Ricci-Yamabe solitons on hyperbolic Kenmotsu manifolds with η -Recurrent Ricci tensor

Let $\mathbb{M}$ be a hyperbolic Kenmotsu manifold with the η -Recurrent Ricci tensor, and admit η -Ricci-Yamabe solitons. Then, from (2.30) and (4.39), we have

$$-\frac{(\mu - 1)}{p} \{g(U, K)\eta(L) + g(K, L)\eta(U) + 2\eta(K)\eta(L)\eta(U)\} = \eta(K)S(L, U). \quad (5.45)$$

Substituting $L = U = \xi$ in equation (5.45), then using (2.8), (2.11) and (3.36), we get $A\eta(K) = 0 \implies$ either $A = 0$, or $\eta(K) = 0$. For the first case $A = 0$, we have $\lambda - \mu = \frac{qr}{2}$. Next, for the second case $\eta(K) = 0$, (2.30) turns to $(\nabla_K S)(L, U) = 0$. Thus, we state the following result:

Theorem 5.1. *In a hyperbolic Kenmotsu manifold $\mathbb{M}$ with η -Recurrent Ricci tensor admitting η -Ricci-Yamabe solitons either $\lambda - \mu = \frac{qr}{2}$, or $\mathbb{M}$ reduces to the Ricci symmetric hyperbolic Kenmotsu manifold.*

6. Ricci semi-symmetric hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons

The notion of Ricci semi-symmetric Riemannian spaces was introduced by Mirzoyan [16] in 1992. In this section we consider a hyperbolic Kenmotsu manifold $\mathbb{M}$ admitting η -Ricci-Yamabe solitons which is Ricci semi-symmetric, i.e., $R(K, L) \cdot S = 0$. This leads to

$$S(R(K, L)W, U) + S(W, R(K, L)U) = 0, \quad (6.46)$$

for K, L, U, W on $\mathbb{M}$.

Putting $K = \xi$ in (6.46), we have

$$S(R(\xi, L)W, U) + S(W, R(\xi, L)U) = 0. \quad (6.47)$$

By using (2.18) in (6.47), we have

$$g(L, W)S(\xi, U) - \eta(W)S(L, U) + g(L, U)S(W, \xi) - \eta(U)S(L, W) = 0.$$

Again putting $W = \xi$, then using (2.11) and (3.36), the foregoing equation turns to

$$S(L, U) = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)g(L, U). \quad (6.48)$$

By replacing $U = \xi$ in (6.48) then using (2.11) and (2.19), we find

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (6.49)$$

Theorem 6.1. *Let $\mathbb{M}$ be a Ricci semi-symmetric hyperbolic Kenmotsu manifold and admit η -Ricci-Yamabe solitons, then $\mathbb{M}$ is an Einstein manifold of the form (6.48). Moreover, the scalars λ and μ are connected by $\lambda - \mu = \frac{qr}{2} - 2np$.*

7. Einstein semi-symmetric hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons

Definition 7.1. *A hyperbolic Kenmotsu manifold $\mathbb{M}$ is called Einstein semi-symmetric if $R \cdot E = 0$, where E is the Einstein tensor given by*

$$E(K, L) = S(K, L) - \frac{r}{n}g(K, L), \quad (7.50)$$

for any K, L on $\mathbb{M}$.

Let a hyperbolic Kenmotsu manifold $\mathbb{M}$ admit η -Ricci-Yamabe solitons, and is Einstein semi-symmetric, i. e., $R \cdot E = 0$. Thus, we have

$$E(R(K, L)U, W) + E(U, R(K, L)W) = 0,$$

which in view of (7.50) takes the form

$$\begin{aligned} S(R(K, L)U, W) + S(U, R(K, L)W) &= \frac{r}{n}\{g(R(K, L)U, W) \\ &+ g(U, R(K, L)W)\}. \end{aligned} \quad (7.51)$$

By putting $K = U = \xi$ in (7.51), we have

$$S(R(\xi, L)\xi, W) + S(\xi, R(\xi, L)W) = \frac{r}{n}\{g(R(\xi, L)\xi, W) + g(\xi, R(\xi, L)W)\}.$$

Making use of (2.17), (2.18) and (3.36), the foregoing equation leads to

$$S(L, W) = -\frac{1}{p}\left(\lambda - \frac{qr}{2} - \mu\right)g(L, W). \quad (7.52)$$

Putting $W = \xi$ in (7.52) and using (2.11), (2.19), we find

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (7.53)$$

Theorem 7.1. *Let $\mathbb{M}$ be an Einstein semi-symmetric hyperbolic Kenmotsu manifold and admit η -Ricci-Yamabe solitons, then $\mathbb{M}$ is an Einstein manifold. Moreover, the scalars λ and μ are connected by $\lambda - \mu = \frac{qr}{2} - 2np$.*

8. Conharmonically flat hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons

The conharmonic curvature tensor C on a (pseudo)-Riemannian manifold $\mathbb{M}$ is defined by [14]

$$\begin{aligned} C(K, L)U &= R(K, L)U - \frac{1}{(n-2)}[S(L, U)K - S(K, U)L \\ &+ g(L, U)QK - g(K, U)QL], \end{aligned} \quad (8.54)$$

for all K, L, U on $\mathbb{M}$.

The conharmonic curvature tensor has also been studied by the authors in [7, 18, 20].

In this section, first we study conharmonically flat hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons, i.e., $C(K, L)U = 0$. Then, (8.54) gives

$$R(K, L)U = \frac{1}{(n-2)}[S(L, U)K - S(K, U)L + g(L, U)QK - g(K, U)QL]. \quad (8.55)$$

Taking the inner product of (8.55) with W , we have

$$\begin{aligned} g(R(K, L)U, W) &= \frac{1}{(n-2)}[S(L, U)g(K, W) - S(K, U)g(L, W) \\ &\quad + g(L, U)S(K, W) - g(K, U)S(L, W)], \end{aligned} \quad (8.56)$$

which by using (3.35) leads to

$$\begin{aligned} g(R(K, L)U, W) &= \frac{2A}{(n-2)}[g(L, U)g(K, W) - g(K, U)g(L, W)] \\ &\quad + \frac{B}{(n-2)}[g(K, W)\eta(L)\eta(U) - g(L, W)\eta(K)\eta(U) \\ &\quad + g(L, U)\eta(K)\eta(W) - g(K, U)\eta(L)\eta(W)], \end{aligned} \quad (8.57)$$

where $A = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)$ and $B = -\frac{1}{p}(\mu - 1)$.

By putting $W = \xi$ in (8.56), then using (2.11), (2.15) and (3.36), we find

$$S(L, U)\eta(K) - S(K, U)\eta(L) = (n-2-A)[g(L, U)\eta(K) - g(K, U)\eta(L)]. \quad (8.58)$$

Putting $K = \xi$ in (8.58), then using (2.8), (2.11) and (3.36), we find

$$S(L, U) = (n-2-A)g(L, U) + (n-2-2A)\eta(U)\eta(L). \quad (8.59)$$

By putting $U = \xi$ in (8.59) and using (3.36), we get

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (8.60)$$

Theorem 8.1. *A conharmonically flat hyperbolic Kenmotsu manifold admitting η -Ricci-Yamabe solitons is an η -Einstein manifold of quasi-constant curvature of the form 8.57. Moreover, the scalars λ and μ are connected by $\lambda - \mu = \frac{qr}{2} - 2np$.*

Next, we study ξ -conharmonically flat hyperbolic Kenmotsu manifolds admitting η -Ricci-Yamabe solitons, i.e., $C(K, L)\xi = 0$. Then, from (8.54) it follows that

$$R(K, L)\xi = \frac{1}{(n-2)}[S(L, \xi)K - S(K, \xi)L + \eta(L)QK - \eta(K)QL],$$

which by using (2.16) and (3.36) takes the form

$$\eta(L)K - \eta(K)L = \frac{1}{(n-2)}[A\eta(L)K - A\eta(K)L + \eta(L)QK - \eta(K)QL], \quad (8.61)$$

where $A = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)$.

Equation (8.61) can be written as

$$[(n-2) - A](\eta(L)K - \eta(K)L) = \eta(L)QK - \eta(K)QL,$$

which by putting $L = \xi$, then using (2.8), (2.20) and the value of A , we find

$$[(n-2) + \frac{1}{p}(\lambda - \frac{qr}{2} - \mu)]K + [(n-2) + \frac{2}{p}(\lambda - \frac{qr}{2} - \mu)]\eta(K)\xi = QK. \quad (8.62)$$

Taking the inner product of (8.62) with L and using (2.11), we find

$$S(K, L) = [(n-2) + \frac{1}{p}(\lambda - \frac{qr}{2} - \mu)]g(K, L) + [(n-2) + \frac{2}{p}(\lambda - \frac{qr}{2} - \mu)]\eta(K)\eta(L). \quad (8.63)$$

Putting $L = \xi$ in (8.63) and using (2.11), we find

$$S(K, \xi) = -\frac{1}{p}(\lambda - \frac{qr}{2} - \mu)\eta(K). \quad (8.64)$$

By using (2.19) in (8.64), we get

$$\lambda - \mu = \frac{qr}{2} - 2np. \quad (8.65)$$

Theorem 8.2. *A ξ -conharmonically flat hyperbolic Kenmotsu manifold admitting η -Ricci-Yamabe solitons is an η -Einstein manifold of the form (8.63). Moreover, the scalars λ and μ are connected by $\lambda - \mu = \frac{qr}{2} - 2np$.*

Next, we consider a conharmonically ϕ -semisymmetric hyperbolic Kenmotsu manifold admitting η -Ricci-Yamabe solitons, i.e., $C(K, L) \cdot \phi = 0$. This implies that

$$C(K, L)\phi U - \phi C(K, L)U = 0, \quad (8.66)$$

which by putting $K = \xi$ takes the form

$$C(\xi, L)\phi U - \phi C(\xi, L)U = 0. \quad (8.67)$$

From (8.54), we find

$$\begin{aligned} C(\xi, L)\phi U &= R(\xi, L)\phi U - \frac{1}{(n-2)}[S(L, \phi U)\xi - S(\xi, \phi U)L \\ &\quad + g(L, \phi U)Q\xi - g(\xi, \phi U)QL]. \end{aligned} \quad (8.68)$$

By using (2.9), (2.18), (3.35) and (3.36), (8.68) reduces to

$$C(\xi, L)\phi U = (1 - \frac{2A}{n-2})g(L, \phi U). \quad (8.69)$$

Also from (8.54), we find

$$C(\xi, L)U = [1 - \frac{2A}{n-2}](g(L, U)\xi - \eta(U)L) - \frac{B}{n-2}\eta(U)\eta(L)\xi,$$

which gives

$$\phi C(\xi, L)U = -[1 - \frac{2A}{n-2}]\eta(U)\phi L. \quad (8.70)$$

From (8.67), (8.69) and (8.70), we arrive at

$$[1 - \frac{2A}{n-2}](g(L, \phi U)\xi - \eta(U)\phi L) = 0. \quad (8.71)$$

By putting $L = \xi$ in (8.71), it follows that

$$\lambda - \mu = \frac{qr}{2} - \frac{(n-2)p}{2}. \quad (8.72)$$

Theorem 8.3. *In a conharmonically ϕ -semisymmetric hyperbolic Kenmotsu manifold admitting η -Ricci-Yamabe solitons the scalars λ and μ are connected by $\lambda - \mu = \frac{qr}{2} - \frac{(n-2)p}{2}$.*

9. Example

Let us consider the 3-dimensional manifold $\mathbb{M}^3 = \{(k, l, u) \in \mathbb{R}^3 : u \neq 0\}$, where (k, l, u) are the standard coordinates in $\mathbb{R}^3$. Let $\{\varrho_1, \varrho_2, \varrho_3\}$ be a set of vector fields on $\mathbb{M}^3$ given by

$$\varrho_1 = u \frac{\partial}{\partial k}, \quad \varrho_2 = u \frac{\partial}{\partial l}, \quad \varrho_3 = u \frac{\partial}{\partial u} = \xi.$$

Clearly, $\{\varrho_1, \varrho_2, \varrho_3\}$ are linearly independent at each point of $\mathbb{M}^3$. Let g be the semi-Riemannian metric defined by

$$g(\varrho_i, \varrho_j) = \begin{cases} 1, & i = j = 1, \\ -1, & 2 \leq i = j \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

Let the 1-form η is defined by $\eta(K) = g(K, \varrho_3)$ for any K on $\mathbb{M}^3$. Let ϕ be the $(1, 1)$ tensor field defined by

$$\phi(\varrho_1) = \varrho_2, \quad \phi(\varrho_2) = -\varrho_1, \quad \phi(\varrho_3) = 0.$$

Then, using the linearity of ϕ and g , we have

$$\phi^2(K) = K + \rho(K)\varrho_3, \quad \eta(\varrho_3) = -1, \quad g(\phi K, \phi L) = -g(K, L) - \eta(K)\eta(L),$$

for all smooth vector fields K, L on $\mathbb{M}^3$. Thus, for $\varrho_3 = \xi$, (ϕ, ξ, ρ, g) defines an almost hyperbolic contact metric structure on $\mathbb{M}^3$.

Let ∇ be the Levi-Civita connection with respect to the metric g . Then, we have

$$[\varrho_1, \varrho_2] = 0, \quad [\varrho_1, \varrho_3] = -\varrho_1, \quad [\varrho_2, \varrho_3] = -\varrho_2.$$

By using Koszul's formula:

$$2g(\nabla_K L, U) = Kg(L, U) + Lg(U, K) - Ug(K, L) - g(K, [L, U]) - g(L, [K, U]) + g(U, [K, L]),$$

we can easily calculate

$$\begin{array}{lll} \nabla_{\varrho_1} \varrho_1 = \varrho_3, & \nabla_{\varrho_1} \varrho_2 = 0, & \nabla_{\varrho_1} \varrho_3 = \varrho_1, \\ \nabla_{\varrho_2} \varrho_1 = 0, & \nabla_{\varrho_2} \varrho_2 = \varrho_3, & \nabla_{\varrho_2} \varrho_3 = \varrho_2, \\ \nabla_{\varrho_3} \varrho_1 = 0, & \nabla_{\varrho_3} \varrho_2 = 0, & \nabla_{\varrho_3} \varrho_3 = 0. \end{array}$$

Now, we check

$$(\nabla_{\varrho_1} \phi) \varrho_2 = \nabla_{\varrho_1} \phi \varrho_2 - \phi(\nabla_{\varrho_1} \varrho_2) = \nabla_{\varrho_1} (-\varrho_1) - \phi(0) = -\varrho_3.$$

On the other hand, we have

$$g(\phi \varrho_1, \varrho_2) \varrho_3 - \eta(\varrho_2) \phi \varrho_1 = g(\varrho_2, \varrho_2) \varrho_3 - 0 = \varrho_3.$$

Hence, $(\nabla_{\varrho_1} \phi) \varrho_2 \neq g(\phi \varrho_1, \varrho_2) \varrho_3 - \rho(\varrho_2) \phi \varrho_1$.

Therefore, the manifold is not a Kenmotsu. But we can show that the manifold is a hyperbolic

Kenmotsu. In fact, we have

$$(\nabla_K \phi)L = g(\phi K, L)\xi - \eta(L)\phi K,$$

for all smooth vector fields K, L on $\mathbb{M}^3$. For example, taking $K = \varrho_1, L = \varrho_2$, we have

$$(\nabla_{\varrho_1} \phi)\varrho_2 = -\varrho_3, \quad g(\phi\varrho_1, \varrho_2)\varrho_3 - \rho(\varrho_2)\phi\varrho_1 = g(\varrho_2, \varrho_2)\varrho_3 - 0 = \varrho_3.$$

Hence, the manifold is not a Kenmotsu but a hyperbolic Kenmotsu.

Now, we compute the curvature tensor. The well-known expression for the curvature tensor R is

$$R(K, L)U = \nabla_K \nabla_L U - \nabla_L \nabla_K U - \nabla_{[K, L]}U.$$

By using the above results, we easily obtain the following components of the curvature tensor:

$$\begin{aligned} R(\varrho_1, \varrho_2)\varrho_1 &= -\varrho_2, & R(\varrho_2, \varrho_1)\varrho_1 &= \varrho_2, & R(\varrho_1, \varrho_3)\varrho_1 &= -\varrho_3, & R(\varrho_3, \varrho_1)\varrho_1 &= \varrho_3, \\ R(\varrho_1, \varrho_2)\varrho_2 &= -\varrho_1, & R(\varrho_2, \varrho_1)\varrho_2 &= \varrho_1, & R(\varrho_2, \varrho_3)\varrho_2 &= \varrho_3, & R(\varrho_3, \varrho_2)\varrho_2 &= -\varrho_3, \\ R(\varrho_1, \varrho_3)\varrho_3 &= -\varrho_1, & R(\varrho_1, \varrho_3)\varrho_3 &= -\varrho_1, & R(\varrho_2, \varrho_3)\varrho_3 &= -\varrho_2, & R(\varrho_3, \varrho_2)\varrho_3 &= \varrho_2. \end{aligned}$$

From these curvature tensors, we can easily calculate

$$S(\varrho_1, \varrho_1) = 2, \quad S(\varrho_2, \varrho_2) = -2, \quad S(\varrho_3, \varrho_3) = -2 \implies r = 6. \quad (9.73)$$

This verifies Remark for the three dimensional case.

Putting $K = L = \xi$ in (3.35) and using (9.73), for a hyperbolic Kenmotsu 3-manifold, we infer

$$\lambda - \mu = \frac{qr}{2} - 2p.$$

Hence λ and μ satisfies (3.35), and so g defines an η -Ricci-Yamabe soliton on the given hyperbolic Kenmotsu 3-manifold.

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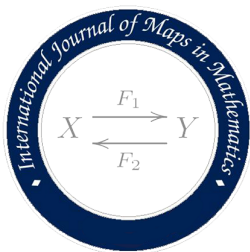
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SYMMETRY VECTOR FIELDS ON THE 4-DIMENSIONAL RIEMANNIAN AND LORENTZIAN LIE GROUP Sol_0^4

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Abstract. In this paper, we investigate various symmetry vector fields on the 4-dimensional Riemannian and Lorentzian Lie group Sol_0^4 . Using Lie derivatives, we characterize affine vector fields with respect to the Levi-Civita connection. We further obtain Ricci collineations vector fields, curvature collineations vector fields, and matter collineations vector fields by computing the Lie derivatives of the Ricci tensor, Riemann curvature tensor, and energy-momentum tensor, respectively. These results contribute to the understanding of geometric and physical symmetries on Sol_0^4 , relevant in both mathematical physics and differential geometry.

Keywords: Lorentzian Sol_0^4 group, affine vector fields, Ricci collineations, curvature collineations, matter collineations.

MSC (2020): 53C50, 53B30.

1. INTRODUCTION

The primary objective of all research in gravitational physics is to construct gravitational potentials that comply with Einstein's field equations, which is frequently accomplished by applying symmetries to the geometry that align with the dynamics of the selected matter distribution. The symmetries of space-time, from a geometric perspective, are expressible through the nullification of the Lie derivative of certain tensors in relation to a vector. The concept of a symmetry of Tensor T involves a one-parameter group of diffeomorphisms that keeps T unchanged. According to this formulation, a vector field X that satisfies $L_X T = 0$, where L is the Lie derivative, corresponds to each symmetry.

Thurston geometries constitute a fascinating subset of Riemannian homogeneous spaces, which have been thoroughly investigated in three dimensions owing to Thurston's geometrization of three-manifolds. This profound result enumerates the potential Riemannian structures of compact orientable three-manifolds, closely resembling the uniformization theorem applicable to compact orientable surfaces. In simple terms, any three-manifold can be partitioned into segments, each of which possesses a Riemannian metric that is locally isometric to one of eight three-dimensional model spaces: the Thurston geometries, i.e., $\mathbb{R}^3$, S^3 , $\mathbb{H}^3$,

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$S^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, $\tilde{SL}(2, \mathbb{R})$, Nil_3 and Sol_3 . As of now, no analogous geometrization has been established in four dimensions.

Based on four-dimensional Thurston geometries, Sol_0^4 is one of them. As stated by Filipkiewicz [13], there exist 19 model spaces in four dimensions i.e., $\mathbb{R}^4$, $\mathbb{H}^4$, $\mathbb{S}^4$, $\mathbb{S}^2 \times \mathbb{S}^2$, $\mathbb{S}^2 \times \mathbb{R}^2$, $\mathbb{S}^2 \times \mathbb{H}^2$, $\mathbb{S}^3 \times \mathbb{R}^1$, $\mathbb{H}^3 \times \mathbb{R}$, Sol_0^4 , Sol_1^4 , $\mathbb{F}^4$, $\mathbb{CP}^2$, $\mathbb{CH}^2$, $\mathbb{R}^2 \times \mathbb{H}^2$, $\mathbb{H}^2 \times \mathbb{H}^2$, $Nil_3 \times \mathbb{R}$, $\tilde{SL}(2, \mathbb{R}) \times \mathbb{R}$, Nil^4 , $Sol_{m,n}^4$. Ishihara [15] demonstrated homogeneous Riemannian 4-spaces are either locally symmetric or Lie groups that possess a left invariant metric. Furthermore, Wall [19] provided a comprehensive list of all 4-dimensional geometries that allow for a compatible complex structure, which are all the abovementioned 19 model spaces, except $\mathbb{H}^4$, $\mathbb{S}^4$, $\mathbb{H}^3 \times \mathbb{R}$, Nil^4 , $Sol_{m,n}^4$. In this paper, symmetries of Sol_0^4 will be investigated.

Some well-known symmetries are the Levi-Civita connection ∇ symmetries that match the affine vector fields; the Ricci tensor Ric symmetries that match the Ricci collineations; and the curvature tensor R symmetries that match the curvature collineations. Research on curvature collineations was initiated by Katzin et al. in 1969 [16].

Matter symmetries offer a distinct and more physically substantiated perspective on the symmetries of spacetimes. A Riemannian manifold (M, g) with matter collineation is represented by a vector field X , which corresponds to a symmetry in the energy-momentum tensor $T = Ric - (\frac{1}{2})\tau g$, where τ is the scalar curvature [11, 12]. Extensive research has been conducted on these symmetries, emphasizing several of them in the past decade that are pertinent to this paper [6, 7, 8, 9, 10, 14, 18]. Batat et al. worked on the symmetries of the three-dimensional Heisenberg group [2] in 2017 and in 2020, Belarbi investigated symmetries on Sol_3 Lie group [4] and symmetries of five-dimensional solvable Lie group [17]. In the same year, he investigated symmetries of three-dimensional generalised symmetric spaces [5] and also on the symmetries of Lorentzian four-dimensional generalised symmetric spaces of type C [3]. Recently, in 2023, Atashpeykar et al. discussed the symmetries of the pseudo-Riemannian manifold $\mathbb{H}^2 \times \mathbb{R}$ [1].

Main Results:

We investigate the Lie derivatives of the Levi-Civita connection ∇ to characterize affine vector fields on the Riemannian Lie group Sol_0^4 .

Theorem 1.1. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Riemannian Lie group Sol_0^4 . Then:*

– *X is an affine vector field if and only if*

$$F_1 = \gamma_1 x e^{-t}, \quad F_2 = \gamma_1 y e^{-t}, \quad F_3 = -2\gamma_1 z e^{2t}, \quad F_4 = \gamma_1, \quad (1.1)$$

where $\gamma_1 \in \mathbb{R}$.

Next, we compute the Lie derivatives of the Ricci tensor Ric and the Riemann curvature tensor R to identify Ricci collineations and curvature collineations on the Riemannian Lie group Sol_0^4 .

Theorem 1.2. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Riemannian Lie group Sol_0^4 . Then:*

(a) X is a Ricci collineation if and only if

$$F_1 = g_1(x, y, z), \quad F_2 = g_2(x, y, z), \quad F_3 = g_3(x, y, z), \quad F_4 = a_2, \quad (1.2)$$

where g_1, g_2, g_3 are smooth functions on Sol_0^4 , and $a_2 \in \mathbb{R}$.

(b) X is a curvature collineation vector field if and only if

$$\begin{cases} F_1 = (a_3x + b_2y + c_4)e^{-t}, & F_2 = (a_3y - b_2x + c_5)e^{-t}, \\ F_3 = (-2a_3z + c_6)e^{2t}, & F_4 = a_3, \end{cases} \quad (1.3)$$

where $a_3, b_2, c_4, c_5, c_6 \in \mathbb{R}$.

We now turn to the study of matter collineations, which are derived from the Lie derivative of the energy-momentum tensor T .

Theorem 1.3. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Riemannian Lie group Sol_0^4 . Then:*

– X is a matter collineation vector field if and only if

$$\begin{cases} F_1 = (a_4x + b_3y + c_7)e^{-t}, & F_2 = (a_4y - b_3x + c_8)e^{-t}, \\ F_3 = (-2a_4z + c_9)e^{2t}, & F_4 = a_4, \end{cases} \quad (1.4)$$

where $a_4, b_3, c_7, c_8, c_9 \in \mathbb{R}$.

We now shift our focus to the Lorentzian structure on the Lie group Sol_0^4 , beginning with affine vector fields under the Lorentzian metric.

Theorem 1.4. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Lorentzian Lie group Sol_0^4 . Then:*

– X is an affine vector field if and only if

$$F_1 = \gamma_2xe^{-t}, \quad F_2 = \gamma_2ye^{-t}, \quad F_3 = -2\gamma_2ze^{2t}, \quad F_4 = \gamma_2, \quad (1.5)$$

where $\gamma_2 \in \mathbb{R}$.

Continuing in the Lorentzian setting, we analyze Ricci and curvature collineations using the Lie derivatives of the Ricci and Riemann curvature tensors.

Theorem 1.5. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Lorentzian Lie group Sol_0^4 . Then:*

(a) X is a Ricci collineation if and only if

$$F_1 = g_4(x, y, z), \quad F_2 = g_5(x, y, z), \quad F_3 = g_6(x, y, z), \quad F_4 = a_6, \quad (1.6)$$

where g_4, g_5, g_6 are smooth functions on Sol_0^4 , and $a_6 \in \mathbb{R}$.

(b) X is a curvature collineation vector field if and only if

$$\begin{cases} F_1 = (a_7x + b_5y + c_{13})e^{-t}, & F_2 = (a_7y - b_5x + c_{14})e^{-t}, \\ F_3 = (-2a_7z + c_{15})e^{2t}, & F_4 = a_7, \end{cases} \quad (1.7)$$

where $a_7, b_5, c_{13}, c_{14}, c_{15} \in \mathbb{R}$.

Finally, we consider the Lie derivative of the energy-momentum tensor T on the Lorentzian Lie group Sol_0^4 to determine matter collineations.

Theorem 1.6. *Let the vector field $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be defined on the Lorentzian Lie group Sol_0^4 . Then:*

– X is a matter collineation vector field if and only if

$$\begin{cases} F_1 = (a_8x + b_6y + c_{15})e^{-t}, & F_2 = (a_8y - b_6x + c_{16})e^{-t}, \\ F_3 = (-2a_8z + c_{17})e^{2t}, & F_4 = a_8, \end{cases} \quad (1.8)$$

where $a_8, b_6, c_{15}, c_{16}, c_{17} \in \mathbb{R}$.

2. PRELIMINARIES

The group operation associated with the Sol_0^4 Lie group is given by:

$$(x, y, z, t) \star (x', y', z', t') = (x + e^t x', y + e^t y', z + e^{-2t} z', t + t'). \quad (2.9)$$

Let g be the left-invariant Riemannian metric, ∇ the Levi-Civita connection, and R the curvature tensor on the four-dimensional solvable Lie group Sol_0^4 . Then the curvature tensor is defined by:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, \quad (2.10)$$

where $X, Y, Z \in \chi(M)$, and the Ricci tensor is given by:

$$\text{Ric}(X, Y) = \sum_{i=1}^4 g(e_i, e_i) g(R(e_i, X)Y, e_i), \quad (2.11)$$

where $\{e_i\}_{i=1}^4$ is an orthonormal basis.

The Riemannian Sol_0^4 Group. The left-invariant Riemannian metric on the Lie group Sol_0^4 is:

$$g = e^{-2t}(dx^2 + dy^2) + e^{4t}dz^2 + dt^2, \quad (2.12)$$

where (x, y, z, t) are the standard Cartesian coordinates on $\mathbb{R}^4$.

The Riemannian Sol_0^4 Lie group possesses the left-invariant basis vector fields $\{e_1, e_2, e_3, e_4\}$ given by:

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^t \frac{\partial}{\partial y}, \quad e_3 = e^{-2t} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \quad (2.13)$$

The Levi-Civita connection ∇ of the Riemannian Lie group Sol_0^4 satisfies:

$$\begin{aligned} \nabla_{e_1} e_1 &= e_4, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= 0, & \nabla_{e_1} e_4 &= -e_1, \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= e_4, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= -e_2, \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= -2e_4, & \nabla_{e_3} e_4 &= 2e_3, \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= 0. \end{aligned} \quad (2.14)$$

The Lie brackets of the basis vector fields are:

$$\begin{aligned} [e_1, e_2] &= [e_1, e_3] = [e_2, e_3] = 0, \\ [e_4, e_1] &= e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3. \end{aligned} \quad (2.15)$$

Using (2.10), (2.14) and (2.15), the non-zero components of the curvature tensor R are:

$$\begin{aligned} R(e_1, e_2)e_2 &= -e_1, & R(e_1, e_3)e_3 &= 2e_1, & R(e_1, e_4)e_4 &= -e_1, \\ R(e_2, e_3)e_3 &= 2e_2, & R(e_2, e_4)e_4 &= -e_2, & R(e_3, e_4)e_4 &= -4e_3, \\ R(e_1, e_2)e_1 &= e_2, & R(e_1, e_3)e_1 &= -2e_3, & R(e_1, e_4)e_1 &= e_4, \\ R(e_2, e_3)e_2 &= -2e_3, & R(e_2, e_4)e_2 &= e_4, & R(e_3, e_4)e_3 &= 4e_4. \end{aligned} \quad (2.16)$$

Using (2.11), (2.12) and (2.16), the components of the Ricci curvature tensor Ric_{ij} are:

$$\begin{aligned} \text{Ric}_{11} &= \text{Ric}_{12} = \text{Ric}_{13} = \text{Ric}_{14} = 0, \\ \text{Ric}_{22} &= \text{Ric}_{23} = \text{Ric}_{24} = \text{Ric}_{33} = \text{Ric}_{34} = 0, \\ \text{Ric}_{44} &= -6. \end{aligned} \quad (2.17)$$

Finally, using (2.12), and (2.17), the scalar curvature τ of the Riemannian Sol_0^4 Lie group is:

$$\tau = \text{trRic} = \sum_{i=1}^4 g(e_i, e_i) \text{Ric}(e_i, e_i) = -6. \quad (2.18)$$

The Lorentzian Sol_0^4 Group. The left-invariant Lorentzian metric of the Lorentzian Sol_0^4 Lie group is given by

$$g = e^{-2t}(dx^2 + dy^2) - e^{4t}dz^2 + dt^2, \quad (2.19)$$

where (x, y, z, t) represents the standard Cartesian coordinates on $\mathbb{R}^4$.

The Lorentzian Sol_0^4 Lie group possesses a left-invariant basis of vector fields $\{e_1, e_2, e_3, e_4\}$ given by:

$$e_1 = e^t \frac{\partial}{\partial x}, \quad e_2 = e^t \frac{\partial}{\partial y}, \quad e_3 = e^{-2t} \frac{\partial}{\partial z}, \quad e_4 = \frac{\partial}{\partial t}. \quad (2.20)$$

The Levi-Civita connection ∇ on the Lorentzian Sol_0^4 Lie group satisfies:

$$\begin{aligned} \nabla_{e_1} e_1 &= e_4, & \nabla_{e_1} e_2 &= 0, & \nabla_{e_1} e_3 &= 0, & \nabla_{e_1} e_4 &= -e_1 \\ \nabla_{e_2} e_1 &= 0, & \nabla_{e_2} e_2 &= e_4, & \nabla_{e_2} e_3 &= 0, & \nabla_{e_2} e_4 &= -e_2 \\ \nabla_{e_3} e_1 &= 0, & \nabla_{e_3} e_2 &= 0, & \nabla_{e_3} e_3 &= 2e_4, & \nabla_{e_3} e_4 &= 2e_3 \\ \nabla_{e_4} e_1 &= 0, & \nabla_{e_4} e_2 &= 0, & \nabla_{e_4} e_3 &= 0, & \nabla_{e_4} e_4 &= 0. \end{aligned} \quad (2.21)$$

The Lie brackets of the basis vector fields are:

$$\begin{aligned} [e_1, e_2] &= [e_1, e_3] = [e_2, e_3] = 0, \\ [e_4, e_1] &= e_1, \quad [e_4, e_2] = e_2, \quad [e_4, e_3] = -2e_3. \end{aligned} \quad (2.22)$$

Using (2.10), (2.21) and (2.22), the non-zero components of the curvature tensor R are:

$$\begin{aligned} R(e_1, e_2)e_2 &= -e_1, & R(e_1, e_3)e_3 &= -2e_1, & R(e_1, e_4)e_4 &= -e_1 \\ R(e_2, e_3)e_3 &= -2e_2, & R(e_2, e_4)e_4 &= -e_2, & R(e_3, e_4)e_4 &= -4e_3 \\ R(e_1, e_2)e_1 &= e_2, & R(e_1, e_3)e_1 &= -2e_3, & R(e_1, e_4)e_1 &= e_4 \\ R(e_2, e_3)e_2 &= -2e_3, & R(e_2, e_4)e_2 &= e_4, & R(e_3, e_4)e_3 &= -4e_4. \end{aligned} \quad (2.23)$$

Using (2.11), (2.19) and (2.23), the components of the Ricci curvature tensor Ric_{ij} are:

$$\begin{aligned} \text{Ric}_{11} &= \text{Ric}_{12} = \text{Ric}_{13} = \text{Ric}_{14} = \text{Ric}_{22} = 0, \\ \text{Ric}_{23} &= \text{Ric}_{24} = \text{Ric}_{33} = \text{Ric}_{34} = 0, \quad \text{Ric}_{44} = -6. \end{aligned} \quad (2.24)$$

Using (2.19), (2.24), the scalar curvature τ associated with the Lorentzian Sol_0^4 Lie group is:

$$\tau = \text{tr}(\text{Ric}) = \sum_{i=1}^4 g(e_i, e_i) \text{Ric}(e_i, e_i) = -6. \quad (2.25)$$

3. PROOF OF THEOREMS

Throughout the paper, we denote the Cartesian basis $\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t} \right\}$ by $\{\partial_x, \partial_y, \partial_z, \partial_t\}$.

Proof of Theorem 1.1. Let $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be a smooth vector field on the Riemannian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

We compute the Lie derivative of the Levi-Civita connection ∇ in order to determine the affine vector fields. Using (2.14) and (2.15), we obtain the following equations:

$$\begin{aligned} (L_X \nabla)(e_1, e_1) &= \left(-(\partial_t F_1 + F_1) - 2e^t \partial_x F_4 + e^{2t} \partial_x^2 F_1 \right) e_1 \\ &\quad + \left(-(F_2 + \partial_t F_2) + e^{2t} \partial_x^2 F_2 \right) e_2 \\ &\quad + \left(-(\partial_t F_3 - 2F_3) + e^{2t} \partial_x^2 F_3 \right) e_3 \\ &\quad + \left(-(\partial_t F_4 + 2F_4) + 2e^t \partial_x F_1 + e^{2t} \partial_x^2 F_4 \right) e_4, \end{aligned} \quad (3.26)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_2) &= \left(-e^t \partial_y F_4 + e^{2t} \partial_x \partial_y F_1 \right) e_1 + \left(-e^t \partial_x F_4 + e^{2t} \partial_x \partial_y F_2 \right) e_2 \\ &\quad + \left(e^{2t} \partial_x \partial_y F_3 \right) e_3 + \left(e^t (\partial_y F_1 + \partial_x F_2) + e^{2t} \partial_x \partial_y F_4 \right) e_4, \end{aligned} \quad (3.27)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_4) &= \left(e^t (\partial_x \partial_t F_2 + \partial_x F_2) \right) e_2 + \left(e^t (\partial_x \partial_t F_3 + \partial_x F_3) \right) e_3 \\ &\quad + \left(-\partial_t F_4 + e^t (\partial_x \partial_t F_1 + \partial_x F_1) \right) e_1 \\ &\quad + \left(F_1 + \partial_t F_1 + e^t (\partial_x \partial_t F_4 + \partial_x F_4) \right) e_4, \end{aligned} \quad (3.28)$$

$$\begin{aligned}
(L_X \nabla)(e_2, e_2) &= \left(-(F_1 + \partial_t F_1) + e^{2t} \partial_y^2 F_1 \right) e_1 + \left(-(\partial_t F_3 - 2F_3) + e^{2t} \partial_y^2 F_3 \right) e_3 \\
&+ \left(-(F_2 + \partial_t F_2) - 2e^t \partial_y F_4 + e^{2t} \partial_y^2 F_2 \right) e_2 \\
&+ \left(-(\partial_t F_4 + 2F_4) + 2e^t \partial_y F_2 + e^{2t} \partial_y^2 F_4 \right) e_4,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
(L_X \nabla)(e_2, e_4) &= \left(e^t (\partial_y F_1 + \partial_y \partial_t F_1) \right) e_1 + \left(e^t (\partial_y F_3 + \partial_y \partial_t F_3) \right) e_3 \\
&+ \left(e^t (\partial_y F_4 + \partial_y \partial_t F_4) + F_2 + \partial_t F_2 \right) e_4 \\
&+ \left(e^t (\partial_y F_2 + \partial_y \partial_t F_2) - \partial_t F_4 \right) e_2,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
(L_X \nabla)(e_3, e_3) &= \left(2(F_1 + \partial_t F_1) + e^{-4t} \partial_z^2 F_1 \right) e_1 + \left(2(F_2 + \partial_t F_2) + e^{-4t} \partial_z^2 F_2 \right) e_2 \\
&+ \left(2\partial_t F_4 - 4(e^{-2t} \partial_z F_3 + 2F_4) + e^{-4t} \partial_z^2 F_4 \right) e_4 \\
&+ \left(2(\partial_t F_3 - 2F_3) + e^{-4t} \partial_z^2 F_3 + 4e^{-2t} \partial_z F_4 \right) e_3,
\end{aligned} \tag{3.31}$$

$$\begin{aligned}
(L_X \nabla)(e_3, e_4) &= \left(e^{-2t} (\partial_z \partial_t F_1 - 2\partial_z F_1) \right) e_1 + \left(e^{-2t} (\partial_z \partial_t F_2 - 2\partial_z F_2) \right) e_2 \\
&+ \left(e^{-2t} (\partial_z \partial_t F_3 - 2\partial_z F_3) + 2\partial_t F_4 \right) e_3 \\
&+ \left(e^{-2t} (\partial_z \partial_t F_4 - 2\partial_z F_4) - 2(\partial_t F_3 - 2F_3) \right) e_4,
\end{aligned} \tag{3.32}$$

$$\begin{aligned}
(L_X \nabla)(e_4, e_4) &= \left(\partial_t^2 F_1 - F_1 \right) e_1 + \left(\partial_t^2 F_2 - F_2 \right) e_2 \\
&+ \left(\partial_t^2 F_4 \right) e_4 + \left(\partial_t^2 F_3 - 4F_3 \right) e_3.
\end{aligned} \tag{3.33}$$

To find affine vector fields F , we solve above system of partial differential equations by setting all the coefficients of the Lie derivative to zero. Therefore, comparing the components of e_4 on both sides in (3.26), (3.29), (3.31) and (3.33) we get

$$-(\partial_t F_4 + 2F_4) + 2e^t \partial_x F_1 + e^{2t} \partial_x^2 F_4 = 0, \tag{3.34}$$

$$-(\partial_t F_4 + 2F_4) + 2e^t \partial_y F_2 + e^{2t} \partial_y^2 F_4 = 0, \tag{3.35}$$

$$2\partial_t F_4 - 4(e^{-2t} \partial_z F_3 + 2F_4) + e^{-4t} \partial_z^2 F_4 = 0, \tag{3.36}$$

and

$$\partial_t^2 F_4 = 0, \tag{3.37}$$

respectively.

Similarly, comparing the components of e_1 on both sides in (3.27), (3.28) and (3.29), we get

$$-e^t \partial_y F_4 + e^{2t} \partial_x \partial_y F_1 = 0, \tag{3.38}$$

$$-\partial_t F_4 + e^t(\partial_x \partial_t F_1 + \partial_x F_1) = 0, \quad (3.39)$$

and

$$-(F_1 + \partial_t F_1) + e^{2t} \partial_y^2 F_1 = 0, \quad (3.40)$$

respectively.

Similarly, comparing the components of e_2 on both sides in (3.27), (3.30), (3.31) and (3.32) we get

$$-e^t \partial_x F_4 + e^{2t} \partial_x \partial_y F_2 = 0, \quad (3.41)$$

$$e^t(\partial_y F_2 + \partial_y \partial_t F_2) - \partial_t F_4 = 0 \quad (3.42)$$

$$2(F_2 + \partial_t F_2) + e^{-4t} \partial_z^2 F_2 = 0 \quad (3.43)$$

and

$$e^{-2t}(\partial_z \partial_t F_2 - 2\partial_z F_2) = 0, \quad (3.44)$$

respectively.

Similarly, comparing the components of e_3 on both sides in (3.26), (3.28), (3.30) and (3.32) we get

$$-(\partial_t F_3 - 2F_3) + e^{2t} \partial_x^2 F_3 = 0, \quad (3.45)$$

$$e^t(\partial_x \partial_t F_3 + \partial_x F_3) = 0, \quad (3.46)$$

$$e^t(\partial_y F_3 + \partial_y \partial_t F_3) = 0, \quad (3.47)$$

and

$$e^{-2t}(\partial_z \partial_t F_3 - 2\partial_z F_3) + 2\partial_t F_4 = 0, \quad (3.48)$$

respectively.

On solving (3.37), we get

$$F_4 = h_1(x, y, z).t + h_2(x, y, z). \quad (3.49)$$

where, $h_1 = h_1(x, y, z)$ and $h_2(x, y, z)$ are real-valued smooth functions on Sol_0^4 .

Using (3.49) in (3.34), we get

$$\partial_x F_1 = \frac{e^{-t} \left(h_1 + 2h_1.t + 2h_2 - e^{2t}(\partial_x^2 h_1.t + \partial_x^2 h_2) \right)}{2}, \quad (3.50)$$

and

$$\partial_x \partial_t F_1 = \frac{e^{-t} \left(h_1 - 2h_1.t - 2h_2 - e^{2t}(\partial_x^2 h_1.t + \partial_x^2 h_2 + \partial_x^2 h_1) \right)}{2}. \quad (3.51)$$

Using (3.49), (3.50) and (3.51) in (3.39), we obtain

$$\partial_x^2 h_1 = 0 = \partial_x^2 h_2. \quad (3.52)$$

Similarly, using (3.49) in (3.35), we get

$$\partial_y F_2 = \frac{e^{-t} \left(h_1 + 2h_1.t + 2h_2 - e^{2t} (\partial_y^2 h_1.t + \partial_y^2 h_2) \right)}{2}, \quad (3.53)$$

and

$$\partial_y \partial_t F_2 = \frac{e^{-t} \left(h_1 - 2h_1.t - 2h_2 - e^{2t} (\partial_y^2 h_1.t + \partial_y^2 h_2 + \partial_y^2 h_1) \right)}{2}. \quad (3.54)$$

Using (3.49), (3.53) and (3.54) in (3.42), we obtain

$$\partial_y^2 h_1 = 0 = \partial_y^2 h_2. \quad (3.55)$$

Similarly, using (3.49) in (3.36), we get

$$\partial_z F_3 = \frac{e^{2t} \left(2h_1 - 8h_1.t - 8h_2 + e^{-4t} (\partial_z^2 h_1.t + \partial_z^2 h_2) \right)}{4}, \quad (3.56)$$

and

$$\partial_z \partial_t F_3 = \frac{e^{2t} \left(-4h_1 - 16h_1.t - 16h_2 - e^{-4t} (2\partial_z^2 h_1.t + 2\partial_z^2 h_2 - \partial_z^2 h_1) \right)}{4}. \quad (3.57)$$

Using (3.49), (3.56) and (3.57) in (3.48), we obtain

$$\partial_z^2 h_1 = 0 = \partial_z^2 h_2. \quad (3.58)$$

On solving (3.52), (3.55) and (3.58), we get

$$h_1(x, y, z) = a_1 x + a_2 y + a_3 z + a_4 xy + a_5 yz + a_6 xz + a_7 xyz + a_8, \quad (3.59)$$

$$h_2(x, y, z) = b_1 x + b_2 y + b_3 z + b_4 xy + b_5 yz + b_6 xz + b_7 xyz + \gamma_1, \quad (3.60)$$

where $a_i, b_j, \gamma_1 \in \mathbb{R}$, $i=1, \dots, 8$ and $j=1, \dots, 7$.

Using (3.49), (3.50), (3.59) and (3.60) in (3.38), we get

$$a_2 = a_4 = a_5 = a_7 = 0. \quad (3.61)$$

Using (3.49), (3.53), (3.59) and (3.60) in (3.41), we get

$$a_1 = a_6 = 0. \quad (3.62)$$

Using (3.52), (3.59), (3.60), (3.61) and (3.62) in (3.50), we get

$$\begin{aligned} F_1 = \alpha_1 + \frac{e^{-t}}{2} & \left((1+2t)(a_3 zx + a_8 x) + (b_1 x^2 + b_4 x^2 y \right. \\ & \left. + b_6 x^2 z + b_7 x^2 yz) + 2(b_2 xy + b_3 xy + b_5 xyz + \gamma_1 x) \right), \end{aligned} \quad (3.63)$$

where $\alpha_1 \in \mathbb{R}$.

Using (3.55), (3.59), (3.60), (3.61) and (3.62) in (3.53), we get

$$\begin{aligned} F_2 = \alpha_2 + \frac{e^{-t}}{2} & \left((1+2t)(a_3 zy + a_8 y) + (b_2 y^2 + b_4 xy^2 \right. \\ & \left. + b_5 y^2 z + b_7 xy^2 z) + 2(b_1 xy + b_3 zy + b_6 xyz + \gamma_1 y) \right), \end{aligned} \quad (3.64)$$

where $\alpha_2 \in \mathbb{R}$.

Using (3.58), (3.59), (3.60), (3.61) and (3.62) in (3.56), we get

$$F_3 = \alpha_3 + \frac{e^{2t}}{4} \left((1-4t)(a_3z^2 + 2a_8z) - 4(b_3z^2 + b_5yz^2 + b_6xz^2 + b_7xyz^2) - 8(b_1xz + b_2yz + b_4xyz + \gamma_1z) \right), \quad (3.65)$$

where $\alpha_3 \in \mathbb{R}$.

Using (3.63) in (3.40), we get

$$\alpha_1 = a_3 = a_8 = 0. \quad (3.66)$$

Using (3.65) in (3.47), we get

$$b_5 = b_2 = b_4 = b_7 = 0. \quad (3.67)$$

Using (3.65) in (3.45), we get

$$\alpha_3 = 0. \quad (3.68)$$

Using (3.65) in (3.46), we get

$$b_1 = b_6 = 0. \quad (3.69)$$

Using (3.64) in (3.43) and (3.44), we get

$$b_3 = \alpha_2 = 0. \quad (3.70)$$

Using (3.61) $\sim$ (3.62) and (3.66) $\sim$ (3.70) in (3.59) $\sim$ (3.60), we find

$$h_1(x, y, z) = 0, \quad (3.71)$$

$$h_2(x, y, z) = \gamma_1. \quad (3.72)$$

Now, using (3.71) $\sim$ (3.72) in (3.49) and (3.66) $\sim$ (3.70) in (3.63) $\sim$ (3.65), we obtain (1.1). Thus the proof of Theorem 1.1 is complete. $\square$

Proof of Theorem 1.2. Let $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be a smooth vector field on the Riemannian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

Using (2.15) and (2.17), we compute the Lie derivative of the Ricci tensor Ric with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\begin{cases} (L_X Ric)(e_1, e_1) = 0, (L_X Ric)(e_1, e_2) = 0, \\ (L_X Ric)(e_1, e_3) = 0, (L_X Ric)(e_1, e_4) = -6e^t \partial_x F_4, \\ (L_X Ric)(e_2, e_2) = 0, (L_X Ric)(e_2, e_3) = 0, \\ (L_X Ric)(e_2, e_4) = -6e^t \partial_y F_4, (L_X Ric)(e_3, e_3) = 0, \\ (L_X Ric)(e_3, e_4) = -6e^{-2t} \partial_z F_4, (L_X Ric)(e_4, e_4) = -12\partial_t F_4. \end{cases} \quad (3.73)$$

These relations (3.73) give the necessary and sufficient conditions for X to be a Ricci collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.2 yields the stated form of X given in (1.2).

Now, to prove Theorem 1.2(b), we compute the Lie derivative of the curvature tensor R in order to determine the curvature collineations. Using (2.15) and (2.16), we obtain the following equations:

$$(L_X R)(e_1, e_2, e_1) = \left(-e^t(\partial_y F_1 + \partial_x F_2) \right) e_1 - 3 \left(e^t \partial_y F_3 \right) e_3 - 2 \left(F_4 - e^t \partial_x F_1 \right) e_2, \quad (3.74)$$

$$(L_X R)(e_1, e_2, e_2) = \left(2(F_4 - e^t \partial_y F_2) \right) e_1 + 3 \left(e^t \partial_x F_3 \right) e_3 + \left(e^t(\partial_y F_1 + \partial_x F_2) \right) e_2, \quad (3.75)$$

$$(L_X R)(e_1, e_2, e_3) = \left(2e^t \partial_y F_3 - e^{-2t} \partial_z F_2 \right) e_1 + \left(e^{-2t} \partial_z F_1 - 2e^t \partial_x F_3 \right) e_2, \quad (3.76)$$

$$(L_X R)(e_1, e_3, e_1) = \left(2(e^{-2t} \partial_z F_1 + e^t \partial_x F_3) \right) e_1 + 3 \left(e^{-2t} \partial_z F_2 \right) e_2 + 4 \left(F_4 - e^t \partial_x F_1 \right) e_3 + 3 \left(e^{-2t} \partial_z F_4 \right) e_4, \quad (3.77)$$

$$(L_X R)(e_1, e_3, e_2) = \left(2e^t \partial_y F_3 - e^{-2t} \partial_z F_2 \right) e_1 - 2e^t \left(\partial_y F_1 + \partial_x F_2 \right) e_3, \quad (3.78)$$

$$(L_X R)(e_1, e_3, e_3) = 4 \left(e^{-2t} \partial_z F_3 + 2F_4 \right) e_1 - 6 \left(e^t \partial_x F_4 \right) e_4 - 2 \left(e^t \partial_x F_3 - e^{-2t} \partial_z F_1 \right) e_3, \quad (3.79)$$

$$(L_X R)(e_2, e_3, e_3) = 4 \left(e^{-2t} \partial_z F_3 + 2F_4 \right) e_2 - 6 \left(e^t \partial_y F_4 \right) e_4 - 2 \left(e^{-2t} \partial_z F_2 - e^t \partial_y F_3 \right) e_3, \quad (3.80)$$

$$(L_X R)(e_2, e_4, e_2) = - \left(\partial_t F_2 + F_2 + e^t \partial_y F_4 \right) e_2 - 3 \left(\partial_t F_3 - 2F_3 \right) e_3 - 2 \left(F_4 - e^t \partial_y F_2 \right) e_4, \quad (3.81)$$

$$(L_X R)(e_2, e_4, e_4) = -2 \left(\partial_t F_4 \right) e_2 - 3e^t \left(\partial_y F_3 \right) e_3 + \left(\partial_t F_2 + F_2 + e^t \partial_y F_4 \right) e_4, \quad (3.82)$$

$$(L_X R)(e_3, e_4, e_1) = 2 \left(\partial_t F_1 + F_1 - 2e^t \partial_x F_4 \right) e_3 + \left(4e^t \partial_x F_3 + e^{-2t} \partial_z F_1 \right) e_4, \quad (3.83)$$

$$(L_X R)(e_3, e_4, e_3) = -6 \left(\partial_t F_1 + F_1 \right) e_1 - 4 \left(\partial_t F_3 - 2F_3 + e^{-2t} \partial_z F_4 \right) e_3 - 6 \left(F_2 + \partial_t F_2 \right) e_2 + \left(8(e^{-2t} \partial_z F_3 + 2F_4) \right) e_4, \quad (3.84)$$

$$(L_X R)(e_3, e_4, e_4) = 3 \left(e^{-2t} \partial_z F_1 \right) e_1 + 3 \left(e^{-2t} \partial_z F_2 \right) e_2 - 8 \left(\partial_t F_4 \right) e_3 + 4 \left(\partial_t F_3 - 2F_3 + e^{-2t} \partial_z F_4 \right) e_4. \quad (3.85)$$

To find curvature collineation vector fields F , we solve above system of partial differential equations by setting all the coefficients of the Lie derivative to zero. Therefore, comparing the components of e_4 on both sides in (3.77), (3.79), (3.80), (3.82) and (3.84), we get

$$3e^{-2t}\partial_z F_4 = 0, \quad (3.86)$$

$$-6e^t\partial_x F_4 = 0, \quad (3.87)$$

$$-6e^t\partial_y F_4 = 0, \quad (3.88)$$

$$\partial_t F_2 + F_2 + e^t\partial_y F_4 = 0, \quad (3.89)$$

and

$$8(e^{-2t}\partial_z F_3 + 2F_4) = 0, \quad (3.90)$$

respectively.

Similarly, comparing the components of e_1 on both sides in (3.74), (3.75) and (3.78), we get

$$-e^t(\partial_y F_1 + \partial_x F_2) = 0, \quad (3.91)$$

$$2(F_4 - e^t\partial_y F_2) = 0, \quad (3.92)$$

and

$$2e^t\partial_y F_3 - e^{-2t}\partial_z F_2 = 0, \quad (3.93)$$

respectively.

Similarly, comparing the components of e_2 on both sides in (3.74) and (3.76), we get

$$-2(F_4 - e^t\partial_x F_1) = 0, \quad (3.94)$$

and

$$e^{-2t}\partial_z F_1 - 2e^t\partial_x F_3 = 0, \quad (3.95)$$

respectively.

Similarly, comparing the components of e_3 on both sides in (3.81), (3.83) and (3.85), we get

$$-3(\partial_t F_3 - 2F_3) = 0, \quad (3.96)$$

$$2(\partial_t F_1 + F_1 - 2e^t\partial_x F_4) = 0, \quad (3.97)$$

and

$$-8\partial_t F_4 = 0, \quad (3.98)$$

respectively.

On solving (3.86), (3.87), (3.88) and (3.98), we get

$$F_4 = a_3, \quad a_3 \in \mathbb{R}. \quad (3.99)$$

Using (3.99) in (3.94), we get

$$F_1 = a_3 x e^{-t} + h_3(y, z, t), \quad (3.100)$$

where $h_3(y, z, t)$ is real valued smooth function.

Using (3.99) in (3.92), we get

$$F_2 = a_3 y e^{-t} + h_4(x, z, t), \quad (3.101)$$

where $h_4(x, z, t)$ is real valued smooth function.

Using (3.99) in (3.90), we get

$$F_3 = -2a_3 z e^{2t} + h_5(x, y, t), \quad (3.102)$$

where $h_5(x, y, t)$ is real valued smooth function.

Using (3.100) in (3.97), we get

$$h_3(y, z, t) = h_6(y, z) e^{-t}, \quad (3.103)$$

where $h_6(y, z)$ is real valued smooth function.

Using (3.101) in (3.89), we get

$$h_4(x, z, t) = h_7(x, z) e^{-t}. \quad (3.104)$$

where $h_7(x, z)$ is real valued smooth function.

Using (3.102) in (3.96), we get

$$h_5(x, z, t) = h_8(x, y) e^{2t}, \quad (3.105)$$

where $h_8(x, y)$ is real valued smooth function.

Using (3.100), (3.101), (3.103) and (3.104) in (3.91) we get

$$h_6(y, z) = b_2 y + h_9(z), \quad (3.106)$$

where $h_9(z)$ is real valued smooth functions.

and

$$h_7(x, z) = -b_2 x + h_{10}(z), \quad (3.107)$$

where $h_{10}(z)$ is real valued smooth functions.

Using (3.100), (3.102), (3.103), (3.105) and (3.106) in (3.95), we get

$$h_9(z) = c_4, \quad c_4 \in \mathbb{R}, \quad (3.108)$$

$$\partial_x h_8(x, y) = 0. \quad (3.109)$$

Using (3.101), (3.102), (3.104), (3.105) and (3.107) in (3.93), we get

$$h_{10}(z) = c_5, \quad c_5 \in \mathbb{R}, \quad (3.110)$$

$$\partial_y h_8(x, y) = 0. \quad (3.111)$$

On solving (3.109) and (3.111), we get

$$h_8(x, y) = c_3, \quad c_3 \in \mathbb{R}. \quad (3.112)$$

Using (3.106) and (3.108) in (3.103), we get

$$h_3(y, z, t) = (b_2y + c_4)e^{-t}, \quad \text{where } c_4 \in \mathbb{R}. \quad (3.113)$$

Using (3.107) and (3.110) in (3.104), we get

$$h_4(x, z, t) = (-b_2x + c_5)e^{-t}, \quad \text{where } c_5 \in \mathbb{R}. \quad (3.114)$$

From (3.105) and (3.112), we get

$$h_5(x, y, t) = c_3e^{2t}, \quad \text{where } c_4 \in \mathbb{R}. \quad (3.115)$$

Using (3.99) and (3.113) $\sim$ (3.115) in (3.100) $\sim$ (3.102), we obtain (1.3). Thus the proof of Theorem 1.2 is complete. $\square$

Proof of Theorem 1.3. Let $X = F_1e_1 + F_2e_2 + F_3e_3 + F_4e_4$ be a smooth vector field on the Riemannian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 . The Energy momentum tensor $T = Ric - (\frac{\tau}{2})g$ is:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Using (2.15), we compute the Lie derivative of the energy-momentum tensor T with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\left\{ \begin{array}{l} (L_X T)(e_1, e_1) = -6[F_4 - e^t \partial_x F_1], \\ (L_X T)(e_1, e_2) = 3e^t [\partial_x F_2 + \partial_y F_1], \\ (L_X T)(e_1, e_3) = 3[e^{-2t} \partial_z F_1 + e^t \partial_x F_3], \\ (L_X T)(e_1, e_4) = 3[\partial_t F_1 + F_1 - e^t \partial_x F_4], \\ (L_X T)(e_2, e_2) = -6[F_4 - e^t \partial_y F_2], \\ (L_X T)(e_2, e_3) = 3[e^t \partial_y F_3 + e^{-2t} \partial_z F_2], \\ (L_X T)(e_2, e_4) = 3[\partial_t F_2 + F_2 - e^t \partial_y F_4], \\ (L_X T)(e_3, e_3) = 6[e^{-2t} \partial_z F_3 + 2F_4], \\ (L_X T)(e_3, e_4) = 3[\partial_t F_3 - 2F_3 - e^{-2t} \partial_z F_4], \\ (L_X T)(e_4, e_4) = -6\partial_t F_4. \end{array} \right. \quad (3.116)$$

These relations (3.116) give the necessary and sufficient conditions for X to be a matter collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.3 yields the stated form of X given in (1.4). $\square$

Proof of Theorem 1.4. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Lorentzian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

We compute the Lie derivative of the Levi-Civita connection ∇ in order to determine the affine vector fields. Using (2.15) and (2.21), we obtain the following equations:

$$\begin{aligned} (L_X \nabla)(e_1, e_1) &= \left(-(\partial_t F_1 + F_1) - 2e^t \partial_x F_4 + e^{2t} \partial_x^2 F_1 \right) e_1 \\ &+ \left(-(F_2 + \partial_t F_2) + e^{2t} \partial_x^2 F_2 \right) e_2 + \left(-(\partial_t F_3 - 2F_3) + e^{2t} \partial_x^2 F_3 \right) e_3 \\ &+ \left(-(\partial_t F_4 + 2F_4) + 2e^t \partial_x F_1 + e^{2t} \partial_x^2 F_4 \right) e_4, \end{aligned} \quad (3.117)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_2) &= \left(-e^t \partial_y F_4 + e^{2t} \partial_x \partial_y F_1 \right) e_1 \\ &+ \left(-e^t \partial_x F_4 + e^{2t} \partial_x \partial_y F_2 \right) e_2 + \left(e^{2t} \partial_x \partial_y F_3 \right) e_3 \\ &+ \left(e^t (\partial_y F_1 + \partial_x F_2) + e^{2t} \partial_x \partial_y F_4 \right) e_4, \end{aligned} \quad (3.118)$$

$$\begin{aligned} (L_X \nabla)(e_1, e_4) &= \left(-\partial_t F_4 + e^t (\partial_x \partial_t F_1 + \partial_x F_1) \right) e_1 \\ &+ \left(e^t (\partial_x \partial_t F_3 + \partial_x F_3) \right) e_3 + \left(e^t (\partial_x \partial_t F_2 + \partial_x F_2) \right) e_2 \\ &+ \left(F_1 + \partial_t F_1 + e^t (\partial_x \partial_t F_4 + \partial_x F_4) \right) e_4, \end{aligned} \quad (3.119)$$

$$\begin{aligned} (L_X \nabla)(e_2, e_2) &= \left(-(F_1 + \partial_t F_1) + e^{2t} \partial_y^2 F_1 \right) e_1 + \left(-(\partial_t F_3 - 2F_3) \right. \\ &\quad \left. + e^{2t} \partial_y^2 F_3 \right) e_3 + \left(-(F_2 + \partial_t F_2) - 2e^t \partial_y F_4 + e^{2t} \partial_y^2 F_2 \right) e_2 \\ &\quad - \left((\partial_t F_4 + 2F_4) - 2e^t \partial_y F_2 - e^{2t} \partial_y^2 F_4 \right) e_4, \end{aligned} \quad (3.120)$$

$$\begin{aligned} (L_X \nabla)(e_2, e_4) &= \left(e^t (\partial_y F_1 + \partial_y \partial_t F_1) \right) e_1 + \left(e^t (\partial_y F_2 + \partial_y \partial_t F_2) - \partial_t F_4 \right) e_2 \\ &+ \left(e^t (\partial_y F_3 + \partial_y \partial_t F_3) \right) e_3 + \left(e^t (\partial_y F_4 + \partial_y \partial_t F_4) + F_2 + \partial_t F_2 \right) e_4, \end{aligned} \quad (3.121)$$

$$\begin{aligned} (L_X \nabla)(e_3, e_3) &= \left(-2(F_1 + \partial_t F_1) + e^{-4t} \partial_z^2 F_1 \right) e_1 + \left(-2(F_2 + \partial_t F_2) \right. \\ &\quad \left. + e^{-4t} \partial_z^2 F_2 \right) e_2 + \left(-2\partial_t F_4 + 4(e^{-2t} \partial_z F_3 + 2F_4) \right. \\ &\quad \left. + e^{-4t} \partial_z^2 F_4 \right) e_4 + \left(-2(\partial_t F_3 - 2F_3) + e^{-4t} \partial_z^2 F_3 + 4e^{-2t} \partial_z F_4 \right) e_3, \end{aligned} \quad (3.122)$$

$$\begin{aligned} (L_X \nabla)(e_3, e_4) &= \left(e^{-2t} (\partial_z \partial_t F_1 - 2\partial_z F_1) \right) e_1 + \left(e^{-2t} (\partial_z \partial_t F_2 - 2\partial_z F_2) \right) e_2 \\ &\quad + \left(e^{-2t} (\partial_z \partial_t F_3 - 2\partial_z F_3) + 2\partial_t F_4 \right) e_3 \\ &\quad + \left(e^{-2t} (\partial_z \partial_t F_4 - 2\partial_z F_4) + 2(\partial_t F_3 - 2F_3) \right) e_4, \end{aligned} \quad (3.123)$$

$$\begin{aligned} (L_X \nabla)(e_4, e_4) &= \left(\partial_t^2 F_1 - F_1 \right) e_1 + \left(\partial_t^2 F_2 - F_2 \right) e_2 \\ &\quad + \left(\partial_t^2 F_4 \right) e_4 + \left(\partial_t^2 F_3 - 4F_3 \right) e_3. \end{aligned} \quad (3.124)$$

Solving the above system of partial differential equations by setting all coefficients of the Lie derivative to zero, we find that the affine vector fields are given by (1.5). Thus the proof of Theorem 1.4 is complete. $\square$

Proof of Theorem 1.5. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Lorentzian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 .

Using (2.12) and (2.15), we compute the Lie derivative of the Ricci tensor Ric with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\begin{cases} (L_X Ric)(e_1, e_1) = 0, (L_X Ric)(e_1, e_2) = 0, \\ (L_X Ric)(e_1, e_3) = 0, (L_X Ric)(e_1, e_4) = -6e^t \partial_x F_4, \\ (L_X Ric)(e_2, e_2) = 0, (L_X Ric)(e_2, e_3) = 0, \\ (L_X Ric)(e_2, e_4) = -6e^t \partial_y F_4, (L_X Ric)(e_3, e_3) = 0, \\ (L_X Ric)(e_3, e_4) = -6e^{-2t} \partial_z F_4, (L_X Ric)(e_4, e_4) = -12\partial_t F_4. \end{cases} \quad (3.125)$$

These relations (3.125) give the necessary and sufficient conditions for X to be a Ricci collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.5 yields the stated form of X given in (1.6).

Now, to prove Theorem 1.5(b), we compute the Lie derivative of the curvature tensor R in order to determine the curvature collineations. Using (2.15) and (2.23), we obtain the following equations:

$$\begin{aligned} (L_X R)(e_1, e_2, e_1) &= \left(-e^t(\partial_y F_1 + \partial_x F_2) \right) e_1 - 3 \left(e^t \partial_y F_3 \right) e_3 \\ &\quad - 2 \left(F_4 - e^t \partial_x F_1 \right) e_2, \end{aligned} \quad (3.126)$$

$$\begin{aligned} (L_X R)(e_1, e_2, e_2) &= \left(2(F_4 - e^t \partial_y F_2) \right) e_1 + 3 \left(e^t \partial_x F_3 \right) e_3 \\ &\quad + \left(e^t(\partial_y F_1 + \partial_x F_2) \right) e_2, \end{aligned} \quad (3.127)$$

$$(L_X R)(e_1, e_2, e_3) = - \left(2e^t \partial_y F_3 + e^{-2t} \partial_z F_2 \right) e_1 + \left(e^{-2t} \partial_z F_1 + 2e^t \partial_x F_3 \right) e_2, \quad (3.128)$$

$$\begin{aligned} (L_X R)(e_1, e_3, e_1) &= \left(2(e^{-2t} \partial_z F_1 - e^t \partial_x F_3) \right) e_1 + 3 \left(e^{-2t} \partial_z F_2 \right) e_2 \\ &\quad + 4 \left(F_4 - e^t \partial_x F_1 \right) e_3 + 3 \left(e^{-2t} \partial_z F_4 \right) e_4, \end{aligned} \quad (3.129)$$

$$(L_X R)(e_1, e_3, e_2) = - \left(2e^t \partial_y F_3 + e^{-2t} \partial_z F_2 \right) e_1 - 2e^t \left(\partial_y F_1 + \partial_x F_2 \right) e_3, \quad (3.130)$$

$$\begin{aligned} (L_X R)(e_1, e_3, e_3) &= -4 \left(e^{-2t} \partial_z F_3 + 2F_4 \right) e_1 + 6 \left(e^t \partial_x F_4 \right) e_4 \\ &\quad + 2 \left(e^t \partial_x F_3 - e^{-2t} \partial_z F_1 \right) e_3, \end{aligned} \quad (3.131)$$

$$(L_X R)(e_2, e_3, e_3) = -4(e^{-2t} \partial_z F_3 + 2F_4)e_2 + 6(e^t \partial_y F_4)e_4 - 2(e^{-2t} \partial_z F_2 - e^t \partial_y F_3)e_3, \quad (3.132)$$

$$(L_X R)(e_2, e_4, e_2) = -(\partial_t F_2 + F_2 + e^t \partial_y F_4)e_2 - 3(\partial_t F_3 - 2F_3)e_3 - 2(F_4 - e^t \partial_y F_2)e_4, \quad (3.133)$$

$$(L_X R)(e_2, e_4, e_4) = -2(\partial_t F_4)e_2 - 3e^t(\partial_y F_3)e_3 + (\partial_t F_2 + F_2 + e^t \partial_y F_4)e_4, \quad (3.134)$$

$$(L_X R)(e_3, e_4, e_1) = 2(\partial_t F_1 + F_1 - 2e^t \partial_x F_4)e_3 + (-4e^t \partial_x F_3 + e^{-2t} \partial_z F_1)e_4, \quad (3.135)$$

$$(L_X R)(e_3, e_4, e_3) = 6(\partial_t F_1 + F_1)e_1 + 4(\partial_t F_3 - 2F_3 - e^{-2t} \partial_z F_4)e_3 + 6(F_2 + \partial_t F_2)e_2 - (8(e^{-2t} \partial_z F_3 + 2F_4))e_4, \quad (3.136)$$

$$(L_X R)(e_3, e_4, e_4) = 3(e^{-2t} \partial_z F_1)e_1 + 3(e^{-2t} \partial_z F_2)e_2 - 8(\partial_t F_4)e_3 + 4(-(\partial_t F_3 - 2F_3) + e^{-2t} \partial_z F_4)e_4. \quad (3.137)$$

Solving the above system of partial differential equations by setting all coefficients of the Lie derivative to zero, we find that the curvature collineation are given by (1.7). Thus the proof of Theorem 1.5 is complete. $\square$

Proof of Theorem 1.6. Let $X = F_1 e_1 + F_2 e_2 + F_3 e_3 + F_4 e_4$ be a smooth vector field on the Lorentzian Lie group Sol_0^4 , where F_1, F_2, F_3, F_4 are infinitely differentiable real-valued functions on Sol_0^4 . The Energy momentum tensor $T = Ric - (\frac{\tau}{2})g$ is:

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

Using (2.15), we compute the Lie derivative of the energy-momentum tensor T with respect to X for each pair of basis vectors $\{e_i\}$, obtaining:

$$\left\{ \begin{array}{l} (L_X T)(e_1, e_1) = -6[F_4 - e^t \partial_x F_1], \\ (L_X T)(e_1, e_2) = 3e^t [\partial_x F_2 + \partial_y F_1], \\ (L_X T)(e_1, e_3) = 3[e^{-2t} \partial_z F_1 - e^t \partial_x F_3], \\ (L_X T)(e_1, e_4) = 3[\partial_t F_1 + F_1 - e^t \partial_x F_4], \\ (L_X T)(e_2, e_2) = -6[F_4 - e^t \partial_y F_2], \\ (L_X T)(e_2, e_3) = 3[-e^t \partial_y F_3 + e^{-2t} \partial_z F_2], \\ (L_X T)(e_2, e_4) = 3[\partial_t F_2 + F_2 - e^t \partial_y F_4], \\ (L_X T)(e_3, e_3) = -6[e^{-2t} \partial_z F_3 + 2F_4], \\ (L_X T)(e_3, e_4) = -3[\partial_t F_3 - 2F_3 + e^{-2t} \partial_z F_4], \\ (L_X T)(e_4, e_4) = -6\partial_t F_4. \end{array} \right. \quad (3.138)$$

These relations (3.138) give the necessary and sufficient conditions for X to be a matter collineation on Sol_0^4 . Solving them under the assumptions of Theorem 1.6 yields the stated form of X given in (1.8). $\square$

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CHARACTERIZATION OF MATRIX CLASSES ASSOCIATED WITH p -ABSOLUTELY SUMMABLE SEQUENCES OF BI-COMPLEX NUMBERS

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Abstract. In this paper, we have characterized the bi-complex matrix transformations on $\ell_p(\mathbb{C}_2)$ space. We provide the necessary and sufficient requirements for an infinite bi-complex matrix to belong to $B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$. Additionally, we study when an infinite bi-complex matrix transforms $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$ and provide a counterexample for the converse statement. We also provide some non trivial examples to support our claims.

Keywords: Bi-complex numbers, $\mathbb{C}_2$ -linear operators, Characterization of matrix classes, p -Absolutely summable sequences, Sequence spaces.

MSC (2020): 40A05, 40C05, 46A45, 46B45, 47L05, 47L25.

1. INTRODUCTION

The concept of bi-complex numbers are four-dimensional vectors that can be recognized similarly to complex numbers, which are two-dimensional vectors. Bi-complex numbers were initially developed using pairs of complex numbers. Why were bi-complex numbers used? What are their advantages and disadvantages? In order to answer those questions, the notion of bi-complex numbers was introduced by Segre [14]. The work of Price [12] contains a detailed review of the bi-complex space. In recent years, many authors [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13, 15] have studied bi-complex numbers. Their theory finds applications in various fields of mathematics and physics, such as electromagnetic theory, fluid dynamics, and special relativity. Now, it has become a subject of research in both physics and mathematics, attracting significant interest from researchers because of its numerous applications in various domains of mathematical sciences and related disciplines.

Throughout the article, we denote the set of all bi-complex sequences by $\omega(\mathbb{C}_2)$.

Segre [14] defined the bi-complex number as follows:

$$z = w_1 + jw_2 = (a + ib) + j(c + id) = a + ib + jc + ijd,$$

where $w_1, w_2 \in \mathbb{C}(i)$; $a, b, c, d \in \mathbb{R}$ and $i^2 = j^2 = -1, ij = ji$.

Here we consider $\mathbb{C}(i)$ as the set of all complex numbers with imaginary units i .

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The set of bi-complex numbers $\mathbb{C}_2$ is defined by

$$\mathbb{C}_2 = \{z = w_1 + jw_2 : w_1, w_2 \in \mathbb{C}(i)\}.$$

The four idempotent elements in $\mathbb{C}_2$ are $0, 1, e_1$ and e_2 , where $e_1 = \frac{1+j}{2}$ and $e_2 = \frac{1-j}{2}$.

Every bi-complex number $z = w_1 + jw_2$ can be uniquely expressed as complex combination of e_1 and e_2 as follows;

$$z = \alpha_1 e_1 + \alpha_2 e_2,$$

where $\alpha_1 = w_1 - iw_2$ and $\alpha_2 = w_1 + iw_2$.

If we consider the two bi-complex numbers $z = \alpha'_1 e_1 + \alpha'_2 e_2$ and $w = \alpha''_1 e_1 + \alpha''_2 e_2$, then we have the following:

$$\begin{aligned} z \pm w &= (\alpha'_1 \pm \alpha''_1) e_1 + (\alpha'_2 \pm \alpha''_2) e_2, \\ zw &= (\alpha'_1 \alpha''_1) e_1 + (\alpha'_2 \alpha''_2) e_2. \end{aligned}$$

A function $\|\cdot\|_{\mathbb{C}_2} : \mathbb{C}_2 \rightarrow \mathbb{R}^+ \cup \{0\}$ such that

$$\begin{aligned} \|z\|_{\mathbb{C}_2} &= \sqrt{|w_1|^2 + |w_2|^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{|\alpha_1|^2 + |\alpha_2|^2} \end{aligned}$$

is a real valued norm on $\mathbb{C}_2$, where $z = w_1 + jw_2 = \alpha_1 e_1 + \alpha_2 e_2$.

This norm satisfies the following properties:

$$\|zw\|_{\mathbb{C}_2} \leq \sqrt{2} \|z\|_{\mathbb{C}_2} \|w\|_{\mathbb{C}_2} \text{ and } \|z \pm w\|_{\mathbb{C}_2} \leq \|z\|_{\mathbb{C}_2} + \|w\|_{\mathbb{C}_2}.$$

With the above norm, $\mathbb{C}_2$ forms a modified Banach space.

Let $(X(\mathbb{C}_2), \|\cdot\|_{X(\mathbb{C}_2)})$ be any bi-complex sequence space, then $X(\mathbb{C}_2)$ can be written as $X(\mathbb{C}_2) = X_1(\mathbb{C})e_1 + X_2(\mathbb{C})e_2$, where $X_1(\mathbb{C}) = \{(\xi_{k,1}) : \xi = (\xi_k) = (\xi_{k,1}e_1 + \xi_{k,2}e_2) \in X(\mathbb{C}_2)\}$ and $X_2(\mathbb{C}) = \{(\xi_{k,2}) : \xi = (\xi_k) = (\xi_{k,1}e_1 + \xi_{k,2}e_2) \in X(\mathbb{C}_2)\}$ and the norm $\|\cdot\|_{X(\mathbb{C}_2)}$ is given by

$$\|\xi\|_{X(\mathbb{C}_2)} = \frac{1}{\sqrt{2}} \sqrt{\left(\|\xi_{k,1}\|_{X_1(\mathbb{C})}^2 + \|\xi_{k,2}\|_{X_2(\mathbb{C})}^2\right)},$$

where $\|\cdot\|_{X_1(\mathbb{C})}$ and $\|\cdot\|_{X_2(\mathbb{C})}$ are the norms over the spaces $X_1(\mathbb{C})$ and $X_2(\mathbb{C})$, respectively.

Again, if $(Y(\mathbb{C}), \|\cdot\|_{Y(\mathbb{C})})$ is any given complex sequence space, we can always define a bi-complex sequence space $(Y(\mathbb{C}_2), \|\cdot\|_{Y(\mathbb{C}_2)})$, where

$Y(\mathbb{C}_2) = \left\{ \xi = (\xi_{k,1}e_1 + \xi_{k,2}e_2) : (\xi_{k,1}), (\xi_{k,2}) \in Y(\mathbb{C}) \right\}$ and the norm is given by

$$\|\xi\|_{Y(\mathbb{C}_2)} = \frac{1}{\sqrt{2}} \sqrt{\left(\|\xi_{k,1}\|_{Y(\mathbb{C})}^2 + \|\xi_{k,2}\|_{Y(\mathbb{C})}^2\right)},$$

Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite matrix of bi-complex numbers $\hat{a}_{nk}$, where

$$\hat{a}_{nk} = a_{nk}^\dagger e_1 + a_{nk}^\ddagger e_2, \quad a_{nk}^\dagger, a_{nk}^\ddagger \in \mathbb{C}(i), \text{ for all } n, k \in \mathbb{N}.$$

Then the matrix $\hat{A}$ can be uniquely written as $\hat{A} = A^\dagger e_1 + A^\ddagger e_2$, where $A^\dagger = [a_{nk}^\dagger]$ and $A^\ddagger = [a_{nk}^\ddagger]$ are infinite matrix of complex numbers.

Now, for any sequence $\xi = (\xi_k) = (\xi_{k,1}e_1 + \xi_{k,2}e_2) \in \omega(\mathbb{C}_2)$, we obtain the sequence $\hat{A}\xi$, the

$\widehat{A}$ -transform of ξ , in the following way;

$$\begin{aligned}
 (\widehat{A}\xi)_n &= \sum_{k=1}^{\infty} \widehat{a}_{nk} \xi_k \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} e_1 + a_{nk}^{\ddagger} e_2) (\xi_{k,1} e_1 + \xi_{k,2} e_2) \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1} e_1 + a_{nk}^{\ddagger} \xi_{k,2} e_2) \\
 &= e_1 \sum_{k=1}^{\infty} a_{nk}^{\dagger} \xi_{k,1} + e_2 \sum_{k=1}^{\infty} a_{nk}^{\ddagger} \xi_{k,2}.
 \end{aligned} \tag{1.1}$$

Therefore,

$$(\widehat{A}\xi)_n = (A^{\dagger} \xi_{k,1})_n e_1 + (A^{\ddagger} \xi_{k,2})_n e_2$$

provided both the series on the right hand side of (1.1) converge for each $n \in \mathbb{N}$. In this way, we transform the $\mathbb{C}_2$ -sequence ξ into the $\mathbb{C}_2$ -sequence $\widehat{A}\xi$.

Let $\omega_1(\mathbb{C}_2)$ and $\omega_2(\mathbb{C}_2)$ be any two bi-complex sequence spaces. For any sequence $\xi \in \omega_1(\mathbb{C}_2)$, if $\widehat{A}\xi$ exist and is in $\omega_2(\mathbb{C}_2)$, then we say that $\widehat{A}$ defines a $\mathbb{C}_2$ -matrix mapping from $\omega_1(\mathbb{C}_2)$ into $\omega_2(\mathbb{C}_2)$. The class of all $\mathbb{C}_2$ -matrices $\widehat{A}$ which transform every sequence $\xi \in \omega_1(\mathbb{C}_2)$ into $\omega_2(\mathbb{C}_2)$ is denoted by $(\omega_1(\mathbb{C}_2), \omega_2(\mathbb{C}_2))$. In additionally, it can easily be verified that this transformation is linear. For more details one may refer to [5].

Definition 1.1. [4]

$$\begin{aligned}
 \ell_{\infty}(\mathbb{C}_2) &= \left\{ \xi = (\xi_k) \in \omega(\mathbb{C}_2) : \sup_{k \in \mathbb{N}} \|\xi_k\|_{\mathbb{C}_2} < \infty \right\}, \\
 \ell_p(\mathbb{C}_2) &= \left\{ \xi = (\xi_k) \in \omega(\mathbb{C}_2) : \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2}^p < \infty \right\}, \quad 0 < p < \infty.
 \end{aligned}$$

Lemma 1.1. [4] $\ell_{\infty}(\mathbb{C}_2)$ is a Banach space with the norm $\|\cdot\|_{\ell_{\infty}(\mathbb{C}_2)}$ defined by

$$\|\xi\|_{\ell_{\infty}(\mathbb{C}_2)} = \sup_{k \in \mathbb{N}} \|\xi_k\|_{\mathbb{C}_2}, \text{ for all } \xi = (\xi_k) \in \ell_{\infty}(\mathbb{C}_2).$$

Lemma 1.2. [4] For $1 \leq p < \infty$, $\ell_p(\mathbb{C}_2)$ is a Banach space with the norm $\|\cdot\|_{\ell_p(\mathbb{C}_2)}$ defined by

$$\|\xi\|_{\ell_p(\mathbb{C}_2)} = \left(\sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}}, \text{ for all } \xi = (\xi_k) \in \ell_p(\mathbb{C}_2).$$

Lemma 1.3. [4] (Bi-complex Hölder's Inequality) Let p and q be two real numbers with $1 < p < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $\xi_k, \eta_k \in \mathbb{C}_2$, for $k \in \{1, 2, 3, \dots, n\}$. Then

$$\sum_{k=1}^n \|\xi_k \cdot \eta_k\|_{\mathbb{C}_2} \leq \sqrt{2} \left(\sum_{k=1}^n \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^n \|\eta_k\|_{\mathbb{C}_2}^q \right)^{\frac{1}{q}}.$$

Lemma 1.4. [4] (Bi-complex Minkowski's Inequality) Let p be a real numbers with $1 < p < \infty$ such that $\xi_k, \eta_k \in \mathbb{C}_2$, for $k \in \{1, 2, 3, \dots, n\}$. Then

$$\left(\sum_{k=1}^n \|\xi_k + \eta_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \leq \left(\sum_{k=1}^n \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} + \left(\sum_{k=1}^n \|\eta_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}}.$$

Theorem 1.1. [10] *Let $A = [a_{nk}]$ be an infinite matrix satisfying the condition*

$$\sup_k \sum_{n=1}^{\infty} |a_{nk}|^2 < \infty.$$

Then $A \in B(\ell_2, \ell_2)$ and $\|A\| = \sup_k \sum_{n=1}^{\infty} |a_{nk}|^2$.

Theorem 1.2. *Let $A = [a_{nk}]$ be an infinite matrix such that $A \in B(\ell_2, \ell_2)$. Then, the bounded linear transformation A assigns a matrix $A = [a_{nk}]$ such that $(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k$, for every $x = (x_k) \in \ell_2$. In addition, the following condition is satisfied:*

$$\sum_{k=1}^{\infty} |a_{nk}|^2 < \infty, \text{ for all } n \in \mathbb{N}.$$

Theorem 1.3. [10] *Let $A = [a_{nk}]$ be an infinite matrix. If $\sum_{k=0}^{\infty} |a_{nk}| < \infty$, for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}| = 0$. Then $\sum_{k=1}^{\infty} |a_{nk}|$ is uniformly convergent in $n \in \mathbb{N}$.*

Theorem 1.4. *Let $A = [a_{nk}]$ be an infinite matrix. The necessary and sufficient condition for the matrix $A \in B(\ell_1, \ell_p)$, $1 \leq p < \infty$ is $\sup_k \sum_{n=1}^{\infty} |a_{nk}|^p < \infty$.*

Theorem 1.5. *Let $A = [a_{nk}]$ be an infinite matrix satisfying the condition $\sup_{n,k} |a_{nk}| < \infty$, then $A \in B(\ell_1, \ell_{\infty})$.*

2. MAIN RESULTS

In this section, we establish the main results of this article.

Theorem 2.1. *Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix where $\hat{a}_{nk} = a_{nk}^{\dagger}e_1 + a_{nk}^{\ddagger}e_2$, for all $n, k \in \mathbb{N}$. If $\hat{A} = A^{\dagger}e_1 + A^{\ddagger}e_2 \in (X(\mathbb{C}_2), Y(\mathbb{C}_2))$, where $X(\mathbb{C}_2) = X_1(\mathbb{C})e_1 + X_2(\mathbb{C})e_2$ and $Y(\mathbb{C}_2) = Y_1(\mathbb{C})e_1 + Y_2(\mathbb{C})e_2$. Then $A^{\dagger} \in (X_1(\mathbb{C}), Y_1(\mathbb{C}))$ and $A^{\ddagger} \in (X_2(\mathbb{C}), Y_2(\mathbb{C}))$.*

Proof. Given that $\hat{A} = A^{\dagger}e_1 + A^{\ddagger}e_2 \in (X(\mathbb{C}_2), Y(\mathbb{C}_2))$. We have, for any sequence $\xi = (\xi_k) \in X(\mathbb{C}_2)$, where $\xi_k = \xi_{k,1}e_1 + \xi_{k,2}e_2$, for all $k \in \mathbb{N}$; $(\xi_{k,1}) \in X_1(\mathbb{C})$ and $(\xi_{k,2}) \in X_2(\mathbb{C})$, which implies that $\hat{A}\xi \in Y(\mathbb{C}_2) = Y_1(\mathbb{C})e_1 + Y_2(\mathbb{C})e_2$.

Then,

$$\begin{aligned}
 (\widehat{A}\xi)_n &= \sum_{k=1}^{\infty} \widehat{a}_{nk} \xi_k \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} e_1 + a_{nk}^{\ddagger} e_2) (\xi_{k,1} e_1 + \xi_{k,2} e_2) \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1} e_1 + a_{nk}^{\ddagger} \xi_{k,2} e_2) \\
 &= \sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1}) e_1 + \sum_{k=1}^{\infty} (a_{nk}^{\ddagger} \xi_{k,2}) e_2.
 \end{aligned} \tag{2.2}$$

Since, both the series of the right hand side of (2.2) are convergent, So we have

$$(\widehat{A}\xi)_n = (A^{\dagger}\xi)_n e_1 + (A^{\ddagger}\xi)_n e_2,$$

where $A^{\dagger}\xi = \left(\sum_{k=1}^{\infty} (a_{nk}^{\dagger} \xi_{k,1}) \right) \in Y_1(\mathbb{C})$ and $A^{\ddagger}\xi = \left(\sum_{k=1}^{\infty} (a_{nk}^{\ddagger} \xi_{k,2}) \right) \in Y_2(\mathbb{C})$.

So, $A^{\dagger}$ transforms the sequence $(\xi_{k,1}) \in X_1(\mathbb{C})$ to the space $Y_1(\mathbb{C})$ and $A^{\ddagger}$ transforms the sequence $(\xi_{k,2}) \in X_2(\mathbb{C})$ to the space $Y_2(\mathbb{C})$. Hence, $A^{\dagger} \in (X_1(\mathbb{C}), Y_1(\mathbb{C}))$ and $A^{\ddagger} \in (X_2(\mathbb{C}), Y_2(\mathbb{C}))$. $\square$

Theorem 2.2. Let $\widehat{A} = [\widehat{a}_{nk}]$ be an infinite bi-complex matrix satisfying the condition $\|M\|_{\mathbb{C}_2} = \sup_k \sum_{n=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 < \infty$. Then $\widehat{A} \in B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$ and $\|\widehat{A}\| = \|M\|_{\mathbb{C}_2}$.

Proof. We know that

$$\begin{aligned}
 \|M\|_{\mathbb{C}_2} &= \sup_k \sum_{n=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 < \infty \\
 &= \sup_k \sum_{n=1}^{\infty} \|a_{nk}^{\dagger} e_1 + a_{nk}^{\ddagger} e_2\|_{\mathbb{C}_2}^2 < \infty \\
 &= \sup_k \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}} \sqrt{|a_{nk}^{\dagger}|^2 + |a_{nk}^{\ddagger}|^2} \right)^2 < \infty \\
 &= \sup_k \frac{1}{2} \sum_{n=1}^{\infty} (|a_{nk}^{\dagger}|^2 + |a_{nk}^{\ddagger}|^2) < \infty.
 \end{aligned}$$

This implies,

$$M_1 = \sup_k \sum_{n=1}^{\infty} |a_{nk}^{\dagger}|^2 < \infty \text{ and } M_2 = \sup_k \sum_{n=1}^{\infty} |a_{nk}^{\ddagger}|^2 < \infty.$$

Then by Theorem 1.1, we have $A^{\dagger}, A^{\ddagger} \in B(\ell_2, \ell_2)$ and $\|A^{\dagger}\| = M_1, \|A^{\ddagger}\| = M_2$. Hence, $\widehat{A} = A^{\dagger} e_1 + A^{\ddagger} e_2 \in (\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$ and it is clear that $\widehat{A}$ is $\mathbb{C}_2$ -linear.

Now, we have to prove that $\widehat{A}$ is a bounded linear operator on $\ell_2(\mathbb{C}_2)$.

As $A^{\dagger}, A^{\ddagger} \in B(\ell_2, \ell_2)$, we can write

$$\|A^{\dagger}\xi_1\|_{\ell_2} \leq k_1 \|\xi_1\|_{\ell_2} \text{ and } \|A^{\ddagger}\xi_2\|_{\ell_2} \leq k_2 \|\xi_2\|_{\ell_2},$$

where $\xi_1 = (\xi_{k,1})$ and $\xi_2 = (\xi_{k,2}) \in \ell_2$.

We have,

$$\begin{aligned} \|\hat{A}\xi\|_{\ell_2(\mathbb{C}_2)} &= \frac{1}{\sqrt{2}} \sqrt{\|A^\dagger \xi_1\|_{\ell_2}^2 + \|A^\ddagger \xi_2\|_{\ell_2}^2} \\ &\leq \frac{1}{\sqrt{2}} \sqrt{k_1^2 \|\xi_1\|_{\ell_2}^2 + k_2^2 \|\xi_2\|_{\ell_2}^2} \\ &\leq \frac{k}{\sqrt{2}} \sqrt{\|\xi_1\|_{\ell_2}^2 + \|\xi_2\|_{\ell_2}^2}, \text{ where } k = \max\{k_1, k_2\}. \end{aligned}$$

Thus, $\|\hat{A}\xi\|_{\ell_2(\mathbb{C}_2)} \leq k\|\xi\|_{\ell_2(\mathbb{C}_2)}$.

On the other hand, if we take

$$M = M_1 e_1 + M_2 e_2,$$

we get

$$\begin{aligned} \|\hat{A}\| &= \frac{1}{\sqrt{2}} \sqrt{\|A^\dagger\|^2 + \|A^\ddagger\|^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{M_1^2 + M_2^2} = \|M\|_{\mathbb{C}_2}. \end{aligned}$$

□

Note: The necessity of the stated condition cannot be relaxed. To illustrate this, we consider the following counterexample. The counterexample below shows that if the condition is relaxed, the conclusion no longer holds. In particular, the matrix $\hat{A} = (\hat{a}_{nk})$ does not belong to $B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$ because it fails to satisfy the required condition.

Example 2.1. Consider the matrix $\hat{A} = (\hat{a}_{nk})$, where

$$\hat{A} = \begin{bmatrix} e_1 + e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \\ \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \end{bmatrix}.$$

Here, for the first column of the matrix $\hat{A}$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} \left\| \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 \right\|_{\mathbb{C}_2}^2 \\ &= \sum_{n=1}^{\infty} \frac{1}{2} \left(\left| \frac{1}{\sqrt{n}} \right|^2 + \left| \frac{1}{\sqrt{n}} \right|^2 \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{1}{n} + \frac{1}{n} \right) \\ &= \sum_{n=1}^{\infty} \frac{1}{n} = \infty. \end{aligned}$$

So, this matrix does not satisfy the given condition.

Now, $e_1 = (1, 0, 0, \dots) \in \ell_2(\mathbb{C}_2)$. Then

$$\widehat{A}e_1 = \begin{bmatrix} e_1 + e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2 & 0 & \dots & 0 \dots \\ \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \\ \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 & 0 & \dots & 0 \dots \\ \vdots & \vdots & \dots & \vdots \dots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} e_1 + e_2 \\ \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2 \\ \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2 \\ \vdots \\ \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2 \\ \vdots \end{bmatrix}.$$

But $\left(e_1 + e_2, \frac{1}{\sqrt{2}}e_1 + \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_1 + \frac{1}{\sqrt{3}}e_2, \frac{1}{\sqrt{n}}e_1 + \frac{1}{\sqrt{n}}e_2, \dots\right) \notin \ell_2(\mathbb{C}_2)$.

Theorem 2.3. Let $\widehat{A} = [\widehat{a}_{nk}]$ be an infinite bi-complex matrix such that $\widehat{A} \in B(\ell_2(\mathbb{C}_2), \ell_2(\mathbb{C}_2))$. Then the bounded linear transformation $\widehat{A}$ assigns a matrix $\widehat{A} = [\widehat{a}_{nk}]$ such that

$$(\widehat{A}\xi)_n = \sum_{k=1}^{\infty} \widehat{a}_{nk}\xi_k, \text{ for every } \xi = (\xi_k) \in \ell_2(\mathbb{C}_2).$$

In addition, the following condition is satisfied

$$\sum_{k=1}^{\infty} \|a_{nk}\|_{\mathbb{C}_2}^2 < \infty, \text{ for all } n \in \mathbb{N}.$$

Proof. Consider the bi-complex sequence (e'_k) defined as $e'_k = (0, 0, \dots, e_1 + e_2, 0, 0, \dots) = e_k^\dagger e_1 + e_k^\dagger e_2$, is a basis for $\ell_2(\mathbb{C}_2)$, where $e_k^\dagger = e_k^\ddagger = (0, 0, \dots, 1, 0, 0, \dots)$, $(e_k^\dagger), (e_k^\ddagger)$ is a basis of ℓ_2 . Therefore, for any sequence $\xi = (\xi_k) \in \ell_2(\mathbb{C}_2)$ such that

$$\xi = \sum_{k=1}^{\infty} \xi_k e'_k = \left(\sum_{k=1}^{\infty} \xi_{k,1} e_k^\dagger\right) e_1 + \left(\sum_{k=1}^{\infty} \xi_{k,2} e_k^\ddagger\right) e_2.$$

Since, $\widehat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation, we have

$$\widehat{A}\xi = \sum_{k=1}^{\infty} \xi_k \widehat{A}e'_k.$$

On the other hand, since, $e'_k \in \ell_2(\mathbb{C}_2)$, for every $k \in \mathbb{N}$. By hypothesis

$$\widehat{A}e'_k = (\widehat{a}_{1k}, \widehat{a}_{2k}, \dots, \widehat{a}_{nk}, \dots) \in \ell_2(\mathbb{C}_2).$$

This implies,

$$\widehat{A}\xi = \sum_{k=1}^{\infty} \xi_k \widehat{a}_{nk}.$$

For the next part, since

$$\|\widehat{a}_{nk}\|_{\mathbb{C}_2} = \frac{1}{\sqrt{2}} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2}.$$

So

$$\begin{aligned} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 &= \frac{1}{2} \left(|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2 \right) \\ &\leq |a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2. \end{aligned}$$

Taking summation on both sides, we get

$$\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 \leq \sum_{k=1}^{\infty} |a_{nk}^\dagger|^2 + \sum_{k=1}^{\infty} |a_{nk}^\ddagger|^2.$$

Then by Theorem 1.2, we can write

$$\sum_{k=1}^{\infty} |a_{nk}^\dagger|^2 < \infty \text{ and } \sum_{k=1}^{\infty} |a_{nk}^\ddagger|^2 < \infty.$$

Hence, $\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}^2 < \infty$, for all $n \in \mathbb{N}$. This completes the proof. $\square$

Theorem 2.4. *Let $\widehat{A} = [\widehat{a}_{nk}]$ be an infinite bi-complex matrix. If $\sum_{k=0}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} < \infty$, for all $n \in \mathbb{N}$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} = 0$. Then $\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2}$ is uniformly convergent in $n \in \mathbb{N}$.*

Proof. Given that

$$\begin{aligned} & \sum_{k=0}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} < \infty, \text{ for each } n \in \mathbb{N}, \\ \Rightarrow & \frac{1}{\sqrt{2}} \sum_{k=1}^{\infty} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} < \infty, \text{ for each } n \in \mathbb{N} \\ \Rightarrow & \sum_{k=1}^{\infty} |a_{nk}^\dagger| < \infty \text{ and } \sum_{k=1}^{\infty} |a_{nk}^\ddagger| < \infty, \text{ for each } n \in \mathbb{N} \end{aligned}$$

where

$$\widehat{a}_{nk} = a_{nk}^\dagger e_1 + a_{nk}^\ddagger e_2.$$

Since $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} = 0$, then for all $\varepsilon > 0$ there exist $N(\varepsilon) \in \mathbb{N}$ such that

$$\sum_{k=1}^{\infty} \|\widehat{a}_{nk}\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } n > N(\varepsilon).$$

As, $|a_{nk}^\dagger| < \frac{1}{\sqrt{2}} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} < \varepsilon$ and $|a_{nk}^\ddagger| < \frac{1}{\sqrt{2}} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} < \varepsilon$.

Then, $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}^\dagger| = 0$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |a_{nk}^\ddagger| = 0$.

Then using the Theorem 1.3, there exist $M_1(\varepsilon)$ and $M_2(\varepsilon)$ such that

$$\sum_{k \geq M_1} |a_{nk}^\dagger| < \frac{\varepsilon}{2} \text{ and } \sum_{k \geq M_2} |a_{nk}^\ddagger| < \frac{\varepsilon}{2}, \text{ for all } n \in \mathbb{N}.$$

Take $M = \max\{M_1, M_2\}$. Then for all $n \in \mathbb{N}$,

$$\begin{aligned} \sum_{k \geq M} \|\hat{a}_{nk}\|_{\mathbb{C}_2} &= \frac{1}{\sqrt{2}} \sum_{k \geq M} \sqrt{|a_{nk}^\dagger|^2 + |a_{nk}^\ddagger|^2} \\ &\leq \frac{1}{\sqrt{2}} \sum_{k \geq M} |a_{nk}^\dagger| + \frac{1}{\sqrt{2}} \sum_{k \geq M} |a_{nk}^\ddagger| \\ &\leq \sum_{k \geq M} |a_{nk}^\dagger| + \sum_{k \geq M} |a_{nk}^\ddagger| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, the theorem is proved. $\square$

Theorem 2.5. Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix. Then the necessary and sufficient condition for the matrix $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$, $1 \leq p < \infty$, is that

$$\sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty.$$

Proof. Necessity: Let us assume that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$. i.e., $\hat{A}$ is a $\mathbb{C}_2$ - bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_p(\mathbb{C}_2)$.

We have to prove that $\sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty$ or $\sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty$, for all $k \in \mathbb{N}$.

Consider the bi-complex sequence (e'_k) defined as $e'_k = (0, 0, \dots, e_1 + e_2, 0, 0, \dots)$, for each $k \in \mathbb{N}$.

Then, $\hat{A}e'_k = (\hat{a}_{1k}, \hat{a}_{2k}, \hat{a}_{3k}, \dots, \hat{a}_{nk}, \dots)$, which is the k -th column of $\hat{A}$.

Now $\hat{A}e'_k \in \ell_p(\mathbb{C}_2)$, then $\|\hat{A}e'_k\|_{\ell_p(\mathbb{C}_2)} < \infty$. This implies that

$$\left(\sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} < \infty, \text{ for all } k \in \mathbb{N} \Rightarrow \sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty.$$

Hence, the necessary condition is proved.

Sufficiency: Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix such that

$$M = \sup_k \sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p < \infty, 1 \leq p < \infty. \quad (2.3)$$

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$, then

$$\|\xi\|_{\ell_1(\mathbb{C}_2)} = \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty.$$

Now,

$$\begin{aligned}
\|\hat{A}\xi\|_{\ell_p(\mathbb{C}_2)} &= \left(\sum_{n=1}^{\infty} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} \|\hat{a}_{nk} \xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} (\sqrt{2})^p \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p \|\xi_k\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&= \sum_{k=1}^{\infty} \sqrt{2} \|\xi_k\|_{\mathbb{C}_2} \left(\sum_{n=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2}^p \right)^{\frac{1}{p}} \\
&\leq \sqrt{2} M^{\frac{1}{p}} \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty.
\end{aligned}$$

This implies that $\hat{A}\xi = \left(\sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right)_n \in \ell_p(\mathbb{C}_2)$, i.e., $\hat{A}$ transforms any sequence in $\ell_1(\mathbb{C}_2)$ into the space $\ell_p(\mathbb{C}_2)$, for $1 \leq p < \infty$. Also, it is clear that $\hat{A}$ is a $\mathbb{C}_2$ -linear transformation. However, $\|\hat{A}\xi\|_{\ell_p(\mathbb{C}_2)} \leq \sqrt{2} M^{\frac{1}{p}} \|\xi\|_{\ell_1(\mathbb{C}_2)}$. This implies that $\hat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_p(\mathbb{C}_2)$. Hence, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$. $\square$

Note: The above condition can not be relaxed. For this we consider the following example.

Example 2.2. Consider the matrix $\hat{A} = (\hat{a}_{nk})$, where

$$\hat{A} = \begin{bmatrix} e_1 + e_2 & e_1 + e_2 & \dots & e_1 + e_2 \dots \\ \frac{1}{2}e_1 + \frac{1}{2}e_2 & \frac{1}{2}e_1 + \frac{1}{2}e_2 & \dots & \frac{1}{2}e_1 + \frac{1}{2}e_2 \dots \\ \frac{1}{3}e_1 + \frac{1}{3}e_2 & \frac{1}{3}e_1 + \frac{1}{3}e_2 & \dots & \frac{1}{3}e_1 + \frac{1}{3}e_2 \dots \\ \vdots & \vdots & \dots & \vdots \dots \\ \frac{1}{n}e_1 + \frac{1}{n}e_2 & \frac{1}{n}e_1 + \frac{1}{n}e_2 & \dots & \frac{1}{n}e_1 + \frac{1}{n}e_2 \dots \\ \vdots & \vdots & \dots & \vdots \ddots \end{bmatrix}.$$

For any column of the matrix $\hat{A}$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\| \frac{1}{n}e_1 + \frac{1}{n}e_2 \right\|_{\mathbb{C}_2}^p \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}} \sqrt{\left| \frac{1}{n} \right|^2 + \left| \frac{1}{n} \right|^2} \right)^p \\
&= \sum_{n=1}^{\infty} \left(\frac{1}{\sqrt{2}} \right)^p \left(\frac{\sqrt{2}}{n} \right)^p \\
&= \sum_{n=1}^{\infty} \frac{1}{n^p}.
\end{aligned}$$

If $p = 1$, then $\sum_{n=1}^{\infty} \frac{1}{n^p} = \infty$. Thus the matrix does not satisfy the given condition. If we take $\xi = (e_1 + e_2, 0, 0, \dots) \in \ell_1(\mathbb{C}_2)$, then we have

$$\hat{A}\xi = \left(e_1 + e_2, \frac{1}{2}e_1 + \frac{1}{2}e_2, \dots, \frac{1}{n}e_1 + \frac{1}{2}e_2, \dots \right), \text{ does not belong to } \ell_p(\mathbb{C}_2).$$

This implies that $\hat{A} \notin B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$.

However, if we take $1 < p < \infty$, then $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent.

Therefore, the matrix $\hat{A}$ satisfies the given condition, i.e., $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_p(\mathbb{C}_2))$.

Theorem 2.6. Let $A = [\hat{a}_{nk}]$ be an infinite bi-complex matrix satisfying the condition $\sup_{n,k} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty$, then $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_{\infty}(\mathbb{C}_2))$. The converse, however, is not true.

Proof. Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix such that

$$M = \sup_{n,k} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty.$$

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$, then $\|\xi\|_{\ell_1(\mathbb{C}_2)} = \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty$.

For all $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right\|_{\mathbb{C}_2} &\leq \sum_{k=1}^{\infty} \|\hat{a}_{nk} \xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} M \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} M \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} \\ &= \sqrt{2} M \|\xi\|_{\ell_1(\mathbb{C}_2)} < \infty. \end{aligned}$$

Therefore, $\sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk} \xi_k \right\|_{\mathbb{C}_2} \leq \sqrt{2} M \|\xi\|_{\ell_1(\mathbb{C}_2)} < \infty$ and hence, $\hat{A}\xi \in \ell_{\infty}(\mathbb{C}_2)$. Also, it is clear that $\hat{A}$ is a $\mathbb{C}_2$ -linear transformation.

On the other hand, $\|\hat{A}\xi\|_{\ell_{\infty}(\mathbb{C}_2)} \leq \sqrt{2} M \|\xi\|_{\ell_1(\mathbb{C}_2)}$, for all $\xi \in \ell_1(\mathbb{C}_2)$, this implies that $\hat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_{\infty}(\mathbb{C}_2)$. Hence, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_{\infty}(\mathbb{C}_2))$. This completes the proof. $\square$

For the converse part, consider the following example.

Example 2.3. Let $\hat{A} = [\hat{a}_{nk}]$ be a infinite matrix bi-complex matrix such that

$$\hat{A} = \begin{bmatrix} e_1 + e_2 & 0 & 0 & \dots & 0\dots \\ 0 & 2e_1 + 2e_2 & 0 & \dots & 0\dots \\ 0 & 0 & 3e_1 + 3e_2 & \dots & 0\dots \\ \vdots & \vdots & \vdots & \dots & \vdots\dots \\ 0 & 0 & 0 & \dots & ne_1 + ne_2\dots \\ \vdots & \vdots & \vdots & \dots & \vdots\dots \end{bmatrix}.$$

Here,

$$\begin{aligned} \|a'_{nn}\|_{\mathbb{C}_2} &= \|ne_1 + ne_2\|_{\mathbb{C}_2} \\ &= \frac{1}{\sqrt{2}} \sqrt{n^2 + n^2} \\ &= \frac{1}{\sqrt{2}} \sqrt{2n^2} \\ &= n. \end{aligned}$$

Then,

$$\sup_{n,k} \|\hat{a}_{nk}\|_{\mathbb{C}_2} = \infty.$$

But we can show that, $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

For any sequence $\xi = (\xi_n) = (\xi_{n,1}e_1 + \xi_{n,2}e_2) \in \ell_1(\mathbb{C}_2)$, we have

$$\|\xi\|_{\ell_1(\mathbb{C}_2)} = \sum_{n=1}^{\infty} \|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} < \infty.$$

Therefore,

$$\lim_{n \rightarrow \infty} \|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} = 0.$$

This implies that, for some real number $c > 0$ and for $n \in \mathbb{N}$, we have

$$\|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} \leq \frac{c}{n^2}.$$

Thus, $\hat{A}\xi = \left(\sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right) = (a'_{nn}\xi_n)$. To show $\hat{A}\xi = (a'_{nn}\xi_n) \in \ell_\infty(\mathbb{C}_2)$, we have to show that

$$\sup_n \|a'_{nn}\xi_n\|_{\mathbb{C}_2} < \infty.$$

Now,

$$\begin{aligned} \|a'_{nn}\xi_n\|_{\mathbb{C}_2} &= \|(ne_1 + ne_2) \cdot (\xi_{n,1}e_1 + \xi_{n,2}e_2)\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \|ne_1 + ne_2\|_{\mathbb{C}_2} \|\xi_{n,1}e_1 + \xi_{n,2}e_2\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \cdot \frac{1}{\sqrt{2}} \sqrt{n^2 + n^2} \cdot \frac{c}{n^2} \\ &= \frac{\sqrt{2}c}{n}. \end{aligned}$$

Therefore, $\hat{A}\xi \in \ell_\infty(\mathbb{C}_2)$. It is also easy to show that $\hat{A}$ is $\mathbb{C}_2$ -linear and bounded transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$. This implies that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

Theorem 2.6 gives only the sufficient condition for an infinite bi-complex matrix $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

In the following theorem, we establish the necessary and sufficient conditions for an infinite bi-complex matrix $\hat{A}$ such that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

Theorem 2.7. *Let $\hat{A} = [\hat{a}_{nk}]$ be an infinite bi-complex matrix. Then the necessary and sufficient condition for the matrix $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$, is $\sup_n \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty$.*

Proof. Necessity: Let $\hat{A} = [a_{nk}]$ be an infinite bi-complex matrix such that $\hat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$.

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$, then $M_1 = \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty$.

Thus, $\hat{A}\xi = \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \in \ell_\infty(\mathbb{C}_2)$. i.e., $\|\hat{A}\xi\|_{\ell_\infty(\mathbb{C}_2)} = \sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} < \infty$.

Now, for any $n \in \mathbb{N}$,

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} &\leq \sum_{k=1}^{\infty} \|\hat{a}_{nk}\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \sum_{k=1}^{\infty} M_1 \cdot \|\hat{a}_{nk}\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \cdot M_1 \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

This implies that, $\sup_n \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty$.

Sufficiency: Let $\hat{A} = [a_{nk}]$ be an infinite bi-complex matrix satisfying the condition

$$M_2 = \sup_n \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} < \infty.$$

Let $\xi = (\xi_k)$ be any sequence in $\ell_1(\mathbb{C}_2)$ such that $\sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty$.

To show $\hat{A}\xi = \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \in \ell_\infty(\mathbb{C}_2)$, we have to show that

$$\|\hat{A}\xi\|_{\ell_\infty(\mathbb{C}_2)} = \sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} < \infty.$$

Now, for any $n \in \mathbb{N}$

$$\begin{aligned} \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} &\leq \sqrt{2} \sum_{k=1}^{\infty} \|\hat{a}_{nk}\|_{\mathbb{C}_2} \|\xi_k\|_{\mathbb{C}_2} \\ &\leq \sqrt{2} \cdot M_2 \sum_{k=1}^{\infty} \|\xi_k\|_{\mathbb{C}_2} < \infty. \end{aligned}$$

This implies that, $\sup_n \left\| \sum_{k=1}^{\infty} \hat{a}_{nk}\xi_k \right\|_{\mathbb{C}_2} \leq \sqrt{2} \cdot M_2 \|\xi_k\|_{\ell_1(\mathbb{C}_2)} < \infty$. i.e., $\hat{A}\xi \in \ell_\infty(\mathbb{C}_2)$. So, $\hat{A}$ transform every sequence of $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$. Also, it is easy to show that $\hat{A}$ is a $\mathbb{C}_2$ -linear

transformation.

On the other hand, since $\|\widehat{A}\xi\|_{\ell_\infty(\mathbb{C}_2)} \leq \sqrt{2} \cdot M_2 \|\xi\|_{\ell_1(\mathbb{C}_2)}$, for all $\xi \in \ell_1(\mathbb{C}_2)$.

Therefore, $\widehat{A}$ is a $\mathbb{C}_2$ -bounded linear transformation from $\ell_1(\mathbb{C}_2)$ into $\ell_\infty(\mathbb{C}_2)$. Hence, $\widehat{A} \in B(\ell_1(\mathbb{C}_2), \ell_\infty(\mathbb{C}_2))$. This completes the proof. $\square$

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A NOTE ON CONNECTED POWER DOMINATION OF CERTAIN CLASSES OF GRAPHS

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Abstract. The efficiency of monitoring the electrical power systems can be optimised with the use of Phasor Measurement Unit (PMU). A complete monitoring of the network can be effectively implemented through the power domination problem, downsizing the count of PMUs. In this paper, we present a detailed study on the Connected Power Domination (CPD) of a graph, a variant of power domination with an additional constraint of being connected. Tight upper bounds on the Connected Power Domination Number of a graph G , denoted $\gamma_{p,c}(G)$ are established and also the nature of the CPD set of unicyclic graphs is explored. Further, this study gives the exact $\gamma_{p,c}$ -set for certain classes of graphs including, Grid graph, Join of graphs, Kragujevac tree and Honeycomb network.

Keywords: Connected Power Domination, Connected component, Unicyclic graph, Pendant path, Grid graph, Join of graphs, Kragujevac tree, Honeycomb network

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1. INTRODUCTION

The need for real-time monitoring and having control over their vast network led the electrical power companies to opt for the use of PMU placements in their networks. These units give a greater understanding of the network performance, which includes the voltage measure and phase angle measure. Taking into account the excessive maintenance cost of the PMUs, the aim is to downsize the quantity of PMUs used. The optimal PMU placement (OPP) problem first appeared in the literature in the early 90's and was widely explored in [1, 5, 6, 15, 16]. This problem was then parametrized into a graph theoretical problem by Haynes in 2002 [13], by associating the vertices of the graph with electrical nodes and the transmission cables with edges. Here, the aim is to find the minimum number of PMU locations to monitor the entire network.

The above discussed problem was formally defined as follows: Let G be the graph model of the given electrical network and let S denote the PMU locations. In line with Ohm's law and Kirchhoff's current law, the system operates with simple monitoring rules as outlined below.

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- (1) The set S is monitored along with its neighbors and incident edges (Ohm's law)
- (2) If a vertex has exactly one unmonitored neighbor, then that neighbor is also monitored (Kirchhoff's current law)

The OPP problem is closely associated with two widely recognised problems, the domination problem and the vertex colouring problem.

Furthermore, in order to improve the smooth operation, response time and efficiency of the system, the electrical power companies might consider including the constraint of positioning the PMU units in a confined and interconnected environment. This added constraint allows the benefit of reduction in the number of processing hubs and data collection infrastructures required. This inspired the problem of Connected Power Domination (CPD), a variation of power domination, where the minimal Power Domination Set (PDS) is required to induce a connected subgraph. Moreover, the CPD of a graph cannot be computed in polynomial time as it is NP-complete to find the PDS of a general graph [4]. Nevertheless, it is polynomial-time solvable for restricted classes such as chain graphs and threshold graphs [10]. Furthermore, the CPD problem has been studied for certain product graphs [8].

2. PRELIMINARIES

The study includes only connected, finite, undirected and simple graphs. For a simple connected graph $G = (V, E)$ with $|V(G)| = n$, the set V comprises of all the vertices and the set E includes the edges of the graph. The adjacency of the vertices (neighbors) is justified by the existence of the edge between them and is denoted by $v \sim w$. Given any arbitrary vertex u , the *open neighborhood* of u , $N(u)$ consists all the adjacent vertices of u , whereas u along with $N(u)$ makes up its *closed neighborhood* $N[u]$. The open neighborhood of the set S is the collection $\{N(w)|w \in S\}$ and $\{N[w]|w \in S\}$ is its closed neighborhood. The *degree* of u , denoted by $d(u)$, refers to the cardinality of $N(u)$, with degree one vertex being referred to as *pendant (leaf)* and the vertex with $|N(u)| = n - 1$ as *universal vertex*. The *maximum degree* $\Delta(G)$ of a graph is obtained by $\max\{d(u)|u \in V(G)\}$ and the *minimum degree* $\delta(G)$ by $\min\{d(u)|u \in V(G)\}$. The distance between two vertices $u, v \in V(G)$, denoted $d(u, v)$, is the length of the shortest $u - v$ path in G . The diameter of G is given by $\text{diam}(G) = \max\{d(u, v)|u, v \in V(G)\}$. The isomorphism between two graphs G_i and G_j is symbolised by $G_i \cong G_j$. $\omega(G)$ denotes the number of connected components of G . Considering any subset $S \subset V(G)$, $G[S]$ represents the *S-induced subgraph* of G , constructed with the vertex set S and the edges that have endpoints in S . A graph having precisely one cycle is termed as an *unicyclic graph*. A spider is a tree in which only one vertex, called the branch vertex is of degree greater than 2. Let $K_n, K_{m,n}, W_{1,n}, S_{1,n}$ and $B_{m,n}$ denote the complete graph, complete bipartite graph, wheel graph, star graph and bistar graph, respectively. Helm graph H_n , is obtained by adjoining a pendant vertex to all the vertices in $W_{1,n}$ except the central vertex. Kayak paddle $KP(k, m, l)$ is obtained by adjoining two cycles C_k and C_m by a path of length l [7].

Given any subset $D \subset V(G)$, D is the dominating set of G provided that $N[D]$ is equivalent to $V(G)$. The domination number of G , denoted by $\gamma(G)$, is the cardinality of the smallest dominating set of G [12]. In addition, if $G[D]$ is connected, then D is called the connected dominating set (CDS), where $\gamma_c(G)$ denotes the cardinality of the minimal CDS of G [17].

In 2002, Haynes et al. [13] formulated a new variation of domination, called the power domination. The set S is called the power dominating set of G if all the vertices are monitored by S , adhering to the rules provided below.

- (1) *Domination*: Initially $N[S]$ is monitored
- (2) *Propagation*: For any monitored vertex u , if $N[u] - w$ is monitored, with $w \in N(u)$ then w is also monitored

The power domination number symbolised by $\gamma_p(G)$ is the minimal cardinality of all such PDS. In this article, we discuss in detail a variant of power domination called the Connected Power Domination (CPD).

Definition 2.1. [4] For a connected graph $G = (V, E)$, a set $S_c \subset V(G)$ is a connected power dominating set (CPDS) of G if S_c is a PDS and $G[S_c]$ is connected. $\gamma_{p,c}(G)$ denotes the connected power domination number of G , which is the order of the smallest CPDS by cardinality.

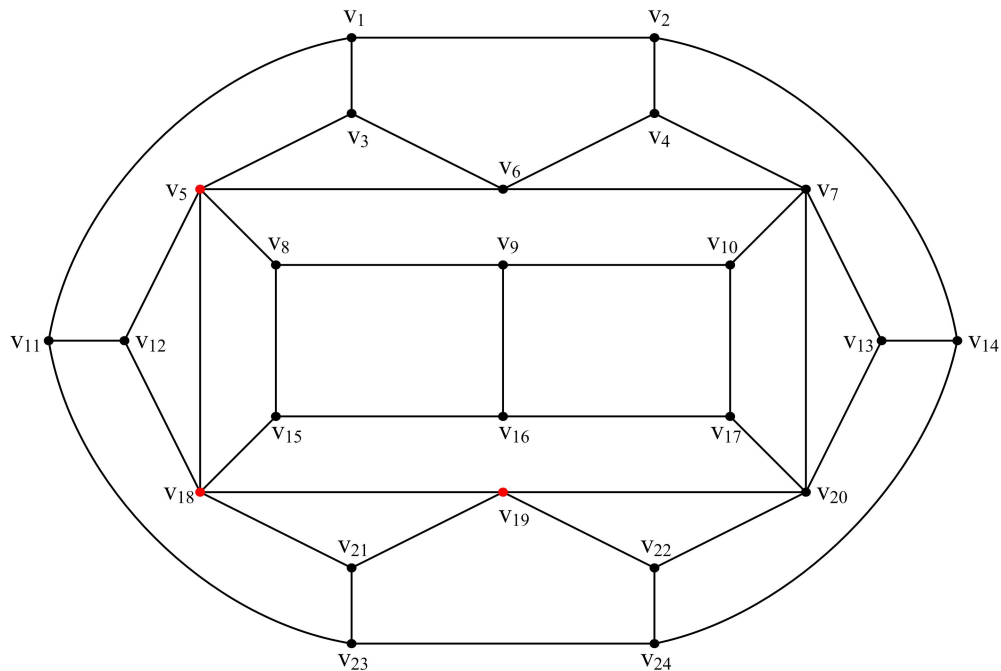


FIGURE 1. A graph G with connected power dominating set

Figure 1 gives a clear demonstration of the CPDS of the graph G . Here, the entire graph can be power dominated by the set $S = \{v_5, v_{19}\}$. However, to incorporate the constraint of connectedness, it is necessary to extend the set S by including the vertex v_{18} , resulting in the minimum CPDS being $S_c = \{v_5, v_{18}, v_{19}\}$. We further observe that, the minimum dominating set is $D = \{v_2, v_5, v_9, v_{18}, v_{20}, v_{23}\}$ and the minimum CDS is $D_c = \{v_2, v_4, v_5, v_6, v_8, v_9, v_{18}, v_{19}, v_{20}, v_{21}, v_{23}\}$. Thus, $\gamma_p(G) = 2$, $\gamma_{p,c}(G) = 3$, $\gamma(G) = 6$ and $\gamma_c(G) = 11$.

For any connected graph $G \not\cong P_n$, consider a vertex $u \in V(G)$ of degree greater than 2. A *pendant path incident to a vertex u* , is a maximal set P of $V(G)$ such that $G[P]$ is a connected

component of $G - u$, which is a path. One of the end vertices of P is adjacent to u . The vertex in $P \cap N(u)$ is the *base* of P . Let $p(u)$ denote the number of pendant paths incident to u [3]. Further, we define

$$\mathcal{P}_1(G) = \{u \in V(G) \mid p(u) = 1 \text{ and } \omega(G - u) = 2\}$$

$$\mathcal{P}_2(G) = \{u \in V(G) \mid p(u) = 0 \text{ and } \omega(G - u) = 2\}$$

$$\mathcal{P}_3(G) = \{u \in V(G) \mid \omega(G - u) > 2\}$$

$$\mathcal{P}_4(G) = \{u \in V(G) \mid d(u) = 1\}$$

$$\mathcal{M}(G) = \mathcal{P}_2(G) \cup \mathcal{P}_3(G)$$

3. BOUNDS ON THE CONNECTED POWER DOMINATION NUMBER OF A GRAPH

In this section, we establish bounds on the connected power domination number for general graphs. Furthermore, we study its behaviour for certain unicyclic graphs.

Proposition 3.1. [17] *For any connected graph G , $\gamma(G) \leq \gamma_c(G)$.*

Proposition 3.2. [13] *For any graph G , $1 \leq \gamma_p(G) \leq \gamma(G)$.*

Lemma 3.1. [4] *Let $G = (V, E)$ be a connected graph different from a path and R be an arbitrary connected power dominating set of G . Then $\mathcal{M}(G) \subset R$.*

Lemma 3.2. [4] *Let G be a graph different from a path. Then, no minimum connected power dominating set of G contains a leaf of G .*

Theorem 3.1. [4] *Let T be a tree. Then, $\gamma_{p,c}(T) = \max\{1, |\mathcal{M}(T)|\}$.*

Theorem 3.2. *For any graph G , $\gamma_p(G) \leq \gamma_{p,c}(G) \leq \gamma_c(G)$.*

Proof. Let G be a graph with $n \geq 2$ vertices. Let D, D_c, S and S_c denote the minimal of dominating set, connected dominating set, power dominating set and connected power dominating set of G respectively. By the definition of connected power domination, we have $S_c = S \cup X$, where $X \subset V(G)$ such that $G[S \cup X]$ is connected. If $X = \emptyset$, then $S_c = S$ and if $X \neq \emptyset$, then $|S_c| > |S|$. Thus, $\gamma_p(G) \leq \gamma_{p,c}(G)$. Note that, S and X are mutually disjoint sets, thus $|S_c| = |S| + |X|$. Similarly, $D_c = D \cup Y$, where $Y \subset V(G)$ such that $G[D \cup Y]$ is connected. Thus, $|D_c| = |D| + |Y|$. From Proposition 3.2, we know that $|S| \leq |D|$. That is,

$$|S_c| - |X| \leq |D_c| - |Y| \quad (3.1)$$

To claim that, $|S_c| \leq |D_c|$, we consider the following cases.

Case 1: If $X = Y = \emptyset$, then eqn. 3.1 implies $|S_c| \leq |D_c|$.

Case 2: Let $X = \emptyset$ and $Y \neq \emptyset$. From eqn. 3.1, we have $|S_c| \leq |D_c| - |Y|$. Note that $|Y| > 0$, thus $|S_c| < |D_c|$.

Case 3: When $X \neq \emptyset$ and $Y = \emptyset$, we have $|S_c| - |X| \leq |D_c|$. Suppose, $|S_c| > |D_c|$. Then, we have

$$\begin{aligned} |S_c| - |X| &> |D_c| - |X| \\ \implies |S| &> |D_c| - |X| \end{aligned}$$

a contradiction, since $|X| > 0$ and $|S| \leq |D| \leq |D_c|$. Therefore, $|S_c| \leq |D_c|$.

Case 4: Let $X \neq \emptyset$ and $Y \neq \emptyset$. Suppose, $|X| \leq |Y|$, then,

$$\begin{aligned} |S_c| - |X| &\leq |D_c| - |Y| \\ \implies |S| + |Y| &\leq |D_c| \\ \implies |S| + |X| + k &\leq |D_c| \quad (\text{since } |Y| = |X| + k, \text{ for some } k) \\ \text{Therefore, } |S_c| &\leq |D_c| \end{aligned}$$

Suppose, $|X| > |Y|$, then it is easy to claim that, $|S_c| \leq |D_c|$. Hence, $\gamma_{p,c}(G) \leq \gamma_c(G)$. $\square$

Theorem 3.3. *Let G be a graph of order $n \geq 3$ with no universal vertex. Then $\gamma_{p,c}(G) \leq n - \Delta(G) - 1$.*

Proof. Let t be a vertex of maximum degree $\Delta(G)$ in G . Construct a spanning tree T of G such that every neighbor of t in G is also a neighbor of t in T . Let $N(t)$ denote the neighbors of t in T .

Case 1 : Suppose all the vertices in the set $V(T) - \{t\}$ is of degree one or two. Define $V(T) - \{t\} = X \cup Y$, where $X = \{v \in V(T) - \{t\} \mid d(v) = 1\}$ and $Y = \{v \in V(T) - \{t\} \mid d(v) = 2\}$. If $Y = \emptyset$, then $\Delta(G) = n - 1$, a contradiction. Therefore, Y is a non-empty subset of $V(T) - \{t\}$. In order to construct a connected power domination set, consider the vertex $t \in T$. Note that this vertex dominates all the vertices of $N(t)$ and the remaining vertices of $V(T) - N[t]$ are power dominated by t , as they all belong to a pendant path with base as one of the vertices of $N(t)$. Thus, $\gamma_{p,c}(G) = 1 \leq n - \Delta(G) - 1$.

Case 2 : Suppose $V(T) - \{t\}$ has at least one vertex of degree greater than 2. Define $V(T) - \{t\} = X \cup Y \cup Z$, where $X = \{v \in V(T) - \{t\} \mid d(v) = 1\}$, $Y = \{v \in V(T) - \{t\} \mid d(v) = 2\}$ and $Z = \{v \in V(T) - \{t\} \mid d(v) > 2\}$ with $|Z| \geq 1$. Now, T has at least $\Delta(G) + |Z|$ leaves. Thus, to monitor all the vertices, we need at most $n - (\Delta(G) + |Z|)$ vertices. Since $|Z| \geq 1$, we have $\gamma_{p,c}(G) \leq n - (\Delta(G) + 1)$. $\square$

Theorem 3.4. *For any tree $T \not\cong P_n$, $\gamma_{p,c}(T) = 1$ if and only if T is a spider.*

Proof. If T is a spider and let v be the branch vertex of T , then clearly $\{v\}$ is a connected power dominating set of T . Conversely, if T is not a spider, then there is a vertex w , such that $v \neq w$ with degree greater than 2. Thus, $\gamma_{p,c}(T) \geq 2$. $\square$

Theorem 3.5. *For any connected graph G of order $n \leq 5$, $\gamma_p(G) = \gamma_{p,c}(G)$.*

Proof. Let G be a connected graph with $n \leq 5$ vertices. Then, $\Delta(G) \leq 4$. Suppose the graph G is of order $n \leq 4$, then G is either P_n or C_n or contains $S_{1,n-1}$. Thus, $\gamma_{p,c}(G) = 1$. When $n = 5$, G either contains $S_{1,4}$ or $\Delta(G) \leq 3 = n - 2$. In either case $\gamma_{p,c}(G) = 1$, with the latter condition being justified by Theorem 3.3. $\square$

Theorem 3.6. *Let T be a tree of order $n \geq 2$ with diameter at most 3. Then, $\gamma_p(T) = \gamma_{p,c}(T)$.*

Proof. Let T be a tree of diameter at most 3. Clearly, T has no more than 2 vertices of degree greater than 2. If there is only one such vertex, then T is a spider, thus by Theorem 3.4,

$\gamma_p(G) = \gamma_{p,c}(G)$. In the case where there are 2 vertices, u and v of degree greater than 2, $S = \{u, v\}$ is a power dominating set of T , since every other vertex in T is power dominated by the vertices in S . Suppose u and v are not adjacent, there is a uv -path P_n ($n \geq 3$) in T , which implies that the neighbor of u and the neighbor of v not in the uv -path are at a distance exceeding 3, a contradiction to the fact that $\text{diam}(T) \leq 3$. Thus, $\gamma_p(T) = \gamma_{p,c}(T)$. $\square$

Theorem 3.7. *Let T be a tree and S_c be the minimum connected power dominating set of T . Then, every vertex $v \in V - S_c$ is either a pendant vertex or a vertex of degree two with $p(v) = 1$.*

Proof. For a path, the result clearly holds. Hence, let us assume that $T \not\cong P_n$ and S_c be the connected power dominating set of T . The vertices in T can be partitioned into four disjoint sets as $\mathcal{P}_1(T)$, $\mathcal{P}_2(T)$, $\mathcal{P}_3(T)$ and $\mathcal{P}_4(T)$. From Theorem 3.1, it is clear that $S_c = \mathcal{M}(T)$. Thus, the vertices in $V - S_c$ are the pendant vertices and the degree 2 vertices which belong to a pendant path. $\square$

In the following results we consider unicyclic graphs G with cycle C such that $V(C) = \{v \in V(G) \mid d(v) \geq 2 \text{ and } p(v) = 0\}$.

Theorem 3.8. *Let G be an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$. If $\langle S \rangle$ is connected, then $\gamma_{p,c}(G) = |\mathcal{M}(G)|$.*

Proof. Let G be an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$. Then, the only connected components of $G - C$ are trees different from a path. We claim that $\mathcal{M}(G)$ is the minimum connected power dominating set of G . The vertices in G can be partitioned into two distinct subsets as $V(G) = X \cup Y$, where $X = \{v \in V(G) \mid \omega(G - v) \geq 2\}$ and $Y = \{v \in V(G) \mid \omega(G - v) = 1\}$. Note that $X = \mathcal{P}_1(G) \cup \mathcal{P}_2(G) \cup \mathcal{P}_3(G)$. Clearly, $Y = V(C - S) \cup \mathcal{P}_4(G)$. Now, deleting the set $\mathcal{P}_1(G) \cup \mathcal{P}_4(G) \cup V(C - S)$ does not disconnect the graph G , as S is connected. Thus, $\mathcal{M}(G)$ is a connected set. Also, $S \subset \mathcal{M}(G)$ initially dominates all its neighbors including the vertices in $N(S) \cap C$. These vertices, being in $V(C - S)$ and having degree exactly two, initiate the propagation phase, which in turn monitors all the vertices in $V(C - S)$. Further, let $u \in \mathcal{P}_3(G) \subset \mathcal{M}(G)$ be a vertex to which a pendant path is attached. Note that u dominates initially all its neighbors which includes the base of each pendant paths and power dominates all the vertices in the path through the propagation step. Also, the vertices in $\mathcal{P}_4(G)$ are either adjacent to $\mathcal{P}_3(G)$ or belong to a pendant path. In both cases, they are dominated and hence power dominated by $\mathcal{M}(G)$. Thus, $\mathcal{M}(G)$ is a connected power dominating set of G . Therefore, $\gamma_{p,c}(G) \leq |\mathcal{M}(G)|$. By Lemma 3.1, we have $|\mathcal{M}(G)| \leq \gamma_{p,c}(G)$ and hence $\gamma_{p,c}(G) = |\mathcal{M}(G)|$. $\square$

Theorem 3.9. *Let G be an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$. If $\{T_i\}$ is the family of connected components of $G - C$, with $\langle S \rangle$ being connected, then $\gamma_{p,c}(G) = \sum_i \gamma_{p,c}(T_i) + |S| + |\mathcal{F}(G)|$, where $\mathcal{F}(G) = \{v \in V(G) \mid d(v) = 2, p(v) = 0 \text{ in } G \text{ and } p(v) = 1 \text{ in } T_i\}$.*

Proof. If G is an unicyclic graph with cycle C and $S = \{v \in V(C) \mid d(v) \geq 3 \text{ and } p(v) = 0\}$, then each connected component T_i of $G - C$ is a tree different from path. From Theorem 3.1, we have $\mathcal{M}(T_i)$ to be the minimum connected power dominating set of T_i . Now if there

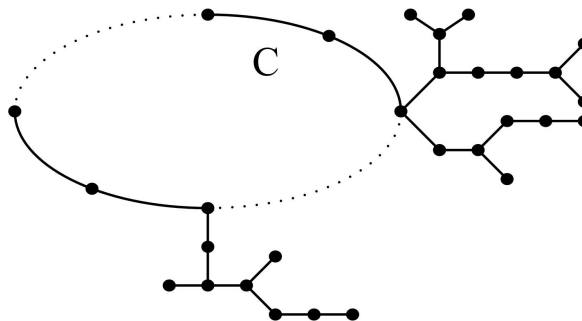


FIGURE 2. An unicyclic graph with cycle C

is a vertex in each $\mathcal{M}(T_i)$ that is adjacent to any vertex in S , then $(\bigcup_i \mathcal{M}(T_i)) \cup S$ is a connected power dominating set of G . Suppose, there is a connected component T_j such that, there is no vertex in $\mathcal{M}(T_j)$ adjacent to any vertex in S , then $(\bigcup_i \mathcal{M}(T_i)) \cup S$ is not connected. That is, there is a vertex u of degree two in T_j such that it belongs to a pendant path in T_j but not in G . Thus, $(\bigcup_i \mathcal{M}(T_i)) \cup S \cup \mathcal{F}(G)$ where $\mathcal{F}(G) = \{v \in V(G) | d(v) = 2, p(v) = 0 \text{ in } G \text{ and } p(v) = 1 \text{ in } T_i\}$ is a connected power dominating set of G . Clearly, $\bigcup_i \mathcal{M}(T_i)$, S and $\mathcal{F}(G)$ are mutually disjoint. Therefore we have, $|\mathcal{M}(G)| \leq \sum_i |\mathcal{M}(T_i)| + |S| + |\mathcal{F}(G)|$. Furthermore, for every i we have $\mathcal{M}(T_i) \subset \mathcal{M}(G)$. By definition, $S \subset \mathcal{M}(G)$ and $\mathcal{F}(G) \subset \mathcal{P}_2(G) \subset \mathcal{M}(G)$. Thus, $\sum_i |\mathcal{M}(T_i)| + |S| + |\mathcal{F}(G)| \leq |\mathcal{M}(G)|$. Therefore, $\gamma_{p,c}(G) = \sum_i \gamma_{p,c}(T_i) + |S| + |\mathcal{F}(G)|$. $\square$

Theorem 3.10. Let G be an unicyclic graph of order n with cycle C , $\mathcal{P}$ be the set of all vertices belonging to the pendant paths of G and $S = \{v \in V(C) | d(v) \geq 3 \text{ and } p(v) = 0\}$. If $|S| > 1$ is not connected, then $\gamma_{p,c}(G) = n - |\mathcal{P}| - \max\{d(u_i, u_{i+1}) - 1 | u_i \in S, i \in \{1, \dots, |S| - 1\}\}$.

Proof. Let G be an unicyclic graph of order n with cycle C and $\mathcal{P}$ be the set of all vertices belonging to the pendant paths of G . Since $S = \{v \in V(C) | d(v) \geq 3 \text{ and } p(v) = 0\}$, the only connected component of $G - C$ are trees not isomorphic to P_n . Therefore, the vertex set of G can be expressed as

$$V(G) = V(C - S) \cup \mathcal{M}(G) \cup \mathcal{P} \quad (3.2)$$

Note that, the sets $V(C - S)$, $\mathcal{M}(G)$ and $\mathcal{P}$ are mutually disjoint. Given that, S is not connected, there is a vertex, say u in S , such that the shortest path between u and $S - \{u\}$ has at least one vertex of the set $V(C - S)$. Now, label the vertices in S as $u_1, u_2, \dots, u_{|S|}$ in a clockwise manner. Then, $d(u_i, u_{i+1}) - 1$ denotes the number of vertices between u_i and u_{i+1} in C . Let Q denote the set of all vertices belonging to the largest path in $C - S$. We claim that $V(G) - (Q \cup \mathcal{P})$ is the minimum connected power dominating set for G . Note that, removing Q from G does not disconnect G , since Q is contained in C . Also, the vertices in Q having degree two are power dominated either by u_i or u_{i+1} . Similarly, the vertices in $V(C - Q)$ are power dominated by S . Further, removing the vertices in the pendant path does not disconnect the graph. Moreover, they are monitored through the propagation step initiated by the vertex in $\mathcal{P}_3(G)$, which is a neighbor of the base of the pendant path. Now, it is clear that the $\mathcal{M}(G)$ is enough to power dominate the entire graph G . Since $S \subset \mathcal{M}(G)$ and S is not connected, $\mathcal{M}(G)$ is also not connected. In order to

achieve the connectivity, we must include the vertices in the set $V(C - (S \cup Q))$ to $\mathcal{M}(G)$. From eqn. 3.2 and $Q \subset V(C - S)$, we have $\mathcal{M}(G) \cup V(C - (S \cup Q)) = V(G) - (Q \cup \mathcal{P})$. Thus, $V(G) - (Q \cup \mathcal{P})$ is a minimum connected power dominating set of G . Consequently, $\gamma_{p,c}(G) = n - |\mathcal{P}| - \max\{d(u_i, u_{i+1}) - 1 | u_i \in S, i \in \{1, \dots, |S| - 1\}\}$. $\square$

4. CONNECTED POWER DOMINATION OF SOME CLASSES OF GRAPHS

In this section, we determine the connected power domination number for certain classes of graphs, including Grid graph, Join of graphs, Kragujevac tree and Honeycomb network.

Definition 4.1. [14] *The Grid graph $G_{m,n}$ is obtained by the cartesian product of the paths P_m and P_n .*

Theorem 4.1. *Let G be a Grid graph $G_{m,n}$, then*

$$\gamma_{p,c}(G_{m,n}) = \begin{cases} \lceil \frac{n}{3} \rceil, & n < 6, n \leq m \\ n - 4, & 6 \leq n \leq m \end{cases}$$

Proof. Let us denote the vertices of $G_{m,n}$ as $v_{i,j}$ where $i \in \{1, \dots, m\}, j \in \{1, \dots, n\}$. It is clear that the vertex set can be partitioned into layers, say, $L_i = \{v_{i,1}, v_{i,2}, \dots, v_{i,n}\}$. To construct the CPDS of $G_{m,n}$, we consider the following cases.

Case 1: Let $n \in \{2, 3\}$ and $m \geq 2$. We consider the set $\{v_{1,2}\}$ to be the CPDS of $G_{m,n}$. Initially, the vertex $v_{1,2}$ dominates the entire L_1 layer and the vertex $v_{2,2}$. Now, all the remaining vertices are dominated through the propagation phase initiated by the vertices in L_1 . Thus, $\gamma_{p,c}(G_{m,n}) = 1 = \lceil \frac{n}{3} \rceil$.

Case 2: Let $n \in \{4, 5\}$; $m \geq n$. We consider the $\gamma_{p,c}$ -set as $S_c = \{v_{2,2}, v_{2,3}\}$. Initially, the vertices in $\gamma_{p,c}$ -set dominates the vertices $\{v_{1,2}, v_{1,3}, v_{2,1}, v_{2,4}, v_{3,2}, v_{3,3}\}$. Now, $v_{1,1}$ and $v_{1,4}$ being the only unmonitored neighbors of $v_{1,2}$ and $v_{1,3}$ respectively, they are also monitored when $n = 4$. Whereas, when $n = 5$, we see that the vertex $v_{1,4}$ power dominates $v_{1,5}$, since $N(v_{1,4}) - \{v_{1,5}\} \in N(S_c)$. Thus, in both cases, the entire L_1 layer is monitored, which in turn power dominates the entire graph. Note that $\langle S_c \rangle \cong P_2$, and hence it is connected. Thus, $\gamma_{p,c}(G_{m,n}) = \lceil \frac{n}{3} \rceil$.

Case 3: When $n \geq 6$, $m \geq n$. Let us consider the set $S_c = \{v_{2,j} | 3 \leq j \leq n - 2\}$ to be the CPDS of $G_{m,n}$. Clearly, $\langle S_c \rangle$ is connected. The set S_c initially dominates $\{v_{2,2}, v_{2,n-1}, v_{1,j}, v_{3,j} | 3 \leq j \leq n - 2\}$. Now, the vertices $v_{1,2}$ and $v_{1,n-1}$ being the only unmonitored neighbors of $N[S_c] \cap L_1$, are monitored in the first step of the propagation phase, which then in turn dominates $v_{1,1}$ and $v_{1,n}$ respectively. As a result, the entire L_1 layer ends up being monitored. This ultimately leads to the entire graph getting monitored. Thus, $\gamma_{p,c}(G_{m,n}) = n - 4$. $\square$

Definition 4.2. [2] *The graph join $G \vee H$ of two graphs G and H is a graph on the vertex set $V(G) \cup V(H)$ with edge connecting every vertex of G to every vertex of H , along with the edges in G and H .*

Theorem 4.2. *Let G and H be any two connected graphs, then*

$$\gamma_{p,c}(G \vee H) = \begin{cases} 1, & \text{if } \min\{\gamma_{p,c}(G), \gamma_{p,c}(H)\} = 1 \\ 2, & \text{otherwise} \end{cases}$$

Proof. Let $G \vee H$ be the join of two graphs G with order $n \geq 2$ and graph H with order $m \geq 2$. Let the vertex set of G be $V(G) = \{x_i | 1 \leq i \leq n\}$ with $d(x_l) = \Delta(G)$ and the vertex set of H be $V(H) = \{y_j | 1 \leq j \leq m\}$ with $d(y_k) = \Delta(H)$. By the definition of $G \vee H$, we have $V(G \vee H) = \{x_i, y_j | 1 \leq i \leq n; 1 \leq j \leq m\}$. Clearly, $S_c = \{x_l, y_k\}$ is a connected power domination set of $G \vee H$, since x_l is adjacent to every vertex of H and y_k is adjacent to every vertex of G . Also, note that the subgraph induced by S_c is connected. Therefore, $\gamma_{p,c}(G \vee H) \leq 2$. For the case $\min\{\gamma_{p,c}(G), \gamma_{p,c}(H)\} = 1$, we claim that $\gamma_{p,c}(G \vee H) = 1$. Without loss of generality, let $\gamma_{p,c}(G) = 1$, then all the vertices in $G \vee H$ are dominated by x_l except the vertices $V(G) - N[x_l]$. These remaining vertices are then power dominated by $V(G) \cap N(x_l)$. Thus, in this case $\gamma_{p,c}(G \vee H) = 1$ $\square$

Definition 4.3. [11] Let P_3 be the 3-vertex tree, rooted at one of its terminal vertices. For $k = 2, 3, \dots$, construct the rooted tree B_k by identifying the roots of k copies of P_3 . The vertex obtained by identifying the roots of P_3 -trees is the root of B_k .

Definition 4.4. [11] Let $t \geq 2$ be an integer. Let $\beta_1, \beta_2, \dots, \beta_t$ be the rooted trees, i.e., $\beta_1, \beta_2, \dots, \beta_t \in \{B_2, B_3, \dots\}$. A Kragujevac tree $Kg_{t,k}$ is a tree possessing a central vertex of degree t , adjacent to the roots of $\beta_1, \beta_2, \dots, \beta_t$.

Theorem 4.3. Let G be a Kragujevac tree $Kg_{t,k}$, then $\gamma_{p,c}(Kg_{t,k}) = t + 1$.

Proof. Let $Kg_{t,k}$ be a Kragujevac tree with a central vertex v of degree $t \geq 2$ and branches B_k . Since each $B_k \cong$ spider, from Theorem 3.4 it is clear that the CPDS of B_k is the root vertex of B_k . Thus, $N(v)$ is enough to power dominate the whole tree. But in order to satisfy the additional constraint of connectedness, we construct the CPDS of $Kg_{t,k}$ to be $N[v]$. Hence, $\gamma_{p,c}(Kg_{t,k}) = |N[v]| = t + 1$ $\square$

Definition 4.5. [9] A n -dimensional Honeycomb network is represented by $HC(n)$, where the parameter n denotes the number of hexagons between the middle to the boundary hexagon.

Theorem 4.4. Let G be a Honeycomb network $HC(n)$, then

$$\gamma_{p,c}(HC(n)) = \begin{cases} 1, & n = 1 \\ 2(n - 1), & n > 1 \end{cases}$$

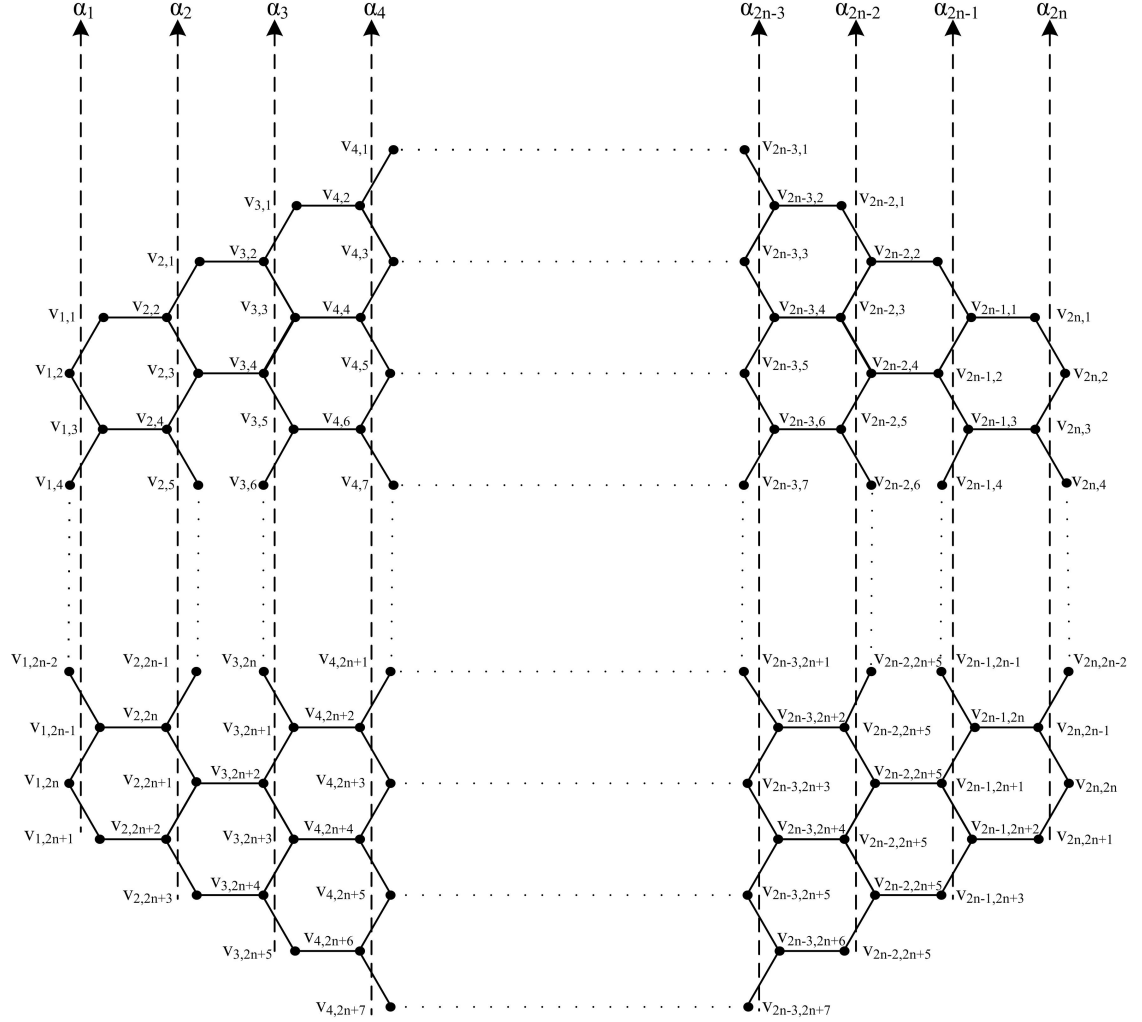
Proof. Let $\alpha_1, \alpha_2, \dots, \alpha_{2n}$ represent the α -lines (vertical lines) of the Honeycomb network $HC(n)$, given from left to right. The vertices on each α_i are named from top to bottom as

$$v_{i,j} \text{ where } 1 \leq i \leq 2n \text{ and } j = \begin{cases} 1, 2, \dots, 2(n + i) - 1; & \text{for } i = 1, 2, \dots, n \\ 1, 2, \dots, 2(3n - i) + 1; & \text{for } i = n + 1, n + 2, \dots, 2n \end{cases}$$

See Figure 3. To construct the CPDS of $HC(n)$, we consider the following cases:

Case 1: When $n = 1$, $HC(n) \cong C_6$. Thus, $\gamma_{p,c}(HC(n)) = 1$.

Case 2: For $n = 2$, it is clear that a single vertex is not enough to power dominate the entire graph $HC(2)$, since each vertex is either of degree 3 or a vertex of degree 2. Let the $\gamma_{p,c}$ -set for the case $n = 2$, be $S_c = \{v_{2,2}, v_{2,3}\}$. Initially, S_c dominates $\{v_{1,1}, v_{2,1}, v_{2,4}, v_{3,3}\}$. Then the vertices $v_{1,1}$ and $v_{2,1}$ being degree 2 vertices initiates the propagation phase, following the monitoring pattern: $v_{1,1} \rightarrow v_{1,2} \rightarrow v_{1,3}$ and $v_{2,1} \rightarrow v_{3,1} \rightarrow v_{3,2}$. Now, each of the degree 3

FIGURE 3. $HC(n)$ with α -lines

vertices in the set $\{v_{1,3}, v_{2,4}, v_{3,2}, v_{3,3}\}$ having only one unmonitored neighbor further proceeds with the propagation phase. Proceeding in a similar way, the entire graph is monitored by the set S_c . Thus, $\gamma_{p,c}(HC(n)) = 2 = 2(n - 1)$.

Case 3: When $n > 2$, the $\gamma_{p,c}$ -set is $S_c = \{v_{i,j} \mid 2 \leq i \leq n; j \in \{2(i-1), 2i-1\}\}$. This set, initially dominates the vertex set $\{v_{1,1}, v_{n+1,2n-1}, v_{i,j} \mid 2 \leq i \leq n; j \in \{2i-3, 2i\}\}$. Similarly, following the propagation phase, the whole graph is monitored. Thus, $\gamma_{p,c}(HC(n)) = 2(n - 1)$. $\square$

Since the proofs of the subsequent theorems are evident, it is left for the understanding of the readers.

Theorem 4.5. *If $G \in \{K_n, K_{m,n}, W_{1,n}, S_{1,n}, B_{m,n}\}$ be a connected graph with $m, n \geq 2$ vertices, then $\gamma(G) = \gamma_p(G) = \gamma_{p,c}(G)$.*

Theorem 4.6. *If $G \in \{P_n, C_n, H_n, P_2 \times P_n\}$ be a connected graph with $n \geq 2$ vertices, then $\gamma_{p,c}(G) < \gamma(G)$.*

In the following table, we present the power domination number and connected power domination number for certain graphs, demonstrating that the connected power domination number is always greater than or equal to the power domination number.

TABLE 4.1. List of Graphs with Power Domination and Connected Power Domination Number

Family of Graphs	$\gamma_p(G)$	$\gamma_{p,c}(G)$
Petersen graph	2	3
Complete Binary Tree	$2^{(n-1)}$	$2^n - 1$
Kayak paddle ($KP(k, m, l)$)	2	$l + 1$

5. CONCLUSION

In this paper, we conducted a comprehensive study of connected power domination in graphs and established bounds on the connected power domination number. Furthermore, the structure and behavior of connected power dominating sets in unicyclic graphs were studied. Moreover, exact values of the connected power domination number were obtained for specific graph classes such as Grid graph, Join of graphs, Kragujevac tree and Honeycomb network. The results obtained in this study provide a foundation for further exploration in this area.

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ON THE QUADRATIC VARIATION OF SOME SEQUENCES OF FRACTIONAL PARTS

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Abstract. Let a, b be real numbers such that $b > a \geq 0$, and let $c = b - a$. Let x be a real number and n a positive integer ($n \in \mathbb{N}^*$). The present study focuses on proving the following asymptotic formula

$$\sum_{n \geq 1} \left(\left\{ \frac{x}{n+b} \right\} - \left\{ \frac{x}{n+a} \right\} \right)^2 = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + O\left(c^{139/378} x^{61/126}\right),$$

with $c = b - a$, uniformly for $x \geq 100c^{-25/3}(1+b)^{29}$ generalization of the following result

$$\sum_{n \geq 1} \left(\left\{ \frac{x}{n} \right\} - \left\{ \frac{x}{n+1} \right\} \right)^2 = \frac{\zeta(3/2)}{\pi} \sqrt{x} + O\left(x^{3/7}\right) \quad (x > 0),$$

established in previous literature for the specific case $a = 0$ and $b = 1$.

Keywords: Arithmetic function, Integer part function, Fractional part function, Quadratic variation, Asymptotic behavior.

MSC (2020): 11N37, 11K38.

1. INTRODUCTION

The sums involving the integer part function were introduced by Dirichlet in the middle of the 19th century when he proved that

$$\sum_{n \leq x} \sum_{d|n} 1 = \sum_{n \leq x} [x/n] = x \log x + x(2\gamma - 1) + O(x^{1/2}), \quad x > 0,$$

where $[t]$ denotes the integer part of $t \in \mathbb{R}$, and γ is Euler's constant (see [1], [7]). For the reader's convenience, we note that the notation $f = O(g)$ (Landau notation) should be understood in the sense that there exists a constant $C > 0$ and a real number x_0 such that for all $x \geq x_0$, $|f(x)| \leq Cg(x)$ (see [9]). Later, this research topic became very popular among mathematicians, and there is a vast amount of literature focusing on estimating sums of the form

$$\sum_{n \leq x} f([x/n]),$$

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where f is an arithmetic function; see [4], [5], [6], [8], [11], [12], [13] and the references therein. These references are considered essential as they provide historical and language-specific insights into sums involving integer part functions. This diverse array of references forms a solid foundation for readers seeking a nuanced understanding of these kinds of sums. The significance of the results obtained in both [6] and especially in [5] sparked our curiosity about the possibility of other similar cases that might offer a broader extension of the established results

$$S_f(x; a) := \sum_{n \leq \frac{x}{1+a}} f\left(\left\lfloor \frac{x}{n} - a \right\rfloor\right), \quad (1.1)$$

where a is a positive real number and $x > 0$. This, in particular, shaped our goal of investigating and resolving all challenges that contribute to formulating an ideal approximate expression for sums of the form (1.1). In our study, f is an arithmetic function satisfying some assumptions to be made precise later on. We recall that some techniques used in the previously cited articles are primarily based on the well-known Ramanujan hypothesis:

$$(H) : |f(n)| \ll n^\varepsilon, \quad n \geq 1, \varepsilon > 0.$$

Here, the notation $f \ll g$ (Vinogradov notation) means that there exists a constant $C > 0$ such that $|f(n)| \leq Cg(n)$ for all sufficiently large n (see [9]). In this work, we opt for the use of another improved version of this hypothesis given by

$$(H') : |f(n)| \ll n(\log en)^{-\alpha}, \quad n \geq 1, \alpha \in \mathbb{R}.$$

Observe here that hypothesis (H') is more general than hypothesis (H) . Additionally, we denote by $\{t\} = t - \lfloor t \rfloor$ the fractional part of $t \in \mathbb{R}$. Before proceeding further in our study, we need to introduce the parameter $m \in [1, x/4]$ chosen according to certain requirements, and the integer $k = \lfloor x/n - a \rfloor$. Immediately, one has

$$x/n - (a + 1) < k \leq x/n - a \quad \text{and} \quad x/(k + a + 1) < n \leq x/(k + a).$$

This allows us to write the sum (1.1) as follows:

$$S_f(x; a) = \sum_{k \leq x-a} f(k) (\lfloor x/(k + a) \rfloor - \lfloor x/(k + a + 1) \rfloor) = S_f^\sharp(x; a, m) + S_f^*(x; a, m), \quad (1.2)$$

where

$$S_f^\sharp(x; a, m) := \sum_{k \leq m} f(k) \left(\left\lfloor \frac{x}{k + a} \right\rfloor - \left\lfloor \frac{x}{k + a + 1} \right\rfloor \right)$$

and

$$S_f^*(x; a, m) := \sum_{m < k \leq x-a} f(k) (\lfloor x/(k + a) \rfloor - \lfloor x/(k + a + 1) \rfloor).$$

Keeping in mind the hypothesis (H') , we easily deduce that

$$S_f^*(x; a, m) \ll \sum_{n \leq x/(m+a)} (x/n)(\log m)^{-\alpha} \ll x(\log m)^{-\alpha} \log(x/(m + a)). \quad (1.3)$$

We will now discuss the estimation of the sum $S_f^\sharp(x; a, m)$. Let $\{t\} = t - \lfloor t \rfloor$ denote the fractional part of t . We can then rewrite

$$S_f^\sharp(x; a, m) = M_f(x; a, m) + R_f(x; a, m), \quad (1.4)$$

where

$$M_f(x; a, m) := \sum_{k \leq m} \frac{xf(k)}{(k+a)(k+a+1)}$$

$$\text{and } R_f(x; a, m) := \sum_{k \leq m} f(k) \left(\left\{ \frac{x}{k+a+1} \right\} - \left\{ \frac{x}{k+a} \right\} \right).$$

As a direct consequence of the hypothesis (H') , we deduce that

$$M_f(x; a, m) = x \sum_{k \geq 1} \frac{f(k)}{(k+a)(k+a+1)} + O(x(\log m)^{-\alpha+1}). \quad (1.5)$$

At this level, the use of the Cauchy–Schwarz inequality leads to

$$R_f(x; a, m) \leq \left(\sum_{k \leq m} |f(k)|^2 \right)^{\frac{1}{2}} \left(\sum_{k \geq 1} \left(\left\{ \frac{x}{k+a+1} \right\} - \left\{ \frac{x}{k+a} \right\} \right)^2 \right)^{\frac{1}{2}}.$$

Observe that if the function f satisfies hypothesis (H') , then

$$R_f(x; a, m) \leq m^{3/2}(\log m)^{-\alpha} \left(\sum_{k \geq 1} \left(\left\{ \frac{x}{k+a+1} \right\} - \left\{ \frac{x}{k+a} \right\} \right)^2 \right)^{1/2}. \quad (1.6)$$

Our strategy now is to provide an explicit estimate for the quantity $R_f(x; a, m)$, referenced in (1.6). Achieving this is equivalent to deriving an asymptotic formula for the sum $Q(x; a, b)$, which is defined as follows

$$Q(x; a, b) := \sum_{n \geq 1} \left(\left\{ \frac{x}{n+b} \right\} - \left\{ \frac{x}{n+a} \right\} \right)^2, \quad b > a \geq 0. \quad (1.7)$$

Certainly, understanding the behavior of the function $Q(x; a, b)$ has been pivotal in deriving an asymptotic formula for the sum referenced in (1.4). This progress would not have been achievable without recognizing the critical role played by the sum in (1.5), which serves as a cornerstone for this development. A notable aspect of this work lies in the investigation of the intermediate sum (1.7). Indeed, in [3, p. 3] and [2], significant results were established for the specific case $Q(x; 0, 1)$, where it was demonstrated that

$$Q(x; 0, 1) = \frac{\zeta(3/2)}{\pi} \sqrt{x} + O(x^{3/7}), \quad x > 0.$$

Building on these findings, our focus shifts to estimating the more general sum $Q(x; a, b)$ as highlighted in (1.7). This sum plays a central role in our analysis, providing the necessary tools to describe the asymptotic behavior of the quantity in (1.1), particularly when considering the estimate given in (1.3). This work can be considered a generalization of M. Balazard’s approach [3]. Our main result is stated as follows

Theorem 1.1. *Let a and b be two real numbers such that $b > a \geq 0$ and $c = b - a$. For $x \geq 100c^{-25/3}(1+b)^{29}$, one has*

$$Q(x; a, b) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + O\left(c^{139/378} x^{61/126}\right).$$

2. PROOF OF THEOREM 1.1

2.1. About the sum $Q(x; a, b)$. In this transitional step, we provide an alternative and useful formulation of the sum $Q(x; a, b)$ defined previously by the formula (1.7). This plays an important role in our study and helps us justify our main results, namely Theorem 1. To simplify the notation, we will use $Q(x)$ to denote the sum $Q(x; a, b)$. Let us introduce the real parameter D (with $D \leq x$), a sufficiently large integer, such that the sum $Q(x)$ can be expressed as follows:

$$Q(x) = Q_0(x) + \sum_{0 < d \leq D} Q_d(x) + \sum_{d > D} Q_d(x),$$

where

$$Q_d(x) = \sum_{n \geq 1} \varrho_d(h, k) \left(\left\{ \frac{x}{n+b} \right\} - \left\{ \frac{x}{n+a} \right\} \right)^2.$$

Here, ϱ_d is the classical indicator function defined by

$$\varrho_d(h, k) = \begin{cases} 1 & \text{if } (h, k) \in E_d, \\ 0 & \text{otherwise.} \end{cases}$$

where

$$k = \left\lfloor \frac{x}{n+a} \right\rfloor, \quad h = \left\lfloor \frac{x}{n+b} \right\rfloor.$$

and

$$E_d = \{(h, k) \in \mathbb{N}^2 : 0 \leq h \leq k \leq \frac{x}{1+a}, k - h = d\}, \quad (2.8)$$

At this level, observe that

$$\max \left(\frac{x}{k+1} - a, \frac{x}{h+1} - b \right) < n \leq \min \left(\frac{x}{k} - a, \frac{x}{h} - b \right),$$

where $0 \leq h \leq k \leq x$ (with the convention $x/0 = \infty$). Furthermore, the introduction of the interval of values of n , denoted by $I_{(h,k)}$, defined as

$$I_{(h,k)} = \mathbb{N}^* \cap \left[\frac{x}{k+1} - a, \frac{x}{k} - a \right] \cap \left[\frac{x}{h+1} - b, \frac{x}{h} - b \right],$$

allows us to write $Q_d(x)$ as follows

$$Q_d(x) = \sum_{(h,k) \in E_d} \sum_{n \in I_{(h,k)}} \left(\frac{cx}{(n+a)(n+b)} - k + h \right)^2; \quad 0 \leq k - h < \frac{cx}{ab} + 1.$$

2.2. Study of the sum $Q_0(x)$. Our main result concerning the quantity $Q_0(x)$ reads as follows.

Proposition 2.1. *For $x \geq \frac{9b^2}{c}$, we have*

$$Q_0(x) = \frac{2}{15} \sqrt{cx} + \delta_0 + O \left(\frac{(1+b)^2}{c} \right),$$

where

$$\delta_0 = \frac{c^2}{x^2} \sum_{1 \leq k \leq K} k^4 \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right), \quad x > 0. \quad (2.9)$$

In order to give the proof of this results, we need to use the following auxiliary results

Lemma 2.1. For $x \geq a^2/c$, one has

$$Q_0(x) = c^2 x^2 F\left(\frac{x}{K+1} - a\right) + c^2 x^2 \sum_{1 \leq k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right)\right), \quad (2.10)$$

where

$$F(t) = \sum_{n>t} \frac{1}{(n+a)^2(n+b)^2}, \quad (2.11)$$

and

$$K = \left\lfloor \sqrt{\frac{x}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor.$$

Proof. Taking into account (2.8), we see that $E_0(x) = \{(k, k), k \in \mathbb{N}, 0 \leq k \leq \frac{x}{1+a}\}$. Therefore,

$$Q_0(x) = \sum_{0 \leq k \leq \frac{x}{1+a}} Q(k, k) = \sum_{0 \leq k \leq \frac{x}{1+a}} \sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k} - b} \frac{(cx)^2}{(n+a)^2(n+b)^2}.$$

Note that if $k(k+1) > \frac{x}{c}$, the sum

$$\sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k} - b} \frac{(cx)^2}{(n+a)^2(n+b)^2}$$

is not well defined. Therefore, let K be the largest integer k such that $k(k+1) \leq \frac{x}{c}$, i.e., $K = \left\lfloor \sqrt{\frac{x}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor$. From which, we observe that $K \leq \sqrt{\frac{x}{c}}$ when $x \geq \frac{a^2}{c}$. Now, for $t \geq 0$, consider the function

$$F(t) = \sum_{n>t} \frac{1}{(n+a)^2(n+b)^2},$$

with $F(\infty) = 0$ and $F(t) = F(0)$ if $t < 0$. Consequently, for $x \geq a$, we deduce that

$$Q_0(x) = c^2 x^2 F\left(\frac{x}{K+1} - a\right) + c^2 x^2 \sum_{1 \leq k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right)\right).$$

□

Lemma 2.2. Let $0 \leq a < b$ and $t > 0$. Then,

$$F(t) = \frac{1}{3t^3} - \frac{\{t\} - 2(a+b) - \frac{1}{2}}{t^4} + O\left(\frac{b^2}{t^5}\right), \quad (2.12)$$

Proof. It suffices to take into account the implication (2.11) and the fact that

$$\frac{1}{(n+a)^2(n+b)^2} = \frac{1}{n^4} - \frac{2(a+b)}{n^5} + O\left(\frac{b^2}{n^6}\right), \quad n \in \mathbb{N}^*.$$

□

Before giving the next technical result, we mention that the quantity $x/(K+1)$ is well defined. In fact, since $K+1 = \sqrt{\frac{x}{c}} + \theta_0$, where $|\theta_0| \leq 1$ and $\sqrt{\frac{x}{c}} \geq 3$, then $\frac{x}{K+1} - a = \frac{x}{\sqrt{\frac{x}{c}} + \theta_0} - a = \sqrt{cx} + O(a)$. More precisely, $\frac{x}{K+1} = \sqrt{cx}(1 + \theta_1)$, with $\theta_1 = O(\sqrt{\frac{c}{x}})$ and $\theta_1 \geq -\frac{5}{6}$. This is justified by the fact that

$$\frac{x}{K+1} - c \geq \frac{\sqrt{cx}}{6}, \quad (2.13)$$

when $x \geq \frac{9b^2}{c}$. Utilizing the result (2.12) we arrive at

Lemma 2.3. *For $x \geq \frac{9b^2}{c}$, we have*

$$F\left(\frac{x}{K+1} - a\right) = \frac{1}{3(cx)^{3/2}} + O\left(\frac{b^2}{(cx)^2}\right).$$

Proof. Taking into account that $F(t) = \frac{1}{3t^3} + O\left(\frac{b^2}{t^4}\right)$, it follows that

$$F\left(\frac{x}{K+1} - a\right) = \frac{1 + O\left(\sqrt{\frac{a}{cx}}\right)}{3(cx)^{3/2}} + O\left(\frac{b^2}{(cx)^2}\right).$$

□

We have also

Lemma 2.4. *Let $x \geq \frac{9b^2}{c}$. Then,*

$$\begin{aligned} \sum_{0 < k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) &= -\frac{1}{5c^{3/2}x^{3/2}} \\ &\quad + x^{-4} \sum_{0 < k \leq K} k^4 \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) \\ &\quad + O\left(\frac{(1+b)^2}{c^3x^2}\right). \end{aligned}$$

Proof. Thanks to the estimate (2.12) and by using the inequality (2.13), we conclude that $\frac{x}{k} - a \geq \frac{\sqrt{ax}}{6}$ and $\frac{x}{k} - b \geq \frac{\sqrt{bx}}{6}$. Consequently, we see that

$$\begin{aligned} \sum_{0 < k \leq K} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) &= \sum_{0 \leq k \leq K} \left(-\frac{ck^4}{x^4} + O\left(\frac{b^2k^5}{x^5}\right) \right) \\ &= -\sum_{0 \leq k \leq K} \frac{ck^4}{x^4} + O\left(\frac{b^2K^6}{x^5}\right), \end{aligned}$$

where

$$\sum_{N < k \leq M} k^4 = \frac{1}{5}(M^5 - N^5) + \frac{1}{2}(M^4 - N^4) + \frac{1}{3}(M^3 - N^3) - \frac{1}{30}(M - N).$$

A particular case when $N = 0$ and $M = K$ gives

$$\sum_{0 < k \leq K} k^4 = \frac{1}{5}K^5 + \frac{1}{2}K^4 + \frac{1}{3}K^3 - \frac{1}{30}K.$$

Now, using the fact that $K = \sqrt{\frac{x}{c}} + O(1)$, we obtain

$$\sum_{0 < k \leq K} k^4 = \frac{x^{5/2}}{5c^{5/2}} + O\left(\frac{x^2}{c^2}\right).$$

Therefore,

$$\sum_{0 < k \leq K} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) = -\frac{1}{5c^{3/2}x^{3/2}} + O\left(\frac{1}{cx^2}\right).$$

On the other hand, one has

$$\begin{aligned} & \sum_{0 < k \leq K} \left(\frac{1}{\left(\frac{x}{k} - a\right)^4} \left(\left\{ \frac{x}{k} \right\} - a - 2(a+b) - \frac{1}{2} \right) - \frac{1}{\left(\frac{x}{k} - b\right)^4} \left(\left\{ \frac{x}{k} \right\} - b - 2(a+b) - \frac{1}{2} \right) \right) \\ &= x^{-4} \sum_{0 < k \leq K} k^4 \left(\left\{ \frac{x}{k} \right\} - a - \left\{ \frac{x}{k} \right\} + b \right) + O\left(\frac{b^2}{c^3 x^2}\right). \end{aligned}$$

This allows us to say that

$$\sum_{0 < k \leq K} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \ll (1+b)^2 \sum_{0 < k \leq K} \frac{k^5}{x^5} \leq (1+b)^2 \frac{K^6}{x^5} \leq \frac{(1+b)^2}{c^3 x^2}.$$

□

2.3. Study of the sum $\sum_{d>D} Q_d(x)$.

Proposition 2.2. *Let D be a sufficiently large real number. Then,*

$$\sum_{d>D} Q_d(x) \leq \sqrt{\frac{cx}{D-1}} + 1. \quad (2.14)$$

Proof. If $d = \left\lfloor \frac{x}{n+a} \right\rfloor - \left\lfloor \frac{x}{n+b} \right\rfloor > D$, it follows that

$$\frac{cx}{n^2} > \frac{cx}{(n+a)(n+b)} = d + \left\{ \frac{x}{n+a} \right\} - \left\{ \frac{x}{n+b} \right\} > D-1,$$

which means that $n < \sqrt{\frac{cx}{D-1}}$. From this, we deduce that

$$\sum_{d>D} Q_d(x) < \sum_{0 \leq n < \sqrt{\frac{cx}{D-1}}} 1 < \sqrt{\frac{cx}{D-1}} + 1.$$

□

2.4. Study of the sum $\sum_{0 < d \leq D} Q_d(x)$. We start this section by the following useful remarks

Remark 2.1. *A direct computation justifies the fact that*

$$\frac{x}{k+1} - a \leq \frac{x}{h+1} - b \implies \frac{x}{k} - a \leq \frac{x}{h} - b$$

and

$$\frac{x}{k} - a > \frac{x}{h} - b \implies \frac{x}{k+1} - a > \frac{x}{h+1} - b.$$

As a consequence,

$$E_d = E_{d,1} \cup E_{d,2} \cup E_{d,3},$$

with

$$E_{d,1} = \left\{ (k-d, k) : d \leq k \leq \frac{x}{a}, (k-d+1)(k+1) \leq \frac{xd}{c} \right\}, \quad (2.15)$$

$$\frac{x}{k-d+1} - b < n \leq \frac{x}{k} - a,$$

$$E_{d,2} = \left\{ (k-d, k) : d \leq k \leq \frac{x}{a}, (k-d)k > \frac{xd}{c} \right\}, \quad (2.16)$$

$$\frac{x}{k+1} - a < n \leq \frac{x}{k-d} - b,$$

$$E_{d,3} = \{(k-d, k) : d \leq k \leq \frac{x}{a}, (k-d)k \leq \frac{xd}{c} < (k-d+1)(k+1)\}, \quad (2.17)$$

$$\frac{x}{k+1} - a < n \leq \frac{x}{k} - a.$$

As a result, we can rearrange Q_d as follows:

$$Q_d = Q_{d,1} + Q_{d,2} + Q_{d,3},$$

with

$$Q_{d,i} = \sum_{(h,k) \in E_{d,i}} \sum_{n \in I_{(h,k)}} \left(\frac{cx}{(n+a)(n+b)} - k + h \right)^2, \quad (1 \leq i \leq 3).$$

At this level, it is suitable to upper bound the quadratic conditions with respect to k . For this purpose, let us define

$$K_d(t) = \left\lfloor \frac{d + \sqrt{d^2 - 4dt}}{2} \right\rfloor, \quad d \in \mathbb{N}, t > 0. \quad (2.18)$$

It is easy to see that $K_d(t)$ is viewed as the largest integer $k \geq d$ such that $(k-d)k \leq dt$, with $K_0(t) = 0$.

Remark 2.2. As a direct consequence, we can upper bound the integer k appearing in the definitions of the sets $E_{d,i}$, $i = 1, 2, 3$. More precisely, one has

(1) For $E_{d,1}$,

$$k \leq K_d(x/c) - 1, \quad (2.19)$$

(2) For $E_{d,2}$,

$$k > K_d(x/c), \quad (2.20)$$

(3) For $E_{d,3}$,

$$k = K_d(x/c). \quad (2.21)$$

There is another remark regarding the integer k that appears in the sets $E_{d,i}$, $i = 1, 2, 3$.

Remark 2.3. The condition $k \leq x/a$ is redundant for $i = 1, 3$ provided that $d \leq cx/ab$. Indeed, the relation $d \leq \frac{cx}{ab}$ can be rewritten as:

$$(K_d(x/c) - d)K_d(x/c) \leq dx/c; \quad d \leq cx/ab \leq x/a,$$

and the fact that $t \mapsto (t-d)t$ is strictly increasing for $t \geq d$, leads to the inequality:

$$K_d(x/c) \leq \frac{x}{a}.$$

2.4.1. Calculation of $Q_{d,1}(x)$. In the sequel, for simplicity, we will write K_d instead of $K_d(x/c)$. In this section, we focus on the quantity $Q_{d,1}(x)$. To this end, we assume that $0 < d \leq \frac{cx}{ab}$. Let us introduce the function G , defined as follows

$$G(t) = \frac{1}{c} \sum_{n>t} \left(\frac{1}{n+a} - \frac{1}{n+b} \right). \quad (2.22)$$

Note that according to conditions (2.15) and (2.19), we get

$$Q_{d,1} = \sum_{d < k \leq K_d - 1} \sum_{\frac{x}{k-d+1} - b < n \leq \frac{x}{k} - a} \left(d - \frac{cx}{(n+a)(n+b)} \right)^2.$$

Furthermore, the sum

$$\sum_{\frac{x}{k-d+1}-b < n \leq \frac{x}{k}-a} \left(d - \frac{cx}{(n+a)(n+b)} \right)^2$$

is well defined if

$$\frac{x}{k-d+1} - b < \frac{x}{k} - a.$$

Immediately, we can conclude that if $k > K_{d-1}$ (recall that $K_0 = 0$), one has

$$Q_{d,1} = \sum_{K_{d-1} < k \leq K_d-1} \sum_{(x/(k-d+1))-b < n \leq (x/k)-a} \left(d - \frac{cx}{(n+a)(n+b)} \right)^2, \quad (2.23)$$

which simplifies to

$$\begin{aligned} Q_{d,1} = & d^2 \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - b \right\rfloor \right) \\ & - 2cdx \sum_{K_{d-1} < k \leq K_d-1} \left(G \left(\frac{x}{k-d+1} - b \right) - G \left(\frac{x}{k} - a \right) \right) \\ & + c^2 x^2 \sum_{K_{d-1} < k \leq K_d-1} \left(F \left(\frac{x}{k-d+1} - b \right) - F \left(\frac{x}{k} - a \right) \right). \end{aligned} \quad (2.24)$$

For the first sum of (2.24), we obtain

$$\begin{aligned} & \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - b \right\rfloor \right) \\ = & \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - a \right\rfloor \right) \\ & + \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k-d+1} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - b \right\rfloor \right). \end{aligned}$$

At this level, we use the following well-known “r-telescopic” identity:

$$\sum_{\alpha < k \leq \beta} (u_k - u_{k-r}) = \sum_{\beta-r < k \leq \beta} u_k - \sum_{\alpha-r < k \leq \alpha} u_k. \quad (2.25)$$

This allows us to write

$$\begin{aligned} \sum_{K_{d-1} < k \leq K_d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k-d+1} - a \right\rfloor \right) = & \sum_{K_{d-1}-d+1 < k \leq K_d-1} \left\lfloor \frac{x}{k} - a \right\rfloor \\ & - \sum_{K_{d-1} < k \leq K_{d-1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \end{aligned}$$

Using a similar argument for the second and third sums of (2.24), we obtain

$$\begin{aligned}
Q_{d,1} = & 2cdx \sum_{K_{d-1}-d+1 < k \leq K_{d-d}} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right) \\
& - c^2x^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-d}} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \\
& + 2cdx \sum_{K_d-d < k \leq K_{d-1}} G\left(\frac{x}{k} - a\right) - 2cdx \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} G\left(\frac{x}{k} - a\right) \\
& - c^2x^2 \sum_{K_d-d < k \leq K_{d-1}} F\left(\frac{x}{k} - a\right) + c^2x^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} F\left(\frac{x}{k} - a\right) \\
& + d^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-d}} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\
& + d^2 \sum_{K_d-d < k \leq K_{d-1}} \left\lfloor \frac{x}{k} - a \right\rfloor - d^2 \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \tag{2.26}
\end{aligned}$$

2.4.2. Calculation of $Q_{d,2}(x)$. Taking into account (2.16) and (2.20) and the condition $0 < d \leq \frac{cx}{ab}$, we get

$$Q_{d,2} = \sum_{K_d < k \leq x} \sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k-d} - b} \left(\frac{cx}{(n+a)(n+b)} - d \right)^2.$$

It is important to note that the sum

$$\sum_{\frac{x}{k+1} - a < n \leq \frac{x}{k-d} - b} \left(\frac{cx}{(n+a)(n+b)} - d \right)$$

is well defined only if the lower bound does not exceed the upper bound, that is, $\frac{x}{k-d} - b \geq \frac{x}{k+1} - a$. This condition can be rewritten equivalently as $k \leq K_{d+1} - 1$, which provides a natural upper bound for the summation index. Then, we deduce that

$$\begin{aligned}
Q_{d,2} = & d^2 \sum_{K_d < k \leq K_{d+1}-1} \left(\left\lfloor \frac{x}{k-d} - b \right\rfloor - \left\lfloor \frac{x}{k+1} - a \right\rfloor \right) \\
& - 2cdx \sum_{K_d < k \leq K_{d+1}-1} \left(G\left(\frac{x}{k+1} - a\right) - G\left(\frac{x}{k-d} - b\right) \right) \\
& + c^2x^2 \sum_{K_d < k \leq K_{d+1}-1} \left(F\left(\frac{x}{k+1} - a\right) - F\left(\frac{x}{k-d} - b\right) \right). \tag{2.27}
\end{aligned}$$

The decomposition in (2.27) follows from expanding the square and expressing the resulting sums in terms of the auxiliary functions F and G previously defined.

To proceed further, we focus on the first sum of (2.27). By introducing an intermediate term, we can split this expression into two more manageable parts

$$\begin{aligned} & \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - b \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right) = \\ & \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - b \right\rfloor - \left\lfloor \frac{x}{k-d-1} - a \right\rfloor \right) \\ & + \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - a \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right). \end{aligned}$$

This decomposition is strategic: the first term on the right-hand side captures the difference between the parameters a and b , while the second term exhibits a telescopic structure.

The use of the telescopic identity, introduced in (2.25), gives us

$$\begin{aligned} \sum_{K_{d+1} < k \leq K_{d+1}} \left(\left\lfloor \frac{x}{k-d-1} - a \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right) = & - \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor \\ & + \sum_{K_d-d < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \end{aligned}$$

This transformation is crucial as it converts the original sum into boundary terms that will be easier to estimate in subsequent sections.

A similar manipulation applies to the second and third sums of (2.27), where we apply the same telescopic technique to the functions G and F . After collecting all terms and reorganizing the summation indices, we obtain

$$\begin{aligned} Q_{d,2} = & c^2 x^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \\ & - 2cdx \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right) \\ & - 2cdx \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} G\left(\frac{x}{k} - a\right) + 2cdx \sum_{K_d-d < k \leq K_{d+1}} G\left(\frac{x}{k} - a\right) \\ & + c^2 x^2 \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} F\left(\frac{x}{k} - a\right) - c^2 x^2 \sum_{K_d-d < k \leq K_{d+1}} F\left(\frac{x}{k} - a\right) \\ & + d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\lfloor \frac{x}{k} - b \right\rfloor - \left\lfloor \frac{x}{k} - a \right\rfloor \right) \\ & - d^2 \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor + d^2 \sum_{K_d-d < k \leq K_{d+1}} \left\lfloor \frac{x}{k} - a \right\rfloor. \end{aligned} \tag{2.28}$$

The final expression (2.28) separates the main terms (involving differences $F(\cdot - a) - F(\cdot - b)$ and $G(\cdot - a) - G(\cdot - b)$) from the boundary terms, which will be shown to contribute to the error estimate.

2.4.3. **Calculation of $Q_{d,3}(x)$.** For $0 < d \leq \frac{cx}{ab}$, and from (2.17) and (2.21), we get

$$Q_{d,3} = d^2 \left(\left\lfloor \frac{x}{K_d} - a \right\rfloor - \left\lfloor \frac{x}{K_d+1} - a \right\rfloor \right) - 2cdx \left(G \left(\frac{x}{K_d+1} - a \right) - G \left(\frac{x}{K_d} - a \right) \right) + c^2x^2 \left(F \left(\frac{x}{K_d+1} - a \right) - F \left(\frac{x}{K_d} - a \right) \right). \quad (2.29)$$

2.4.4. **Calculation of $Q_d(x)$.** Keeping in mind (2.26), (2.28), (2.29) and the fact that for $0 < d \leq \frac{cx}{ab}$, we write

$$\begin{aligned} Q_d(x) = & d^2 \left(\sum_{K_{d-1}-d+1 < k \leq K_d-d} - \sum_{K_d-d < k \leq K_{d+1}-d-1} \right) \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\ & + 2cdx \left(\sum_{K_{d-1}-d+1 < k \leq K_d-d} - \sum_{K_d-d < k \leq K_{d+1}-d-1} \right) \left(G \left(\frac{x}{k} - a \right) - G \left(\frac{x}{k} - b \right) \right) \\ & - c^2x^2 \left(\sum_{K_{d-1}-d+1 < k \leq K_d-d} - \sum_{K_d-d < k \leq K_{d+1}-d-1} \right) \left(F \left(\frac{x}{k} - a \right) - F \left(\frac{x}{k} - b \right) \right) \\ & + d^2 \left(\sum_{K_{d+1}-d-1 < k \leq K_{d+1}} - \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \right) \left\lfloor \frac{x}{k} - a \right\rfloor \\ & + 2cdx \left(2 \sum_{K_d-d < k \leq K_d} - \sum_{K_{d+1}-d-1 < k \leq K_{d+1}} - \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} \right) G \left(\frac{x}{k} - a \right) \\ & + c^2x^2 \left(\sum_{K_{d+1}-d-1 < k \leq K_{d+1}} + \sum_{K_{d-1}-d+1 < k \leq K_{d-1}} - 2 \sum_{K_d-d < k \leq K_d} \right) F \left(\frac{x}{k} - a \right). \quad (2.30) \end{aligned}$$

2.4.5. **Estimate of $Q_d(x)$ for $d \geq 1$.**

Proposition 2.3. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$Q_d = \vartheta(d)\sqrt{cx} + \delta(d) + O \left(\frac{(1+b)^4 d^3}{c} \right),$$

with

$$\vartheta(d) = \frac{32d^2}{5}\sqrt{d} - \frac{48d^2 + 16d - 2}{15}\sqrt{d+1} - \frac{48d^2 - 16d - 2}{15}\sqrt{d-1},$$

and

$$\delta(d) = \left(\sum_{K_d-d < k \leq K_{d+1}-d-1} - \sum_{K_{d-1}-d+1 < k \leq K_d-d} \right) \left(\frac{c^2k^4}{x^2} - \frac{2cd}{x} + d^2 \right) \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right). \quad (2.31)$$

Remark 2.4. In this section, it is necessary to recall some useful technical results established in [3, p. 12], [4, p. 421], and [2, p. 4-6].

(1) For $0 < d \leq x/c$ and $x > c$, one has:

$$K_{d+1} - K_d = \left(\sqrt{d+1} - \sqrt{d} \right) \sqrt{\frac{x}{c}} + O(1). \quad (2.32)$$

$$\sum_{K_d < k \leq K_{d+1}} k^2 = \frac{(d+1)\sqrt{d+1} - d\sqrt{d}}{3} \left(\frac{x}{c}\right)^{3/2} + O\left((d+1)\frac{x}{c}\right). \quad (2.33)$$

$$K_{d+1} - K_d \ll \left(\frac{x}{cd}\right)^{1/2} \quad (0 < d \leq x/c). \quad (2.34)$$

$$\sqrt{\frac{dx}{c}} + \frac{d}{2} - 1 < K_d \leq \sqrt{\frac{dx}{c}} + d. \quad (2.35)$$

(2) For $0 \leq d \leq x/c$ and $x \geq c$, one has:

$$K_d = \sqrt{\frac{dx}{c}} + \frac{d}{2} + O\left(d^{3/2}c^{1/2}x^{-1/2}\right) + O(1). \quad (2.36)$$

$$\sum_{K_d - d < k \leq K_d} k = d\sqrt{\frac{dx}{c}} + O\left(d^{5/2}x^{-1/2}c^{1/2}\right) + O(d). \quad (2.37)$$

$$\sum_{K_d - d < k \leq K_d} k^3 = \frac{x}{c} \left(d^2 \sqrt{\frac{dx}{c}} + O\left(d^{7/2}x^{-1/2}c^{1/2}\right) + O(d^2) \right). \quad (2.38)$$

$$\sum_{K_d - d < k \leq K_{d+1} - (d+1)} k^4 = \frac{(d+1)^2 \sqrt{d+1} - d^2 \sqrt{d}}{5} \left(\frac{x}{c}\right)^{5/2} + O\left((d+1)^2 c^{-2} x^2\right). \quad (2.39)$$

Lemma 2.5. For $t > 0$ and $b > a > 0$, we have

$$G(t) = \frac{1}{t} + \frac{\{t\} - \frac{a+b+1}{2}}{t^2} + O\left(\frac{(1+b)^2}{t^3}\right). \quad (2.40)$$

Proof. For $t > 0$, $n \in \mathbb{N}^*$ and $b > a > 0$, we have according to (2.22)

$$G(t) = \sum_{n>t} \frac{1}{(n+a)(n+b)},$$

Then

$$\begin{aligned} \sum_{n>t} \frac{1}{n^2} &= \frac{1}{t} + \frac{\{t\} - \frac{1}{2}}{t^2} + O(t^{-3}), \quad \sum_{n>t} \frac{1}{n^3} = \frac{1}{2t^2} + O(t^{-3}), \\ \sum_{n>t} \frac{1}{n^4} &= O(t^{-3}), \end{aligned}$$

which allows us to conclude that

$$G(t) = \frac{1}{t} + \frac{\{t\} - \frac{a+b+1}{2}}{t^2} + O\left(\frac{1+b^2}{t^3}\right).$$

□

Lemma 2.6. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$\begin{aligned} & d^2 \sum_{K_d - d < k \leq K_{d+1} - d - 1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\ &= d^2 \left(\sqrt{d+1} - \sqrt{d} \right) \sqrt{cx} \\ &+ d^2 \sum_{K_d - d < k \leq K_{d+1} - d - 1} \left(\left\{ \frac{x}{k} - b \right\} - \left\{ \frac{x}{k} - a \right\} \right) \\ &+ O(cd^2). \end{aligned} \quad (2.41)$$

Proof. By a direct computation, we obtain

$$d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) = cd^2 (K_{d+1} - K_d - 1) \\ + d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\{ \frac{x}{k} - b \right\} - \left\{ \frac{x}{k} - a \right\} \right).$$

Then we use the result noted (2.32), and get

$$d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left\lfloor \frac{x}{k} - a \right\rfloor - \left\lfloor \frac{x}{k} - b \right\rfloor \right) \\ = d^2 \left(\sqrt{d+1} - \sqrt{d} \right) \sqrt{cx} + d^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\left(\frac{x}{k} - b \right) - \left(\frac{x}{k} - a \right) \right) + O(cd^2).$$

□

Lemma 2.7. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$2cdx \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right) \\ = 2d \left(\frac{d\sqrt{d} - (d+1)\sqrt{d+1}}{3} \right) \sqrt{cx} \\ + 2d \sum \left(\frac{ck^2}{x} \right) \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) + O\left(\frac{(1+b)^2 d^2}{c} \right). \quad (2.42)$$

Proof. For practical reasons, we set

$$S = \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(G\left(\frac{x}{k} - a\right) - G\left(\frac{x}{k} - b\right) \right).$$

According to (2.34), the contribution to the sum S of the error term of (2.40) is expressed as follows:

$$S \ll (1+b)^2 (K_{d+1} - K_d) \left(\frac{d}{cx} \right)^{3/2} \leq (1+b)^2 \sqrt{\frac{x}{c(d+1)}} \left(\frac{d}{cx} \right)^{3/2} \leq \frac{(1+b)^2 d}{xc^2}.$$

According to formulas (2.33), (2.34), and (2.35), the contribution to the sum S of the $1/t$ term of (2.40) is

$$\sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{\frac{x}{k} - a} - \frac{1}{\frac{x}{k} - b} \right) \\ = \frac{d\sqrt{d} - (d+1)\sqrt{d+1}}{3} x^{-1/2} c^{-1/2} + O\left(\frac{d+1}{x} \right) + O\left(\frac{abd}{xc} \right).$$

Finally, the contribution to the sum S of the term $\frac{\{t\} - \frac{a+b+1}{2}}{t^2}$ of (2.40) is expressed as follows:

$$\begin{aligned} & \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{1}{\left(\frac{x}{k} - a\right)^2} \left(\left\{ \frac{x}{k} - a \right\} - \frac{a+b+1}{2} \right) \\ & - \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{1}{\left(\frac{x}{k} - b\right)^2} \left(\left\{ \frac{x}{k} - b \right\} - \frac{a+b+1}{2} \right) \\ & = \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{k^2}{x^2} \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) + O\left(\frac{(b+1)^2 d}{c^2 x}\right). \end{aligned}$$

We obtain relation (2.42) by collecting these estimates. $\square$

Lemma 2.8. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, one has

$$\begin{aligned} & c^2 x^2 \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(F\left(\frac{x}{k} - a\right) - F\left(\frac{x}{k} - b\right) \right) \\ & = \frac{d^2 \sqrt{d} - (d+1)^2 \sqrt{d+1}}{5} \sqrt{cx} \\ & + \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{c^2 k^4}{x^2} \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) \\ & + O\left(cb^2 d^2\right). \end{aligned} \tag{2.43}$$

Proof. For practical reasons, we set

$$S' = \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(F\left(\frac{x}{k}\right) - F\left(\frac{x}{k} - a\right) \right).$$

According to (2.34), the contribution of the error term of (2.12) is estimated as follows:

$$\ll (1+b)^2 (K_{d+1} - K_d) \left(\frac{d}{cx}\right)^{5/2} \leq (1+b)^2 \sqrt{\frac{x}{c(d+1)}} \left(\frac{d}{cx}\right)^{5/2} \leq \frac{(1+b)^2 d^2}{c^3 x^2}.$$

The contribution of the term $\frac{1}{3t^3}$ of (2.12) is given by

$$\sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) = \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(-\frac{ck^4}{x^4} + O\left(\frac{b^2 k^5}{x^5}\right) \right),$$

then from (2.34) and (2.39), it follows that

$$\sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{3\left(\frac{x}{k} - a\right)^3} - \frac{1}{3\left(\frac{x}{k} - b\right)^3} \right) = \frac{d^2 \sqrt{d} - (d+1)^2 \sqrt{d+1}}{5} (cx)^{-3/2} + O\left(\frac{b^2 d^2}{c^3 x^2}\right).$$

Finally, according to (2.34) and (2.39), the contribution of the term $\frac{\{t\} - \frac{1+4(a+b)}{2}}{t^4}$ of (2.12) is expressed as follows:

$$\begin{aligned} & \sum_{K_d-d < k \leq K_{d+1}-d-1} \left(\frac{1}{\left(\frac{x}{k} - a\right)^4} \left(\left\{ \frac{x}{k} - a \right\} - \frac{1+4(a+b)}{2} \right) \right. \\ & \quad \left. - \frac{1}{\left(\frac{x}{k} - b\right)^4} \left(\left\{ \frac{x}{k} - b \right\} - \frac{1+4(a+b)}{2} \right) \right) \\ &= \sum_{K_d-d < k \leq K_{d+1}-d-1} \frac{k^4}{x^4} \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right) + O\left(\frac{b^2 d^2}{c^3 x^2}\right). \end{aligned}$$

Summing up, we obtain the desired result, namely the relation (2.43). $\square$

Lemma 2.9. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, and $K_d - d < k \leq K_d$, we have

$$\left\lfloor \frac{x}{k} - a \right\rfloor = \sqrt{\frac{cx}{d}} + O(a). \quad (2.44)$$

Proof. Since $k - \sqrt{\frac{dx}{c}} \ll d$ according to (2.35), one has

$$\left\lfloor \frac{x}{k} - a \right\rfloor = \frac{x}{k} + O(a) = \sqrt{\frac{cx}{d}} + O(a).$$

$\square$

Lemma 2.10. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, one has

$$2cdx \sum_{K_d-d < k \leq K_d} G\left(\frac{x}{k} - a\right) = 2d^2 \sqrt{cdx} + O(d^3) + O\left(\frac{c^{3/2}(1+b)^2 d^{7/2}}{x^{1/2}}\right). \quad (2.45)$$

Proof. According to (2.40), we see that

$$\sum_{K_d-d < k \leq K_d} G\left(\frac{x}{k} - a\right) = \sum_{K_d-d < k \leq K_d} \left(\frac{k}{x} + \frac{k^2}{x^2} \left(\frac{x}{k} - a - \frac{a+b+1}{2} \right) + O\left(\frac{(1+b)^2 k^2}{x^2}\right) \right).$$

Using (2.37), we obtain

$$\sum_{K_d-d < k \leq K_d} \frac{k}{x} = \frac{d}{x} \sqrt{\frac{dx}{c}} + O\left(\frac{d}{x}\right) + O\left(\frac{c^{1/2} d^{5/2}}{x^{3/2}}\right).$$

From (2.35), we can see that

$$\begin{aligned} \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} \left(\frac{x}{k} - a \right) &\ll \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} \leq \frac{dK_d^2}{x^2} \leq \frac{d^2}{cx}, \\ \frac{a+b+1}{2} \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} &\ll \frac{(a+b+1)d^2}{cx}, (1+b)^2 \sum_{K_d-d < k \leq K_d} \frac{k^2}{x^2} \ll \frac{(1+b)^2 d^2}{cx}. \end{aligned}$$

Then, the lemma's result is deduced. $\square$

Lemma 2.11. For $0 < d \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$c^2 x^2 \sum_{K_d-d < k \leq K_d} F\left(\frac{x}{k} - a\right) = \frac{d^2}{3} \sqrt{dcx} + O(b^2 d^3) + O\left(c^{3/2} d^{7/2} x^{-1/2}\right). \quad (2.46)$$

Proof. According to (2.12), one has

$$\sum_{K_d-d < k \leq K_d} F\left(\frac{x}{k} - a\right) = \sum_{K_d-d < k \leq K_d} \left(\frac{k^3}{3x^3} + \frac{k^4}{x^4} \left(\left\{ \frac{x}{k} - a \right\} - 2(a+b) - \frac{1}{2} \right) + O\left(\frac{b^2 k^4}{x^4}\right) \right).$$

Using (2.38), we get

$$\sum_{K_d-d < k \leq K_d} \frac{k^3}{3x^3} = \frac{1}{3x^2} \left(\frac{d^2}{c} \sqrt{\frac{dx}{c}} + O\left(\frac{d^3 \sqrt{\frac{d}{cx}}}{1}\right) + O\left(\frac{d^2}{c}\right) \right).$$

By using the fact that $K_d \ll \sqrt{\frac{dx}{c}}$, it follows that

$$\sum_{K_d-d < k \leq K_d} \left(\frac{k^4}{x^4} \left(\left\{ \frac{x}{k} - a \right\} - 2(a+b) - \frac{1}{2} \right) + O\left(\frac{b^2 k^4}{x^4}\right) \right) \leq \frac{b^2 d^3}{c^2 x^2}.$$

□

The combination of formulas (2.41) through (2.46) and (2.30) leads to the proposition 2.3

2.5. The estimate of the sum $Q(x)$.

2.5.1. **About the main term in $Q(x)$.** In order to appear the main term and the error term involving the parameter D , we give another formulation of $Q(x)$.

Proposition 2.4. For two real numbers x and D such that $3 \leq D \leq \frac{cx}{9(b+1)^2} - 1$, we have

$$Q(x) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + R(D) + O\left(\frac{(1+b)^4 D^4}{c}\right) + O\left(\sqrt{\frac{cx}{D}}\right),$$

with

$$R(D) = \delta_0 + \sum_{1 \leq d \leq D} \delta(d).$$

Here, the quantities δ_0 and $\delta(d)$ are defined by (2.9) for $d = 0$ (resp. by (2.31) for $d > 0$).

Proof. According to the propositions 2.1, 2.2 and 2.3, the use of the estimate $\vartheta(d) \ll d^{-3/2}$, and the following equality (see [2, p. 11] and the equations (12)–(13)), we conclude that $\sum_{d \leq D} \vartheta(d) = -\frac{2}{15} + \frac{\zeta(3/2)}{\pi} + O(D^{-1/2})$, and we get

$$Q(x) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + \delta_0 + \sum_{1 \leq d \leq D} \delta(d) + O\left(\frac{(1+b)^4 D^4}{c}\right) + O\left(\sqrt{\frac{cx}{D}}\right).$$

□

2.5.2. **The estimate of the sum $R(D)$.** To bound the sum $R(D)$, we will exploit the following result due to van der Corput.

Lemma 2.12. Let a and b be real numbers such that $b - a \geq 1$, and let $f : [a, b] \rightarrow \mathbb{R}$ be a twice continuously differentiable function with a monotonic second derivative of constant sign. Then we have

$$\sum_{a < n \leq b} \left(\{f(n)\} - \frac{1}{2} \right) \ll \int_a^b |f''(t)|^{1/3} dt + \frac{1}{\sqrt{|f''(a)|}} + \frac{1}{\sqrt{|f''(b)|}}. \quad (2.47)$$

Proof. See [10, p. 32].

□

By partial summation, we derive the following estimate from (2.47).

Lemma 2.13. *Let a and b be real numbers such that $b - a > 0$, and let $f : [a, b] \rightarrow \mathbb{R}$ be a twice continuously differentiable function with a monotonic second derivative of constant sign. Let $(v_n)_{a < n \leq b}$ be a sequence of real numbers. Then we have*

$$\sum_{a < n \leq b} v_n \left(\{f(n)\} - \frac{1}{2} \right) \ll U \int_a^b |f''(t)|^{1/3} dt + \frac{U}{\sqrt{|f''(a)|}} + \frac{U}{\sqrt{|f''(b)|}}, \quad (2.48)$$

where

$$U = \max |v_n| + \sum_{a < n \leq b} |v_{n+1} - v_n|.$$

And one has

Proposition 2.5. *For $x \geq 100(1+b)^{21/10}/c$ and $1 \leq D \leq (x/c)^{1/21}$, one has*

$$R(D) \ll x^{1/3} \ln(x/c) + c^{-2/21} x^{3/7} D^{7/4}.$$

The proof of this proposition is based on the following lemmas.

Lemma 2.14. *For $0 < d \leq \frac{x}{c}$ and $K_d - d < k \leq K_{d+1} - (d+1)$, one has*

$$-\frac{2cd}{x} \leq \frac{c^2 k^4}{x^2} - \frac{2cd}{x} + d^2 \leq \frac{64d^3 c}{x} + 65d^2.$$

Proof. It suffices to show that

$$d^2 \left(\frac{c^2 k^2}{x^2} - 8 \frac{ck}{x} + 17 \right) - \frac{2cd}{x} \leq \frac{c^2 k^4}{x^2} - \frac{2cd}{x} + d^2 \leq (d+1)^2 \left(\frac{c^2 k^2}{x^2} - 8 \frac{ck}{x} + 16 \right) - \frac{2cd}{x} + d^2.$$

In fact, we choose $K_d - d$ as the largest integer verifying $k^2 + dk - \frac{4dx}{c} \leq 0$, which allows us to say that $k^2 \in \left[-dk + \frac{4dx}{c}, -(d+1)k + \frac{4(d+1)x}{c} \right]$. $\square$

Lemma 2.15. *For $x \geq 100(1+b)^{21/10}/c$ and $1 \leq d \leq (x/c)^{1/21}$, one has*

$$\delta(d) \ll c^{-2/21} d^{-1} x^{3/7} + c^{-71/84} d^{3/4} x^{29/84}.$$

Proof. A simple computation shows that if $x \geq \frac{100(1+b)^{21/10}}{c}$, then $\left(\frac{x}{c}\right)^{1/21} \leq \frac{cx}{9(b+1)^2} - 1$.

Let us define $v_k = \frac{c^2 k^4}{x^2} - \frac{2cd}{x} + d^2$. Since the sequence (v_k) is increasing, Lemma 2.14 implies that

$$\max |v_k| + \sum_{K_d - d < k \leq K_{d+1} - (d+1)} |v_{k+1} - v_k| \ll \left(\frac{x}{c}\right)^{2/21}.$$

Applying (2.48) to the two functions $f(t) = \frac{x}{t} - a$ and $f(t) = \frac{x}{t} - b$, and taking into account (2.34) and (2.35), we obtain

$$\sum_{K_d - d < k \leq K_{d+1} - (d+1)} v_k \left(\{f(n)\} - \frac{1}{2} \right) \ll c^{-2/21} d^{-1} x^{3/7} + c^{-71/84} d^{3/4} x^{29/84}, \quad (2.49)$$

In the case where $K_d - d < k \leq K_{d+1} - d - 1$, we obtain the same result by following all the same arguments. $\square$

Lemma 2.16. *For $x \geq 1 + c$, one has*

$$\delta_0 \ll x^{1/3} \ln(2x/c) + c^{-3/4} x^{1/4}.$$

Proof. We have $\delta_0 = \frac{c^2}{x^2} \sum_{1 \leq k \leq K} k^4 \left(\left\{ \frac{x}{k} - a \right\} - \left\{ \frac{x}{k} - b \right\} \right)$, with $K = \left\lfloor \sqrt{\frac{x}{c} + \frac{1}{4}} - \frac{1}{2} \right\rfloor$. Using the same reasoning as in the proof of Lemma 2.15, we deduce that

$$\delta_0 \ll x^{1/3} \ln \left(\frac{2x}{c} \right) + c^{-3/4} x^{1/4}. \quad (2.50)$$

According to Proposition 2.4 as well as Lemmas 2.15 and 2.16, we get

$$R(D) = \delta_0 + \sum_{1 \leq d \leq D} \delta(d) \ll x^{1/3} \ln \left(\frac{x}{c} \right) + c^{-2/21} x^{3/7} D^{7/4}.$$

□

2.5.3. *The estimate of the sum $Q(x)$.* Combining the results obtained in propositions 2.4 and 2.5, we give the following result.

Proposition 2.6. *For $x \geq 100(1+b)^{21/10}/c$ and $1 \leq d \leq (x/c)^{1/21}$, we have*

$$Q(x) = \frac{\zeta(3/2)}{\pi} \sqrt{cx} + O \left(\sqrt{\frac{cx}{D}} \right) + O \left(c^{-2/21} x^{3/7} D^{7/4} \right) + O \left(\frac{(1+b)^4 D^4}{c} \right) + O \left(x^{1/3} \ln \left(\frac{x}{c} \right) \right).$$

Proof of theorem 1.1. We are now ready to assemble all the previous estimates to establish our main result. For

$$x \geq 100(1+b)^{21/10} c^{-1}, \quad (2.51)$$

and $1 \leq d \leq (x/c)^{1/21}$, the combination of the results from propositions 2.5 and 2.6 indicates that

$$Q(x) - \frac{\zeta(3/2)}{\pi} \sqrt{cx} \ll \sqrt{\frac{cx}{D}} + c^{-2/21} x^{3/7} D^{7/4} + \frac{(1+b)^4 D^4}{c} + x^{1/3} \ln \left(\frac{x}{c} \right).$$

This inequality represents the total error bound as a sum of four distinct terms, each arising from different stages of our analysis.

To optimize this bound, we must choose the parameter D carefully. We choose $D = c^{50/189} x^{2/63}$ to balance the first two error terms. This strategic choice ensures that neither term dominates the other, leading to the sharpest possible estimate. This quantity satisfies the bounds $1 \leq D \leq (x/c)^{1/21}$ if

$$x \geq \max(c^{20}, c^{-25/3}). \quad (2.52)$$

We now verify that all remaining error terms are dominated by the main error term. On the other hand, under the assumptions (2.51) and (2.52), it follows that

$$x^{1/3} \ln \left(\frac{x}{c} \right) \ll x^{1/3} \left(\frac{x}{c} \right)^{19/126} \ll c^{139/378} x^{61/126},$$

which shows that the logarithmic term is absorbed into the main error bound.

It remains to control the term involving the parameter b . Keeping in mind and that

$$x \geq (1+b)^{14} c^{-13/15}, \quad (2.53)$$

we conclude that

$$\frac{(1+b)^4 D^4}{c} \asymp (1+b)^4 c^{11/89} x^{8/63} \ll c^{139/378} x^{61/126}.$$

This demonstrates that the dependence on b is also properly controlled under our assumptions.

Finally, we consolidate all conditions into a single requirement. Further, observe that if $x \geq 100c^{-25/3}(1+b)^{29}$, the three conditions (2.51), (2.52), and (2.53) are satisfied, and the result is demonstrated. This completes the proof of Theorem 1.1. $\square$

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DOUBLE-DEPRESSED EUCLIDEAN-DESCARTES POLYNOMIALS AND ASSOCIATED CUBIC CURVES

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Abstract. The aim of this paper is to continue the study [5] of quartic real polynomials P having the same Euclidean norm as their Descartes resolvent $DR(P)$. The polynomial P is given in its reduced form P_d which means without a cubic term while double-depressed means the lack of the quadratic term. The class of these polynomials is found to consists in two 1-parameter families and the discriminant of cubic curve defined naturally by $DR(P_d)$ is computed. We consider also the double-depressed Descartes resolvent as a self-map of $\mathbb{R}^2$ and its fixed points are determined. Some special angles appear throughout this papers since the coefficients of P_d are trigonometrical functions of the real parameter t .

Keywords: Quartic real polynomial, Descartes resolvent, Euclidean-Descartes polynomial, Cubic curve, Discriminant.

MSC (2020): 12D05, 51N35, 65H04.

1. INTRODUCTION

The well-known Descartes method associates to a given monic (and reduced) quartic polynomial P_d its Descartes resolvent $DR(P_d)$ as a monic cubic polynomial. In the paper [5] we have considered both polynomials P_d and $DR(P_d)$ as vectors in $\mathbb{R}^3$ and hence we introduce and study the class of *Euclidean-Descartes polynomials* by asking the equality $\|P_d\| = \|DR(P_d)\|$ where $\|\cdot\|$ is the usual Euclidean norm of $\mathbb{R}^n$, $n \in \mathbb{N}^*$. The method of Lagrange is the subject of the paper [3].

The defining condition of isometry of the Descartes map $P_d \rightarrow DR(P_d)$ was expressed as an equation in terms of the coefficients (p, q, r) of P_d and to solve it we have fixed q in the set $\{0, \pm 1\}$. The present paper continues this line of research with another condition, namely $p = 0$; therefore P_d and $DR(P_d)$ belong now to $\mathbb{R}^2$. Since p appears in the coefficients of x^2 and z^2 of P_d respectively $DR(P_d)$ we call this case as being *double-depressed*. It follows a quadratic equation for the pair (q^2, r) which is naturally interpreted as the equation of an ellipse for which we provide the eccentricity e .

Since an ellipse is usually parametrised with the trigonometrical functions $(\cos t, \sin t)$ we obtain that $q = \cos t$ while r comes in pairs $r_{\pm} = \pm \frac{\sin 2t}{2\sqrt{15}}$ for the arbitrary real number t . It

follows two 1-parametric families of double-depressed quartic polynomials. A second interest in the cubic curves $y^2 = DR(P_d(t))(x)$ for which we compute the discriminant Δ ; on this way we continue the studies of [2], [4] and [6]. Hence, we discuss also the vanishing of Δ which gives a special angle $\simeq 80.96^\circ$.

The last section deals with the fixed points of the double-depressed Descartes map $ddDR : (q, r) \in \mathbb{R}^2 \rightarrow (-4r, -q^2) \in \mathbb{R}^2$. In addition to the obvious $(0, 0)$ we get the fixed point $(1/4, -1/16)$ implying a second remarkable angle, namely $\arccos(\frac{1}{4})$. We point out that the associated cubic curve is not an elliptic one over the field $\mathbb{R}$, see e.g. [7], since its discriminant Δ is strictly positive. Another important conclusion is the non-existence of double-depressed Euclidean-Descartes quartic polynomials with multiple roots.

2. DOUBLE DEPRESSED EUCLIDEAN-DISCARTES QUARTIC POLYNOMIALS

Fix a natural number $n \in \mathbb{N}^*$. The general framework of this paper is provided by the n -dimensional real linear space of monic polynomials of grade n :

$$\mathbb{R}_n^{monic}[x] := \{P(x) = x^n + a_1x^{n-1} + \dots + a_n; a_1, \dots, a_n \in \mathbb{R}\}.$$

Since $\mathbb{R}^n$ is an Euclidean space with respect the usual scalar product $\langle \cdot, \cdot \rangle$ the polynomial P has the Euclidean norm:

$$\|P\| := (a_1^2 + \dots + a_n^2)^{\frac{1}{2}}. \quad (2.1)$$

Fix now the quartic polynomial $P \in \mathbb{R}_4^{monic}[x]$:

$$P(x) := y^4 + Ay^3 + By^2 + Cy + D$$

to which we apply the Tschirnhausen transformation $y = x - \frac{A}{4}$ have the *reduced* form (here the subscript d means *depressed*):

$$P_d(x) := x^4 + px^2 + qx + r. \quad (2.2)$$

The Descartes approach for P_d is one of (relative many) methods to deal with P_d ; see e.g. [8, p. 174] as well as [9] for an universal method to solve the quartic equation. More precisely, Descartes performs a product decomposition of P_d into two quadratic polynomials:

$$P_d(x) = (x^2 + ax + b)(x^2 + \tilde{a}x + \tilde{b}) \quad (2.3)$$

and therefore:

$$a + \tilde{a} = 0, \quad b + \tilde{b} + a\tilde{a} = p, \quad a\tilde{b} + \tilde{a}b = q, \quad b\tilde{b} = r. \quad (2.4)$$

It follows immediately:

$$4r = 4b\tilde{b} = (b + \tilde{b})^2 - (b - \tilde{b})^2 = (p + a^2)^2 - \left(\frac{q}{a}\right)^2 \quad (2.5)$$

and hence, denoting $a^2 = z$, it follows a cubic equation in u :

$$DR(P_d)(z) := z^3 + 2pz^2 + (p^2 - 4r)z - q^2 = 0 \quad (2.6)$$

and let us call the polynomial $DR(P_d) \in \mathbb{R}_3^{monic}[z]$ as being the *Descartes resolvent* of P_d . From:

$$DR(P_d)(0) = -q^2 \leq 0 < +\infty = \lim_{z \rightarrow +\infty} DR(P_d)(z)$$

the Descartes resolvent have at least one positive root $z_0 = a^2$ which yields the whole data $(a, \tilde{a} = -a, b, \tilde{b})$ through the relations (2.4). In addition to the previous decomposition of P_d the Descartes resolvent is also useful in the following irreducibility characterization: suppose that $P_d \in \mathbb{Q}[x]$. Then P_d is irreducible over $\mathbb{Q}$ if and only if P_d does not have a rational root and $DR(P_d)$ does not have a root that is a square of a rational number.

In a previous study [5] of quartic polynomials we introduce:

Definition 2.1. *i) P (or equivalently P_d) is an Euclidean-Descartes polynomial if it preserves the Euclidean norm with respect to Descartes transformation DR :*

$$\|P_d\| = \|DR(P_d)\|. \quad (2.7)$$

So, the restriction of DR to the class of Euclidean-Descartes polynomials is an Euclidean isometry.

ii) The Descartes-cubic curve associated to (arbitrary) P_d is:

$$DC(P_d) : y^2 = DR(P_d)(x). \quad (2.8)$$

A straightforward computation yields:

Proposition 2.1. *P_d is an Euclidean-Descartes polynomial if and only if:*

$$p^4 + q^4 + 15r^2 - 8p^2r + 3p^2 - q^2 = 0. \quad (2.9)$$

With the 3D-transformation ($X := p^2, Y := q^2, Z = r$) we have the equation of a two-sheets hyperboloid:

$$H_2(P_d) : X^2 + Y^2 + 15Z^2 - 8XZ + 3X - Y = 0. \quad (2.10)$$

Proof. The hypothesis means that:

$$\|P_d\|^2 = \|DR(P_d)\|^2 \rightarrow p^2 + q^2 + r^2 = 4p^2 + (p^2 - 4r)^2 + q^4$$

which is exactly (2.9). □

In [5] we point out an easy case of reduction namely $q \in \{0, \pm 1\}$ and hence P_d which such a coefficient was be called *reduced*. Now, we remark that the coefficient of x^2 in P_d is half the coefficient of z^2 in $DR(P_d)$ and then we introduce a new class:

Definition 2.2. *The depressed polynomial P_d is called double-depressed if $p = 0$.*

Remark 2.1. *i) If we look at the initial quartic polynomial P since $p = B - \frac{3A^2}{8}$ it follows that the double-depressed condition means:*

$$B = \frac{3A^2}{8}. \quad (2.11)$$

Concerning the Descartes coefficients $(a, \tilde{a} = -a, b, \tilde{b})$ the vanishing of p means: $b + \tilde{b} = a^2 = z_0$.

ii) The Descartes-cubic curve is:

$$DC(P_d) : y^2 = x^3 + Px + Q, \quad P = -4r, \quad Q = -q^2 \quad (2.12)$$

and we recall its discriminant: $\Delta := 4P^3 + 27Q^2$.

iii) The intersection of the hyperboloid $H_2(P_d)$ with the plane $X = 0$ is the ellipse:

$$E : Y^2 + 15Z^2 - Y = 0 \quad (2.13)$$

with the eccentricity:

$$e = \sqrt{\frac{14}{15}} \simeq 0.966092. \quad (2.14)$$

□

Since it is easy to give a parametrization of E it follows a complete characterization of double-depressed quartic polynomials:

Theorem 2.1. *The class of double-depressed quartic polynomials consists in two 1-parametric families with the coefficients:*

$$q(t) = \cos t, \quad r_{\pm}(t) = \pm \frac{\sin 2t}{2\sqrt{15}}, \quad t \in \mathbb{R}. \quad (2.15)$$

The discriminant of the Descartes-cubic curve $DC(P_d)(t)$ is:

$$\begin{cases} DC(P_d)(t) : y^2 = x^3 \mp \left(\frac{2\sin 2t}{\sqrt{15}} \right) x - (\cos t)^2, \\ \Delta_{\pm}(t) = (\cos t)^3 \left[27 \cos t \mp \frac{256}{15\sqrt{15}} (\sin t)^3 \right]. \end{cases} \quad (2.16)$$

Proof. The equation (2.9) with $p = 0$ means:

$$15r^2 = q^2(1 - q^2) \quad (2.17)$$

forcing q to be a cosine of a real parameter t . Hence, the claimed expressions (2.15) follow immediately. Also, the value of the discriminant being:

$$\Delta = 27q^4 - 16^2r^3 \quad (2.18)$$

the formulae (2.16) is directly. □

We are interested now in the vanishing of $\Delta_{\pm}$:

- i) $\Delta_- = 0$ implies $\cos t \in \{0, \simeq 0.1572\}$; this means $t \in \{90^\circ, \simeq 80.96^\circ\}$
- ii) $\Delta_+ = 0$ implies $\cos t \in \{0, \simeq -0.1572\}$.

3. THE FIXED POINTS OF THE DOUBLE-DEPRESSED DESCARTES MAP

A second way to find double-depressed Euclidean-Descartes polynomials is to find the fixed points of the *dd-Descartes map*:

$$ddDR : (q, r) \in \mathbb{R}^2 \rightarrow (-4r, -q^2) \in \mathbb{R}^2 \quad (3.1)$$

or, in a matrix form:

$$ddDR : \begin{pmatrix} q \\ r \end{pmatrix} \in \mathbb{R}^2 \rightarrow \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} q^2 \\ r^2 \end{pmatrix} + \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} q \\ r \end{pmatrix} \in \mathbb{R}^2. \quad (3.2)$$

By denoting Γ_1 and Γ_2 the 2×2 matrices above it results that $\Gamma_2 = 4(\Gamma_1)^T$. The Descartes map is a bijective map on its image $\mathbb{R}_-^2 := \{(A, B) \in \mathbb{R}^2; B \leq 0\}$ with:

$$(ddDR)^{-1}(A, B) = \left(\pm\sqrt{-B}, -\frac{A}{4} \right) \quad (3.3)$$

and the iteration of the map $ddDR$ produces the interesting quadratic map:

$$(q, r) \in \mathbb{R}^2 \rightarrow (4q^2, -16r^2) \in \mathbb{R}^2. \quad (3.4)$$

A direct computation yields:

Proposition 3.1. *The dd -Descartes map (3.1) has only two fixed points: $(0, 0)$ which corresponds via (2.15) to $t_1 = \frac{\pi}{2}$ and $(\frac{1}{4}, -\frac{1}{16})$ which corresponds to $r_-(t_2 = \arccos(0.25) \simeq 75.52^\circ)$.*

It results:

- a) $DC(P_d)(t_1) : y^2 = x^3$, which is the *semicubical parabola*, $\Delta(t_1) = 0$ for both signs $\pm$,
- b) $DR(P_d(t_2))(z) = z^3 + \frac{z}{4} - \frac{1}{16}$, $\|P_d(t_2)\| = \frac{\sqrt{17}}{16}$, $DC(P_d)(t_2) : y^2 = x^3 + \frac{x}{4} - \frac{1}{16}$, $\Delta_-(t_2) = \frac{43}{256} > 0$. $DR(P_d(t_2))$ has an unique real root $z_0 \simeq 0.2119$. We recall that a real polynomial P with all real roots is called *hyperbolic*, see e.g. [1]; so, $DR(P_d(t_2))$ is not a hyperbolic polynomial.

The fixed point $(\frac{1}{4}, -\frac{1}{16})$ corresponds to the point $(Y, Z) = (\frac{1}{16}, -\frac{1}{16})$ which is the intersection of the ellipse E with the second bisectrix $B_2 : y = -x$.

It is worth to point out $P_d(t_1)(x) = x^4$ have multiple roots. We finish this section with the problem of existence of double-depressed Euclidean-Descartes polynomials with multiple roots. In [5] we have obtained:

Proposition 3.2. *Excepting $P_d(x) = x^4$ there are only two Euclidean-Descartes polynomials with multiple roots:*

$$\begin{cases} P_d^\pm(x) = x^4 \pm 4\sqrt{3}x^2 + 12 = (x^2 \pm 2\sqrt{3})^2, \\ DR(P_d^\pm)(z) = z^2(z \pm 8\sqrt{3}), \quad \|P_d^\pm\| = 8\sqrt{3}. \end{cases} \quad (3.5)$$

The value $p = \pm 4\sqrt{3} \neq 0$ means that there are no double-depressed Euclidean-Descartes polynomials with multiple roots.

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NEW RESULTS ON RIEMANNIAN CONCIRCULAR STRUCTURE MANIFOLDS

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Abstract. In this paper, we study Riemannian concircular structure manifolds. Some basic concepts and some basic properties such manifolds are established. Also in the last section, some flat and some symmetric properties of a Riemannian concircular structure manifold are studied.

Keywords: Riemannian manifolds, Riemannian concircular structure manifolds, curvature tensors, symmetric spaces, torse-forming vector field, concircular vector field

MSC (2020): 53C21, 58D17, 58J60, 70G45.

1. INTRODUCTION

In the study of the geometry of Riemannian manifolds, intrinsic vector fields such as torse-forming, torqued, concircular, recurrent, and parallel vector fields play a fundamental role. In the year 1939, in a Riemannian manifold M the notion of concircular vector fields introduced by Fialkow [13]. Indeed, in a Riemannian manifold a concircular vector field seems the theory of a transformation. For example, a concircular vector field plays an important role in the study of a concircular mapping [25]. Subsequently, in 1965, Tashiro provided a classification of Riemannian manifolds admitting concircular vector fields, which has since become a cornerstone in this field [24]. Following these foundational works, the study of manifolds equipped with specific vector fields was expanded by various authors. For instance, Yano developed the theory of torse-forming vector fields [26]. Then Mihai and Mihai [18] investigated some applications of the existence of torse formations on Sasakian and Kenmotsu manifolds. Furthermore, Chen and his co-authors extensively studied the properties of torqued and parallel vector fields, particularly in the context of Ricci solitons ([9], [10], [12]). The Riemannian manifolds with different vector fields have been studied by several authors ([4], [13], [17], [16], [19], [24], [25], [26]). Recently, Chaubey and Suh introduced a new class of Riemannian manifolds known as Riemannian concircular structures manifolds (briefly, $(R_iCS_t)_n$ -manifolds [6]. They also studied of semisymmetric and the

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conformally flat $(R_iCS_t)_n$ -manifolds. Also Chen et al. studied almost Ricci solitons on $(R_iCS_t)_3$ -manifolds [11].

In differential geometry, the Riemann curvature tensor $\mathcal{R}_{cur}$ is the most common way used to express the curvature of Riemannian manifolds. It assigns a tensor to each point of a Riemannian manifold (i.e., it is a tensor field). The nature of a manifold mostly depends on its curvature tensor. Using the tools of conformal transformation geometers have deduced conformal curvature tensor. In the similar way with the help of projective and conharmonical transformations, the notion of projective curvature and conharmonical curvature have been defined [21]. Apart from conformal curvature tensor, the projective curvature tensor and conharmonical curvature tensor are another important tensors from the differential geometric point of view.

A central theme in the analysis of Riemannian manifolds is the classification of spaces based on curvature symmetry conditions. The most basic of these is "flatness" ($\mathcal{R}_{cur} = 0$) and "local symmetry" ($\nabla \mathcal{R}_{cur} = 0$) [23]. Here $\mathcal{R}_{cur}$ denotes the curvature tensor of the manifold M , ∇ is the Levi-Civita connection of the Riemannian metric $\tilde{g}$ and $\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)$ acts as a derivation on $\mathcal{R}_{cur}$ for all vector fields ϑ_1 and ϑ_2 on M . If the manifold M satisfies the condition $\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur} = 0$, then M is called a semisymmetric manifold and M satisfies the condition $\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{S}_{Ric} = 0$, then M is called a Ricci semisymmetric manifold. Hence, every Ricci semisymmetric manifold automatically satisfies the Ricci symmetric condition $\nabla \mathcal{S}_{Ric} = 0$, although the converse does not hold in general. Also every semisymmetric manifold is Ricci semisymmetric, but in general the converse part is not true. The properties of symmetric spaces have been studied by several authors in Riemannian and semi-Riemannian manifolds ([2], [3], [7], [15], [19], [22]). Apart from, using these definitions, if a (semi-)Riemannian manifold satisfying $\mathcal{P}_{cur} \cdot \mathcal{R}_{cur} = 0$ and $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = 0$ is called projective symmetric and projective Ricci symmetric. In an n -dimensional Riemannian manifold $(M, \tilde{g})$ the projective tensor $\mathcal{P}_{cur}$ is defined by

$$\begin{aligned} \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3 &= \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3 \\ &\quad - \frac{1}{n-1} \{ \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\vartheta_1 - \mathcal{S}_{Ric}(\vartheta_1, \vartheta_3)\vartheta_2 \}. \end{aligned}$$

We also define the endomorphism $\vartheta_1 \wedge_{\tilde{A}} \vartheta_2$ of $\chi(M)$ by

$$(\vartheta_1 \wedge_{\tilde{A}} \vartheta_2)\vartheta_3 = \tilde{A}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{A}(\vartheta_1, \vartheta_3)\vartheta_2,$$

where $\tilde{A}$ is a $(0, 2)$ -tensor field and symmetric. For a $(0, k)$ -tensor field $\tilde{T}$, $k < 1$ and a $(0, 2)$ -tensor field $\tilde{A}$ on $(M, \tilde{g})$, we define the tensor $\tilde{Q}(\tilde{A}, \tilde{T})$ by

$$\begin{aligned} \tilde{Q}(\tilde{A}, \tilde{T})(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \dots, \tilde{\vartheta}_k; \vartheta_1, \vartheta_2) &= -\tilde{T}((\vartheta_1 \wedge_{\tilde{A}} \vartheta_2)\tilde{\vartheta}_1, \tilde{\vartheta}_2, \dots, \tilde{\vartheta}_k) - \dots - \\ &\quad -\tilde{T}(\tilde{\vartheta}_1, \tilde{\vartheta}_2, \dots, \tilde{\vartheta}_{k-1}, (\vartheta_1 \wedge_{\tilde{A}} \vartheta_2)\tilde{\vartheta}_k). \end{aligned}$$

In particular, a (semi-)Riemannian manifold satisfying $\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = L_{\mathcal{R}_{cur}} \tilde{Q}(\tilde{g}, \mathcal{R}_{cur})$, $\mathcal{R}_{cur} \cdot \mathcal{S}_{Ric} = L_{\mathcal{S}_{Ric}} \tilde{Q}(\tilde{g}, \mathcal{S}_{Ric})$, $\mathcal{R}_{cur} \cdot \mathcal{P}_{cur} = L_{\mathcal{P}_{cur}} \tilde{Q}(\tilde{g}, \mathcal{P}_{cur})$, $\mathcal{P}_{cur} \cdot \mathcal{R}_{cur} = L_1 \tilde{Q}(\tilde{g}, \mathcal{R}_{cur})$ and $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = L_2 \tilde{Q}(\tilde{g}, \mathcal{S}_{Ric})$ is respectively called pseudosymmetric, Ricci pseudosymmetric, projective pseudosymmetric, projective Ricci pseudosymmetric and Ricci pseudosymmetric due to the projective curvature tensor, where $L_{\mathcal{R}_{cur}}$, $L_{\mathcal{S}_{Ric}}$, $L_{\mathcal{P}_{cur}}$, L_1 , L_2 are the associated scalars.

On the other hand, a Riemannian manifold is flat if and only if its curvature tensor vanishes identically. As is well known that the necessary and sufficient condition for some curvature tensors defined with the help of the curvature tensor to be flat is that these tensors vanish identically. In fact, a Riemannian manifold is conformally flat if each point has a neighborhood that can be mapped to flat space by a conformal transformation. Thus, a Riemannian manifold is conformally flat if and only if its conformal curvature tensor vanishes identically. The projective curvature tensor is an invariant under transformations preserving geodesics on semi-Riemannian manifolds. Moreover, it exhibits geometric features distinct from other generalized curvature tensors. Also a Riemannian manifold is projectively flat (respectively conharmonically flat) if and only if its projective curvature tensor (respectively conharmonic curvature tensor) vanishes identically.

In the year 1987, the notion of quasi-Einstein metric introduced by Koiso [14]. For an n -dimensional Riemannian manifold M if the Ricci tensor $\mathcal{S}_{Ric}$ satisfies

$$\mathcal{S}_{Ric} = \check{a}\check{g} + \check{b}\hat{\eta} \otimes \hat{\eta}, \quad (1.1)$$

where $\check{a}$ and $\check{b}$ are non-zero smooth functions on M and $\hat{\eta}$ is a 1-form ([8], [5]). If the non-vanishing Ricci tensor $\mathcal{S}_{Ric}$ of M satisfies equation (1.1), then the manifold M is referred to as a quasi-Einstein manifold [5]. In deed, if $\check{b} = 0$, then the manifold M reduces to the Einstein manifold, while the Ricci flat manifold is obtained in the particular case $\check{a} = 0$. In [5], Chaki and Maity studied manifolds under the following assumptions: (i) $\check{a}$ and $\check{b}$ are constants and the generator of $(QE)_n$ -manifold is recurrent, and (ii) $\check{a} + \check{b} = 0$, $\vartheta_4 = \frac{1}{2\check{a}}grad\check{a}$ and $\nabla_{\vartheta_1}\vartheta_4 = -\vartheta_1 + A(\vartheta_1)\vartheta_4$, where A is a dual 1-form of ϑ_4 .

The present paper is organized as follows: After introduction in section 2, we provide the basic concepts and properties of $(R_iCS_t)_n$ -manifolds. In section 3, we investigate the flatness and symmetry properties of these manifolds, specifically examining conditions such as projective flatness, projective symmetry, and various forms of pseudosymmetry within the framework of Riemannian concircular structures. Also, we give a non-trivial example.

2. Basic concepts of a $(R_iCS_t)_n$ -manifold

In this section, we present the fundamental definitions and geometric properties of the $(R_iCS_t)_n$ -manifold, as established in the foundational works of Chaubey and Suh [6].

Let M be an n -dimensional Riemannian manifold, $\check{g}$ is a Riemannian metric and ξ a torse-forming vector field on M satisfying

$$(\nabla_{\vartheta_1}\hat{\eta})(\vartheta_2) = \check{\alpha}\check{g}(\vartheta_1, \vartheta_2) + \pi(\vartheta_1)\hat{\eta}(\vartheta_2), \quad \forall \vartheta_1, \vartheta_2 \in \chi_\nu(M), \quad (2.2)$$

where $\hat{\eta}(\cdot) = \check{g}(\cdot, \xi)$ is a 1-form associated with ξ and π is also a 1-form. If 1-form π is closed on M , then ξ is said to be a concircular vector field ([13, 25, 26]). Also, for the torse-forming vector field ξ on M , we can say the followings:

- * if $\pi(\xi) = 0$, then ξ is a torqued vector field [10],
- * if $\pi = 0$, then ξ is a concircular vector field (in Fialkow's sense) [9, 13],
- * if the 1-form π is closed, then ξ is a concircular vector field (in Yano's sense) [25],
- * if $\alpha = 0$, then ξ is a recurrent vector field [20],
- * if $\pi = 0$, then ξ is a concurrent vector field and $\alpha = 1$ [12],

* if $\pi = 0$ and $\alpha = 0$, then ξ is a parallel vector field [12, 13].

Set $\vartheta_2 = \xi$ in the equation (2.2), we find

$$\check{\alpha}\hat{\eta}(\vartheta_1) + \pi(\vartheta_1) = 0, \quad (2.3)$$

Using the equation (2.3) in the equation (2.2), we obtain

$$(\nabla_{\vartheta_1}\hat{\eta})(\vartheta_2) = \check{\alpha}\{\tilde{g}(\vartheta_1, \vartheta_2) - \hat{\eta}(\vartheta_1)\hat{\eta}(\vartheta_2)\}, \quad (2.4)$$

which implies that

$$\nabla_{\vartheta_1}\xi = \check{\alpha}\{\vartheta_1 - \hat{\eta}(\vartheta_1)\xi\}, \quad (2.5)$$

where $\check{\alpha}$ is a non-zero scalar and $\nabla_{\vartheta_1}\check{\alpha} = \tilde{g}(\vartheta_1, \tilde{D}\check{\alpha}) = \vartheta_1(\check{\alpha}) = \mu u(\vartheta_1)$ for a certain smooth function $\tilde{\mu}$ on M . Here $\tilde{D}$ denotes the gradient operator of $\tilde{g}$. From the equation (2.4), it follows directly that the 1-form $\hat{\eta}$ is closed. Taking the covariant derivative of the equation (2.3) along ϑ_2 and using the relation $\vartheta_1(\alpha) = \mu u(\vartheta_1)$ together with the equation (2.4), we infer that π is also closed. Hence, the unit torse-forming vector field ξ defined in the equation (2.2) can be regarded as a unit concircular vector field on M in the sense of Yano.

Let the Riemannian manifold M admit a $(1, 1)$ tensor field ϕ such that

$$\check{\alpha}\phi\vartheta_1 = \nabla_{\vartheta_1}\xi, \quad \check{\alpha} \neq 0,$$

which gives

$$\phi\vartheta_1 = \vartheta_1 - \hat{\eta}(\vartheta_1)\xi, \quad (2.6)$$

where the equation (2.5) is used. Thus, we obtain

$$\phi^2 = I - \hat{\eta} \otimes \xi.$$

In view of the equation (2.6), we have

$$\begin{aligned} \tilde{g}(\phi\vartheta_1, \phi\vartheta_2) &= \tilde{g}(\vartheta_1, \vartheta_2) - \hat{\eta}(\vartheta_1)\hat{\eta}(\vartheta_2), \\ \hat{\eta}(\vartheta_2) &= \tilde{g}(\vartheta_2, \xi). \end{aligned}$$

Also, it follows that if M admits a unit concircular vector field ξ , a $(1, 1)$ tensor field ϕ and a 1-form $\hat{\eta}$, then we get

$$\hat{\eta}(\xi) = 1, \quad \tilde{g}(\phi\cdot, \phi\cdot) = \tilde{g}(\cdot, \cdot) - \hat{\eta} \otimes \hat{\eta}. \quad (2.7)$$

On an n -dimensional Riemannian manifold M , if quadruple $(\phi, \xi, \hat{\eta}, \tilde{g})$ satisfies the equation (2.7), then the manifold M is called $(R_iCS_t)_n$ -manifold. Hence the structure $(\phi, \xi, \hat{\eta}, \tilde{g})$ defines a (R_iCS_t) on M . Also we have the followings:

$$\begin{aligned} \phi\xi &= 0, \quad \hat{\eta}(\phi\vartheta_1) = 0, \quad \text{rank}(\phi) = n - 1, \\ (\nabla_{\vartheta_1}\phi)(\vartheta_2) &= \check{\alpha}[2\hat{\eta}(\vartheta_1)\hat{\eta}(\vartheta_2)\xi - \tilde{g}(\vartheta_1, \vartheta_2)\xi - \hat{\eta}(\vartheta_2)\vartheta_1], \quad \forall \vartheta_1, \vartheta_2 \in \chi_\nu(M). \end{aligned}$$

Also the curvature identities

$$\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\xi = (\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\vartheta_2 - \hat{\eta}(\vartheta_2)\vartheta_1\}, \quad (2.8)$$

and

$$\mathcal{R}_{cur}(\xi, \vartheta_1)\vartheta_2 = (\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_2)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_2)\xi\}, \quad (2.9)$$

hold for any vector fields ϑ_1, ϑ_2 . Using the equations (2.8) and (2.9), we obtain

$$\hat{\eta}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3) = (\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_2)\tilde{g}(\vartheta_1, \vartheta_3) - \hat{\eta}(\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_3)\}, \quad (2.10)$$

and

$$\mathcal{S}_{Ric}(\vartheta_1, \xi) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\hat{\eta}(\vartheta_1), \quad (2.11)$$

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in \chi_\nu(M)$ and $(\check{\alpha}^2 + \tilde{\mu}) \neq 0$ [6].

3. Main results

In this section, we investigate the geometric properties of the $(R_iCS_t)_n$ -manifolds under various curvature conditions, including projective flatness, projective symmetry and pseudosymmetry. Firstly, we consider a projectively flat $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{P}_{cur} = 0, \quad (3.12)$$

where $\mathcal{P}_{cur}$ is the projective curvature tensor of dimension n and is defined by

$$\mathcal{P}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 = \mathcal{R}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 - \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\vartheta_5 - \mathcal{S}_{Ric}(\vartheta_5, \vartheta_4)\vartheta_6\}. \quad (3.13)$$

From (3.12) and (3.13), it follows that

$$\mathcal{R}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 = \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\vartheta_5 - \mathcal{S}_{Ric}(\vartheta_5, \vartheta_4)\vartheta_6\}. \quad (3.14)$$

Putting $\vartheta_5 = \xi$ in (3.14), we have

$$\mathcal{R}_{cur}(\xi, \vartheta_6)\vartheta_4 = \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\xi - \mathcal{S}_{Ric}(\xi, \vartheta_4)\vartheta_6\}. \quad (3.15)$$

Taking the inner product of (3.15) with ϑ_7 , we have

$$\tilde{g}(\mathcal{R}_{cur}(\xi, \vartheta_6)\vartheta_4, \vartheta_7) = \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\hat{\eta}(\vartheta_7) - \mathcal{S}_{Ric}(\xi, \vartheta_4)\tilde{g}(\vartheta_6, \vartheta_7)\}. \quad (3.16)$$

From (3.13), we find

$$\begin{aligned} (\check{\alpha}^2 + \tilde{\mu})\{\tilde{g}(\vartheta_6, \vartheta_7)\hat{\eta}(\vartheta_4) - \tilde{g}(\vartheta_6, \vartheta_4)\hat{\eta}(\vartheta_7)\} &= \frac{1}{n-1}\{\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4)\hat{\eta}(\vartheta_7) \\ &\quad + (n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_6, \vartheta_7)\hat{\eta}(\vartheta_4)\}. \end{aligned} \quad (3.17)$$

Putting $\vartheta_7 = \xi$ in (3.17), we have

$$\mathcal{S}_{Ric}(\vartheta_6, \vartheta_4) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_6, \vartheta_4),$$

which means that the manifold M is an Einstein manifold. Thus we can state the following:

Theorem 3.1. *Let M be a projectively flat $(R_iCS_t)_n$ -manifold. Then the manifold M is an Einstein manifold.*

As is well known that a Riemannian manifold of dimension n , ($n > 3$) is projectively flat if and only if it is of constant curvature $\tilde{c}$. Then we can write

$$\mathcal{R}_{cur}(\vartheta_5, \vartheta_6)\vartheta_4 = \tilde{c}\{\tilde{g}(\vartheta_6, \vartheta_4)\vartheta_5 - \tilde{g}(\vartheta_5, \vartheta_4)\vartheta_6\}. \quad (3.18)$$

Comparing (3.14) and (3.18), we get

$$\tilde{c} = -(\check{\alpha}^2 + \tilde{\mu}),$$

which means that the manifold is of constant curvature $-(\check{\alpha}^2 + \tilde{\mu})$. Since $\check{\alpha}$ is constant scalar then the smooth function $\tilde{\mu}$ must be a constant. Thus we have the following:

Corollary 3.1. *A $(R_iCS_t)_n$ -manifold is projectively flat if and only if the smooth function $\tilde{\mu}$ is a constant.*

Now we consider the condition $\mathcal{P}_{cur} \cdot \mathcal{R}_{cur} = 0$ on a $(R_iCS_t)_n$ -manifold. Thus we have

$$(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4 = 0,$$

which turns to

$$\begin{aligned} & \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 = 0. \end{aligned} \quad (3.19)$$

Putting $\vartheta_1 = \xi$ in (3.19), we obtain

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2 \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\xi + \tilde{g}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4)\xi \right. \\ & - \tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4)\xi + \tilde{g}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\vartheta_5 - \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\vartheta_3 \\ & - \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\vartheta_3 + \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\vartheta_5 \} \\ & + \frac{(\check{\alpha}^2 + \tilde{\mu})}{n-1} \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\xi - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4)\xi \right. \\ & + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\vartheta_5 - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\vartheta_3 + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4)\xi \\ & \left. - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\vartheta_3 + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\vartheta_5 \right\} = 0. \end{aligned} \quad (3.20)$$

Taking the inner product of (3.20) with ξ , we have

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2 \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4) - \tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4) \right. \\ & + \tilde{g}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_5) - \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_3) + \tilde{g}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4) \\ & - \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\hat{\eta}(\vartheta_3) + \tilde{g}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\hat{\eta}(\vartheta_5) \} \\ & + \frac{(\check{\alpha}^2 + \tilde{\mu})}{n-1} \left\{ -\frac{1}{(\check{\alpha}^2 + \tilde{\mu})} \mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4) - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_5, \vartheta_4) \right. \\ & + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_5) - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4)\hat{\eta}(\vartheta_3) \\ & \left. + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\tilde{g}(\vartheta_3, \vartheta_4) - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_5)\hat{\eta}(\vartheta_3) + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\hat{\eta}(\vartheta_3)\hat{\eta}(\vartheta_5) \right\} = 0. \end{aligned} \quad (3.21)$$

Putting $\vartheta_3 = \xi$ in (3.21), we have

$$(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\xi, \vartheta_5)\vartheta_4) + \frac{1}{n-1}\mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\xi, \vartheta_5)\vartheta_4) = 0.$$

Using the equation (2.9), we get

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2 \{ \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4) - \tilde{g}(\vartheta_4, \vartheta_5)\hat{\eta}(\vartheta_2) \} \\ & + \frac{1}{n-1}(\check{\alpha}^2 + \tilde{\mu}) \{ \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_4) - \mathcal{S}_{Ric}(\vartheta_4, \vartheta_5)\hat{\eta}(\vartheta_2) \} = 0. \end{aligned} \quad (3.22)$$

Putting $\vartheta_4 = \xi$ in (3.22), we have

$$\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_5)$$

which means that the manifold is an Einstein manifold. Thus we can state the following:

Theorem 3.2. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{P} \cdot \mathcal{R} = 0$. Then the manifold M is an Einstein manifold.*

Now we consider the condition $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = 0$ on a $(R_iCS_t)_n$ -manifold. Thus we have

$$(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{S}_{Ric})(\vartheta_3, \vartheta_5) = 0,$$

which turns to

$$\mathcal{S}_{Ric}(\mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, \mathcal{P}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5) = 0. \quad (3.23)$$

Putting $\vartheta_1 = \xi$ in (3.23), we have

$$\mathcal{S}_{Ric}(\mathcal{P}_{cur}(\xi, \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, \mathcal{P}_{cur}(\xi, \vartheta_2)\vartheta_5) = 0. \quad (3.24)$$

Using (3.13) in (3.24), we obtain

$$\begin{aligned} & (\alpha^2 + \tilde{\mu})\{\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_5) + \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_3)\} \\ & + (n-1)(\alpha^2 + \tilde{\mu})^2\{\tilde{g}(\vartheta_2, \vartheta_3)\hat{\eta}(\vartheta_5) + \tilde{g}(\vartheta_2, \vartheta_5)\hat{\eta}(\vartheta_3)\} = 0. \end{aligned} \quad (3.25)$$

Putting $\vartheta_5 = \xi$ in (3.25), we get

$$\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3) = -(n-1)(\alpha^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_3),$$

which means that the manifold is an Einstein manifold. Thus we can state the following:

Theorem 3.3. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{P}_{cur} \cdot \mathcal{S}_{Ric} = 0$. Then the manifold M is an Einstein manifold.*

Now we consider pseudo-symmetric $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = \mathcal{L}_{\mathcal{R}_{cur}}Q(\tilde{g}, \mathcal{R}_{cur})$$

which turns to

$$(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4 = \mathcal{L}_{\mathcal{R}_{cur}}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4,$$

and

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_{\mathcal{R}_{cur}}\{(\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, (\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)(\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_4\}, \end{aligned}$$

or,

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_{\mathcal{R}_{cur}}\{\tilde{g}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_1 - \tilde{g}(\vartheta_1, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_2 \\ & - \tilde{g}(\vartheta_2, \vartheta_3)\mathcal{R}_{cur}(\vartheta_1, \vartheta_5)\vartheta_4 + \tilde{g}(\vartheta_1, \vartheta_3)\mathcal{R}_{cur}(\vartheta_2, \vartheta_5)\vartheta_4 \\ & - \tilde{g}(\vartheta_2, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_1)\vartheta_4 + \tilde{g}(\vartheta_1, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_2)\vartheta_4 \\ & - \tilde{g}(\vartheta_2, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_1 + \tilde{g}(\vartheta_1, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\} \end{aligned} \quad (3.26)$$

Putting $\vartheta_1 = \vartheta_4 = \xi$ in (3.26), we get

$$(\check{\alpha}^2 + \tilde{\mu})(1 + \mathcal{L}_{\mathcal{R}_{cur}})\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_5 - \tilde{g}(\vartheta_2, \vartheta_5)\vartheta_3\} = 0,$$

which means that

$$\mathcal{L}_{\mathcal{R}_{cur}} = -1.$$

Thus we can state the following:

Theorem 3.4. *Every $(R_iCS_t)_n$ -manifold M is a pseudo-symmetric manifold satisfying*

$$\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = -Q(\tilde{g}, \mathcal{R}_{cur}).$$

Now we consider pseudo-symmetric $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = \mathcal{L}_q Q(\mathcal{S}_{Ric}, \mathcal{R}_{cur}),$$

which turns to

$$(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4 = \mathcal{L}_q((\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2) \cdot \mathcal{R}_{cur})(\vartheta_3, \vartheta_5)\vartheta_4,$$

and

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_q\{(\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}((\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, (\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)(\vartheta_1 \wedge_{\mathcal{S}_{Ric}} \vartheta_2)\vartheta_4\}, \end{aligned}$$

or,

$$\begin{aligned} & \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5)\vartheta_4 \\ & - \mathcal{R}_{cur}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5)\vartheta_4 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_4 \\ = & \mathcal{L}_q\{\mathcal{S}_{Ric}(\vartheta_2, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_1 - \mathcal{S}_{Ric}(\vartheta_1, \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_4)\vartheta_2 \\ & - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\mathcal{R}_{cur}(\vartheta_1, \vartheta_5)\vartheta_4 + \mathcal{S}_{Ric}(\vartheta_1, \vartheta_3)\mathcal{R}_{cur}(\vartheta_2, \vartheta_5)\vartheta_4 \\ & - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_1)\vartheta_4 + \mathcal{S}_{Ric}(\vartheta_1, \vartheta_5)\mathcal{R}_{cur}(\vartheta_3, \vartheta_2)\vartheta_4 \\ & - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_1 + \mathcal{S}_{Ric}(\vartheta_1, \vartheta_4)\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\}. \end{aligned} \quad (3.27)$$

Putting $\vartheta_1 = \vartheta_4 = \xi$ in (3.27), we get

$$\begin{aligned} & (\check{\alpha}^2 + \tilde{\mu})^2\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_5 - \tilde{g}(\vartheta_2, \vartheta_5)\vartheta_3 - \mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\} \\ = & \mathcal{L}_q(\check{\alpha}^2 + \tilde{\mu})\{\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5)\vartheta_3 - \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\vartheta_5 - (n-1)(\check{\alpha}^2 + \tilde{\mu})\mathcal{R}_{cur}(\vartheta_3, \vartheta_5)\vartheta_2\}. \end{aligned} \quad (3.28)$$

Again putting $\vartheta_3 = \xi$ and taking the inner product with ξ in (3.28), we obtain

$$\begin{aligned} \mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) &= \frac{-(n-1)(\alpha^2 + \tilde{\mu})\mathcal{L}_{\check{\alpha}} - (1 - \check{\alpha}^2 - \tilde{\mu})}{\mathcal{L}_q}\tilde{g}(\vartheta_2, \vartheta_5) \\ &+ \frac{(1 - \check{\alpha}^2 - \tilde{\mu})}{\mathcal{L}_q}\hat{\eta}(\vartheta_2)\hat{\eta}(\vartheta_5), \end{aligned}$$

which means that the manifold M is an $\hat{\eta}$ -Einstein manifold. Thus we can state the following:

Theorem 3.5. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{R}_{cur} \cdot \mathcal{R}_{cur} = \mathcal{L}_q Q(\mathcal{S}_{Ric}, \mathcal{R}_{cur})$. Then the manifold M is an $\hat{\eta}$ -Einstein manifold.*

Now we consider pseudo-Riccisymmetric $(R_iCS_t)_n$ -manifold, which satisfies the condition

$$\mathcal{R}_{cur} \cdot \mathcal{S}_{Ric} = \mathcal{L}_{\mathcal{S}_{Ric}} Q(\tilde{g}, \mathcal{S}_{Ric}),$$

which turns to

$$(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2) \cdot \mathcal{S}_{Ric})(\vartheta_3, \vartheta_5) = \mathcal{L}_{\mathcal{S}_{Ric}}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2) \cdot \mathcal{S}_{Ric})(\vartheta_3, \vartheta_5),$$

and

$$\begin{aligned} & \mathcal{S}_{Ric}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_5) \\ &= \mathcal{L}_{\mathcal{S}}\{\mathcal{S}_{Ric}((\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_3, \vartheta_5) + \mathcal{S}_{Ric}(\vartheta_3, (\vartheta_1 \wedge_{\tilde{g}} \vartheta_2)\vartheta_5)\}. \end{aligned} \quad (3.29)$$

Putting $\vartheta_1 = \xi$ in (3.29), we get

$$\begin{aligned} & (\alpha^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_3)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) - \tilde{g}(\vartheta_2, \vartheta_3)\mathcal{S}_{Ric}(\xi, \vartheta_5) \\ & + \hat{\eta}(\vartheta_5)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3) - \tilde{g}(\vartheta_2, \vartheta_5)\mathcal{S}_{Ric}(\xi, \vartheta_3)\} \\ &= \mathcal{L}_{\mathcal{S}_{Ric}}\{\tilde{g}(\vartheta_2, \vartheta_3)\mathcal{S}_{Ric}(\xi, \vartheta_5) - \hat{\eta}(\vartheta_3)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) \\ & + \tilde{g}(\vartheta_2, \vartheta_5)\mathcal{S}_{Ric}(\xi, \vartheta_3) - \hat{\eta}(\vartheta_5)\mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\}. \end{aligned} \quad (3.30)$$

Again putting $\vartheta_3 = \xi$ in (3.30), we have

$$(\check{\alpha}^2 + \tilde{\mu} + \mathcal{L}_{\mathcal{S}_{Ric}})\{\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) + (n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_5)\} = 0.$$

Since $\alpha^2 + \tilde{\mu} + \mathcal{L}_{\mathcal{S}_{Ric}} \neq 0$, we have

$$\mathcal{S}_{Ric}(\vartheta_2, \vartheta_5) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_2, \vartheta_5),$$

which means that the manifold M is an Einstein manifold. Thus we have the following:

Theorem 3.6. *Let M be a $(R_iCS_t)_n$ -manifold satisfying the condition $\mathcal{R}_{cur} \cdot \mathcal{S}_{Ric} = \mathcal{L}_{\mathcal{S}_{Ric}} Q(\tilde{g}, \mathcal{S}_{Ric})$. Then the manifold M is an Einstein manifold.*

Let M be a ϕ -symmetric $(R_iCS_t)_n$ -manifold which is satisfying the condition

$$\phi^2((\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = 0. \quad (3.31)$$

Then we have

$$-(\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3) + \hat{\eta}((\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = 0. \quad (3.32)$$

Putting $\vartheta_3 = \xi$ in (3.32), we have

$$\begin{aligned} (\nabla_{\vartheta_4} \mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \xi) &= \nabla_{\vartheta_4} \mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \xi) - \mathcal{R}_{cur}(\nabla_{\vartheta_4} \vartheta_1, \vartheta_2, \xi) \\ &\quad - \mathcal{R}_{cur}(\vartheta_1, \nabla_{\vartheta_4} \vartheta_2, \xi) - \mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \nabla_{\vartheta_4} \xi), \end{aligned}$$

which turns to

$$\begin{aligned} & \nabla_{\vartheta_4}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\vartheta_2 - \hat{\eta}(\vartheta_2)\vartheta_1\} \\ & + (\check{\alpha}^2 + \tilde{\mu})\{\nabla_{\vartheta_4}(\hat{\eta}(\vartheta_1))\vartheta_2 + \hat{\eta}(\vartheta_1) \nabla_{\vartheta_4} \vartheta_2 \\ & - \nabla_{\vartheta_4}(\hat{\eta}(\vartheta_2))\vartheta_1 - \hat{\eta}(\nabla_{\vartheta_4} \vartheta_1)\vartheta_2\} \\ & - \check{\alpha}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\vartheta_2 - \hat{\eta}(\vartheta_2)\vartheta_1\} - \check{\alpha}\mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \vartheta_4) = 0. \end{aligned} \quad (3.33)$$

Taking the inner product with ϑ_5 in (3.33), we get

$$\begin{aligned} & \nabla_{\vartheta_4}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_5) - \hat{\eta}(\vartheta_2)\tilde{g}(\vartheta_1, \vartheta_5)\} \\ & + (\check{\alpha}^2 + \tilde{\mu})\{\nabla_{\vartheta_4}(\hat{\eta}(\vartheta_1))\tilde{g}(\vartheta_2, \vartheta_5) + \hat{\eta}(\vartheta_1)\tilde{g}(\nabla_{\vartheta_4}\vartheta_2, \vartheta_5) \\ & - \nabla_{\vartheta_4}(\hat{\eta}(\vartheta_2))\tilde{g}(\vartheta_1, \vartheta_5) - \hat{\eta}(\nabla_{\vartheta_4}\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_5)\} \\ & - \check{\alpha}(\check{\alpha}^2 + \tilde{\mu})\{\hat{\eta}(\vartheta_1)\tilde{g}(\vartheta_2, \vartheta_5) - \hat{\eta}(\vartheta_2)\tilde{g}(\vartheta_1, \vartheta_5)\} - \check{\alpha}\tilde{g}(\mathcal{R}_{cur}(\vartheta_1, \vartheta_2, \vartheta_4), \vartheta_5) = 0. \end{aligned} \quad (3.34)$$

Putting $\vartheta_2 = \vartheta_5 = e_i$ in (3.34), we obtain

$$\mathcal{S}_{Ric}(\vartheta_1, \vartheta_4) = -(n-1)(\check{\alpha}^2 + \tilde{\mu})\tilde{g}(\vartheta_1, \vartheta_4)$$

which means that the manifold M is an Einstein manifold. So we have the following:

Theorem 3.7. *Let M be a ϕ -symmetric $(R_iCS_t)_n$ -manifold. Then the manifold M is an Einstein manifold.*

Now let M be a locally ϕ -symmetric $(R_iCS_t)_3$ -manifold which is satisfying the condition $\phi^2((\nabla_{\vartheta_4}\mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = 0$, which means that all vector fields are orthogonal to ξ . Firstly as is well known that if the manifold M is of dimension 3, we always have

$$\begin{aligned} \mathcal{R}_{cur}(\vartheta_1, \vartheta_2)\vartheta_3 &= \mathcal{S}_{Ric}(\vartheta_2, \vartheta_3)\vartheta_1 - \mathcal{S}_{Ric}(\vartheta_1, \vartheta_3)\vartheta_2 \\ &+ \tilde{g}(\vartheta_2, \vartheta_3)Q\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)Q\vartheta_2 - \frac{r}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\}. \end{aligned} \quad (3.35)$$

Using (3.35) in (3.32), we get

$$\begin{aligned} & \tilde{g}((\nabla_{\vartheta_4}Q)\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}((\nabla_{\vartheta_4}Q)\vartheta_1, \vartheta_3)\vartheta_2 \\ & + \tilde{g}(\vartheta_2, \vartheta_3)(\nabla_{\vartheta_4}Q)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)(\nabla_{\vartheta_4}Q)\vartheta_2 \\ & - \frac{\vartheta_4(r)}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\} = 0. \end{aligned}$$

From the last equation, we obtain

$$\begin{aligned} & \frac{\vartheta_4(r)}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\} \\ & - (\frac{r}{2} + 2(\check{\alpha}^2 + \tilde{\mu}))\check{\alpha}\{\tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_1, \vartheta_4)\xi - \tilde{g}(\vartheta_1, \vartheta_3)\tilde{g}(\vartheta_2, \vartheta_4)\xi\}, \end{aligned} \quad (3.36)$$

and

$$\hat{\eta}((\nabla_{\vartheta_4}\mathcal{R}_{cur})(\vartheta_1, \vartheta_2, \vartheta_3)) = -(\frac{r}{2} + 2(\check{\alpha}^2 + \tilde{\mu}))\check{\alpha}\{\tilde{g}(\vartheta_2, \vartheta_3)\tilde{g}(\vartheta_1, \vartheta_4) - \tilde{g}(\vartheta_1, \vartheta_3)\tilde{g}(\vartheta_2, \vartheta_4)\}, \quad (3.37)$$

respectively. Thus from (3.36) and (3.37), we have

$$\frac{\vartheta_4(r)}{2}\{\tilde{g}(\vartheta_2, \vartheta_3)\vartheta_1 - \tilde{g}(\vartheta_1, \vartheta_3)\vartheta_2\} = 0,$$

which means that $\vartheta_4(r) = 0$. Thus, the scalar curvature r is a constant. So, we have the following:

Theorem 3.8. *A $(R_iCS_t)_3$ -manifold is locally ϕ -symmetric if and only if the scalar curvature r is constant.*

Now, we give a non-trivial example of a $(R_iCS_t)_3$ -manifold.

Example 3.1. We consider the three dimensional manifold $M = \{(x, y, z) \in \mathbb{R}^3 : z \neq 0\}$ where (x, y, z) are standard coordinates in $\mathbb{R}^3$. The vector fields

$$\theta_1 = e^{-f(z)} \frac{\partial}{\partial x}, \quad \theta_2 = e^{-f(z)} \frac{\partial}{\partial y}, \quad \theta_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M . Here, $f(z)$ is a non-vanishing smooth function of z and $f'(z) \neq 0$. Suppose that the associated metric $\tilde{g}$ of M^3 is defined by

$$\tilde{g} = e^{2f(z)} \{dx \otimes dx + dy \otimes dy\} + dz \otimes dz$$

and

$$\tilde{g}(\theta_1, \theta_1) = \tilde{g}(\theta_2, \theta_2) = \tilde{g}(\theta_3, \theta_3) = 1,$$

$$\tilde{g}(\theta_1, \theta_2) = \tilde{g}(\theta_1, \theta_3) = \tilde{g}(\theta_2, \theta_3) = 0.$$

Since the metric $\tilde{g}$ is a Riemannian metric, thus M equipped with $\tilde{g}$ is a three dimensional Riemannian manifold. Let u be a 1-form associated with the unit vector field $\xi = \theta_3$ such that $\tilde{g}(\theta_i, \theta_3) = \tilde{u}(\theta_i)$. Then we have

$$\nabla_{\theta_i} \theta_3 = f'(z) \{\theta_i - \tilde{g}(\theta_i, \xi) \xi\}, \quad (3.38)$$

for $i = 1, 2, 3$, where ∇ is the Levi-Civita connection of the metric $\tilde{g}$. This infers that $(M, \tilde{g})$ admits a concircular vector field ξ . Let φ is a $(1, 1)$ -tensor field on M such that $\phi(\theta_1) = \theta_1$, $\phi(\theta_2) = \theta_2$, $\phi(\theta_3) = 0$ and $f'(z)\phi\theta_i = \nabla_{\theta_i} \theta_3$, then by straight forward calculations we notice that the relations $\phi^2 = I - \tilde{u} \otimes \theta_3$, $\tilde{g}(\theta_3, \theta_3) = 1$ and $\tilde{g}(\phi \cdot, \phi \cdot) = \tilde{g}(\cdot, \cdot) - \tilde{u} \otimes \tilde{u}$ for all $\theta_i, i = 1, 2, 3$ hold on M . Here I is the identity transformation [6]. Now, by direct calculations, we have

$$[\theta_1, \theta_2] = 0, \quad [\theta_1, \theta_3] = f'(z)\theta_1, \quad [\theta_2, \theta_3] = f'(z)\theta_2.$$

Also, non-zero components of ∇ on M are

$$\nabla_{\theta_1} \theta_1 = -f'(z)\theta_3, \quad \nabla_{\theta_1} \theta_3 = f'(z)\theta_1,$$

$$\nabla_{\theta_2} \theta_2 = -f'(z)\theta_3, \quad \nabla_{\theta_2} \theta_3 = f'(z)\theta_2.$$

From (3.38), we conclude that M endowed with the structure $(\phi, \theta_3, \tilde{u}, \tilde{g})$ is a three-dimensional Riemannian concircular structure manifold.

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RELATIVE UNIFORM CONVERGENCE OF SEQUENCE OF FUNCTIONS RELATED TO ℓ_p - SPACE

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Abstract. The concept of relative uniform convergence of sequence of Sargent type functions spaces $m(\phi, ru)$ and $n(\phi, ru)$ are defined in this article. Here we established a bounded, p -absolutely summable, convergent, null sequence of functions using the concept of relative uniform convergent for the scale function and these are denoted by $\ell_\infty(ru)$, $\ell_p(ru)$, $c(ru)$, $c_0(ru)$ respectively. We established a relationship between the new notion $m(\phi, ru)$, $n(\phi, ru)$, and other known concepts. Also, we established some algebraic and geometrical characteristics. Many geometric concepts and difficulties may be solved through an understanding of the characteristics of these special spaces.

Keywords: Relative uniform convergent, BK-space, Convergence free, Solidness, modular, convexity

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1. INTRODUCTION

Sequence spaces have undergone several types of research over the years. The symbols ω , ℓ_∞ , c , c_0 , and ℓ_p are used to indicate the classes of all, bounded, convergent, null, and p -absolutely summable sequences of real or complex terms, respectively. Throughout the article ω_f , Z_f represent the space of all sequences of functions and the subspace of sequences of functions space respectively. Several researchers have contributed significantly to the study of sequence spaces and summability theory, including Cai et al. [2], Ibrahim et al. [10], Ibrahim et al. [11], Ibrahim et al. [12], and Bayram et al. [1], among others.

The study of different modes of convergence plays a central role in functional analysis and sequence space theory. Beyond classical notions such as pointwise, uniform, and norm convergence, relative uniform convergence has proved to be an important intermediate concept, particularly in spaces of functions and sequences. Introduced to capture uniform behavior with respect to a controlling function, relative uniform convergence provides a flexible framework that is weaker than uniform convergence but stronger than pointwise convergence. This notion has found applications in summability theory, approximation theory, and the study of locally convex and modular spaces.

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Inspired by this idea, several sequence spaces related to relative uniform convergence have been studied. These spaces enable the analysis of convergence qualities that are not sufficiently characterized by classical norms alone, as well as the behavior of sequences under scaling by a positive control function. In this context, sequence spaces defined via relative uniform convergence offer a natural setting for extending results from classical spaces such as ℓ_p and ℓ_∞ .

The idea of relative uniform convergence in relation to a scale function was first proposed by Moore [14]. The concept of relative uniform convergence of a sequence of functions was further developed by Chittenden [3] in his formulation. Several other researchers conducted additional research on the concept, such as Demirci et al. [5], Demirci and Orhan [6], Sahin and Dirik [16], Devi and Tripathy ([7], [8]), Debbarma et al. [4], Gurdal et al. [9] and so on. The space $m(\phi)$ generalizes several well-known sequence spaces and provides a unified framework for studying boundedness, summability, and geometric properties. By appropriate choices of the sequence ϕ , the space $m(\phi)$ reduces to familiar spaces, while in the general case it exhibits richer structural and topological behaviour.

The space $m(\phi)$, which is related to ℓ_p space, was made in 1960 by Sargent [17], he also investigated a few $m(\phi)$ space properties. Later, it was examined from the perspective of sequence space, and Rath and Tripathy [15], Tripathy [18], and Tripathy and Sen [19] characterized a number of matrix classes with one member as $m(\phi)$.

The purpose of the present work is to study the structural and topological properties of the space $m(\phi, ru)$. We specifically investigate its normed structure, convergence behavior under relative uniform convergence, and embedding connections with classical sequence spaces. Our findings deepen our knowledge of relative uniform convergence in contemporary functional analysis and build upon previous research on generalized sequence spaces.

2. PRELIMINARIES

Definition 2.1. *Given a compact domain D , a sequence of functions (ζ_n) converges relatively uniformly to a function ζ , if for every positive number C , there exists an integer n_C such that, for every $n \geq n_C$, the following inequality holds:*

$$|\zeta(x) - \zeta_n(x)| \leq C|\mu(x)|.$$

The function $\mu(x)$ of the definition above is called scale function which is on the same compact domain D .

According to Moore [14], Chittenden developed the concept of relative uniform convergence sequence of functions that was shown above.

Example 2.1. *Let $[0, 1]$ be a compact domain in $\mathbb{R}$ and consider the sequence of functions $\zeta_n : [0, 1] \rightarrow \mathbb{R}$, for $n \in \mathbb{N}$ defined by*

$$\zeta_n(x) = \begin{cases} \frac{1}{n^2x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Consequently, it is evident that (ζ_n) does not converge uniformly, but rather converges somewhat uniformly to the zero function in relation to a scale function $\mu(x)$ defined by

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Kizmaz [13] proposed the idea of difference sequence space, he also explored several properties of the sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$, and $c_0(\Delta)$. The difference sequence, $\Delta x = (\Delta x_n)$, is defined as $\Delta x_n = x_n - x_{n+1}, n \in \mathbb{N}$.

Throughout, the space ξ_s will represent the class of all subsets of $\mathbb{N}$, those contain a maximum of s elements and the space Φ consists of all non-decreasing sequences $\phi = (\phi_n)$ of real numbers, such that for every positive integer n , $0 < \phi_1 \leq \phi_n \leq \phi_{n+1} < \infty$ and $(n+1)\phi_n \geq n\phi_{n+1}$. i.e.,

$$\frac{\phi_n}{n} \geq \frac{\phi_{n+1}}{n+1}, \text{ for all } n \in \mathbb{N}. \quad (2.1)$$

Example 2.2. (1) Let $\phi_n = 1$, for all $n \in \mathbb{N}$, then $(\phi_n) \in \Phi$.

(2) Let $\phi_n = n$, for all $n \in \mathbb{N}$, then $(\phi_n) \in \Phi$.

According to Sargent [17], the spaces $m(\phi)$ and $n(\phi)$ are defined as follows,

Definition 2.2. The class of sequence $m(\phi)$ is defined by

$$m(\phi) = \left\{ (x_n) \in \omega : \sup_{s \geq 1, \sigma \in \xi_s, x \in D} \frac{1}{\phi_s} \sum_{n \in \sigma} |x_n| < \infty \right\}.$$

The class of sequence $n(\phi)$ is defined by,

$$n(\phi) = \left\{ (x_n) \in \omega : \sup_{u \in \Lambda(x_n)} \sum_{n=1}^{\infty} |u_n| \Delta \phi_n < \infty \right\},$$

where $\Delta \phi_n = \phi_n - \phi_{n-1}$, for all $n \in \mathbb{N}$.

Definition 2.3. Let $\phi = (\phi_n)$ be a sequence of real numbers satisfying equation (2.1). Then the following inequality holds if a sequence of functions (ζ_n) constructed on a compact domain D converge relative uniformly to a limit function ζ for each tiny positive value $C > 0$.

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x) - \zeta(x)| < C|\mu(x)|,$$

where $\mu(x)$ represents the scale function defined in $D/\mu^{-1}(0)$.

The classes of all relative uniform convergence sequences of Sargent-type function are denoted by $m(\phi, ru)$ and $n(\phi, ru)$, respectively are defined as,

$$m(\phi, ru) = \left\{ (\zeta_n) \in \omega_f : \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| < C|\mu(x)| \right\},$$

we also introduce another class of sequence of functions:

$$n(\phi, ru) = \left\{ (\zeta_n) \in \omega_f : \sup_{(\eta_n) \in S(\zeta)} \sum_{n=1}^{\infty} |\eta_n(x)| \Delta \phi_n < C|\mu(x)| \right\},$$

where, $S(\zeta)$ represent the rearrangement of (ζ_n) .

Example 2.3. let $\phi_n = n$, for all $n \in \mathbb{N}$. Consider a sequence of function (ζ_n) defined on the range $[0, 1]$ by,

$$\zeta_n(x) = \begin{cases} \frac{n^2 x}{1+n^2 x^2}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Now,

$$m(\phi, ru) = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| \geq \frac{x}{s(1+x^2)},$$

which is not convergent uniformly. However, it is convergent relative uniformly if we consider a scale function defined by,

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then, $(\zeta_n) \in m(\phi, ru)$ w.r.t. the scale function $\mu(x)$.

Example 2.4. Consider, $\phi_n = n$, for all $n \in \mathbb{N}$. The function (ζ_n) defined on the interval $[0, 1]$ is considered by,

$$\zeta_n(x) = \begin{cases} \frac{1}{n^2 x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Now,

$$\sup_{(\eta_n) \in S(\zeta)} \sum_{n=1}^{\infty} |\eta_n(x)| \Delta \phi_n \geq \frac{1}{x}.$$

Where, (η_n) is the rearrangement of (ζ_n) , which does not exist, as $x \rightarrow 0$. So, it is not uniformly convergent. However, it is relatively uniformly convergent for a scaling function that is specified by

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Then, $(\zeta_n) \in n(\phi, ru)$.

Definition 2.4. The norm of the space $m(\phi, ru)$ is defined as follows,

$$\|\zeta\|_{m(\phi, ru)} = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|}.$$

The following well-known relative uniform convergence sequence of function spaces will be used in this article:

$$\begin{aligned} \ell_p(ru) &= \left\{ (\zeta_n) \in \omega_f : \sum_{n=1}^{\infty} |\zeta_n(x)|^p < C|\mu(x)| \right\} \\ \ell_1(ru) &= \left\{ (\zeta_n) \in \omega_f : \sum_{n=1}^{\infty} |\zeta_n(x)| < C|\mu(x)| \right\}, \text{ for } p = 1 \\ \ell_{\infty}(ru) &= \left\{ (\zeta_n) \in \omega_f : \sup_{n \geq 1} |\zeta_n(x)| < C|\mu(x)| \right\} \\ c(ru) &= \left\{ (\zeta_n) \in \omega_f : \frac{|\zeta_n(x)| - L}{|\mu(x)|} \rightarrow 0, \text{ as } n \rightarrow \infty \right\} \\ c_0(ru) &= \left\{ (\zeta_n) \in \omega_f : \frac{|\zeta_n(x)|}{|\mu(x)|} \rightarrow 0, \text{ as } n \rightarrow \infty \right\}. \end{aligned}$$

Note: If $\phi_n = 1$, $n \in \mathbb{N}$, then $m(\phi, ru) = \ell_1(ru)$; if $\phi_n = n$, $n \in \mathbb{N}$, then $m(\phi, ru) = \ell_{\infty}(ru)$.

Definition 2.5. A subset $Z_f \subset \omega_f$ is considered solid or normal if for every $n \in \mathbb{N}$, $|\eta_n(x)| \leq |\zeta_n(x)|$, implies $(\eta_n) \in Z_f$.

Definition 2.6. Convergence-free subsets are defined as $Z_f \subset \omega_f$ if $(\zeta_n) \in Z_f$, then $\zeta_n(x) = 0 \implies \eta_n(x) = 0$, on $x \in D$ and $(\eta_n) \in Z_f$ if $\zeta_n(x) \neq 0 \implies \eta_n(x)$ may be anything.

Definition 2.7. Suppose $Z_f \subset \omega_f$ is a linear domain. Then $\mathcal{M} : Z_f \rightarrow [0, \infty)$ is a modular in a real sequence of functions space if for $(\zeta_n), (\eta_n) \in Z_f$, the following conditions are hold.

- (1) $\mathcal{M}_{Z_f}(\zeta_n(x)) = 0$ iff $\zeta_n(x) = \bar{\theta}$, where $\bar{\theta}$ represents null sequence of function, for all $x \in Z \subseteq \mathbb{R}$.
- (2) $\mathcal{M}_{Z_f}(-\zeta_n(x)) = \mathcal{M}_{Z_f}(\zeta_n(x))$, for all $x \in Z \subseteq \mathbb{R}$.
- (3) $\mathcal{M}_{Z_f}(\lambda \zeta_n(x)) = \mathcal{M}_{Z_f}(\zeta_n(x))$, for all the scalar $\lambda = 1$.
- (4) $\mathcal{M}_{Z_f}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \leq \mathcal{M}_{Z_f}(\zeta_n(x)) + \mathcal{M}_{Z_f}(\eta_n(x))$,
for all $(\zeta_n), (\eta_n) \in Z_f$ and $\lambda_1, \lambda_2 \geq 0, |\lambda_1| + |\lambda_2| = 1$.
Farther the modular $\mathcal{M}$ is convex if
- (5) $\mathcal{M}_{Z_f}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \leq \lambda_1 \mathcal{M}_{Z_f}(\zeta_n(x)) + \lambda_2 \mathcal{M}_{Z_f}(\eta_n(x))$,
for all $(\zeta_n), (\eta_n) \in Z_f$ and $\lambda_1, \lambda_2 \geq 0, |\lambda_1| + |\lambda_2| = 1$. The modular $\mathcal{M}$ is called $\tilde{s}$ -convex if
- (6) $\mathcal{M}_{Z_f}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \leq \lambda_1^{\tilde{s}} \mathcal{M}_{Z_f}(\zeta_n(x)) + \lambda_2^{\tilde{s}} \mathcal{M}_{Z_f}(\eta_n(x))$, for all $\lambda_1, \lambda_2 \geq 0, \lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1, 0 < \tilde{s} \leq 1$.

Definition 2.8. A sequence of functions $(\zeta_n) \in \mathcal{M}_{Z_f}$ is modular convergent to $\zeta \in \mathcal{M}_{Z_f}$ occurs if there is a $\lambda > 0$ such that $\mathcal{M}(\lambda \zeta_n(x)) - \zeta(x) \rightarrow 0$ as $n \rightarrow \infty$, for all $x \in D$.

Definition 2.9. In the normed space $m(\phi, ru)$, the unit ball and the unit sphere are defined by

$$\begin{aligned} B_{m(\phi, ru)} &= \left\{ (\zeta_n) \in m(\phi, ru) : \|\zeta_n\|_{m(\phi, ru)} = 1 \right\} \\ S_{m(\phi, ru)} &= \left\{ (\zeta_n) \in m(\phi, ru) : \|\zeta_n\|_{m(\phi, ru)} = 1 \right\}. \end{aligned}$$

Definition 2.10. A Banach space is considered uniformly convex if, for any $\varepsilon > 0$, such that for $(\zeta_n), (\eta_n) \in S_{Z_f}$, where S_{Z_f} represents the unit sphere. There exists a $\mathfrak{U}(\varepsilon) > 0$ such that,

$$\|\zeta_n\|_{Z_f} = \|\eta_n\|_{Z_f} = 1, \|\zeta_n - \eta_n\|_{Z_f} \geq \varepsilon \implies \left\| \frac{\zeta_n + \eta_n}{2} \right\|_{Z_f} \leq 1 - \mathfrak{U}(\varepsilon).$$

The purpose of this research is to investigate this form of convergence of a sequence of the Sargent type-related function and investigate some of its topological and geometrical characteristics.

3. MAIN RESULTS

This section establishes the article's results.

Theorem 3.1. The relative uniform convergent sequence space of functions $m(\phi, ru)$ is a liner space.

Proof. Let (ζ_n) and $(\eta_n) \in m(\phi, ru)$, then for $C_1, C_2 > 0$, we have

$$\begin{aligned} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| &< C_1 |\mu(x)|, \text{ for all } x \in D/\mu^{-1}(0), \\ \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| &< C_2 |\mu(x)|, \text{ for all } x \in D/\mu^{-1}(0). \end{aligned}$$

For both the sequences of functions (ζ_n) and (η_n) , we can take into consideration the same scale function μ without losing generality. Consider two scalars λ_1 and λ_2 , We have

$$\begin{aligned} &\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)| \\ &\leq \lambda_1 \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| + \lambda_2 \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| \\ &\leq \lambda_1 C_1 |\mu(x)| + \lambda_2 C_2 |\mu(x)| \\ &\leq (\lambda_1 C_1 + \lambda_2 C_2) |\mu(x)| \\ &= C |\mu(x)|, \text{ where } C = \lambda_1 C_1 + \lambda_2 C_2. \end{aligned}$$

Hence, $(\lambda_1 \zeta_n + \lambda_2 \eta_n) \in m(\phi, ru)$. Therefore, $m(\phi, ru)$ is a linear space. $\square$

Theorem 3.2. With the norm $\|\cdot\|_{m(\phi, ru)}$, the space $m(\phi, ru)$ is a BK-space, as defined by

$$\|\zeta_n\|_{m(\phi, ru)} = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|}.$$

Proof. Assuming throughout that D is compact domain and that $\mu : D \rightarrow (0, \infty)$ is continuous. We first show that $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ is a normed space and then establish its completeness. The proof of completeness proceeds by showing pointwise convergence of each coordinate of a Cauchy sequence, defining the corresponding candidate limit sequence, and finally using uniform boundedness to justify passing the limit under the supremum involved

in the norm. Under the same assumptions, the continuity of the coordinate projections follows immediately by choosing $s = 1$ and $\sigma = \{n\}$ in the defining inequality of the norm.

Firstly we show that $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ is a normed space.

- (1) Clearly $\|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} = 0$ if and only if $\zeta_n = \bar{\theta}$, for all $n \in \mathbb{N}$, where $(\zeta_n) = \bar{\theta} = (\theta, \theta, \theta, \dots)$, the null sequence of functions.
- (2) Let λ be a scalar, then we have

$$\begin{aligned} \|(\lambda\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\lambda\zeta_n(x)|}{|\mu(x)|} \\ &= |\lambda| \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}. \end{aligned}$$

- (3) Let $(\zeta_n), (\eta_n) \in m(\phi, ru)$, then

$$\begin{aligned} &\|(\zeta_n + \eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x) + \eta_n(x)|}{|\mu(x)|} \\ &\leq \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} + \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\eta_n(x)|}{|\mu(x)|} \\ &= \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} + \|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}. \end{aligned}$$

Therefore,

$$\|(\zeta_n + \eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \leq \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} + \|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}.$$

Hence, $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ is a normed space.

We now demonstrate the completeness of $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$.

Let $\left\{\frac{\zeta_n^i}{\mu^i}\right\}$ be a Cauchy sequence in $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$. Such that $\sup_i \mu^i(x)$ exists over $x \in D/\mu^{-1}(0)$.

Let $\sup_i \mu^i(x) = \mu(x)$, for all $x \in D/\mu^{-1}(0)$. Where, $\frac{\zeta_n^i}{\mu} = \left\{\frac{\zeta_n^i(x)}{\mu(x)}\right\}_{n=1}^\infty$, for each fixed $i \in \mathbb{N}$.

Then for a given $C > 0$, there exist a positive integer $n_0(C) > 0$ such that,

$$\begin{aligned} &\left\|\frac{\zeta_n^i - \zeta_n^j}{\mu}\right\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n^i(x) - \zeta_n^j(x)|}{|\mu(x)|} < C, \text{ for all } i, j > n_0(C). \end{aligned}$$

Taking $s = 1$ varying $\sigma \in \xi_s$,

$$\frac{|\zeta_n^i(x) - \zeta_n^j(x)|}{|\mu(x)|} < C\phi_1, \text{ for each fixed } n \in \mathbb{N} \text{ and all } i, j \geq n_0(C).$$

Therefore,

$$\frac{|\zeta_n^i(x)|}{|\mu(x)|} - \frac{|\zeta_n^j(x)|}{|\mu(x)|} < C\phi_1, \text{ for all } i, j \geq n_0(C).$$

i.e., $\left\{\left|\frac{\zeta_n^i(x)}{\mu(x)}\right|\right\}_{i \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$ for every fixed $n \in \mathbb{N}$.

Since, $\mathbb{R}$ is complete it converges for each $n \in \mathbb{N}$ and can find a function ξ_n such that

$$\lim_{i \rightarrow \infty} \left| \frac{\zeta_n^i(x)}{\mu(x)} \right| = \left| \frac{\zeta_n(x)}{\mu(x)} \right|.$$

Then, we have

$$\lim_{i \rightarrow \infty} \left\| \frac{\zeta_n^i - \zeta_n^j}{\mu} \right\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} = 0.$$

Using uniform boundedness and passing to the limit under the supremum, it follows that $\zeta = (\zeta_n) \in m(\phi, ru)$.

Hence $m(\phi, ru)$ is complete.

Now we prove that, $m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)$ has continuous coordinate projections p_n , where

$$p_k : \omega_f \rightarrow K \text{ and } p_k(\zeta_n(x)) = \zeta_n(x), \quad x \in D/\mu^{-1}(0).$$

Indeed by taking, $s = 1$ and $\sigma = \{n\}$ in the defining inequality of the norm, we obtain

$$|\zeta_n(x)| \leq \sup_{x \in D/\mu^{-1}(0), \sigma \in \xi_s, s \geq 1} \phi_s \|\zeta\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} |\mu(x)|, \text{ for each } n \in \mathbb{N},$$

which proves the continuity of the coordinate projections. $\square$

In light of Theorem 3.2, we propose the following theorem without providing evidence.

Theorem 3.3. *The sequence space of functions $n(\phi, ru)$ is a BK-space.*

Theorem 3.4. *The sequence space of functions $m(\phi, ru)$ is solid.*

Proof. Let $(\zeta_n) \in m(\phi, ru)$. Therefore,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| < C |\mu(x)|. \quad (3.2)$$

Consider another sequence of functions $(\eta_n) \in \omega_f$ such that,

$$|\eta_n(x)| \leq |\zeta_n(x)|. \quad (3.3)$$

Now from (3.2) and (3.3) we can get,

$$\begin{aligned} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| &\leq \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| < C |\mu(x)| \\ \implies (\eta_n) &\in m(\phi, ru). \end{aligned}$$

Hence, $m(\phi, ru)$ is solid. $\square$

Theorem 3.5. *The sequence space of functions $m(\phi, ru)$ is a continuous embedding between the sequence space of functions $\ell_p(ru)$, where $1 \leq p < \infty$.*

Proof. We consider a transformation defined by $\mathcal{T} : m(\phi, ru) \rightarrow \ell_p(ru)$, such that

$$\mathcal{T}(\zeta_n(x)) = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \left(\sum_{n=1}^{\infty} \frac{|\zeta_n(x)|^p}{|\mu(x)|} \right)^{\frac{1}{p}}.$$

(1) Let $(\zeta_n), (\eta_n) \in m(\phi, ru)$

The linearity of $\mathcal{T}$ is obvious here. Now, it's clear that, $(\zeta_n(x)) = \theta$, whenever $\mathcal{T}(\zeta_n(x)) = \theta$, where $x \in D/\mu^{-1}(0)$, (a compact domain), $\theta = (0, 0, 0, \dots)$ and hence $\mathcal{T}$ is injective.

Let us take any $(\eta_n) \in \ell_p(ru)$ such that

$$\begin{aligned} \mathcal{T}(\zeta_n(x)) &= \eta_n(x), \\ \Rightarrow \frac{1}{\phi_s} \sum_{n \in \sigma} \left(\sum_{n=1}^{\infty} \frac{|\zeta_n(x)|^p}{|\mu(x)|} \right)^{\frac{1}{p}} &= \eta_n(x) \\ \Rightarrow \left(\sum_{n=1}^{\infty} \frac{|\zeta_n(x)|^p}{|\mu(x)|} \right)^{\frac{1}{p}} &= \sum_{n \in \sigma} \phi_s \eta_n(x) \\ \Rightarrow \zeta_n(x) &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \sum_{n \in \sigma} \left(\sum_{n=1}^{\infty} (\phi_s \eta_n(x))^p |\mu(x)| \right)^{\frac{1}{p}} < C |\mu'(x)|. \end{aligned}$$

Where $\mu'(x)$ is the new scale function w.r.t. the sequence of function (η_n) .

Hence, the sequence space of functions $m(\phi, ru)$ is a continuous embedding with the sequence space of function $\ell_p(ru)$, where $1 \leq p < \infty$. $\square$

Given the explanation above, we assert the following corollary.

Corollary 3.1. *The sequence space of function $m(\phi, ru)$ is continuous embedded with the sequence space of function $\ell_\infty(ru)$, if $\phi_n = n$.*

Theorem 3.6. *The space $m(\phi, ru)$ is isomorphic to the space $\ell_1(ru)$, i.e., $m(\phi, ru) \cong \ell_1(ru)$, if $\phi_n = 1$.*

Proof. The proof follows directly from the definition of isomorphism by verifying linearity, bijectivity, and norm equivalence. $\square$

Result 3.1. *Convergence is not free in the sequence space of functions $m(\phi, ru)$.*

The following example leads to the stated conclusion.

Proof. Let us take a sequence of functions on a compact domain $[0, 1]$ defined by,

$$\zeta_n(x) = \begin{cases} \frac{1}{nx}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

With this established, it becomes clear that (ζ_n) does not converge uniformly, but rather converges relative uniformly to the zero function with regard to a scale function μ defined by,

$$\mu(x) = \begin{cases} \frac{1}{x}, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0. \end{cases}$$

Consider the sequence (ϕ_n) defined by $\phi_n = 1$, for all $n \in \mathbb{N}$.

Then,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\frac{1}{nx}|}{|\frac{1}{x}|} < C.$$

Hence, $(\zeta_n) \in m(\phi, ru)$.

Now we consider another sequence of functions on the same compact domain

$[0, 1]$, defined by

$$\eta_n(x) = \begin{cases} nx, & \text{for } x \neq 0; \\ 0, & \text{for } x = 0, \end{cases}$$

using $\mu(x)$ as the scaling function. But in that case,

$$\sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|nx|}{|x|} \rightarrow \infty, \text{ as } n \rightarrow \infty.$$

Therefore, $(\eta_n) \notin m(\phi, ru)$.

Hence, $m(\phi, ru)$ is not convergence-free in general. $\square$

Theorem 3.7. (1) $m(\phi, ru)$ is a symmetric space. If $(\zeta_n) \in m(\phi, ru)$ and $(\eta_n) \in m(\phi, ru)$, then

$$\|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} = \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}.$$

(2) Assuming $(\zeta_n) \in m(\phi, ru)$ and $|(\eta_n)| \leq |(\zeta_n)|$; for each positive number n , then $(\eta_n) \in m(\phi, ru)$ and $\|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \leq \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}.$

Proof. (1) The proof of this lemma can be established by using Lemma 5(a) of Sargent[17].

(2) Let, $(\zeta_n) \in m(\phi, ru)$ and also, consider a sequence of function (η_n) such that $|\eta_n(x)| \leq |\zeta_n(x)|$.

Now,

$$\begin{aligned} & \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\eta_n(x)| \\ & \leq \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} |\zeta_n(x)| \leq C|\mu(x)|, (\text{ since } (\zeta_n) \in m(\phi, ru)) \\ & \implies \|(\eta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)} \leq \|(\zeta_n)\|_{m(\|\cdot\|_{D/\mu^{-1}(0)}, \phi, ru)}. \end{aligned}$$

Hence, $(\eta_n) \in m(\phi, ru)$. $\square$

Theorem 3.8. $m(\phi, ru) \subseteq m(\psi, ru)$ iff $\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty$.

Proof. Let,

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} = H(< \infty).$$

Then,

$$\phi_s \leq H\psi_s, \text{ for all } s \in \mathbb{N}. \quad (3.4)$$

Let, $(\zeta_n) \in m(\phi, ru)$. Then,

$$\begin{aligned} & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{H\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C, \text{ [from 3.4]} \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < HC \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < H', \text{ where } H' = HC \\ \implies & (\zeta_n) \in m(\phi, ru). \end{aligned}$$

Therefore, The expression $m(\phi, ru) \subseteq m(\psi, ru)$.

Assume, on the other hand, that $m(\phi, ru) \subseteq m(\psi, ru)$.

Next consider $\sup_{s \geq 1} \frac{\phi_s}{\psi_s} = \infty$.

Let,

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} = \sup_{s \geq 1} \delta_s, \text{ and } (\zeta_n) \in m(\phi, ru).$$

Therefore,

$$\sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C.$$

Now,

$$\begin{aligned} & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\psi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} \\ &= \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \delta_s \cdot \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} = \infty. \end{aligned}$$

i.e, $(\zeta_n) \notin m(\phi, ru)$. Here we arrive at a contradiction.

Hence,

$$\sup_{s \geq 1} \frac{\phi_s}{\psi_s} < \infty.$$

□

Theorem 3.9. $\ell_p(ru) \subseteq m(\phi, ru) \subseteq \ell_\infty(ru)$.

Proof. Assuming $\phi_n = 1$, for all $n \in \mathbb{N}$, we have $\ell_p(ru) = m(\phi, ru)$.

Next we suppose that $(\zeta_n) \in m(\phi, ru)$.

Then,

$$\begin{aligned} & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} < C \\ \implies & \sup_{x \in D/\mu^{-1}(0), s \geq 1, \sigma \in \xi_s} \frac{|\zeta_n(x)|}{|\mu(x)|} < C\phi_s, \text{ for all } n \in \mathbb{N}. \end{aligned}$$

Thus, $(\zeta_n) \in \ell_\infty(ru)$.

Hence, $\ell_p(ru) \subseteq m(\phi, ru) \subseteq \ell_\infty(ru)$

□

Theorem 3.10. $m(\phi, ru) = \ell(\phi, ru)$ if and only if $\lim_{n \rightarrow \infty} \phi_s < \infty$.

Proof. The theorem follow from lemma 5(c) of Sargent[17].

□

4. GEOMETRIC PROPERTIES

The geometric characteristics of the sequence space $m(\phi, ru)$ and $n(\phi, ru)$ of functions are the subject of certain theorems in this section.

Theorem 4.1. *The function $\mathcal{M}_{m(\phi, ru)}$ denotes the modular function of the space $m(\phi, ru)$, with modular,*

$$\mathcal{M}_{m(\phi, ru)}(\zeta_n(x)) = \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\zeta_n(x)|}{|\mu(x)|} \right).$$

Proof. Consider a function $\mathcal{M}_{m(\phi, ru)}$ defined by,

$\mathcal{M}_{m(\phi, ru)} : m(\phi, ru) \rightarrow [0, \infty)$. Let $(\zeta_n), (\eta_n) \in m(\phi, ru)$.

- (1) Clearly, $\mathcal{M}_{m(\phi, ru)}(\zeta_n(x)) = 0 \implies \zeta_n(x) = \bar{\theta}$, where $x \in D/\mu^{-1}(0)$ and $\bar{\theta} = (\theta, \theta, \theta, \dots)$.
- (2) $\mathcal{M}_{m(\phi, ru)}(-\zeta_n(x)) = \mathcal{M}_{m(\phi, ru)}(\zeta_n(x))$, for all $(\zeta_n) \in m(\phi, ru), x \in D/\mu^{-1}(0)$.
- (3) For λ a scalar with $|\lambda| = 1$, we have

$$\begin{aligned} & \mathcal{M}_{m(\phi, ru)}(\lambda \zeta_n(x)) \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\lambda \zeta_n(x)|}{|\mu(x)|} \\ &= \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)), \text{ for } x \in D/\mu^{-1}(0). \end{aligned} \tag{4.5}$$

- (4) Now for any two scalar $\lambda_1, \lambda_2 \geq 0$ with $\lambda_1 + \lambda_2 = 1$ we get

$$\begin{aligned} & \mathcal{M}_{m(\phi, ru)}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)|}{|\mu(x)|} \\ &\leq |\lambda_1| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} + |\lambda_2| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\eta_n(x)|}{|\mu(x)|} \\ &\leq \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)) + \mathcal{M}_{m(\phi, ru)}(\eta_n(x)) \\ & \text{(Since, } \lambda_1 + \lambda_2 = 1, \text{ thus either both the scalars are less than 1).} \end{aligned}$$

□

The preceding theorem's consequence is the proof of the ensuing findings.

Theorem 4.2. *The function $\mathcal{M}_{n(\phi, ru)}$ denotes the modular function of the space $n(\phi, ru)$, with modular,*

$$\mathcal{M}_{n(\phi, ru)} = \sup_{(\eta_n) \in \Lambda(\zeta_n), x \in D/\mu^{-1}(0)} \sum_{n=1}^{\infty} \mathcal{M}_n \left(\frac{|\eta_n(x)| \Delta \phi_n}{|\mu(x)|} \right), \text{ for all } x \in D/\mu^{-1}(0).$$

Proof. The theorem follows from the Theorem 4.2. □

In light of the previous debate, we assert the following corollary.

Corollary 4.1. *The modular $\mathcal{M}_m(\phi, ru)$ and $\mathcal{M}_n(\phi, ru)$ are 1-convex or a convex modular.*

Theorem 4.3. *The Modular function $\mathcal{M}_{m(\phi, ru)}$ on the space $m(\phi, ru)$ is $\tilde{s}$ -convex.*

Proof. Consider $(\zeta_n), (\eta_n) \in \mathcal{M}_{m(\phi, ru)}$ and $\lambda_1, \lambda_2 \geq 0, \lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1$, where $\tilde{s} \in (0, 1]$.

Now,

$$\begin{aligned} & \mathcal{M}_{m(\phi, ru)}(\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)) \\ &= \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\lambda_1 \zeta_n(x) + \lambda_2 \eta_n(x)|}{|\mu(x)|} \right) \\ &\leq |\lambda_1| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\zeta_n(x)|}{|\mu(x)|} \right) + |\lambda_2| \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \mathcal{M}_n \left(\frac{|\eta_n(x)|}{|\mu(x)|} \right) \\ &\leq |\lambda_1|^{\tilde{s}} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\zeta_n(x)|}{|\mu(x)|} + |\lambda_2|^{\tilde{s}} \sup_{s \geq 1, \sigma \in \xi_s, x \in D/\mu^{-1}(0)} \frac{1}{\phi_s} \sum_{n \in \sigma} \frac{|\eta_n(x)|}{|\mu(x)|} \end{aligned}$$

(Since, $\lambda_1, \lambda_2 \geq 0, \lambda_1^{\tilde{s}} + \lambda_2^{\tilde{s}} = 1$, where $\tilde{s} \in (0, 1]$, so λ_1, λ_2 must belongs to $[0, 1]$).

Hence the space $\mathcal{M}_{m(\phi, ru)}$ is $\tilde{s}$ -modular. $\square$

We present the following theorem in light of the explanation above.

Theorem 4.4. *The Modular function $\mathcal{M}_{n(\phi, ru)}$ on the space $n(\phi, ru)$ is $\tilde{s}$ -convex.*

Proof. The theorem 4.4 is a consequence of the theorem's proof. $\square$

Theorem 4.5. *The modular spaces $\mathcal{M}_{m(\phi, ru)}$ and $\mathcal{M}_{n(\phi, ru)}$ are continuous modular on the space $m(\phi, ru)$ and $n(\phi, ru)$ respectively.*

Proof. Since the function $\mathcal{M}_{m(\phi, ru)}$ is modular on the space $m(\phi, ru)$, thus we can take limit as $\lambda \rightarrow 1^+$ in (4.5) we observe that,

$$\lim_{\lambda \rightarrow 1^+} \mathcal{M}_{m(\phi, ru)}(\lambda \zeta_n(x)) = \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)), \text{ for all } (\zeta_n) \in \mathcal{M}_{m(\phi, ru)},$$

and for all $x \in D/\mu^{-1}(0)$. Hence, $\mathcal{M}_{m(\phi, ru)}$ is right continuous. Similarly taking limit as $\lambda \rightarrow 1^-$ in (4.5), we observe that

$$\begin{aligned} & \lim_{\lambda \rightarrow 1^-} \mathcal{M}_{m(\phi, ru)}(\lambda \zeta_n(x)) \\ &= \mathcal{M}_{m(\phi, ru)}(\zeta_n(x)), \text{ for all } (\zeta_n) \in \mathcal{M}_{m(\phi, ru)} \text{ and for all } x \in D/\mu^{-1}(0). \end{aligned}$$

These two consequences give us the desired fact that the function $\mathcal{M}_{m(\phi, ru)}$ is continuous modular.

Similarly, we can establish for the modular $\mathcal{M}_{n(\phi, ru)}$ on the space $n(\phi, ru)$. $\square$

The following corollary is obvious for the rest of the proof and is the consequence of the previous theorem.

Corollary 4.2. *The modular $\mathcal{M}_{m(\phi, ru)}$ is modular convergent to $\zeta \in \mathcal{M}_{m(\phi, ru)}$.*

5. APPLICATION & FUTURE WORK

The framework developed in this paper for the sequence space $m(\phi, ru)$ provides a useful tool for studying convergence and boundedness under scale function. Such spaces naturally arise in summability theory, where relative uniform convergence offers a finer mode of convergence than pointwise convergence. The results obtained here may be applied to the study of

generalised summability methods, approximation of function sequences, dual sequence space and the analysis of operators acting between weighted sequence spaces.

6. CONCLUSION

In this article we define relative uniform convergence of Sargent type sequence of functions concerning the scale function $\mu(x)$ defined on a compact domain $D/\mu^{-1}(0) \subseteq \mathbb{R}$, we have introduced the classes of sequences of functions $m(\phi, ru)$ and $n(\phi, ru)$. We investigated their fundamental topological and geometric properties, including completeness, embedding relations, norm-related characteristics and also studied some geometric properties of convexity. Several illustrative examples were provided to clarify the theoretical developments. The results obtained not only extend and unify a number of known sequence spaces but also open avenues for further research. We believe that the framework developed here can serve as a foundation for further investigations in the theory of generalised sequence spaces.

7. OUTLINE OF THE ARTICLE

The sequence space $m(\phi, ru)$ associated with relative uniform convergence is studied in this paper, and its fundamental structural properties are established. The main results include the characterization of the normed structure of $m(\phi, ru)$, embedding relations with classical sequence spaces, and convergence properties under relative uniform control. The paper is structured as follows: Section 2 provides the necessary definitions and preliminaries; Section 3 presents the main theorems and their proofs; Section 4 discusses related consequences and further observations; and the final section concludes the paper and suggests future research directions.

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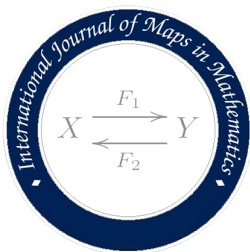
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RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS SATISFYING SEMI-SYMMETRY CURVATURE CONDITIONS

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Abstract. The main objective of this paper is to study Ricci solitons on a class of indefinite almost paracontact metric manifolds, namely indefinite para-Kenmotsu (also termed, (ϵ) -para Kenmotsu) manifolds satisfying semi-symmetry curvature conditions. In the conclusion, we develop an example of an indefinite para-Kenmotsu manifold with dimension of 3, which confirms the results described in this paper. A comparison table with the results obtained for (ϵ) -para Sasakian manifolds is included in the final version of the results.

Keywords: Indefinite para-Kenmotsu manifolds, Curvature tensor, Ricci solitons, Shrinking, Space-like vector field.

MSC (2020): 53C15, 53C50, 53C25.

1. INTRODUCTION

A Ricci soliton, defined on a Riemannian manifold, is a natural generalization of an Einstein metric, as well as a self-similar solution to Hamilton's Ricci flow presented by Hamilton [4] in 1988. For a manifold M_n , a Ricci soliton is a triplet (g, V, λ) if there is a smooth vector field V , a Riemannian metric g , and a real scalar λ such that

$$(L_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \quad (1.1)$$

where L_V is the Lie-derivative of the metric g along the vector field V and S is the Ricci tensor of M_n . The Ricci soliton is stated to be shrinking, steady, and expanding, depending on whether λ is negative, zero, or positive. If the potential vector field V disappears uniformly, the Ricci soliton becomes trivial, and the manifold is an Einstein one.

Motivated by Hamilton's findings, many geometers have conducted extensive research on the generalizations of Ricci soliton. Various classes of Ricci solitons have been investigated and studied due to their numerous applications in both physics and Riemannian geometry [1, 2, 19, 18, 20].

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The index of the indefinite metric, on the other hand, creates diverse vector fields, including space-like, time-like, and light-like fields, rendering the examination of indefinite structures on manifolds exceedingly significant and stimulating. With an emphasis on Sasakian manifolds, in 1969, Takahashi [15] presented pseudo-Riemannian metrics on almost contact manifolds. In 2009, De and Sarkar [3] presented a (ϵ) -Kenmotsu manifold, which is the Kenmotsu manifold [5] that admits an indefinite metric. Several geometers have extensively studied these manifolds and have produced a number of intriguing findings [9, 12, 17].

In 1976, Sato [11] defined an almost paracontact structure, which is analogous to the almost contact structure. Tripathi *et al.*, [16] presented the notion of an indefinite almost paracontact metric structure, also referred as (ϵ) -almost paracontact structure, by linking an almost paracontact structure with a semi-Riemannian metric, which need not be the Lorentzian. In this instance, $(\epsilon = 1)$ or $(\epsilon = -1)$ indicates that the vector field ξ can be either space-like or time-like. In addition, they introduced and examined the characteristics of (ϵ) -para Sasakian [16] and (ϵ) -para Sasakian 3-manifolds [8].

Analogous to these, in 2025, Sunitha and Sai Prasad [14] introduced a novel category of indefinite almost paracontact metric manifolds known as (ϵ) -para Kenmotsu manifolds, focusing specifically on the curvature characteristics of (ϵ) -para Kenmotsu 3-manifolds. These manifold structures are derived from a category of almost paracontact metric manifolds, specifically para-Kenmotsu manifolds, which were delineated in 1995 by Sinha and Sai Prasad [13]. Further, Sai Prasad along with Sunitha and Satyanarayana have recently explored the Z-symmetries of three-dimensional indefinite para-Kenmotsu manifolds [10].

Inspired by the prior study, this current paper continues the study of indefinite para-Kenmotsu manifolds. Apart from a few requirements about indefinite para-Kenmotsu manifolds in section 2, subsequently in sections 3, 4 and 5, we examine Ricci solitons within the indefinite para-Kenmotsu manifolds admitting the following semi-symmetry curvature conditions:

$$R(\xi, X).R = 0, \quad (1.2)$$

$$P(\xi, X).R = 0, \quad (1.3)$$

$$S(\xi, X).P = 0, \quad (1.4)$$

where $R(X, Y)$ and $P(X, Y)$ denote the Riemann and projective curvature tensors, respectively. In order to validate the findings presented in this study, we also develop an example of a 3-dimensional (ϵ) -para Kenmotsu manifold. At the end, the results are provided in a more comprehensive form, including a comparison table with the results obtained for (ϵ) -para Sasakian manifolds [7].

2. PRELIMINARIES

Assuming that an n -dimensional differentiable manifold M_n is capable of accepting a vector field ξ , a $(1, 1)$ tensor field ϕ , and a 1-form η that satisfy the requirements, then it is considered to have a structure that is an almost paracontact:

$$\phi^2 X = X - \eta(X) \xi, \quad (2.5)$$

$$\eta(\xi) = 1, \quad (2.6)$$

$$\eta(\phi X) = 0, \quad \phi\xi = 0. \quad (2.7)$$

By taking into account the semi-Riemannian metric $g(X, Y)$, which has an index of $g(X, Y) = \nu$, it is possible to assert that

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y), \quad (2.8)$$

for any X, Y belongs to $\chi(M_n)$, which is a group of vector fields on M_n , and (ϵ) is either 1 or -1 . Then M_n is considered as a (ϵ) -almost paracontact metric manifold endowed with (ϵ) -almost paracontact metric structure $(\phi, \xi, \eta, g, \epsilon)$.

According to the information provided in reference [6], the manifold M_n is regarded to be a Lorentzian manifold that is almost paracontact if the index of the function $g(X, Y)$ is one. Due to the fact that the metric $g(X, Y)$ is positive definite, the manifold M_n is regarded as almost paracontact metric manifold [11]. A clear and convincing demonstration of the assertion is provided by equation (2.8) that

$$g(\phi X, Y) = g(X, \phi Y) \text{ and} \quad (2.9)$$

$$g(X, \xi) = \epsilon \eta(X). \quad (2.10)$$

In addition, the equations (2.5), (2.6), (2.7) and (2.9), give us the following:

$$g(\xi, \xi) = \epsilon. \quad (2.11)$$

The structure of the (ϵ) -paracontact metric is also known as the (ϵ) -para Kenmotsu metric if

$$(\nabla_X \phi)Y = g(X, \phi Y)\xi - \epsilon \eta(Y) \phi X. \quad (2.12)$$

Moreover, the manifold M_n that contains the structure $(\phi, \xi, \eta, g, \epsilon)$ is understood to be the (ϵ) -para Kenmotsu manifold [14]. The symbol ∇ is used to denote the Levi-Civita connection for the indefinite metric $g(X, Y)$. In a particular instance, for the Riemannian metric $g(X, Y)$ and $\epsilon = 1$, the manifold is para Kenmotsu [13]. We may change the manifold M_n into a Lorentzian para-Kenmotsu manifold by substituting $-\xi$ for ξ and $\epsilon = -1$. This manifold is characterised by the Lorentzian metric $g(X, Y)$ [6].

Among the features that the curvature tensor $R(X, Y)$ possesses in an (ϵ) -para Kenmotsu manifold are as follows [14]:

$$\nabla_X \xi = \epsilon[X - \eta(X)\xi], \quad (2.13)$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \quad (2.14)$$

$$R(\xi, X)Y = \epsilon\eta(Y)X - g(X, Y)\xi, \quad (2.15)$$

$$R(\xi, X)\xi = \epsilon X - \epsilon\eta(X)\xi, \quad (2.16)$$

$$\eta(R(X, Y)Z) = \epsilon[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \quad (2.17)$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \quad (2.18)$$

for all X, Y , and $Z \in \chi(M_n)$.

We can get the following from the equation (2.18):

$$S(X, \phi Y) = S(\phi X, Y) \text{ and} \quad (2.19)$$

$$S(X, \xi) = -(n - 1)\eta(X). \quad (2.20)$$

Consider a Ricci soliton (ξ, g, λ) in a (ϵ) -para Kenmotsu manifold. Using (2.13), we get

$$(L_{\xi}g)(X, Y) = 2 \epsilon [g(X, Y) - \epsilon \eta(X)\eta(Y)]. \quad (2.21)$$

With reference to equations (1.1) and (2.21), we have

$$S(X, Y) = -(\epsilon + \lambda) g(X, Y) + \eta(X)\eta(Y). \quad (2.22)$$

This gives

$$S(\xi, Y) = -\lambda \epsilon \eta(Y), \quad (2.23)$$

$$QX = -(\epsilon + \lambda)X + \epsilon \eta(X)\xi, \text{ and} \quad (2.24)$$

$$r = -n(\epsilon + \lambda) + \epsilon. \quad (2.25)$$

The definition of the projective curvature tensor $P(X, Y)$ is

$$P(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [S(X, Z)Y - S(Y, Z)X]. \quad (2.26)$$

Equations (2.17), (2.23), and (2.25) allow $P(X, Y)$ to reduce to

$$\begin{aligned} P(\xi, Y)Z &= \left[1 - \frac{\lambda}{n-1}\right] [\epsilon \eta(Z)Y - g(Y, Z)\xi] \\ &\quad + \left[\frac{1}{n-1}\right] [\epsilon g(Y, Z) - \eta(Y)\eta(Z)]\xi. \end{aligned} \quad (2.27)$$

Additionally, by using the equations (2.6), (2.17), (2.23), and (2.24) and adding $X = \xi$ to (2.26), we get

$$\begin{aligned} P(Y, \xi)Z &= \left[-1 + \frac{\lambda}{n-1}\right] [\epsilon \eta(Z)Y - g(Y, Z)\xi] \\ &\quad - \left[\frac{1}{n-1}\right] [\epsilon g(Y, Z) - \eta(Y)\eta(Z)]\xi. \end{aligned} \quad (2.28)$$

Using the equations (2.6), (2.17), (2.21), and (2.24), for $Z = \xi$, we can obtain from (2.26)

$$P(X, Y)\xi = \left[1 - \frac{\lambda \epsilon}{n-1}\right] [\eta(X)Y - \eta(Y)X]. \quad (2.29)$$

3. RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS ADMITTING

$$R(\xi, X).R = 0$$

Assume that $R(\xi, X).R = 0$. This yields

$$\begin{aligned} R(\xi, X)R(Y, Z)U - R(Y, Z)R(\xi, X)U \\ - R(Y, R(\xi, X)Z)U - R(R(\xi, X)Y, Z)U = 0. \end{aligned} \quad (3.30)$$

Using equation (2.17), the given equation can be reduced to

$$\begin{aligned} \epsilon \eta(R(Y, Z)U)X - g(X, R(Y, Z)U)\xi - \epsilon \eta(U)R(Y, Z)X \\ + g(X, Y)R(\xi, Z)U - \epsilon \eta(Z)R(Y, X)U + g(X, Z)R(Y, \xi)U \\ - \epsilon \eta(Y)R(X, Z)U + g(X, U)R(Y, Z)\xi = 0. \end{aligned} \quad (3.31)$$

By applying (2.6) and (2.7) to the inner product of the previous equation with ξ , we arrive at

$$\begin{aligned} & \eta(X) \eta(R(Y, Z) U) - \epsilon g(X, R(Y, Z) U) + g(X, Y) \eta(R(\xi, Z) U) \\ & - \epsilon \eta(Y) \eta(R(X, Z) U) + g(X, Z) \eta(R(Y, \xi) U) - \epsilon \eta(Z) \eta(R(Y, X) U) \\ & + \epsilon g(X, U) \eta(R(Y, Z) \xi) - \epsilon \eta(U) \eta(R(Y, Z) X) = 0. \end{aligned} \quad (3.32)$$

Using (2.22), the equation above takes the form:

$$\begin{aligned} \epsilon g(X, R(Y, Z) U) &= (1 - \epsilon)[g(Z, U) \eta(X) \eta(Y) - g(Y, U) \eta(X) \eta(Z)] \\ &+ \epsilon[g(X, Z) g(Y, U) - g(X, Y) g(Z, U)]. \end{aligned} \quad (3.33)$$

After entering $U = X = e_i$ in the equation above and calculating the sum over i ($1 \leq i \leq n$), we obtain

$$S(Y, Z) = \eta(Y) \eta(Z) - n \epsilon g(Y, Z). \quad (3.34)$$

Using $Z = \xi$ in (3.34), we get

$$S(Y, \xi) = -(n - 1) \eta(Y). \quad (3.35)$$

Then by using (2.23), the above equation gives $\lambda > 0$, showing that the λ is expanding. Based on this, we can conclude that:

Theorem 3.1. *A Ricci soliton (g, ξ, λ) on the indefinite para-Kenmotsu manifold under the semi-symmetry condition $R(\xi, X) \cdot R = 0$ are expanding, and ξ is a vector field that is space-like.*

4. RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS ADMITTING

$$P(\xi, X) \cdot R = 0$$

Assume that $P(\xi, X) \cdot R = 0$. This yields

$$\begin{aligned} & P(\xi, X) R(Y, Z) W - R(Y, Z) P(\xi, X) W \\ & - R(Y, P(\xi, X) Z) W - R(P(\xi, X) Y, Z) W = 0. \end{aligned} \quad (4.36)$$

Equation (2.28) allows the equation above to be reduced to

$$\begin{aligned} & \epsilon \eta(R(Y, Z) W) X - \epsilon \eta(Z) R(Y, X) W + \frac{\epsilon \lambda}{n - 1} \eta(Y) R(X, Z) W \\ & + \frac{\epsilon + \lambda}{n - 1} g(X, R(Y, Z) W) \xi - \frac{1}{n - 1} \eta(X) \eta(R(Y, Z) W) \xi \\ & - \epsilon \eta(Y) R(X, Z) W + g(X, Y) R(\xi, Z) W - \epsilon \frac{\lambda}{n - 1} \eta(R(Y, Z) W) X \\ & - \frac{\epsilon + \lambda}{n - 1} g(X, Y) R(\xi, Z) W + \frac{1}{n - 1} \eta(X) \eta(Y) R(\xi, Z) W \\ & - \frac{\epsilon + \lambda}{n - 1} g(X, Z) R(Y, \xi) W + \frac{1}{n - 1} \eta(X) \eta(Z) R(Y, \xi) W \\ & + \frac{\epsilon \lambda}{n - 1} \eta(Z) R(Y, X) W + g(X, Z) R(Y, \xi) W - g(X, R(Y, Z) W) \xi \\ & - R(Y, Z) R(\xi, X) W - \left[1 - \frac{\lambda}{n - 1}\right] \epsilon \eta(W) R(Y, Z) X \\ & + \left[1 - \frac{\lambda + \epsilon}{n - 1}\right] g(X, W) R(Y, Z) \xi + \frac{1}{n - 1} \eta(X) \eta(W) R(Y, Z) \xi = 0. \end{aligned} \quad (4.37)$$

Taking the inner product of (4.37) with ξ and applying (2.6) and (2.7) yield

$$\begin{aligned}
 & \eta(R(Y, Z) W) \eta(X) - \epsilon g(X, R(Y, Z) W) - \eta(Y) \eta(R(X, Z) W) \\
 & + \frac{\epsilon + \lambda}{n-1} g(X, R(Y, Z) W) \epsilon - \frac{1}{n-1} \eta(X) \eta(R(Y, Z) W) \epsilon \\
 & + \epsilon g(X, Y) \eta(R(\xi, Z) W) - \frac{\lambda}{n-1} \eta(R(Y, Z) W) \eta(X) \\
 & + \frac{\lambda}{n-1} \eta(Y) \eta(R(X, Z) W) - \frac{\epsilon + \lambda}{n-1} g(X, Y) \epsilon \eta(R(\xi, Z) W) \\
 & + \frac{1}{n-1} \eta(X) \eta(Y) \epsilon \eta(R(\xi, Z) W) - \eta(Z) \eta(R(Y, X) W) \\
 & + \epsilon g(X, Z) \eta(R(Y, \xi) W) + \frac{1}{n-1} \epsilon \eta(X) \eta(Z) \eta(R(Y, \xi) W) \\
 & - \epsilon \eta(R(Y, Z) R(\xi, X) W) - \left[1 - \frac{\lambda}{n-1}\right] \eta(W) \eta(R(Y, Z) X) \\
 & + \left[1 - \frac{\epsilon + \lambda}{n-1}\right] g(X, W) \epsilon \eta(R(Y, Z) \xi) \\
 & + \frac{1}{n-1} \eta(X) \eta(W) \epsilon \eta(R(Y, Z) \xi) = 0.
 \end{aligned} \tag{4.38}$$

Equations (2.10) and (2.17) allow us to derive from the equation above that

$$\left[\frac{\lambda}{n-1} - 1\right] [g(Y, X) \eta(Z) - g(Z, X) \eta(Y)] \epsilon \eta(W) = 0, \tag{4.39}$$

which gives λ as $n-1$ as the term $[g(Y, X) \eta(Z) - g(Z, X) \eta(Y)] \epsilon \eta(W) \neq 0$. This indicates that as λ is expanding as $\lambda > 0$. We can therefore assert the following.

Theorem 4.1. *A Ricci soliton (g, ξ, λ) on indefinite para-Kenmotsu manifolds admitting the condition $P(\xi, X) \cdot R = 0$ are expanding, and ξ is a vector field that is space-like.*

5. RICCI SOLITONS ON INDEFINITE PARA-KENMOTSU MANIFOLDS ADMITTING

$$S(\xi, X) \cdot P = 0$$

Let $S(X, \xi) \cdot P(Y, Z) W = 0$. Then

$$\begin{aligned}
 & S(X, \xi) P(Y, Z) W - P(Y, Z) S(\xi, X) W \\
 & - P(Y, S(X, \xi) Z) W - P(S(X, \xi) Y, Z) W = 0.
 \end{aligned} \tag{5.40}$$

Equation (2.17) allows the equation above to be reduced to

$$\begin{aligned}
 & S(X, \xi) R(Y, Z) W - R(Y, S(X, \xi) Z) W \\
 & - R(Y, Z) S(\xi, X) W - R(S(X, \xi) Y, Z) W \\
 & - \frac{1}{n-1} S(S(X, \xi) Y, W) Z + \frac{1}{n-1} S(Z, W) S(X, \xi) Y \\
 & + \frac{1}{n-1} S(X, \xi) S(Y, W) Z - \frac{1}{n-1} S(Y, W) S(X, \xi) Z \\
 & + \frac{1}{n-1} S(S(X, \xi) Z, W) Y - \frac{1}{n-1} S(X, \xi) S(Z, W) Y \\
 & - \frac{1}{n-1} S(Y, S(\xi, X) W) Z + \frac{1}{n-1} S(Z, S(\xi, X) W) Y = 0.
 \end{aligned} \tag{5.41}$$

Taking the above equation's inner product with ξ and applying equations (2.7) and (2.10), we obtain

$$\begin{aligned}
& S(X, \xi) \eta(R(Y, Z)W) + \frac{1}{n-1} S(X, \xi) S(Z, W) \epsilon \eta(Y) \\
& - \frac{1}{n-1} S(X, \xi) S(Y, W) \epsilon \eta(Z) - \eta(R(Y, S(X, \xi)Z)W) \\
& - \eta(R(S(X, \xi)Y, Z)W) - \eta(R(Y, Z) S(\xi, X)W) \\
& - \frac{1}{n-1} S(\xi, X)S(Y, W) \epsilon \eta(Z) + \frac{1}{n-1} S(Z, W) S(\xi, X) \epsilon \eta(Y) = 0,
\end{aligned} \tag{5.42}$$

which further reduces to

$$\frac{2\epsilon}{n-1} S(X, \xi)[S(Z, W) \eta(Y) - S(Y, W) \eta(Z)] - 2S(X, \xi) \eta(R(Y, Z)W) = 0. \tag{5.43}$$

Using (2.20) and (2.22), (5.43) gives

$$-\frac{\lambda \epsilon}{n-1} [-2 - \lambda \epsilon + n] [g(Y, W) \eta(Z) - g(Z, W) \eta(Y)] \eta(X) = 0. \tag{5.44}$$

Applying the equations (2.6), (2.10), and (2.11) to the preceding equation and substituting $Z = W = \xi$, we obtain

$$\lambda \epsilon = 0 \text{ (or) } -2 - \lambda \epsilon + n = 0, \tag{5.45}$$

which gives

$$\lambda = 0 \text{ (or) } \lambda = (n-2) \epsilon. \tag{5.46}$$

Now, let us suppose that ξ be a space-like vector field in indefinite para-Kenmotsu manifolds. From the equation above, we can find that either $\lambda = 0$, (or) $\lambda > 0$ for $n > 2$. This indicates that λ is either steady or expanding. As a result, we can say this.

Theorem 5.1. *A Ricci soliton (g, ξ, λ) on indefinite para-Kenmotsu manifolds admitting the condition $S(\xi, X).P = 0$ are either steady or expanding, and ξ is a vector field that is space-like.*

Furthermore, the vector field ξ is assumed to be a time-like on the (ϵ) -para Kenmotsu manifold. We therefore have $\lambda = 0$ (or) $\lambda < 0$ in light of (5.46). This suggests that λ is either steady or shrinking the same. Therefore, we can say that:

Theorem 5.2. *A Ricci soliton (g, ξ, λ) on indefinite para-Kenmotsu manifolds that satisfy $S(\xi, X).P = 0$ and possess ξ as a time-like vector field are characterised as either steady or shrinking.*

6. AN EXAMPLE OF A THREE-DIMENSIONAL INDEFINITE PARA-KENMOTSU MANIFOLD

We look at a three-dimensional manifold $M_3 = \{(x, y, z) \in R^3\}$, where (x, y, z) are the standard coordinates in R^3 . For each point in M_3 , let e_1, e_2 , and e_3 be a set of linearly independent vectors provided by

$$e_1 = z \frac{\partial}{\partial x}, \quad e_2 = z \frac{\partial}{\partial y}, \quad e_3 = -z \frac{\partial}{\partial z}. \tag{6.47}$$

The definition of the Riemannian metric g is

$$g(e_i, e_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j; i, j = 1, 2, 3. \end{cases}$$

The 1-form, denoted by η , is defined as:

$$g(X, e_3) = \epsilon \eta(X), \text{ for each } X \in \chi(M_3).$$

Let ϕ be a $(1, 1)$ -tensor field on M_3 with the following definition:

$$\phi^2(e_1) = e_1, \phi^2(e_2) = e_2, \phi^2(e_3) = 0; \text{ and}$$

$$\phi(e_1) = -\epsilon e_1, \phi(e_2) = -\epsilon e_2, \phi(e_3) = 0.$$

We can get this by using the linearity of ϕ and g :

$$\eta(e_3) = 1, g(\xi, \xi) = \epsilon \text{ where } \epsilon = \pm 1,$$

$$\phi^2(X) = X - \eta(X) e_3,$$

$$\text{and } g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X) \eta(Y),$$

for all $X, Y \in \chi(M_3)$. Therefore, for $e_3 = \xi$, the structure $(\phi, \xi, \eta, g, \epsilon)$ defines an indefinite almost paracontact structure on M_3 .

Now, from (6.47) we have

$$[e_1, e_3] = e_1, [e_2, e_3] = e_2, \text{ and } [e_1, e_2] = 0.$$

Koszul's formula gives us the Levi-Civita connection ∇ of the metric tensor g as follows:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X g(Y, Z) + Y g(Z, X) - Z g(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \quad (6.48)$$

Next, we obtain the following using the Koszul formula mentioned above:

$$\nabla_{e_1} e_1 = -e_3, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = \epsilon e_1;$$

$$\nabla_{e_2} e_1 = 0, \nabla_{e_2} e_2 = -e_3, \nabla_{e_2} e_3 = \epsilon e_2;$$

$$\nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0.$$

Based on the parameters listed above, the manifold under evaluation satisfies the conditions (2.12) and (2.13). This demonstrates that M_3 is an indefinite para-Kenmotsu manifold with the structure $(\phi, \xi, \eta, g, \epsilon)$.

Additionally, using (1.1) and the values mentioned before, we obtain

$$(L_\xi g)(e_1, e_1) = 2, (L_\xi g)(e_2, e_2) = 2 \text{ and } (L_\xi g)(e_3, e_3) = 0.$$

For a three-dimensional manifold, using (2.20), we get

$$S(X, \xi) = -2 \eta(X). \quad (6.49)$$

Using (2.22), we obtain

$$S(X, Y) = -(\epsilon + \lambda) \epsilon \eta(X) + \eta(X). \quad (6.50)$$

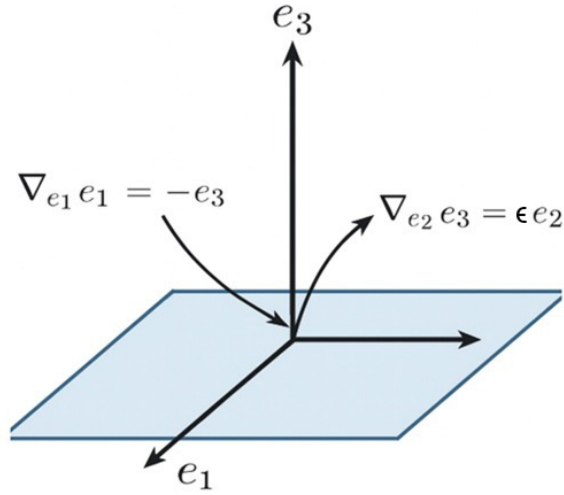


FIGURE 1. Classification of 3-Manifold with Covariant Derivatives.

From the preceding two equations, we obtain

$$\lambda = 2 \epsilon. \quad (6.51)$$

For $\epsilon = 1$, we have $\lambda = 2$. From (2.22), we have

$$S(X, Y) = -3 g(X, Y) + \eta(X) \eta(Y). \quad (6.52)$$

Similarly, for $\epsilon = -1$, we get $\lambda = -2$, and from (2.22), we get

$$S(X, Y) = 3 g(X, Y) + \eta(X) \eta(Y). \quad (6.53)$$

Hence from (2.22), we have

$$S(e_1, e_1) = -(\epsilon + \lambda) \epsilon, \quad S(e_2, e_2) = -(\epsilon + \lambda) \epsilon, \quad S(e_3, e_3) = -(\epsilon + \lambda) \epsilon + 1,$$

where $r = \sum_{i=1}^3 S(e_i, e_i) g(e_i, e_i)$. This gives

$$r = -3 (\epsilon + \lambda) + \epsilon. \quad (6.54)$$

From the above equation, we get $r = -8$, for $\epsilon = 1$ and $\lambda = 2$; and for $\epsilon = -1$ and $\lambda = -2$, we get $r = 8$.

Clearly for $\epsilon = 1$, the manifold has negative scalar curvature shows that the Ricci soliton is expanding, and the manifold has positive scalar curvature for $\epsilon = -1$. This proves that the Ricci soliton is steady.

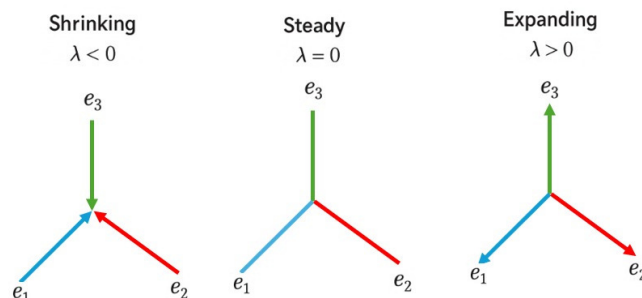


FIGURE 2. Classification of Ricci solitons based on the sign of λ

7. COMPARISON

TABLE 7.1. Comparison of Ricci solitons on indefinite para-Sasakian and para-Kenmotsu manifolds

Property	indefinite para-Sasakian	indefinite para-Kenmotsu
Structure equations	$\nabla_X \phi Y = -g(X, Y)\xi + \epsilon \eta(Y) X$	$\nabla_X \phi Y = -g(X, \phi Y)\xi - \epsilon \eta(Y) \phi X$
Nature of ξ	$\nabla_X \xi = \epsilon \phi X$	$\nabla_X \xi = \epsilon(X - \eta(X)) \xi$
Ricci Soliton form	$L_V g + 2S + 2\lambda g = 0$	$L_V g + 2S + 2\lambda g = 0$
Ricci Soliton equation	$S(X, \xi) = -(n-1)\eta(X)$	$S(X, Y) = -\lambda \epsilon \eta(X)$
Type of Soliton	$\lambda = (1-n)\epsilon$	$\lambda = (n-2)\epsilon$
Geometric behaviour	Shrinking or steady depending on ϵ	Space-like steady or expanding depending on ϵ

8. APPLICATION

Geometric Flow Modeling: The potential vector field $V = \beta \xi$ indicates that the metric evolves along the vector field ξ .

1. For $\epsilon = 1$, the soliton expands. So the manifold stretches along ξ , whereas orthogonal directions evolve differently or remain constant.
2. This simulates geometric flows that are anisotropic, meaning that one preferred direction changes noticeably.

Practical use: Anisotropic cosmologies and manifolds with directed deformation are examples of spaces or spacetimes with directional expansion.

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THE PROBABILITY DISTRIBUTION OF THE NUMBER OF DISTINCT OUTCOMES IN REPEATED WEIGHTED EXPERIMENTS

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Abstract. The number of distinct outcomes arising in repeated weighted experiments is a fundamental quantity in probability theory, with natural applications in statistics and game theory. In this paper, we pursue two complementary objectives. First, we establish explicit and recursive formulas for the probability distribution governing this quantity, then derive its expected value and variance via two independent methods: a classical combinatorial approach and a probabilistic method based on indicator random variables, joint probabilities, and covariance matrices. We demonstrate the practical relevance of this distribution through applications in weighted bootstrap sampling with minority classes and in an analysis of the NBA Draft Lottery. Second, as a central theoretical contribution, we resolve a conjecture proposed in [5] asserting that the expected value of this distribution is maximized when all outcomes are equiprobable, a result consistent with the maximum-entropy principle from information theory. We first support the conjecture through Monte Carlo simulations, which motivate and guide a complete formal proof confirming its validity.

Keywords: Probability Distribution; Weighted Experiments; Distinct Outcomes; Expected Value; Entropy.

MSC (2020): 60C05, 62E17, 91A80, 65C05.

1. INTRODUCTION

Historically, weighted probabilistic models have contributed heavily to engineering and the sciences, arising because their unweighted counterparts often lack the functionality to accurately simulate many real-world scenarios. These weighted models necessitate a concept in information theory [11] called the entropy of a random variable, which quantifies the average amount of “information,” “surprise,” or “uncertainty” associated with a random outcome. Entropy has emerged as a burgeoning concept in probability and statistics, with work by [6, 11] highlighting its capabilities. Weighted experiments also arise naturally in machine learning, where the probability of distinct outcomes is influenced by reward structures, class imbalance, or feature relevance. Applications include outcome-weighted learning for optimal decision rules [9], probability estimation for multicategory outcomes [12], and probability-weighted feature selection in high-dimensional models [1].

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In this manuscript, we explore weighted probabilistic models and entropy through the problem of determining a probability distribution for the number of distinct outcomes that occur in repeated weighted experiments. This distribution allows us to analyze how randomness behaves in weighted settings and illustrates that the highest level of entropy occurs when all outcomes are equiprobable.

To motivate our study, consider a scenario from incentive design, a construct from the Nobel Prize-winning field of contract theory. Suppose a firm manager oversees twelve employees and has twelve equal-value incentive checks to distribute at the year-end banquet. To determine the recipient of each check, the manager conducts a weighted random selection with replacement, aiming both to reward superior employees and to avoid accusations of bias. Over the year, employees who meet a universal standard earn personalized achievement cards that are placed into a jar, and during the banquet the CEO makes twelve selections with replacement from this jar to determine the recipients. Because different employees contribute different numbers of achievement cards, the selection process is conducted in a weighted and randomized manner. It is therefore pertinent to determine the probability of k distinct employees receiving at least one incentive check, for $1 \leq k \leq 12$, as well as the expected number of distinct employees that receive a check. Our model answers this question directly.

Interestingly, our results can also be employed to analyze the annual NBA Draft Lottery. The lottery determines the selection order for that year’s NBA Draft through a weighted random draw among each of the league’s non-postseason teams, where the odds of winning correlate negatively with league standing [17]. Here, we restrict our focus to the winner of the Draft Lottery, as the value of the top pick has led some teams to be accused of “tanking”, purposely losing games to increase their probability of securing the first pick [4]. In 2019, the NBA significantly overhauled their lottery odds to mitigate this issue [4], as expressed in Table 1.1. These new odds theoretically increase the diversity of teams that can obtain the top pick.

Standing	1	2	3	4	5	6	7	8	9	10	11	12	13	14
2018	0.25	0.199	0.138	0.137	0.088	0.053	0.053	0.028	0.017	0.011	0.008	0.007	0.006	0.005
2019	0.14	0.14	0.14	0.125	0.105	0.09	0.06	0.06	0.06	0.03	0.02	0.01	0.01	0.01

TABLE 1.1. Probabilities of getting the top pick in 2018 versus 2019 based on league record [18]. (1 denotes the team with the worst league standing.)

As an application of our distribution, we examine the impact of the NBA’s revised lottery odds on the randomness of the Draft by modeling the expected number of distinct pre-lottery positions that secure the top pick over a 14-year horizon (corresponding to the 14 teams in the lottery). To quantify this effect, we compute the percent increase in the expected number of distinct outcomes between the 2018 and 2019 lottery systems, which we find to be approximately 19%.

Mathematically, the NBA Draft Lottery problem represents the expected number of distinct outcomes that occur in repeated weighted experiments. We can model this event by obtaining a concise, explicit formula for a probability distribution representing this scenario, which can then be used to compute its expected value and answer our Draft Lottery question. It is fascinating how mathematics weaves its structure into everyday scenarios.

Before we embark on this journey of discovery, it is important to note that the historical foundations of our work trace back to early analyses of games of chance, beginning with Cardano's *Liber de Ludo Aleae* [2], composed around 1560 and published posthumously in 1663. Although Cardano's treatment lacked the formal structure of later developments, it introduced many of the fundamental ideas of probability in the context of gambling. These ideas were later refined by Pascal and Fermat [14] in their correspondence on equiprobable events. Huygens, in his 1657 manuscript *De Ratiociniis in Ludo Aleae* [7], synthesized and extended this earlier work, providing the first systematic exposition of expectations and unequal-probability outcomes.

In [5], the authors obtained explicit formulas for the probability distribution, expected value, and variance of the number of distinct outcomes in the setting of repeated equiprobable events (see Corollary 3.1 and Corollary 3.2). Based on these results, they further proposed a conjecture asserting that, among all possible probability assignments, the expected number of distinct outcomes is maximized in the equiprobable case.

Our work advances this line of inquiry by deriving an explicit distribution for the number of distinct outcomes produced by repeated weighted experiments, thereby generalizing the results of [5] and extending Huygens' perspective into contemporary probabilistic modeling. This explicit distribution also enables us to prove the conjecture proposed in [5].

We develop the technical setup with the following definition:

Definition 1.1. Assume that n independent trials are performed in which a random integer from 1 to n is chosen. Let the probabilities of selecting each number be $p_1, p_2, \dots, p_n$ where $\sum_{i=1}^n p_i = 1$. We define the random variable X to be the number of distinct integers that will be seen during this procedure.

As convention, we assume $\binom{0}{0} = 1$. For $k_1 + k_2 + \dots + k_m = n$, we set

$$\binom{n}{k_1, k_2, \dots, k_m} = \frac{n!}{k_1! k_2! \dots k_m!}.$$

By the Multinomial Theorem, we have

$$(p_1 + p_2 + \dots + p_m)^n = \sum_{\substack{k_1, k_2, \dots, k_m \geq 0 \\ k_1 + k_2 + \dots + k_m = n}} \binom{n}{k_1, k_2, \dots, k_m} p_1^{k_1} p_2^{k_2} \dots p_m^{k_m}.$$

2. HEURISTIC DERIVATIONS FOR THE PROBABILITY DISTRIBUTION OF X

In this section, we develop two separate derivations. Both will serve as foundational tools for establishing further theorems in the subsequent sections.

2.1. Derivations via the Complementary Principle. To derive a probability distribution for X , we leverage the *complementary principle*. In this context, we take X to be the number of distinct pre-lottery positions that receive the top draft pick, in order to provide a more tangible example for the reader's motivation.

As an example, we model a hypothetical draft lottery containing five teams performed annually over five years. We specify probabilities p_1, p_2, p_3, p_4, p_5 , where $\sum_{i=1}^5 p_i = 1$, and each p_i denotes the probability that the i -th pre-lottery position wins the lottery and receives the top draft pick.

Case: $P[X = 1]$. The probability that exactly one distinct pre-lottery position receives the top pick across all five years is

$$P[X = 1] = \sum_{1 \leq i \leq 5} p_i^5. \quad (2.1)$$

Case $P[X = 2]$. To derive $P[X = 2]$, we again use the complementary principle. Fix a pair of distinct teams (i_1, i_2) . By the Binomial Theorem, the probability that the top pick falls to two pre-lottery positions i_1 and i_2 is $(p_{i_1} + p_{i_2})^5$. Within this event, we must exclude the cases where *only one* team appears among the five picks. Using our earlier expression for $P[X = 1]$, the total probability that all five picks come from a single team in $\{i_1, i_2\}$ can be written as $p_{i_1} P[X = 1] + p_{i_2} P[X = 1]$ or $(p_{i_1} + p_{i_2}) P[X = 1]$. Thus, for this fixed pair, the probability that *exactly two* distinct teams (namely i_1 and i_2) appear is $(p_{i_1} + p_{i_2})^5 - (p_{i_1} + p_{i_2}) P[X = 1]$. Summing over all possible pairs of teams yields

$$P[X = 2] = \sum_{1 \leq i_1 < i_2 \leq 5} \left[(p_{i_1} + p_{i_2})^5 - (p_{i_1} + p_{i_2}) P[X = 1] \right].$$

The correction term scales $P[X = 1]$ by the proportion of the total probability mass concentrated in $\{i_1, i_2\}$, since $P[X = 1]$ was computed over all five positions.

Case $P[X = 3]$. To derive an expression for $P[X = 3]$, we fix a subset of three teams and isolate the probability that *exactly* these three teams appear among the five outcomes. Let $S = \{i_1, i_2, i_3\}$ be a fixed triple of distinct pre-lottery positions to which the top pick may fall.

Step 1: *At most three distinct teams from S .* The probability that all five outcomes are drawn from this triple is

$$P(\text{all draws in } S) = (p_{i_1} + p_{i_2} + p_{i_3})^5. \quad (2.2)$$

This event includes all configurations with exactly one, exactly two, or exactly three distinct teams from S .

Step 2: *Decomposition by the number of distinct teams.* Focus first on the index i_1 in S . Among the three teams in S , the number of k -element subsets that contain i_1 is $\binom{3-1}{k-1}$, since we choose the remaining $k-1$ teams from $S \setminus \{i_1\}$. Among all five teams, the number of k -element subsets that contain i_1 is $\binom{5-1}{k-1}$, corresponding to choosing the remaining $k-1$ teams from $\{1, \dots, 5\} \setminus \{i_1\}$. Thus, the fraction of k -subsets that contain i_1 and lie entirely within S is $\binom{3-1}{k-1} / \binom{5-1}{k-1}$. Consequently, the portion of the event “exactly k distinct teams occur and i_1 is among them” that is compatible with all k teams lying in S is weighted by

$$\frac{\binom{3-1}{k-1}}{\binom{5-1}{k-1}} p_{i_1} P[X = k].$$

The same reasoning applies to i_2 and i_3 .

Step 3: *Isolating exactly three distinct teams from S .*

$$P[\text{exactly 3 distinct teams from } S] = (p_{i_1} + p_{i_2} + p_{i_3})^5 - \sum_{k=1}^2 \frac{\binom{3-1}{k-1}}{\binom{5-1}{k-1}} (p_{i_1} + p_{i_2} + p_{i_3}) P[X = k].$$

Finally, summing over all triples $S = \{i_1, i_2, i_3\}$ with $1 \leq i_1 < i_2 < i_3 \leq 5$ yields

$$P[X = 3] = \sum_{1 \leq i_1 < i_2 < i_3 \leq 5} \left[(p_{i_1} + p_{i_2} + p_{i_3})^5 - \sum_{k=1}^{3-1} \frac{\binom{3-1}{k-1}}{\binom{5-1}{k-1}} (p_{i_1} + p_{i_2} + p_{i_3}) P[X = k] \right].$$

General pattern: $P[X = r]$. The structure above extends naturally.

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq 5} \left[(p_{i_1} + \dots + p_{i_r})^5 - \sum_{k=1}^{r-1} \frac{\binom{r-1}{k-1}}{\binom{5-1}{k-1}} (p_{i_1} + \dots + p_{i_r}) P[X = k] \right],$$

where the inner sum subtracts, via complementary counting, the contributions corresponding to exactly k distinct teams. The factor $\binom{r-1}{k-1} / \binom{5-1}{k-1}$ is the proportion of k -element subsets of $\{1, \dots, 5\}$ (containing a fixed index) that lie entirely within the chosen r -subset $\{i_1, \dots, i_r\}$. This weighting ensures that only the portion of $P[X = k]$ attributable to subsets contained in $\{i_1, \dots, i_r\}$ is removed, thereby isolating the probability that exactly r distinct positions are observed.

2.2. Derivations via the Inclusion-Exclusion Principle. To derive the probability distribution for X , the number of distinct pre-lottery positions that receive the top draft pick, we leverage the *Inclusion-Exclusion Principle*. As a concrete example, we model a hypothetical draft lottery containing $n = 5$ teams performed annually over 5 years. We specify probabilities p_1, p_2, p_3, p_4, p_5 , where $\sum_{i=1}^5 p_i = 1$, and each p_i denotes the probability that the i -th pre-lottery position wins the top draft pick in any given year.

To find the probability $P[X = r]$ that exactly r distinct teams win the top pick, we first choose a specific subset of r teams, denoted by $S := \{i_1, \dots, i_r\} \subset \{1, 2, 3, 4, 5\}$ where $|S| = r$. We calculate the probability that the winners over all 5 years come *only* from this subset S , and use inclusion-exclusion to ensure all r teams are chosen at least once.

Case $P[X = 1]$. The probability that exactly one distinct pre-lottery position receives the top pick across all five years is simply the sum of the probabilities of each single team winning all 5 years:

$$P[X = 1] = \sum_{1 \leq i \leq 5} p_i^5.$$

Case $P[X = 2]$. For a fixed pair of teams $\{i_1, i_2\}$, the probability that all 5 picks come from this pair is $(p_{i_1} + p_{i_2})^5$. However, this includes the cases where only i_1 wins all 5 years, or only i_2 wins all 5 years. By the Inclusion-Exclusion Principle, we subtract these single-team outcomes:

$$P[X = 2] = \sum_{1 \leq i_1 < i_2 \leq 5} [(p_{i_1} + p_{i_2})^5 - p_{i_1}^5 - p_{i_2}^5].$$

Case $P[X = 3]$. For a fixed triplet of teams $\{i_1, i_2, i_3\}$, the baseline probability that all picks fall within this triplet is $(p_{i_1} + p_{i_2} + p_{i_3})^5$. To isolate the event where *all three* appear at least once, we subtract the cases where at most two appear, and add back the cases where only

one appears:

$$P[X = 3] = \sum_{1 \leq i_1 < i_2 < i_3 \leq 5} \left[(p_{i_1} + p_{i_2} + p_{i_3})^5 - \sum_{1 \leq j < k \leq 3} (p_{i_j} + p_{i_k})^5 + \sum_{1 \leq j \leq 3} p_{i_j}^5 \right].$$

General Pattern. The structure generalizes cleanly. For any $1 \leq r \leq 5$, the probability that exactly r distinct positions are observed is found by summing over all possible subsets S of size r , and applying the complementary principle internally to each subset:

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq 5} \sum_{j=0}^{r-1} (-1)^j \sum_{\substack{T \subset \{i_1, \dots, i_r\} \\ |T|=r-j}} \left(\sum_{i \in T} p_i \right)^5.$$

Alternatively, setting $k = r - j$, we get

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq 5} \sum_{k=1}^r (-1)^{r-k} \sum_{\substack{T \subset \{i_1, \dots, i_r\} \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^5. \quad (2.3)$$

3. ESTABLISHING THE PROBABILITY DISTRIBUTION OF X

Guided by the previous observations in Section 2.1, we examine the emerging patterns and use them to formulate a recursive characterization of the probability distribution of X .

Theorem 3.1. *Let N be a random variable with probability distribution*

$$P[N = k] = p_k, \quad 1 \leq k \leq n,$$

and X be the number of distinct integers observed in n independent observations of N . The probability of obtaining r distinct integers in n trials is given by

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} \left((p_{i_1} + \dots + p_{i_r})^n - \sum_{k=1}^{r-1} \frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} (p_{i_1} + \dots + p_{i_r}) P[X = k] \right). \quad (3.4)$$

Proof. As shown in Section 2.1, the complementary principle yields the structure required to isolate exact cardinalities, and Theorem 3.1 follows as a direct application of this framework. To this end, let $1 \leq r \leq n$ and fix an r -subset $S := \{i_1, \dots, i_r\} \subset \{1, \dots, n\}$, and consider the event that all observed integers lie in S and that exactly r distinct integers from S appear among the n draws.

Step 1: *At most r distinct integers from S .* By the Binomial Theorem, the probability that all n draws come from S (i.e., no integer outside S appears) is

$$P[\text{all draws in } S] = (p_{i_1} + \dots + p_{i_r})^n. \quad (3.5)$$

This event includes all configurations with exactly k distinct integers from S for $1 \leq k \leq r$.

Step 2: *Decomposing by the number of distinct integers.* Let $1 \leq k \leq r$ and let $X_k(S)$ denote the event that *exactly* k distinct integers appear among the outcomes, and that all of them belong to the subset S . Then the event in (3.5) can be decomposed as

$$\{\text{all draws in } S\} = \bigcup_{k=1}^r X_k(S),$$

and these events are disjoint. Hence

$$\sum_{k=1}^r P[X_k(S)] = (p_{i_1} + \cdots + p_{i_r})^n. \quad (3.6)$$

Our goal in the decomposition above is to single out the probability $P[X_r(S)]$, which captures the event that precisely r distinct integers from S are observed.

Step 3: *Relating $P[X_k(S)]$ to $P[X = k]$.* Fix k with $1 \leq k \leq r-1$. We now express $P[X_k(S)]$ as a fraction of the global probability $P[X = k]$. Focus first on a single index $i_1 \in S$. Among the r integers in S , the number of k -element subsets that contain i_1 is $\binom{r-1}{k-1}$, since we choose the remaining $k-1$ integers from $S \setminus \{i_1\}$. Among all n integers, the number of k -element subsets that contain i_1 is $\binom{n-1}{k-1}$, corresponding to choosing the remaining $k-1$ integers from $\{1, \dots, n\} \setminus \{i_1\}$. Thus, the fraction of k -subsets (containing i_1) that lie entirely in S is $\binom{r-1}{k-1} / \binom{n-1}{k-1}$. Consequently, the portion of the event “exactly k distinct integers occur and i_1 is among them” that is compatible with all k integers lying in S is weighted by

$$\frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} p_{i_1} P[X = k].$$

The same reasoning applies to each $i_j \in S$, $j = 1, \dots, r$. Summing over all indices in S yields

$$P[X_k(S)] = \frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} (p_{i_1} + \cdots + p_{i_r}) P[X = k], \quad 1 \leq k \leq r-1. \quad (3.7)$$

Step 4: *Isolating exactly r distinct integers from S .* Substituting (3.7) into (3.6) gives

$$(p_{i_1} + \cdots + p_{i_r})^n = \sum_{k=1}^{r-1} \frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} (p_{i_1} + \cdots + p_{i_r}) P[X = k] + P[X_r(S)].$$

Rearranging, we obtain

$$P[X_r(S)] = (p_{i_1} + \cdots + p_{i_r})^n - \sum_{k=1}^{r-1} \frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} (p_{i_1} + \cdots + p_{i_r}) P[X = k].$$

That is,

$$P[\text{exactly } r \text{ distinct integers from } S] = (p_{i_1} + \cdots + p_{i_r})^n - \sum_{k=1}^{r-1} \frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} (p_{i_1} + \cdots + p_{i_r}) P[X = k].$$

Summing over all r -subsets $S = \{i_1, \dots, i_r\}$ with $1 \leq i_1 < \cdots < i_r \leq n$ yields the desired result for $P[X = r]$. $\square$

Consider the summation of all possible $p_{i_1} + \cdots + p_{i_r}$, where $1 \leq i_1 < \cdots < i_r \leq n$. The total number of terms here is $r \binom{n}{r}$. Since each p_i appears equally, we can rearrange them to form $\frac{1}{n} [r \binom{n}{r}] = \binom{n-1}{r-1}$ blocks of $p_1 + \cdots + p_n$. Mathematically, we can express this formula as follows:

Lemma 3.1. *For any $1 \leq r \leq n$, we have*

$$\sum_{1 \leq i_1 < \cdots < i_r \leq n} (p_{i_1} + \cdots + p_{i_r}) = \binom{n-1}{r-1} (p_1 + \cdots + p_n) = \binom{n-1}{r-1}.$$

Proof. We establish the proof by mathematical induction on $n \geq 1$. The base case where $n = 1$ is simple as $n = 1$ implies that $r = 1$. Hence,

$$\sum_{1 \leq i_1 < \dots < i_r \leq 1} (p_{i_1} + \dots + p_{i_r}) = \sum_{r=1}^1 p_r = 1 = \binom{1-1}{1-1}.$$

We carry on with our inductive assumption that for some n and any $1 \leq r \leq n$,

$$\sum_{1 \leq i_1 < \dots < i_r \leq n} (p_{i_1} + \dots + p_{i_r}) = \binom{n-1}{r-1}.$$

For our inductive step with $p_1 + p_2 + \dots + p_n + p_{n+1} = 1$, let us set $q_i = \frac{p_i}{1-p_{n+1}}$ and apply the inductive hypothesis with $q_1 + q_2 + \dots + q_n = 1$ to ascertain

$$\sum_{1 \leq i_1 < \dots < i_r \leq n} (q_{i_1} + \dots + q_{i_r}) = \binom{n-1}{r-1}, \quad \sum_{1 \leq i_1 < \dots < i_{r-1} \leq n} (q_{i_1} + \dots + q_{i_{r-1}}) = \binom{n-1}{r-2}.$$

Now, we consider r -combinations of $n+1$ integers. There are two cases regarding the inclusion or exclusion of p_{n+1} .

Case 1. Let p_{n+1} be included in the sum's r -combination. We have

$$\begin{aligned} \Sigma_1 &:= \sum_{\substack{1 \leq i_1 < \dots < i_{r-1} \leq n \\ \text{including } p_{n+1}}} (p_{i_1} + \dots + p_{i_{r-1}} + p_{n+1}) \\ &= \sum_{1 \leq i_1 < \dots < i_{r-1} \leq n} (p_{i_1} + \dots + p_{i_{r-1}}) + \sum_{1 \leq i_1 < \dots < i_{r-1} \leq n} p_{n+1} \\ &= (1 - p_{n+1}) \sum_{1 \leq i_1 < \dots < i_{r-1} \leq n} (q_{i_1} + \dots + q_{i_{r-1}}) + \sum_{1 \leq i_1 < \dots < i_{r-1} \leq n} p_{n+1} \\ &= (1 - p_{n+1}) \binom{n-1}{r-2} + \binom{n}{r-1} p_{n+1}. \end{aligned}$$

Case 2. Let p_{n+1} be excluded in the sum's r -combination. We have

$$\begin{aligned} \Sigma_2 &:= \sum_{\substack{1 \leq i_1 < \dots < i_r \leq n \\ \text{excluding } p_{n+1}}} (p_{i_1} + \dots + p_{i_{r-1}} + p_{i_r}) \\ &= (1 - p_{n+1}) \sum_{1 \leq i_1 < \dots < i_r \leq n} (q_{i_1} + \dots + q_{i_{r-1}} + q_{i_r}) \\ &= (1 - p_{n+1}) \binom{n-1}{r-1}. \end{aligned}$$

Let

$$\Sigma := \sum_{1 \leq i_1 < \dots < i_r \leq n+1} (p_{i_1} + \dots + p_{i_r}) = \Sigma_1 + \Sigma_2.$$

Employing Pascal's identity,

$$\binom{n-1}{r-2} + \binom{n-1}{r-1} = \binom{n}{r-1},$$

and algebraic manipulations produces the desired result:

$$\begin{aligned}
 \Sigma &= (1 - p_{n+1}) \binom{n-1}{r-2} + \binom{n}{r-1} p_{n+1} + (1 - p_{n+1}) \binom{n-1}{r-1} \\
 &= (1 - p_{n+1}) \left[\binom{n-1}{r-2} + \binom{n-1}{r-1} \right] + \binom{n}{r-1} p_{n+1} \\
 &= (1 - p_{n+1}) \binom{n}{r-1} + \binom{n}{r-1} p_{n+1} = \binom{n}{r-1} = \binom{(n+1)-1}{r-1}.
 \end{aligned}$$

This completes the proof. $\square$

In our manuscript, Lemma 3.1 serves two purposes: simplifying our recursive formula and connecting our recursive formula to an explicit formula shown in a later theorem. Indeed, applying Lemma 3.1 after distributing the outer summation in Theorem 3.1 produces the following expression:

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} (p_{i_1} + \dots + p_{i_r})^n - \sum_{k=1}^{r-1} \frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} \binom{n-1}{r-1} P[X = k]. \quad (3.8)$$

The clarity of this equation is increased by applying the popular subcommittee identity, $\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$ along with the identity $\binom{m}{\ell} = \frac{m}{\ell} \binom{m-1}{\ell-1}$, which yields

$$\frac{\binom{r-1}{k-1}}{\binom{n-1}{k-1}} \binom{n-1}{r-1} = \frac{\binom{n-1}{r-1} \binom{r-1}{k-1}}{\binom{n-1}{k-1}} = \frac{\binom{n-1}{k-1} \binom{n-k}{r-k}}{\binom{n-1}{k-1}} = \binom{n-k}{r-k},$$

which can be substituted into (3.8) to obtain:

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} (p_{i_1} + \dots + p_{i_r})^n - \sum_{k=1}^{r-1} \binom{n-k}{r-k} P[X = k]. \quad (3.9)$$

To demonstrate the second use of Lemma 3.1, we first prove the following fact, which results in an explicit formula for the probability distribution of X . While this should signal the end of our search, we continue on, believing this explicit formula's complexity makes calculating the expected value and variance unnecessarily difficult. Though this explicit formula lacks the brevity we desire, its value stems from its connection to (3.9), which will prove invaluable to confirming our recursive formula's truth.

Theorem 3.2. *Let N be a random variable with probability distribution*

$$P[N = k] = p_k, \quad 1 \leq k \leq n,$$

and X be the number of distinct integers observed in n independent observations of N . Then the probability distribution of X is given by

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} \left[\sum_{\substack{y_{i_1} + \dots + y_{i_r} = n \\ \text{all } y_{i_j} \geq 1}} \frac{n!}{y_{i_1}! \dots y_{i_r}!} p_{i_1}^{y_{i_1}} \dots p_{i_r}^{y_{i_r}} \right], \quad r = 1, 2, \dots, n. \quad (3.10)$$

Proof. Let $Y_k = B(n, p_k)$ be a binomial random variable expressing the number of times that k occurs in the n observations. Then $Y = (Y_1, \dots, Y_n)$ has a multinomial distribution

with probabilities $p_1, \dots, p_n$, number of observations n , and joint probability function

$$p(y) = p(y_1, \dots, y_n) = \binom{n}{y_1, \dots, y_n} p_1^{y_1} \cdots p_n^{y_n} = \frac{n!}{y_1! \cdots y_n!} p_1^{y_1} \cdots p_n^{y_n}$$

where $y_1 + \cdots + y_n = n$. We now define the vectors

$$e_1 = (1, 0, \dots, 0) \quad e_2 = (0, 1, \dots, 0) \quad \dots \quad e_n = (0, 0, \dots, 1), \quad (3.11)$$

These vectors enable us to alternatively represent our joint probability function as

$$p(y_1, \dots, y_n) = p(y_1 e_1 + \cdots + y_n e_n) = \frac{n!}{y_1! \cdots y_n!} p_1^{y_1} \cdots p_n^{y_n}.$$

Hence, for $r = 1, \dots, n$, we have

$$\begin{aligned} P[X = r] &= \sum_{1 \leq i_1 < \cdots < i_r \leq n} \left[\sum_{\substack{y_{i_1} + \cdots + y_{i_r} = n \\ \text{all } y_{i_j} \geq 1}} p(y_{i_1} e_{i_1} + \cdots + y_{i_r} e_{i_r}) \right] \\ &= \sum_{1 \leq i_1 < \cdots < i_r \leq n} \left[\sum_{\substack{y_{i_1} + \cdots + y_{i_r} = n \\ \text{all } y_{i_j} \geq 1}} \frac{n!}{y_{i_1}! \cdots y_{i_r}!} p_{i_1}^{y_{i_1}} \cdots p_{i_r}^{y_{i_r}} \right], \end{aligned}$$

which is exactly what we desired to prove. $\square$

The connection between (3.9) and Theorem 3.2 can be observed in Remark 3.1 below.

Remark 3.1. We observe the following probabilities for $P[X = r]$:

$$\begin{aligned} (a) \quad r = 1: \quad P[X = 1] &= \sum_{k=1}^n p(n e_k) = \sum_{k=1}^n p_k^n. \\ (b) \quad r = 2: \quad P[X = 2] &= \sum_{1 \leq i < j \leq n} \left[\sum_{\substack{y_i + y_j = n \\ y_i \geq 1, y_j \geq 1}} p(y_i e_i + y_j e_j) \right] \\ &= \sum_{1 \leq i < j \leq n} \left[\left(\sum_{y_i + y_j = n} p(y_i e_i + y_j e_j) \right) - p(n e_i) - p(n e_j) \right] \\ &= \sum_{1 \leq i < j \leq n} \left[\left(\sum_{y_i + y_j = n} \frac{n!}{y_i! y_j!} p_i^{y_i} p_j^{y_j} \right) - p_i^n - p_j^n \right] \\ &= \sum_{1 \leq i < j \leq n} (p_i + p_j)^n - \sum_{1 \leq i < j \leq n} (p_i^n + p_j^n) \\ &= \sum_{1 \leq i < j \leq n} (p_i + p_j)^n - (n-1)P[X = 1]. \\ (c) \quad r = n: \quad P[X = n] &= p(1, \dots, 1) = n!(p_1 \cdots p_n). \end{aligned}$$

We highlight the connection between our recursive and explicit formulas through the calculation of $P[X = 2]$. Note how the vectorial representation in (3.11) clarifies the terms taken out of the inner summation. From this, we apply Lemma 3.1 to acquire our final result, connecting the findings from Theorem 3.2 to our recursive formula in (3.9). We leverage this connection to verify our recursive formula with the ensuing theorem.

Theorem 3.3. *Let N be a random variable with probability distribution*

$$P[N = k] = p_k, \quad 1 \leq k \leq n,$$

and X be the number of distinct integers observed in n independent observations of N . We have the following recursive formula for the probabilities:

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} (p_{i_1} + \dots + p_{i_r})^n - \sum_{k=1}^{r-1} \binom{n-k}{r-k} P[X = k]. \quad (3.12)$$

Proof. We establish this result in two parts: the explicit formula provided by Theorem 3.2 and the complementary counting method used to derive Theorem 3.1. To find $P[X = r]$, we leverage the multinomial theorem to calculate $P[X = s]$ for any s in $\{1, 2, \dots, r\}$. We then use the explicit formula from Theorem 3.2 to subtract the probability $P[X = t]$ for any t in $\{1, 2, \dots, r-1\}$. Calculating this for every r -combination in the set $\{1, 2, \dots, n\}$ gives the expression

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{\substack{y_{i_1} + \dots + y_{i_r} = n \\ \text{all } y_{i_j} \geq 1}} \frac{n!}{y_{i_1}! \dots y_{i_r}!} p_{i_1}^{y_{i_1}} \dots p_{i_r}^{y_{i_r}} := P_1 - P_2$$

where

$$P_1 = \sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{y_{i_1} + \dots + y_{i_r} = n} \frac{n!}{y_{i_1}! \dots y_{i_r}!} p_{i_1}^{y_{i_1}} \dots p_{i_r}^{y_{i_r}} = \sum_{1 \leq i_1 < \dots < i_r \leq n} (p_{i_1} + \dots + p_{i_r})^n,$$

and

$$P_2 = \sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{k=1}^{r-1} \sum_{1 \leq j_1 < \dots < j_{r-k} \leq r} \sum_{\substack{y_{i_1} + \dots + y_{i_r} = n \\ y_{j_1} = \dots = y_{j_{r-k}} = 0}} \frac{n!}{y_{i_1}! \dots y_{i_r}!} p_{i_1}^{y_{i_1}} \dots p_{i_r}^{y_{i_r}}.$$

While one may be fairly familiar with P_1 from our recursive formula, P_2 may seem daunting at first glance due to its four summations. To simplify P_2 , we first swap the outermost two sums to obtain

$$P_2 = \sum_{k=1}^{r-1} \sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{1 \leq j_1 < \dots < j_{r-k} \leq r} \sum_{\substack{y_{i_1} + \dots + y_{i_r} = n \\ y_{j_1} = \dots = y_{j_{r-k}} = 0}} \frac{n!}{y_{i_1}! \dots y_{i_r}!} p_{i_1}^{y_{i_1}} \dots p_{i_r}^{y_{i_r}}.$$

We now use (3.10) to turn two sums into one, yielding

$$P_2 = \sum_{k=1}^{r-1} \binom{n}{r} \binom{r}{r-k} \left[\frac{1}{\binom{n}{k}} \sum_{1 \leq l_1 < \dots < l_k \leq n} \sum_{\substack{y_{l_1} + \dots + y_{l_k} = n \\ \text{all } y_{l_j} \geq 1}} \frac{n!}{y_{l_1}! \dots y_{l_k}!} p_{l_1}^{y_{l_1}} \dots p_{l_k}^{y_{l_k}} \right].$$

To obtain $P[X = k]$ in the above without over-counting, it is necessary to divide by $\binom{n}{k}$. Next, we apply (3.10) and the subcommittee identity to P_2 , which produces the expression

$$P_2 = \sum_{k=1}^{r-1} \frac{\binom{n}{r} \binom{r-k}{r-k}}{\binom{n}{k}} P[X = k] = \sum_{k=1}^{r-1} \frac{\binom{n}{k} \binom{n-k}{r-k}}{\binom{n}{k}} P[X = k] = \sum_{k=1}^{r-1} \binom{n-k}{r-k} P[X = k].$$

Putting P_1 and P_2 together, we get our desired result:

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} (p_{i_1} + \dots + p_{i_r})^n - \sum_{k=1}^{r-1} \binom{n-k}{r-k} P[X = k].$$

This completes the proof of Theorem 3.3. $\square$

Remark 3.2. Lemma 3.1 furnishes the link between Theorem 3.1 and Theorem 3.3, thereby yielding an alternative proof of Theorem 3.3 via Theorem 3.1.

Because the laborious nature of the recursive formula makes it unsuitable for actual calculations, we instead utilize it to derive a clear, explicit formula that enables us to calculate $P[X = r]$ directly for $1 \leq r \leq n$. For brevity, we let

$$Q(n, k) := \sum_{1 \leq i_1 < \dots < i_k \leq n} (p_{i_1} + \dots + p_{i_k})^n. \quad (3.13)$$

We now embark on our journey of discovery by employing (3.9) to calculate $P[X = r]$ for $n = 5$ and $1 \leq r \leq 5$, leading to the pattern:

$$\begin{aligned} P[X = 1] &= \binom{4}{0} Q(5, 1), \\ P[X = 2] &= -\binom{4}{1} Q(5, 1) + \binom{3}{0} Q(5, 2), \\ P[X = 3] &= \binom{4}{2} Q(5, 1) - \binom{3}{1} Q(5, 2) + \binom{2}{0} Q(5, 3), \\ P[X = 4] &= -\binom{4}{3} Q(5, 1) + \binom{3}{2} Q(5, 2) - \binom{2}{1} Q(5, 3) + \binom{1}{0} Q(5, 4), \\ P[X = 5] &= \binom{4}{4} Q(5, 1) - \binom{3}{3} Q(5, 2) + \binom{2}{2} Q(5, 3) - \binom{1}{1} Q(5, 4) + \binom{0}{0} Q(5, 5). \end{aligned}$$

A careful observation of the above findings culminates our search with a simple explicit formula for the probability distribution of X .

Theorem 3.4. Let N be a random variable with probability distribution

$$P[N = k] = p_k, \quad 1 \leq k \leq n,$$

and X be the number of distinct integers observed in n independent observations of N . The probability of obtaining r distinct integers in n trials is explicitly given by

$$P[X = r] = \sum_{k=1}^r (-1)^{r-k} \binom{n-k}{r-k} \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} (p_{i_1} + \dots + p_{i_k})^n \right). \quad (3.14)$$

Proof of Theorem 3.4 by Induction. To prove this result, we shall use induction on r and Theorem 3.3. As in (3.13), we use

$$Q(n, k) = \sum_{1 \leq i_1 < \dots < i_k \leq n} (p_{i_1} + \dots + p_{i_k})^n.$$

Note that $\binom{n-1}{0} = 1$. When $r = 1$, (3.12) becomes

$$P[X = 1] = Q(n, 1) = \sum_{k=1}^1 (-1)^{1-k} \binom{n-k}{1-k} Q(n, k),$$

which is consistent with (3.14). Now assume that

$$P[X = m] = \sum_{k=1}^m (-1)^{m-k} \binom{n-k}{m-k} Q(n, k) \quad (3.15)$$

holds true for any m with $1 \leq m \leq r < n$. Let's prove this for $m = r + 1 \leq n$. Recall

$$\sum_{t=1}^r \left[\sum_{k=1}^t a(k, t) \right] = \sum_{k=1}^r \left[\sum_{t=k}^r a(k, t) \right] = \sum_{k=1}^r \left[\sum_{t=0}^{r-k} a(k, t+k) \right]. \quad (3.16)$$

Another well-known fact from Pascal's Triangle is the symmetry property:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}. \quad (3.17)$$

Furthermore, using the subcommittee identity, we have

$$\binom{n-k}{n-i} \binom{n-i}{n-r-1} = \binom{n-k}{n-r-1} \binom{r+1-k}{r+1-i} \quad (3.18)$$

for integers $n \geq r + 1 \geq i \geq k \geq 0$. Finally, using the Binomial Theorem we get

$$\sum_{i=0}^{r+1-k} (-1)^i \binom{r+1-k}{i} = (1-1)^{r+1-k} = 0,$$

and hence we arrive at

$$\sum_{i=0}^{r-k} (-1)^i \binom{r+1-k}{i} = -(-1)^{r+1-k} \binom{r+1-k}{r+1-k}. \quad (3.19)$$

Also, we can write

$$Q(n, r+1) = (-1)^{r+1-(r+1)} \binom{n-(r+1)}{r+1-(r+1)} Q(n, r+1). \quad (3.20)$$

Using Theorem 3.3 and taking into account the inductive hypothesis in (3.15) together with the identities in (3.16), (3.17), (3.18), (3.19) and (3.20), we observe that

$$\begin{aligned} P[X = r+1] &= \sum_{1 \leq i_1 < \dots < i_{r+1} \leq n} (p_{i_1} + \dots + p_{i_{r+1}})^n - \sum_{i=1}^{(r+1)-1} \binom{n-i}{(r+1)-i} P[X = i] \\ &\stackrel{(3.17)}{=} Q(n, r+1) - \sum_{i=1}^r \binom{n-i}{n-r-1} \left[\sum_{k=1}^i (-1)^{i-k} \binom{n-k}{n-i} Q(n, k) \right] \\ &\stackrel{(3.16) \& (3.18)}{=} Q(n, r+1) - \sum_{k=1}^r Q(n, k) \binom{n-k}{n-r-1} \left[\sum_{i=k}^r (-1)^{i-k} \binom{r+1-k}{r+1-i} \right] \\ &\stackrel{(3.16) \& (3.17)}{=} Q(n, r+1) - \sum_{k=1}^r Q(n, k) \binom{n-k}{n-r-1} \left[\sum_{i=0}^{r-k} (-1)^i \binom{r+1-k}{i} \right] \\ &\stackrel{(3.19) \& (3.20)}{=} \sum_{k=1}^{r+1} (-1)^{r+1-k} \binom{n-k}{r+1-k} Q(n, k) \\ &= \sum_{k=1}^{(r+1)} (-1)^{(r+1)-k} \binom{n-k}{(r+1)-k} \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} (p_{i_1} + \dots + p_{i_k})^n \right), \end{aligned}$$

which completes the proof by induction. $\square$

We now present an alternative proof of the explicit formula in (3.14). This argument relies on the inclusion-exclusion principle introduced in Section 2.2. Since the steps closely parallel those already given, we leave the details to the reader.

Theorem 3.5. *Let N be a random variable with probability distribution*

$$P[N = k] = p_k, \quad 1 \leq k \leq n,$$

and X be the number of distinct integers observed in n independent observations of N . The probability of obtaining r distinct integers in n trials is given by

$$P[X = r] = \sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{k=1}^r (-1)^{r-k} \sum_{\substack{T \subset \{i_1, \dots, i_r\} \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n. \quad (3.21)$$

Proof of Theorem 3.4 by the Inclusion-Exclusion Principle. Fix k with $0 \leq k \leq r \leq n$. Let

$$J = \{1, \dots, n\} \quad \text{and} \quad S = \{i_1, \dots, i_r\} \subset J.$$

Note that we can write

$$\sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{\substack{T \subset \{i_1, \dots, i_r\} \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n = \sum_{\substack{S \subset J \\ |S|=r}} \sum_{\substack{T \subset S \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n \quad (3.22)$$

The double sum $\sum_{\substack{S \subset J \\ |S|=r}} \sum_{\substack{T \subset S \\ |T|=k}}$ ranges over all pairs (S, T) with $T \subset S \subset J$, $|T| = k$, and $|S| = r$. Interchanging the order of summation, and grouping these pairs by T first gives

$$\sum_{\substack{S \subset J \\ |S|=r}} \sum_{\substack{T \subset S \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n = \sum_{\substack{T \subset J \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n \left| \{ S : T \subset S \subset J, |S| = r \} \right|. \quad (3.23)$$

Each pair (S, T) is counted exactly once on either side. For a fixed T with $|T| = k$, an admissible S is obtained by adjoining $r - k$ further elements chosen from $J \setminus T$, a set of size $n - k$. Hence

$$\left| \{ S : T \subset S \subset J, |S| = r \} \right| = \binom{n-k}{r-k}, \quad (3.24)$$

independently of T . Therefore, combining (3.22), (3.23), and (3.24), we have

$$\sum_{1 \leq i_1 < \dots < i_r \leq n} \sum_{\substack{T \subset \{i_1, \dots, i_r\} \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n = \binom{n-k}{r-k} \sum_{\substack{T \subset J \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n. \quad (3.25)$$

Using (3.21) together with (3.25), we obtain

$$P[X = r] = \sum_{k=1}^r (-1)^{r-k} \binom{n-k}{r-k} \sum_{\substack{T \subset J \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n.$$

Writing the inner sum $\sum_{\substack{T \subset J \\ |T|=k}}$ explicitly over all k -tuples $1 \leq i_1 < \dots < i_k \leq n$ yields (3.14). $\square$

Remark 3.3. Yet another way of seeing (3.25) is to regroup the terms. Let $J = \{1, \dots, n\}$ and $S = \{i_1, \dots, i_r\} \subset J$, and fix k with $0 \leq k \leq r$. We shall count the terms of the form $(\sum_{i \in T} p_i)^n$. Summing over every admissible pair (first choosing an r -subset $S \subset J$, then a k -subset $T \subset S$) produces $\binom{n}{r} \binom{r}{k}$ terms of the form $(\sum_{i \in T} p_i)^n$, counted with repetition, since the same T can arise from several different S . Rather than collecting these terms by S , that is, into the inner sums

$$\sum_{\substack{T \subset \{i_1, \dots, i_r\} \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n,$$

each containing $\binom{r}{k}$ terms, we collect them into copies of the full sum

$$\sum_{\substack{T \subset J \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n,$$

each containing $\binom{n}{k}$ terms. By symmetry every k -subset $T \subset J$ occurs equally often, so the terms assemble into whole copies of this full sum. The number of copies is therefore

$$\frac{\binom{n}{r} \binom{r}{k}}{\binom{n}{k}},$$

and consequently

$$\sum_{\substack{S \subset J \\ |S|=r}} \sum_{\substack{T \subset S \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n = \frac{\binom{n}{r} \binom{r}{k}}{\binom{n}{k}} \sum_{\substack{T \subset J \\ |T|=k}} \left(\sum_{i \in T} p_i \right)^n.$$

Using the identity $\binom{n}{r} \binom{r}{k} = \binom{n}{k} \binom{n-k}{r-k}$ together with (3.22), we arrive at (3.25).

Remark 3.4. Observe that if in particular $p_i = 1/n$ for all $i = 1, \dots, n$, then

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} (p_{i_1} + \dots + p_{i_k})^n = \sum_{1 \leq i_1 < \dots < i_k \leq n} \left(\frac{k}{n} \right)^n = \binom{n}{k} \left(\frac{k}{n} \right)^n.$$

Using Remark 3.4, we can match our results with the findings in [5] as follows.

Corollary 3.1. If n events are independent and equally likely to occur (i.e., $p_i = 1/n$ for all $i = 1, \dots, n$), then

$$P[X = r] = \sum_{k=1}^r (-1)^{r-k} \binom{n-k}{r-k} \left[\binom{n}{k} \left(\frac{k}{n} \right)^n \right].$$

Theorem 3.4 simplifies calculating the expected value and variance of X , which will be utilized to answer the Draft Lottery question posed at the start of this paper.

Theorem 3.6. Let N be a random variable with probability distribution

$$P[N = k] = p_k, \quad 1 \leq k \leq n,$$

and X be the number of distinct integers observed in n observations of N with repetition. Then

(a) The expected value of X is given by

$$E(X) = \sum_{k=1}^n (1 - (1 - p_k)^n) = n - \sum_{k=1}^n (1 - p_k)^n.$$

(b) The variance of X is given by

$$\text{Var}(X) = \sum_{k=1}^n (1 - p_k)^n - (1 - p_k)^{2n} + 2 \sum_{1 \leq j < k \leq n} (1 - p_j - p_k)^n - (1 - p_j)^n (1 - p_k)^n.$$

Proof of (a). We will provide two proofs: one using the probability distribution from Theorem 3.4 along with combinatorial identities, and one using the indicator random variable, which proves more efficient in deriving the distribution's expected value.

Proof 1. Here we use Theorem 3.4 to derive the expected value formula:

$$P[X = r] = \sum_{k=1}^r (-1)^{r-k} \binom{n-k}{r-k} Q(n, k), \quad Q(n, k) := \sum_{1 \leq i_1 < \dots < i_k \leq n} (p_{i_1} + \dots + p_{i_k})^n.$$

Employing Theorem 3.4, we can rewrite $E(X) = \sum_{r=1}^n r P[X = r]$ as:

$$E(X) = \sum_{r=1}^n (n - (n - r)) \left(\sum_{k=1}^r (-1)^{r-k} \binom{n-k}{r-k} Q(n, k) \right).$$

Upon applying the distributive property and the fact that $n = \sum_{r=1}^n n P[X = r]$ yields

$$E(X) = n - \sum_{r=1}^{n-1} (n - r) \left(\sum_{k=1}^r (-1)^{r-k} \binom{n-k}{n-r} Q(n, k) \right).$$

Since $\binom{n-k}{n-r} = \frac{n-k}{n-r} \binom{n-1-k}{n-1-r}$, re-indexing the inner summation enables us to obtain

$$E(X) = n - \sum_{k=1}^{n-1} (n - k) Q(n, k) \left(\sum_{r=k}^{n-1} (-1)^{r-k} \binom{n-1-k}{n-1-r} \right).$$

The following identity then allows for the removal of the inner summation:

$$\sum_{r=k}^{n-1} (-1)^{r-k} \binom{n-1-k}{n-1-r} = \begin{cases} 0 & \text{if } k < n-1 \\ 1 & \text{if } k = n-1 \end{cases} := \delta_k(n-1),$$

which allows us to reduce the expected value formula to

$$E(X) = n - \sum_{k=1}^{n-1} (n - k) Q(n, k) \delta_k(n-1) = n - (n - (n-1)) Q(n, n-1).$$

Finally, utilizing the equality

$$Q(n, n-1) = \sum_{1 \leq i_1 < \dots < i_{n-1} \leq n} (p_{i_1} + \dots + p_{i_{n-1}})^n = \sum_{k=1}^n (1 - p_k)^n$$

provides our desired result for the expected value of X :

$$E(X) = \sum_{k=1}^n [1 - (1 - p_k)^n].$$

Proof 2. Let I_k be the indicator random variable in n observations of N . Then

$$X = \sum_{k=1}^n I_k.$$

Let $Y_k = B(n, p_k)$, the number of times that k occurs in n observations. Thus, we obtain $P(I_k = 0) = P(Y_k = 0) = (1 - p_k)^n$ and $P(I_k = 1) = 1 - (1 - p_k)^n$. Hence, we acquire $E(I_k) = 1 - (1 - p_k)^n$, allowing us to arrive at the desired result

$$E(X) = E\left(\sum_{k=1}^n I_k\right) = \sum_{k=1}^n E(I_k) = \sum_{k=1}^n [1 - (1 - p_k)^n].$$

Proof of (b). Similar to (a), we will provide two proofs: one utilizing combinatorial arguments and the other utilizing the indicator random variable.

Proof 1. Using the identity in [5],

$$\sum_{r=0}^m (-1)^r r^p \binom{m}{r} = 0 \quad \text{if } 0 \leq p \leq m-1, \quad m \geq 1,$$

we uncover the following three results that we will employ to derive $E(X^2)$.

$$\sum_{r=0}^m (-1)^r \binom{m}{r} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \geq 1 \end{cases}, \quad \sum_{r=0}^m (-1)^r r \binom{m}{r} = \begin{cases} 0 & \text{if } m = 0 \\ -1 & \text{if } m = 1 \\ 0 & \text{if } m \geq 2 \end{cases}, \quad (3.26)$$

$$\sum_{r=0}^m (-1)^r r^2 \binom{m}{r} = \begin{cases} 0 & \text{if } m = 0 \\ -1 & \text{if } m = 1 \\ 2 & \text{if } m = 2 \\ 0 & \text{if } m \geq 3 \end{cases}. \quad (3.27)$$

We begin by observing that

$$E(X^2) = \sum_{r=1}^n r^2 P[X = r] = \sum_{r=1}^n r^2 \left(\sum_{k=1}^r (-1)^{r-k} \binom{n-k}{r-k} Q(n, k) \right).$$

Applying the idea from (3.16) to rearrange the double summation then yields:

$$E(X^2) = \sum_{k=1}^n Q(n, k) \left(\sum_{r=0}^{n-k} (-1)^r (r+k)^2 \binom{n-k}{r} \right).$$

Whence expansion of terms allows us to get

$$\begin{aligned} E(X^2) &= \sum_{k=1}^n Q(n, k) \left(\sum_{r=0}^{n-k} (-1)^r r^2 \binom{n-k}{r} \right) + \sum_{k=1}^n 2k Q(n, k) \left(\sum_{r=0}^{n-k} (-1)^r r \binom{n-k}{r} \right) \\ &\quad + \sum_{k=1}^n k^2 Q(n, k) \left(\sum_{r=0}^{n-k} (-1)^r \binom{n-k}{r} \right). \end{aligned}$$

Use of (3.26) and (3.27) allows us to conclude that

$$\begin{aligned} E(X^2) &= Q(n, n-2) \left(\sum_{r=0}^2 (-1)^r r^2 \binom{2}{r} \right) + Q(n, n-1) \left(\sum_{r=0}^1 (-1)^r r^2 \binom{2}{r} \right) \\ &\quad + 2(n-1) Q(n, n-1) \left(\sum_{r=0}^1 (-1)^r r \binom{1}{r} \right) + n^2 Q(n, n) \left(\sum_{r=0}^0 (-1)^r \binom{0}{r} \right). \end{aligned}$$

Invoking (3.26) and (3.27) again produces

$$E(X^2) = n^2 Q(n, n) - (2n-1)Q(n, n-1) + 2Q(n, n-2).$$

Knowing that $Q(n, n) = 1$, $Q(n, n-1) = \sum_{k=1}^n (1-p_k)^n$, and

$$Q(n, n-2) = \sum_{1 \leq i_1 < \dots < i_{n-2} \leq n} (p_{i_1} + \dots + p_{i_{n-2}})^n = \sum_{1 \leq j < k \leq n} (1-p_j - p_k)^n,$$

we arrive at

$$E(X^2) = n^2 + (1-2n) \sum_{k=1}^n (1-p_k)^n + 2 \sum_{1 \leq j < k \leq n} (1-p_j - p_k)^n.$$

Additionally, we can use the expected value formula derived in (a) to say that

$$\begin{aligned} E(X)^2 &= \left(n - \sum_{k=1}^n (1-p_k)^n \right)^2 = n^2 - 2n \sum_{k=1}^n (1-p_k)^n + \left(\sum_{k=1}^n (1-p_k)^n \right)^2 \\ &= n^2 - 2n \sum_{k=1}^n (1-p_k)^n + \sum_{k=1}^n (1-p_k)^{2n} + 2 \sum_{1 \leq j < k \leq n} (1-p_j)^n (1-p_k)^n. \end{aligned}$$

With the formulas for $E(X^2)$ and $E(X)^2$ calculated, we are able to derive the variance of X as follows:

$$\begin{aligned} \text{Var}(X) &= E(X^2) - E(X)^2 \\ &= \sum_{k=1}^n [(1-p_k)^n - (1-p_k)^{2n}] + 2 \sum_{1 \leq j < k \leq n} [(1-p_j - p_k)^n - (1-p_j)^n (1-p_k)^n]. \end{aligned}$$

Proof 2. Using the indicator random variable, we observe $P(I_k = 0) = (1-p_k)^n$ and

$$P(I_j = 0 \text{ and } I_k = 0) = (1-p_j - p_k)^n \text{ for any } 1 \leq j < k \leq n.$$

Since $P(I_k = 0) = P(I_j = 0 \text{ and } I_k = 0) + P(I_j = 1 \text{ and } I_k = 0)$, we acquire

$$\begin{aligned} P(I_j = 1 \text{ and } I_k = 0) &= P(I_k = 0) - P(I_j = 0 \text{ and } I_k = 0) \\ &= (1-p_k)^n - (1-p_j - p_k)^n. \end{aligned}$$

Similarly, we can calculate the probabilities

$$\begin{aligned} P(I_j = 1, I_k = 0) &= (1-p_k)^n - (1-p_j - p_k)^n, \\ P(I_j = 1, I_k = 1) &= 1 - (1-p_j)^n - (1-p_k)^n + (1-p_j - p_k)^n, \end{aligned}$$

and utilize them to construct the joint probability table of (I_j, I_k) below.

	$I_k = 0$	$I_k = 1$	$P(I_j)$
$I_j = 0$	$(1 - p_j - p_k)^n$	$(1 - p_j)^n - (1 - p_j - p_k)^n$	$(1 - p_j)^n$
$I_j = 1$	$(1 - p_k)^n - (1 - p_j - p_k)^n$	$1 - (1 - p_j)^n - (1 - p_k)^n + (1 - p_j - p_k)^n$	$1 - (1 - p_j)^n$
$P(I_k)$	$(1 - p_k)^n$	$1 - (1 - p_k)^n$	1

Employing $E(I_k) = 1 - (1 - p_k)^n$, we procure $E(I_j I_k)$, $\text{Cov}(I_j, I_k)$, and $\text{Cov}(I_k, I_k)$:

$$E(I_j I_k) = \sum_{I_j=0}^1 \sum_{I_k=0}^1 (I_j I_k) P(I_j, I_k) = 1 - (1 - p_j)^n - (1 - p_k)^n + (1 - p_j - p_k)^n,$$

$$\text{Cov}(I_j, I_k) = E(I_j I_k) - E(I_j)E(I_k) = (1 - p_j - p_k)^n - (1 - p_j)^n(1 - p_k)^n,$$

$$\text{Cov}(I_k, I_k) = \text{Var}(I_k) = E(I_k^2) - E(I_k)^2 = (1 - p_k)^n - (1 - p_k)^{2n}.$$

Using the findings above, we acquire the variance $\text{Var}(X) = \text{Var}(\sum_{k=1}^n I_k)$ of X :

$$\begin{aligned} \text{Var}(X) &= \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \text{Cov}(I_1, I_1) & \text{Cov}(I_1, I_2) & \cdots & \text{Cov}(I_1, I_n) \\ \text{Cov}(I_2, I_1) & \text{Cov}(I_2, I_2) & \cdots & \text{Cov}(I_2, I_n) \\ \vdots & \vdots & \ddots & \vdots \\ \text{Cov}(I_n, I_1) & \text{Cov}(I_n, I_2) & \cdots & \text{Cov}(I_n, I_n) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\ &= \sum_{k=1}^n \text{Cov}(I_k, I_k) + 2 \sum_{1 \leq j < k \leq n} \text{Cov}(I_j, I_k) \\ &= \sum_{k=1}^n [(1 - p_k)^n - (1 - p_k)^{2n}] + 2 \sum_{1 \leq j < k \leq n} [(1 - p_j - p_k)^n - (1 - p_j)^n(1 - p_k)^n], \end{aligned}$$

as desired. This completes the proof. $\square$

Theorem 3.6 leads to the subsequent corollary previously obtained in [5].

Corollary 3.2. *Let N be a random variable with probability distribution*

$$P[N = i] = \frac{1}{n}, \quad 1 \leq i \leq n,$$

and X be the number of distinct integers that occur in n observations of N with repetition. Then

$$\begin{aligned} (a) \quad E(X) &= n \left[1 - \left(1 - \frac{1}{n} \right)^n \right]. \\ (b) \quad \text{Var}(X) &= n(n-1) \left(\frac{n-2}{n} \right)^n + n \left(\frac{n-1}{n} \right)^n - n^2 \left(\frac{n-1}{n} \right)^{2n}. \end{aligned}$$

Proof of (a). Setting $p_k = 1/n$ for $1 \leq k \leq n$ in part (a) of Theorem 3.6 yields

$$E(X) = \sum_{k=1}^n [(1 - (1 - p_k)^n)] = \sum_{k=1}^n \left[1 - \left(1 - \frac{1}{n} \right)^n \right] = n \left[1 - \left(1 - \frac{1}{n} \right)^n \right].$$

Proof of (b). Setting $p_k = 1/n$ for $1 \leq k \leq n$ in part (b) of Theorem 3.6 gives

$$\begin{aligned} \text{Var}(X) &= \sum_{k=1}^n [(1 - p_k)^n - (1 - p_k)^{2n}] + 2 \sum_{1 \leq j < k \leq n} [(1 - p_j - p_k)^n - (1 - p_j)^n (1 - p_k)^n] \\ &= n \left[\left(1 - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^{2n} \right] + 2 \binom{n}{2} \left[\left(1 - \frac{1}{n} - \frac{1}{n}\right)^n - \left(1 - \frac{1}{n}\right)^n \left(1 - \frac{1}{n}\right)^n \right] \\ &= n(n-1) \left(\frac{n-2}{n}\right)^n + n \left(\frac{n-1}{n}\right)^n - n^2 \left(\frac{n-1}{n}\right)^{2n}, \end{aligned}$$

as desired. $\square$

4. APPLICATIONS

4.1. Weighted Minority Representation in Bootstrap Sampling. The distribution of distinct outcomes plays a meaningful role in analyzing the behavior of bootstrap sampling procedures commonly used in ensemble methods such as Bagging. In bootstrap resampling, observations are drawn with replacement, which naturally leads to variability in the number of unique data points represented in each resampled dataset. Characterizing this distribution provides insight into the expected diversity of the bootstrap samples and, consequently, the degree of heterogeneity among the models in the ensemble. Moreover, in settings with class imbalance, the number of distinct observations from each class that appear in a bootstrap sample can deviate substantially from their proportions in the original dataset. Understanding the distribution of distinct outcomes therefore helps quantify how bootstrap sampling may amplify or mitigate imbalance across resamples. This perspective clarifies both the statistical properties of Bagging and the mechanisms through which bootstrap diversity contributes to model robustness.

Example 4.1 (Expected Representation of Weighted Minority Classes). *Consider a categorical random variable N taking values in $\{1, 2, 3\}$ with the imbalanced probability distribution*

$$P(N = 1) = p_1 = 0.80, \quad P(N = 2) = p_2 = 0.15, \quad P(N = 3) = p_3 = 0.05.$$

Here, class 1 is the majority class, while classes 2 and 3 represent minority groups with different weights. Suppose we draw $n = 3$ observations of N with replacement, and let X denote the number of distinct classes observed in these three draws. By Theorem 3.6, the expected number of distinct outcomes is

$$E(X) = \sum_{k=1}^3 (1 - (1 - p_k)^3) = 1 - (1 - p_1)^3 + 1 - (1 - p_2)^3 + 1 - (1 - p_3)^3.$$

We compute each term:

$$\begin{aligned} 1 - (1 - p_1)^3 &= 1 - (0.2)^3 = 0.992, \\ 1 - (1 - p_2)^3 &= 1 - (0.85)^3 \approx 0.386, \\ 1 - (1 - p_3)^3 &= 1 - (0.95)^3 \approx 0.143. \end{aligned}$$

Thus,

$$E(X) \approx 0.992 + 0.385875 + 0.142625 = 1.5205.$$

This calculation shows that, in expectation, a bootstrap sample of size 3 contains only about 1.52 distinct classes. The majority class appears with probability 0.992, meaning it is almost always represented. The heavier minority class ($p_2 = 0.15$) appears with probability 0.386, while the rarest minority class ($p_3 = 0.05$) appears with probability only 0.143. The theorem therefore quantifies how weighted minority classes contribute to the expected diversity of bootstrap samples and highlights how rare classes may be systematically underrepresented in small resamples.

4.2. Impact of Lottery Weighting on Pre-Lottery Position Diversity. Before turning to the NBA data, it is useful to note that the number of distinct outcomes in a weighted lottery provides a natural measure of how widely the lottery spreads opportunity among participants. In a system where each participant (or team) is assigned a probability weight, the distribution of distinct outcomes captures the likelihood that multiple participants appear among the winning positions, rather than a small subset repeatedly occupying them. This perspective is especially relevant in settings where the weighting scheme is intentionally designed to balance fairness, unpredictability, and competitive incentives.

The following example examines the NBA Draft Lottery in the 2018 and 2019 seasons, illustrating how teams occupy distinct pre-lottery positions under the league’s weighted lottery structure and how the distribution of these positions changes across the two years due to modifications in the lottery rules.

Example 4.2 (Distinct pre-lottery positions in the NBA). *Using Remark 3.1, Theorem 3.4, and Theorem 3.6, we compute the probabilities and expected values for the number of distinct pre-lottery positions observed in the NBA Draft Lottery for the 2018 and 2019 seasons:*

	2018	2019
$P[X = 1]$	3.880×10^{-9}	3.583×10^{-12}
$P[X = 6]$	0.3214	0.1171
$P[X = 7]$	0.3096	0.2826
$P[X = 8]$	0.1461	0.3280
$P[X = 14]$	1.783×10^{-10}	3.662×10^{-8}
$E(X)$	6.5059	7.7297

TABLE 4.2. Probabilities and expected values for the number of distinct pre-lottery positions that receive the top draft pick over 14 years using both 2018 and 2019 odds.

We conclude that the NBA’s changes to the draft lottery increased the expected number of pre-lottery positions receiving the top pick over a 14-year period from 6.5059 to 7.7297, representing an increase of approximately 19%. This finding is supported by the following data in Figure 1 garnered by sampling a 14-year period of the NBA Draft Lottery 10^5 times using both 2018 and 2019 lottery probabilities.

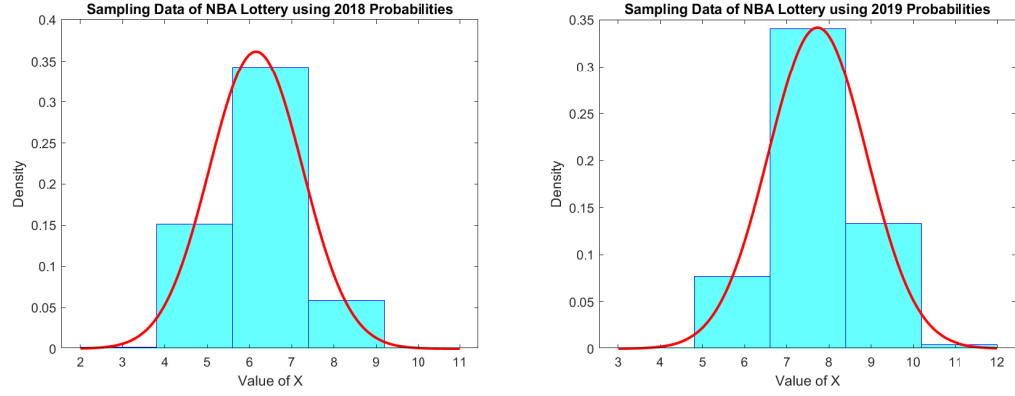


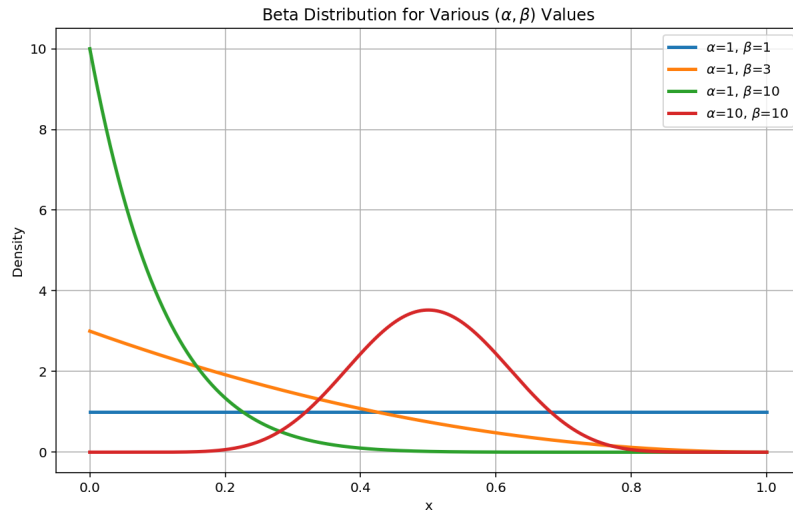
FIGURE 1. Simulations of the NBA Draft Lottery using 2018 and 2019 probabilities.

5. MONTE CARLO SIMULATIONS

We strive to gain a better understanding of both weighted and unweighted experiments by utilizing Monte Carlo simulations (MCS), a powerful computational tool. Here, our MCS utilize a beta random variable B with the density function

$$B(\alpha, \beta) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}.$$

Note that the beta random variable can represent both a weighted and uniform continuous distribution over $(0, 1)$.

FIGURE 2. Beta distribution for various (α, β) values.

To generate a discrete weighted distribution across n categories indexed from 1 to n , we map a continuous Beta distribution into discrete intervals using the floor function. Let $Y \sim \text{Beta}(\alpha, \beta)$ be a continuous random variable bounded on the interval $(0, 1)$. We define the discrete random variable $T(\alpha, \beta)$ as:

$$T(\alpha, \beta) = \lfloor n \cdot Y \rfloor + 1. \quad (5.28)$$

This transformation scales the continuous support by n , partitions it into n equal-width intervals via the floor operator $\lfloor \cdot \rfloor$, and shifts the resulting support from $[0, n-1]$ to $[1, n]$. By varying the shape parameters α and β , we alter the underlying probability density function of Y , which directly dictates the probability weights assigned to each discrete outcome. For instance, $(\alpha = 1, \beta = 1)$ yields a uniform discrete distribution; $(\alpha = 1, \beta = 3)$ and $(\alpha = 1, \beta = 10)$ introduce progressively stronger right-skewed profiles where lower-indexed categories receive higher weights; and $(\alpha = 10, \beta = 10)$ produces a symmetric, bell-shaped discrete distribution centered around the middle categories, as seen in Figure 2. Sampling from $T(\alpha, \beta)$ to generate a sequence of random numbers allows for a direct tally of distinct outcomes to analyze the empirical distribution under these varied weighting schemes. To produce sufficient data on this experiment, we executed this process m times and took the data's mean and variance at the end of the m -th experiment. Letting $n = 100$ and $m = 10^5$, we ran experiments (MCS) that varied the parameters α and β , which yielded the following data used to calculate the sample mean and sample variance for the trials.

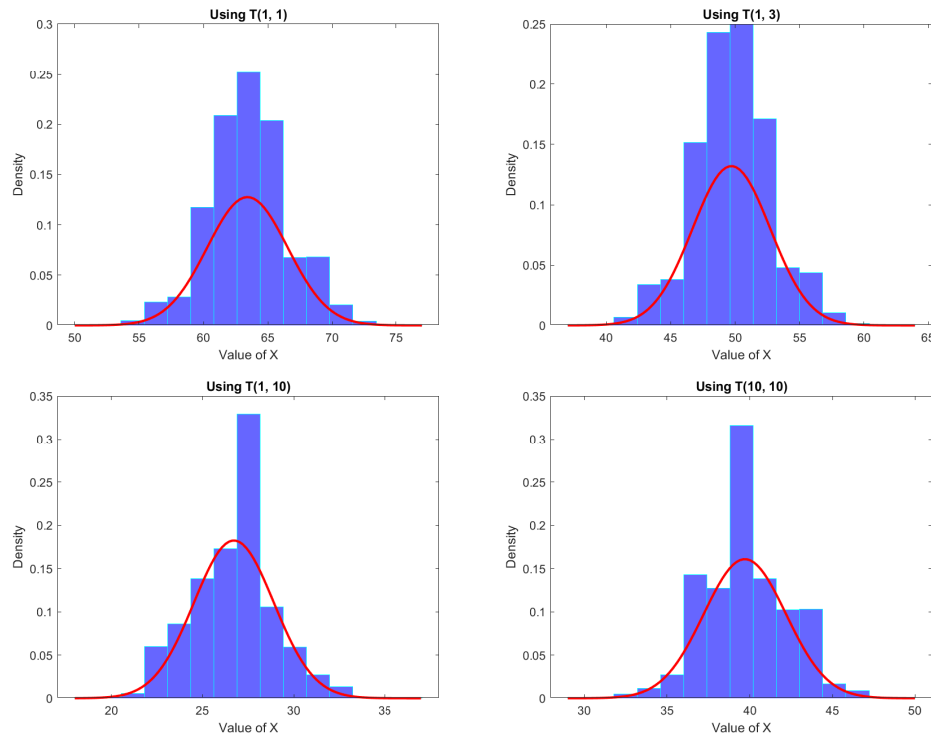


FIGURE 3. Histograms illustrating $m = 10^5$ MCS of X for $n = 100$ outcomes.

The mean and variance from the MCS are expressed in Table 5.3 below.

(α, β)	(1, 1)	(1, 3)	(1, 10)	(10, 10)
$\bar{X}$	63.4092	49.7069	26.7178	39.6961
$\text{Var}(X)$	9.7719	9.1289	4.7803	6.1499

TABLE 5.3. Mean and variance of $m = 10^5$ MCS for α and β with $n = 10^5$.

From the results of our MCS, we observe that the mean of our experiment is maximized when $\alpha = \beta = 1$, which represents the uniform distribution on $(0, 1)$ where each $p_i = 1/n$.

6. RESPONSE TO THE CONJECTURE IN [5]

The authors of [5] employed similar MCS results to conjecture that the expected value of the random variable X , representing the number of distinct outcomes in repeated experiments, attains its maximum when the events are equiprobable. We address this question by establishing the following theorem.

Theorem 6.1. *Let $n \geq 2$. The expected value of the number of distinct outcomes observed in a repeated weighted experiment with event probabilities $p_1, p_2, \dots, p_n$ is maximized when the events are equiprobable, i.e.,*

$$p_1 = \dots = p_n = \frac{1}{n}.$$

Proof. First, note that the expected value $E(X) = \sum_{k=1}^n [1 - (1 - p_k)^n] := \mathcal{E}(p_1, \dots, p_n)$ is concave down as its Hessian matrix $H(p_1, \dots, p_n) = \nabla^2 \mathcal{E}(p_1, \dots, p_n)$ is strictly negative definite with entries

$$h_{ii} = -n(n-1)(1-p_i)^{n-2} < 0 \quad \text{and} \quad h_{ij} = 0 \quad \text{when } i \neq j.$$

Therefore, $\mathcal{E}(p_1, \dots, p_n)$ has a maximizer. Now, we employ the Lagrange multiplier method to maximize $\mathcal{E}(p_1, \dots, p_n)$ subject to the probability constraint $\sum_{k=1}^n p_k = 1$. To this end, consider the function

$$F(p_1, \dots, p_n, \lambda) = \sum_{k=1}^n [1 - (1 - p_k)^n] + \lambda \left(1 - \sum_{k=1}^n p_k \right).$$

Observe that for each $1 \leq i \leq n$, we have

$$\frac{\partial F}{\partial p_i} = n(1 - p_i)^{n-1} - \lambda.$$

Setting each $\frac{\partial F}{\partial p_i} = 0$ we obtain

$$\lambda = n(1 - p_1)^{n-1} = \dots = n(1 - p_n)^{n-1},$$

implying that $p_1 = \dots = p_n$. Using $\sum_{k=1}^n p_k = 1$, we get $p_1 = \dots = p_n = 1/n$, as desired. $\square$

For a simple example, we use Figure 4 to illustrate that the expected value of two events is maximized at the point $(1/2, 1/2)$. This is done by considering $E(X)$ as a three-dimensional surface where $x = p_1$, $y = p_2$, and $z = E(X)$. Applying our expected value formula in Theorem 3.6 yields the surface $z = 2 - (1 - x)^2 - (1 - y)^2$, which we maximize subject to the constraint that $x + y = 1$ to produce Figure 4. Note that the probabilities in Figure 4 are represented by the parabolic intersection of the expected value surface and its constraining plane.

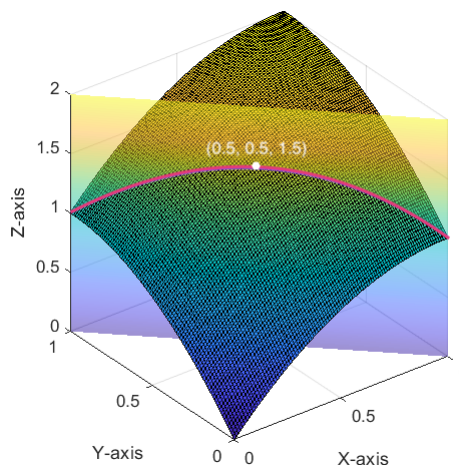


FIGURE 4. The expected value $z = 2 - (1 - x)^2 - (1 - y)^2$ is maximized at $(1/2, 1/2)$ subject to $x + y = 1$, where $x = p_1$ and $y = p_2$.

Remark 6.1. *The connection between our results and information theory is best understood through the framework developed by Kendall [11], who traces the entropy concept from Boltzmann’s statistical mechanics through Shannon’s foundational communication theory. Shannon’s measure of entropy,*

$$H = -K \sum_{i=1}^S p_i \log p_i,$$

is maximized when all outcomes are equiprobable. Shannon characterized H axiomatically through three requirements: continuity in the p_i , monotone increase in S under equiprobability (i.e., when $p_i = 1/S$), and an additivity postulate ensuring that H decomposes consistently across successive selections. Our Theorem 6.1 establishes an analogous maximization result in the discrete probabilistic setting of repeated weighted experiments: the expected number of distinct outcomes

$$E(X) = \sum_{k=1}^n [1 - (1 - p_k)^n]$$

is maximized when $p_1 = \dots = p_n = 1/n$. This result is consistent with the maximum-entropy principle, since the uniform distribution is precisely the distribution that maximizes Shannon entropy over a finite alphabet. Kendall emphasizes that entropy arises naturally as an additive quantity in thermodynamic and communication-theoretic contexts, and our expected value formula reflects a parallel additivity: $E(X)$ decomposes as a sum of contributions $1 - (1 - p_k)^n$ from each outcome, each representing the probability that outcome k appears at least once in n trials. Just as concentrating probability mass on fewer outcomes reduces Shannon entropy below its uniform maximum, concentrating probability weights reduces the expected diversity of observed outcomes below the equiprobable benchmark, a phenomenon our Monte Carlo simulations in Section 5 illustrate concretely across several Beta-distributed weighting schemes.

7. FURTHER APPLICATIONS

These games fit our model as each outcome in poker and craps has a well-defined probability, which increases our model's accuracy. To conclude, our findings can also apply to a different set of situations where probabilities are approximated using a sample set of data or a statistical model. This can be used to estimate the number of distinct seeds that win a knockout tournament [13], the number of distinct days that contain the winning ticket of a sweepstakes giveaway [3], and the number of distinct selection positions that win "The Singled Out Game" [19]. Furthermore, we believe our models can be used to analyze new variations of classic problems such as "The Birthday Problem" [10, 15], the problem of playing a fair game of craps [16] with a loaded pair of dice, and the De Moivre probability problem on the sum of numerical results in equally likely trials [8]. The endless possibilities to which our findings can be applied make them a versatile result that will provide great mathematical insights for future generations.

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Conflict of interest. The authors declare no conflict of interest.

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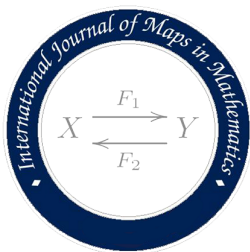
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A COMPARATIVE STUDY OF SOFT 2-NORMS: LIMITATIONS, REFINED CONCEPTS, AND STRUCTURAL INTEGRATION

ESRA SELEN ÖNDER  AND HACER BOZKURT  *

Abstract. The present work aims to further contribute to the study of soft linear structures by analyzing key definitions and providing illustrative examples that highlight both the strengths and limitations of existing approaches. In particular, similarly to classical linear functional analysis, we establish a connection between the refined soft 2-norm and the soft norm by means of a suitable soft basis. This relationship clarifies the structural coherence between these two notions and supports the development of a consistent mathematical framework within soft set theory. Refined soft 2-norm studies contribute to a globally consistent way to measure linear dependence across all parameters, establish connections with soft 2-inner products, and provide concrete constructions that extend classical 2-norm theory to soft vector spaces with parametric structure. In this work, a careful comparison between the soft 2-norm and the refined soft 2-norm is presented to highlight their conceptual and structural differences. Furthermore, while both norms were attempted to be related to Gram-type 2-inner products, the refined soft 2-norm was found to naturally provide a consistent derivation from the soft 2-inner product thanks to its parameter-coordinated structure. Therefore, the refined version not only strengthens the link between geometry and algebra in soft vector spaces, but also fills a gap in the literature by providing an example where global parametric dependence is clearly characterized. Another aim of this paper is to investigate an extension theorem for soft linear functionals in refined soft 2-normed spaces. By introducing an induced soft norm obtained from a fixed soft vector, we establish a Hahn–Banach type extension theorem in the framework of refined soft 2-normed spaces. Furthermore, we discuss consequences related to soft bounded linear functionals and the dual structure of these spaces. Our results reinforce the importance of refined concepts in soft set theory and contribute to the ongoing effort to establish a solid and coherent mathematical framework for handling uncertainty.

Keywords: Soft set, Soft normed space, Refined soft 2-normed space, Refined soft 2-inner product space.

MSC (2020): 03E72, 08A72, 46B99.

1. INTRODUCTION

The theory of soft sets, introduced by Molodtsov in 1999 [16], has emerged as an effective mathematical framework for dealing with uncertainty, vagueness, and incomplete information. Unlike classical approaches such as fuzzy sets, rough sets, and intuitionistic fuzzy sets, soft set theory avoids the need for additional structures such as membership functions or probability measures. Instead, it is based on a parameterized family of subsets, which provides greater flexibility in modeling real-world problems involving uncertain or approximate data.

Since its introduction, soft set theory has attracted considerable attention, and numerous extensions and applications have been developed. Researchers have investigated algebraic structures of soft sets, including soft groups, soft rings, and soft vector spaces, as well as topological and analytical aspects such as soft topology, soft metric spaces, and soft normed spaces. These developments have significantly enriched the theoretical foundation of the field and expanded its applicability to areas such as decision-making, optimization, data analysis, and engineering sciences [5], [4], [7], [6], [2], [19], [18], [3], [20].

In particular, the study of soft linear structures has played a central role in the advancement of soft set theory. Concepts such as soft linear dependence, soft bases, and soft inner products have been proposed to extend classical linear algebra into the soft setting. However, transferring classical notions directly into the soft framework often leads to subtle difficulties. Several examples in the literature demonstrate that naive generalizations may fail to preserve fundamental properties, thereby necessitating refined definitions and careful theoretical analysis.

The development of soft set theory continued with Kadhim's introduction of the concept of a soft 2-normed space [11]. However, after some time, Sanabria addressed the deficiency in Kadhim's definition by redefining the soft 2-normed space within the refined framework [11], [17]. Recent works have focused on addressing these challenges by introducing more robust notions, such as refined soft linear dependence and improved formulations of soft inner product and soft 2-inner product spaces. These refinements have proven essential in preventing inconsistencies and ensuring that the soft analogues faithfully reflect the behavior of their classical counterparts. In particular, examples provided by Sanabria and other authors have clarified critical issues and guided the development of a consistent soft theory [17].

The concept of 2-normed spaces was first investigated by Gähler in 1965 [10]. Later, Lewandowska studied the properties of 2-normed sets and constructed two 2-normed sets of linear operators in 2003 [14]. He also proved several important theorems on generalized 2-normed spaces [13], [12]. Furthermore, in 2007, he introduced a new structure called generalized 2-normed spaces [1].

Motivated by these developments, the present work aims to further contribute to the study of soft linear structures by analyzing key definitions and providing illustrative examples that highlight both the strengths and limitations of existing approaches. Specifically, it establishes a link between the soft 2-norm and the soft norm through a suitable soft basis, similar to classical linear functional analysis. This relationship clarifies the structural consistency between these two notions and supports the development of a coherent soft framework within

soft set theory. Furthermore, both norms were attempted to be related to Gram-type 2-inner products, and it was observed that the refined soft 2-norm naturally provides a consistent derivation from the soft 2-inner product thanks to its parameter-coordinate structure. Therefore, it was seen that the refined version not only strengthens the link between geometry and algebra in soft vector spaces but also clearly characterizes global parametric dependence by means of an example. Our results reinforce the importance of refined concepts in soft set theory and support the ongoing effort to establish a solid and coherent mathematical framework for handling uncertainty.

In summary, this study presents a careful comparison between the soft 2-norm and the refined soft 2-norm is presented to highlight their conceptual and structural differences. The soft 2-norm is defined parameterwise, allowing each parameter to be treated independently; as a result, linear dependence in a single parameter does not necessarily yield a zero norm, and its construction is often simpler and softer for computational or illustrative purposes. In contrast, the refined soft 2-norm imposes a stronger, global condition: the norm vanishes only if the soft vectors are linearly dependent across all parameters. This global perspective captures the notion of refined linear dependence more rigorously, providing a more theoretically consistent framework. Further, while both norms can be connected to Gram-type 2-inner products, the refined soft 2-norm naturally ensures a coherent derivation from a soft 2-inner product due to its parameter-coordinated structure. Therefore, the refined version not only strengthens the link between geometry and algebra in soft vector spaces but also fills a gap in the literature by offering an example where global parametric dependence is explicitly captured and visualized. Furthermore, we know that one of the most fundamental results in functional analysis is the Hahn–Banach Theorem, which guarantees the extension of bounded linear functionals from a subspace to the whole space without increasing the norm. This theorem has numerous applications in analysis, optimization, and operator theory. Motivated by these developments, in this work we study the extension of soft bounded linear functionals in refined soft 2-normed spaces and establish a refined soft 2-normed Hahn–Banach type theorem for such spaces.

2. PRELIMINARIES

This section is devoted to recalling various concepts from soft set theory together with essential notions such as refined soft 2-normed spaces and refined soft 2-inner product spaces.

Let Q be a universe and P be a set of parameters, $P(Q)$ denotes the power set of Q and B be a non-empty subset of P .

Definition 2.1. [16] *We define a soft set over Q as a pair (G, P) where G maps each element of P to a subset of Q , i.e. $G : P \rightarrow P(Q)$.*

Definition 2.2. [15] *Consider a soft set (G, P) defined over the universe Q . If for every parameter $p \in P$, $G(p) = Q$, then (G, P) is called an absolute soft set, often denoted by $\tilde{Q}$.*

Definition 2.3. [15] *For a soft set (G, P) over Q if each parameter $p \in P$ satisfies $G(p) = \emptyset$, then $G(p)$ is referred to as a null soft set, usually written as Φ .*

Definition 2.4. [15] *Let Q be a non-empty set and P be a nonempty parameter set. A function $q : P \rightarrow Q$ is called a soft element of Q . We say that a soft element q belongs to a*

soft set G over Q , denoted $q \in Q$, if for every parameter $p_1 \in P$, the valued q_{p_1} is an element of $G(p_1)$. In other words, a soft set G can be viewed as assigning to each parameter of $p_1 \in P$ a subset of Q , namely $G(p_1)$, and a soft element belongs to G if its values are contained in these subsets for all parameters. Furthermore, if (G, P) is a soft set in which every $G(p_1)$ contains exactly one element (i.e., is a singleton) for each $p_1 \in P$, then the entire soft set can be uniquely represented by a single soft element. In this case, knowing the soft element is enough to reconstruct the soft set.

Definition 2.5. [5] Let Q be a linear space over a field $\mathbb{R}$, and the parameter set P be the real number set $\mathbb{R}$. The soft set (G, P) is defined as a soft vector and is denoted by $\tilde{q}(p_1)$ if there exists one $p_1 \in P$ such that $G(p_1) = \{q\}$ for some $q \in Q$ and $G(p'_1) = \emptyset$ for all $p'_1 \in P \setminus \{p_1\}$. The set of all soft vectors over $\tilde{Q}$ will be defined by $SV(\tilde{Q})$.

Definition 2.6. [4] Let $\tilde{Q}$ be a soft linear space, P be a nonempty set of parameters and (G, P) be a soft linear space on Q . A finite set of soft vectors $\{\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n\}$ of (G, P) is said to be linearly dependent in G if there exist soft scalars $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n$ not all equal to $\tilde{0}$, such that $\tilde{\alpha}_1 \tilde{q}_1 + \tilde{\alpha}_2 \tilde{q}_2 + \dots + \tilde{\alpha}_n \tilde{q}_n = \tilde{\theta}$.

Definition 2.7. [8] Let $F : SV(\tilde{Q}) \rightarrow SV(\tilde{Q})$ be an operator. Then F is said to be soft linear if

- i) $F(\tilde{q} + \tilde{w}) = F(\tilde{q}) + F(\tilde{w})$ for every soft vectors $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- ii) for every soft scalar $\tilde{\alpha}$, $F(\tilde{\alpha}\tilde{q}) = \tilde{\alpha}F(\tilde{q})$ for every soft vector $\tilde{q} \in \tilde{Q}$.

Definition 2.8. [9] A soft linear functional F is a soft linear operator such that $F : SV(\tilde{Q}) \rightarrow K$ where $\tilde{Q}$ is a soft linear space and $K = \mathbb{R}(P)$ if $\tilde{Q}$ is a real soft linear space and $K = \mathbb{C}(P)$ if $\tilde{Q}$ is a complex soft linear space.

Definition 2.9. [11] Let $\tilde{Q}$ be an absolute soft linear space. A soft 2-norm on $\tilde{Q}$ is a mapping $\|\cdot, \cdot\| : SV(\tilde{Q}) \times SV(\tilde{Q}) \rightarrow \mathbb{R}(P)$ which satisfies the following conditions:

- i) $\|\tilde{q}, \tilde{w}\| \geq \tilde{0}$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and $\|\tilde{q}, \tilde{w}\| = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are linearly dependent in $\tilde{Q}$,
- ii) $\|\tilde{q}, \tilde{w}\| = \|\tilde{w}, \tilde{q}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- iii) $\|\tilde{q}, \tilde{\alpha}\tilde{w}\| = |\tilde{\alpha}| \|\tilde{q}, \tilde{w}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and for all soft scalar $\tilde{\alpha}$,
- iv) $\|\tilde{q}, \tilde{w} + \tilde{z}\| \leq \|\tilde{q}, \tilde{w}\| + \|\tilde{q}, \tilde{z}\|$ for all $\tilde{q}, \tilde{w}, \tilde{z} \in \tilde{Q}$,

The absolute soft linear space $\tilde{Q}$ with a refined soft 2-norm $\|\cdot, \cdot\|$ on $\tilde{Q}$ is said to be a soft 2-normed space and is denoted by $(\tilde{Q}, \|\cdot, \cdot\|, P)$ or $(\tilde{Q}, \|\cdot, \cdot\|)$.

However, it is not possible to show that this definition yields the soft 2-inner product. Inspired by Sanabria's example, we present a new example that explains and illustrates this situation:

Example 2.1. Let $Q = \mathbb{R}^2$ be a vector space over $\mathbb{R}$ and let the set of parameters be $P = \{1, 2\}$. Define the soft vectors $\tilde{q}, \tilde{w} : P \rightarrow Q$

$$\tilde{q}(p) = \begin{cases} (1, 0), & p = 1 \\ (1, 2), & p = 2 \end{cases}; \quad \tilde{w}(p) = \begin{cases} (2, 0), & p = 1 \\ (2, 3), & p = 2 \end{cases}.$$

Observe that $p = 1$ we have $\tilde{w}(1) = 2\tilde{q}(1)$; hence, the soft vectors $\{\tilde{q}(p), \tilde{w}(p)\}$ are linearly dependent in the soft vector space $\tilde{Q}$ over the parameter set P . Consider now the standard 2-inner product induced by the usual inner product on $\mathbb{R}^2$, $\langle \tilde{q}(p), \tilde{q}(p) | \tilde{w}(p) \rangle = \|\tilde{q}(p)\|^2 \|\tilde{w}(p)\|^2 - |\langle \tilde{q}(p), \tilde{w}(p) \rangle|^2$. For $p = 2$, $\tilde{q}(2) = (1, 2)$ and $\tilde{w}(2) = (2, 3)$, and straightforward computations yield $\|\tilde{q}(2)\|^2 = 5$, $\|\tilde{w}(2)\|^2 = 13$, $|\langle \tilde{q}(2), \tilde{w}(2) \rangle| = 8$. Thus,

$$\langle \tilde{q}(2), \tilde{q}(2) | \tilde{w}(2) \rangle = 5.13 - 64 = 1 > 0.$$

Therefore, although the value of the induced soft 2-inner product is strictly positive for $p = 2$, the set of soft vectors $\{\tilde{q}(p), \tilde{w}(p)\}$ is soft linearly dependent, which contradicts the first condition in the definition of a soft 2-inner product given in [11]. This example demonstrates that the standard 2-inner product does not necessarily generate a soft 2-inner product that satisfies the linear independence requirement stated in Definition 5.1 of [11], thereby supporting the observation presented in Example 5.2.

This issue was identified by Sanabria in [17], where he introduced the notion of refined linear dependence, thereby preventing the emergence of a significant problem in the development of soft theory.

Definition 2.10. [17] Suppose Q is a linear space, P is a nonempty parameter set, and (G, P) is a soft linear space on Q . A finite set of soft vectors $\{\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n\}$ in (G, P) is called refined linearly dependent in G if there exist soft scalars $\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_n$ with $\tilde{\alpha}_i \neq \tilde{0}$ for at least one $i \in \{1, 2, \dots, n\}$, such that $\tilde{\alpha}_1 \tilde{q}_1 + \tilde{\alpha}_2 \tilde{q}_2 + \dots + \tilde{\alpha}_n \tilde{q}_n = \tilde{\theta}$ for every $p \in P$.

Definition 2.11. [17] Suppose $\tilde{Q}$ is an absolute soft linear space. We define a refined soft 2-norm on $\tilde{Q}$ to be a mapping $\|\cdot, \cdot\| : SV(\tilde{Q}) \times SV(\tilde{Q}) \rightarrow \mathbb{R}(P)$, provided that it satisfies the conditions listed below:

- i) $\|\tilde{q}, \tilde{w}\| \geq \tilde{0}$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and $\|\tilde{q}, \tilde{w}\| = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent in $\tilde{Q}$,
- ii) $\|\tilde{q}, \tilde{w}\| = \|\tilde{w}, \tilde{q}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- iii) $\|\tilde{q}, \tilde{\alpha} \tilde{w}\| = |\tilde{\alpha}| \|\tilde{q}, \tilde{w}\|$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and for all soft scalar $\tilde{\alpha}$,
- iv) $\|\tilde{q}, \tilde{w} + \tilde{z}\| \leq \|\tilde{q}, \tilde{w}\| + \|\tilde{q}, \tilde{z}\|$ for all $\tilde{q}, \tilde{w}, \tilde{z} \in \tilde{Q}$.

A refined soft 2-normed space is an absolute soft linear space $\tilde{Q}$ together with a refined soft 2-norm $\|\cdot, \cdot\|$ and it is represented by $(\tilde{Q}, \|\cdot, \cdot\|, P)$ or $(\tilde{Q}, \|\cdot, \cdot\|)$.

Definition 2.12. [17] For an absolute soft linear space $\tilde{Q}$, a refined soft 2-inner product is a function $\langle \cdot, \cdot | \cdot \rangle : SV(\tilde{Q}) \times SV(\tilde{Q}) \times SV(\tilde{Q}) \rightarrow \mathbb{R}(P)$ that fulfills the conditions listed below:

- i) $\langle \tilde{q}, \tilde{q} | \tilde{w} \rangle \geq \tilde{0}$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and $\langle \tilde{q}, \tilde{q} | \tilde{w} \rangle = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent in $\tilde{Q}$,
- ii) $\langle \tilde{q}, \tilde{q} | \tilde{w} \rangle = \langle \tilde{w}, \tilde{w} | \tilde{q} \rangle$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$,
- iii) $\langle \tilde{q}, \tilde{w} | \tilde{z} \rangle = \langle \tilde{w}, \tilde{q} | \tilde{z} \rangle$ for all $\tilde{q}, \tilde{w}, \tilde{z} \in \tilde{Q}$,
- iv) $\langle \tilde{\alpha} \tilde{q}, \tilde{w} | \tilde{z} \rangle = |\tilde{\alpha}| \langle \tilde{q}, \tilde{w} | \tilde{z} \rangle$ for all $\tilde{q}, \tilde{w} \in \tilde{Q}$ and for all soft scalar $\tilde{\alpha}$,
- v) $\langle \tilde{q}_1 + \tilde{q}_2, \tilde{y} | \tilde{z} \rangle = \langle \tilde{q}_1, \tilde{y} | \tilde{z} \rangle + \langle \tilde{q}_2, \tilde{y} | \tilde{z} \rangle$ for all $\tilde{q}_1, \tilde{q}_2, \tilde{w}, \tilde{z} \in \tilde{Q}$.

Let $\tilde{Q}$ be an absolute soft linear space. If $\tilde{Q}$ is endowed with a refined soft 2-inner product $\langle \cdot, \cdot | \cdot \rangle$, then $\tilde{Q}$ is said to be a refined soft 2-inner product space and it is represented by $(\tilde{Q}, \langle \cdot, \cdot | \cdot \rangle, P)$ or $(\tilde{Q}, \langle \cdot, \cdot | \cdot \rangle)$.

3. MAIN RESULTS

Before introducing new definitions and theorems related to the concept of refined soft 2-norm, we present two new examples to better understand the relationship between the concepts of soft 2-norm and refined soft 2-norm. Then, based on the definition of refined soft 2-normed spaces given by Sanabria, we present new definitions and results concerning refined soft sequences [17]. We begin by extending certain classical convergence results from functional analysis to the framework of refined soft sequences.

Example 3.1. Let $Q = \mathbb{R}^2$ be a real vector space, and let the parameter set be $P = \{\text{optimism}, \text{pessimism}, \text{stability}\}$. A soft vector is a mapping

$$\tilde{q} : P \rightarrow \mathbb{R}^2.$$

Let the base vector be $x = (2, 3)$. Define a soft scalar function $\tilde{\alpha} : P \rightarrow \mathbb{R}$ by $\tilde{\alpha}(\text{optimism}) = 1.5$, $\tilde{\alpha}(\text{pessimism}) = 0.5$, $\tilde{\alpha}(\text{stability}) = 1.0$. Define the refined soft vector

$$\tilde{q}(p) = \tilde{\alpha}(p)x$$

for all $p \in P$. Explicitly, $\tilde{q}(\text{optimism}) = 1.5(2, 3) = (3, 4.5)$, $\tilde{q}(\text{pessimism}) = 0.5(2, 3) = (1, 1.5)$, $\tilde{q}(\text{stability}) = 1.0(2, 3) = (2, 3)$.

Now define another refined soft vector by $\tilde{w}(p) = 2\tilde{q}(p)$ for all $p \in P$. Clearly, the soft vectors $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent because they are linearly dependent for every parameter. Now, define the refined soft 2-norm by

$$\|\tilde{q}, \tilde{w}\|(p) = \left| \det \begin{pmatrix} \tilde{x}_1(p) & \tilde{y}_1(p) \\ \tilde{x}_2(p) & \tilde{y}_2(p) \end{pmatrix} \right|.$$

Equivalently, $\|\tilde{q}, \tilde{w}\|(p) = |\tilde{x}_1(p)\tilde{y}_2(p) - \tilde{x}_2(p)\tilde{y}_1(p)|$. Explicitly:

Parameter	$\tilde{q}(p)$	$\tilde{w}(p)$
optimism	(3, 4.5)	(6, 9)
pessimism	(1, 1.5)	(2, 3)
stability	(2, 3)	(4, 6)

In all cases optimism, pessimism or stability, we find $\|\tilde{q}, \tilde{w}\|(p) = 0$. For every parameter $p \in P$, $\|\tilde{q}, \tilde{w}\|(p) = 0$. Therefore, $\|\tilde{q}, \tilde{w}\| = \tilde{0}$. Since the vectors are linearly dependent for all parameters, $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent. Thus, in this example: $\|\tilde{q}, \tilde{w}\| = \tilde{0}$ if and only if $\tilde{q}$ and $\tilde{w}$ are refined linearly dependent. Furthermore, this determinant-based definition also satisfies all other axioms of the refined soft 2-norm.

Now define another soft vector: $\tilde{z}(\text{optimism}) = (6, 9)$, $\tilde{z}(\text{pessimism}) = (1, 0)$, $\tilde{z}(\text{stability}) = (4, 6)$. Then we obtain: the soft vectors $\tilde{w}$ and $\tilde{z}$ are dependent at optimism, dependent at stability, and independent at pessimism. Therefore, $\tilde{w}$ and $\tilde{z}$ are not refined linearly dependent, but according to Kadhim's approach, they are linearly dependent. Recall, $\|\tilde{w}, \tilde{z}\|(p) = |\tilde{w}_1(p)\tilde{z}_2(p) - \tilde{w}_2(p)\tilde{z}_1(p)|$, we get

$$\|\tilde{w}, \tilde{z}\| = \begin{cases} 0, & \text{optimism} \\ 3, & \text{pessimism} \\ 0, & \text{stability} \end{cases}.$$

This also contradicts Kadhim's definition of soft 2-normed space. This example shows clearly: Some parameters may produce linear dependence, but unless dependence holds for every parameter, the vectors are not refined linearly dependent, and therefore the refined soft 2-norm does not vanish.

Example 3.2. Let $\tilde{Q} = \widetilde{\mathbb{R}^2}$, $P = \{1, 2\}$. We define a refined soft 2-norm parameterwise as:

$$\|\tilde{q}, \tilde{w}\|(p) = \|\tilde{q}(p)\| \|\tilde{w}(p)\| \sin \theta(\tilde{q}(p)\tilde{w}(p)),$$

where $\cos \theta(\tilde{q}(p)\tilde{w}(p)) = \frac{\langle \tilde{q}(p)\tilde{w}(p) \rangle}{\|\tilde{q}(p)\| \|\tilde{w}(p)\|}$. Define soft vectors $\tilde{q}, \tilde{w} : P \rightarrow \tilde{Q}$ as

$$\tilde{q}(p) = \begin{cases} (1, 2), & p = 1 \\ (1, 0), & p = 2 \end{cases}$$

and

$$\tilde{w}(p) = \begin{cases} (2, 4), & p = 1 \\ (0, 1), & p = 2 \end{cases}.$$

They are not refined linearly dependent, because refined linear dependence requires linear dependence for all parameters. But they are linearly dependent. On the other hand, we obtain $\|\tilde{q}, \tilde{w}\|(1) = 0$ for $p = 1$ and $\|\tilde{q}, \tilde{w}\|(2) = 1$ for $p = 2$. This would also contradict Kadhim's approach.

Therefore, these two examples show us that Sanabria's definition of refined linear dependence is a crucial requirement for soft 2-norm spaces.

Definition 3.1. A soft sequence $\tilde{q}_n(p)$ in the refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$ is said to be soft Cauchy if for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Definition 3.2. A refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$ is bounded if one can find a sequence of soft elements $\{\tilde{q}_n(p)\}$ in $\tilde{Q}$ for which, for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$, there exists a non-negative constant L such that

$$\|\tilde{q}_n(p), \tilde{w}(p)\| \lesssim L$$

for all natural numbers n .

Theorem 3.1. In a refined soft 2-normed space, any sequence that is soft convergent must be a soft Cauchy sequence.

Proof. Let $\{\tilde{q}_n(p)\}$ be a soft convergent sequence in the refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$ that is

$$\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| \rightarrow 0 \text{ for every } \tilde{w} \in SV(\tilde{Q}) \text{ and } p \in P,$$

which implies $\tilde{q}_n(p) \rightarrow \tilde{q}(p)$. Since the sequence $\{\tilde{q}_n(p)\}$ is soft convergent, it follows that

$$\begin{aligned} \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| &= \|\tilde{q}_n(p) + \tilde{q}(p) - \tilde{q}(p) - \tilde{q}_m(p), \tilde{w}(p)\| \\ &\lesssim \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| + \|\tilde{q}_m(p) - \tilde{q}(p), \tilde{w}(p)\| \\ &\rightarrow 0. \end{aligned}$$

for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$. $\square$

Theorem 3.2. *If $\{\tilde{q}_n(p)\}$ and $\{\tilde{w}_n(p)\}$ are soft Cauchy sequences in the refined soft 2-normed space $\tilde{Q}$, and $\{\tilde{\beta}_n(p)\}$ is a soft real Cauchy sequence in $\tilde{\mathbb{R}}$, then both sequences $\{(\tilde{q}_n + \tilde{w}_n)(p)\}$ and $\{(\tilde{\beta}_n \tilde{q}_n)(p)\}$ are soft Cauchy sequences in $\tilde{Q}$.*

Proof. Let $\{\tilde{q}_n(p)\}$ and $\{\tilde{w}_n(p)\}$ be soft Cauchy sequences in $\tilde{Q}$. By Definition 3.1, for every $\tilde{z}(p) \in SV(\tilde{Q})$ and $p \in P$

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{z}(p)\| \rightarrow 0 \text{ and } \|\tilde{w}_n(p) - \tilde{w}_m(p), \tilde{z}(p)\| \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Consider the sequence $\{(\tilde{q}_n + \tilde{w}_n)(p)\}$. For every $\tilde{z}(p) \in SV(\tilde{Q})$ and $p \in P$, we have

$$\begin{aligned} & \|(\tilde{q}_n(p) + \tilde{w}_n(p)) - (\tilde{q}_m(p) + \tilde{w}_m(p)), \tilde{z}(p)\| \\ & \leq \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{z}(p)\| + \|\tilde{w}_n(p) - \tilde{w}_m(p), \tilde{z}(p)\| \\ & \rightarrow 0 \text{ as } n, m \rightarrow \infty. \end{aligned}$$

Hence, $\{(\tilde{q}_n + \tilde{w}_n)(p)\}$ is a soft Cauchy sequence. Let $\{\tilde{q}_n(p)\}$ be soft Cauchy sequences in $\tilde{Q}$ and $\{\tilde{\beta}_n(p)\}$ is a soft real Cauchy sequence in $\tilde{\mathbb{R}}$. From Theorem 3.3, there exists a constant $L > 0$ such that $\|\tilde{q}_n(p), \tilde{w}(p)\| \leq L$ for all $n \in \mathbb{N}$ and $\tilde{w}(p) \in SV(\tilde{Q})$. Moreover, since $\{\tilde{\beta}_n(p)\}$ is soft real Cauchy in $\mathbb{R}$, it converges, and hence $\{\tilde{\beta}_n(p)\}$ is bounded. We now consider the sequence $\{(\tilde{\beta}_n \tilde{q}_n)(p)\}$. For any $\tilde{w}(p) \in SV(\tilde{Q})$,

$$\begin{aligned} \|\tilde{\beta}_n \tilde{q}_n(p) - \tilde{\beta}_m \tilde{q}_m(p), \tilde{w}(p)\| &= \|\tilde{\beta}_n \tilde{q}_n(p) + \tilde{\beta}_n \tilde{q}_m(p) - \tilde{\beta}_n \tilde{q}_m(p) - \tilde{\beta}_m \tilde{q}_m(p), \tilde{w}(p)\| \\ &\leq |\tilde{\beta}_n - \tilde{\beta}_m| \|\tilde{q}_m(p), \tilde{w}(p)\| + |\tilde{\beta}_n| \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \\ &\rightarrow 0 \end{aligned}$$

This shows that $\{(\tilde{\beta}_n \tilde{q}_n)(p)\}$ is a soft Cauchy sequences in $\tilde{Q}$. Hence, the set of soft Cauchy sequences in a soft normed space is closed under addition and scalar multiplication. $\square$

Theorem 3.3. *Let $\{\tilde{q}_n(p)\}$ be a soft Cauchy sequence in the refined soft 2-normed space $(\tilde{Q}, \|\cdot, \cdot\|, P)$. Then, for each $\tilde{w}(p) \in SV(\tilde{Q})$, the sequence $\{\tilde{q}_n(p)\}$ is bounded.*

Proof. Since $\{\tilde{q}_n(p)\}$ is a soft Cauchy sequence in $\tilde{Q}$, for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$, we have

$$\lim_{m, n \rightarrow \infty} \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| = 0.$$

which means: $\forall \tilde{\varepsilon} > 0 \exists n_0 \in \mathbb{N}$ such that $\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \leq \tilde{\varepsilon}$ for all $n, m > n_0$. Take $\tilde{\varepsilon} = \tilde{1}$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \leq \tilde{1}$$

for all $n, m > n_0$. Now, consider the finitely many terms before n_0 . Define

$$L = \max \left\{ \|\tilde{q}_1(p) - \tilde{q}_{n_0}(p), \tilde{w}(p)\|, \|\tilde{q}_2(p) - \tilde{q}_{n_0}(p), \tilde{w}(p)\|, \dots, \|\tilde{q}_{n_0-1}(p) - \tilde{q}_{n_0}(p), \tilde{w}(p)\| \right\}.$$

Note that L is finite because it is the maximum of finitely many real numbers. For any $n \in \mathbb{N}$, we estimate $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\|$. If $n < n_0$, then $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| \lesssim L$ from definition of L . If $n \geq n_0$, then $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| \lesssim 1$. Thus, for all n , $\|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| < L + 1$. Therefore, we obtain

$$\begin{aligned} \|\widetilde{q}_n(p) - \widetilde{q}_m(p), \widetilde{w}(p)\| &= \|\widetilde{q}_n(p) + \widetilde{q}_{n_0}(p) - \widetilde{q}_{n_0}(p) - \widetilde{q}_m(p), \widetilde{w}(p)\| \\ &\leq \|\widetilde{q}_n(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| + \|\widetilde{q}_m(p) - \widetilde{q}_{n_0}(p), \widetilde{w}(p)\| \\ &< 2(L + 1) \end{aligned}$$

for any $n, m \in \mathbb{N}$. Thus, the sequence $\{\widetilde{q}_n(p)\}$ is bounded by the constant $2(L + 1)$. $\square$

Theorem 3.4. *Let $\widetilde{Q}$ be a refined soft 2-normed space and let $\{\widetilde{q}_n(p)\}$ be a soft Cauchy sequence in $\widetilde{Q}$. If a soft subsequence $\{\widetilde{q}_{n_k}(p)\}$ of $\{\widetilde{q}_n(p)\}$ converges to $\{\widetilde{q}(p)\}$ then the sequence $\{\widetilde{q}_n(p)\}$ also converges to $\{\widetilde{q}(p)\}$ in $\widetilde{Q}$.*

Proof. For any $\widetilde{w}(p) \in SV(\widetilde{Q})$ and $p \in P$, we can write

$$\|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{w}(p)\| = \|\widetilde{q}_n(p) + \widetilde{q}_{n_k}(p) - \widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\|.$$

Now let n be large and choose k such that $n_k > n$. Because $\{\widetilde{q}_n(p)\}$ is Cauchy, $\|\widetilde{q}_n(p) - \widetilde{q}_{n_k}(p), \widetilde{w}(p)\| \rightarrow 0$ as $n, n_k \rightarrow \infty$. And since the subsequence $\{\widetilde{q}_{n_k}(p)\}$ converges to $\{\widetilde{q}(p)\}$, $\|\widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \rightarrow 0$ for every $\widetilde{w}(p) \in SV(\widetilde{Q})$. Hence, we have

$$\begin{aligned} \|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{w}(p)\| &= \|\widetilde{q}_n(p) + \widetilde{q}_{n_k}(p) - \widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \\ &\lesssim \|\widetilde{q}_n(p) - \widetilde{q}_{n_k}(p), \widetilde{w}(p)\| + \|\widetilde{q}_{n_k}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \\ &\rightarrow 0. \end{aligned}$$

Hence, the sequence $\{\widetilde{q}_n(p)\}$ converges to $\{\widetilde{q}(p)\}$ in $\widetilde{Q}$. $\square$

Lemma 3.1. *Let $\widetilde{Q}$ be a refined soft 2-normed space and let $\{\widetilde{q}(p)\}, \{\widetilde{w}(p)\}, \{\widetilde{z}(p)\} \in SV(\widetilde{Q})$. Then, the following inequality holds for every $p \in P$:*

$$|||\widetilde{q}(p), \widetilde{w}(p)\| - \|\widetilde{z}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\|.$$

Proof. Let $\{\widetilde{q}(p)\}, \{\widetilde{w}(p)\}, \{\widetilde{z}(p)\} \in SV(\widetilde{Q})$ and $p \in P$. By the properties of a refined soft 2-norm, we have

$$\begin{aligned} \|\widetilde{q}(p), \widetilde{w}(p)\| &= \|\widetilde{q}(p) + \widetilde{z}(p) - \widetilde{z}(p), \widetilde{w}(p)\| \\ &\lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\| + \|\widetilde{z}(p), \widetilde{w}(p)\|. \end{aligned}$$

This implies $|||\widetilde{q}(p), \widetilde{w}(p)\| - \|\widetilde{z}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\|$. Similarly, we obtain

$$\begin{aligned} \|\widetilde{z}(p), \widetilde{w}(p)\| &= \|\widetilde{z}(p) + \widetilde{q}(p) - \widetilde{q}(p), \widetilde{w}(p)\| \\ &\lesssim \|\widetilde{z}(p) - \widetilde{q}(p), \widetilde{w}(p)\| + \|\widetilde{q}(p), \widetilde{w}(p)\|. \end{aligned}$$

Hence, $|||\widetilde{z}(p), \widetilde{w}(p)\| - \|\widetilde{q}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{z}(p) - \widetilde{q}(p), \widetilde{w}(p)\|$. Combining the above two inequalities, we obtain

$$|||\widetilde{q}(p), \widetilde{w}(p)\| - \|\widetilde{z}(p), \widetilde{w}(p)\||| \lesssim \|\widetilde{q}(p) - \widetilde{z}(p), \widetilde{w}(p)\|$$

for every $p \in P$. $\square$

Theorem 3.5. Let $\tilde{Q}$ be a refined soft 2-normed space and let $\{\tilde{q}_n(p)\}$ be soft sequence with $\{\tilde{q}_n(p)\} \rightarrow \{\tilde{q}(p)\}$ in $\tilde{Q}$ for $n \rightarrow \infty$. Then, the following statements hold:

- a) The soft sequence $\{\tilde{q}_n(p)\}$ is bounded,
- b) Every subsequence of $\{\tilde{q}_n(p)\}$ converges to $\{\tilde{q}(p)\}$,
- c) If the soft scalar sequence $\{\tilde{\alpha}_n(p)\}$ converges to $\tilde{\alpha}(p)$, then the sequence $\{\tilde{\alpha}_n(p) \tilde{q}_n(p)\}$ converges to $\tilde{\alpha}(p) \tilde{q}(p)$,
- d) If the sequence $\{\tilde{w}_n(p)\}$ converges to $\{\tilde{w}(p)\}$, then the sequence $\{\tilde{q}_n(p) + \tilde{w}_n(p)\}$ converges to $\tilde{q}(p) + \tilde{w}(p)$,
- e) $\|\tilde{q}_n(p), \tilde{w}(p)\| \rightarrow \|\tilde{q}(p), \tilde{w}(p)\|$ for every $\{\tilde{w}(p)\} \in SV(\tilde{Q})$ and $p \in P$.

Proof. The proof can be easily obtained by a method similar to the above proofs. □

Example 3.3. Let $\tilde{Q} = \mathbb{R}^2$ be equipped with the 2-norm:

$$\|\tilde{q}(p), \tilde{w}(p)\| = \left| \sum_{i=1}^2 \tilde{q}_i(p), \tilde{w}_i(p) \right|$$

where $\tilde{q}(p) = (\tilde{q}_1(p), \tilde{q}_2(p)) \in SV(\mathbb{R}^2)$, $\tilde{w}(p) = (\tilde{w}_1(p), \tilde{w}_2(p)) \in SV(\mathbb{R}^2)$ and $p \in P$ is refined soft 2-norm.

Theorem 3.6. Let $\tilde{Q}$ be a refined soft 2-normed space. Assume that $\{\tilde{q}_n(p)\}$ and $\{\tilde{w}_n(p)\}$ are soft sequences such that $\{\tilde{q}_n(p)\} \rightarrow \{\tilde{q}(p)\}$ and $\{\tilde{w}_n(p)\} \rightarrow \{\tilde{w}(p)\}$ in $\tilde{Q}$ as $n \rightarrow \infty$. Further, assume that the soft scalar sequences $\{\tilde{\alpha}_n(p)\} \rightarrow \{\tilde{\alpha}(p)\}$ and $\{\tilde{\beta}_n(p)\} \rightarrow \{\tilde{\beta}(p)\}$ as $n \rightarrow \infty$. Then, it follows that $\tilde{\alpha}_n(p) \tilde{q}_n(p) + \tilde{\beta}_n(p) \tilde{w}_n(p) \rightarrow \tilde{\alpha}(p) \tilde{q}(p) + \tilde{\beta}(p) \tilde{w}(p)$ in $\tilde{Q}$ as $n \rightarrow \infty$.

Proof. For every $\epsilon > 0$, we can state $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{z}(p)\| \rightarrow 0$ for every $\tilde{z} \in SV(\tilde{Q})$ and $p \in P$ and $\|\tilde{w}_n(p) - \tilde{w}(p), \tilde{z}(p)\| \rightarrow 0$ for every $\tilde{z} \in SV(\tilde{Q})$ and $p \in P$. Further, for every $\epsilon > 0$, we can write $|\tilde{\alpha}_n(p) - \tilde{\alpha}(p)| < \epsilon$ and $|\tilde{\beta}_n(p) - \tilde{\beta}(p)| < \epsilon$. Then, it follows that

$$\begin{aligned} & \left\| \left(\tilde{\alpha}_n(p) \tilde{q}_n(p) + \tilde{\beta}_n(p) \tilde{w}_n(p) \right) - \left(\tilde{\alpha}(p) \tilde{q}(p) + \tilde{\beta}(p) \tilde{w}(p) \right), \tilde{z}(p) \right\| \\ & \leq \left\| \tilde{\alpha}_n(p) \tilde{q}_n(p) - \tilde{\alpha}(p) \tilde{q}(p), \tilde{z}(p) \right\| + \left\| \tilde{\beta}_n(p) \tilde{w}_n(p) - \tilde{\beta}(p) \tilde{w}(p), \tilde{z}(p) \right\| \\ & \rightarrow 0 \end{aligned}$$

from Theorem 3.5. Thus, the proof is complete. □

Theorem 3.7. Let $\tilde{Q}$ be a refined soft 2-normed space. If $\{\tilde{q}_n(p)\}$ is a soft Cauchy sequence in $\tilde{Q}$, then for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$, the soft real sequence $(\|\tilde{q}_n(p), \tilde{w}(p)\|_{n=1}^\infty)$ is a real soft Cauchy sequence.

Proof. Let $\tilde{Q}$ be a refined soft 2-normed space and let $\{\tilde{q}_n(p)\}$ be a soft Cauchy sequence in $\tilde{Q}$. By definition, for every soft real number $\tilde{\epsilon} \geq \tilde{0}$, there exists a natural number $k \in \mathbb{N}$ such that

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \leq \tilde{\epsilon}$$

for all $n, m > k$ and for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$. Fix an arbitrary element $\tilde{w}(p) \in SV(\tilde{Q})$. Then, for all $n, m > k$, we have

$$||\|\tilde{q}_n(p), \tilde{w}(p)\| - \|\tilde{q}_m(p), \tilde{w}(p)\||| \leq \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\|.$$

Using the soft Cauchy property of $\{\tilde{q}_n(p)\}$, it follows that $||\|\tilde{q}_n(p), \tilde{w}(p)\| - \|\tilde{q}_m(p), \tilde{w}(p)\||| < \tilde{\epsilon}$ for all $n, m > k$. Hence, the real soft sequence $\|\tilde{q}_n(p), \tilde{w}(p)\|$ satisfies the Cauchy condition. Therefore, it is a real soft Cauchy sequence. $\square$

Definition 3.3. Let $\tilde{Q}$ be a refined soft 2-normed space. If every soft Cauchy sequence in $\tilde{Q}$ converges to a soft element of $\tilde{Q}$, then $\tilde{Q}$ is called a complete refined soft 2-normed space, or a refined soft 2-Banach space.

Definition 3.4. Let $\tilde{Q}$ be a refined soft 2-normed space. Recall that we assume that $\tilde{Q}$ has finite dimension d , with $2 \leq d < \infty$, unless stated otherwise. Let $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ be a soft basis of $\tilde{Q}$. Then every soft vector $\tilde{q}(p) \in SV(\tilde{Q})$ can be expressed uniquely in the form

$$\tilde{q}(p) = \sum_{i=1}^d \tilde{\alpha}_i(p) \cdot \tilde{a}_i(p) \text{ where } \tilde{\alpha}_i(p) \text{ are soft real scalars for } i = 1, 2, \dots, d.$$

Further, since $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ is a soft basis of $\tilde{Q}$, it is soft refined linearly independent and $\text{span}\tilde{Q}$. Consequently, the above representation of $\tilde{q}(p)$ is unique. Let $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$ be given by $\tilde{q}(p) = \sum_{i=1}^d \tilde{\alpha}_i(p) \cdot \tilde{a}_i(p)$ and $\tilde{w}(p) = \sum_{i=1}^d \tilde{\beta}_i(p) \cdot \tilde{a}_i(p)$ for $p \in P$. Then the soft 2-norm $\|\tilde{q}(p), \tilde{w}(p)\|$ can be expressed in terms of the soft basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ and the corresponding soft coordinates $\tilde{\alpha}_i, \tilde{\beta}_i$.

Furthermore, it follows from the results of [4] that if $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ is a soft basis of $\tilde{Q}$, then the set $\{\tilde{a}_1(p), \tilde{a}_2(p), \dots, \tilde{a}_d(p)\}$ constitutes a basis of $\tilde{Q}(p) = Q$ for all $p \in P$.

Lemma 3.2. A soft sequence $\{\tilde{q}_n(p)\}$ in $\tilde{Q}$ is said to soft converge to $\tilde{q}(p)$ if and only if $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$.

Proof. If $\{\tilde{q}_n(p)\}$ is soft convergent to $\tilde{q}(p)$ in $\tilde{Q}$, then it is clear that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. Conversely, suppose that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. We now show that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| \rightarrow 0$ for every $\tilde{w}(p) \in SV(\tilde{Q})$ and $p \in P$. Since $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ is a soft basis of $\tilde{Q}$, there exist soft scalars $\{\tilde{\alpha}_1, \tilde{\alpha}_2, \dots, \tilde{\alpha}_d\}$ such that $\tilde{w}(p) = \tilde{\alpha}_1(p) \cdot \tilde{a}_1(p) + \tilde{\alpha}_2(p) \cdot \tilde{a}_2(p) + \dots + \tilde{\alpha}_d(p) \cdot \tilde{a}_d(p)$. Using the properties of the refined soft 2-norm, we obtain

$$\begin{aligned} \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{w}(p)\| &= \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{\alpha}_1(p) \cdot \tilde{a}_1(p) + \tilde{\alpha}_2(p) \cdot \tilde{a}_2(p) + \dots + \tilde{\alpha}_d(p) \cdot \tilde{a}_d(p)\| \\ &\leq |\tilde{\alpha}_1(p)| \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_1(p)\| + |\tilde{\alpha}_2(p)| \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_2(p)\| + \dots + \\ &\quad |\tilde{\alpha}_d(p)| \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_d(p)\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

This completes the proof. $\square$

From the above lemma, we can also derive the following result.

Lemma 3.3. A soft sequence $\{\tilde{q}_n(p)\}$ in $\tilde{Q}$ is said to soft converge to $\tilde{q}(p)$ if and only if $\max \{\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. The proof is immediate. $\square$

Lemma 3.4. For a $\tilde{q}(p) = \sum_{i=1}^2 \tilde{\alpha}_i(p) \cdot \tilde{e}_i(p)$ and $\tilde{w}(p) = \sum_{i=1}^2 \tilde{\beta}_i(p) \cdot \tilde{e}_i(p)$, we obtain $\|\tilde{q}(p), \tilde{w}(p)\| = \left| \tilde{\alpha}_1(p) \tilde{\beta}_1(p) - \tilde{\beta}_2(p) \tilde{\alpha}_2(p) \right| \|\tilde{e}_1(p), \tilde{e}_2(p)\|$, $\tilde{e}_1(p) = \left(\tilde{1}(p), \tilde{0}(p) \right) = (1, 0)$ and $\tilde{e}_2(p) = \left(\tilde{0}(p), \tilde{1}(p) \right) = (0, 1)$.

Proof. For every $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$, we find

$$\begin{aligned} \|\tilde{q}(p), \tilde{w}(p)\| &= \left\| \sum_{i=1}^2 \tilde{\alpha}_i(p) \cdot \tilde{e}_i(p), \sum_{i=1}^2 \tilde{\beta}_i(p) \cdot \tilde{e}_i(p) \right\| \\ &= \left\| \tilde{\alpha}_1(p) \cdot \tilde{e}_1(p) + \tilde{\alpha}_2(p) \cdot \tilde{e}_2(p), \tilde{\beta}_1(p) \cdot \tilde{e}_1(p) + \tilde{\beta}_2(p) \cdot \tilde{e}_2(p) \right\| \\ &= \left| \tilde{\alpha}_1(p) \tilde{\beta}_1(p) - \tilde{\beta}_2(p) \tilde{\alpha}_2(p) \right| \|\tilde{e}_1(p), \tilde{e}_2(p)\|. \end{aligned}$$

$\square$

Theorem 3.8. A two-dimensional refined soft linear 2-normed space is said to be a refined soft 2-Banach space if it is complete over the underlying field.

Proof. Let $\tilde{Q}$ be a refined soft 2-normed space with standard soft basis $\{\tilde{e}_1, \tilde{e}_2\}$. Let $\{\tilde{q}_n(p)\}$ be a soft Cauchy sequence in $\tilde{Q}$. Thus there exists a natural number $k \in \mathbb{N}$ such that

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| \lesssim \tilde{\epsilon} \text{ and } \|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}'(p)\| \lesssim \tilde{\epsilon}$$

for all $n, m > k$ and for every $\tilde{w}(p), \tilde{w}'(p) \in SV(\tilde{Q})$ and $p \in P$. Here, $\tilde{w}$ and $\tilde{w}'$ are assumed to be refined linearly independent. If they were refined linearly dependent, this representation would not be necessary. Let $\tilde{q}_n(p) = \tilde{q}_{n1}(p) \cdot \tilde{e}_1(p) + \tilde{q}_{n2}(p) \cdot \tilde{e}_2(p)$, $\tilde{w}(p) = \tilde{w}_1(p) \cdot \tilde{e}_1(p) + \tilde{w}_2(p) \cdot \tilde{e}_2(p)$ and $\tilde{w}'(p) = \tilde{w}'_1(p) \cdot \tilde{e}_1(p) + \tilde{w}'_2(p) \cdot \tilde{e}_2(p)$. From Lemma 3.4, we obtain

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}(p)\| = |\tilde{w}_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \|\tilde{e}_1(p), \tilde{e}_2(p)\|$$

and similarly

$$\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{w}'(p)\| = |\tilde{w}'_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}'_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \|\tilde{e}_1(p), \tilde{e}_2(p)\|.$$

$\{\tilde{e}_1, \tilde{e}_2\}$ is refined linearly independent; therefore

$$|\tilde{w}_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

and

$$|\tilde{w}'_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}'_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

as $n, m \rightarrow \infty$. From here, we obtain

$$|\tilde{w}'_2(p) \tilde{w}_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) - \tilde{w}_1(p) \tilde{w}'_2(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

and

$$|-\tilde{w}_2(p) \tilde{w}'_2(p) (\tilde{\alpha}_{n1}(p) - \tilde{\alpha}_{m1}(p)) + \tilde{w}_2(p) \tilde{w}'_1(p) (\tilde{\alpha}_{n2}(p) - \tilde{\alpha}_{m2}(p))| \rightarrow 0$$

as $n, m \rightarrow \infty$. By adding the two equations, we get

$$\left| \left(\widetilde{w}_2(p) \widetilde{w}'_1(p) - \widetilde{w}_1(p) \widetilde{w}'_2(p) \right) (\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_{m2}(p)) \right| \rightarrow 0$$

as $n, m \rightarrow \infty$. Since $\widetilde{w}$ and $\widetilde{w}'$ are linearly independent, this condition holds only when $|\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_{m2}(p)| \rightarrow 0$ as $n, m \rightarrow \infty$. This implies that $\{\widetilde{\alpha}_{n2}\}$ is a Cauchy sequence of soft real numbers. By a similar argument, $\{\widetilde{\alpha}_{n1}\}$ can also be shown to be real soft Cauchy. Since $\{\widetilde{\alpha}_{n1}\}$ and $\{\widetilde{\alpha}_{n2}\}$ are real soft Cauchy sequences, there are real soft numbers $\widetilde{b}_1$ and $\widetilde{b}_2$ such that $\widetilde{\alpha}_{n1}(p) \rightarrow \widetilde{b}_1(p)$ and $\widetilde{\alpha}_{n2}(p) \rightarrow \widetilde{b}_2(p)$ as $n \rightarrow \infty$. If we take $\widetilde{q}(p) = \widetilde{\alpha}_1(p) \cdot \widetilde{e}_1(p) + \widetilde{\alpha}_2(p) \cdot \widetilde{e}_2(p) \in SV(\widetilde{Q})$. Then we get

$$\begin{aligned} \left\| \widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{w}''(p) \right\| &= \left\| \begin{aligned} &(\widetilde{\alpha}_{n1}(p) - \widetilde{\alpha}_1(p)) \widetilde{e}_1(p) + (\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_2(p)) \widetilde{e}_2(p), \widetilde{w}_1''(p) \widetilde{e}_1(p) \\ &+ \widetilde{w}_2''(p) \widetilde{e}_2(p) \end{aligned} \right\| \\ &= \left| \widetilde{w}_2''(p) (\widetilde{\alpha}_{n1}(p) - \widetilde{\alpha}_1(p)) - \widetilde{w}_1''(p) (\widetilde{\alpha}_{n2}(p) - \widetilde{\alpha}_2(p)) \right| \\ &\rightarrow 0 \end{aligned}$$

as $n \rightarrow \infty$. Thus, we obtain $\widetilde{q}_n(p) - \widetilde{q}(p)$ as $n \rightarrow \infty$. Therefore, $\widetilde{Q}$ is a soft 2-Banach space. $\square$

Similarly to classical 2-normed spaces, one can relate a refined soft 2-normed space to a soft normed linear space through a basis, using the norm. If $\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}$ is a soft basis of the refined soft 2-normed space $\widetilde{Q}$, we can define the following norm on $\widetilde{Q}$:

$$\|\widetilde{q}(p)\|_0 = \max \{ \|\widetilde{q}(p), \widetilde{a}_i(p)\| : i = 1, 2, \dots, d \}. \quad (3.1)$$

One can observe that the norm $\|\cdot\|_0$ defined above is a soft norm on $\widetilde{Q}$.

Example 3.4. Let $\widetilde{Q}$ be a refined soft 2-normed space with refined 2-norm $\|\cdot, \cdot\|$. We can define another soft norm $\|\cdot\|_1$ on $\widetilde{Q}$ by $\|\widetilde{q}(p)\|_1 = \sum_{i=1}^d \|\widetilde{q}_i(p), \widetilde{a}_i(p)\|$. This norm is well-defined and satisfies all the properties of a soft norm.

Theorem 3.9. Let $\widetilde{Q}$ be a refined soft 2-normed space with refined 2-norm $\|\cdot, \cdot\|$. The soft sequence $\{\widetilde{q}_n\}$ is soft convergent to $\widetilde{q}$ in $\widetilde{Q}$ if and only if $\|\widetilde{q}_n(p) - \widetilde{q}(p)\|_0 \rightarrow 0$ as $n \rightarrow \infty$.

Proof. If $\{\widetilde{q}_n\}$ is soft convergent to $\widetilde{q}$ in $\widetilde{Q}$, then it is clear that $\|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. From 3.1, we can write

$$\|\widetilde{q}_n(p) - \widetilde{q}(p)\|_0 = \max \{ \|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{a}_i(p)\| : i = 1, 2, \dots, d \} \rightarrow 0$$

as $n \rightarrow \infty$. Conversely, suppose that $\|\widetilde{q}_n(p) - \widetilde{q}(p)\|_0 \rightarrow 0$ as $n \rightarrow \infty$. From 3.1, we get $\|\widetilde{q}_n(p) - \widetilde{q}(p), \widetilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$. $\square$

Let $\widetilde{Q}$ be a refined soft 2-normed space with a soft basis $\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}$. Using the $\|\cdot\|_0$ norm generated by the two-norm, the open ball centered at $\widetilde{q}$ in $\widetilde{Q}$ with soft radius $\widetilde{r} \succ \widetilde{0}$ is defined as

$$S_{\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}}(\widetilde{q}(p), \widetilde{r}(p)) = \{ \widetilde{w}(p) : \|\widetilde{q}(p) - \widetilde{w}(p)\|_0 \prec \widetilde{r}(p) \}.$$

This is called the soft open ball generated by the soft basis $\{\widetilde{a}_1, \widetilde{a}_2, \dots, \widetilde{a}_d\}$. We can also express the definition of convergence in a refined soft 2-normed space as follows:

Classical: $\{\tilde{q}_n\} \rightarrow \tilde{q} \Leftrightarrow \forall \tilde{\epsilon} > \tilde{0}, \exists n \in \mathbb{N}$ such that $\forall n \geq N \Rightarrow \tilde{q}_n(p) \in S_{\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}}(\tilde{q}(p), \tilde{r}(p))$.

Using the 0–norm: $\{\tilde{q}_n\} \rightarrow \tilde{q} \Leftrightarrow \forall \tilde{\epsilon} > \tilde{0}, \exists n \in \mathbb{N}$ such that $\forall n \geq N \Rightarrow \|\tilde{q}_n(p) - \tilde{q}(p)\|_0 < \tilde{\epsilon}$.

Component-wise via the basis vectors: $\{\tilde{q}_n\} \rightarrow \tilde{q} \Leftrightarrow \forall i = 1, 2, \dots, d, \|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$.

Thus, convergence in a refined soft 2-normed space can be described either in terms of the 0–norm generated by the basis or equivalently via the convergence of the 2-norms with respect to each basis vector.

Example 3.5. Let $\tilde{Q} = \widetilde{\mathbb{R}^2}$ with the refined soft 2-norm defined by

$$\|\tilde{q}(p), \tilde{w}(p)\| = |\tilde{q}_1(p), \tilde{w}_2(p) - \tilde{q}_2(p), \tilde{w}_1(p)|,$$

for $\tilde{q}(p) = (\tilde{q}_1(p), \tilde{q}_2(p)) \in SV(\widetilde{\mathbb{R}^2})$, $\tilde{w} = (\tilde{w}_1(p), \tilde{w}_2(p)) \in SV(\widetilde{\mathbb{R}^2})$. Let $\tilde{i} = (\tilde{1}, \tilde{0}), \tilde{j} = (\tilde{0}, \tilde{1})$ be the soft standard basis of $\widetilde{\mathbb{R}^2}$. Then for $\tilde{q}(p) = (\tilde{q}_1(p), \tilde{q}_2(p))$ we have $\|\tilde{q}(p), \tilde{i}(p)\| = |\tilde{q}_2(p)|$ and $\|\tilde{q}(p), \tilde{j}(p)\| = |\tilde{q}_1(p)|$. This generates a soft norm. Using these, we can define $\|\tilde{q}(p)\|_1 = \sum_{i=1}^2 \|\tilde{q}_i(p), \tilde{e}_i(p)\| = |\tilde{q}_1(p)| + |\tilde{q}_2(p)|$.

Lemma 3.5. $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space if and only if $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space, where $\|\cdot\|$ is the soft norm generated by the refined soft 2-norm $\|\tilde{q}(p)\|_0 = \max\{\|\tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\}$ for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$.

Proof. Assume that $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space and that $\{\tilde{q}_n\}$ is a soft Cauchy sequence in the soft normed space $(\tilde{Q}, P, \|\cdot\|)$. Then, for every $\tilde{\epsilon} > \tilde{0}$, there exists a natural number $k \in \mathbb{N}$ such that $\|\tilde{q}_n(p) - \tilde{q}_m(p)\| < \tilde{\epsilon}$ for all $n, m > k$, and for every $p \in P$. From the equality $\|\tilde{q}(p)\|_0 = \max\{\|\tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\}$, we obtain $\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{a}_i(p)\| < \tilde{\epsilon}$ for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$, for every $p \in P$, and $i = 1, 2, \dots, d$. Hence, since $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space, it follows that $\tilde{q}_n(p) \rightarrow \tilde{q}(p) \in SV(\tilde{Q})$ as $n \rightarrow \infty$. If $\{\tilde{q}_n\}$ is soft convergent to $\tilde{q}$ in $\tilde{Q}$, then it is clear that $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for every $i = 1, 2, \dots, d$. Consequently, $\|\tilde{q}_n(p) - \tilde{q}(p)\|_0 \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space.

Conversely, assume that $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space and that $\{\tilde{q}_n\}$ is a soft Cauchy sequence in the refined soft 2-normed space $(\tilde{Q}, P, \|\cdot, \cdot\|)$. Then, for every $\tilde{\epsilon} > \tilde{0}$, there exists a natural number $k \in \mathbb{N}$ such that $\|\tilde{q}_n(p) - \tilde{q}_m(p), \tilde{a}_i(p)\| < \tilde{\epsilon}$ for all $n, m > k$, and for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$, and for every $p \in P$. From the equality $\|\tilde{q}(p)\|_0 = \max\{\|\tilde{q}(p), \tilde{a}_i(p)\| : i = 1, 2, \dots, d\}$, we obtain $\|\tilde{q}_n(p) - \tilde{q}_m(p)\|_0 < \tilde{\epsilon}$ for every $p \in P$. Hence, since $(\tilde{Q}, P, \|\cdot\|)$ is a soft Banach space, it follows that $\tilde{q}_n(p) \rightarrow \tilde{q}(p) \in SV(\tilde{Q})$ as $n \rightarrow \infty$. If $\{\tilde{q}_n\}$ is soft convergent to $\tilde{q}$ in $\tilde{Q}$, then it is clear that $\|\tilde{q}_n(p) - \tilde{q}(p)\|_0 \rightarrow 0$. Consequently, $\|\tilde{q}_n(p) - \tilde{q}(p), \tilde{a}_i(p)\| \rightarrow 0$ as $n \rightarrow \infty$ for a basis $\{\tilde{a}_1, \tilde{a}_2, \dots, \tilde{a}_d\}$ for every $i = 1, 2, \dots, d$. as $n \rightarrow \infty$. Therefore, $(\tilde{Q}, P, \|\cdot, \cdot\|)$ is a refined soft 2-Banach space. $\square$

Any refined soft 2-normed space $(\tilde{Q}, P, \|\cdot, \cdot\|)$ whose soft 2-norm satisfies the condition

$$\|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\|^2 + \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\|^2 = 2 \left(\|\tilde{q}(p), \tilde{z}(p)\|^2 + \|\tilde{w}(p), \tilde{z}(p)\|^2 \right)$$

will be regarded as a refined soft 2-inner product space, with the refined soft 2-inner product defined by

$$\langle \tilde{q}(p), \tilde{w}(p) | \tilde{z}(p) \rangle = \frac{1}{4} \left(\|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\|^2 - \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\|^2 \right)$$

[17].

Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space. For any nonzero soft vector $\tilde{z}(p) \in SV(\tilde{Q})$, let $M(\tilde{z}(p))$ denote the soft subspace of $\tilde{Q}$ generated by $\tilde{z}(p)$. Let $\tilde{Q}_{\tilde{z}(p)} = \tilde{Q}/M(\tilde{z}(p))$ be the corresponding soft quotient space. For each $\tilde{q}(p) \in SV(\tilde{Q})$, we denote the equivalence class of $\tilde{q}(p)$ with respect to $M(\tilde{z}(p))$ by $(\tilde{q}(p))_{\tilde{z}(p)}$. The space $\tilde{Q}_{\tilde{z}(p)}$ becomes a soft linear space under the operations defined by soft scalar multiplication

$$\tilde{\alpha}(p) (\tilde{q}(p))_{\tilde{z}(p)} = (\tilde{\alpha}(p) \tilde{q}(p))_{\tilde{z}(p)}$$

and addition

$$(\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} = (\tilde{q}(p) + \tilde{w}(p))_{\tilde{z}(p)}.$$

If $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$ satisfy $(\tilde{q}(p))_{\tilde{z}(p)} = (\tilde{w}(p))_{\tilde{z}(p)}$, then $\|\tilde{q}(p), \tilde{z}(p)\| = \|\tilde{w}(p), \tilde{z}(p)\|$ since $\|\tilde{q}(p), \tilde{z}(p)\| - \|\tilde{w}(p), \tilde{z}(p)\| \leq \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\| = 0$. Thus, the function $\|\cdot\|_{\tilde{z}(p)}$ on $\tilde{Q}_{\tilde{z}(p)}$ defined by $\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{q}(p), \tilde{z}(p)\|$ is well defined.

Theorem 3.10. $\|\cdot\|_{\tilde{z}(p)}$ is a soft norm on $\tilde{Q}_{\tilde{z}(p)}$.

Proof. $\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = 0$ if and only if $\|\tilde{q}(p), \tilde{z}(p)\| = 0$ if and only if $\tilde{q}(p) \in M(\tilde{z}(p))$ i.e., if and only if $(\tilde{q}(p))_{\tilde{z}(p)} = (\tilde{0}(p))_{\tilde{z}(p)}$.

$\left\| \tilde{\alpha}(p) (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \left\| (\tilde{\alpha}(p) \tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{\alpha}(p) \tilde{q}(p), \tilde{z}(p)\| = |\tilde{\alpha}(p)| \|\tilde{q}(p), \tilde{z}(p)\| = |\tilde{\alpha}(p)| \left\| \tilde{q}(p)_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}$ for $\tilde{\alpha}(p) \in \mathbb{R}$.

$\left\| (\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \left\| (\tilde{q}(p) + \tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\| \leq \|\tilde{q}(p), \tilde{z}(p)\| + \|\tilde{w}(p), \tilde{z}(p)\| = \left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} + \left\| (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}.$ $\square$

Theorem 3.11. Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space. This refined soft 2-normed space is a refined soft 2-inner product space if and only if for every non-zero soft vector $\tilde{z}(p) \in SV(\tilde{Q})$, $(\tilde{Q}_{\tilde{z}(p)}, \|\cdot\|_{\tilde{z}(p)})$ is a soft inner product space.

Proof. Assume that $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space where the refined soft 2-norm obeys the parallelogram identity. Let $\tilde{z}(p)$ be an arbitrary non-zero vector in $SV(\tilde{Q})$. For any elements $(\tilde{q}(p))_{\tilde{z}(p)}, (\tilde{w}(p))_{\tilde{z}(p)}$ in the quotient space $\tilde{Q}_{\tilde{z}(p)}$, we utilize the functional definition of the induced norm $\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)} = \|\tilde{q}(p), \tilde{z}(p)\|$. By applying the refined soft 2-norm properties, the following identity holds:

$$\begin{aligned} & \left\| (\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{q}(p))_{\tilde{z}(p)} - (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 \\ &= \|\tilde{q}(p) + \tilde{w}(p), \tilde{z}(p)\|^2 + \|\tilde{q}(p) - \tilde{w}(p), \tilde{z}(p)\|^2 \\ &= 2 \left(\|\tilde{q}(p), \tilde{z}(p)\|^2 + \|\tilde{w}(p), \tilde{z}(p)\|^2 \right) \end{aligned}$$

$$= 2 \left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2.$$

This result demonstrates that the induced soft norm $\|\cdot\|_{\tilde{z}(p)}$ on the space $\tilde{Q}_{\tilde{z}(p)}$ satisfies the standard parallelogram law, effectively reducing the refined soft 2-normed structure to a classical soft normed structure. Conversely, assume that for every non-zero $\tilde{z}(p) \in SV(\tilde{Q})$, the induced soft norm $\|\cdot\|_{\tilde{z}(p)}$ on $\tilde{Q}_{\tilde{z}(p)}$ satisfies the parallelogram law. For any $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$, we can write:

$$\begin{aligned} & \|(\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)}, \tilde{z}(p)\|^2 + \|(\tilde{q}(p))_{\tilde{z}(p)} - (\tilde{w}(p))_{\tilde{z}(p)}, \tilde{z}(p)\|^2 \\ &= \left\| (\tilde{q}(p))_{\tilde{z}(p)} + (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{q}(p))_{\tilde{z}(p)} - (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 \\ &= 2 \left(\left\| (\tilde{q}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 + \left\| (\tilde{w}(p))_{\tilde{z}(p)} \right\|_{\tilde{z}(p)}^2 \right) \\ &= 2 \left(\| \tilde{q}(p), \tilde{z}(p) \|^2 + \| \tilde{w}(p), \tilde{z}(p) \|^2 \right). \end{aligned}$$

□

Now we will give the Hahn-Banach extension theorem for refined soft 2-norm spaces. This theorem is related to the generalization of the Hahn-Banach extension theorem-one of the fundamental results of linear functional analysis-to the refined 2-norm, as well as to the properties of the dual space. This theorems is presented through the connection established between the refined soft 2-norm and the soft norm in a way that remains consistent with the context of the study.

Theorem 3.12. (Refined Soft 2-Normed Hahn-Banach Extension) Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space and let $\tilde{w}(p) \in SV(\tilde{Q})$. Let $\tilde{M} \subset \tilde{Q}$ be a soft linear subspace and let $\tilde{f} : \tilde{M} \rightarrow \tilde{\mathbb{R}}$ be a soft linear functional satisfying $|\tilde{f}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \|$ for all $\tilde{q}(p) \in SV(\tilde{M})$ and some soft constant $\tilde{K} \gtrsim \tilde{0}$. Then there exists a soft linear functional $\tilde{F} : \tilde{Q} \rightarrow \tilde{\mathbb{R}}$ such that $\tilde{f}(\tilde{q}) = \tilde{F}(\tilde{q})$ and $|\tilde{F}(\tilde{q})(p)| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \|$ for all $\tilde{q}(p) \in SV(\tilde{M})$.

Proof. Define $\| \tilde{q}(p) \|_{\tilde{w}(p)} = \| \tilde{q}(p), \tilde{w}(p) \|$. From Theorem 3.11, $(\tilde{Q}, \|\cdot, \cdot\|_{\tilde{w}(p)})$ is a soft normed linear space. From the assumption, we have $|\tilde{f}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \| = \tilde{K} \| \tilde{q}(p) \|_{\tilde{w}(p)}$ for all $\tilde{q}(p) \in SV(\tilde{M})$. Thus, $\tilde{f}$ is a bounded soft linear functional on the soft normed space $(\tilde{M}, \|\cdot, \cdot\|_{\tilde{w}(p)})$. By the Hahn-Banach Theorem, the soft functional $\tilde{f}$ can be extended to a soft linear functional $\tilde{F} : \tilde{Q} \rightarrow \tilde{\mathbb{R}}$ such that $|\tilde{F}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p) \|_{\tilde{w}(p)}$ for all $\tilde{q}(p) \in SV(\tilde{Q})$. Since $\| \tilde{q}(p) \|_{\tilde{w}(p)} = \| \tilde{q}(p), \tilde{w}(p) \|$, we obtain $|\tilde{F}(\tilde{q}(p))| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \|$. Therefore, the soft linear functional $\tilde{f}$ admits an extension $\tilde{F}$ on $\tilde{Q}$ preserving the bound. □

Theorem 3.13. Let $(\tilde{Q}, P, \|\cdot, \cdot\|)$ be a refined soft 2-normed space and let $\tilde{w}(p) \in SV(\tilde{Q})$ be a fixed nonzero soft vector. Define

$$\tilde{Q}_{\tilde{w}(p)}^* = \left\{ \tilde{F} : \tilde{Q} \rightarrow \tilde{\mathbb{R}} : \tilde{F} \text{ is a soft linear and } |\tilde{F}(\tilde{q})(p)| \lesssim \tilde{K} \| \tilde{q}(p), \tilde{w}(p) \| \text{ for some soft constant } \tilde{K} \gtrsim \tilde{0} \right\}.$$

Then $\tilde{Q}_{\tilde{w}(p)}^*$ forms the soft dual space of the soft normed space $(\tilde{Q}, \|\cdot, \cdot\|_{\tilde{w}(p)})$, where $\| \tilde{q}(p) \|_{\tilde{w}(p)} = \| \tilde{q}(p), \tilde{w}(p) \|$ for all $\tilde{q}(p), \tilde{w}(p) \in SV(\tilde{Q})$.

Proof. From the definition, $\|\tilde{q}(p)\|_{\tilde{w}(p)} = \|\tilde{q}(p), \tilde{w}(p)\|$ defines a soft norm on $(\tilde{Q}, \|\cdot\|_{\tilde{w}(p)})$, which is a soft normed linear space. By the Hahn–Banach Theorem, every bounded soft linear functional defined on a subspace of $\tilde{Q}$ can be extended to the whole space while preserving the bound. Therefore, the soft set $\tilde{Q}_{\tilde{w}(p)}^*$ coincides with the dual space of the soft normed space $(\tilde{Q}, \|\cdot\|_{\tilde{w}(p)})$. $\square$

4. CONCLUSION

In conclusion, this work contributes to the development of a consistent soft linear framework by clarifying the relationship between soft norms and soft 2-norms through an appropriate soft basis. The analysis demonstrates that refined definitions are essential for preserving fundamental structural properties in soft linear spaces. These findings provide a solid foundation for future research, particularly in the study of completeness, convergence, continuity, and linear operators in soft normed and soft 2-normed spaces. Further investigations may also explore applications of these refined structures in optimization, decision-making, and other uncertainty-based mathematical models.

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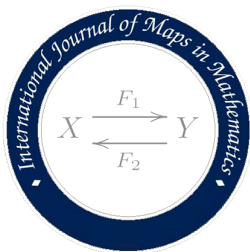
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LINEAR DIOPHANTINE FUZZY IDEAL THEORY IN d -ALGEBRAS

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Abstract. This study presents a comprehensive investigation into the integration of linear Diophantine fuzzy sets within the framework of d -algebraic structures, offering a refined perspective on the interaction between fuzzy logic and abstract algebra. The concepts of (weak) linear Diophantine fuzzy d -subalgebras and (weak) linear Diophantine fuzzy d -ideals are rigorously defined and formalized, supported by illustrative examples to enhance conceptual clarity. A thorough examination is conducted to identify and characterize the algebraic properties of these structures, with a particular focus on establishing the necessary and sufficient conditions under which a structural transformation between the two notions is feasible. Furthermore, the study delves into two distinguished classes of ideals in d -algebras through the lens of linear Diophantine fuzzy theory, emphasizing their structural alignment and interaction with classical d -ideals.

Keywords: d -algebra, Linear Diophantine fuzzy set, Weak linear Diophantine fuzzy set, Linear Diophantine fuzzy subalgebra, Linear Diophantine fuzzy ideal

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1. INTRODUCTION

In real world problems, the limitations of classical mathematical tools become apparent due to the inherent vagueness and uncertainty present in many complex systems. The introduction of fuzzy set theory by Zadeh [23] marked a significant shift in mathematical modeling, enabling more flexible and nuanced representations of imprecise information. Fuzzy logic, derived from fuzzy sets, provides a framework where truth values range in the interval $[0, 1]$, offering an alternative to classical binary logic and probability theory for managing uncertainty and ambiguity. Atanassov [5] extended the concept of fuzzy sets by introducing intuitionistic fuzzy sets, in which the sum of an element's membership degree and non-membership degree does not exceed 1.

Two significant classes of logic algebras, namely BCK-algebras and BCI-algebras, were introduced in the 1960s by Imai and Iseki [11], [12], and have since been the subject of extensive study by numerous researchers. It is well established that BCK-algebras constitute

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a proper subclass of BCI-algebras. Among the generalizations of BCK-algebras, the concept of d -algebras was introduced by Neggers and Kim [16]. Furthermore, the notions of d -subalgebras, d -ideals, and several related concepts, along with the relationships among them, were defined and examined by Neggers et al. [17]. Several studies integrating different types of ideals and fuzzy structures in the context of BCK-algebras and d -algebras can be found in [2], [3], [14], [21], [22].

Riaz and Hashmi [19] proposed the concept of a linear Diophantine fuzzy set by incorporating reference parameters inspired by the classical linear Diophantine equation $ax + by = c$ from number theory. This approach eliminates the constraints typically imposed on membership and non-membership degrees, thereby enabling decision-makers to assign values freely without predefined restrictions. Diophantine equations, particularly linear Diophantine equations, play a fundamental role in number theory and algebra due to their simplicity and wide range of applications in cryptography, coding theory, and optimization. Recently, there has been increasing interest in combining the flexibility of fuzzy logic with the structural rigor of Diophantine equations, leading to the notion of linear Diophantine fuzzy sets. These sets aim to represent collections of fuzzy elements subject to linear Diophantine constraints, enriching fuzzy modeling with arithmetic consistency.

The interplay between algebraic structures and linear Diophantine fuzzy sets was first introduced by Kamacı [13] in 2021. In his pioneering work, he examined finite linear Diophantine fuzzy substructures within classical algebraic systems such as groups, rings, and fields. Subsequently, Al-Tahan et al. [4] extended this line of research by investigating linear Diophantine fuzzy subpolygroups of a polygroup. Inspired by recent developments on linear Diophantine fuzzy substructures and the basic literature on fuzzy algebraic structures, Muhiuddin et al. focused on the study of linear Diophantine fuzzy sets in the context of BCK/BCI algebras [15]. Other researchers applied linear Diophantine fuzzy sets to different algebraic structures [1], [6], [20]. Building upon the application of linear Diophantine fuzzy sets to BE-algebras, Öztürk et al. [18] introduced weak linear Diophantine fuzzy BE-algebras and investigated their fundamental properties. However, extending the theory to d -algebras is not merely a direct translation. The order relation in d -algebras, defined by $\nu * \omega = 0$, together with the additional d -condition, yields structural properties that do not arise in the BE-algebra setting. These features enable us to establish new relationships between linear Diophantine fuzzy d -subalgebras and d -ideals, particularly in d -algebras and edge d -algebras, and to investigate the behavior of weak linear Diophantine level sets within this distinct algebraic framework.

Recently, linear Diophantine fuzzy theory has attracted considerable attention due to its applicability in decision-making and uncertainty modeling. Several aggregation operators and decision-making approaches have been developed in this framework [7, 9, 10]. In particular, roughness and approximation-based extensions of linear Diophantine fuzzy sets have also been investigated, providing new tools for handling uncertainty in more complex environments [8]. These studies demonstrate the growing importance of linear Diophantine fuzzy structures in both theoretical and applied research, further motivating their study in different algebraic systems.

The primary motivation of the present study is to investigate the intersection between linear Diophantine fuzzy sets and d -algebraic structures. This integration not only enhances the theoretical framework of fuzzy algebra but also introduces novel directions for applying fuzzy arithmetic logic in contexts characterized by uncertainty within a structurally defined setting. This work aims to contribute to the ongoing synthesis of fuzzy set theory and algebraic systems, paving the way for further advancements in the theoretical foundations and applications of fuzzy algebra.

Linear Diophantine fuzzy sets generalize several existing fuzzy models, including intuitionistic and Pythagorean fuzzy sets, by introducing the reference parameters τ and φ . These parameters provide additional flexibility in handling membership and non-membership information. Motivated by this feature, we investigate linear Diophantine fuzzy ideals in d -algebras and establish their basic properties.

The structure of this paper is organized as follows. In Section 2, we provide the necessary preliminaries and recall fundamental definitions related to d -algebras and linear Diophantine fuzzy sets. Section 3 is devoted to the introduction of linear Diophantine fuzzy d -subalgebras and linear Diophantine fuzzy d -ideals in d -algebras. In this section, illustrative examples are provided and several basic properties are examined. In Section 4, we define the notions of linear Diophantine fuzzy $d^\#$ -ideal and linear Diophantine fuzzy commutative d -ideal in d -algebras. In Section 5, we present the definitions of weak linear Diophantine fuzzy d -subalgebras and weak linear Diophantine fuzzy d -ideals with illustrative examples, and further provide the necessary condition for nonempty weak linear Diophantine level sets to constitute d -subalgebras.

2. PRELIMINARIES

In this section, we provide a foundational background by presenting several fundamental results concerning fuzzy sets and intuitionistic fuzzy sets, linear Diophantine fuzzy sets, d -algebras and some types of d -ideals. These preliminary concepts will serve as a basis for the subsequent theoretical developments in the paper.

Definition 2.1. ([16]) *A non-empty set $(\Upsilon; *, 0)$ of type $(2, 0)$ is called a d -algebra if $\forall v, \omega \in \Upsilon$, the following conditions hold:*

- (1) $v * v = 0$,
- (2) $0 * v = 0$,
- (3) $v * \omega = 0$ and $\omega * v = 0$ implies $v = \omega$.

A BCK-algebra is a d -algebra $(\Upsilon; *, 0)$ satisfying the following additional axioms: $\forall v, \omega, \eta \in \Upsilon$,

- (4) $((v * \omega) * (v * \eta)) * (\eta * \omega) = 0$,
- (5) $(v * (v * \omega)) * \omega = 0$.

Definition 2.2. ([17]) *Let $(\Upsilon; *, 0)$ be a d -algebra, and $\emptyset \neq \Phi \subseteq \Upsilon$. If for any $v, \omega \in \Phi$, the element $v * \omega$ also belong to Φ , then Φ is called a d -subalgebra of Υ . Moreover, Φ is called a d -ideal of Υ if it satisfies the following conditions:*

- (dI-1) $v * \omega \in \Phi$ and $\omega \in \Phi$ imply $v \in \Phi$,
 (dI-2) $v \in \Phi$ and $\omega \in \Upsilon$ imply $v * \omega \in \Phi$, i.e., $\Phi * \Upsilon \subseteq \Phi$.

Lemma 2.1. ([17]) *If Φ is a d -ideal of a d -algebra Υ , then $0 \in \Phi$.*

Definition 2.3. ([17]) *Let Υ be a d -algebra. A d -ideal Φ of Υ is called a $d^\#$ -ideal of Υ , if $\nu * \omega \in \Phi$ and $\omega * \eta \in \Phi$ imply $\nu * \eta \in \Phi$ for all $\nu, \omega, \eta \in \Upsilon$.*

Definition 2.4. ([16]) *Let $(\Upsilon; *, 0)$ be a d -algebra, and $\nu, \omega \in \Upsilon$. Define the operation $\nu \wedge \omega = \omega * (\omega * \nu)$. Furthermore, a d -algebra Υ is said to be commutative if $\nu \wedge \omega = \omega \wedge \nu$ holds for all $\nu, \omega \in \Upsilon$.*

Definition 2.5. ([22]) *A non-empty subset Φ of a d -algebra Υ is called a commutative d -ideal of Υ if it satisfies $0 \in \Phi$ and $(\nu * \omega) * \eta \in \Phi$, $\eta \in \Phi \Rightarrow \nu * (\omega \wedge \nu) \in \Phi$ for all $\nu, \omega, \eta \in \Phi$.*

Definition 2.6. ([16]) *Let Υ be a d -algebra and $v \in \Upsilon$. Define $v * \Upsilon = \{v * \omega \mid \omega \in \Upsilon\}$. If for any $v \in \Upsilon$, $v * \Upsilon = \{v, 0\}$, then Υ is called an edge d -algebra.*

Remark 2.1. ([16]) *If $(\Upsilon, \leq)$ is an ordered set, then the operation $*$ on Υ given by $\nu * \omega = 0$ iff $\nu \leq \omega$.*

Lemma 2.2. ([16]) *If Υ is an edge d -algebra, then $v * 0 = v$ for any $v \in \Upsilon$.*

Lemma 2.3. ([16],[17]) *Let Υ be a d -algebra. Then, for all $\nu, \omega \in \Upsilon$, the inequality $(\nu * \omega) * (\eta * \omega) \leq \nu * \eta$ holds.*

Moreover, if Υ is an edge d -algebra, then the following conditions hold: For all $\nu, \omega, \eta \in \Upsilon$

- i. $(\nu * (\nu * \omega)) * \omega = 0$,
- ii. $(\nu * \omega) * \eta = (\nu * \eta) * \omega$.

Definition 2.7. ([17]) *A d -algebra Υ is called d^* -algebra if it satisfies the identity $(\nu * \omega) * \nu = 0$ for all $\nu, \omega \in \Upsilon$.*

Definition 2.8. ([23]) *Let S be a universal set, $I = [0, 1]$, and $\mu : S \rightarrow I$ be a function. Thus $X = \{(s, \mu(s)) \mid s \in S\}$ is called a fuzzy set.*

Definition 2.9. ([5]) *Let S be a universal set, $I = [0, 1]$, and $\mu, \theta : S \rightarrow I$ be the membership and non-membership functions, respectively. Thus $X = \{(s, \mu(s), \theta(s)) \mid s \in S\}$ is called an intuitionistic fuzzy set. Here, $\mu(s) + \theta(s) \in I$ for all $s \in S$.*

Definition 2.10. ([19]) *Let S be a non-empty reference set. A linear Diophantine fuzzy set on S is defined to be an object of the form:*

$$\mathcal{L} = \{(s; (\xi(s), \hbar(s)), (\tau, \varphi)) \mid s \in S\},$$

where ξ and $\hbar$ are fuzzy sets in S , and $\tau, \varphi \in [0, 1]$ such that $0 \leq \tau + \varphi \leq 1$ and $0 \leq \tau \xi(s) + \varphi \hbar(s) \leq 1$ for all $s \in S$.

We say that $\xi(s)$ and $\hbar(s)$ are the degrees of membership and non-membership at a point $s \in S$, respectively, and τ, φ are reference parameters. The linear Diophantine fuzzy set $\mathcal{L}$ simply denoted by $\mathcal{L} := (\xi, \hbar; \tau, \varphi)$.

Remark 2.2. *In the definition of a linear Diophantine fuzzy set $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$ introduced in [19], since $\xi(s), \bar{h}(s), \tau$, and φ take values in $[0, 1]$, it follows that whenever $0 \leq \tau + \varphi \leq 1$, the condition*

$$0 \leq \tau\xi(s) + \varphi\bar{h}(s) \leq 1$$

is automatically satisfied for all $s \in S$.

Although this observation may suggest that the reference parameters do not impose additional restrictions, their significance lies in weighting the relative influence of membership and non-membership degrees. Different choices of (τ, φ) may yield different evaluations of the same fuzzy information and consequently affect the defining inequalities of LDF-subalgebras and LDF-ideals. Therefore, linear Diophantine fuzzy sets provide a parameter-dependent extension of intuitionistic and Pythagorean fuzzy frameworks.

Let S be a non-empty reference set, $\tau, \varphi \in [0, 1]$ be reference parameters, and ξ and $\bar{h}$ be functions from S to $[0, 1]$.

Consider an object of the form:

$$\mathcal{L} = \{(s; (\xi(s), \bar{h}(s)), (\tau, \varphi)) \mid s \in S\}$$

and it is simply denoted by $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$, and we say it is a Diophantine structure in S . In the definition of a linear Diophantine fuzzy set in [19], the reference parameters $\tau, \varphi \in [0, 1]$ are chosen independently of the membership and non-membership degrees $\xi(s)$ and $\bar{h}(s)$. For convenience, the parameters assigned to each $s \in S$ will be denoted by $\tau(s)$ and $\varphi(s)$.

Definition 2.11. ([18]) *Let S be a non-empty reference set. A weak linear Diophantine fuzzy set on S is defined to be an object of the form:*

$$\mathcal{L} = \{(s; (\xi(s), \bar{h}(s)), (\tau(s), \varphi(s))) \mid s \in S\},$$

where ξ and $\bar{h}$ are fuzzy sets in S , and $\tau(s), \varphi(s) \in [0, 1]$ are reference parameters such that $\tau(s) \cdot \xi(s) + \varphi(s) \cdot \bar{h}(s) \in [0, 1]$ for all $s \in S$.

For simplicity, the weak linear Diophantine fuzzy set is denoted by $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$, in the above definition.

In a weak linear Diophantine fuzzy set $\mathcal{L} := (\xi, \bar{h}; \tau, \varphi)$ on S , we consider the following mappings:

$$\begin{aligned} \tau\xi &: S \rightarrow [0, 1], s \mapsto \tau(s) \cdot \xi(s), \\ \varphi\bar{h} &: S \rightarrow [0, 1], s \mapsto \varphi(s) \cdot \bar{h}(s). \end{aligned}$$

Note that $\tau\xi$ and $\varphi\bar{h}$ are fuzzy sets in S . So we can establish two level sets

$$U(\tau\xi; \alpha) := \{s \in S \mid \tau\xi(s) \geq \alpha\} \text{ and } L(\varphi\bar{h}; \beta) := \{s \in S \mid \varphi\bar{h}(s) \leq \beta\},$$

of $\tau\xi$ and $\varphi\bar{h}$ respectively, where $\alpha, \beta \in [0, 1]$. We call $U(\tau\xi; \alpha)$ and $L(\varphi\bar{h}; \beta)$ weak linear Diophantine level sets of $\tau\xi$ and $\varphi\bar{h}$, respectively (see [18]).

Definition 2.12. ([19]) *Let S be a universal set and $\mathcal{L}_1, \mathcal{L}_2$ be linear Diophantine fuzzy sets on S . Thus*

(1) The intersection $\mathcal{L}_1 \cap \mathcal{L}_2$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined as

$$\{(s; (\xi_{\mathcal{L}_1}(s) \wedge \xi_{\mathcal{L}_2}(s), \hbar_{\mathcal{L}_1}(s) \vee \hbar_{\mathcal{L}_2}(s)), (\tau_{\mathcal{L}_1}(s) \wedge \tau_{\mathcal{L}_2}(s), \varphi_{\mathcal{L}_1}(s) \vee \varphi_{\mathcal{L}_2}(s))) \mid s \in S\}.$$

(2) The union $\mathcal{L}_1 \cup \mathcal{L}_2$ of $\mathcal{L}_1$ and $\mathcal{L}_2$ is defined as

$$\{(s; (\xi_{\mathcal{L}_1}(s) \vee \xi_{\mathcal{L}_2}(s), \hbar_{\mathcal{L}_1}(s) \wedge \hbar_{\mathcal{L}_2}(s)), (\tau_{\mathcal{L}_1}(s) \vee \tau_{\mathcal{L}_2}(s), \varphi_{\mathcal{L}_1}(s) \wedge \varphi_{\mathcal{L}_2}(s))) \mid s \in S\}.$$

(3) $\mathcal{L}_1$ is subset of $\mathcal{L}_2$, denoted by $\mathcal{L}_1 \subseteq_D \mathcal{L}_2$, if $\mathcal{L}_1(s) \leq \mathcal{L}_2(s)$ for all $s \in S$, i.e., $\xi_{\mathcal{L}_1}(s) \leq \xi_{\mathcal{L}_2}(s)$, $\hbar_{\mathcal{L}_1}(s) \geq \hbar_{\mathcal{L}_2}(s)$, $\tau_{\mathcal{L}_1}(s) \leq \tau_{\mathcal{L}_2}(s)$ and $\varphi_{\mathcal{L}_1}(s) \geq \varphi_{\mathcal{L}_2}(s)$ for all $s \in S$.

(4) $\mathcal{L}_1 = \mathcal{L}_2$ if $\mathcal{L}_1 \subseteq_D \mathcal{L}_2$ and $\mathcal{L}_2 \subseteq_D \mathcal{L}_1$.

(5) The complement $\mathcal{L}_1^c$ of $\mathcal{L}_1$ is defined as

$$\mathcal{L}_1^c = \{(s; \hbar_{\mathcal{L}_1}(s), \xi_{\mathcal{L}_1}(s)), (\varphi_{\mathcal{L}_1}(s), \tau_{\mathcal{L}_1}(s))) \mid s \in S\}.$$

Here, “ $\vee$ ” and “ $\wedge$ ” represent the maximum and minimum, respectively.

3. LINEAR DIOPHANTINE FUZZY d -SUBALGEBRAS AND d -IDEALS

In this section, we present the definitions and illustrative examples of linear Diophantine fuzzy d -subalgebras and linear Diophantine fuzzy d -ideals. We investigate several properties satisfied by these structures and provide proofs regarding the conditions under which a transition between them is possible.

Definition 3.1. Let Υ be a d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a linear Diophantine fuzzy set of Υ . Thus $\mathcal{L}$ is called a linear Diophantine fuzzy d -subalgebra (briefly, LDFd-sub) if

$$(LDFd\text{-sub1}) (\forall \nu, \omega \in \Upsilon) (\xi(\nu * \omega) \geq \xi(\nu) \wedge \xi(\omega), \hbar(\nu * \omega) \leq \hbar(\nu) \vee \hbar(\omega))$$

$$(LDFd\text{-sub2}) (\forall \nu, \omega \in \Upsilon) (\tau(\nu * \omega) \geq \tau(\nu) \wedge \tau(\omega), \varphi(\nu * \omega) \leq \varphi(\nu) \vee \varphi(\omega)).$$

Example 3.1. Consider a set $\Upsilon = \{0, \nu, \omega, \eta\}$ with a binary operation $*$ given in the following table:

$*$	0	ν	ω	η
0	0	0	0	0
ν	ν	0	0	ω
ω	ω	ω	0	0
η	η	η	ν	0

Thus $(\Upsilon, *, 0)$ is a d -algebra. Define a linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as follows:

Υ	$\mathcal{L} = (\xi, \hbar; \tau, \varphi)$
0	(0.86, 0.53; 0.7, 0.4)
ν	(0.66, 0.72; 0.6, 0.5)
ω	(0.74, 0.62; 0.3, 0.5)
η	(0.52, 0.84; 0.1, 0.7)

It is routine to verify that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is an LDFd-sub of Υ .

Remark 3.1. In Example 3.1, the functions τ and φ play an essential role in satisfying condition (LDFd-sub2). Indeed, if the membership and non-membership functions ξ and $\hbar$ are kept unchanged while the reference function τ is modified by setting

$$\tau(\omega) = 0.8$$

instead of $\tau(\omega) = 0.3$, then

$$\tau(\omega * \omega) = \tau(0) = 0.7 < 0.8 = \tau(\omega) \wedge \tau(\omega),$$

and consequently condition (LDFd-sub2) fails. Therefore, the resulting structure is no longer a linear Diophantine fuzzy d -subalgebra. This shows that the reference functions may affect the algebraic behaviour of the structure, even when the membership and non-membership functions remain unchanged.

Proposition 3.1. Let Υ be a d -algebra. Thus every LDFd-sub of Υ $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ satisfies:

$$(\forall \nu \in \Upsilon)(\mathcal{L}(\nu) \leq \mathcal{L}(0)),$$

where the relation “ $\leq$ ” is understood in the sense of the component-wise partial order defined in Definition 2.12(3).

Proof. Let $\nu \in \Upsilon$. Thus, we have

$$\xi(0) = \xi(\nu * \nu) \geq \xi(\nu) \wedge \xi(\nu) = \xi(\nu),$$

$$\hbar(0) = \hbar(\nu * \nu) \leq \hbar(\nu) \vee \hbar(\nu) = \hbar(\nu),$$

$$\tau(0) = \tau(\nu * \nu) \geq \tau(\nu) \wedge \tau(\nu) = \tau(\nu),$$

and

$$\varphi(0) = \varphi(\nu * \nu) \leq \varphi(\nu) \vee \varphi(\nu) = \varphi(\nu).$$

Therefore, we get $\mathcal{L}(\nu) \leq \mathcal{L}(0)$ for all $\nu \in \Upsilon$. $\square$

Definition 3.2. Let Υ be a d -algebra. A linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ is called a linear Diophantine fuzzy d -ideal (briefly, LDFd-id) if it satisfies:

$$(LDFd-id1) (\forall \nu \in \Upsilon)(\xi(0) \geq \xi(\nu), \hbar(0) \leq \hbar(\nu), \tau(0) \geq \tau(\nu), \varphi(0) \leq \varphi(\nu)),$$

$$(LDFd-id2) (\forall \nu, \omega \in \Upsilon)(\xi(\nu) \geq \xi(\nu * \omega) \wedge \xi(\omega), \hbar(\nu) \leq \hbar(\nu * \omega) \vee \hbar(\omega)),$$

$$(LDFd-id3) (\forall \nu, \omega \in \Upsilon)(\tau(\nu) \geq \tau(\nu * \omega) \wedge \tau(\omega), \varphi(\nu) \leq \varphi(\nu * \omega) \vee \varphi(\omega)).$$

Example 3.2. Consider a d -algebra $\Upsilon = \{0, \nu, \omega, \eta\}$, which is not a BCK-algebra, defined as follows.

$*$	0	ν	ω	η
0	0	0	0	0
ν	ν	0	0	ν
ω	ω	ω	0	0
η	η	η	η	0

Define a linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as follows:

Υ	$\mathcal{L} = (\xi, \hbar; \tau, \varphi)$
0	(0.62, 0.36; 0.8, 0.1)
ν	(0.53, 0.45; 0.5, 0.4)
ω	(0.51, 0.45; 0.4, 0.5)
η	(0.35, 0.51; 0.1, 0.7)

It is easy to show that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is an LDFd-id of Υ .

Proposition 3.2. Let Υ be a d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be an LDFd-id of Υ , and $\nu, \omega \in \Upsilon$ such that $\nu \leq \omega$. Then $\mathcal{L}(\omega) \leq \mathcal{L}(\nu)$.

Proof. Let $\nu, \omega \in \Upsilon$ such that $\nu \leq \omega$. Hence, we get

$$\begin{aligned} \xi(\nu) &\geq \xi(\nu * \omega) \wedge \xi(\omega) = \xi(0) \wedge \xi(\omega) = \xi(\omega), \\ \hbar(\nu) &\leq \hbar(\nu * \omega) \vee \hbar(\omega) = \hbar(0) \vee \hbar(\omega) = \hbar(\omega), \\ \tau(\nu) &\geq \tau(\nu * \omega) \wedge \tau(\omega) = \tau(0) \wedge \tau(\omega) = \tau(\omega), \\ \varphi(\nu) &\leq \varphi(\nu * \omega) \vee \varphi(\omega) = \varphi(0) \vee \varphi(\omega) = \varphi(\omega), \end{aligned}$$

and so $\mathcal{L}(\omega) \leq \mathcal{L}(\nu)$. $\square$

Proposition 3.3. Let Υ be a d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be an LDFd-id of Υ , and $\nu, \omega, \eta \in \Upsilon$ such that $\nu * \omega \leq \eta$. Thus, we have $\mathcal{L}(\eta) \wedge \mathcal{L}(\omega) \leq \mathcal{L}(\nu)$.

Proof. Let $\nu, \omega, \eta \in \Upsilon$ such that $\nu * \omega \leq \eta$. Hence, we get

$$\begin{aligned} \xi(\nu) &\geq \xi(\nu * \omega) \wedge \xi(\omega) \geq (\xi((\nu * \omega) * \eta) \wedge \xi(\eta)) \wedge \xi(\omega) \\ &= \xi(0) \wedge \xi(\eta) \wedge \xi(\omega) \\ &= \xi(\eta) \wedge \xi(\omega), \\ \hbar(\nu) &\leq \hbar(\nu * \omega) \vee \hbar(\omega) \leq (\hbar((\nu * \omega) * \eta) \vee \hbar(\eta)) \vee \hbar(\omega) \\ &= \hbar(0) \vee \hbar(\eta) \vee \hbar(\omega) \\ &= \hbar(\eta) \vee \hbar(\omega), \\ \tau(\nu) &\geq \tau(\nu * \omega) \wedge \tau(\omega) \geq (\tau((\nu * \omega) * \eta) \wedge \tau(\eta)) \wedge \tau(\omega) \\ &= \tau(0) \wedge \tau(\eta) \wedge \tau(\omega) \\ &= \tau(\eta) \wedge \tau(\omega), \\ \varphi(\nu) &\leq \varphi(\nu * \omega) \vee \varphi(\omega) \leq (\varphi((\nu * \omega) * \eta) \vee \varphi(\eta)) \vee \varphi(\omega) \\ &= \varphi(0) \vee \varphi(\eta) \vee \varphi(\omega) \\ &= \varphi(\eta) \vee \varphi(\omega). \end{aligned}$$

Therefore, we have $\mathcal{L}(\eta) \wedge \mathcal{L}(\omega) \leq \mathcal{L}(\nu)$. $\square$

Remark 3.2. There is no implication between as LDFd-subalgebra and LDFd-ideal in d -algebras. In Example 3.1, the LDFd-subalgebra $\mathcal{L}$ is not an LDFd-ideal, because $\xi(\eta) \not\geq \xi(\eta * \omega) \wedge \xi(\omega)$. Furthermore, Example 3.3 shows that LDFd-ideals may not necessarily be LDFd-subalgebras. This situation holds in the case of d^* -algebras (see Theorem 3.1).

Example 3.3. Let $\Upsilon = \{0, \nu, \omega, \eta\}$ be a d -algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	ν	ω	η
0	0	0	0	0
ν	ν	0	0	ω
ω	ω	η	0	0
η	η	η	η	0

We define a linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as:

Υ	$\mathcal{L} = (\xi, \hbar; \tau, \varphi)$
0	(0.85, 0.12; 0.7, 0.14)
ν	(0.73, 0.32; 0.5, 0.41)
ω	(0.52, 0.16; 0.5, 0.36)
η	(0.36, 0.44; 0.2, 0.44)

It is easy to verify that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is an LDFd-id of Υ but not an LDFd-sub of Υ , because $\xi(\omega * \nu) = \xi(\eta) \not\geq \xi(\omega) \wedge \xi(\nu)$ or $\tau(\omega * \nu) = \tau(\eta) \not\geq \tau(\omega) \wedge \tau(\nu)$.

Theorem 3.1. Let Υ be a d^* -algebra. Then every LDFd-id of Υ is an LDFd-sub of Υ .

Proof. Assume that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is an LDFd-id of Υ and $\nu, \omega \in \Upsilon$. Since Υ is a d^* -algebra, we have $(\nu * \omega) * \nu = 0$ and so $\nu * \omega \leq \nu$. Using Proposition 3.2, we get $\mathcal{L}(\nu) \leq \mathcal{L}(\nu * \omega)$. Hence,

(LDFd-sub1) :

$$\begin{aligned} \xi(\nu * \omega) &\geq \xi(\nu) \geq \xi(\nu * \omega) \wedge \xi(\omega) \geq \xi(\nu) \wedge \xi(\omega), \\ \hbar(\nu * \omega) &\leq \hbar(\nu) \leq \hbar(\nu * \omega) \vee \hbar(\omega) \leq \hbar(\nu) \vee \hbar(\omega). \end{aligned}$$

(LDFd-sub2) :

$$\begin{aligned} \tau(\nu * \omega) &\geq \tau(\nu) \geq \tau(\nu * \omega) \wedge \tau(\omega) \geq \tau(\nu) \wedge \tau(\omega), \\ \varphi(\nu * \omega) &\leq \varphi(\nu) \leq \varphi(\nu * \omega) \vee \varphi(\omega) \leq \varphi(\nu) \vee \varphi(\omega). \end{aligned}$$

Therefore, $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is an LDFd-sub of Υ . □

Theorem 3.2. Let Υ be an edge d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be an LDFd-sub of Υ . Then $\mathcal{L}$ is an LDFd-id of $\Upsilon \Leftrightarrow$ For all $\nu, \omega, \eta \in \Upsilon$, $\nu * \omega \leq \eta$ implies $\xi(\nu) \geq \xi(\omega) \wedge \xi(\eta)$, $\hbar(\nu) \leq \hbar(\omega) \vee \hbar(\eta)$, $\tau(\nu) \geq \tau(\omega) \wedge \tau(\eta)$, and $\varphi(\nu) \leq \varphi(\omega) \vee \varphi(\eta)$.

Proof. ($\Rightarrow$) : Follows from Proposition 3.3.

($\Leftarrow$) : Since Υ is an edge d -algebra, we have $\nu * (\nu * \omega) \leq \omega$. Using our hypothesis, $\xi(\nu) \geq \xi(\nu * \omega) \wedge \xi(\omega)$, $\hbar(\nu) \leq \hbar(\nu * \omega) \vee \hbar(\omega)$, $\tau(\nu) \geq \tau(\nu * \omega) \wedge \tau(\omega)$ and $\varphi(\nu) \leq \varphi(\nu * \omega) \vee \varphi(\omega)$. Moreover, it follows from Proposition 3.1 $\mathcal{L}(\nu) \leq \mathcal{L}(0)$ for all $\nu \in \Upsilon$. Therefore, $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is an LDFd-id of Υ . □

4. TYPES AND PROPERTIES OF LINEAR DIOPHANTINE FUZZY d -IDEALS

In this section, we examined two specific types of ideals in d -algebras in relation to linear Diophantine fuzzy sets, and investigated their connection with the structure of d -ideals.

Definition 4.1. Let Υ be a d -algebra. An $LDFd$ -id $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is called a linear Diophantine fuzzy $d^\#$ -ideal (briefly, $LDFd^\#$ -id) of Υ if it satisfies:

$$\begin{aligned} (LDFd^\#-id1) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\xi(\nu * \eta) \geq \xi(\nu * \omega) \wedge \xi(\omega * \eta), \hbar(\nu * \eta) \leq \hbar(\nu * \omega) \vee \hbar(\omega * \eta)), \\ (LDFd^\#-id2) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\tau(\nu * \eta) \geq \tau(\nu * \omega) \wedge \tau(\omega * \eta), \varphi(\nu * \eta) \leq \varphi(\nu * \omega) \vee \varphi(\omega * \eta)). \end{aligned}$$

Example 4.1. Let $\Upsilon = \{0, \nu, \omega, \eta\}$ be a set with the following Cayley table:

$*$	0	ν	ω	η
0	0	0	0	0
ν	ν	0	0	ν
ω	η	ω	0	η
η	η	ω	ω	0

Then $(\Upsilon, *, 0)$ is a d -algebra, but not a BCK-algebra. We define a linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as:

Υ	$\mathcal{L} = (\xi, \hbar; \tau, \varphi)$
0	(0.55, 0.14; 0.37, 0.28)
ν	(0.42, 0.36; 0.21, 0.63)
ω	(0.25, 0.52; 0.14, 0.85)
η	(0.25, 0.52; 0.14, 0.85)

Some computations show that $\mathcal{L}$ is an $LDFd^\#$ -id of Υ .

Remark 4.1. Clearly, while every $LDFd^\#$ -ideal is an $LDFd$ -ideal, the reverse implication does not always apply.

Example 4.2. Let Υ be a d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a $LDFd$ -id of Υ as in Example 3.2. We see that $\mathcal{L}$ is not an $LDFd^\#$ -id of Υ , since $\hbar(\nu * \eta) = \hbar(\nu) = 0.45 \not\leq \hbar(\nu * \omega) \vee \hbar(\omega * \eta) = \hbar(0) = 0.36$.

Definition 4.2. Let Υ be a d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a linear Diophantine fuzzy set on Υ . Then $\mathcal{L}$ is called a linear Diophantine fuzzy commutative d -ideal (briefly, $LDFCd$ -id) if it satisfies ($LDFd$ -id1) and the following conditions:

$$\begin{aligned} (LDFCd-id1) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\xi(\nu * (\omega * (\omega * \nu))) \geq \xi((\nu * \omega) * \eta) \wedge \xi(\eta), \hbar(\nu * (\omega * (\omega * \nu))) \leq \hbar((\nu * \omega) * \eta) \vee \hbar(\eta)), \\ (LDFCd-id2) \quad & (\forall \nu, \omega, \eta \in \Upsilon) (\tau(\nu * (\omega * (\omega * \nu))) \geq \tau((\nu * \omega) * \eta) \wedge \tau(\eta), \varphi(\nu * (\omega * (\omega * \nu))) \leq \varphi((\nu * \omega) * \eta) \vee \varphi(\eta)). \end{aligned}$$

Example 4.3. Consider a d -algebra Υ of Example 4.1. Now define a linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as:

Υ	$\mathcal{L} = (\xi, \hbar; \tau, \varphi)$
0	(0.7, 0.3; 0.51, 0.41)
ν	(0.6, 0.5; 0.33, 0.63)
ω	(0.2, 0.8; 0.17, 0.74)
η	(0.2, 0.8; 0.17, 0.74)

It is easy to verify that $\mathcal{L}$ is an LDFCd-id of Υ .

Remark 4.2. In general, LDFd-ideals are not necessarily LDFCd-ideals.

Example 4.4. The LDFd-id structure given in Example 3.3 is not an LDFCd-id of Υ since $\xi(\omega * (\eta * (\eta * \omega))) = \xi(\omega) \not\geq \xi((\omega * \eta) * \nu) \wedge \xi(\nu) = \xi(\nu)$.

Theorem 4.1. In edge d -algebras, the structures of LDFd-ideal and LDFCd-ideal coincide.

Proof. Let Υ be an edge d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be an LDFd-id of Υ . Then, for all $\nu, \omega, \eta \in \Upsilon$, we have $((\nu * (\omega * (\omega * \nu))) * ((\nu * \omega) * \eta)) \leq \eta$. Since $\mathcal{L}$ is an LDFd-id of Υ ,

$$\xi(\nu * (\omega * (\omega * \nu))) \geq \xi((\nu * \omega) * \eta) \wedge \xi(\eta), \hbar(\nu * (\omega * (\omega * \nu))) \leq \hbar((\nu * \omega) * \eta) \vee \hbar(\eta),$$

$$\tau(\nu * (\omega * (\omega * \nu))) \geq \tau((\nu * \omega) * \eta) \wedge \tau(\eta), \varphi(\nu * (\omega * (\omega * \nu))) \leq \varphi((\nu * \omega) * \eta) \vee \varphi(\eta)$$

by Proposition 3.3. Therefore, $\mathcal{L}$ is an LDFCd-id of Υ .

Now, we assume that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be an LDFCd-id of Υ . Then, we have $\nu = \nu * (0 * (0 * \nu))$ and so

$$\begin{aligned} \xi(\nu) &= \xi(\nu * (0 * (0 * \nu))) \geq \xi((\nu * 0) * \omega) \wedge \xi(\omega) = \xi(\nu * \omega) \wedge \xi(\omega), \\ \hbar(\nu) &= \hbar(\nu * (0 * (0 * \nu))) \leq \hbar((\nu * 0) * \omega) \vee \hbar(\omega) = \hbar(\nu * \omega) \vee \hbar(\omega), \\ \tau(\nu) &= \tau(\nu * (0 * (0 * \nu))) \geq \tau((\nu * 0) * \omega) \wedge \tau(\omega) = \tau(\nu * \omega) \wedge \tau(\omega), \\ \varphi(\nu) &= \varphi(\nu * (0 * (0 * \nu))) \leq \varphi((\nu * 0) * \omega) \vee \varphi(\omega) = \varphi(\nu * \omega) \vee \varphi(\omega). \end{aligned}$$

Thus we have $\mathcal{L}$ is an LDFd-id of Υ . □

5. WEAK LINEAR DIOPHANTINE FUZZY d -SUBALGEBRAS AND d -IDEALS

In this section, we present the definitions and illustrative examples of weak linear Diophantine fuzzy d -subalgebras and weak linear Diophantine fuzzy d -ideals.

Definition 5.1. Let Υ be a d -algebra and $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a weak linear Diophantine fuzzy set of Υ . Thus $\mathcal{L}$ is called a weak linear Diophantine fuzzy d -subalgebra (briefly, WLDFd-sub) if

$$\begin{aligned} (WLDFd\text{-sub1}) \quad & (\forall \nu, \omega \in \Upsilon) (\tau\xi(\nu * \omega) \geq \min\{\tau(\nu), \tau(\omega)\} \cdot \min\{\xi(\nu), \xi(\omega)\}), \\ (WLDFd\text{-sub2}) \quad & (\forall \nu, \omega \in \Upsilon) (\varphi\hbar(\nu * \omega) \leq \max\{\varphi(\nu), \varphi(\omega)\} \cdot \max\{\hbar(\nu), \hbar(\omega)\}). \end{aligned}$$

Example 5.1. Let $\Upsilon = \{0, \nu, \omega, \eta\}$ be a set with the following Cayley table:

$*$	0	ν	ω	η
0	0	0	0	0
ν	ν	0	0	ω
ω	ω	ω	0	0
η	η	η	η	0

Then $(\Upsilon, *, 0)$ is a d -algebra which is not a BCK-algebra. We define a weak linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as:

$s \in \Upsilon$	$\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$
0	(0.85, 0.22; 0.7, 0.5)
ν	(0.64, 0.56; 0.5, 0.8)
ω	(0.81, 0.35; 0.6, 0.7)
η	(0.78, 0.41; 0.2, 0.6)

It is routine to verify that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is a WLDFd-sub of Υ .

Theorem 5.1. Let Υ be a d -algebra. Every LDFd-sub of Υ is a WLDFd-sub of Υ .

Proof. The result is straightforward from the definitions. $\square$

The following example shows that the converse of Theorem 5.1 may not be true. That is, if $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is a WLDFd-sub of Υ , then there are $\nu, \omega \in \Upsilon$ that satisfy $\xi(\nu * \omega) \not\geq \xi(\nu) \wedge \xi(\omega)$, $\hbar(\nu * \omega) \not\leq \hbar(\nu) \vee \hbar(\omega)$, $\tau(\nu * \omega) \not\geq \tau(\nu) \wedge \tau(\omega)$, $\varphi(\nu * \omega) \not\leq \varphi(\nu) \vee \varphi(\omega)$.

Example 5.2. Consider the d -algebra given in Example 3.1. We define the following weak linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ as:

$s \in \Upsilon$	$\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$
0	(0.62, 0.51; 0.7, 0.1)
ν	(0.41, 0.63; 0.2, 0.9)
ω	(0.38, 0.84; 0.6, 0.8)
η	(0.44, 0.84; 0.2, 0.4)

It is routine to verify that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is a WLDFd-sub of Υ and $\xi(\nu * \eta) = \xi(\omega) = 0.38 \not\geq 0.41 = \xi(\nu) \wedge \xi(\eta)$. Moreover, $\varphi(\eta * \omega) = \varphi(\nu) = 0.9 \not\leq 0.8 = \varphi(\eta) \vee \varphi(\omega)$.

Proposition 5.1. Let Υ be a d -algebra. Thus every WLDFd-sub $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ of Υ satisfies:

$$(\forall \nu \in \Upsilon)(\tau\xi(0) \geq \tau(\nu) \cdot \xi(\nu), \varphi\hbar(0) \leq \varphi(\nu) \cdot \hbar(\nu)).$$

Proof. Let $\nu \in \Upsilon$. Thus we have

$$\begin{aligned} \tau\xi(0) &= \tau\xi(\nu * \nu) \geq \min\{\tau(\nu), \tau(\nu)\} \cdot \min\{\xi(\nu), \xi(\nu)\} = \tau(\nu) \cdot \xi(\nu), \\ \varphi\hbar(0) &= \varphi\hbar(\nu * \nu) \leq \max\{\varphi(\nu), \varphi(\nu)\} \cdot \max\{\hbar(\nu), \hbar(\nu)\} = \varphi(\nu) \cdot \hbar(\nu). \end{aligned}$$

$\square$

Theorem 5.2. Let Υ be a d -algebra, and let $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a WLDFd-sub of Υ . If the reference parameters τ and φ satisfy:

$$\begin{aligned} \min \{ \tau(\nu), \tau(\omega) \} \cdot \min \{ \xi(\nu), \xi(\omega) \} &\geq \min \{ \tau(\nu) \cdot \xi(\nu), \tau(\omega) \cdot \xi(\omega) \}, \\ \max \{ \varphi(\nu), \varphi(\omega) \} \cdot \max \{ \hbar(\nu), \hbar(\omega) \} &\leq \max \{ \varphi(\nu) \cdot \hbar(\nu), \varphi(\omega) \cdot \hbar(\omega) \}, \end{aligned}$$

for all $\nu, \omega \in \Upsilon$, then the non-empty weak linear Diophantine level sets of $\tau\xi$ and $\varphi\hbar$ are d -subalgebras of Υ .

Proof. Let $\nu, \omega \in \Upsilon$ and $\alpha, \beta \in [0, 1]$ be such that $\nu, \omega \in U(\tau\xi; \alpha) \cap L(\varphi\hbar; \beta)$. Thus $\tau\xi(\nu) \geq \alpha, \tau\xi(\omega) \geq \alpha, \varphi\hbar(\nu) \leq \beta$ and $\varphi\hbar(\omega) \leq \beta$. Consequently,

$$\begin{aligned} \tau\xi(\nu * \omega) &\geq \min \{ \tau(\nu), \tau(\omega) \} \cdot \min \{ \xi(\nu), \xi(\omega) \} \\ &\geq \min \{ \tau(\nu) \cdot \xi(\nu), \tau(\omega) \cdot \xi(\omega) \} \\ &= \min \{ \tau\xi(\nu), \tau\xi(\omega) \} \geq \alpha \end{aligned}$$

and

$$\begin{aligned} \varphi\hbar(\nu * \omega) &\leq \max \{ \varphi(\nu), \varphi(\omega) \} \cdot \max \{ \hbar(\nu), \hbar(\omega) \} \\ &\leq \max \{ \varphi(\nu) \cdot \hbar(\nu), \varphi(\omega) \cdot \hbar(\omega) \} \\ &= \max \{ \varphi\hbar(\nu), \varphi\hbar(\omega) \} \leq \beta. \end{aligned}$$

Hence $\nu * \omega \in U(\tau\xi; \alpha) \cap L(\varphi\hbar; \beta)$. Therefore $U(\tau\xi; \alpha)$ and $L(\varphi\hbar; \beta)$ are d -subalgebras of Υ . $\square$

Remark 5.1. These conditions are introduced as sufficient conditions and are not claimed to be necessary. They ensure the closure properties required for weak linear Diophantine fuzzy level sets to form d -subalgebras in a constructive manner.

The following example illustrates Theorem 5.2.

Example 5.3. Consider the d -algebra $(\Upsilon, *, 0)$ in Example 3.2. Let $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a weak linear Diophantine fuzzy set on Υ , $\tau\xi$ and $\varphi\hbar$ are calculated by the following table.

$s \in \Upsilon$	$\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$	$\tau\xi(s)$	$\varphi\hbar(s)$
0	(0.83, 0.31; 0.8, 0.1)	0.664	0.031
ν	(0.64, 0.48; 0.7, 0.4)	0.448	0.192
ω	(0.52, 0.47; 0.4, 0.3)	0.208	0.141
η	(0.39, 0.58; 0.2, 0.6)	0.078	0.348

It is routine to check that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is a WLDFd-sub of Υ and the reference parameters satisfy the conditions of Theorem 5.2. The weak linear Diophantine level sets of $\tau\xi$ and $\varphi\hbar$ are calculated as follows:

$$U(\tau\xi; \alpha) = \begin{cases} \Upsilon & \text{if } \alpha \in [0, 0.078] \\ \{0, \nu, \omega\} & \text{if } \alpha \in (0.078, 0.208] \\ \{0, \nu\} & \text{if } \alpha \in (0.208, 0.448] \\ \{0\} & \text{if } \alpha \in (0.448, 0.664] \\ \emptyset & \text{if } \alpha \in (0.664, 1] \end{cases}$$

and

$$L(\varphi\hbar; \beta) = \begin{cases} \Upsilon & \text{if } \beta \in [0.348, 1] \\ \{0, \nu, \omega\} & \text{if } \beta \in [0.192, 0.348) \\ \{0, \omega\} & \text{if } \beta \in [0.141, 0.192) \\ \{0\} & \text{if } \beta \in [0.031, 0.141) \\ \emptyset & \text{if } \beta \in [0, 0.031) \end{cases}$$

which are d -subalgebras of Υ .

The following example shows that the converse of Theorem 5.2 may not be true, that is, there exists a weak linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ such that the weak linear Diophantine level sets of $\tau\xi$ and $\varphi\hbar$ are d -subalgebras of Υ , but the reference parameters τ and φ don't satisfy the conditions of Theorem 5.2.

Example 5.4. Consider the d -algebra $(\Upsilon, *, 0)$ in Example 3.2. Let $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a weak linear Diophantine fuzzy set on Υ , $\tau\xi$ and $\varphi\hbar$ are calculated by the following table.

$s \in \Upsilon$	$\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$	$\tau\xi(s)$	$\varphi\hbar(s)$
0	(0.7, 0.4; 0.8, 0.3)	0.56	0.12
ν	(0.8, 0.3; 0.2, 0.7)	0.16	0.21
ω	(0.6, 0.5; 0.4, 0.5)	0.24	0.25
η	(0.3, 0.6; 0.5, 0.7)	0.15	0.42

Then the non-empty the weak linear Diophantine level sets of $\tau\xi$ and $\varphi\hbar$ are d -subalgebras of Υ , and they are given as follows:

$$U(\tau\xi; \alpha) = \begin{cases} \Upsilon & \text{if } \alpha \in [0, 0.15] \\ \{0, \nu, \omega\} & \text{if } \alpha \in (0.15, 0.16] \\ \{0, \omega\} & \text{if } \alpha \in (0.16, 0.24] \\ \{0\} & \text{if } \alpha \in (0.24, 0.56] \\ \emptyset & \text{if } \alpha \in (0.56, 1] \end{cases}$$

and

$$L(\varphi\hbar; \beta) = \begin{cases} \Upsilon & \text{if } \beta \in [0.42, 1] \\ \{0, \nu, \omega\} & \text{if } \beta \in [0.25, 0.42) \\ \{0, \nu\} & \text{if } \beta \in [0.21, 0.25) \\ \{0\} & \text{if } \beta \in [0.12, 0.21) \\ \emptyset & \text{if } \beta \in [0, 0.12) \end{cases}$$

It is routine to check that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is a WLDFd-sub of Υ , but the reference parameters τ and φ do not satisfy the conditions of Theorem 5.2, since

$$\begin{aligned} \min \{\tau(\nu), \tau(\omega)\} \cdot \min \{\xi(\nu), \xi(\omega)\} &= \min \{0.2, 0.4\} \cdot \min \{0.8, 0.6\} \\ &= 0.12 \not\geq 0.16 \\ &= \min \{0.2 \cdot 0.8, 0.4 \cdot 0.6\} = \min \{\tau(\nu) \cdot \xi(\nu), \tau(\omega) \cdot \xi(\omega)\} \end{aligned}$$

and/or

$$\begin{aligned}
 \max \{ \varphi(\omega), \varphi(\nu) \} \cdot \max \{ \hbar(\omega), \hbar(\nu) \} &= \max \{ 0.5, 0.7 \} \cdot \max \{ 0.5, 0.3 \} \\
 &= 0.35 \not\leq 0.25 \\
 &= \max \{ 0.5 \cdot 0.5, 0.7 \cdot 0.3 \} = \max \{ \varphi(\omega) \cdot \hbar(\omega), \varphi(\nu) \cdot \hbar(\nu) \}.
 \end{aligned}$$

Definition 5.2. Let Υ be a d -algebra. A weak linear Diophantine fuzzy set $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ on Υ is called a weak linear Diophantine fuzzy d -ideal (briefly, *WLDFd-id*) if it satisfies:

$$\begin{aligned}
 (WLDFd-id1) \quad &(\forall \nu \in \Upsilon) (\tau \xi(0) \geq \tau \xi(\nu), \varphi \hbar(0) \leq \varphi \hbar(\nu)), \\
 (WLDFd-id2) \quad &(\forall \nu, \omega \in \Upsilon) (\tau \xi(\nu) \geq \min \{ \tau(\nu * \omega), \tau(\omega) \} \cdot \min \{ \xi(\nu * \omega), \xi(\omega) \}), \\
 (WLDFd-id3) \quad &(\forall \nu, \omega \in \Upsilon) (\varphi \hbar(\nu) \leq \max \{ \varphi(\nu * \omega), \varphi(\omega) \} \cdot \max \{ \hbar(\nu * \omega), \hbar(\omega) \}).
 \end{aligned}$$

Example 5.5. Consider the d -algebra $(\Upsilon, *, 0)$ in Example 3.2. Let $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ be a weak linear Diophantine fuzzy set on Υ , $\tau \xi$ and $\varphi \hbar$ are calculated by the following table.

$s \in \Upsilon$	$\mathcal{L}(s) = (\xi(s), \hbar(s); \tau(s), \varphi(s))$	$\tau \xi(s)$	$\varphi \hbar(s)$
0	(0.8, 0.2; 0.6, 0.4)	0.48	0.08
ν	(0.7, 0.6; 0.5, 0.7)	0.35	0.42
ω	(0.3, 0.5; 0.8, 0.9)	0.24	0.45
η	(0.4, 0.3; 0.3, 0.5)	0.12	0.15

It is easy to show that $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is a *WLDFd-id* of Υ .

Note that every *LDFd-id* of Υ is a *WLDFd-id*, but the converse need not be true. In Example 5.5, $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ is not an *LDFd-id* of Υ , since

$$0.5 = \tau(\nu) \not\geq \tau(\nu * \omega) \wedge \tau(\omega) = \tau(0) \wedge \tau(\omega) = 0.6$$

and/or

$$0.5 = \hbar(\omega) \not\leq \hbar(\omega * \eta) \vee \hbar(\eta) = \hbar(0) \vee \hbar(\eta) = 0.3.$$

Remark 5.2. It has been observed in the examples that there is no transition between the definitions of *WLDFd-sub* and *WLDFd-id*. In fact, the *WLDFd-sub* $\mathcal{L} = (\xi, \hbar; \tau, \varphi)$ given in Example 5.1 is not a *WLDFd-id*, because

$$\begin{aligned}
 0.32 &= \tau \xi(\nu) \not\geq \min \{ \tau(\nu * \omega), \tau(\omega) \} \cdot \min \{ \xi(\nu * \omega), \xi(\omega) \} \\
 &= \min \{ \tau(0), \tau(\omega) \} \cdot \min \{ \xi(0), \xi(\omega) \} \\
 &= \min \{ 0.7, 0.6 \} \cdot \min \{ 0.85, 0.81 \} \\
 &= 0.6 \cdot 0.81 = 0.486.
 \end{aligned}$$

Moreover, Example 5.5 shows that a *WLDFd-id* may not be a *WLDFd-sub*, because

$$\begin{aligned}
 0.35 &= \tau \xi(\nu) \not\geq \min \{ \tau(\nu * \eta), \tau(\eta) \} \cdot \min \{ \xi(\nu * \eta), \xi(\eta) \} \\
 &= \min \{ \tau(\nu), \tau(\eta) \} \cdot \min \{ \xi(\tau), \xi(\eta) \} \\
 &= \min \{ 0.5, 0.3 \} \cdot \min \{ 0.7, 0.4 \} \\
 &= 0.3 \cdot 0.4 = 0.12.
 \end{aligned}$$

Remark 5.3. To clarify the relationships among the introduced linear Diophantine fuzzy structures, we summarize the main implications established in this paper for d -algebras.

<i>Structure Type</i>	<i>Relation</i>
<i>LDFd-sub</i>	$\Rightarrow$ <i>WLDFd-sub</i> (Theorem 5.1)
<i>WLDFd-sub</i>	$\nRightarrow$ <i>LDFd-sub</i> (Example 5.2)
<i>WLDFd-sub</i>	$\Leftrightarrow$ <i>WLDFd-id</i>

6. Conclusion

In this study, we investigated the structural relationship between linear Diophantine fuzzy sets and d -algebraic structures. We introduced the notions of linear Diophantine fuzzy d -subalgebras and linear Diophantine fuzzy d -ideals, and analyzed their fundamental properties and interrelations. In particular, we established conditions under which these structures are compatible and examined their behavior under different algebraic settings.

Furthermore, we studied weak linear Diophantine fuzzy d -subalgebras and weak linear Diophantine fuzzy d -ideals, and provided examples illustrating the difference between the classical and weak formulations. We also derived conditions ensuring that weak linear Diophantine fuzzy level sets generate d -subalgebras, thereby clarifying the structural role of level-based constructions in this framework.

The results presented in this paper contribute to the development of fuzzy ideal theory in d -algebras by incorporating the flexibility of linear Diophantine fuzzy sets. The proposed framework may be extended in future work to other algebraic structures such as BCI/BCK-type algebras and related logical systems, particularly in the study of uncertainty within algebraic reasoning models.

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ON THE GEOMETRY OF LP-SASAKIAN MANIFOLDS ASSOCIATED WITH THE EXTENDED QUASI-CONFORMAL CURVATURE TENSOR

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Abstract. The geometry of LP-Sasakian manifolds and their interaction with various curvature tensors—particularly conformal, quasi-conformal, and concircular tensors—has been extensively studied in the literature. However, most existing works focus on classical curvature tensors or standard quasi-conformal curvature structures under general or well-known curvature restrictions. The present study introduces and investigates an extended quasi-conformal curvature tensor, which generalizes the traditional quasi-conformal curvature tensor by incorporating additional structural components adapted to LP-Sasakian manifolds.

In particular, we focus on LP-Sasakian manifolds with in the frame work of the extended quasi-conformal curvature tensor satisfying the vanishing conditions $K_e(U_1, Y_1)Z_1 = 0$ and $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$. Furthermore, the existence and properties of Ricci soliton structures are also examined.

Keywords: LP-Sasakian manifolds, extended quasi conformal curvature tensor, η -Einstein manifold, scalar curvature.

MSC (2020): 53C15, 53C25, 53C50.

1. INTRODUCTION

The extended quasi-conformal curvature tensor is a generalization of several well-known curvature tensors in differential geometry, designed to unify and extend notions like conformal, concircular, and quasi-conformal curvature. Its importance becomes clearer when you look at both its structural role and its applications in special manifolds, including LP-Sasakian manifolds. Its relevance becomes evident once we understand the limitations of the usual curvature measures.

In 1989, Matsumoto [8] first introduced the concept of LP-Sasakian manifolds. Subsequently, Mihai and Rosca [10] independently presented the same notion and established several significant results. Since then, LP-Sasakian manifolds have been extensively investigated by various

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researchers, including Bagewadi et al.,[3], De et al.,[4], Mihai et al.,[9], Shaikh et al.,[12, 14] and many others. Recently generalized Ricci-type solitons and almost Ricci-Bourguignon solitons on Lorentzian para-Kenmotsu geometry has been studied by [11, 15].

On the other hand, in [17] Yano and Sawaki introduced and investigated a tensor field K on an $(2n + 1)$ -dimensional Riemannian manifold that generalizes both the conformal curvature tensor and the concircular curvature tensor as particular instances. This tensor field K is called the quasi-conformal curvature tensor and is defined as follows:

$$\begin{aligned} K(U_1, Y_1)Z_1 &= aR(U_1, Y_1)Z_1 \\ &+ b[S(Y_1, Z_1)U_1 - S(U_1, Z_1)Y_1 + g(Y_1, Z_1)QU_1 - g(U_1, Z_1)QY_1] \\ &- \frac{r}{2n+1} \left[\frac{a}{2n} + 2b \right] \{g(Y_1, Z_1)U_1 - g(U_1, Z_1)Y_1\}, \end{aligned} \quad (1.1)$$

where each of U_1, Y_1, Z_1 belongs to the tangent space $T_p M$, while a and b are constants and r denotes the scalar curvature. In particular, when $a = 1$ and $b = -\frac{1}{n-2}$, then the quasi conformal curvature tensor coincides with the conformal curvature tensor.

Subsequently, De et al. [5] introduced and formulated an extended version of the quasi-conformal curvature tensor, referred to as the extended quasi-conformal curvature tensor, which is defined by:

$$\begin{aligned} K_e(U_1, Y_1)Z_1 &= K(U_1, Y_1)Z_1 - \eta(U_1)K(\xi, Y_1)Z_1 \\ &- \eta(Y_1)K(U_1, \xi)Z_1 - \eta(Z_1)K(U_1, Y_1)\xi. \end{aligned} \quad (1.2)$$

Recently, authors Al Hussein and Habeeb [1, 2] investigated and studied the geometric properties and applications of Quasi Sasakian manifolds endowed with generalized Sasakian space form structures in the framework of the pseudo-quasi conformal curvature tensor.

Motivated by the above, the present work investigates certain aspects of LP-Sasakian manifold admitting specific geometric properties associated with the extended quasi-conformal curvature tensor. The organization of the paper is as follows. After presenting the necessary preliminaries, Section 3 examines LP-Sasakian manifolds for which the extended quasi-conformal curvature tensor vanishes, that is, $K_e(U_1, Y_1)Z_1 = 0$. In Section 4, we analyze LP-Sasakian manifolds satisfying the condition $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$. The final section summarizes and concludes the findings established throughout the paper.

2. PRELIMINARIES

An $(2n + 1)$ -dimensional differentiable manifold M is said to be an LP-Sasakian manifold [8] if it possesses a $(1, 1)$ -tensor field ϕ , a unit time like vector field ξ , a 1-form η and a Lorentzian metric g that fulfill the following conditions:

$$\eta(\xi) = -1, \phi^2 U_1 = U_1 + \eta(U_1)\xi, \quad (2.3)$$

$$g(\phi U_1, \phi Y_1) = g(U_1, Y_1) + \eta(U_1)\eta(Y_1), \quad (2.4)$$

$$g(U_1, \xi) = \eta(U_1), \quad \nabla_{U_1} \xi = \phi U_1 \quad (2.5)$$

$$(\nabla_{U_1} \phi)(Y_1) = g(U_1, Y_1)\xi + \eta(Y_1)U_1 + 2\eta(U_1)\eta(Y_1)\xi, \quad (2.6)$$

for all vectors $U_1, Y_1 \in T_p M$, where ∇ represents the covariant derivative associated with the Lorentzian metric g .

It is readily observed that the following relations are satisfied in an LP-Sasakian manifold [8]:

$$\phi\xi = 0, \quad \eta(\phi U_1) = 0, \quad (2.7)$$

$$\text{rank}\phi = 2n. \quad (2.8)$$

As before, if we assume

$$\Omega(U_1, Y_1) = g(U_1, \phi Y_1) \text{ and } \nabla_{U_1}\xi = \phi U_1, \quad (2.9)$$

for arbitrary vector fields U_1, Y_1 , then the tensor $\Omega(U_1, Y_1)$ defines a symmetric $(0, 2)$ -type tensor field [8].

Moreover, since the vector field η is closed in an $(2n + 1)$ -dimensional LP-Sasakian manifold, it follows that [8, 10]:

$$(\nabla_{U_1}\eta)(Y_1) = \Omega(U_1, Y_1), \quad (2.10)$$

$$\Omega(U_1, \xi) = 0, \quad (2.11)$$

for any $U_1, Y_1 \in T_pM$.

Consider an $(2n + 1)$ -dimensional LP-Sasakian manifold M equipped with the structure (ϕ, ξ, η, g) . The following relations are satisfied [8]:

$$R(U_1, Y_1)Z_1 = g(Y_1, Z_1)U_1 - g(U_1, Z_1)Y_1, \quad (2.12)$$

$$R(\xi, Y_1)Z_1 = g(Y_1, Z_1)\xi - \eta(Z_1)Y_1, \quad (2.13)$$

$$R(U_1, Y_1)\xi = \eta(Y_1)U_1 - \eta(U_1)Y_1, \quad (2.14)$$

$$S(U_1, \xi) = 2n\eta(U_1), \quad (2.15)$$

$$S(\phi U_1, \phi Y_1) = S(U_1, Y_1) + 2n\eta(U_1)\eta(Y_1), \quad (2.16)$$

for any $U_1, Y_1, Z_1 \in T_pM$, where R is the Riemannian curvature tensor and S is the Ricci tensor of the manifold.

Additionally, for an $(2n + 1)$ -dimensional LP-Sasakian manifold with extended quasi-conformal curvature, the following holds:

$$\begin{aligned} K_e(U_1, Y_1)Z_1 = & \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] \{ g(Y_1, Z_1)U_1 - g(U_1, Z_1)Y_1 \\ & + g(U_1, Z_1)\eta(Y_1)\xi - g(Y_1, Z_1)\eta(U_1)\xi + 2\eta(U_1)\eta(Z_1)Y_1 \\ & - 2\eta(Y_1)\eta(Z_1)U_1 \} + b \{ S(Y_1, Z_1)U_1 - S(U_1, Z_1)Y_1 \\ & + \eta(Y_1)S(U_1, Z_1)\xi - \eta(U_1)S(Y_1, Z_1)\xi + g(Y_1, Z_1)QU_1 \\ & - g(U_1, Z_1)QY_1 + 2\eta(U_1)\eta(Z_1)QY_1 - 2\eta(Y_1)\eta(Z_1)QU_1 \\ & + 2n\eta(Y_1)g(U_1, Z_1)\xi - 2n\eta(U_1)g(Y_1, Z_1)\xi \\ & + 4n\eta(U_1)\eta(Z_1)Y_1 - 4n\eta(Y_1)\eta(Z_1)U_1 \}, \end{aligned} \quad (2.17)$$

$$\begin{aligned}
K_e(\xi, Y_1)Z_1 &= \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] \{ 2g(Y_1, Z_1)\xi - 3\eta(Z_1)Y_1 \\
&\quad - \eta(Y_1)\eta(Z_1)\xi \} + b[2S(Y_1, Z_1)\xi - 6n\eta(Z_1)Y_1 \\
&\quad + 4ng(Y_1, Z_1)\xi - 4n\eta(Y_1)\eta(Z_1)\xi - 3\eta(Z_1)QY_1] \\
&= -K_e(Y_1, \xi)Z_1,
\end{aligned} \tag{2.18}$$

$$\begin{aligned}
K_e(U_1, Y_1)\xi &= \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] [3\eta(Y_1)U_1 - 3\eta(U_1)Y_1] \\
&\quad + b[6n\eta(Y_1)U_1 - 6n\eta(U_1)Y_1 + 3\eta(Y_1)QU_1 - 3\eta(U_1)QY_1],
\end{aligned} \tag{2.19}$$

$$K_e(\xi, \xi)Z_1 = 0. \tag{2.20}$$

Definition 2.1. An $(2n+1)$ -dimensional LP-Sasakian manifold is called η -Einstein, if its Ricci tensor S of type $(0, 2)$ satisfies

$$S(U_1, Y_1) = \alpha g(U_1, Y_1) + \beta \eta(U_1)\eta(Y_1), \tag{2.21}$$

for all $U_1, Y_1 \in T_p M$, where α and β are smooth functions. Moreover, if $\beta = 0$, the manifold is referred to as an Einstein manifold.

Definition 2.2. In an $(2n+1)$ -dimensional LP-Sasakian manifold, if the Ricci tensor S satisfies

$$(\nabla_{W_1} S)(\phi U_1, \phi Y_1) = 0, \tag{2.22}$$

for all $U_1, Y_1, W_1 \in T_p M$, then S is called η -parallel.

A Ricci soliton extends the concept of an Einstein metric and is formulated on a Riemannian manifold (M, g) [6]. Specifically, a Ricci soliton is represented by a triple (g, ξ, λ) consisting of a Riemannian metric g , a unit timelike vector field ξ , and a real scalar λ , satisfying

$$L_V g + 2S + 2\lambda g = 0. \tag{2.23}$$

Here S denotes the Ricci tensor and L_V represents the Lie derivative taken along the solenoidal vector field V .

A Ricci soliton on a Riemannian manifold is classified as shrinking, steady, or expanding depending on whether the scalar λ is negative, zero, or positive, respectively.

3. LP-SASAKIAN MANIFOLDS ADMITTING VANISHING EXTENDED QUASI-CONFORMAL CURVATURE TENSOR

Consider an $(2n+1)$ -dimensional LP-Sasakian manifold in which the extended quasi-conformal curvature tensor is identically zero i.e.,

$$K_e(U_1, Y_1)Z_1 = 0, \tag{3.24}$$

for any choice of U_1, Y_1, Z_1 in the tangent space $T_p M$.

It follows directly from (3.24) that

$$K(U_1, Y_1)Z_1 - \eta(U_1)K(\xi, Y_1)Z_1 - \eta(Y_1)K(U_1, \xi)Z_1 - \eta(Z_1)K(U_1, Y_1)\xi = 0. \tag{3.25}$$

Using (3.25) and then taking the inner product with W_1 , we obtain

$$\begin{aligned} & \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] [g(Y_1, Z_1)g(U_1, W_1) - g(U_1, Z_1)g(Y_1, W_1) \\ & - \eta(U_1)\eta(W_1)g(Y_1, Z_1) + \eta(Y_1)\eta(W_1)g(U_1, Z_1) + 2\eta(U_1)\eta(Z_1)g(Y_1, W_1) \\ & - 2\eta(Y_1)\eta(Z_1)g(U_1, W_1)] + b[S(Y_1, Z_1)g(U_1, W_1) - S(U_1, Z_1)g(Y_1, W_1) \\ & + g(Y_1, Z_1)S(U_1, W_1) - g(U_1, Z_1)S(Y_1, W_1) - \eta(U_1)\eta(W_1)S(Y_1, Z_1) \\ & + \eta(Y_1)\eta(W_1)S(U_1, Z_1) + 4n\eta(U_1)\eta(Z_1)g(Y_1, W_1) - 4n\eta(Y_1)\eta(Z_1)g(U_1, W_1) \\ & - 2n\eta(U_1)\eta(W_1)g(Y_1, Z_1) + 2n\eta(Y_1)\eta(W_1)g(U_1, Z_1) \\ & + 2\eta(U_1)\eta(Z_1)S(Y_1, W_1) - 2\eta(Y_1)\eta(Z_1)S(U_1, W_1)] = 0. \end{aligned} \quad (3.26)$$

By substituting $U_1 = W_1 = e_i$, where e_i forms an orthonormal basis for the tangent space at each point on the manifold, and summing over i , $i = 1, 2, \dots, 2n+1$, we get

$$\begin{aligned} & \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] [2ng(Y_1, Z_1) - g(Y_1, Z_1) + g(Y_1, Z_1) \\ & + \eta(Y_1)\eta(Z_1) + 2\eta(Y_1)\eta(Z_1) - 4n\eta(Y_1)\eta(Z_1)] + b[2nS(Y_1, Z_1) \\ & - S(Y_1, Z_1) + rg(Y_1, Z_1) - S(Y_1, Z_1) + S(Y_1, Z_1) + 2n\eta(Y_1)\eta(Z_1) \\ & + 4n\eta(Y_1)\eta(Z_1) - 8n^2\eta(Y_1)\eta(Z_1) + 2ng(Y_1, Z_1) + 2n\eta(Y_1)\eta(Z_1) \\ & + 4n\eta(Y_1)\eta(Z_1) - 2r\eta(Y_1)\eta(Z_1)] = 0. \end{aligned} \quad (3.27)$$

The equation above can be transformed into

$$S(Y_1, Z_1) = l_1g(Y_1, Z_1) + m_1\eta(Y_1)\eta(Z_1), \quad (3.28)$$

where

$$\begin{aligned} l_1 &= \frac{a2n(r - 2n - 1) + b2n[r(2n - 1) - 2n(2n + 1)]}{b2n(2n + 1)(2n - 1)}, \\ m_1 &= \frac{a(4n - 3)[2n(2n + 1) - r] + b4n[r(-2n + 4) - (2n + 1)(4n^2 - 6n)]}{b2n(2n + 1)(2n - 1)}. \end{aligned}$$

Thus we can state the following:

Theorem 3.1. *If an LP-Sasakian manifold has a vanishing extended quasi-conformal curvature tensor, then it is an η -Einstein manifold.*

Applying $Y_1 = Z_1 = e_i$ once again in (3.28) results in

$$r = \frac{a(2n - 1) - b(2n - 3)}{a(2n + 1)(2n - 1) + b2n(2n + 1)(2n - 2)}. \quad (3.29)$$

This leads to the following:

Corollary 3.1. *In an LP-Sasakian manifold where the extended quasi-conformal curvature tensor vanishes, the scalar curvature remains constant.*

Similarly, by replacing Y_1 by ϕY_1 and Z_1 by ϕZ_1 in (3.28) and then using (2.5), we obtain

$$S(\phi Y_1, \phi Z_1) = \frac{a(r - 2n - 1) + b[r(2n - 1) - 2n(2n + 1)]}{b(2n + 1)(2n - 1)}g(\phi Y_1, \phi Z_1). \quad (3.30)$$

Applying covariant differentiation to (3.30) with respect to X_1 , we arrive at

$$(\nabla_{X_1} S)(\phi Y_1, \phi Z_1) = dr(X_1) \frac{a + b(2n - 1)}{b(2n + 1)(2n - 1)} g(\phi Y_1, \phi Z_1). \quad (3.31)$$

Given that an LP-Sasakian manifold with a vanishing extended quasi-conformal curvature tensor has constant scalar curvature, it follows from the above equation that

$$(\nabla_{X_1} S)(\phi Y_1, \phi Z_1) = 0. \quad (3.32)$$

This results in:

Corollary 3.2. *An LP-Sasakian manifold with vanishing extended quasi-conformal curvature tensor and constant scalar curvature possesses an η -parallel Ricci tensor.*

Taking an account of (3.28) in (2.23), we get

$$\begin{aligned} L_V g(Y_1, Z_1) + \frac{a2n(r - 2n - 1) + b2n[r(2n - 1) - 2n(2n + 1)]}{b2n(2n + 1)(2n - 1)} 2g(Y_1, Z_1) \\ + \frac{a(4n - 3)[2n(2n + 1) - r] + b4n[r(-2n + 4) - n(8n^2 - 8n - 6)]}{b2n(2n + 1)(2n - 1)} 2\eta(Y_1)\eta(Z_1) \\ + 2\lambda g(Y_1, Z_1) = 0. \end{aligned} \quad (3.33)$$

By setting $Y_1 = Z_1 = e_i$, the above equation becomes

$$\begin{aligned} \operatorname{div}(V) + \lambda b8n^2(2n + 1)(2n - 1) \\ + 2a(2n - 1)[r(2n + 3) - 6n(2n + 1)] \\ + 4bn[r(4n^2 + 2n - 8) - 3(2n + 1)(4n^2 - 4n)] = 0. \end{aligned} \quad (3.34)$$

Because V is a solenoidal vector field, it follows that

$$\lambda = - \left\{ \frac{a(2n - 1)[r(2n + 3) - 6n(2n + 1)] + b2n[r(4n^2 + 2n - 8) - 3(2n + 1)(4n^2 - 4n)]}{b4n^2(2n + 1)(2n - 1)} \right\} \quad (3.35)$$

Consequently, the following occurs:

Remark 3.1. *In an LP-Sasakian manifold with a vanishing extended quasi-conformal curvature tensor, the Ricci soliton is of shrinking type.*

4. LP-SASAKIAN MANIFOLDS ADMITTING EXTENDED QUASI-CONFORMAL CURVATURE TENSOR SATISFYING $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$

Let M be an $(2n + 1)$ -dimensional LP-Sasakian manifold for which the curvature condition $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$ holds. The preceding condition implies that

$$\begin{aligned} g(Y_1, K_e(U_1, V_1)W_1)\xi - \eta(K_e(U_1, V_1)W_1)Y_1 - g(Y_1, U_1)K_e(\xi, V_1)W_1 \\ + \eta(U_1)K_e(Y_1, V_1)W_1 - g(Y_1, V_1)K_e(U_1, \xi)W_1 + \eta(V_1)K_e(U_1, Y_1)W_1 \\ - g(Y_1, W_1)K_e(U_1, V_1)\xi + \eta(W_1)K_e(U_1, V_1)Y_1 = 0. \end{aligned} \quad (4.36)$$

In view of (4.36), we have

$$\begin{aligned} & -g(Y_1, K_e(U_1, V_1)W_1) - \eta(K_e(U_1, V_1)W_1)\eta(Y_1) - g(Y_1, U_1)\eta(K_e(\xi, V_1)W_1) \quad (4.37) \\ & + \eta(U_1)\eta(K_e(Y_1, V_1)W_1) + \eta(W_1)\eta(K_e(U_1, V_1)Y_1) + \eta(V_1)\eta(K_e(U_1, Y_1)W_1) \\ & - g(Y_1, W_1)\eta(K_e(U_1, V_1)\xi) - g(Y_1, V_1)\eta(K_e(U_1, \xi)W_1) = 0. \end{aligned}$$

Making use of (2.17)-(2.20) in (4.37), we derive

$$\begin{aligned} & \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] [g(V_1, W_1)g(Y_1, U_1) - g(U_1, W_1)g(Y_1, V_1) \quad (4.38) \\ & + g(V_1, W_1)\eta(U_1)\eta(Y_1) - g(U_1, W_1)\eta(V_1)\eta(Y_1) + 2g(Y_1, U_1)\eta(V_1)\eta(W_1) \\ & - 2g(V_1, Y_1)\eta(U_1)\eta(W_1)] + b[S(V_1, W_1)g(Y_1, U_1) - S(U_1, W_1)g(V_1, Y_1) \\ & + S(V_1, Y_1)g(U_1, W_1) - S(Y_1, U_1)g(V_1, W_1) + S(V_1, W_1)\eta(U_1)\eta(Y_1) \\ & - S(U_1, W_1)\eta(V_1)\eta(Y_1) + 2ng(V_1, W_1)\eta(U_1)\eta(Y_1) - 2ng(U_1, W_1)\eta(V_1)\eta(Y_1) \\ & + 8ng(Y_1, U_1)\eta(V_1)\eta(W_1) - 8ng(Y_1, V_1)\eta(U_1)\eta(W_1) + 4ng(V_1, W_1)g(Y_1, U_1) \\ & - 4ng(U_1, W_1)g(V_1, Y_1)] = 0. \end{aligned}$$

Putting $U_1 = W_1 = e_i$, where e_i is an orthonormal basis of the tangent space at each point of the manifold, and taking summation over i , $i = 1, 2, \dots, 2n + 1$, we obtain

$$\begin{aligned} & \left[a - \frac{r}{2n+1} \left(\frac{a}{2n} + 2b \right) \right] [g(Y_1, V_1) - 2ng(Y_1, V_1) + \eta(Y_1)\eta(V_1) \quad (4.39) \\ & - 2n\eta(Y_1)\eta(V_1) + 2\eta(Y_1)\eta(V_1) + 2g(Y_1, V_1)] + b[S(Y_1, V_1) \\ & - rg(Y_1, V_1) + 2nS(V_1, Y_1) - S(Y_1, V_1) + 2n\eta(Y_1)\eta(V_1) \\ & - r\eta(Y_1)\eta(V_1) + 2n\eta(Y_1)\eta(V_1) - 4n^2\eta(Y_1)\eta(V_1) + 8ng(Y_1, V_1) \\ & + 8n\eta(Y_1)\eta(V_1) + 4ng(Y_1, V_1) - 8n^2g(Y_1, V_1)] = 0. \end{aligned}$$

The preceding equation can be simplified as

$$S(Y_1, V_1) = l_1g(Y_1, V_1) + m_1\eta(Y_1)\eta(V_1), \quad (4.40)$$

where

$$\begin{aligned} l_2 &= \frac{a(2n-3)[2n(2n+1)-r] + b2n[4n(2n+1)(2n-3) - r(2n-7)]}{b4n^2(2n+1)}, \\ m_2 &= \frac{a(2n-3)[2n(2n+1)-r] + b2n[2n(2n-6) - r(2n-7)]}{b4n^2(2n+1)}. \end{aligned}$$

Therefore, we can conclude that:

Theorem 4.1. *An $(2n + 1)$ -dimensional LP-Sasakian manifold that admits an extended quasi-conformal curvature tensor and satisfies $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$ is an η -Einstein manifold.*

Plugging $Y_1 = V_1 = e_i$ again into (4.40) gives

$$r = \frac{a2n(2n+1)(2n-1) + b8n^2[2n(2n+1)(2n-3) - (2n-)]}{a(2n-1)(2n-3) + b2n[2n(2n+1) + (2n-1)(2n-7)]}. \quad (4.41)$$

As a result, we can assert that:

Corollary 4.1. *If an LP-Sasakian manifold has an extended quasi-conformal curvature tensor satisfies $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$, then its scalar curvature is constant.*

Now replacing Y_1 by ϕY_1 and V_1 by ϕV_1 in (4.40), gives

$$S(\phi Y_1, \phi V_1) = Ag(\phi Y_1, \phi V_1). \quad (4.42)$$

where,

$$A = \frac{a(2n-3)[2n(2n+1)-r] + b2n[4n^2(2n-5) - r(2n-7)]}{b4n^2(2n+1)}.$$

Computing the covariant derivative of (4.42) along X_1 , we find

$$(\nabla_{X_1} S)(\phi Y_1, \phi V_1) = \frac{[-a(2n-3) - b2n(2n-7)]dr(X_1)}{b4n^2(2n+1)}g(\phi Y_1, \phi V_1). \quad (4.43)$$

Since an LP-Sasakian manifold endowed with an extended quasi-conformal curvature tensor and satisfying $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$ has constant scalar curvature, it follows that

$$(\nabla_{X_1} S)(\phi Y_1, \phi V_1) = 0. \quad (4.44)$$

From this, it follows that:

Corollary 4.2. *An LP-Sasakian manifold whose extended quasi-conformal curvature tensor satisfies $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$ necessarily has a Ricci tensor that is η -parallel.*

Upon substituting (4.40) into (2.23), we arrive at

$$\begin{aligned} & L_V g(V_1, Y_1) \\ & + \frac{a(2n-3)[2n(2n+1)-r] + b2n[4n(2n+1)(2n-3) - r(2n-7)]}{b4n^2(2n+1)} 2g(V_1, Y_1) \\ & + \frac{a(2n-3)[2n(2n+1)-r] + b2n[2n(2n-6) - r(2n-7)]}{b4n^2(2n+1)} 2\eta(V_1)\eta(Y_1) \\ & + 2\lambda g(V_1, Y_1) = 0. \end{aligned} \quad (4.45)$$

If we substitute $V_1 = Y_1 = e_i$ into (4.45), we obtain

$$\begin{aligned} & \text{div}(V) + \lambda b16n^3(2n+1) + a2(2n+1)(2n-3)[2n(2n+1)-r] \\ & + b4n[2n(8n^3 - 16n^2 - 5n - 3) - r(2n+1)(2n-7)] = 0. \end{aligned} \quad (4.46)$$

Owing to the fact that V is solenoidal, we get

$$\lambda = - \left\{ \frac{a(2n+1)(2n-3)[2n(2n+1)-r] + b2n[2n(8n^3 - 16n^2 - 5n - 3) - r(2n+1)(2n-7)]}{b8n^3(2n+1)} \right\} \quad (4.47)$$

This leads us to conclude that:

Remark 4.1. *On an LP-Sasakian manifold whose extended quasi-conformal curvature tensor satisfies $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$, the Ricci soliton is shrinking.*

5. Conclusion:

In general, the extended form of the quasi-conformal curvature tensor aligns with the extended quasi-conformal curvature tensor defined by De and Sarkar [5]. In this work, we investigate an n -dimensional LP-Sasakian manifold equipped with the extended quasi-conformal curvature tensor. Imposing the curvature conditions $K_e(U_1, Y_1)Z_1 = 0$ and $(R(\xi, Y_1) \cdot K_e)(U_1, V_1)W_1 = 0$, we derive the following geometric results:

- The LP-Sasakian manifold to take on a highly ordered, Einstein-type configuration.
- The manifold has uniform curvature distribution at every point.
- The Ricci tensor is stable along the contact distribution.
- The geometry of a manifold does not remain fixed or increase—it diminishes over time under Ricci flow.

Future research directions:

- Consider non-vanishing or bounded cases, this would help determine whether the Einstein-type configuration obtained is stable or if small perturbations lead to entirely different geometric behavior.
- Examining analogous results in Sasakian, Kenmotsu, or trans-Sasakian manifolds would show whether the conclusions depend heavily on the LP-structure or reflect a broader phenomenon in contact geometry.

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CONFORMAL SUBMERSIONS FROM RICCI-YAMABE SOLITONS

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Abstract. The concept of Ricci-Yamabe (RY) soliton is such an approach that unifies the geometric properties of Ricci solitons and Yamabe solitons. Because of the wealth of geometric properties and applications in differential geometry and mathematical physics, the research on RY solitons has gained increasing interest. This paper is devoted to conformal Riemannian submersions whose total spaces possess Ricci-Yamabe (RY) solitons and gives some examples for clarification.

Keywords: Ricci-Yamabe soliton; Conformal Riemannian submersions; Scalar curvatures; Totally geodesic

MSC (2020): Primary 53B20; Secondary 53C12, 53C25, 53C43.

1. INTRODUCTION

Physically occurring waves which maintain their shape after interaction with other waves are known as solitons. Solitons are viewed as self-similar solutions of some wave equations in mathematics. From the idea of solitons, Richard S. Hamilton [4] defined Ricci solitons as self-similar solutions of nonlinear partial differential equations in the context of geometric flows. Recently, solitons have gained much significance and attention in differential geometry due to its important link with curvature flow and geometrical analysis on manifolds. More specifically, Ricci solitons naturally occur while studying Ricci flows, invented by Richard Hamilton, in terms of their singularity formation and asymptotic behavior. After the advent of the Ricci flow by Hamilton, several geometers started investigating geometric solitons. Moreover, the spectacular application of Ricci flow by Grigori Perelman [12] to prove the famous Poincaré Conjecture made the study of solitons a very important field in modern geometry. Thus, soliton structures become important in the study of manifolds admitting different geometric structures.

The topic of geometric flows became one of the most important in contemporary differential geometry after the introduction of Ricci flow and Yamabe flow by Hamilton in the 1980s. The self-similar solutions to such flows generate Ricci solitons and Yamabe solitons, respectively. Thus, it is reasonable to consider RY solitons, which are the fixed points of the RY flow.

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A Riemannian manifold (M, g) equipped with a vector field ζ is said to admit a RY soliton (RY soliton) if

$$\frac{1}{2}\mathcal{L}_\zeta g + \alpha \text{Ric} + \left(\lambda - \frac{\beta}{2}\kappa\right)g = 0, \quad (1.1)$$

where Ric is the Ricci tensor, κ is the scalar curvature, λ, α, β are constants, and $\mathcal{L}_\zeta$ denotes the Lie derivative along ζ . The soliton is called expanding, steady, or shrinking according to whether λ is positive, zero, or negative.

Note that

- (1) if ζ is gradient of some smooth function f on M , then the above notion is called gradient RY soliton.
- (2) if we replace λ in equation (1.1) by a smooth function, then we say that (M, g) is an almost RY soliton.

On the other hand, conformal Riemannian submersions (CRSs) provide a rich geometric framework that generalizes Riemannian submersions, in which the differential restricted to the horizontal distribution is conformal with a positive dilation function rather than an isometry (see [2]).

A surjective smooth map between two Riemannian manifolds

$$F : (M, g) \rightarrow (N, h)$$

is said to be Riemannian submersion if it has maximal rank at every point $p \in M$ and the differential F_{*p} maps the horizontal space $(\text{Ker } F_*)^\perp$ isometrically onto $T_{F(p)}N$. The fibers of F over $q \in N$ is defined by $F^{-1}(q)$. Moreover, the submersion F is called a conformal (horizontally conformal) submersion (CS) if the restriction of F_* to the horizontal distribution $(\text{Ker } F_*)^\perp$ is conformal, that is, there exists a positive smooth function σ (the dilation) such that [1]

$$h(F_*X, F_*Y) = \sigma^2 g(X, Y), \quad (1.2)$$

for all $X, Y \in \Gamma((\text{Ker } F_*)^\perp)$.

Note that F is called horizontally weakly conformal map at $p \in M$ if either

- (1) $F_{*p} = 0$, or
- (2) F_{*p} maps the horizontal space conformally onto $T_{F(p)}N$.

A horizontally weakly conformal map $F : M \rightarrow N$ is said to be horizontally homothetic if the gradient of its dilation σ is vertical at regular points. Equivalently, σ is constant along horizontal directions.

In recent times, an increased amount of research is being conducted on the relationship between the theory of solitons and submersions. The work of Meriç and Kılıç [9] involved studying Riemannian submersions with total manifolds possessing Ricci solitons, and Meena and Yadav [6] further expanded the study to include conformal Riemannian submersions. In [8], Meriç discussed Riemannian submersions whose total manifolds are almost Yamabe solitons to Riemannian manifolds. But still, there is no study about the CSs whose total manifolds admit RY solitons.

This paper aims to fill this gap. We organize our work as follows. Section 2 recalls essential preliminaries on the study. The main results have been discussed in Section 3, where we have

considered CSs from a RY soliton and established a condition for the fibers/base manifold to be (almost) RY solitons/Einstein manifolds. Section 4 deals with the examples of these mappings, and conclusions are drawn in Section 5.

2. PRELIMINARIES

In this section, we give a brief review of basic facts; cf. [15].

Let

$$F : (M^r, g) \rightarrow (N^r, h)$$

be a Riemannian submersion. The vertical distribution $\mathcal{V} = \text{Ker } F_*$ and the horizontal distribution $\mathcal{H} = (\text{Ker } F_*)^\perp$ are orthogonal complements. For any vector field E on M , we write $E = \mathcal{V}E + \mathcal{H}E$. The O'Neill tensors A and T are defined by [11]

$$A_E F = \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H}F, \quad (2.3)$$

$$T_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H}F. \quad (2.4)$$

These tensors satisfy $A_E = A_{\mathcal{H}E}$, $T_E = T_{\mathcal{V}E}$, and are skew-symmetric. Their properties include

$$\nabla_U V = T_U V + \mathcal{V} \nabla_U V, \quad \nabla_X U = A_X U + \mathcal{V} \nabla_X U, \quad \nabla_X Y = A_X Y + \mathcal{H} \nabla_X Y, \quad (2.5)$$

for $U, V \in \Gamma(\mathcal{V})$ and $X, Y \in \Gamma(\mathcal{H})$.

The submersion F is known to be of totally umbilical fibers if; cf. [19], [20]

$$T_U V = g(U, V)H \text{ or } T_U X = -g(H, X)U, \quad (2.6)$$

where H denotes the mean curvature vector field of the fibers.

For the map F , the second fundamental form is expressed as [3]

$$(\nabla F_*)(X, Y) = D_{F_* X} F_* Y - F_*(\nabla_X Y), \quad (2.7)$$

where D is the Levi-Civita connection on N .

Let

$$F : (M^r, g) \rightarrow (N^s, h)$$

be a CS with dilation σ . The following fundamental relation holds for basic vector fields [3]:

$$\begin{aligned} F_*(\mathcal{H} \nabla_X Y) &= \nabla_{F_* X}^N F_* Y + \frac{\sigma^2}{2} \{X(\sigma^{-2}) F_* Y \\ &\quad + Y(\sigma^{-2}) F_* X - g(X, Y) F_*(\text{grad}_{\mathcal{H}} \sigma^{-2})\}. \end{aligned} \quad (2.8)$$

The tension field $\mathfrak{T}(F)$ of F is defined as the trace of the second fundamental form. For a CS, we have [7]

$$\mathfrak{T}(F) = (s - 2) \frac{\sigma^2}{2} F_*(\text{grad}_{\mathcal{H}} \sigma^{-2}) - (r - s) F_*(H), \quad (2.9)$$

where H is the mean curvature vector of the fibers. The map F is harmonic if and only if $\mathfrak{T}(F) = 0$.

Now, we recall the components of the Ricci tensor on the total manifold of a CS. These formulas are essential for studying RY solitons.

Let $\{U_i\}_{s+1 \leq i \leq r}$ be an orthonormal basis of $\mathcal{V}$ and $\{X_j\}_{1 \leq j \leq s}$ an orthonormal basis of $\mathcal{H}$. Denote by $\text{Ric}^{\mathcal{V}}$ the Ricci tensor of the fibers and by Ric^N the Ricci tensor of the base

manifold N . If horizontal distribution $(\ker F_*)^\perp$ is integrable, then the horizontal space is totally umbilical, that is,

$$A_X Y = g(X, Y) H^{hor},$$

where mean curvature vector H^{hor} is given by [3]

$$H^{hor} = -\frac{\sigma^2}{2} (\nabla_V \sigma^{-2}). \quad (2.10)$$

Proposition 2.1. [6] *For a CS*

$$F : (M, g) \rightarrow (N, h)$$

with dilation σ , the Ricci tensor components on M are as follows:

$$\begin{aligned} \text{Ric}_{UV} &= \text{Ric}_{UV}^\mathcal{V} - (r-s)g(T_U V, H) + \sum_{j=1}^s g((\nabla_U A)_{X_j} X_j, V) \\ &\quad + \sum_{j=1}^s g(A_{X_j} U, A_{X_j} V) - \sum_{j=1}^s g((\nabla_{X_j} T)_U X_j, V) \\ &\quad - \frac{s\sigma^4}{2} g(U, \nabla_V \sigma^{-2}) g(V, \nabla_V \sigma^{-2}), \end{aligned} \quad (2.11)$$

$$\begin{aligned} \text{Ric}_{UX} &= (r-s)g(\nabla_U H, X) - \sum_{i=s+1}^r g((\nabla_{U_i} T)_U U_i, X) \\ &\quad + \sum_{j=1}^s g((\nabla_X A)_{X_j} X_j, U) - \sum_{j=1}^s g((\nabla_{X_j} A)_X X_j, U) \\ &\quad - \sum_{j=1}^s g(T_U X_j, \mathcal{V}[X, X_j]), \end{aligned} \quad (2.12)$$

$$\begin{aligned} \text{Ric}_{XY} &= \sum_{i=s+1}^r g((\nabla_{U_i} A)_X Y, U_i) + \sum_{i=s+1}^r g(A_X U_i, A_Y U_i) \\ &\quad - \sum_{i=s+1}^r g((\nabla_X T)_{U_i} Y, U_i) - \sum_{i=s+1}^r g(T_{U_i} X, T_{U_i} Y) \\ &\quad + \sigma^2 g(A_X Y, \nabla_V \sigma^{-2}) + \sigma^{-2} \text{Ric}_{X^\mathcal{H} Y^\mathcal{H}}^N \\ &\quad + \frac{3}{4} \sum_{j=1}^s g(\mathcal{V}[X, X_j], \mathcal{V}[X_j, Y]) - \frac{(s-2)}{2} \sigma^2 g(\nabla_X \text{grad } \sigma^{-2}, Y) \\ &\quad - \frac{\sigma^2}{2} g(X, Y) \left\{ \Delta^\mathcal{V} \sigma^{-2} - s \left(H^{hor} \sigma^{-2} \right) \right\} + \frac{s\sigma^4}{4} g(X, Y) \|\text{grad } \sigma^{-2}\|^2 \\ &\quad + \frac{\sigma^4}{4} (s-2) (X \sigma^{-2}) (Y \sigma^{-2}). \end{aligned} \quad (2.13)$$

Immediate consequences of Proposition 2.1 are as follows:

Corollary 2.1. [6] *Let*

$$F : (M^r, g) \rightarrow (N^s, h)$$

be a homothetic CS with dilation σ such that the fibers of F are totally geodesic and $(\ker F_*)^\perp$ is integrable. Then

$$\begin{aligned} \operatorname{Ric}_{XY} &= g(X, Y) \delta(H^{hor}) + \sigma^{-2} \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N \\ &\quad - \frac{1}{4} \sigma^4 g(X, Y) |\nabla_{\mathcal{V}} \sigma^{-2}|^2 + \frac{s \sigma^2}{2} g(X, Y) (H^{hor} \sigma^{-2}), \end{aligned}$$

here δ is used for divergence.

Corollary 2.2. [6] Suppose that

$$F : (M^r, g) \rightarrow (N^s, h)$$

defines a horizontally CS which is totally geodesic and admits dilation σ . Then

$$\operatorname{Ric}_{UV} = \operatorname{Ric}_{UV}^{\mathcal{V}}, \quad (2.14)$$

$$\operatorname{Ric}_{XY} = \sigma^{-2} \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N, \quad (2.15)$$

$$\operatorname{Ric}_{UX} = 0, \quad (2.16)$$

where X and Y are the horizontal lifts of $X^{\mathcal{H}}$ and $Y^{\mathcal{H}}$, respectively.

Theorem 2.1. [6] Consider a CS

$$F : (M, g) \rightarrow (N, h)$$

which is totally geodesic and admits dilation σ . Then the scalar curvature κ of M is given by

$$\kappa = \kappa^{\mathcal{V}} + \sigma^{-2} \kappa^N, \quad (2.17)$$

where $\kappa^{\mathcal{V}}$ and κ^N denote the scalar curvatures of the fibers and the base manifold, respectively.

Theorem 2.2. [6] Let

$$F : (M, g) \rightarrow (N, h)$$

be a horizontally CS. Then F is totally geodesic if and only if

- (1) fibers of F are totally geodesic,
- (2) $(\ker F_*)^\perp$ is totally geodesic,
- (3) F is homothetic.

3. CONFORMAL SUBMERSIONS FROM RY SOLITONS

Throughout this section, we assume that (M, g) admits a RY soliton with potential vector field ζ , constants λ, α, β , and that F is a CS.

3.1. Potential Field Vertical. We first consider the case where the potential field ζ is vertical, i.e., $\zeta \in \Gamma(\operatorname{Ker} F_*)$.

Theorem 3.1. Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ vertical, and let

$$F : (M, g) \rightarrow (N, h)$$

be a totally geodesic CS. If N is Ricci-flat, then each fiber admits a RY soliton. Moreover, if ζ is unitary then the corresponding soliton is classified as follows:

- (1) shrinking, if $\beta \kappa^{\mathcal{V}} > 0$,

- (2) *steady*, if $\beta\kappa^\mathcal{V} = 0$,
- (3) *expanding*, if $\beta\kappa^\mathcal{V} < 0$.

In particular, if $\beta > 0$, then the nature of the soliton depends on the sign of the vertical scalar curvature $\kappa^\mathcal{V}$, whereas for $\beta < 0$ the classification is reversed.

Proof. Since ζ is vertical, for any $U, V \in \Gamma(\text{Ker } F_*)$, the RY soliton equation gives

$$\frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) = 0.$$

Using the expression for Ric_{UV} from Proposition 2.1 and the total geodesicity assumption, many terms vanish. Then by Theorem 2.1, we obtain

$$0 = \frac{1}{2}\mathcal{L}_\zeta g + \alpha \text{Ric}^\mathcal{V} + \left(\lambda - \frac{\beta}{2}\kappa^\mathcal{V}\right)g - \frac{\beta}{2}\sigma^{-2}\kappa^N g. \quad (3.18)$$

As N is Ricci-flat, we have $\kappa^N = 0$, and thus we get the RY soliton equation on the fiber. In addition, when ζ is assumed to be a unit vector field, we have $\frac{1}{2}\mathcal{L}_\zeta g(U, V) = 0$ for vertical vector fields. From (3.18) and the assumption of Ricci-flatness of N , we obtain

$$\lambda - \frac{\beta}{2}\kappa^\mathcal{V} = 0,$$

that is,

$$\lambda = \frac{\beta}{2}\kappa^\mathcal{V}.$$

This completes the classification. □

Theorem 3.2. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ vertical, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a CS such that fibers of F are totally umbilical and $(\text{Ker } F_)^\perp$ is totally geodesic. If N is Ricci-flat, then each fiber admits an almost RY soliton.*

Proof. Under the hypotheses that fibers of F are totally umbilical $T_U V = g(U, V)H$ and $(\text{Ker } F_*)^\perp$ is totally geodesic, we obtain

$$\text{Ric}_{UV} = \text{Ric}_{UV}^\mathcal{V} - (r - s)\|H\|^2 g(U, V) + \delta(H)g(U, V). \quad (3.19)$$

From (1.1) and (3.19), we get

$$\begin{aligned} 0 &= \frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \left(\text{Ric}_{UV}^\mathcal{V} - (r - s)\|H\|^2 g(U, V) + \delta(H)g(U, V) \right) \\ &\quad + \left(\lambda - \frac{\beta}{2}\kappa \right) g(U, V) \\ &= \frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV}^\mathcal{V} + \left(\lambda - \frac{\beta}{2}\kappa^\mathcal{V} \right) g(U, V) \\ &\quad + \left(-\alpha(r - s)\|H\|^2 + \alpha\delta(H) - \frac{\beta}{2\sigma^2}\kappa^N \right) g(U, V). \end{aligned}$$

Since N is Ricci-flat, $\kappa^N = 0$. Then the above equation reduces to

$$0 = \frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV}^\mathcal{V} + \left(\mu - \frac{\beta}{2}\kappa^\mathcal{V} \right) g(U, V),$$

where $\mu = \alpha(- (r - s)\|H\|^2 + \delta(H)) + \lambda$ is a smooth function on M . Thus, each fiber of F is an almost RY soliton. $\square$

Theorem 3.3. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ vertical, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a totally geodesic CS. If M has constant scalar curvature κ , then N is Einstein.

Proof. Since ζ is vertical, for any $X, Y \in \Gamma(\text{Ker } F_*)^\perp$, equation (1.1) gives

$$\begin{aligned} 0 &= \frac{1}{2} \{g(\nabla_X \zeta, Y) + g(\nabla_Y \zeta, X)\} + \alpha \text{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y) \\ &= -\frac{1}{2} \{g(\nabla_X Y, \zeta) + g(\nabla_Y X, \zeta)\} + \alpha \text{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y) \\ &= -\frac{1}{2} g(A_X Y + A_Y X, \zeta) + \alpha \text{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y). \end{aligned}$$

Using the assumption that the submersion is totally geodesic and Corollary 2.2, we obtain

$$\alpha \sigma^{-2} \text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N + \left(\lambda - \frac{\beta}{2} \kappa\right) \sigma^{-2} h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = 0,$$

which can be re-written as

$$\text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N = -\frac{1}{\alpha} \left(\lambda - \frac{\beta}{2} \kappa\right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}),$$

showing that N is Einstein. $\square$

Theorem 3.4. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ vertical, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a homothetic CS such that fibers of F are totally geodesic and $(\text{Ker } F_)^\perp$ is integrable. Then N is Einstein.*

Proof. From Corollary 2.1 and (3.1), we derive

$$\begin{aligned} 0 &= -\frac{1}{2} g(A_X Y + A_Y X, \zeta) + \alpha \left\{ g(X, Y) \delta(H^{hor}) + \sigma^{-2} \text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N \right. \\ &\quad \left. - \frac{1}{4} \sigma^4 g(X, Y) \|\nabla_{\mathcal{V}} \sigma^{-2}\|^2 + \frac{s \sigma^2}{2} g(X, Y) (H^{hor} \sigma^{-2}) \right\} \\ &\quad + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y). \end{aligned}$$

In the case of homothetic submersions, the tensor A is equal to zero (see Theorem 2.2), so the first term is eliminated. This gives us

$$\text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N = \mu h(X^{\mathcal{H}}, Y^{\mathcal{H}}),$$

where

$$\mu = -\sigma^2 \delta(H^{hor}) + \frac{\sigma^4}{4} \|\nabla_{\mathcal{V}} \sigma^{-2}\|^2 - \frac{s \sigma^2}{2} (H^{hor} \sigma^{-2}) - \frac{\sigma^2}{\alpha} \left(\lambda - \frac{\beta}{2} \kappa\right)$$

is a smooth function on M . Hence N is Einstein. $\square$

3.2. Potential Field Horizontal. Now we move on to the situation when the potential field ζ is horizontal, that is, $\zeta \in \Gamma((\text{Ker } F_*)^\perp)$.

Theorem 3.5. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a totally geodesic CS. Then each fiber of F is Einstein.

Proof. Since ζ is horizontal, for any $U, V \in \Gamma(\text{Ker } F_*)$, the RY soliton equation yields

$$\frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) = 0.$$

Since ζ is horizontal vector field and the submersion is totally geodesic, we have $\nabla_U \zeta = -A_\zeta U$ (vertical). Now the Lie derivative term becomes

$$-g(A_\zeta U, V) - g(A_\zeta V, U) = 0$$

due to the skew-symmetry of A . Using Corollary 2.2, $\text{Ric}_{UV} = \text{Ric}_{UV}^\mathcal{V}$. Thus

$$\alpha \text{Ric}_{UV}^\mathcal{V} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) = 0,$$

which gives

$$\text{Ric}_{UV}^\mathcal{V} = \mu g(U, V),$$

so each fiber is Einstein with $\mu = -\frac{1}{\alpha}\left(\lambda - \frac{\beta}{2}\kappa\right)$. □

Theorem 3.6. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a CS such that fibers of F are totally umbilical and $(\text{Ker } F_)^\perp$ is totally geodesic. Then each fiber of F is Einstein.*

Proof. From (1.1) and Proposition 2.1, we find

$$\begin{aligned} & \alpha \left\{ \text{Ric}_{UV}^\mathcal{V} - (r - s)\|H\|^2 g(U, V) + \delta(H)g(U, V) \right\} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) \\ &= -\frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} = g(T_U V, \zeta). \end{aligned}$$

Given that the fibers of F are totally umbilical, we have $T_U V = g(U, V)H$. Thus, the RHS simplifies to $g(H, \zeta)g(U, V)$. Moreover, since $(\text{Ker } F_*)^\perp$ is totally geodesic, the mixed curvature terms vanish and the scalar curvature splits. Hence

$$\begin{aligned} \alpha \text{Ric}_{UV}^\mathcal{V} &= [g(H, \zeta) + \alpha(r - s)\|H\|^2 - \alpha\delta(H) \\ &\quad - \left(\lambda - \frac{\beta}{2}\kappa\right)]g(U, V). \end{aligned}$$

Thus $\text{Ric}_{UV}^\mathcal{V} = \tilde{\mu}g(U, V)$, where $\tilde{\mu}$ is a smooth function, proving each fiber is Einstein. □

Theorem 3.7. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a totally geodesic CS. Then N admits a RY soliton with potential field $\zeta^\mathcal{H} = F_\zeta$.*

Proof. In the case of X, Y, ζ being horizontal, from the property of F as a totally geodesic CS, we obtain $F_*(\nabla_X Y) = D_{F_*X} F_*Y$ and $F_*(\nabla_X \zeta) = D_{F_*X} F_*\zeta$. Then from (1.1),

$$\begin{aligned} 0 &= \frac{1}{2} \{g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta)\} + \alpha \operatorname{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y) \\ &= \frac{1}{2} \sigma^{-2} \{h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}})\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N \\ &\quad + \sigma^{-2} \left(\lambda - \frac{\beta}{2} \kappa\right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}). \end{aligned}$$

The multiplication by σ^2 will give us the RY soliton on N :

$$\frac{1}{2} \{h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}})\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N + \left(\lambda - \frac{\beta}{2} \kappa\right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = 0.$$

Thus $(N, h, \zeta^{\mathcal{H}}, \lambda, \alpha, \beta)$ is a RY soliton (with the same constants λ, α, β but note the scalar curvature term uses κ from M , not κ^N). $\square$

Theorem 3.8. *Let $(M, g, \zeta, \lambda, \alpha, \beta)$ be a RY soliton with ζ horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

be a homothetic CS such that fibers of F are totally geodesic and $(\operatorname{Ker} F_)^\perp$ is integrable. Then N is an almost RY soliton.*

Proof. Using (1.1) and the homothetic condition together with Corollary 2.1, and (2.8), we obtain after simplification

$$\begin{aligned} 0 &= \frac{1}{2} \{h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}})\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N \\ &\quad + \left\{ \delta(H^{hor}) - \frac{1}{4} \sigma^4 \|\nabla_V \sigma^{-2}\|^2 + \frac{s}{2} \sigma^2 (H^{hor} \sigma^{-2}) + \lambda - \frac{\beta}{2} \kappa \right\} h(X^{\mathcal{H}}, Y^{\mathcal{H}}). \end{aligned}$$

This shows that N satisfies

$$\frac{1}{2} \left\{ h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}}) \right\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N + \left(\mu - \frac{\beta}{2} \kappa \right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = 0,$$

where $\mu = \delta(H^{hor}) - \frac{1}{4} \sigma^4 \|\nabla_V \sigma^{-2}\|^2 + \frac{s}{2} \sigma^2 (H^{hor} \sigma^{-2}) + \lambda$ is a smooth function on M . Hence N is an almost RY soliton. $\square$

Remark 3.1. *We obtain the scalar curvature of the Ricci–Yamabe soliton manifold M as*

$$\kappa = \frac{-2\lambda r}{2\alpha - \beta r}. \quad (3.20)$$

4. EXAMPLES

We next provide explicit examples to demonstrate our findings.

Example 4.1. *Let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ with metric*

$$g = dx_1^2 + dx_2^2 + \frac{1}{x_3^2} dx_3^2,$$

and Let $N = \mathbb{R}^2$ with the Euclidean metric

$$h = dy_1^2 + dy_2^2.$$

The map

$$F : M \rightarrow N$$

defined as

$$F(x_1, x_2, x_3) = (x_1, x_2)$$

yields

$$\text{Ker } F_* = \text{span}\{\partial_{x_3}\}$$

and

$$\mathcal{H} = \text{span}\{\partial_{x_1}, \partial_{x_2}\}.$$

The dilation is $\sigma = 1$, implying that F is a Riemannian submersion. The fibers are totally geodesic, and N is flat (and therefore Ricci-flat). We take the potential vector field for the vertical direction $\zeta = \partial_{x_3}$. It can be easily shown that (M, g) admits Ricci Yamabe soliton structure with ζ being vertical, $\alpha = 1$, $\beta = 2$, $\lambda = \frac{1}{2}$ (the scalar curvature of the fiber $\kappa^\mathcal{V} = 1/2$). The fiber is one-dimensional, therefore, Ricci-flat and there exists an induced RY soliton. This supports Theorem 3.1.

Example 4.2. Let $M = \mathbb{R}^3$ with metric

$$g = e^{2x_3} dx_1^2 + e^{2x_3} dx_2^2 + dx_3^2,$$

and let $N = \mathbb{R}^2$ with metric

$$h = dy_1^2 + dy_2^2.$$

Consider the function

$$F(x_1, x_2, x_3) = (x_1, x_2).$$

Here (x_1, x_2) gives a CS with dilation $\sigma = e^{x_3}$. The fibers are vertical lines

$$\{x_1 = \text{constant}, x_2 = \text{constant}\}$$

and their metrics are dx_3^2 and are totally geodesic. The horizontal potential field

$$\zeta = \partial_{x_1}.$$

By using corollary 2.2, the Ricci tensor of M can be expressed as

$$\text{Ric}_{XY} = \sigma^{-2} \text{Ric}_{X^\mathcal{H}Y^\mathcal{H}}^N = 0$$

since N is flat. So the scalar curvature $\kappa = 0$. The RY soliton equation becomes

$$\frac{1}{2} \mathcal{L}_\zeta g + \lambda g = 0.$$

Since a straightforward calculation yields

$$\mathcal{L}_{\partial_{x_1}} g = 0,$$

then we get that the equation is satisfied with $\lambda = 0$. This means that $(M, g, \zeta, 0, \alpha, \beta)$ is a steady RY soliton regardless of the values α, β . Since N is a flat manifold, hence a RY soliton (Theorem 3.7). Moreover, the fibers are also Einstein (Theorem 3.5).

Example 4.3. Let $M = \mathbb{R}^3$ with metric

$$g = dx_1^2 + e^{2x_1} dx_2^2 + dx_3^2,$$

and let $N = \mathbb{R}^2$ with metric

$$h = dy_1^2 + dy_2^2.$$

Let us define

$$F : M \rightarrow N$$

such that

$$F(x_1, x_2, x_3) = (x_1, x_3)$$

with dilation factor $\sigma = e^{x_1}$. Here the fibers are the lines $\{x_1 = \text{constant}, x_3 = \text{constant}\}$ having the metric $e^{2x_1} dx_2^2$. They are non-totally geodesic since their mean curvature vector does not vanish. We choose the horizontal potential field

$$\zeta = \partial_{x_2}.$$

Because ζ is horizontal and Killing, we have $\mathcal{L}_\zeta g = 0$. The scalar curvature κ is a constant (indeed, $\kappa = 0$, since the metric is a warped product with both base and fiber having zero scalar curvatures). We verify that the RY soliton equation is satisfied for $\lambda = 0$. The base manifold N is a flat manifold, but by Theorem 3.8 the base manifold N admits an almost RY soliton since the dilation and mean curvature produce a non-constant function μ . Thus, if the submersion is not totally geodesic, the induced structure is broader than an RY soliton structure.

To support the theoretical results established in the preceding sections, we now revisit the examples from [6] and extend them to the framework of RY solitons.

Example 4.4. Let $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$ be a Riemannian manifold with metric

$$g = e^{-2x_2} dx_1^2 + dx_2^2. \quad (4.21)$$

Let $N = \mathbb{R}$ with metric $h = dy_1^2$. Define $F : M \rightarrow N$ by

$$F(x_1, x_2) = x_1. \quad (4.22)$$

Define the orthonormal basis

$$e_1 = e^{x_2} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}. \quad (4.23)$$

Then,

$$\text{Ker } F_* = \text{span}\{e_2\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_1\}. \quad (4.24)$$

The non-zero Christoffel symbols for the metric $g = e^{-2x_2} dx_1^2 + dx_2^2$ are

$$\Gamma_{11}^2 = e^{-2x_2}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = -1. \quad (4.25)$$

Using the Christoffel symbols, we compute

$$\nabla_{e_1} e_1 = e_2, \quad (4.26)$$

$$\nabla_{e_1} e_2 = -e_1, \quad (4.27)$$

$$\nabla_{e_2} e_1 = 0, \quad (4.28)$$

$$\nabla_{e_2} e_2 = 0. \quad (4.29)$$

For the diagonal metric $g = e^{-2x_2}dx_1^2 + dx_2^2$, the Gaussian curvature is

$$K = -\frac{1}{2\sqrt{|g|}}\partial_{x_2}\left(\frac{\partial_{x_2}g_{11}}{\sqrt{|g|}}\right). \quad (4.30)$$

Since $\sqrt{|g|} = e^{-x_2}$ and $\partial_{x_2}g_{11} = -2e^{-2x_2}$, we get

$$K = -\frac{1}{2e^{-x_2}}\partial_{x_2}\left(\frac{-2e^{-2x_2}}{e^{-2x_2}e^{-x_2}}\right) \quad (4.31)$$

$$= -\frac{1}{2e^{-x_2}}\partial_{x_2}(-2e^{x_2}) \quad (4.32)$$

$$= -\frac{1}{2e^{-x_2}}(-2e^{x_2}) = -1. \quad (4.33)$$

Therefore,

$$\kappa = 2K = -2, \quad (4.34)$$

and

$$\text{Ric}(e_1, e_1) = \text{Ric}(e_2, e_2) = -1, \quad \text{Ric}(e_1, e_2) = 0. \quad (4.35)$$

Thus for arbitrary $X_1, Y_1 \in \Gamma(TM)$,

$$\text{Ric}(X_1, Y_1) = -(a_1a_3 + a_2a_4), \quad (4.36)$$

where $X_1 = a_1e_1 + a_2e_2$ and $Y_1 = a_3e_1 + a_4e_2$.

Let $Z_1 = a_5e_1 + a_6e_2$. Using (4.26)-(4.29), we have

$$\nabla_{X_1}Z_1 = \nabla_{a_1e_1+a_2e_2}(a_5e_1 + a_6e_2) \quad (4.37)$$

$$= a_1a_5\nabla_{e_1}e_1 + a_1a_6\nabla_{e_1}e_2 + a_2a_5\nabla_{e_2}e_1 + a_2a_6\nabla_{e_2}e_2 \quad (4.38)$$

$$= a_1a_5e_2 - a_1a_6e_1. \quad (4.39)$$

Thus,

$$g(\nabla_{X_1}Z_1, Y_1) = a_1a_5a_4 - a_1a_6a_3, \quad (4.40)$$

and

$$g(\nabla_{Y_1}Z_1, X_1) = a_3a_5a_2 - a_3a_6a_1. \quad (4.41)$$

Therefore,

$$\begin{aligned} \frac{1}{2}(L_{Z_1}g)(X_1, Y_1) &= \frac{1}{2}\{g(\nabla_{X_1}Z_1, Y_1) + g(\nabla_{Y_1}Z_1, X_1)\} \\ &= \frac{1}{2}(a_1a_4a_5 + a_2a_3a_5 - 2a_1a_3a_6). \end{aligned} \quad (4.42)$$

Also,

$$g(X_1, Y_1) = a_1a_3 + a_2a_4. \quad (4.43)$$

The RY soliton equation is

$$\frac{1}{2}(L_{Z_1}g)(X_1, Y_1) + \alpha \text{Ric}(X_1, Y_1) + \beta \kappa g(X_1, Y_1) = 0. \quad (4.44)$$

Substituting the computed values

$$\frac{1}{2}(a_1a_4a_5 + a_2a_3a_5 - 2a_1a_3a_6) - \alpha(a_1a_3 + a_2a_4) - 2\beta(a_1a_3 + a_2a_4) = 0. \quad (4.45)$$

For the special choice $a_1 = 1, a_3 = 1, a_4 = 0, a_5 = 1$, this simplifies to

$$-a_6 - \alpha - 2\beta = 0. \quad (4.46)$$

Hence,

$$\alpha + 2\beta = -a_6. \quad (4.47)$$

Remark 4.1. The metric $g = e^{-2x_2} dx_1^2 + dx_2^2$ represents the hyperbolic plane with constant curvature $K = -1$.

Example 4.5. Let $M = \mathbb{R}^2$ with metric

$$g = e^{2x_1} dx_1^2 + dx_2^2. \quad (4.48)$$

Let $N = \mathbb{R}$ with metric $h = dy_1^2$. Define $F : M \rightarrow N$ by

$$F(x_1, x_2) = x_2. \quad (4.49)$$

Define the orthonormal basis

$$e_1 = e^{-x_1} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}. \quad (4.50)$$

Then

$$\text{Ker } F_* = \text{span}\{e_2\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_1\}. \quad (4.51)$$

For the metric $g = e^{2x_1} dx_1^2 + dx_2^2$, the non-zero Christoffel symbol is

$$\Gamma_{11}^1 = 1. \quad (4.52)$$

For the orthonormal basis, we compute:

$$\nabla_{e_1} e_1 = \nabla_{e^{-x_1} \partial_{x_1}} (e^{-x_1} \partial_{x_1}) \quad (4.53)$$

$$= e^{-x_1} [e^{-x_1} \nabla_{\partial_{x_1}} \partial_{x_1} + \partial_{x_1} (e^{-x_1}) \partial_{x_1}] \quad (4.54)$$

$$= e^{-x_1} [e^{-x_1} \partial_{x_1} - e^{-x_1} \partial_{x_1}] = 0, \quad (4.55)$$

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 0. \quad (4.56)$$

Thus all connections vanish

$$\nabla_{e_i} e_j = 0 \quad \text{for all } i, j \in \{1, 2\}. \quad (4.57)$$

For a diagonal metric $g = \text{diag}(e^{2x_1}, 1)$, the Gaussian curvature $K = 0$.

Therefore,

$$\kappa = 0, \quad \text{Ric} = 0. \quad (4.58)$$

Using (4.57) and (4.58), the RY soliton equation becomes

$$\frac{1}{2} (L_{Z_1} g)(X_1, Y_1) = 0. \quad (4.59)$$

Remark 4.2. The metric $g = e^{2x_1} dx_1^2 + dx_2^2$ is flat. Under the coordinate transformation $u = e^{x_1}$, the metric becomes $g = du^2 + dx_2^2$, which is the standard Euclidean metric.

Any vector field Z_1 gives a RY soliton with arbitrary constants α, β . In particular, any constant vector field is a Killing vector field for this flat metric.

Example 4.6. Let $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 1\}$ with metric

$$g = x_3^{-2}(dx_1^2 + dx_2^2 + dx_3^2). \quad (4.60)$$

Let $N = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 > 1\}$ with metric $h = dy_1^2 + dy_2^2$. Define $F : M \rightarrow N$ by

$$F(x_1, x_2, x_3) = (x_2, x_3). \quad (4.61)$$

Define

$$e_1 = x_3 \frac{\partial}{\partial x_1}, \quad e_2 = x_3 \frac{\partial}{\partial x_2}, \quad e_3 = x_3 \frac{\partial}{\partial x_3}. \quad (4.62)$$

Then,

$$\text{Ker } F_* = \text{span}\{e_1\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_2, e_3\}. \quad (4.63)$$

The non-zero Christoffel symbols are

$$\Gamma_{13}^1 = \Gamma_{31}^1 = -x_3^{-1}, \quad \Gamma_{23}^2 = \Gamma_{32}^2 = -x_3^{-1}, \quad (4.64)$$

$$\Gamma_{33}^3 = -x_3^{-1}, \quad \Gamma_{11}^3 = x_3^{-1}, \quad \Gamma_{22}^3 = x_3^{-1}. \quad (4.65)$$

Using the Christoffel symbols

$$\nabla_{e_1} e_1 = e_3, \quad (4.66)$$

$$\nabla_{e_2} e_2 = e_3, \quad (4.67)$$

$$\nabla_{e_1} e_3 = -e_1, \quad (4.68)$$

$$\nabla_{e_3} e_1 = -e_1, \quad (4.69)$$

$$\nabla_{e_2} e_3 = -e_2, \quad (4.70)$$

$$\nabla_{e_3} e_2 = -e_2, \quad (4.71)$$

$$\nabla_{e_3} e_3 = -e_3. \quad (4.72)$$

For $U = e_1 \in \Gamma(\text{Ker } F_*)$, we have

$$T_U U = \nabla_{e_1} e_1 = e_3. \quad (4.73)$$

For totally umbilical fibers, $T_U U = g(U, U)H = H$. Thus,

$$H = e_3. \quad (4.74)$$

So the fibers are totally umbilical with mean curvature $H = e_3$.

$$[e_2, e_3] = [x_3 \partial_{x_2}, x_3 \partial_{x_3}] \quad (4.75)$$

$$= x_3[\partial_{x_2}, \partial_{x_3}] + x_3 \partial_{x_2}(x_3) \partial_{x_3} - x_3 \partial_{x_3}(x_3) \partial_{x_2} \quad (4.76)$$

$$= -x_3 \partial_{x_2} = -e_2. \quad (4.77)$$

Since $[e_2, e_3] = -e_2 \in (\text{Ker } F_*)^\perp$, the horizontal distribution is integrable.

For $X = ae_2 + be_3 \in \Gamma(\text{Ker } F_*)^\perp$, we have

$$\nabla_X X = \nabla_{ae_2+be_3}(ae_2 + be_3) \quad (4.78)$$

$$= a^2 \nabla_{e_2} e_2 + ab \nabla_{e_2} e_3 + ab \nabla_{e_3} e_2 + b^2 \nabla_{e_3} e_3 \quad (4.79)$$

$$= a^2 e_3 - abe_2 - abe_2 - b^2 e_3 \quad (4.80)$$

$$= (a^2 - b^2)e_3 - 2abe_2. \quad (4.81)$$

The horizontal part is

$$\mathcal{H}\nabla_X X = (a^2 - b^2)e_3. \quad (4.82)$$

For $(\text{Ker } F_*)^\perp$ to be totally geodesic, we need $\mathcal{H}\nabla_X X = 0$ for all a, b . This requires $a^2 = b^2$ for all a, b , which is false.

Conclusion: The horizontal distribution is NOT totally geodesic.

The metric $g = x_3^{-2}(dx_1^2 + dx_2^2 + dx_3^2)$ is the standard hyperbolic metric on $\mathbb{H}^3$ with constant curvature $K = -1$. Thus:

$$\kappa = -6, \quad \text{Ric} = -2g. \quad (4.83)$$

For $\zeta = W \in \Gamma(\text{Ker } F_*)$ and $U, V \in \Gamma(\text{Ker } F_*)$, we have

$$\frac{1}{2}(L_W g)(U, V) + \alpha(-2)g(U, V) + \beta(-6)g(U, V) = 0. \quad (4.84)$$

Thus,

$$\frac{1}{2}(L_W g)(U, V) - (2\alpha + 6\beta)g(U, V) = 0. \quad (4.85)$$

For the 1-dimensional fiber, this equation can be solved for appropriate W and parameters α, β .

Example 4.7. Let $M = \mathbb{R}^3$ with metric

$$g = 4(dx_1^2 + dx_2^2 + dx_3^2). \quad (4.86)$$

Let $N = \mathbb{R}^2$ with metric $h = dy_1^2 + dy_2^2$. Define $F : M \rightarrow N$ by

$$F(x_1, x_2, x_3) = (x_1, x_3). \quad (4.87)$$

Define

$$e_1 = \frac{1}{2} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{1}{2} \frac{\partial}{\partial x_2}, \quad e_3 = \frac{1}{2} \frac{\partial}{\partial x_3}. \quad (4.88)$$

Then

$$\text{Ker } F_* = \text{span}\{e_2\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_1, e_3\}. \quad (4.89)$$

Since the metric is flat, all Christoffel symbols vanish

$$\nabla_{e_i} e_j = 0 \quad \text{for all } i, j \in \{1, 2, 3\}. \quad (4.90)$$

Thus both $\text{Ker } F_*$ and $(\text{Ker } F_*)^\perp$ are totally geodesic.

For $X_1 = e_1$ and $X_2 = e_3$:

$$F_*(e_1) = F_*(\frac{1}{2}\partial_{x_1}) = \frac{1}{2}\partial_{y_1}, \quad (4.91)$$

$$F_*(e_3) = F_*(\frac{1}{2}\partial_{x_3}) = \frac{1}{2}\partial_{y_2}, \quad (4.92)$$

$$F_*(e_2) = 0. \quad (4.93)$$

Then

$$h(F_*e_1, F_*e_1) = h(\frac{1}{2}\partial_{y_1}, \frac{1}{2}\partial_{y_1}) = \frac{1}{4} = \lambda^2 g(e_1, e_1) = \lambda^2 \cdot 1. \quad (4.94)$$

So $\lambda = \frac{1}{2}$.

Thus F is a horizontally conformal submersion with constant dilation $\lambda = \frac{1}{2}$.

- Fibers are totally geodesic: $\nabla_{e_2}e_2 = 0$
- Horizontal distribution is totally geodesic: $\mathcal{H}\nabla_X X = 0$ for all $X \in \Gamma(\text{Ker } F_*)^\perp$
- F is homothetic (constant dilation)

Therefore, F is totally geodesic.

The metric is flat, so

$$\kappa = 0, \quad \text{Ric} = 0. \quad (4.95)$$

Using (4.90) and (4.95), the RY soliton equation becomes

$$\frac{1}{2}(L_\zeta g) = 0. \quad (4.96)$$

So ζ must be a Killing vector field.

For the flat metric $g = 4(dx_1^2 + dx_2^2 + dx_3^2)$, the Killing vector fields are

- Translations: $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}$
- Rotations: $x_i\partial_{x_j} - x_j\partial_{x_i}$

Any constant vector field

$$\zeta = c_1\partial_{x_1} + c_2\partial_{x_2} + c_3\partial_{x_3}, \quad c_1, c_2, c_3 \in \mathbb{R} \quad (4.97)$$

gives a RY soliton $(M, g, \zeta, \alpha, \beta)$ for arbitrary α, β .

5. CONCLUSION

In this paper, we considered conformal Riemannian submersions whose total manifolds possess RY solitons. The curvatures of such submersions have been analyzed and some curvature relations have been derived, using which we obtained sufficient conditions for the fibers as well as the base manifold of a given submersion to inherit Einstein, almost Einstein, and Ricci Yamabe solitons. Thus, this paper is a continuation of the research performed on Riemannian submersions admitting Ricci solitons ([17], [18]) or RY solitons ([10]). In fact, our results include those in [6] if $\alpha = 1$ and $\beta = 0$.

One of the significant aspects of this paper is the comparison between Ricci soliton and Ricci–Yamabe soliton cases. As the equation of RY soliton includes a scalar curvature term, it may be considered an extension of the equation of Ricci soliton. This implies that when a Riemannian submersion is taken into account, in addition to the soliton constant, the scalar curvature of the manifold is included in the induced Einstein structure on the base manifold. Unlike Ricci soliton, the constancy of the scalar curvature turns out to be one of the essential conditions for the base manifold to be Einstein. Particularly, whenever the scalar curvature is a constant, the RY case becomes similar to the Ricci soliton case, while variable scalar curvature gives almost Einstein manifolds.

The findings in this study have shown that the use of CS geometry offers a natural environment for the discussion of interaction between RY solitons and the geometric flow structure. There are still several possible research areas that one could explore in the future. One might

analyze the situation when the potential vector field has horizontal and vertical components, or explore gradient RY solitons using the technique of CSs. One could also explore the impact of the dilation factor on the resulting geometric structure and explore the connection between conformal harmonic morphisms and RY solitons or η -Ricci-Yamabe solitons [16]. Another approach would be to conduct analogous analysis for statistical manifolds, semi-Riemannian manifolds, contact manifolds, or even warped products. These results open up new directions for studying conformal submersions in the context of generalized solitons and geometric flows.

The conformal Riemannian map was introduced in [14], generalizing conformal submersions. Recently, Clairaut conformal submersions and Clairaut conformal Riemannian maps whose total manifolds possess Ricci solitons were explored in [13] and [5]. As in the present study, these mappings may be explored in the near future for total manifolds possessing RY solitons.

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ON QUASI BI-SLANT SUBMANIFOLDS OF A KÄHLER MANIFOLD IMMERSED AS SEQUENTIAL WARPED PRODUCTS

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Abstract. The present paper focuses on the study of a new class of sequential warped product submanifolds considered by harmonizing one invariant submanifold and two proper slant submanifolds of Kähler manifolds with the concept of sequential warped products. In particular, we examine the existence (or non-existence) of proper quasi bi-slant warped product submanifolds of Kähler manifolds, conceptualized as sequential warped products. Our claim demonstrates that a Kähler manifold does not admit any proper quasi bi-slant warped product submanifolds (in the sequential warped product sense). The study advances the understanding of submanifold geometry in complex manifolds, offering significant insights into the non-existence of such warped product submanifolds and their geometric implications in Kähler geometry.

Keywords: Warped product submanifolds, Kähler manifolds, slant submanifold, quasi bi-slant submanifolds.

MSC (2020): 53B25, 53B30, 53B35.

1. INTRODUCTION

The geometry of submanifolds, studying subsets of higher-dimensional spaces with geometric structures, emerged in the 19th century through the work of Gauss and Riemann. Their foundational contributions in differential geometry enabled the analysis of submanifolds' intrinsic and extrinsic properties. The study of invariant, anti-invariant, and CR -submanifolds of almost Hermitian manifolds began in the mid-20th century, building on Hermitian geometry foundations from Riemann's theory. Invariant and anti-invariant submanifolds, defined by their behavior under the almost complex structure, along with CR -submanifolds combining both properties, were formalized and advanced by Bejancu in the 1970s through studies on submanifolds in Kähler and complex manifolds [2, 3].

The study of slant submanifolds in almost Hermitian manifolds originated with B.Y. Chen's [8] introduction of the concept, defining them as submanifolds where the angle between the tangent space and its image under the almost complex structure is constant. This

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notion was further extended by N. Papaghiuc [14] to semi-slant submanifolds, which decompose the tangent space into invariant and slant distributions. In 2000, A. Carriazo [6] introduced bi-slant submanifolds, featuring two orthogonal slant distributions with potentially different angles, encompassing semi-slant and hemi-slant types as special cases, where hemi-slant involves one totally real and one slant distribution. These generalizations have since been explored in various geometric settings [23], providing a unified framework for intermediate submanifold classes between invariant and anti-invariant ones.

On the other hand, warped product manifolds were first introduced by R. L. Bishop and B. O'Neill [5], generalizing Riemannian products to create manifolds with negative sectional curvature, facilitating the study of geometric properties such as completeness and embeddings. In Kähler manifolds, Chen [9] work on CR -warped product submanifolds established key inequalities and classifications. B. Şahin [17] furthered investigating slant warped product submanifolds in Kähler manifolds, proving non-existence for non-trivial cases through inequalities tied to the warping function and slant angle. Further, Şahin [17, 18] studied semi-slant and pointwise semi-slant submanifolds of Kähler manifolds immersed as warped products. Later, several authors have investigated such warped products from a geometric perspective and obtained a variety of interesting results [13, 12, 21, 1].

Moreover, sequential warped product manifolds, initially introduced by Shenawy [22], represent a generalization of traditional warped product manifolds. These structures were thoroughly investigated by De, Shenawy, and Ünal [10, 24], focusing on their properties as submanifolds in various geometric settings. Additionally, sequential warped products have been recognized as effective frameworks for modeling generalized Robertson–Walker space-time and standard static space-time, as demonstrated in further studies. Building on this, Şahin [19], advanced their application in almost Hermitian manifolds, particularly Kähler manifolds.

Inspired by this concept, this paper proposes a sequential warped product characterized by two proper slant distributions and one invariant distribution, which we designate as a quasi bi-slant sequential warped product. Additionally, we explore the existence of such proper sequential warped product manifolds of Kähler manifolds that feature two proper slant distributions and one invariant distribution. The structure of this paper is organized as follows: Section 2 presents the foundational concepts and preliminaries concerning almost Hermitian manifolds, with a particular emphasis on Kähler manifolds. In Section 3, we provide an overview of essential properties and characteristics of sequential warped product submanifolds of almost Hermitian manifolds. Additionally, this section investigates the existence of such sequential warped product submanifolds specifically in the context of Kähler manifolds.

2. PRELIMINARIES

An almost complex manifold is defined as a pair (M, J) , where M is a real differentiable manifold and J is a smooth field of endomorphisms on the tangent bundle (i.e., $J_p : T_p M \rightarrow T_p M$, for each $p \in M$) satisfying $J^2 = -I$. When M is paracompact, there always exists a Riemannian metric g that respects this structure. Such a metric, characterized by the invariance condition $g(JU_1, JU_2) = g(U_1, U_2)$, is referred to as a Hermitian metric on the almost complex manifold. A differentiable manifold is called almost Hermitian if it is almost

complex and possesses a Hermitian metric. In the integrable case, such a manifold is simply called a Hermitian manifold. By definition, a Hermitian manifold (M, J, g) therefore carries a canonical geometric structure, which includes the complex structure J , the Hermitian metric g and its unique torsion-free metric connection, the Levi-Civita connection $\bar{\nabla}$.

The fundamental 2-form Ω associated with an almost complex structure J and Hermitian metric g of an almost Hermitian manifold $\bar{M}$ is defined by

$$\Omega(U_1, U_2) = g(U_1, JU_2), \quad (2.1)$$

for all vector fields U_1, U_2 on $\bar{M}$. Since g is invariant under J so is Ω , i.e.

$$\Omega(JU_1, JU_2) = \Omega(U_1, U_2). \quad (2.2)$$

Definition 2.1. [25] *An almost complex manifold endowed with a Hermitian metric is called a Kähler manifold if its fundamental 2-form Ω is closed. Equivalently, a Hermitian manifold $\bar{M}$ is Kähler if and only if its almost complex structure J is parallel with respect to the Hermitian connection $\bar{\nabla}$, i.e.,*

$$(\bar{\nabla}_{U_1} J)U_2 = 0 \quad \text{for all } U_1, U_2 \in T\bar{M}. \quad (2.3)$$

When this condition holds, the connection $\bar{\nabla}$ is referred to as the Kähler connection.

Let M and $\bar{M}$ be differentiable manifolds of dimensions n and m respectively. Hence, for every point $p \in M$, the tangent space $T_p\bar{M}$ of $\bar{M}$ admits the following decomposition

$$T_p\bar{M} = T_pM \oplus T_p^\perp M.$$

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^\perp$ on TM and on the normal bundle $T^\perp M$ respectively governed by the Gauss and Weingarten formulae, viz.

$$\bar{\nabla}_{U_1} U_2 = \nabla_{U_1} U_2 + h(U_1, U_2), \quad (2.4)$$

$$\bar{\nabla}_{U_1} N = -A_N U_1 + \nabla_{U_1}^\perp N, \quad (2.5)$$

where U_1, U_2 are tangent vector fields on M and $N \in T^\perp M$. h and A are the second fundamental form and the shape operator respectively and are related as

$$g(h(U_1, U_2), N) = g(A_N U_1, U_2). \quad (2.6)$$

A submanifold M isometrically immersed in a Kähler manifold $\bar{M}$ with an almost complex structure J is said to be CR -submanifold, if there exists invariant $\mathcal{H}^T$ and anti-invariant distribution $\mathcal{H}^\perp$ such that $J\mathcal{H}^T = \mathcal{H}^T$ and $J\mathcal{H}^\perp \subset \mathcal{H}^\perp$ [4]. A submanifold M of a Kähler manifold $\bar{M}$ is called a slant submanifold [7] if, for every point $p \in M$ and every nonzero tangent vector $X \in T_pM$, the angle $\theta(X)$ (known as the Wirtinger angle or slant angle) between JX and T_pM is constant, independent of the choice of p and $X \in T_pM$. This constant angle $\theta \in [0, \pi/2]$ characterizes the submanifold globally. Special cases include, $\theta = 0$: M is invariant (holomorphic), $\theta = \pi/2$: M is anti-invariant (totally real) and $0 < \theta < \pi/2$: Proper slant. A submanifold M of a Kähler manifold $\bar{M}$ with almost complex structure J is called a semi-slant submanifold [14] if the tangent bundle TM admits an orthogonal decomposition $TM = \mathcal{H}^T \oplus \mathcal{H}^\theta$, where $\mathcal{H}^T$ is an invariant distribution, i.e., $J(\mathcal{H}^T) = \mathcal{H}^T$ and $\mathcal{H}^\theta$ is a slant distribution. A submanifold M of a Kähler manifold $\bar{M}$ said to be hemi-slant

submanifold if $\mathcal{H}^\perp$ is an anti-invariant distribution, i.e., $J(\mathcal{H}^\perp) \subseteq T^\perp M$ and $\mathcal{H}^\theta$ is slant distribution [20]. A submanifold M of a Kähler manifold $\bar{M}$ is called a bi-slant submanifold if there exists two distributions $\mathcal{H}^{\theta_1}, \mathcal{H}^{\theta_2}$ on M such that the distribution $\mathcal{H}^{\theta_1}$ (resp. $\mathcal{H}^{\theta_2}$) is slant, i.e., for every point $p \in M$ (resp. $q \in M$) and every nonzero vector $X \in (\mathcal{H}^{\theta_1})_p$, (resp. $Y \in (\mathcal{H}^{\theta_2})_q$) the Wirtinger angle $\theta_1(X)$ (resp. $\theta_2(Y)$) between JX and $T_p M$ (resp. $T_q M$) is constant, independent of p (resp. q) in M and $X \in T_p M$ (resp. $T_q M$) [6].

In this context, we revisit the definition of quasi bi-slant submanifolds of almost Hermitian manifolds as established by R. Prasad et al., [15].

Definition 2.2. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a quasi bi-slant submanifold if there exists distribution $\mathcal{H}^T, \mathcal{H}^{\theta_1}$ and $\mathcal{H}^{\theta_2}$ such that

$$(i) \quad TM = \mathcal{H}^T \oplus \mathcal{H}^{\theta_1} \oplus \mathcal{H}^{\theta_2},$$

$$(ii) \quad J\mathcal{H}^T = \mathcal{H}^T,$$

$$(iii) \quad J\mathcal{H}^{\theta_1} \perp \mathcal{H}^{\theta_2}, \quad J\mathcal{H}^{\theta_2} \perp \mathcal{H}^{\theta_1},$$

$$(iv) \quad \text{for every point } p \in M \text{ and every nonzero vector field } Y_1 \in (\mathcal{H}^{\theta_1})_p, \text{ the angle } \theta_1 \text{ between } JY_1 \text{ and } (\mathcal{H}^{\theta_1})_p \text{ is constant, independent of the choice of } p \in M \text{ and } Y_1 \in (\mathcal{H}^{\theta_1})_p,$$

$$(v) \quad \text{for every point } q \in M \text{ and every nonzero vector field } Y_2 \in (\mathcal{H}^{\theta_2})_q, \text{ the angle } \theta_2 \text{ between } JY_2 \text{ and } (\mathcal{H}^{\theta_2})_q \text{ is constant, independent of the choice of } q \in M \text{ and } Y_2 \in (\mathcal{H}^{\theta_2})_q.$$

The angles θ_1 and θ_2 are called the slant angles.

As the generalization of warped product and doubly warped product, here is the definition of sequential warped product manifolds.

Definition 2.3. [11] For pseudo-Riemannian manifolds (M_i, g_i) ($i = 1, 2, 3$), let $f : M_1 \rightarrow (0, \infty)$ and $h : M_1 \times M_2 \rightarrow (0, \infty)$ be smooth positive functions. The sequential warped product, denoted $(M_1 \times_f M_2) \times_h M_3$, is the product manifold $M_1 \times M_2 \times M_3$ endowed with the metric $\bar{g} = g_1 \oplus f^2 g_2 \oplus h^2 g_3$. Here, f and h are the warping functions. Notably, if all (M_i, g_i) are Riemannian, then this construction yields a Riemannian manifold.

Proposition 2.1. [11] Consider a sequential warped product manifold $(M_1 \times_f M_2) \times_h M_3$ equipped with the metric $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ denote vector fields on the respective factor manifolds for $i = 1, 2, 3$. Then

$$(i) \quad \bar{\nabla}_{W_1} W_2 = \bar{\nabla}_{W_2} W_1 = W_1(\ln f)W_2,$$

$$(ii) \quad \bar{\nabla}_{W_3} W_1 = \bar{\nabla}_{W_1} W_3 = W_1(\ln h)W_3,$$

$$(iii) \quad \bar{\nabla}_{W_2} W_3 = \bar{\nabla}_{W_3} W_2 = W_2(\ln h)W_3,$$

$$(iv) \quad \bar{R}(W_i, W_3)W_j = \frac{-1}{h}\mathcal{H}^h(W_i, W_j)W_3, \text{ for } i, j = 1, 2.$$

Remark 2.1. Consider smooth positive functions $f : M_1 \rightarrow (0, \infty)$ and $h : M_1 \times M_2 \rightarrow (0, \infty)$ defined on the Riemannian manifolds M_1 and $M_1 \times M_2$, respectively. Then the sequential warped product $M = (M_1 \times_f M_2) \times_h M_3$ is said to be proper if the following three conditions hold for all non-zero vector fields $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$:

$$X_1(\ln f) \neq 0, \quad X_1(\ln h) \neq 0, \quad X_2(\ln h) \neq 0.$$

3. QUASI BI-SLANT WARPED PRODUCT SUBAMNIFOLDS AS SEQUENTIAL WARPED PRODUCT SUBAMANIFOLDS

B. Sahin [19] introduced the concept of sequential warped products in Kähler manifolds, considering factors that are holomorphic, totally real, and pointwise slant. He further established fundamental inequalities and a characterization for the associated warping functions.

As a natural consequence, it is obvious to seek an answer to the following question.

Question: Does a Kähler manifold admit any proper sequential warped product submanifold with one holomorphic factor and two proper slant factors?

To answer this affirmative, we investigate the geometry of a quasi bi-slant submanifold of a Kähler manifold immersed as a sequential warped product.

By definition of a quasi bi-slant submanifold M of a Kähler manifold $\bar{M}$, considering the decomposition of the tangent bundle of M , we have $TM = \mathcal{H}^T \oplus \mathcal{H}^{\theta_1} \oplus \mathcal{H}^{\theta_2}$, which indicates that TM is decomposed into three distributions of which one is holomorphic (invariant) while the other two are proper slant distributions. This yields that a quasi bi-slant submanifold of a Kähler manifold can be immersed either as a doubly warped product submanifold or as a sequential warped product submanifold. From the stated two type of warped products, the most simple possible warped products for quasi bi-slant submanifolds are sequential warped products and in this context, the possible sequential warped products for this class of submanifolds are as follows:

$$(i) \quad (M_T \times_{f_1} M_{\theta_i}) \times_{f_2} M_{\theta_j},$$

$$(ii) \quad (M_{\theta_i} \times_{f_1} M_T) \times_{f_2} M_{\theta_j},$$

$$(iii) \quad (M_{\theta_i} \times_{f_1} M_{\theta_j}) \times_{f_2} M_T,$$

where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are proper slant submanifolds of $\bar{M}$ for $i, j = 1, 2$.

For any vector fields $U_1 \in \Gamma(TM)$ and $U_2 \in \Gamma(TM^\perp)$, we have

$$JU_1 = tU_1 + nU_1 \tag{3.7}$$

and

$$JU_2 = bU_2 + cU_2, \tag{3.8}$$

where tU_1 and bU_2 are the tangential components of JU_1 , whereas nU_1 and cU_2 are the normal components of JU_2 .

For any vector fields $U_1 \in TM_T$ and $X_1 \in TM_{\theta_i}$, by using Proposition 2.1, we have

$$g(\nabla_{tX_1} U_1, X_1) = (U_1 \ln f_1)g(tX_1, X_1) = 0.$$

From Eqs. (2.3), (2.4) and (3.7), we have

$$\begin{aligned} 0 &= g(\nabla_{tX_1} U_1, X_1) \\ &= g(\bar{\nabla}_{tX_1} U_1 - h(tX_1, U_1), X_1) \\ &= -g(JU_1, \bar{\nabla}_{tX_1} tX_1) - g(JU_1, \bar{\nabla}_{tX_1} nX_1). \end{aligned}$$

In the light of Eqs. (2.5), (2.6) and Proposition 2.1, we get

$$(JU_1 \ln f_1) \cos^2 \theta_i g(X_1, X_1) = -g(h(JU_1, tX_1), nX_1). \quad (3.9)$$

On replacing U_1 by JU_1 in Eq. (3.9), we have

$$(U_1 \ln f_1) \cos^2 \theta_i g(X_1, X_1) = -g(h(U_1, tX_1), nX_1). \quad (3.10)$$

By replacing X_1 by tX_1 in Eq. (3.10), we acquire

$$g(h(U_1, X_1), ntX_1) = \cos^2 \theta_i (U_1 \ln f_1)g(X_1, X_1). \quad (3.11)$$

On the other hand, for any $U_1 \in TM_T$ and $X_1, X_2 \in TM_{\theta_i}$, by using Eqs. (2.3), (2.4) and (3.7), we have

$$\begin{aligned} g(h(tX_1, U_1), nX_2) &= g(\bar{\nabla}_{U_1} tX_1 - \nabla_{U_1} tX_1, nX_2) \\ &= -g(\bar{\nabla}_{U_1} nX_2, tX_1) \\ &= -g(\bar{\nabla}_{U_1} JX_2, tX_1) + g(\bar{\nabla}_{U_1} tX_2, tX_1) \\ &= g(t^2 X_1 + ntX_1, \bar{\nabla}_{U_1} X_2) + g(tX_1, \nabla_{U_1} tX_2) \\ &= g(t^2 X_1, \bar{\nabla}_{U_1} X_2) + g(ntX_1, \bar{\nabla}_{U_1} X_2) + g(tX_1, \nabla_{U_1} tX_2). \end{aligned}$$

In the light of Eq. (2.4) with Proposition 2.1, we get

$$\begin{aligned} g(h(tX_1, U_1), nX_2) &= -\cos^2 \theta_i g(X_1, \bar{\nabla}_{U_1} X_2) + g(ntX_1, \bar{\nabla}_{U_1} X_2) + g(tX_1, \nabla_{U_1} tX_2) \\ &= -\cos^2 \theta_i g(X_1, \nabla_{U_1} X_2) + g(ntX_1, h(U_1, X_2)) + g(tX_1, \nabla_{U_1} tX_2) \\ &= -\cos^2 \theta_i (U_1 \ln f_1)g(X_1, X_2) + g(ntX_1, h(U_1, X_2)) + (U_1 \ln f_1)g(tX_1, tX_2) \\ &= g(ntX_1, h(U_1, X_2)) - \cos^2 \theta_i (U_1 \ln f_1)g(X_1, X_2) + (U_1 \ln f_1)g(tX_1, tX_2). \end{aligned}$$

From the above equation, it is easy to obtain that

$$g(h(tX_1, U_1), nX_2) = g(ntX_1, h(U_1, X_2)). \quad (3.12)$$

Taking $X_1 = X_2$ in Eq. (3.12), we get

$$g(h(tX_1, U_1), nX_1) = g(ntX_1, h(U_1, X_1)). \quad (3.13)$$

By using Eqs. (3.10), (3.11) and (3.13), we have

$$(U_1 \ln f_1) \cos^2 \theta_i g(X_1, X_1) = 0.$$

As M_{θ_i} is proper slant submanifold and X_1 is non-null vector, therefore we get $U_1 \ln f_1 = 0$, which implies that f_1 is constant on M_T .

Based on the above, we can assert the following result:

Theorem 3.1. *There do not exist any proper sequential warped product submanifold of a Kähler manifold $\bar{M}$ of the form $(M_T \times_{f_1} M_{\theta_i}) \times_{f_2} M_{\theta_j}$ where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are the slant submanifolds of a Kähler manifold $\bar{M}$ for $i, j = 1, 2$.*

Moreover, from the computation in the prior section, with $U_1 \in TM_T$ and $X_1, X_2 \in TM_{\theta_i}$, we infer that

Theorem 3.2. *There do not exist any proper sequential warped product submanifold of a Kähler manifold $\bar{M}$ of the form $(M_{\theta_i} \times_{f_1} M_T) \times_{f_2} M_{\theta_j}$, where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are the slant submanifolds of a Kähler manifold $\bar{M}$ for $i, j = 1, 2$.*

The subsequent theorem demonstrates that a Kähler manifold $\bar{M}$ does not admit a quasi bi-slant sequential warped product submanifold of the form $(M_{\theta_i} \times_{f_1} M_{\theta_j}) \times_{f_2} M_T$.

Theorem 3.3. *There do not exist any proper sequential warped product submanifold of a Kähler manifold $\bar{M}$ of the form $(M_{\theta_i} \times_{f_1} M_{\theta_j}) \times_{f_2} M_T$ where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are the slant submanifolds of a Kähler manifold $\bar{M}$ for $i, j = 1, 2$.*

Proof. Let us suppose for any $U_1 \in TM_T$ and $X_1 \in TM_{\theta_j}$, by using Proposition 2.1, we get

$$g(\nabla_{JU_1} X_1, U_1) = (X_1 \ln f_2)g(JU_1, U_1) = 0.$$

From Gauss formula and considering the above fact, we have

$$g(\nabla_{JU_1} X_1, U_1) = g(\bar{\nabla}_{JU_1} X_1, U_1).$$

By using Eqs. (2.3) and (3.7), we may write

$$g(\nabla_{JU_1} X_1, U_1) = -g(tX_1 + nX_1, \bar{\nabla}_{JU_1} JU_1).$$

Considering Eq. (2.4) and Proposition 2.1, we get

$$(tX_1 \ln f_2)g(U_1, U_1) = g(h(JU_1, JU_1), nX_1). \quad (3.14)$$

By using the polarization identity for any $U_1, V_1 \in TM_T$ and $X_1 \in TM_{\theta_j}$, we have

$$(tX_1 \ln f_2)g(U_1, V_1) = g(h(JU_1, JV_1), nX_1). \quad (3.15)$$

On the other hand, considering

$$g(A_{nX_1} JU_1, JV_1) = g(-\bar{\nabla}_{JU_1} nX_1 + \nabla_{JU_1}^\perp nX_1, JV_1),$$

by using Eq. (2.5). Further, in the light of fact from Eqs. (2.3) and (3.7), we can write

$$\begin{aligned} g(A_{nX_1} JU_1, JV_1) &= g(JX_1, J\bar{\nabla}_{JU_1} V_1) - g(tX_1, \bar{\nabla}_{JU_1} JV_1) \\ &= g(X_1, \nabla_{JU_1} V_1 + h(JU_1, V_1)) + g(\bar{\nabla}_{JU_1} tX_1, JV_1) \\ &= g(X_1, \nabla_{JU_1} V_1) + g(\nabla_{JU_1} tX_1, JV_1). \end{aligned}$$

In the light of Proposition 2.1 and Eq. (2.4), we get

$$\begin{aligned} g(A_{nX_1}JU_1, JV_1) &= g(X_1, \bar{\nabla}_{JU_1}V_1) + (tX_1 \ln f_2)g(JU_1, JV_1) \\ &= -g(\nabla_{JU_1}X_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1) \\ &= -g(\nabla_{JU_1}X_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1) \\ &= -(X_1 \ln f_2)g(JU_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1), \end{aligned}$$

which implies that

$$g(h(JU_1, JV_1), nX_1) = -(X_1 \ln f_2)g(JU_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1). \quad (3.16)$$

From Eqs. (3.15) and (3.16), we have

$$(X_1 \ln f_2)g(JU_1, V_1) = 0.$$

Then taking $U_1 = JU_1$ in the last equation, we get $(X_1 \ln f_2)g(U_1, V_1) = 0$. As M_T is non-degenerate, thus we have $\ln f_2 = 0$, which implies that f_2 is constant on M_{θ_j} . $\square$

4. CONCLUSION

Inspired by B. Şahin's [19] investigation into the existence of sequential warped product submanifolds in almost Hermitian manifolds, where such warped products were found to exist, we explored the existence of proper sequential warped product submanifolds in Kähler manifolds, which we name this type of warped products as proper quasi bi-slant warped product submanifolds. Our findings reveal that no such proper warped product submanifolds exist in Kähler manifolds. This opens a scope for future research to investigate whether warped product submanifolds of this type exist in nearly Kähler manifolds or almost contact metric manifolds. On a similar note, one may look to characterize the notion of sequential warped products for submanifolds involving one or two pointwise slant factors in different ambient space settings.

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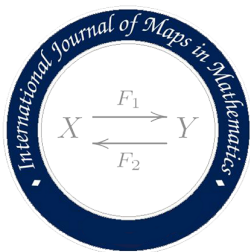
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VARIATION-OF-PARAMETERS AND LIPSCHITZ-TYPE STABILITY IN UNPERTURBED DIFFERENTIAL SYSTEMS WITH INITIAL TIME DIFFERENCE

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Abstract. In this study, we explore the connection between unperturbed differential systems that exhibit variations in both their initial conditions and the starting time. Through the application of variation-of-parameters methods, we construct integral representations that reveal this relationship. To formulate Lipschitz-type stability criteria in the presence of an initial time shift in nonlinear differential systems, it is essential to employ the variational framework derived from the unperturbed model.

Keywords: differential equations, initial time difference, Lipschitz stability, variation-of-parameters

MSC (2020): 34D10, 34D20, 34D99.

1. INTRODUCTION

The variation-of-parameters formula (VPF) is a classical and powerful tool in the qualitative analysis of nonlinear differential equations, particularly in describing the structure and behavior of solutions. In recent years, VPF-based approaches have been extended to investigate nonlinear systems in the absence of an initial time difference (ITD), as well as to develop associated differential inequality frameworks (see [2, 4, 5]). Motivated by these developments, the present paper establishes VPF-based representations that relate unperturbed systems with different initial data and incorporates the effects of initial time shifts.

The qualitative theory of dynamical systems has a long history, especially in the study of stability properties of differential equations. Classical results mainly consider perturbations in the initial state while keeping the initial time fixed, as discussed in [2, 4, 5, 17, 19]. More recently, attention has shifted toward models where both the initial state and the initial time are perturbed. This leads to the notion of initial time difference (ITD) stability. In [10, 14], variation-of-parameters techniques were used to analyze relationships between solutions corresponding to different initial times and initial conditions, leading to the concept of ITD-based stability structures.

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A key feature distinguishing ITD stability from classical Lipschitz stability is the reference framework. Classical Lipschitz stability is formulated with respect to the zero solution, whereas ITD-based stability is defined relative to a nontrivial unperturbed trajectory that incorporates both initial state perturbations and initial time shifts. As shown in Section 4, standard transformations used in classical stability theory do not directly carry over to the ITD setting, which highlights a structural difference between the two frameworks.

To develop stability criteria in this generalized setting, we employ the variational system associated with the underlying unperturbed dynamics. Within this framework, we introduce several notions of ITD-Lipschitz stability, including variation-based and time-shifted formulations. Our results demonstrate that classical stability theory is insufficient to fully characterize dynamics under ITD perturbations, and that ITD stability requires a distinct analytical treatment.

Furthermore, we show that ITD-Lipschitz stability implies ITD stability, while the converse implication does not hold in general. Additional results are obtained for exponential asymptotic ITD-Lipschitz stability and uniform ITD stability, providing a unified framework for analyzing stability under simultaneous perturbations in state and initial time.

The study of Lipschitz and exponential stability for nonlinear differential systems has been extensively developed in the literature. Classical works such as [3, 7, 11] focus on stability with respect to the zero solution under fixed initial time assumptions. These approaches typically rely on variation-of-parameters formulas, Lyapunov techniques, and differential inequalities.

More recent contributions, including [1, 13, 16], extend these ideas to systems with time-dependent perturbations and employ Gronwall-type inequalities to derive stability results. However, these studies still evaluate stability relative to a fixed reference solution or the zero equilibrium.

In contrast, the present paper introduces Lipschitz stability under Initial Time Difference (ITD), where stability is measured with respect to an unperturbed trajectory defined at a shifted initial time. This framework generalizes classical stability notions by simultaneously incorporating spatial perturbations and initial time shifts. Unlike previous approaches, we provide a unified treatment of variation-based, uniform, and exponential asymptotic ITD-stability. In particular, we establish that ITD-Lipschitz stability implies ITD stability, while the reverse implication does not generally hold.

2. PRELIMINARIES

We consider the following nonlinear differential models characterized by different initial conditions:

$$x' = f(t, x), \quad x(t_0) = x_0 \text{ for } t \geq t_0, \quad t_0 \in \mathbb{R}_+ \quad (2.1)$$

$$x' = f(t, x), \quad x(\rho_0) = y_0 \text{ for } t \geq \rho_0, \quad \rho_0 \in \mathbb{R}_+ \quad (2.2)$$

Assume that the function f is continuous and possesses continuous partial derivatives with respect to x , i.e., $\frac{\partial f}{\partial x}$ is defined on $\mathbb{R}_+ \times \mathbb{R}^n$. Let $\bar{X} = x(t, t_0, x_0)$ denote the solution of system (2.1) for $t \geq t_0$. Under these assumptions, the derivative $\gamma = \frac{\partial \bar{X}}{\partial x_0}$ exists and satisfies

$$z' = \frac{\partial f(t, \bar{X})}{\partial \bar{X}} z \quad z(t_0) = I_n. \quad (2.3)$$

where I_n is $n \times n$ the identity matrix. Moreover, $\frac{\partial \tilde{\mathcal{X}}}{\partial \tilde{\mathcal{X}}}$ not only exists but also fulfills the identity:

$$\frac{\partial \tilde{\mathcal{X}}}{\partial x_0} + \gamma(t, t_0, x_0) f(t_0, x_0) = 0, \quad \text{for } t \geq t_0 \quad (2.4)$$

For a fixed initial point (ρ_0, y_0) , in the rest of the paper, we shall denote by $X = x(t - \tilde{\zeta}, t_0, x_0)$ the solution of (2.1) for $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ and $t \geq \rho_0$.

The following stability concepts are introduced by extending classical notions of stability, Lipschitz stability, asymptotic stability, and exponential stability available in the literature, particularly those developed through variation-of-parameters methods and nonlinear differential systems; see, for example, [19]. Motivated by this reference, we formulate their corresponding counterparts under the Initial Time Difference (ITD) framework.

Definition 2.1. We say that $\tilde{\mathcal{X}} = x(t, \rho_0, y_0)$ exhibits ITD stability relative to the solution $\mathcal{X}$, provided that $\forall \varepsilon > 0$, $\exists$ positive constants $\delta_1 = \delta_1(\rho_0, \varepsilon)$ and $\delta_2 = \delta_2(\rho_0, \varepsilon)$ such that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \varepsilon \quad (2.5)$$

when $\|y_0 - x_0\| < \delta_1$ and $|\rho_0 - t_0| < \delta_2$, $\forall t \geq \rho_0$.

Furthermore, if the constants δ_1 and δ_2 can be chosen independently of ρ_0 , then the solution $\tilde{\mathcal{X}}$ is uniformly ITD stable with respect to $\mathcal{X}$ for $t \geq \rho_0$.

Definition 2.2. We say that $\tilde{\mathcal{X}}$ is ITD Lipschitz stable relative to the solution $\mathcal{X}$ if and only if there exists a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|), \quad \text{for all } t \geq \rho_0 \quad (2.6)$$

Moreover, when the Lipschitz constant $\mathcal{L}^*$ does not depend on the initial time ρ_0 , the solution $\tilde{\mathcal{X}}$ is uniformly Lipschitz stable with regard to ITD, As compared with the corresponding solution $\mathcal{X}$ of system (2.1).

Definition 2.3. We say that $\tilde{\mathcal{X}}$ exhibits ITD asymptotically LS with respect to the solution $\mathcal{X}$, if there exists $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) \nu(t - \rho_0), \quad t \geq \rho_0, \quad (2.7)$$

where $\nu(t - \rho_0) \rightarrow 0$ as $t \rightarrow \infty$.

If the constant $\mathcal{L}^*$ is independent of ρ_0 , then the solution $\tilde{\mathcal{X}}$ is said to be uniformly asymptotically LS with respect to ITD.

Definition 2.4. We say that $\tilde{\mathcal{X}}$ exhibits ITD asymptotically LS in variation compared to $\mathcal{X}$ if the following two conditions are fulfilled:

(i) $\exists$ a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) \nu(t - \rho_0), \quad t \geq \rho_0 \quad (2.8)$$

where $\nu(t - \rho_0) \rightarrow 0$ as $t \rightarrow \infty$.

(ii) For every $\beta > 0$, $\exists$ a constant $\mathcal{N}^* > 0$ such that the fundamental matrix solution $\gamma(t, \rho_0, y_0)$ of the associated variational system (as defined in Theorem 3.1) satisfies:

$$\|\gamma(t, \rho_0, y_0)\| \leq \mathcal{N}^*, \quad \forall t \geq \rho_0, \text{ whenever } \|y_0\| \leq \beta \quad (2.9)$$

If the constant $\mathcal{L}^*$ does not depend on ρ_0 , then the system is said to exhibit uniform ITD asymptotic LS in variation.

Additionally, when $\mathcal{N}^*$ remains independent of ρ_0 , the system satisfies the condition of uniform ITD asymptotic Lipschitz stability in variation.

Definition 2.5. We say that $\tilde{\mathcal{X}}$ exhibits generalized exponential asymptotic LS under ITD, with respect to the solution $\mathcal{X}$ if $\exists$ a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) \exp[\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)], \quad t \geq \rho_0 \quad (2.10)$$

where the function $\mathcal{S}^*$ belongs to the class

$$\mathcal{K}^* = \{\mathcal{S}^* \in C[\mathbb{R}_+, \mathbb{R}_+] : \mathcal{S}^* \text{ is strictly increasing and } \mathcal{S}^*(0) = 0\}$$

and $\mathcal{S}^*(t) \rightarrow \infty$ as $t \rightarrow \infty$.

If the constant $\mathcal{L}^*$ is independent of ρ_0 , then the solution is called uniformly generalized exponentially asymptotically LS.

In the special case where $\tilde{p}(t) = \beta t$ for some $\beta > 0$, the system exhibits ITD exponential asymptotic Lipschitz stability.

In the subsequent section, we present several established lemmas from the literature [17] that are essential for the formulation of the main results.

3. VARIATION-OF-PARAMETERS IN NONLINEAR DIFFERENTIAL SYSTEMS UNDER INITIAL TIME DIFFERENCE

In this section, we examine the stability behavior of the solutions associated with systems (2.1) and (2.2) as detailed in the subsequent results.

Throughout the remainder of this paper, we shall use the following notations:

$\tilde{\mathcal{X}} = x(t, \rho_0, y_0)$ denotes the solution of (2.2) passing through (ρ_0, y_0) ,

$\bar{\mathcal{X}} = x(t, t_0, x_0)$ denotes the solution of (2.1) passing through (t_0, x_0) ,

$\mathcal{X}^* = x(t, \rho_0, x_0)$ denotes the solution of (2.2) passing through (ρ_0, x_0) .

Theorem 3.1. Assume that f and its partial derivative with respect to x , denoted by $\partial f / \partial x$, are continuous on the domain $\mathbb{R}_+ \times \mathbb{R}^{n \times n}$. Let $\tilde{\mathcal{X}}$ be the unique solution of (2.2). Define the matrix-valued function

$$\mathcal{H}(t, \rho_0, y_0) := \frac{\partial f}{\partial x}(t, x(t, \rho_0, y_0)).$$

Under these conditions, the following result holds:

- (i) The derivative $\gamma(t, \rho_0, y_0) := \frac{\partial \tilde{\mathcal{X}}}{\partial y_0}$ exists and satisfies the matrix differential equation

$$z' = \mathcal{H}(t, \rho_0, y_0)z \quad (3.11)$$

for $\gamma(\rho_0, \rho_0, y_0) = \mathcal{I}_{n \times n}$, $\mathcal{I}_{n \times n}$ means the identity matrix.

- (ii) The partial derivative $\frac{\partial \tilde{\mathcal{X}}}{\partial \rho_0}$ also exists and solves the same differential equation (3.11), with the initial condition

$$\frac{\partial x(\rho_0, \rho_0, y_0)}{\partial \rho_0} = -f(\rho_0, y_0),$$

and satisfies the identity

$$\frac{\partial \tilde{\mathcal{X}}}{\partial \rho_0} + \gamma(t, \rho_0, y_0) f(\rho_0, y_0) = 0, \quad \forall t \geq \rho_0.$$

(iii) Any solution $\tilde{\mathcal{X}}$ of the perturbed system (3.12) satisfies the integral identity known as Alekseev's formula:

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_{\rho_0}^t \gamma(t, s, y(s, \rho_0, y_0)) \mathcal{R}^*(s, y(s, \rho_0, y_0)) ds \quad (3.12)$$

where $\gamma(t, \rho_0, y_0) = \frac{\partial \tilde{\mathcal{X}}}{\partial y_0}$, and $t \geq \rho_0$.

Proof. A detailed justification of this theorem is available in the literature; see, for instance, [19, 18] for a comprehensive exposition. $\square$

This section focuses on establishing a formal link between the unperturbed differential models.

Theorem 3.2. Suppose that the nonlinear differential systems described by (2.1) and (2.2) possess unique solutions denoted by $\mathcal{X}$ and $\tilde{\mathcal{X}}$, respectively. Under these assumptions, the following integral identity holds:

$$\begin{aligned} \tilde{\mathcal{X}} = \mathcal{X} &+ \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0) \\ &- \int_{\rho_0}^t \left[\frac{\partial x}{\partial \rho_0}(t, s, x(s - \tilde{\zeta})) + \gamma(t, s, x(s - \tilde{\zeta})) f(s - \tilde{\zeta}, x(s - \tilde{\zeta})) \right] ds, \end{aligned} \quad (3.13)$$

for all $t \geq \rho_0 \geq 0$, where the interpolating function is defined by $\tilde{\Omega}(s) := sy_0 + (1 - s)x_0$, $s \in [0, 1]$.

Proof. Assume that both systems (2.1) and (2.2) admit unique solutions: $\mathcal{X}$, associated with the initial pair (t_0, x_0) , and $\tilde{\mathcal{X}}$, corresponding to (ρ_0, y_0) , where $t \geq \rho_0 \geq 0$ and $\rho_0 \in \mathbb{R}_+$.

For consistency of notation, define the auxiliary solution

$$\mathcal{X}^* := x(t, \rho_0, x_0),$$

that is, the solution evaluated at time t , starting from the shifted initial time ρ_0 with the original initial state x_0 .

Next, define the auxiliary function

$$\mathcal{E}(s) := x(t, s, x(s - \tilde{\zeta})), \quad s \in [\rho_0, t],$$

where $\tilde{\zeta} = \rho_0 - t_0$, and

$$x(\rho_0 - \tilde{\zeta}, t_0, x_0) = x_0.$$

Applying the chain rule, we obtain

$$\frac{d}{ds} \mathcal{E}(s) = \frac{\partial x}{\partial \rho_0}(t, s, x(s - \tilde{\zeta})) + \frac{\partial x}{\partial y_0}(t, s, x(s - \tilde{\zeta})) x'(s - \tilde{\zeta}). \quad (3.14)$$

Integrating (3.14) from ρ_0 to t , we obtain

$$\mathcal{X} = \mathcal{X}^* + \int_{\rho_0}^t \left[\frac{\partial x}{\partial \rho_0}(t, s, x(s - \tilde{\zeta})) + \gamma(t, s, x(s - \tilde{\zeta})) f(s - \tilde{\zeta}, x(s - \tilde{\zeta})) \right] ds. \quad (3.15)$$

Let us now consider

$$\mathcal{S}^*(r) := x(t, \rho_0, \tilde{\Omega}(r)), \quad r \in [0, 1],$$

where

$$\tilde{\Omega}(r) := ry_0 + (1 - r)x_0.$$

Differentiating with respect to r , we get

$$\frac{d}{dr}\mathcal{S}^*(r) = \frac{\partial x}{\partial y_0}(t, \rho_0, \tilde{\Omega}(r))(y_0 - x_0). \quad (3.16)$$

Integrating (3.16) over $r \in [0, 1]$, we obtain

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(r)) dr (y_0 - x_0). \quad (3.17)$$

Finally, substituting (3.17) into (3.15), we obtain the desired identity (3.13). This completes the proof. $\square$

Remark 3.1. Suppose that the nonlinear differential system (2.2) admits unique solutions $\mathcal{X}^*$ and $\tilde{\mathcal{X}}$, respectively. In the special case when $\tilde{\zeta} = 0$, these solutions are related by the following integral expression:

$$\tilde{\mathcal{X}} = \mathcal{X}^* + \int_0^1 \gamma(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0), \quad t \geq \rho_0, \quad (3.18)$$

where the function $\tilde{\Omega}(s)$ is given by the convex combination $\tilde{\Omega}(s) = sy_0 + (1 - s)x_0$, with $s \in [0, 1]$.

Equation (3.18) thus characterizes the dependence between solutions of the same unperturbed system (2.2) that differ only in their initial values.

Theorem 3.3. Let the system (2.1) satisfy the assumptions stated in Lemma 2.1, and let $\bar{\mathcal{X}}$ denote its solution. Then, for any solution $\tilde{\mathcal{X}}$ of (2.2) and the corresponding solution $\mathcal{X}$, the following relation holds:

$$\begin{aligned} \tilde{\mathcal{X}} = \mathcal{X} &- \int_{\rho_0}^t \frac{\partial y}{\partial \rho_0} \left(t, \nu, x(\nu - \tilde{\zeta}) \right) d\nu \\ &- \int_{\rho_0}^t \frac{\partial y}{\partial y_0} \left(t, \nu, x(\nu - \tilde{\zeta}) \right) f \left(\nu - \tilde{\zeta}, x(\nu - \tilde{\zeta}) \right) d\nu \\ &+ \int_0^1 \frac{\partial y}{\partial y_0} \left(t, \rho_0, \tilde{\Omega}(s) \right) ds \cdot (y_0 - x_0), \end{aligned} \quad (3.19)$$

where $\tilde{\Omega}(s) := sy_0 + (1 - s)x_0$ for $s \in [0, 1]$.

Proof. Let $\tilde{\mathcal{X}}$ be the unique solution of the unperturbed system (2.2), defined for $t \geq \rho_0$, and let $\mathcal{X}$ be the solution of system (2.1) corresponding to the initial value (t_0, x_0) , where $t_0 \in \mathbb{R}_+$. Assume further that all systems admit unique solutions in their respective domains.

Define the auxiliary function

$$\bar{\mathcal{E}}(\nu) := x(t, \nu, x(\nu - \tilde{\zeta})), \quad \text{for } \nu \in [\rho_0, t],$$

with the condition $x(\rho_0 - \tilde{\zeta}, t_0, x_0) = x_0$. By applying the chain rule, the total derivative of $\bar{\mathcal{E}}$ with respect to ν yields:

$$\begin{aligned}
\frac{d}{d\nu}\bar{\mathcal{E}}(\nu) &= \frac{\partial x}{\partial \nu}(t, \nu, x(\nu - \tilde{\zeta})) + \frac{\partial x}{\partial x}(t, \nu, x(\nu - \tilde{\zeta})) \cdot \frac{dx(\nu - \tilde{\zeta})}{d\nu} \\
&= \frac{\partial x}{\partial \nu}(t, \nu, x(\nu - \tilde{\zeta})) + \frac{\partial x}{\partial x}(t, \nu, x(\nu - \tilde{\zeta}))f(\nu - \tilde{\zeta}, x(\nu - \tilde{\zeta})) \\
&=: \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}).
\end{aligned} \tag{3.20}$$

By integrating (3.20),

$$\mathcal{X} = x(t, \rho_0, x_0) + \int_{\rho_0}^t \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}) d\nu. \tag{3.21}$$

Now define another auxiliary function

$$\tilde{\mathcal{S}}^*(s) := x(t, \rho_0, \tilde{\Omega}(s)), \quad \text{where } \tilde{\Omega}(s) := sy_0 + (1-s)x_0, \quad s \in [0, 1].$$

Then, the derivative with respect to s is

$$\frac{d}{ds}\tilde{\mathcal{S}}^*(s) = \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) \cdot (y_0 - x_0). \tag{3.22}$$

Integrating (3.22) over the interval $s \in [0, 1]$ yields:

$$\tilde{\mathcal{X}} = x(t, \rho_0, x_0) + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0). \tag{3.23}$$

Combining the expressions in (3.21) and (3.23), we deduce the following identity:

$$\begin{aligned}
\tilde{\mathcal{X}} &= \mathcal{X} + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0) \\
&\quad - \int_{\rho_0}^t \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}) d\nu.
\end{aligned} \tag{3.24}$$

The proof is completed. $\square$

Remark 3.2. Let $\bar{\mathcal{X}}$ denote the solution of the unperturbed system (2.1). Then, for any solution $\tilde{\mathcal{X}}$ of system (2.2), the corresponding solution $\mathcal{X}$ satisfies the following integral representation:

$$\begin{aligned}
\tilde{\mathcal{X}} &= \mathcal{X} + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0) \\
&\quad - \int_{\rho_0}^t \tilde{f}(\nu, x(\nu - \tilde{\zeta}), \tilde{\zeta}) d\nu,
\end{aligned} \tag{3.25}$$

where $\tilde{\Omega}(s) := sy_0 + (1-s)x_0$ for $s \in [0, 1]$.

Remark 3.3. Suppose that the solution $\tilde{\mathcal{X}}$ of the non-linear unperturbed system (2.2) possesses continuous partial derivatives with respect to both ρ_0 and y_0 , that is, $\frac{\partial x}{\partial \rho_0}$ and $\frac{\partial x}{\partial x_0}$ exist and are continuous in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n$. Assume further that the systems (2.2) admits a unique solution $\tilde{\mathcal{X}}$ for all $t \geq \rho_0$, initialized at the same time.

For $\tilde{\zeta} = 0$, the solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}^*$ satisfy the integral relation:

$$\begin{aligned}\tilde{\mathcal{X}} - \mathcal{X}^* &= - \int_{\rho_0}^t \left[\frac{\partial x}{\partial \rho_0}(t, \nu, x(\nu)) + \frac{\partial x}{\partial x_0}(t, \nu, x(\nu)) \mathcal{F}^*(\nu, x(\nu)) \right] d\nu \\ &\quad + \int_0^1 \frac{\partial x}{\partial x_0}(t, \rho_0, \tilde{\Omega}(s)) ds \cdot (y_0 - x_0),\end{aligned}\tag{3.26}$$

where $\tilde{\Omega}(s) := sy_0 + (1-s)x_0$ for $s \in [0, 1]$, and $t \geq \rho_0$.

This identity characterizes the deviation between (2.1) and (2.2), when the two trajectories begin at the same initial time but from different spatial positions.

Theorem 3.4. Assume that the system (2.1) satisfies the assumptions stated in Theorem 3.2, and let $\tilde{\mathcal{X}}$ denote its unique solution. Then, for any solution $\tilde{\mathcal{X}}$ of system (2.2), the difference between $\tilde{\mathcal{X}}$ and the shifted unperturbed solution $\mathcal{X}$ can be expressed as:

$$\tilde{\mathcal{X}} - \mathcal{X} = x(t, \rho_0, y_0 - x_0) + \int_{\rho_0}^t \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}) ds,\tag{3.27}$$

where $\overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta})$ represents the integrand that depends on the perturbed dynamics, and $\tilde{d}(s)$ is a trajectory determined by the system's state evolution.

Proof. Assume that all involved systems admit unique solutions in their respective domains. Define

$$\mathcal{S}^*(s) := x(t, s, \tilde{d}(s)), \quad \text{where} \quad \tilde{d}(s) := x(s, \rho_0, y_0) - x(s - \tilde{\zeta}, t_0, x_0), \quad s \in (\rho_0, t).$$

Differentiating $\mathcal{S}^*(s)$ with respect to s , we compute:

$$\begin{aligned}\frac{d}{ds} \mathcal{S}^*(s) &= \frac{\partial x}{\partial s}(t, s, \tilde{d}(s)) + \frac{\partial x}{\partial z}(t, s, \tilde{d}(s)) \cdot \frac{d\tilde{d}(s)}{ds} \\ &= \frac{\partial x}{\partial s}(t, s, \tilde{d}(s)) + \frac{\partial x}{\partial z}(t, s, \tilde{d}(s)) \cdot \mathcal{H}(s, \tilde{d}(s), \tilde{\zeta}) \\ &=: \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}),\end{aligned}\tag{3.28}$$

where the function $\mathcal{H}(s, \tilde{d}(s), \tilde{\zeta})$ is defined by

$$\mathcal{H}(s, \tilde{d}(s), \tilde{\zeta}) = \mathcal{F}^*(s, \tilde{d}(s) + x(s - \tilde{\zeta})) - f(s - \tilde{\zeta}, x(s - \tilde{\zeta})).$$

Integrating both sides of (3.28) over the interval $[\rho_0, t]$, and noting that $\tilde{d}(t) = \tilde{\mathcal{X}} - \mathcal{X}$, we obtain the following representation:

$$\tilde{\mathcal{X}} - \mathcal{X} = x(t, \rho_0, y_0 - x_0) + \int_{\rho_0}^t \overline{\mathcal{H}}(s, \tilde{d}(s), \tilde{\zeta}) ds.\tag{3.29}$$

This completes the proof. $\square$

In recent studies, the qualitative dynamics of nonlinear differential systems involving discrepancies in both initial time and position have garnered increasing attention [14, 20]. Notably, the concept of Lipschitz stability employed in this context is defined not relative to the trivial solution, but in reference to the corresponding unperturbed system, which itself incorporates deviations in both the initial state and starting time. These generalized stability notions concerning ITDs are employed in Section 4 to derive further analytical results.

4. ON LS ANALYSIS OF NONLINEAR DIFFERENTIAL MODELS WITH INITIAL TIME DIFFERENCE

In what follows, we demonstrate that, in the nonlinear setting, LS under ITD provides a stronger notion of stability than ITD stability alone, regardless of whether it is uniform or non-uniform.

Theorem 4.1 (Lipschitz stability). *For nonlinear differential systems exhibiting an ITD, Lipschitz stability with respect to the reference solution implies ITD stability—whether uniform or non-uniform—for the corresponding nonlinear differential system.*

Proof. We consider the notion of Lipschitz stability under ITD as given in Definition 2.1. Suppose that the solution $\tilde{\mathcal{X}}$ of system (2.2), initiated at (ρ_0, y_0) , satisfies the Lipschitz stability criterion with respect to the reference solution $\mathcal{X}$ of system (2.1), where $\tilde{\zeta} = \rho_0 - t_0 \geq 0$ and $t \geq \rho_0$.

Then, by definition, $\exists$ a constant $\mathcal{L}^* = \mathcal{L}^*(\rho_0) > 0$ such that:

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|), \quad \forall t \geq \rho_0. \quad (4.30)$$

Let $\varepsilon > 0$ be arbitrary. If we select

$$\|y_0 - x_0\| < \frac{\varepsilon}{2L} =: \delta_1(\varepsilon, \rho_0), \quad \text{and} \quad |\rho_0 - t_0| < \frac{\varepsilon}{2L} =: \delta_2(\varepsilon, \rho_0),$$

then it follows that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| < \varepsilon, \quad \forall t \geq \rho_0,$$

which confirms that $\tilde{\mathcal{X}}$ is stable w.r.t ITD to the unperturbed solution $\mathcal{X}$.

Moreover, if the constant $\mathcal{L}^*$ is independent of ρ_0 , then the associated thresholds δ_1 and δ_2 are also independent of ρ_0 , yielding uniform stability in the sense of ITD. $\square$

Theorem 4.2. *ITD stability in nonlinear differential systems does not necessarily imply Lipschitz stability with respect to initial time shifts.*

For a rigorous discussion and proof, see [19, 20]. A specific counterexample illustrating the failure of this implication in the special case $y_0 = x_0$ and $\rho_0 = t_0$ is presented in [2], Example 1.4.

Theorem 4.3. *Consider the nonlinear differential systems (2.1) and (2.2), which admit unique solutions $\bar{\mathcal{X}}$ and $\tilde{\mathcal{X}}$, respectively, defined for $t \geq \rho_0$, where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Assume that the following conditions hold:*

- (i) *There exist finite, nonnegative constants $\mathcal{K}^*(\rho_0)$ and $\mathcal{M}^*(\rho_0)$ such that*

$$\int_{\rho_0}^{\infty} \mathcal{N}^*(s) ds \leq \mathcal{K}^*(\rho_0), \quad \text{and} \quad \exp \left(\int_{\rho_0}^{\infty} \tilde{\mu}(s) ds \right) \leq \mathcal{M}^*(\rho_0),$$

where $\mathcal{N}^(s), \tilde{\mu}(s) \in C([0, \infty), \mathbb{R}_+)$.*

- (ii) *The function f is Lipschitz continuous and*

$$\begin{aligned} & \left\| f \left(t, \tilde{b}(t, \rho_0, \tilde{b}_0) + \mathcal{X} \right) - f \left(t - \tilde{\zeta}, \mathcal{X} \right) \right\| \\ & \leq \mu(t) \|\tilde{b}(t)\| + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} N(t), \end{aligned} \quad (4.31)$$

where $\tilde{b}(t, \rho_0, \tilde{b}_0) := \tilde{\mathcal{X}} - \mathcal{X}$, and $\tilde{b}_0 := \tilde{b}(\rho_0) = y_0 - x_0$.

Then $\tilde{\mathcal{X}}$ is LS under ITD with respect to the reference solution $\mathcal{X}$.

Proof. Let $\tilde{\mathcal{X}}$ and $\mathcal{X}$ be the unique solutions to (2.2) and (2.1), respectively. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}, \quad \tilde{b}_0 := \tilde{b}(\rho_0) = y_0 - x_0.$$

Applying the assumptions in (i) and (ii) of Theorem 4.3, we obtain the inequality:

$$\|\tilde{b}(t)\| \leq \|\tilde{b}_0\| + \int_{\rho_0}^t \left[\tilde{\mu}(s) \|\tilde{b}(s)\| + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} \mathcal{N}^*(s) \right] ds. \quad (4.32)$$

We define

$$\tilde{d}(t) := \|\tilde{b}(t)\|, \quad z(\rho_0) := \|\tilde{b}_0\|.$$

Then inequality (4.32) becomes:

$$\tilde{d}(t) \leq z(\rho_0) + \int_{\rho_0}^t \tilde{\mu}(s) \tilde{d}(s) ds + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} \int_{\rho_0}^t \mathcal{N}^*(s) ds.$$

By applying Gronwall's inequality, we deduce:

$$\tilde{d}(t) \leq \left(z(\rho_0) + \frac{|\tilde{\zeta}|}{\mathcal{K}^*(\rho_0)} \int_{\rho_0}^t \mathcal{N}^*(s) ds \right) \exp \left(\int_{\rho_0}^t \tilde{\mu}(s) ds \right).$$

So, we have

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{M}^*(\rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|),$$

which confirms LS under ITD.

Furthermore, if $\mathcal{M}^*(\rho_0)$ is uniformly bounded with respect to ρ_0 , then the same inequality holds with a constant independent of ρ_0 , implying uniform Lipschitz stability. $\square$

Theorem 4.4. Consider the nonlinear differential systems (2.1) and (2.2), which admit unique solutions given by:

- $\tilde{\mathcal{X}}$, defined for $t \geq t_0$, corresponding to the unperturbed system (2.1);
- $\tilde{\mathcal{X}}$, defined for $t \geq \rho_0$, corresponding to the shifted system (2.2);

where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$ and $\mathcal{X}$ are well-defined in $t \geq \rho_0$. Assume further that the following hold:

- The solution $\tilde{\mathcal{X}}$ is uniformly LS under ITD with respect to $\mathcal{X}$;
- $\exists \mathcal{M}_1^* > 0$ and $\lambda_1 \in C([0, \infty), \mathbb{R}_+)$, such that for the fundamental matrix $\gamma(t, s, x(s))$ as in Theorem 3.2, the following bounds hold for $\|x(s)\| \leq \beta$

$$\|\gamma(t, s, x(s))\| \leq \mathcal{M}_1^*, \quad \|\mathcal{R}^*(s, x(s))\| \leq \lambda_1(s) \|x(s)\|.$$

Then $\tilde{\mathcal{X}}$ is uniformly LS in variation with respect to the solution $\mathcal{X}$, under ITD.

Proof. We start from the representation given in Theorem 3.1 together with (3.12). For $t \geq \rho_0$, the difference between the solutions can be written as

$$\tilde{\mathcal{X}}(t) - \mathcal{X}(t) = \tilde{\mathcal{X}}(\rho_0) - \mathcal{X}(\rho_0) + \int_{\rho_0}^t \gamma(t, s, \tilde{\mathcal{X}}(s)) \mathcal{R}^*(s, \tilde{\mathcal{X}}(s)) ds. \quad (4.33)$$

Taking norms and applying the triangle inequality, we obtain

$$\begin{aligned} \|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| &\leq \|\tilde{\mathcal{X}}(\rho_0) - \mathcal{X}(\rho_0)\| \\ &\quad + \int_{\rho_0}^t \|\gamma(t, s, \tilde{\mathcal{X}}(s))\| \|\mathcal{R}^*(s, \tilde{\mathcal{X}}(s))\| ds. \end{aligned} \quad (4.34)$$

From Theorem 4.3, the initial perturbation satisfies

$$\|\tilde{\mathcal{X}}(\rho_0) - \mathcal{X}(\rho_0)\| \leq \mathcal{L}^*(\rho_0)(\|y_0 - x_0\| + |\rho_0 - t_0|).$$

Using assumption (ii), we have

$$\|\gamma(t, s, x)\| \leq \mathcal{M}_1^*, \quad \|\mathcal{R}^*(s, x)\| \leq \lambda_1(s)\|x\|,$$

and assume boundedness of the solutions:

$$\|\tilde{\mathcal{X}}(s)\| \leq \beta, \quad \|\mathcal{X}(s)\| \leq \beta.$$

Hence,

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C + \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) \|\tilde{\mathcal{X}}(s)\| ds, \quad (4.35)$$

where

$$C := \mathcal{L}^*(\rho_0)(\|y_0 - x_0\| + |\rho_0 - t_0|).$$

Using

$$\|\tilde{\mathcal{X}}(s)\| \leq \|\tilde{\mathcal{X}}(s) - \mathcal{X}(s)\| + \|\mathcal{X}(s)\| \leq \mathcal{A}^*(s) + \beta,$$

we obtain

$$\mathcal{A}^*(t) \leq C + \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) \mathcal{A}^*(s) ds + \beta \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) ds, \quad (4.36)$$

where

$$\mathcal{A}^*(t) := \|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\|.$$

Setting

$$\hat{C}(t) := C + \beta \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) ds,$$

we arrive at

$$\mathcal{A}^*(t) \leq \hat{C}(t) + \mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) \mathcal{A}^*(s) ds.$$

Applying Gronwall's inequality yields

$$\mathcal{A}^*(t) \leq \hat{C}(t) \exp\left(\mathcal{M}_1^* \int_{\rho_0}^t \lambda_1(s) ds\right), \quad t \geq \rho_0.$$

Since $\lambda_1 \in L^1([\rho_0, t])$, it follows that there exists a constant $\mathcal{M}^* > 0$ such that

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq \mathcal{M}^*(\|y_0 - x_0\| + |\rho_0 - t_0|),$$

which shows that $\tilde{\mathcal{X}}$ is uniformly LS-stable with respect to $\mathcal{X}$ under ITD. $\square$

Theorem 4.5. *Let the nonlinear differential systems (2.1) and (2.2), which admit unique solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}$, respectively, defined for $t \geq \rho_0$, with $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Moreover, define $\tilde{b}(t, \rho_0, \tilde{b}_0) := \tilde{\mathcal{X}} - \mathcal{X}$. Assume that the following conditions hold:*

(i) If $\mathcal{S}^* \in \bar{K}$, where

$$\bar{K} := \{\mathcal{S}^* \in C([0, \infty), \mathbb{R}_+) \mid \mathcal{S}^* \text{ strictly increasing, } \mathcal{S}^*(0) = 0\}$$

$\lim_{t \rightarrow \infty} \tilde{s}(t) = \infty$, and $\tilde{s}$ is an antiderivative of $\mathcal{S}^*$, then

$$\lim_{h \rightarrow 0^+} \inf \frac{\|\tilde{b}(t+h) - \tilde{b}(t)\|}{h} \leq \lim_{h \rightarrow 0^-} \frac{\|\tilde{b}(t) + [f(t, x(t-\tilde{\zeta}) + \tilde{b}(t)) - f(t, x(t-\tilde{\zeta}))]h - \tilde{b}(t)\|}{h} \leq -\tilde{s}(t)\|\tilde{b}(t)\|. \quad (4.37)$$

(ii) The nonlinear f is Lipschitz continuous, and satisfies

$$\|f(t, x(t-\tilde{\zeta})) - f(t-\tilde{\zeta}, x(t-\tilde{\zeta}))\| \leq \frac{|\tilde{\zeta}|}{\mathcal{L}_{2}^*(\rho_0)} \mathcal{L}_1^*(t),$$

where $\mathcal{L}_1^* \in C([0, \infty), \mathbb{R}_+)$ and $\mathcal{L}_2^* \in C(\mathbb{R}_+, \mathbb{R}_+)$ are such that

$$\int_{\rho_0}^{\infty} \exp(\mathcal{S}^*(u) - \mathcal{S}^*(\rho_0)) \mathcal{L}_1^*(u) du = \mathcal{L}_2^*(\rho_0).$$

Then, $\tilde{\mathcal{X}}$ is generalized exponentially asymptotically LS with regard to the solution $\mathcal{X}$.

Proof. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}, \quad \mathcal{A}^*(t) := \|\tilde{b}(t)\|.$$

Under assumption (i),

$$\mathcal{A}_+^{*'}(t) \leq -\tilde{p}(t)\mathcal{A}^*(t) + \frac{|\tilde{\zeta}|}{\mathcal{L}_2^*(\rho_0)} \mathcal{L}_1^*(t), \quad \mathcal{A}^*(\rho_0) = \|y_0 - x_0\|.$$

This is a linear inhomogeneous differential inequality. Applying the variation-of-parameters method with the integrating factor

$$\exp\left(\int_{\rho_0}^t \mathcal{S}^*(s) ds\right),$$

we obtain

$$\mathcal{A}^*(t) \leq \exp\left(-\int_{\rho_0}^t \mathcal{S}^*(s) ds\right) \left[\|y_0 - x_0\| + \frac{|\tilde{\zeta}|}{\mathcal{L}_2^*(\rho_0)} \int_{\rho_0}^t \exp\left(\int_{\rho_0}^u \mathcal{S}^*(s) ds\right) \mathcal{L}_1^*(u) du \right].$$

Using assumption (ii), it follows that

$$\int_{\rho_0}^{\infty} \exp(\mathcal{S}^*(u) - \mathcal{S}^*(\rho_0)) \mathcal{L}_1^*(u) du = \mathcal{L}_2^*(\rho_0),$$

and hence

$$\mathcal{A}^*(t) \leq \exp(\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)) (\|y_0 - x_0\| + |\tilde{\zeta}|).$$

Therefore,

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \exp(\mathcal{S}^*(\rho_0) - \mathcal{S}^*(t)) (\|y_0 - x_0\| + |\rho_0 - t_0|), \quad t \geq \rho_0.$$

Hence, $\tilde{\mathcal{X}}$ is exponentially asymptotically LS under ITD with respect to the reference solution $\mathcal{X}$, in the sense of Definition 2.5. $\square$

Remark 4.1. Let the systems (2.2) and (2.1) admit unique solutions $\tilde{\mathcal{X}}$ and $\mathcal{X}$, respectively, for $t \geq \rho_0$, where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Define

$$\tilde{b}(t) := \tilde{\mathcal{X}} - \mathcal{X}.$$

(i) $\exists$ a constant $\theta > 0$ such that $\tilde{b}(t)$ satisfies the following

$$\lim_{h \rightarrow 0^+} \sup \frac{\|\tilde{b}(t+h)\| - \|\tilde{b}(t)\|}{h} \quad (4.38)$$

$$\leq \lim_{h \rightarrow 0^-} \frac{\|\tilde{b}(t) + [f(t, x(t-\tilde{\zeta}) + \tilde{b}(t)) - f(t, x(t-\tilde{\zeta}))]h\| - \|\tilde{b}(t)\|}{h} \leq -\theta \|\tilde{b}(t)\|. \quad (4.39)$$

(ii) The nonlinear f is LS in time, and satisfies

$$\|f(t, x(t-\tilde{\zeta})) - f(t-\tilde{\zeta}, x(t-\tilde{\zeta}))\| \leq \frac{|\tilde{\zeta}|}{\mathcal{L}^*_{2}(\rho_0)} \mathcal{L}^*_{1}(t),$$

for some functions $\mathcal{L}^*_{1} \in C([0, \infty), \mathbb{R}_+)$ and $\mathcal{L}^*_{2} \in C(\mathbb{R}_+, \mathbb{R}_+)$ satisfying

$$\int_{\rho_0}^{\infty} \exp(\theta(s - \rho_0)) \mathcal{L}^*_{1}(s) ds = \mathcal{L}^*_{2}(\rho_0).$$

Under these conditions, the solution $\tilde{\mathcal{X}}$ is exponentially asymptotically LS w.r.t $\mathcal{X}$.

Proof. Define the difference function $\tilde{\xi}(t) := \tilde{\mathcal{X}} - \mathcal{X}$, and set

$$\mathcal{A}^*(t) := \|\tilde{b}(t)\|.$$

Under the assumptions, $\mathcal{A}^*(t)$ satisfies the inequality

$$D^+ \mathcal{A}^*(t) \leq -\theta \mathcal{A}^*(t) + \frac{|\tilde{\zeta}|}{\mathcal{L}^*_{2}(\rho_0)} \mathcal{L}^*_{1}(t), \quad T(\rho_0) = \|y_0 - x_0\|.$$

This leads to a linear differential inequality, whose solution yields

$$\mathcal{A}^*(t) \leq e^{-\theta(t-\rho_0)} \left[T(\rho_0) + \frac{|\tilde{\zeta}|}{\mathcal{L}^*_{2}(\rho_0)} \int_{\rho_0}^{\infty} e^{\theta(s-\rho_0)} \mathcal{L}^*_{1}(s) ds \right].$$

By applying assumption (ii), namely

$$\int_{\rho_0}^{\infty} e^{\theta(s-\rho_0)} \mathcal{L}^*_{1}(s) ds = \mathcal{L}^*_{2}(\rho_0),$$

we obtain

$$\mathcal{A}^*(t) \leq e^{-\theta(t-\rho_0)} [\|y_0 - x_0\| + |\tilde{\zeta}|],$$

which implies

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq e^{-\theta(t-\rho_0)} [\|y_0 - x_0\| + |\rho_0 - t_0|].$$

So, we get

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq e^{-\theta(t-\rho_0)} [\|y_0 - x_0\| + |\rho_0 - t_0|].$$

Therefore, $\exists$ a constant $\mathcal{L}^*$ such that

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^* (\|y_0 - x_0\| + |\rho_0 - t_0|),$$

and the exponential decay is governed by the constant $\mathcal{M}^* := \mathcal{L}^* > 0$. By applying Definition 2.5 with $\tilde{p}(t) = \theta t$ for some constant $\theta > 0$, $\tilde{\mathcal{X}}$ is exponentially asymptotically LS under ITD, with respect to the reference trajectory $\mathcal{X}$. $\square$

Theorem 4.6. *Let the nonlinear differential systems (2.1) and (2.2) admit unique solutions $\tilde{\mathcal{X}}$ and $\tilde{\mathcal{X}}$, corresponding to the initial values (t_0, x_0) and (ρ_0, y_0) , respectively. Let $\mathcal{X}$ be well-defined for $t \geq \rho_0$, where $\tilde{\zeta} := \rho_0 - t_0 \geq 0$. Suppose that the assumptions of Theorem 4.4 hold:*

- (i) *The solution $\tilde{\mathcal{X}}$ to the unperturbed system (2.2) is exponentially asymptotically LS under ITD, with respect to the trajectory $\mathcal{X}$.*
- (ii) *$\exists$ a constant $\mathcal{N}^* > 0$ and $\lambda \in C([0, \infty), \mathbb{R}_+)$ such that*

$$\|\gamma(t, s, x(s))\| \leq \mathcal{N}^* \exp[-\beta(t-s)] \quad \text{and} \quad \|\mathcal{R}^*(s, x(s))\| \leq \lambda(s)\|x(s)\|,$$

where $\|x(s)\| \leq \beta$ for some $\beta > 0$.

Then, the solution $\tilde{\mathcal{X}}$ of the unperturbed system (2.2) is exponentially asymptotically LS in variation under ITD, with respect to the solution $\mathcal{X}$.

Proof. Under the assumptions of Theorem 4.4 and hypotheses (i)–(ii), we consider the following decomposition:

$$\begin{aligned} \|\tilde{\mathcal{X}} - \mathcal{X}\| &\leq \|\tilde{b}(t, \rho_0, \tilde{b}_0)\| + \int_{\rho_0}^t \|\gamma(t, s, x(s))\| \cdot \|\mathcal{R}^*(s, x(s))\| ds \\ &\leq \mathcal{M}^* (\|y_0 - x_0\| + |\rho_0 - t_0|) e^{-\beta(t-\rho_0)} + \int_{\rho_0}^t \mathcal{N}^* \lambda(s) e^{-\beta(t-s)} \|x(s)\| ds, \end{aligned}$$

where $\tilde{b}(t) = \tilde{\mathcal{X}} - \mathcal{X}$, and $\|\gamma(t, s, x(s))\| \leq N e^{-\beta(t-s)}$ by assumption.

Now, define

$$\mathcal{A}^*(t) := e^{-\beta(t-\rho_0)} \|\tilde{\mathcal{X}} - \mathcal{X}\|.$$

Then,

$$\mathcal{A}^*(t) \leq \mathcal{M}^* (\|y_0 - x_0\| + |\rho_0 - t_0|) + \int_{\rho_0}^t \mathcal{N}^* \lambda(s) \mathcal{A}^*(s) ds.$$

By Gronwall's inequality, this yields

$$\mathcal{A}^*(t) \leq [\mathcal{M}^* (\|y_0 - x_0\| + |\rho_0 - t_0|)] \cdot \exp \left(\mathcal{N}^* \int_{\rho_0}^t \lambda(s) ds \right).$$

Then we have

$$\|\tilde{\mathcal{X}} - \mathcal{X}\| \leq \mathcal{L}^*(\beta, \rho_0) (\|y_0 - x_0\| + |\rho_0 - t_0|) e^{-\beta(t-\rho_0)},$$

where the constant $\mathcal{L}^*(\beta, \rho_0)$ is given by

$$\mathcal{L}^*(\beta, \rho_0) := \mathcal{M}^* \cdot \exp \left(\mathcal{N}^* \int_{\rho_0}^{\infty} \lambda(s) ds \right).$$

Thus, $\tilde{\mathcal{X}}$ is exponentially asymptotically LS in variation with respect to the reference trajectory $\mathcal{X}$, under ITD. $\square$

Theorem 4.7. *If a nonlinear differential system exhibits exponentially asymptotic (uniform) LS under ITD, then the corresponding nonlinear differential system is also asymptotically (uniform) LS under ITD.*

Proof. The result follows directly from Definition 2.3, which characterizes asymptotic LS in the presence of an initial time shift. Since exponential decay implies asymptotic behavior, the claim is immediate. For brevity, the detailed steps are omitted. $\square$

Theorem 4.8. *If a nonlinear differential system satisfies exponentially asymptotic (uniform) LS in variation under ITD, then it also satisfies asymptotic (uniform) LS in variation under ITD.*

Proof. This result is a direct consequence of Definition 2.4, which characterizes LS in variation with respect to initial time shifts. Since exponential decay implies asymptotic decay, the conclusion follows immediately. Detailed verification is omitted for brevity. $\square$

5. NUMERICAL VALIDATION OF THE MAIN THEORETICAL RESULTS

In this section, we numerically illustrate the theoretical results obtained for Lipschitz stability under Initial Time Difference (ITD). The purpose of these examples is to compare two solutions of the *same unperturbed differential system* corresponding to different initial states and shifted initial times.

More precisely, let

$$\mathcal{X}(t) = \mathcal{X}(t; t_0, x_0)$$

and

$$\tilde{\mathcal{X}}(t) = \mathcal{X}(t; \rho_0, y_0),$$

where

$$\rho_0 = t_0 + \zeta.$$

Then the theoretical results imply that there exists a constant $C > 0$ such that

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C(\|y_0 - x_0\| + |\zeta|), \quad t \geq \rho_0.$$

The following examples confirm this estimate numerically.

EXAMPLE 1: PREDATOR–PREY SYSTEM

Consider the nonlinear unperturbed system

$$\begin{cases} \frac{dx}{dt} = x(1 - x) - \alpha xy, \\ \frac{dy}{dt} = -\beta y + \gamma xy, \end{cases}$$

where $x(t)$ and $y(t)$ denote prey and predator populations.

Choose parameters

$$\alpha = 0.5, \quad \beta = 0.3, \quad \gamma = 0.2.$$

Let the reference solution be generated by

$$t_0 = 0, \quad x(0) = 0.60, \quad y(0) = 0.40.$$

Let the second solution of the same system be generated by

$$\rho_0 = 0.05, \quad \tilde{x}(0.05) = 0.61, \quad \tilde{y}(0.05) = 0.41.$$

Hence,

$$\|y_0 - x_0\| + |\zeta| = \sqrt{(0.01)^2 + (0.01)^2} + 0.05.$$

Therefore,

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C \left(\sqrt{2} \cdot 0.01 + 0.05 \right).$$

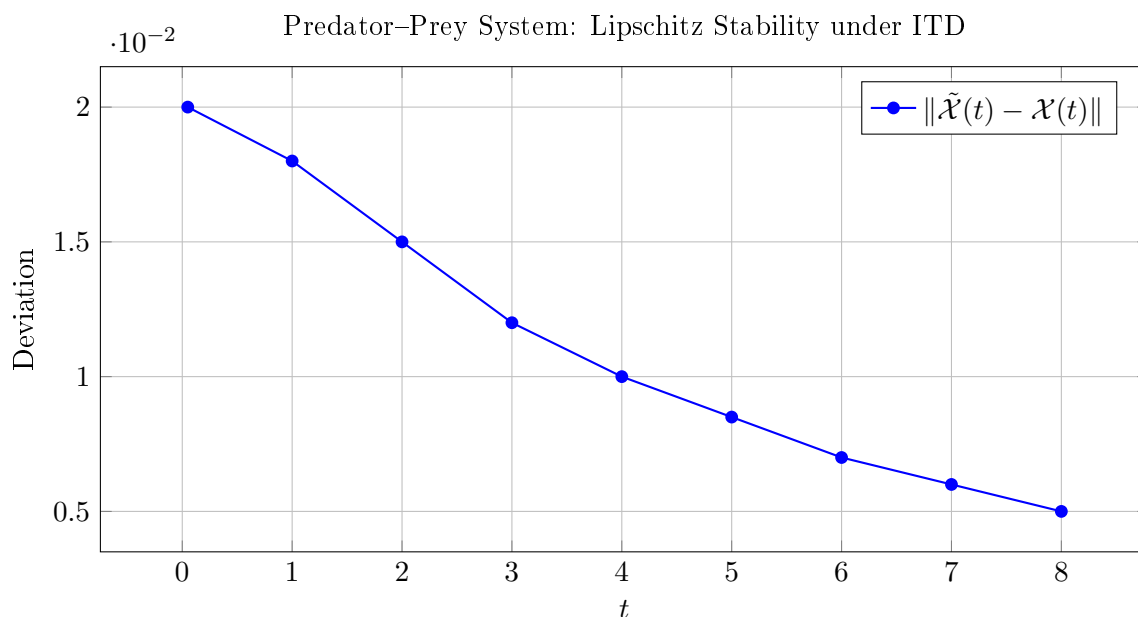


FIGURE 1. Deviation between two solutions of the same unperturbed predator–prey system under different initial data and shifted initial times.

EXAMPLE 2: SIR EPIDEMIC MODEL

Consider the unperturbed SIR system

$$\begin{cases} S' = -\beta SI, \\ I' = \beta SI - \gamma I, \\ R' = \gamma I. \end{cases}$$

Choose

$$\beta = 0.40, \quad \gamma = 0.20.$$

Let the reference solution satisfy

$$S(0) = 0.90, \quad I(0) = 0.10, \quad R(0) = 0.$$

Let the second solution of the same system satisfy

$$\rho_0 = 0.03,$$

$$\tilde{S}(0.03) = 0.89, \quad \tilde{I}(0.03) = 0.11, \quad \tilde{R}(0.03) = 0.$$

Then

$$\|\tilde{\mathcal{X}}(t) - \mathcal{X}(t)\| \leq C (\|y_0 - x_0\| + 0.03).$$

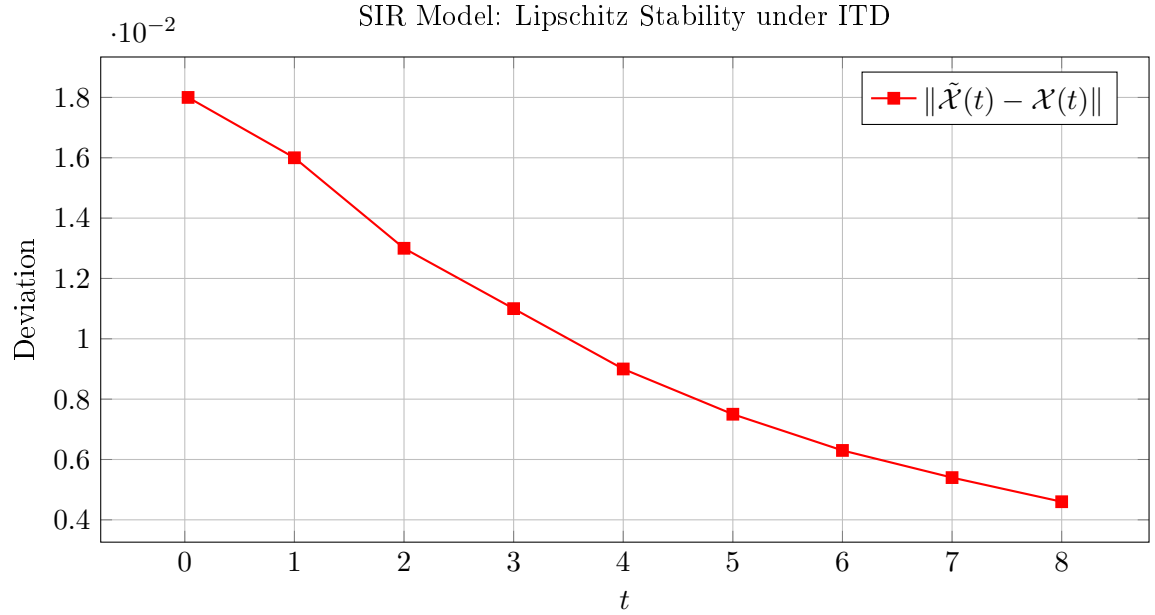


FIGURE 2. Deviation between two solutions of the same unperturbed SIR system with shifted initial times.

EXAMPLE 3: EFFECT OF PERTURBATION MAGNITUDE

We now compare two cases:

$$\text{Case A: } \|y_0 - x_0\| + |\zeta| = 0.08, \quad \text{Case B: } \|y_0 - x_0\| + |\zeta| = 0.03.$$

The theoretical estimate predicts linear dependence of the deviation with respect to perturbation size.

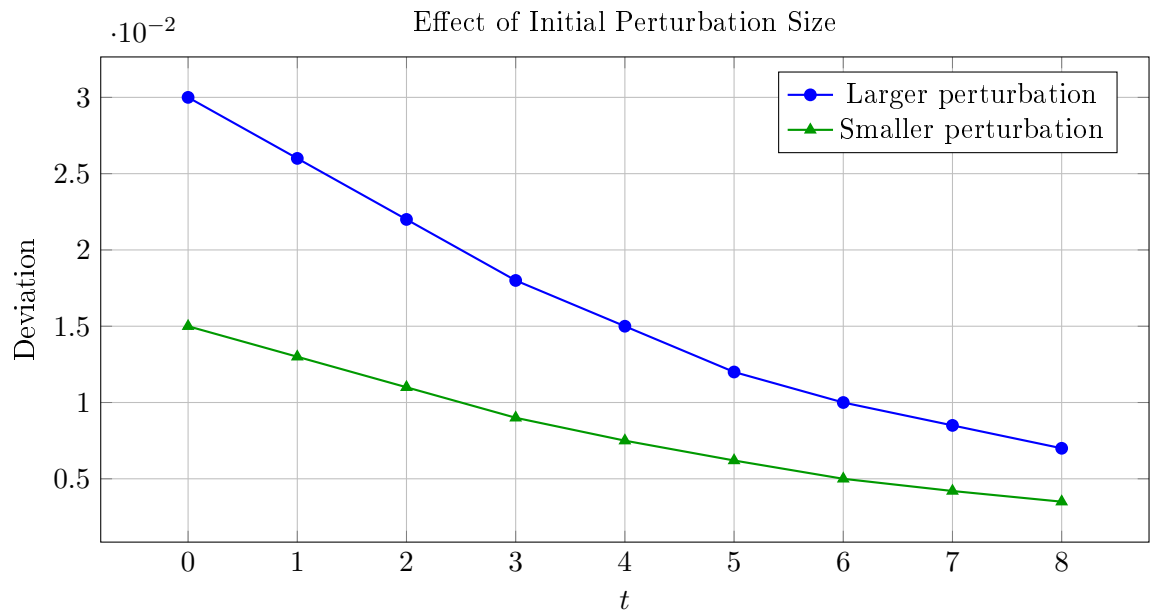


FIGURE 3. The deviation scales proportionally with the size of the initial perturbation, confirming Lipschitz dependence.

DISCUSSION

The numerical experiments demonstrate that:

- the deviation between two solutions of the same unperturbed system remains bounded;
- smaller changes in initial states and initial times produce proportionally smaller deviations;
- the observed behavior is fully consistent with the theoretical Lipschitz stability estimates under ITD.

Hence, the numerical results validate the main theoretical conclusions of this paper.

6. CONCLUSION

In this paper, we investigated Lipschitz stability properties of nonlinear unperturbed differential systems under Initial Time Difference (ITD). By using the variation-of-parameters formula together with suitable differential and integral inequality techniques, several new criteria were obtained for comparing solutions generated by different initial states and shifted initial times.

The main results demonstrate that the classical notion of Lipschitz stability, which is generally formulated with respect to the zero solution under a fixed initial time, is fundamentally different from the ITD framework considered here. In the present setting, stability is measured relative to a nontrivial reference trajectory while simultaneously incorporating perturbations in both the initial state and the initial time. This provides a broader perspective for the qualitative analysis of nonlinear systems.

Compared with the classical works of Lakshmikantham and coauthors [9, 10, 11], where variation-of-parameters methods and differential inequalities were developed mainly for systems with fixed initial time, the present paper extends these ideas to a more general framework involving time-shifted initial configurations. In particular, the current results show how the interaction between state perturbations and initial time discrepancies can be handled within a unified Lipschitz stability structure.

Furthermore, in comparison with the studies of Yakar and collaborators [14, 15, 18, 19, 20], where ITD-based variation formulas and several stability concepts were introduced, the present work advances the literature by establishing new Lipschitz stability criteria for nonlinear unperturbed systems, together with uniform, variation-based, and exponential asymptotic formulations. In addition, the implication from ITD-Lipschitz stability to ITD stability was rigorously proved, while the converse implication was shown not to hold in general.

These contributions indicate that the proposed framework not only extends earlier ITD studies, but also provides a clearer structural distinction between classical stability theory and time-shifted stability analysis. Hence, the present approach offers a more flexible mathematical tool for studying nonlinear dynamical systems arising in applications where delays, synchronization errors, or uncertain starting times naturally occur.

To support the theoretical developments, numerical illustrations involving predator–prey and SIR-type models were presented. The numerical results confirmed that deviations between trajectories remain bounded proportionally to the magnitude of the initial perturbations, in full agreement with the derived Lipschitz estimates.

The methods developed in this work may be extended to broader classes of systems such as delay differential equations, impulsive systems, integro-differential equations, fractional-order models, and stochastic dynamical systems. It is expected that the present study will motivate further investigations on generalized stability theories involving shifted initial configurations and time-dependent uncertainties.

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Conflict of interest. The authors declare no potential conflict of interests.

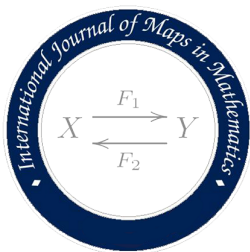
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FOURIER TRANSFORM OF BICOMPLEX-VALUED HYPERBOLIC NORM INTEGRABLE FUNCTIONS

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Abstract. For the purpose of this investigation, the Fourier transforms of hyperbolic norm integrable functions with bicomplex values that are constructed on locally compact abelian groups are utilized. Performing an analysis of these transformations is the purpose of this study. The bicomplex-valued Fourier transform is referred to as the $\mathbb{BC}$ -Fourier transform as a result of the nomenclature it utilizes. It is well defined and unique. There is a relationship between this transform and the convolution operator. The translation and character operators are sometimes included in this transform, which results in outcomes that are comparable to those of the well-known Fourier transform. Many fields employ the Fourier transform to convert a time-dependent function to a frequency-dependent function for time-frequency analysis, including signal processing. Using complex-valued signals is more convenient here. Thus, the $\mathbb{BC}$ -Fourier transform for time-frequency analysis of signals with bicomplex values would be more efficient since bicomplex numbers constitute an extended form of ordinary complex numbers.

Keywords: $\mathbb{BC}$ -Fourier transform, convolution, bicomplex Lebesgue space

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1. INTRODUCTION

Bicomplex numbers, a novel number system, have recently been introduced as an extension of the established complex number system. Price's book [18] is the primary recommended introductory text for bicomplex numbers. We recommend the references [1] and [17], which provide comprehensive explanations of the fundamental principles of bicomplex numbers, with an emphasis on functional analysis. These publications encompass a broad spectrum of applications and offer a thorough introduction to the subject.

The following sources are used to obtain the fundamental information regarding bicomplex numbers: [18], [1], and [17].

In this research, the symbols $\mathbb{R}$ and $\mathbb{C}$ represent real numbers and complex numbers, respectively. Let us now examine two independent imaginary numbers, i and j , where $i^2 = j^2 = -1$. This results in two sets, $\mathbb{C}(i)$ and $\mathbb{C}(j)$, which are identical to the set $\mathbb{C}$. Now, let

us examine an imaginary number k , defined as the product of these two imaginary numbers, possessing the following characteristics:

$$k^2 = 1, ij = ji = k, jk = kj = -i \text{ and } ik = ki = -j.$$

The collection of bicomplex numbers, designated as $\mathbb{BC}$, is characterized as

$$\mathbb{BC} = \{w_1 + jw_2 \mid w_1, w_2 \in \mathbb{C}(i)\}.$$

It is possible to interpret the set $\mathbb{BC}$ as

$$\mathbb{BC} = \{a_1 + a_2i + a_3j + a_4k \mid a_1, a_2, a_3, a_4 \in \mathbb{R}\}$$

in accordance with the considerations mentioned previously. In the following ways, the operations of addition and multiplication are, respectively, defined:

$$(w_1 + jw_2) + (w_3 + jw_4) = (w_1 + w_3) + j(w_2 + w_4)$$

and

$$(w_1 + jw_2)(w_3 + jw_4) = (w_1w_3 - w_2w_4) + j(w_1w_4 + w_2w_3).$$

Through the execution of these operations, the set $\mathbb{BC}$ is transformed into a commutative ring and, consequently, a module over itself. The collection of hyperbolic numbers, denoted $\mathbb{D}$, a notable subset of $\mathbb{BC}$, is characterized as

$$\mathbb{D} = \{a_1 + ka_2 \mid a_1, a_2 \in \mathbb{R}\}$$

through the substitution of w_1 and w_2 with a_1 and ia_2 , respectively, within the definition of $\mathbb{BC}$, such that $a_1, a_2 \in \mathbb{R}$. The conditions stated below are satisfied by the crucial elements; these are zero divisors in $\mathbb{BC}$: $e_1 = \frac{1+ij}{2}$ and $e_2 = \frac{1-ij}{2}$. Here,

$$e_1 + e_2 = 1, e_1e_2 = 0, e_2e_2 = e_2, e_1e_1 = e_1.$$

It is also possible to see e_1 and e_2 as hyperbolic numbers. The above components make up the idempotent basis for bicomplex numbers. The bicomplex number $w_1 + jw_2$ is characterized in a specific form by the expression

$$w_1 + jw_2 = z_1e_1 + z_2e_2,$$

such that $z_1 = w_1 - iw_2, z_2 = w_1 + iw_2 \in \mathbb{C}(i)$. This particular statement is referred to as the "idempotent representation of bicomplex numbers." In that case, each hyperbolic number is assigned the idempotent representation

$$a_1 + ka_2 = e_1b_1 + e_2b_2,$$

where $b_1 = a_1 + a_2, b_2 = a_1 - a_2 \in \mathbb{R}$. The collection of non-negative hyperbolic numbers is denoted by

$$\mathbb{D}^+ = \{a_1e_1 + a_2e_2 \mid a_1, a_2 \geq 0\}.$$

Let a_1 and a_2 belong to $\mathbb{D}$. In the context of hyperbolic numbers, the partial order relation $\preceq$ is stated as

$$a_1 \preceq a_2 \text{ if and only if } a_2 - a_1 \in \mathbb{D}^+.$$

Additionally,

$$a_1 \prec a_2 \text{ if and only if } a_1 \lesssim a_2 \text{ but } a_1 \neq a_2.$$

For further information on the characteristics of the partial order relation $\lesssim$, refer to Section 2.6.2 of [17]. As stated in Section 2.6.3 of [17], let us now give a brief summary of the $\mathbb{D}$ -boundedness. Consider $V \subset \mathbb{D}$. Given that V has a $\mathbb{D}$ -lower bound (or a $\mathbb{D}$ -upper bound) b , it follows that for any $v \in V$, v is comparable to b and $b \lesssim v$ (or $v \lesssim b$). It is said that a set is $\mathbb{D}$ -bounded iff it is $\mathbb{D}$ -bounded from both below and above. We define the concept of $\mathbb{D}$ -infimum, represented by $\inf_{\mathbb{D}} V$, as the greatest $\mathbb{D}$ -lower bound for V if V is a set $\mathbb{D}$ -bounded from below, and the $\mathbb{D}$ -supremum, represented by $\sup_{\mathbb{D}} V$, as the least $\mathbb{D}$ -upper bound for V , if V is a set $\mathbb{D}$ -bounded from above. Here, the greatest $\mathbb{D}$ -lower bound is described as $b \lesssim \inf_{\mathbb{D}} V$ for any $\mathbb{D}$ -lower bound b , despite the fact that not every $\mathbb{D}$ -lower bound is comparable. In this context, the least $\mathbb{D}$ -upper bound is determined as $\sup_{\mathbb{D}} V \lesssim b$ for any $\mathbb{D}$ -upper bound b , regardless of whether all $\mathbb{D}$ -upper bounds are comparable. Let us take V_1 and V_2 that are defined by

$$V_1 = \{v_1 : e_1 v_1 + e_2 v_2 \in V\} \text{ and } V_2 = \{v_2 : e_1 v_1 + e_2 v_2 \in V\},$$

respectively. Assuming that V is $\mathbb{D}$ -bounded from below, it follows that

$$\inf_{\mathbb{D}} V = e_1 \inf V_1 + e_2 \inf V_2.$$

Assuming that V is $\mathbb{D}$ -bounded from above, it follows that

$$\sup_{\mathbb{D}} V = e_1 \sup V_1 + e_2 \sup V_2.$$

Although different modules are defined for bicomplex numbers, the concept of a hyperbolic modulus to be addressed in this study is as follows. For a bicomplex number $w_1 + jw_2$, the hyperbolic modulus $|\cdot|_k$ is represented by the equation

$$|w_1 + jw_2|_k^2 = (w_1 + jw_2)(\overline{w_1} - j\overline{w_2}).$$

For the idempotent representation of a bicomplex number $w_1 + jw_2$, the hyperbolic modulus is given in the form

$$|w_1 + jw_2|_k = |e_1 z_1 + e_2 z_2|_k = e_1 |z_1| + e_2 |z_2|. \quad (1.1)$$

Modules are established over rings, analogous to how vector spaces are constructed over fields. A module U is called a $\mathbb{BC}$ -module if it is defined over a ring $\mathbb{BC}$ (see [8, 20]). Assume that U is a $\mathbb{BC}$ -module. If the following requirements are satisfied, a map $\|u\|_{\mathbb{D}} : U \rightarrow \mathbb{D}^+$ is considered a hyperbolic norm ($\mathbb{D}$ -valued) on U .

- (a) $\|u\|_{\mathbb{D}} = 0$ iff $u = 0$,
- (b) $\|wu\|_{\mathbb{D}} = |w|_k \|u\|_{\mathbb{D}}$ for all $u \in U$, $w \in \mathbb{BC}$,
- (c) $\|u + v\|_{\mathbb{D}} \lesssim \|u\|_{\mathbb{D}} + \|v\|_{\mathbb{D}}$ for all $u, v \in U$.

To describe the hyperbolic ($\mathbb{D}$ -valued) norm of $w_1 + jw_2 \in \mathbb{BC}$, we have the hyperbolic modulus $|\cdot|_k$ with expression (1.1). Assume that $w, \omega \in \mathbb{BC}$ and $\zeta \in \mathbb{D}^+$. The following properties are provided by the hyperbolic ($\mathbb{D}$ -valued) norm:

- (i)

$$|w|_k = 0 \text{ iff } w = 0,$$

(ii)

$$|w\omega|_k = |w|_k |\omega|_k,$$

(iii)

$$|w + \omega|_k \lesssim |w|_k + |\omega|_k,$$

(iv)

$$|\zeta|_k = \zeta,$$

(v)

$$|\zeta w|_k = \zeta |w|_k.$$

Suppose that U is a $\mathbb{BC}$ -module. As a result, we get

$$U = e_1 U_1 + e_2 U_2,$$

where $U_1 = e_1 U$ and $U_2 = e_2 U$ are complex vector spaces and $\mathbb{BC}$ -modules. The term given to this equation is the idempotent decomposition of U . Consequently, any u in U can be uniquely expressed as $u = e_1 u_1 + e_2 u_2$ such that u_1 is a member of U_1 and u_2 is a member of U_2 (see [7]). It is assumed that U_1 and U_2 are normed spaces with norms of $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Therefore,

$$\|u\|_{\mathbb{D}} = e_1 \|u_1\|_1 + e_2 \|u_2\|_2$$

is a way to endow the $\mathbb{BC}$ -module U with the hyperbolic ($\mathbb{D}$ -valued) norm. In the context of the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$, a sequence (u_n) in U converges to $u_0 \in X$ iff for every $0 \prec \epsilon$, there exists $n_0 \in \mathbb{N}$, where $\|u_n - u_0\|_{\mathbb{D}} \prec \epsilon$ for all $n \geq n_0$. This convergence is denoted by the term $\mathbb{D}$ -convergence (or $\mathbb{D}$ -converges). Additionally, sequences that are $\mathbb{D}$ -non-convergent will be referred to as $\mathbb{D}$ -divergent. According to the hyperbolic ($\mathbb{D}$ -valued) norm $\|\cdot\|_{\mathbb{D}}$, a sequence (u_n) in U is a Cauchy sequence iff for every $0 \prec \epsilon$ there exists $n_0 \in \mathbb{N}$, where $\|u_n - u_m\|_{\mathbb{D}} \prec \epsilon$ for all $n, m \geq n_0$. In this context, completeness is defined in a manner that is analogous to the conventional definition, that every Cauchy sequence in U $\mathbb{D}$ -converges to an element in U with regard to the hyperbolic ($\mathbb{D}$ -valued) norm. A bicomplex Banach module is described as a complete real-valued or hyperbolic normed $\mathbb{BC}$ -module, ([13, 7, 8, 16]).

Assume B to be any measurable space with a positive measure μ . Hence, for all values of p from 0 to infinity, the usual Lebesgue space is denoted as $(L^p(B, \mu), \|\cdot\|_p)$, [25, 4, 19]. Consider $\widehat{G}$ as the dual group of G , where G is a locally compact abelian group. Let h be a member of $L^1(G, \nu)$ such that ν is a Haar measure on G . Hence, the Fourier transform $\mathcal{F}h$ is indicated on $\widehat{G}$ by

$$\mathcal{F}h(\gamma) = \int_G h(x) \langle -x, \gamma \rangle d\nu(x)$$

for all $\gamma \in \widehat{G}$, [21].

Consider a σ -algebra Σ constituted by a set B . A bicomplex-valued function $\lambda = e_1 \lambda_1 + e_2 \lambda_2$ on B is referred to as a $\mathbb{BC}$ -measure on Σ if both λ_1 and λ_2 are complex measures on Σ . Specifically, with respect to positive measures λ_1 and λ_2 on Σ , λ is referred to as a $\mathbb{D}$ -measure on Σ , [10, Definition 3.3]. If both h_1 and h_2 are complex measurable functions on B , then the bicomplex-valued function h on B , defined as $h = e_1 h_1 + e_2 h_2$, is said to

be $\mathbb{BC}$ -measurable on B . Notably, h is a $\mathbb{D}$ -measurable function on B if h_1 and h_2 are real measurable functions on B , [10].

For any measurable space B , let $\lambda = e_1\lambda_1 + e_2\lambda_2$ be a $\mathbb{D}$ -measure. Assume that $0 < p < \infty$. Let $L^p(B, \lambda_1)$ and $L^p(B, \lambda_2)$ represent the linear spaces that consist of all equivalence classes of complex-valued measurable functions h_1 and h_2 defined on the set B , where

$$\int_B |h_1(x)|^p d\lambda_1 < \infty$$

and

$$\int_B |h_2(x)|^p d\lambda_2 < \infty,$$

respectively. Hence, $L_{\mathbb{BC}}^p(B, \lambda)$ represents all equivalence classes of bicomplex-valued $\mathbb{BC}$ -measurable functions on B in which the form is $h = e_1h_1 + e_2h_2$, with h_1 belonging to $L^p(B, \lambda_1)$ and h_2 to $L^p(B, \lambda_2)$, [24].

An essential integral operator, the Fourier transform is implemented in numerous disciplines, including signal analysis, time-frequency analysis, quantum mechanics, and the solution of partial and ordinary differential equations. In addition, the Fourier transform, which is defined on locally compact abelian groups G , serves as the foundation for both functional analysis and harmonic analysis, [21, 19]. Recently, the Fourier transform of functions of bicomplex variables was defined in [2]. Subsequently, the inverse of this transform as well as some examples of it were studied in [3]. Afterwards, bicomplex revisions of significant integral transforms, notably the Fourier and Laplace transforms, were examined in [11]. In particular, an examination of the Fourier transforms employed in the study of harmonic analysis reveals that they are defined on locally compact abelian groups. A $\mathbb{BC}$ -Fourier transform of bicomplex-valued hyperbolic norm integrable functions that are defined on the locally compact abelian group will be defined in this paper, and some examples will be provided.

2. BC-FOURIER TRANSFORM

Assume that G is an abelian group that is locally compact. Consequently, a unique $\mathbb{D}$ -Haar measure λ can be expressed as $\lambda = e_1\lambda + e_2\lambda$, where λ represents a unique Haar measure on G . The space $L_{\mathbb{BC}}^p(G, \lambda)$ is established as a $\mathbb{BC}$ -module, for $0 < p < \infty$, [23]. Given that G is a locally compact abelian group, its dual group $\widehat{G}$ is also a locally compact abelian group (see [21]). Therefore, there exists a unique $\mathbb{D}$ -Haar measure $\nu = e_1\nu + e_2\nu$, where ν is a unique Haar measure on $\widehat{G}$.

Definition 2.1. Let $h \in L_{\mathbb{BC}}^1(G, \lambda)$, where $h = e_1h_1 + e_2h_2$. Let $\widehat{G}$ be the dual group of G . Then $\mathbb{BC}$ -Fourier transform of h is defined by

$$\mathcal{F}_{\mathbb{BC}}h(\gamma) = e_1\mathcal{F}h_1(\gamma) + e_2\mathcal{F}h_2(\gamma).$$

for all $\gamma \in \widehat{G}$. Since the Fourier transforms of the functions $h_1, h_2 \in L^1(G, \mu)$ are well defined, then the $\mathbb{BC}$ -Fourier transform of any function $h = e_1h_1 + e_2h_2 \in L_{\mathbb{BC}}^1(G, \lambda)$ is well defined.

Remark 2.1. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $h = e_1 h_1 + e_2 h_2$. Take $\widehat{G}$ as the dual group corresponding to G . It is established that

$$|\mathcal{F}h_1(\gamma)| \leq \|h_1\|_1 \quad (2.2)$$

and

$$|\mathcal{F}h_2(\gamma)| \leq \|h_2\|_1 \quad (2.3)$$

for all $\gamma \in \widehat{G}$ (see [19, inequality (4.4.2)]). Combining (2.2), (2.3) and using the definition of $\mathbb{BC}$ -Fourier transform of h , we may write

$$\begin{aligned} |\mathcal{F}_{\mathbb{BC}}h(\gamma)|_k &= e_1 |\mathcal{F}h_1(\gamma)| + e_2 |\mathcal{F}h_2(\gamma)| \\ &\lesssim e_1 \|h_1\|_1 + e_2 \|h_2\|_1 = \|h\|_{1, \mathbb{D}} \end{aligned}$$

for all $\gamma \in \widehat{G}$.

Definition 2.2. Let X be any locally compact Hausdorff space. Then we define $C_0(X, \mathbb{BC})$ as the collection of bicomplex valued functions $h = e_1 h_1 + e_2 h_2$ on X such that $h_1 \in C_0(X)$ and $h_2 \in C_0(X)$.

Proposition 2.1. The set of functions $C_0(X, \mathbb{BC})$ is taken as a $\mathbb{BC}$ -module, utilizing the standard addition operation for functions along with bicomplex scalar multiplication.

Proof. Let $g, h \in C_0(X, \mathbb{BC})$ such that $g = e_1 g_1 + e_2 g_2$ and $h = e_1 h_1 + e_2 h_2$. Consider $\alpha = e_1 \alpha_1 + e_2 \alpha_2 \in \mathbb{BC}$. Obviously, $\alpha_1, \alpha_2 \in \mathbb{C}$. By the definition of $C_0(X, \mathbb{BC})$, we may write $g_i, h_i \in C_0(X)$ for $i = 1, 2$. Since the space $C_0(X)$ is a linear space, then we have

$$g_1 + h_1, g_2 + h_2, \alpha_1 g_1, \alpha_2 g_2 \in C_0(X). \quad (2.4)$$

Besides, we get

$$(g + h) = e_1 (g_1 + h_1) + e_2 (g_2 + h_2) \quad (2.5)$$

and

$$\alpha g = e_1 \alpha_1 g_1 + e_2 \alpha_2 g_2. \quad (2.6)$$

Combining (2.4), (2.5) and (2.6), we obtain $g + h \in C_0(X, \mathbb{BC})$ and $\alpha g \in C_0(X, \mathbb{BC})$. Similarly, it is easy to exhibit various characteristics of the module. $\square$

Remark 2.2. By the definition of the $\mathbb{BC}$ -module $C_0(X, \mathbb{BC})$, we may write (see [1, 16, 7])

$$C_0(X, \mathbb{BC}) = e_1 C_0(X) + e_2 C_0(X).$$

Obviously, the space $C_0(X)$ is a normed space with norm

$$\|h_i\|_{\infty} = \sup_{x \in X} |h_i(x)|$$

for $h_i \in C_0(X)$ with $i = 1, 2$. Let $h \in C_0(X, \mathbb{BC})$, where $h = e_1 h_1 + e_2 h_2$. Hence we can endow the $\mathbb{BC}$ -module $C_0(X, \mathbb{BC})$ with hyperbolic ($\mathbb{D}$ -valued) norm that given by

$$\|h\|_{\infty, \mathbb{D}} = e_1 \|h_1\|_{\infty} + e_2 \|h_2\|_{\infty}.$$

Theorem 2.1. The $\mathbb{BC}$ -Fourier transform of any function $h = e_1 h_1 + e_2 h_2 \in L^1_{\mathbb{BC}}(G, \lambda)$ belongs to $C_0(\widehat{G}, \mathbb{BC})$ and the inequality $\|\mathcal{F}_{\mathbb{BC}}h\|_{\infty, \mathbb{D}} \lesssim \|h\|_{1, \mathbb{D}}$ holds.

Proof. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $h = e_1 h_1 + e_2 h_2$. Then $h_1, h_2 \in L^1(G, \mu)$. By the definition of the $\mathbb{BC}$ -Fourier transform, we write

$$\mathcal{F}_{\mathbb{BC}} h = e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2. \quad (2.7)$$

It is known that Fourier transforms of the functions h_1, h_2 belong to $C_0(\widehat{G})$, [21, 1.2.4. Theorem]. Therefore the $\mathbb{BC}$ -Fourier transform of the function h belongs to $C_0(\widehat{G}, \mathbb{BC})$ by Definition 2.2. Since $h_1, h_2 \in L^1(G, \mu)$, then the inequalities

$$\|\mathcal{F} h_1\|_{\infty} \leq \|h_1\|_1 \quad (2.8)$$

and

$$\|\mathcal{F} h_2\|_{\infty} \leq \|h_2\|_1 \quad (2.9)$$

hold (see [21, Theorem 1.2.4]). Combining (2.7), (2.8) and (2.9), we get

$$\begin{aligned} \|\mathcal{F}_{\mathbb{BC}} h\|_{\infty, \mathbb{D}} &= \|e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2\|_{\infty, \mathbb{D}} \\ &= e_1 \|\mathcal{F} h_1\|_{\infty} + e_2 \|\mathcal{F} h_2\|_{\infty} \\ &\lesssim e_1 \|h_1\|_1 + e_2 \|h_2\|_1 = \|h\|_{1, \mathbb{D}}. \end{aligned}$$

□

Proposition 2.2. *The $\mathbb{BC}$ -Fourier transform from $L^1_{\mathbb{BC}}(G, \lambda)$ into $C_0(\widehat{G}, \mathbb{BC})$ is linear.*

Proof. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$ and $v, \eta \in \mathbb{BC}$. Then we have $g = e_1 g_1 + e_2 g_2$ and $h = e_1 h_1 + e_2 h_2$, where $g_i, h_i \in L^1(G, \mu)$ with $i = 1, 2$. Hence, we can write

$$\mathcal{F}_{\mathbb{BC}}(g + h) = e_1 \mathcal{F}(g_1 + h_1) + e_2 \mathcal{F}(g_2 + h_2)$$

By using the linearity of the Fourier transform $\mathcal{F}$, we obtain

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(g + h) &= e_1 \mathcal{F}(g_1 + h_1) + e_2 \mathcal{F}(g_2 + h_2) \\ &= e_1 (\mathcal{F} g_1 + \mathcal{F} h_1) + e_2 (\mathcal{F} g_2 + \mathcal{F} h_2) \\ &= (e_1 \mathcal{F} g_1 + e_2 \mathcal{F} g_2) + (e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2) \\ &= \mathcal{F}_{\mathbb{BC}} g + \mathcal{F}_{\mathbb{BC}} h. \end{aligned}$$

Let us take $\mathfrak{c} = e_1 c_1 + e_2 c_2 \in \mathbb{BC}$. Therefore, we have

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\mathfrak{c}h) &= \mathcal{F}_{\mathbb{BC}}((e_1 c_1 + e_2 c_2)(e_1 h_1 + e_2 h_2)) \\ &= \mathcal{F}_{\mathbb{BC}}(e_1(c_1 h_1) + e_2(c_2 h_2)) \\ &= e_1 \mathcal{F}(c_1 h_1) + e_2 \mathcal{F}(c_2 h_2). \end{aligned}$$

By the linearity property of the Fourier transform $\mathcal{F}$, we conclude

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\mathfrak{c}h) &= e_1 \mathcal{F}(c_1 h_1) + e_2 \mathcal{F}(c_2 h_2) \\ &= e_1 c_1 \mathcal{F} h_1 + e_2 c_2 \mathcal{F} h_2 \\ &= (e_1 c_1 + e_2 c_2)(e_1 \mathcal{F} h_1 + e_2 \mathcal{F} h_2) \\ &= \mathfrak{c} \mathcal{F}_{\mathbb{BC}} h. \end{aligned}$$

□

Definition 2.3. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$, where $g = e_1g_1 + e_2g_2$ and $h = e_1h_1 + e_2h_2$. Then the $\ast'$ convolution of the functions g, h is

$$g \ast' h = e_1 (g_1 \ast h_1) + e_2 (g_2 \ast h_2), \quad (2.10)$$

where $\ast$ is usual convolution on $L^1(G, \mu)$.

Proposition 2.3. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$. Then we have

$$\mathcal{F}_{\mathbb{BC}}(g \ast' h) = (\mathcal{F}_{\mathbb{BC}}g)(\mathcal{F}_{\mathbb{BC}}h).$$

Proof. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$. Then we can write $g = e_1g_1 + e_2g_2$ and $h = e_1h_1 + e_2h_2$, where $g_i, h_i \in L^1(G, \mu)$ with $i = 1, 2$. It is known that the Fourier transform of usual convolution between two functions in $L^1(G, \mu)$ equals the product of their Fourier transforms (see [21]). Therefore, we have

$$\mathcal{F}(g_1 \ast h_1) = \mathcal{F}g_1\mathcal{F}h_1 \quad (2.11)$$

and

$$\mathcal{F}(g_2 \ast h_2) = \mathcal{F}g_2\mathcal{F}h_2. \quad (2.12)$$

By using the definition of $\mathbb{BC}$ -Fourier transform, (2.10), (2.11) and (2.12), we get

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(g \ast' h) &= e_1\mathcal{F}(g_1 \ast h_1) + e_2\mathcal{F}(g_2 \ast h_2) \\ &= e_1\mathcal{F}g_1\mathcal{F}h_1 + e_2\mathcal{F}g_2\mathcal{F}h_2 \\ &= (e_1\mathcal{F}g_1 + e_2\mathcal{F}g_2)(e_1\mathcal{F}h_1 + e_2\mathcal{F}h_2) \\ &= (\mathcal{F}_{\mathbb{BC}}g)(\mathcal{F}_{\mathbb{BC}}h). \end{aligned}$$

□

Theorem 2.2 (Uniqueness Theorem). A function $h \in L^1_{\mathbb{BC}}(G, \lambda)$ is uniquely determined almost everywhere by its $\mathbb{BC}$ -Fourier transform.

Proof. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$. Then we have idempotent representations $h = e_1h_1 + e_2h_2$ and $\mathcal{F}_{\mathbb{BC}}h = e_1\mathcal{F}h_1 + e_2\mathcal{F}h_2$, where $h_1, h_2 \in L^1(G, \mu)$ and $\mathcal{F}h_1, \mathcal{F}h_2 \in C_0(\widehat{G})$. It is enough to show that if $\mathcal{F}_{\mathbb{BC}}h(\gamma) = 0$ for all $\gamma \in \widehat{G}$, then $h(x) = 0$ almost everywhere on G . Suppose that $\mathcal{F}_{\mathbb{BC}}h(\gamma) = 0$ for all $\gamma \in \widehat{G}$. By the definition of the $\mathbb{BC}$ -Fourier transform, we have

$$\mathcal{F}_{\mathbb{BC}}h(\gamma) = e_1\mathcal{F}h_1(\gamma) + e_2\mathcal{F}h_2(\gamma) = 0,$$

for all $\gamma \in \widehat{G}$. This means that

$$\mathcal{F}h_1(\gamma) = 0$$

and

$$\mathcal{F}h_2(\gamma) = 0,$$

for all $\gamma \in \widehat{G}$. By the uniqueness theorem of Fourier transform (see, [21]), we may write

$$h_1(x) = 0 \quad (2.13)$$

and

$$h_2(x) = 0, \quad (2.14)$$

almost everywhere on G . Using idempotent representation of the function h and combining (2.13) and (2.14), we obtain

$$h(x) = e_1 h_1(x) + e_2 h_2(x) = 0,$$

almost everywhere on G . □

Remark 2.3. Let us take $\mathbb{D}$ -Haar measure $\nu = e_1 \nu + e_2 \nu$ on $\widehat{G}$ such that ν is a unique Haar measure on $\widehat{G}$. Therefore, normalization of $\mathbb{D}$ -Haar measure ν means normalization of Haar measure ν due to the idempotent representation of $\mathbb{D}$ -Haar measure ν .

Theorem 2.3 (Inversion Theorem). Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$ and $\mathcal{F}_{\mathbb{BC}} h \in L^1_{\mathbb{BC}}(\widehat{G}, \nu)$. If a suitable normalized $\mathbb{D}$ -Haar measure ν is obtained for a fixed $\mathbb{D}$ -Haar measure λ , then

$$\begin{aligned} h(x) &= e_1 \int_{\widehat{G}} \mathcal{F}h_1(\gamma) \langle x, \gamma \rangle d\nu(\gamma) + e_2 \int_{\widehat{G}} \mathcal{F}h_2(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \\ &= e_1 \mathcal{F}^{-1}h_1(x) + e_2 \mathcal{F}^{-1}h_2(x), \end{aligned}$$

holds almost everywhere on G .

Proof. Let $h \in L^1_{\mathbb{BC}}(G, \lambda)$ and $\mathcal{F}_{\mathbb{BC}} h \in L^1_{\mathbb{BC}}(\widehat{G}, \nu)$. Thus we have idempotent representations $h = e_1 h_1 + e_2 h_2$ and $\mathcal{F}_{\mathbb{BC}} h = e_1 \mathcal{F}h_1 + e_2 \mathcal{F}h_2$, where $h_1, h_2 \in L^1(G, \mu)$ and $\mathcal{F}h_1, \mathcal{F}h_2 \in L^1_{\mathbb{BC}}(\widehat{G}, \nu)$. Assume that there exists a suitable normalized $\mathbb{D}$ -Haar measure ν is obtained for a fixed $\mathbb{D}$ -Haar measure λ . By the inversion theorem in [19, Remark 4.4.8], the equalities

$$h_1(x) = \int_{\widehat{G}} \mathcal{F}h_1(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \quad (2.15)$$

and

$$h_2(x) = \int_{\widehat{G}} \mathcal{F}h_2(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \quad (2.16)$$

hold almost everywhere on G . We utilize the proofs of Fourier Inversion Theorem I and Fourier Inversion Theorem II in [6] to understand the proof of the inversion theorem for the ordinary Fourier transform on the locally compact abelian groups. Using idempotent representation of the function h and combining (2.15) and (2.16), we get

$$\begin{aligned} h(x) &= e_1 \int_{\widehat{G}} \mathcal{F}h_1(\gamma) \langle x, \gamma \rangle d\nu(\gamma) + e_2 \int_{\widehat{G}} \mathcal{F}h_2(\gamma) \langle x, \gamma \rangle d\nu(\gamma) \\ &= e_1 \mathcal{F}^{-1}h_1(x) + e_2 \mathcal{F}^{-1}h_2(x), \end{aligned}$$

almost everywhere on G . □

Definition 2.4. The character operator is defined in the following manner using Duality theorem (for Duality theorem see, [19, Theorem 4.2.11]):

- (1) Let us take an element γ from $\widehat{G}$ and fix it. Then the function χ_γ from G into $\mathbb{C}$ is defined as

$$\chi_\gamma(x) = \langle x, \gamma \rangle \quad (2.17)$$

for all $x \in G$.

- (2) Let us take an element x from G and fix it. Then the function χ_x from $\widehat{G}$ into $\mathbb{C}$ is defined as

$$\chi_x(\gamma) = \langle x, \gamma \rangle \quad (2.18)$$

for all $\gamma \in \widehat{G}$.

Let us now provide the following proposition expressing some algebraic properties of the $\mathbb{BC}$ -Fourier transform:

Proposition 2.4. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$.

- (1) Let $x \in G$. Then

$$\mathcal{F}_{\mathbb{BC}}(T_x g) = \chi_{-x} \mathcal{F}_{\mathbb{BC}} g. \quad (2.19)$$

- (2) Let $\gamma \in \widehat{G}$. Then

$$\mathcal{F}_{\mathbb{BC}}(\chi_\gamma g) = T_\gamma \mathcal{F}_{\mathbb{BC}} g. \quad (2.20)$$

Proof. Let $g, h \in L^1_{\mathbb{BC}}(G, \lambda)$.

- (1) Let $x \in G$. The equality

$$\mathcal{F}(T_x g)(\gamma) = \chi_{-x} \mathcal{F}g(\gamma)$$

holds for all $\gamma \in \widehat{G}$, which is well known by [19, Proposition 4.4.10]. By applying the equality (2.18), we may write

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(T_x g)(\gamma) &= e_1 \mathcal{F}(T_x g_1)(\gamma) + e_2 \mathcal{F}(T_x g_2)(\gamma) \\ &= e_1 \chi_{-x} \mathcal{F}g_1(\gamma) + e_2 \chi_{-x} \mathcal{F}g_2(\gamma) \\ &= \langle -x, \gamma \rangle (e_1 \mathcal{F}g_1(\gamma) + e_2 \mathcal{F}g_2(\gamma)) \\ &= \langle -x, \gamma \rangle \mathcal{F}_{\mathbb{BC}} g(\gamma) = \chi_{-x} \mathcal{F}_{\mathbb{BC}} g(\gamma) \end{aligned}$$

for all $\gamma \in \widehat{G}$. Hence, the equation (2.19) is satisfied.

- (2) Let $\gamma \in \widehat{G}$. As stated in [19, Proposition 4.4.10], the equality

$$\mathcal{F}(\chi_\gamma g) = T_\gamma \mathcal{F}g$$

is valid. Then we have

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\chi_\gamma g) &= e_1 \mathcal{F}(\chi_\gamma g_1) + e_2 \mathcal{F}(\chi_\gamma g_2) \\ &= e_1 T_\gamma \mathcal{F}g_1 + e_2 T_\gamma \mathcal{F}g_2. \end{aligned}$$

Therefore, we can write

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}}(\chi_\gamma g)(\eta) &= e_1 (T_\gamma \mathcal{F}g_1)(\eta) + e_2 (T_\gamma \mathcal{F}g_2)(\eta) \\ &= e_1 (T_\gamma \mathcal{F}g_1)(\eta) + e_2 (T_\gamma \mathcal{F}g_2)(\eta) \\ &= (e_1 \mathcal{F}g_1 + e_2 \mathcal{F}g_2)(\eta - \gamma) \\ &= \mathcal{F}_{\mathbb{BC}} g(\eta - \gamma) = T_\gamma \mathcal{F}_{\mathbb{BC}} g(\eta) \end{aligned}$$

for all $\eta \in \widehat{G}$. Therefore, the equation (2.20) is fulfilled.

□

Consider the additive group of real numbers $(\mathbb{R}, +)$ as a locally compact abelian group G . Let's use the Lebesgue measure on $\mathbb{R}$ instead of the Haar measures in the previously

mentioned $\mathbb{D}$ -Haar measure. Let's determine the $\mathbb{BC}$ -Fourier transforms of certain functions in $L^1_{\mathbb{BC}}(G, \lambda)$ space. Regarding the kernel of the Fourier transform, the literature is varied. In particular, we will consider the $K(w, t) = \exp(-iwt)$ function as the kernel for $t \in \mathbb{R}$ and $w \in \widehat{\mathbb{R}} = \mathbb{R}$.

Example 2.1. Let $a, b \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{C}$, $c_1 \neq c_2$. Consider a function $f(t) = c_1 e_1 + c_2 e_1$ for all $t \in [a, b]$. Let us take functions $f_1(t) = c_1$ and $f_2(t) = c_2$ for all $t \in [a, b]$. Therefore, we have $f(t) = c_1 e_1 + c_2 e_2$ for all $t \in [a, b]$. Let dt be a Lebesgue measure on $\mathbb{R}$. Since constant functions are measurable functions in the sense of the Lebesgue measure on $\mathbb{R}$ and

$$\int_a^b |f_j(t)| dt = \int_a^b |c_j| dt = |c_j| (b - a) < \infty,$$

for $j = 1, 2$, then these functions are elements of the space $L^1(\mathbb{R}, dt)$. Hence, the function f is a member of the space $L^1_{\mathbb{BC}}(\mathbb{R}, dx)$, and its $\mathbb{BC}$ -Fourier transform can be described. Then we obtain

$$\begin{aligned} \mathcal{F}_{\mathbb{BC}} f(w) &= e_1 \mathcal{F} f_1(w) + e_2 \mathcal{F} f_2(w) \\ &= e_1 \int_a^b f_1(t) \exp(-iwt) dt + e_2 \int_a^b f_2(t) \exp(-iwt) dt \\ &= e_1 c_1 \int_a^b \exp(-iwt) dt + e_2 c_2 \int_a^b \exp(-iwt) dt \\ &= e_1 c_1 \frac{i}{w} (\exp(-ibt) - \exp(-iat)) + e_2 c_2 \frac{i}{w} (\exp(-ibt) - \exp(-iat)) \\ &= \frac{i}{w} (\exp(-ibt) - \exp(-iat)) (e_1 c_1 + e_2 c_2) \end{aligned}$$

for all $w \in \mathbb{R} - \{0\}$.

Example 2.2. Let a and b are positive real numbers such that $a \neq b$. Let us take the Gaussian functions $h_1(t) = \exp(-at^2)$ and $h_2(t) = \exp(-bt^2)$ for all $t \in \mathbb{R}$. Gaussian functions are also measurable functions in the sense of the Lebesgue measure on $\mathbb{R}$, as they are continuous on the set of all real numbers. The integrals of the Gaussian functions over $\mathbb{R}$ are calculated as

$$\int_{-\infty}^{+\infty} \exp(-at^2) dt = \sqrt{\frac{\pi}{a}} \text{ and } \int_{-\infty}^{+\infty} \exp(-bt^2) dt = \sqrt{\frac{\pi}{b}}.$$

Therefore, $h_1, h_2 \in L^1(\mathbb{R}, dt)$. By utilizing these functions, let us construct a function h as

$$h(t) = e_1 h_1(t) + e_2 h_2(t) = e_1 \exp(-at^2) + e_2 \exp(-bt^2)$$

for all $t \in \mathbb{R}$. We will now compute the ordinary Fourier transform of the function h_1 . We write

$$\begin{aligned}\mathcal{F}h_1(w) &= \int_{-\infty}^{+\infty} \exp(-at^2) \exp(-iwt) dt \\ &= \int_{-\infty}^{+\infty} \exp(-(at^2 + iwt)) dt \\ &= \int_{-\infty}^{+\infty} \exp\left(-\left(\sqrt{a}t^2 + \frac{iw}{2\sqrt{a}}\right)^2 - \frac{w^2}{4a}\right) dt \\ &= \exp\left(-\frac{w^2}{4a}\right) \int_{-\infty}^{+\infty} \exp\left(-\left(\sqrt{a}t^2 + \frac{iw}{2\sqrt{a}}\right)^2\right) dt.\end{aligned}$$

If we make the substitution $u = \sqrt{a}t^2 + \frac{iw}{2\sqrt{a}}$, then we have

$$\mathcal{F}h_1(w) = \frac{1}{\sqrt{a}} \exp\left(-\frac{w^2}{4a}\right) \int_{-\infty}^{+\infty} \exp(-u^2) du.$$

Thus, the integral $\int_{-\infty}^{+\infty} \exp(-u^2) du$ provides the area of the region under the bell-curve $\exp(-u^2)$, and its value is known as $\sqrt{\pi}$. Hence, we obtain the Fourier transforms of the Gaussian function h_1 is given as

$$\mathcal{F}h_1(w) = \sqrt{\frac{\pi}{a}} \exp\left(\frac{-w^2}{4a}\right),$$

and so

$$\mathcal{F}h_2(w) = \sqrt{\frac{\pi}{b}} \exp\left(\frac{-w^2}{4b}\right)$$

is meaningful. Therefore, we get

$$\begin{aligned}\mathcal{F}_{\mathbb{BC}}h(w) &= e_1\mathcal{F}h_1(w) + e_2\mathcal{F}h_2(w) \\ &= e_1\sqrt{\frac{\pi}{a}} \exp\left(\frac{-w^2}{4a}\right) + e_2\sqrt{\frac{\pi}{b}} \exp\left(\frac{-w^2}{4b}\right)\end{aligned}$$

for all $w \in \mathbb{R}$.

We designate $\mathbb{R}$ as the additive group of real numbers and $\mathbb{Z}$ as the subgroup that consists of the integers. The definition of the group $\mathbb{T}$ is $\mathbb{Z}$ the quotient $\mathbb{R}/2\pi\mathbb{Z}$, where $2\pi\mathbb{Z}$ is the group of integer multiples of 2π , as shown in the notation. The functions on $\mathbb{T}$ and 2π -periodic functions on $\mathbb{R}$ are readily identifiable. For comprehensive details regarding the properties of the group $\mathbb{T}$, please consult [12]. Additionally, the n -dimensional form of the group $\mathbb{T}$ is denoted as $\mathbb{T}^n$ and is described as $\mathbb{T}^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$. Besides, the restriction of the Lebesgue measure of $\mathbb{R}^n$ to $[0, 2\pi)^n$ is the Lebesgue measure dt on $\mathbb{T}^n$. Now, consider $\mathbb{T}^2$ as a locally compact abelian group G . Additionally, the dual group of group $G = \mathbb{T}^2$ turns into group

$\widehat{G} = \mathbb{Z}^2$. The Fourier transform of a function f in the space $L^1(\mathbb{T}^2, dt)$ is of the form

$$\mathcal{F}g(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} g(t_1, t_2) \exp(-i(t_1 k_1 + t_2 k_2)) dt_1 dt_2$$

for all $k = (k_1, k_2) \in \mathbb{Z}^2$ (see, [22, Chapter 7]).

Example 2.3. Let $l, m \in \mathbb{Z}$ and g be a bicomplex trigonometric wave of type

$$g(t_1, t_2) = e_1 \exp(it_1 l) + e_2 \exp(it_2 m)$$

for all $t_1, t_2 \in \mathbb{T}$ that is given in [5, Subsection 3.1]. Since the functions $g_1(t_1, t_2) = \exp(it_1 l)$ and $g_2(t_1, t_2) = \exp(it_2 m)$ are continuous functions, then these functions are measurable functions on $\mathbb{T}^2$. Also, we have

$$\int_0^{2\pi} \int_0^{2\pi} |\exp(it_1 l)| dt_1 dt_2 = 4\pi^2 \text{ and } \int_0^{2\pi} \int_0^{2\pi} |\exp(it_2 m)| dt_1 dt_2 = 4\pi^2.$$

Therefore, the functions g_1 and g_2 belong to $L^1(\mathbb{T}^2, dt)$. Now, we will calculate the $\mathbb{BC}$ -Fourier transform of the bicomplex trigonometric wave g . Firstly, we compute the ordinary Fourier transform of the function g_1 . Let us take $k = (k_1, k_2) \in \mathbb{Z}^2$. We write

$$\mathcal{F}g_1(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \int_0^{2\pi} \exp(it_1 l) \exp(-i(t_1 k_1 + t_2 k_2)) dt_1 dt_2.$$

By the Fubini's theorem, we get

$$\mathcal{F}g_1(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \exp(-it_2 k_2) \left(\int_0^{2\pi} \exp(it_1 (l - k_1)) dt_1 \right) dt_2$$

Let $k_2 \neq 0$ and $k_1 \neq l$. Then we have

$$\mathcal{F}g_1(k) = \frac{1}{4\pi^2} \left(\frac{i}{k_1 - l} (\exp(2\pi i (l - k_1)) - 1) \right) \int_0^{2\pi} \exp(-it_2 k_2) dt_2.$$

Since $l - k_1 \in \mathbb{Z}$ and $k_2 \in \mathbb{Z}$, then we conclude

$$\mathcal{F}g_1(k_1, k_2) = 0.$$

If $k_2 = 0$ and $k_1 = l$. Then we have

$$\mathcal{F}g_1(k_1, k_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \exp(-it_2 k_2) \left(\int_0^{2\pi} \exp(it_1 (l - k_1)) dt_1 \right) dt_2 = 1.$$

Hence, we obtain

$$\mathcal{F}g_1(k_1, k_2) = \begin{cases} 1, & k_1 = l \text{ and } k_2 = 0 \\ 0, & \text{otherwise} \end{cases}. \quad (2.21)$$

Similarly, the ordinary Fourier transform of the function g_2 is calculated as

$$\mathcal{F}g_2(k) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \exp(-it_1 k_1) \left(\int_0^{2\pi} \exp(it_2(m - k_2)) dt_2 \right) dt_1.$$

Let $k_1 \neq 0$ and $k_2 \neq m$. Then we have

$$\mathcal{F}g_2(k) = \frac{1}{4\pi^2} \left(\frac{i}{k_2 - m} (\exp(2\pi i(m - k_2)) - 1) \right) \int_0^{2\pi} \exp(-it_1 k_1) dt_1.$$

Since $m - k_2 \in \mathbb{Z}$ and $k_1 \in \mathbb{Z}$, then we conclude

$$\mathcal{F}g_2(k_1, k_2) = 0.$$

If $k_1 = 0$ and $k_2 = m$. Then we have

$$\mathcal{F}g_1(k_1, k_2) = \frac{1}{4\pi^2} \int_0^{2\pi} \exp(-it_1 k_1) \left(\int_0^{2\pi} \exp(it_2(m - k_2)) dt_2 \right) dt_1 = 1.$$

Hence, we obtain

$$\mathcal{F}g_2(k_1, k_2) = \begin{cases} 1, & k_1 = 0 \text{ and } k_2 = m \\ 0, & \text{otherwise} \end{cases}. \quad (2.22)$$

By combining (2.21) and (2.22), we calculate the $\mathbb{BC}$ -Fourier transform of the bicomplex trigonometric wave g as

$$\mathcal{F}_{\mathbb{BC}}g(k_1, k_2) = e_1 \mathcal{F}g_1(k_1, k_2) + e_2 \mathcal{F}g_2(k_1, k_2).$$

Remark 2.4. Let us take the complex-valued function $g \in L^1(G, \mu)$. We can write this function as the following idempotent representation:

$$g(x) = e_1 g(x) + e_2 g(x)$$

almost everywhere on G . By the definition of the $\mathbb{BC}$ -Fourier transform, we write

$$\mathcal{F}_{\mathbb{BC}}g(\gamma) = e_1 \mathcal{F}g(\gamma) + e_2 \mathcal{F}g(\gamma) = \mathcal{F}g(\gamma)(e_1 + e_2) = \mathcal{F}g(\gamma)$$

for all $\gamma \in \widehat{G}$. Hence, the $\mathbb{BC}$ -Fourier transform of the complex-valued function $g \in L^1(G, \mu)$ corresponds to the ordinary Fourier transform of the function g .

3. DISCUSSION

The kernel function of the known Fourier transform is generalized in the bicomplex sense when looking at [2], which is the fundamental work that defines the bicomplex Fourier transform. Furthermore, the bicomplex Fourier transform is exclusively defined for real-valued functions in [2, Definition 3.2]. Nevertheless, the literature does not contain a valid Fourier transform for bicomplex-valued functions. In particular, the Fourier transform of the bicomplex trigonometric wave that is a bicomplex-valued function, presented in Example 2.3, which means that it cannot be calculated using the bicomplex Fourier transform defined in [2]. As a result, this paper's definition of the $\mathbb{BC}$ -Fourier transform is entirely distinct from that of [2]. Additionally, as the examples show, both functions defined on $\mathbb{T}$ and $\mathbb{R}$ are applicable to the $\mathbb{BC}$ -Fourier transform defined in this paper. The $\mathbb{BC}$ -Fourier transform

provided here yields the answer for bicomplex-valued functions defined on $\mathbb{T}^n$ and $\mathbb{R}^n$ in its most general form. Moreover, Remark 2.4 asserts that the BC-Fourier transform of a complex-valued function defined on a locally compact abelian group G corresponds to the known Fourier transform, which implies that the BC-Fourier transform is a generalization of the known Fourier transform.

4. CONCLUSION

Specifically, an analysis of the Fourier transforms used in the study of harmonic analysis shows that they are defined on locally compact abelian groups. This paper introduces the definition of a BC-Fourier transform for hyperbolic norm integrable functions that have bicomplex values and are specified on a locally compact abelian group. Additionally, the paper includes several examples to illustrate this concept. Furthermore, the paper explores the properties of the BC-Fourier transform in harmonic analysis. Within the field of harmonic analysis, this research will serve as a foundation for forecasting that the results achieved with the well-known Fourier transform can also be achieved with the BC-Fourier transform, providing insights for future investigations.

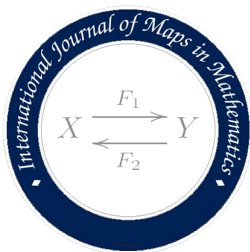
Conflict of interest. The authors declare no potential conflict of interests.

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EULER AND CHEN-INVARIANT TYPE INEQUALITIES ON SUBMANIFOLDS IN GOLDEN STATISTICAL MANIFOLDS

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Abstract. The purpose of this research paper is to analyze the statistical manifolds coupled with Golden structure. On golden-like statistical manifolds, we derive several basic attributes for the intrinsic and extrinsic invariants of statistical submanifolds in golden-like statistical manifolds. Also, we derive an Euler inequality for submanifolds in golden-like statistical manifolds. Moreover, we used particular types of second fundamental forms of Chen-invariant type inequalities for these classes of submanifolds in golden-like statistical manifolds to characterize the equality case. We strengthen our idea with an example as well.

Keywords: Statistical manifolds, Submanifolds, Euler inequality, Chen-invariant.

MSC (2020): 53C15, 53C20, 53C25, 53C42.

1. INTRODUCTION

Statistical manifolds are fundamental in the field of information geometry, a branch of mathematics that studies the geometric forms of probability distribution. These manifolds provide a framework for analyzing statistical models and estimating parameters in a system that considers the core geometry of the statistical space. This is the situation of normal linear regression that is statistically significant. The statistical framework of classical information geometry can be seen as a means of expanding the geometric basis of regression. Refer to Vos and Marriott (2010) for wider settings [46].

Statisticians, have mostly concentrated on applications pertaining to contemporary statistical practice, however, as the introduction illustrates, information geometry has expanded to a wide range of fields and has significantly influenced numerous other fields. Naturally, there are many applications for the general concepts of geometric dualistic structures and ideas of divergence that we have explored here [46].

Amari [3, 4] examines how divergences can be effectively utilized to fit neural network models. It estimates hidden layers in a neural network using a dual geometric version in

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information geometry . It specifically makes use of the normal mixture model, expert mixtures, and stochastic multilayer perceptron models.

Statistical manifolds act for the compilation of probability distributions, constantly equipped with a Riemannian metric that quantifies the directions and distances in this space. The Riemannian metric determines the curvature and structure of the manifold, activating the development of logical statistical methods adapted to the underlying geometry. One of the central persons in the evolution of information geometry and statistical manifolds is Shun-ichi Amari [3, 4]. Amari's study has been instrumental in establishing the theoretical basics of this pasture and has paved the method for applications in numerous fields, including machine learning, signal processing and physics. Golden-like statistical manifolds are a specialized category of statistical manifolds that have possession similar to the golden geometry, known for its exceptional mathematical attributes, and the golden structure itself has its algebraic roots in the broader theory of polynomial structures on manifolds [18]. The phrase "golden-like" mentions to structures with certain geometric possessions or connections to the golden ratio, a mathematical constant that become visible in numerous natural occurrence and creative endeavors, with further applications of the golden ratio on Riemannian manifolds discussed in [21]. Statistical submanifolds in golden-like statistical structure has practical inferences. Firstly, in model selection, by scouting submanifolds, practitioners can better choose statistical models that describe data without overfitting, especially in high-dimensional spaces. Secondly, in Bayesian inference, Bayesian techniques can benefit from the geometric viewpoints provided by submanifolds, particularly in specifying prior distributions that regard the geometric form of the underlying statistical manifold. Thirdly, in machine learning, algorithms operating manifold learning methods can utilize submanifolds to catch complex data patterns while staying computationally feasible. [3, 4, 8, 34].

Intrinsic invariants are possessions that are inherent to a geometric item and are independent of how the item appears from the outside. For instance, the curvature properties of a surface or the length of a curve. Such invariants are directly related to the geometric structure of the surface or curve. Extrinsic invariants are related to the position of a geometric item in space. That is, these possessions are related to the item's position and association to other items in its domain. For instance, how a surface is laid out in space or the position of a curve in a plane. Likewise invariants can be identified by outer observations and measurements. Presently, in the comparison relationship problems between the intrinsic and extrinsic invariants, statistical manifolds are being studied very actively. Golden semi-Riemannian manifolds are an interesting section of study in modern mathematics, combining classical geometrical structures with the unique possessions of the golden ratio. This structure generates a major field for further research, promising insights that transcend conventional frontiers in mathematics and physics. In [9], B.-Y. Chen discovered new intrinsic curvature invariants, very popularly known as Chen's invariants and derived optimal relationships between these intrinsic invariants and the extrinsic invariants, a line of study later extended to the Lagrangian setting [32] and to pseudo-Riemannian ambient spaces via δ -invariants [12]. Currently, this is a very active area of research [9, 42, 43].

In [2], Amari introduced the concept of statistical manifolds via information geometry. An analogue to conjugate connections in affine geometry (see [29]), statistical manifolds

are endowed with a pair of dual torsion-free connections. Generally sectional curvature of a statistical manifold is not very easy to define. Thus, in [31], B. Opozda showed how to define sectional curvature of a statistical manifold, building on Bochner-type techniques for statistical structures developed earlier [30]. The relationships between the intrinsic and extrinsic curvature invariants are very important in submanifold theory. In this connection, B.-Y. Chen obtained the relationship between the Gaussian and the mean curvatures [10], [11]. In [40], [41], K. Takano studied the statistical submersions with certain structures. In [44], A.D. Vîlcu and G.E. Vîlcu examined the statistical submanifolds of quaternionic statistical manifolds. In the same paper, they proposed several open problems. For similar studies on curvature inequalities for submanifolds and Casorati curvatures in various ambient spaces, we refer [5], [26], [27], [7], [14], [38], [39], [25], [29], [36], [1], [37], [6], [13], [16], [17], [19], [22], [23], [24], [33], [45], [47]. Now coming towards the actual theme of the current paper. In one of their earlier submitted papers, the first and second author of this paper defined the Golden-like statistical manifolds and obtained the optimal inequalities using Casorati curvatures.

In this paper, we obtain some sharp relationships between extrinsic and intrinsic invariants for statistical submanifolds of Golden-like statistical manifolds. At the end, we talk about their equality cases.

2. SOME BASIC CONCEPTS

Definition 2.1. [15, 20, 35] *Let (F, μ) be a semi-Riemannian manifold and let ϱ be $(1, 1)$ tensor field on F . If ϱ satisfies the following equality*

$$\varrho^2 = \varrho + I. \quad (2.1)$$

At that point the tensor field ϱ is named a Golden structure on F . On condition that the metric μ is ϱ compatible, the (F, μ, ϱ) is named a Golden semi-Riemannian manifold.

The following equalities are true for the ϱ compatible metric g

$$\mu(\varrho X_1, Y_1) = \mu(X_1, \varrho Y_1), \quad (2.2)$$

$$\begin{aligned} \mu(\varrho X_1, \varrho Y_1) &= \mu(\varrho^2 X_1, Y_1) \\ &= \mu(\varrho X_1, Y_1) + \mu(X_1, Y_1), \quad X_1, Y_1 \in TF. \end{aligned} \quad (2.3)$$

Let F be a semi-Riemannian manifold and $\hat{\nabla}$ be the torsion-free affine connection. The triple $(F, \hat{\nabla}, \mu)$ is named a statistical manifold if $\hat{\nabla}\mu$ is symmetric. Let's define a new ∇^* affine connection as follows:

$$X_1\mu(Y_1, Z_1) = \mu(\hat{\nabla}_{X_1} Y_1, Z_1) + \mu(\hat{\nabla}_{X_1}^* Z_1, Y_1), \quad (2.4)$$

for any vector fields X_1, Y_1 and Z_1 on F . The affine connection $\hat{\nabla}^*$ is called conjugate to $\hat{\nabla}$ in accordance with μ . The affine connection $\hat{\nabla}^*$ is torsion-free, $\hat{\nabla}^*\mu$ is symmetric and provides $\hat{\nabla}^0 = \frac{\hat{\nabla} + \hat{\nabla}^*}{2}$. We indicate by Ξ and Ξ^* the curvature tensors on F in accordance with the affine connection $\hat{\nabla}$ and its conjugate $\hat{\nabla}^*$, respectively. Moreover, the curvature tensor field $\hat{\Xi}^0$ associated with $\hat{\nabla}^0$ is called semi-Riemannian curvature tensor. In this case the following

relationship is given

$$\mu(\hat{\Xi}(X_1, Y_1)Z_1, W_1) = -\mu(Z_1, \hat{\Xi}^*(X_1, Y_1)W_1),$$

, where $\hat{\Xi}(X_1, Y_1)Z_1 = [\hat{\nabla}_{X_1}, \hat{\nabla}_{Y_1}]Z_1 - \hat{\nabla}_{[X_1, Y_1]}Z_1$.

The following definition is given for the statistical structure $(\hat{\nabla}, g)$ on F :

$$\hat{S}(X_1, Y_1)Z_1 = \frac{1}{2}[\hat{\Xi}(X_1, Y_1)Z_1 + \hat{\Xi}^*(X_1, Y_1)Z_1], \quad (2.5)$$

for any $X_1, Y_1, Z_1 \in TF$.

Usually, dual connections are not metric. as in the instance of semi-Riemannian geometry, section curvature cannot be defined in the statistical setting. Therefore, Oposda proposed two presumptions of section curvature in this type manifolds (see [30, 31]).

Let F be a statistical manifold and π a plane section in TF with orthonormal basis $\{X_1, Y_1\}$, in that case the sectional K -curvature is defined in [31] as

$$\hat{K}(\pi) = \frac{1}{2} \left[\mu(\hat{\Xi}(X_1, Y_1)Y_1, X_1) + \mu(\hat{\Xi}^*(X_1, Y_1)Y_1, X_1) - 2\mu(\hat{\Xi}^0(X_1, Y_1)Y_1, X_1) \right].$$

Definition 2.2. Suppose that (F, μ, ϱ) is a Golden semi-Riemannian manifold equipped with a tensor field ϱ^* of type $(1, 1)$ providing

$$\mu(\varrho X_1, Y_1) = \mu(X_1, \varrho^* Y_1), \quad (2.6)$$

for vector fields X_1 and Y_1 on F . In view of (2.6), we can easily obtain

$$(\varrho^*)^2 X_1 = \varrho^* X_1 + X_1, \quad (2.7)$$

$$\mu(\varrho X_1, \varrho^* Y_1) = \mu(\varrho X_1, Y_1) + \mu(X_1, Y_1). \quad (2.8)$$

In that case (F, μ, ϱ) is called Golden-like statistical manifold.

In accordance with equations (2.6), (2.7) and (2.8), we easily derive the following proposition:

Proposition 2.1. (F, μ, ϱ) is a Golden-like statistical manifold if and only if so is (F, μ, ϱ^*) .

A remarkable fact is that, choosing $\varrho = \varrho^*$ in the Golden-like statistical manifold yields a Golden semi-Riemannian manifold.

Definition 2.3. Let $(F = F_a(\zeta_a) \times F_b(\zeta_b), g, \varrho)$ be a Golden-like statistical manifold. In that case the Riemannian curvature tensor $\hat{\Xi}$ of F is given as follows([35]):

$$\begin{aligned} \hat{\Xi}(X_1, Y_1)Z_1 &= \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) \{ \mu(Y_1, Z_1)X_1 - \mu(X_1, Z_1)Y_1 + \mu(\varrho Y_1, Z_1)\varrho X_1 \\ &\quad - \mu(\varrho X_1, Z_1)\varrho Y_1 \} + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) \{ \mu(\varrho Y_1, Z_1)X_1 - \mu(\varrho X_1, Z_1)Y_1 \\ &\quad + \mu(Y_1, Z_1)\varrho X_1 - \mu(X_1, Z_1)\varrho Y_1 \}. \end{aligned} \quad (2.9)$$

We can get the curvature tensor $\hat{\Xi}^*$ in accordance with dual connection precisely by modifying ϱ by ϱ^* .

By using the sectional curvature, the scalar curvature of N for orthonormal tangent frame $\{\varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ and normal frame $\{\varsigma_{d+1}, \dots, \varsigma_q\}$, respectively, is defined as

$$\tau = 2 \sum_{1 \leq i < j \leq d} K(\varsigma_i \wedge \varsigma_j). \quad (2.10)$$

The average or mean curvature vector fields of N are also provided by

$$\Upsilon = \frac{1}{d} \sum_{i=1}^d \sigma(\varsigma_i, \varsigma_i) = \frac{1}{d} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^\gamma \right) \xi_\gamma, \quad \sigma_{ij}^\gamma = \mu(\sigma(\varsigma_i, \varsigma_j), \varsigma_\gamma)$$

and

$$\Upsilon^* = \frac{1}{d} \sum_{i=1}^d \sigma^*(\varsigma_i, \varsigma_i) = \frac{1}{d} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^{*\gamma} \right) \xi_\gamma, \quad \sigma_{ij}^{*\gamma} = \mu(\sigma^*(\varsigma_i, \varsigma_j), \varsigma_\gamma),$$

for $d+1 \leq \gamma \leq q$.

Moreover, we have $2\Upsilon^0 = \Upsilon + \Upsilon^*$, and $2\sigma^0 = \sigma + \sigma^*$, wherein the the mean curvature Υ^0 and second fundamental form σ^0 are obtained in accordance with Levi-Civita connection ∇^0 on N .

For N , the squared mean curvatures can be expressed as

$$\begin{aligned} \|\Upsilon\|^2 &= \frac{1}{d^2} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^\gamma \right)^2, \\ \|\Upsilon^*\|^2 &= \frac{1}{d^2} \sum_{\gamma=d+1}^q \left(\sum_{i=1}^d \sigma_{ii}^{*\gamma} \right)^2. \end{aligned}$$

Let $\hat{\Xi}$ and $\hat{\Xi}^*$ be the curvature tensors according to $\hat{\nabla}$ and $\hat{\nabla}^*$, respectively. In that case, we have the following relations on statistical manifolds.

$$\begin{aligned} \mu(\hat{\Xi}(X_1, Y_1)Z_1, W_1) &= \mu(\Xi(X_1, Y_1)Z_1, W_1) + \mu(\sigma(X_1, Z_1), \sigma^*(Y_1, W_1)) \\ &\quad - \mu(\sigma^*(X_1, W_1), \sigma(Y_1, Z_1)), \end{aligned} \quad (2.11)$$

$$\begin{aligned} \mu(\hat{\Xi}^*(X_1, Y_1)Z_1, W_1) &= \mu(\Xi^*(X_1, Y_1)Z_1, W_1) + \mu(\sigma^*(X_1, Z_1), \sigma(Y_1, W_1)) \\ &\quad - \mu(\sigma(X_1, W_1), \sigma^*(Y_1, Z_1)), \end{aligned} \quad (2.12)$$

for all vector fields X_1, Y_1, Z_1, W_1 .[44]

3. MAIN INEQUALITIES

With a double of conjugate connections, here we derive some sharp relationships for statistical submanifolds of Golden-like statistical manifolds.

Lemma 3.1. [27] *Let $x_1, x_2, x_3, \dots, x_d$ be $d \geq 4$ real numbers. In that case, we have*

$$\sum_{1 \leq \alpha < \beta \leq d} x_\alpha x_\beta - x_1 x_2 - x_3 x_4 \leq \frac{d-3}{2(d-2)} \left(\sum_{\alpha=1}^d x_\alpha \right)^2.$$

Equality holds if and only if $x_1 + x_2 = x_3 + x_4 = x_5 = \dots = x_d$.

Theorem 3.1. Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case, we have the following inequality

$$\begin{aligned} & (\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\ \geq & \left(-\frac{(1-\beta)\zeta_p - \beta\zeta_q}{2\sqrt{5}} \right) ([d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] - [1 + \Psi(\pi_1) + \Theta(\pi_1)] \\ & - [1 + \Psi(\pi_2) + \Theta(\pi_2)]) + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\ & - \frac{d^2(d-3)}{4(d-2)} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + 2(\hat{K}^0(\pi_1) + \hat{K}^0(\pi_2) - \hat{\tau}^0). \end{aligned} \quad (3.13)$$

Proof. Let $\{\varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ and $\{\varsigma_{d+1}, \dots, \varsigma_q\}$ be the orthonormal frames of $T_x N$ and $T_x^\perp N$, respectively. The scalar curvature of N corresponding to the sectional K -curvature is

$$\begin{aligned} \tau = & \frac{1}{2} \sum_{1 \leq i < j \leq d} \left[g(\Xi(\varsigma_i, \varsigma_j)\varsigma_j), \varsigma_i) + g(\Xi^*(\varsigma_i, \varsigma_j)\varsigma_j), \varsigma_i) \right. \\ & \left. - 2g(\Xi^0(\varsigma_i, \varsigma_j)\varsigma_j), \varsigma_i) \right]. \end{aligned}$$

Make use of (2.9) and Gauss equation for Ξ and Ξ^* . After discharging basic simple computations, we obtain

$$\begin{aligned} \tau = & \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-1) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - \tau^0 \\ & + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} \left[\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} + \sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - 2\sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma \right]. \end{aligned}$$

Using (2.7), we get

$$\begin{aligned} \tau = & \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) - \tau^0 \\ & + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} \left[\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} + \sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - 2\sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma \right], \end{aligned}$$

which can be put down as follows

$$\begin{aligned} \tau = & \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) - \tau^0 \\ & + 2 \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\sigma_{ii}^{0\gamma} \sigma_{jj}^{0\gamma} - (\sigma_{ij}^{0\gamma})^2] - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^\gamma \\ & + (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}]. \end{aligned}$$

By utilizing Gauss equation for the Levi-Civita connection, we obtain

$$\begin{aligned} \tau &= \tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ &\quad + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) - 2\hat{\tau}^0 \\ &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^\gamma - (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}]. \end{aligned} \quad (3.14)$$

Let π_1 and π_2 be mutually orthogonal two-planes spanned by $\{\varsigma_1, \varsigma_2\}$ and $\{\varsigma_3, \varsigma_4\}$, respectively. We denote

$$\mu(\phi_{\varsigma_1, \varsigma_1})\mu(\phi_{\varsigma_2, \varsigma_2}) = \Psi(\pi_1)$$

and

$$\mu(\phi_{\varsigma_3, \varsigma_3})\mu(\phi_{\varsigma_4, \varsigma_4}) = \Psi(\pi_2).$$

Also, as in [28], for an orthonormal bases $\{\varsigma_1, \varsigma_2\}$ and $\{\varsigma_3, \varsigma_4\}$ of two-plane sections, we denote

$$\Theta(\pi_1) = \mu(\phi_{\varsigma_1, \varsigma_2})\mu(\phi_{\varsigma_1, \varsigma_2}^*)$$

and

$$\Theta(\pi_2) = \mu(\phi_{\varsigma_3, \varsigma_4})\mu(\phi_{\varsigma_3, \varsigma_4}^*),$$

where $\Theta(\pi_1)$ and $\Theta(\pi_2)$ are real numbers in $[0, 1]$. In that case, the sectional curvature $K(\pi_1)$ of the plane π_1 is given by

$$\begin{aligned} K(\pi_1) &= \frac{1}{2} \left[\mu(\Xi(\varsigma_1, \varsigma_2)\varsigma_2), \varsigma_1) + \mu(\Xi^*(\varsigma_1, \varsigma_2)\varsigma_2), \varsigma_1) \right. \\ &\quad \left. - 2g(\Xi^0(\varsigma_1, \varsigma_2)\varsigma_2), \varsigma_1) \right]. \end{aligned} \quad (3.15)$$

By utilizing (2.9) and Gauss equation for Ξ and Ξ^* and set down systems in (3.15), we derive

$$\begin{aligned} K(\pi_1) &= \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \mu(\phi_{e_1, e_1})\mu(\phi_{e_2, e_2}) - \mu(\phi_{e_1, e_2})\mu(\phi_{e_2, e_1})] \\ &\quad - K^0(\pi_1) + \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^{*\gamma} + \sigma_{11}^{*\gamma} \sigma_{22}^\gamma - 2\sigma_{12}^{*\gamma} \sigma_{12}^\gamma \right]. \end{aligned}$$

By utilizing $\sigma + \sigma^* = 2\sigma^0$, we obtain

$$\begin{aligned} K(\pi_1) &= \left(-\frac{(1-\beta)\zeta_p - \beta\zeta_q}{2\sqrt{5}} \right) [1 + g(\phi_{e_1, e_1})g(\phi_{e_2, e_2}) \\ &\quad - g(\phi_{e_1, e_2})g(\phi_{e_2, e_1})] - K^0(\pi_1) + 2 \sum_{\gamma=d+1}^q \left[\sigma_{11}^{0\gamma} \sigma_{22}^{0\gamma} - (\sigma_{12}^{0\gamma})^2 \right] \\ &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left\{ \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] + \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right] \right\}. \end{aligned}$$

By utilizing Gauss equation for Levi-Civita connection, we obtain

$$\begin{aligned}
 K(\pi_1) &= K^0(\pi_1) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \mu(\phi_{\varsigma_1, \varsigma_1})\mu(\phi_{\varsigma_2, \varsigma_2}) \\
 &\quad + \mu(\phi_{\varsigma_1, \varsigma_2})\mu(\phi^*_{\varsigma_1, \varsigma_2})] - 2\hat{K}^0(\pi_1) - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right].
 \end{aligned} \tag{3.16}$$

The on the top equation can be noted in the following form:

$$\begin{aligned}
 K(\pi_1) &= K^0(\pi_1) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] - 2\hat{K}^0(\pi_1) \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right].
 \end{aligned} \tag{3.17}$$

Similarly, the sectional curvature $K(\pi_2)$ of the plane π_2 can be written as

$$\begin{aligned}
 K(\pi_2) &= K^0(\pi_2) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] - 2\hat{K}^0(\pi_2) \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^\gamma \sigma_{44}^\gamma - (\sigma_{34}^\gamma)^2 \right] \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^{*\gamma} \sigma_{44}^{*\gamma} - (\sigma_{34}^{*\gamma})^2 \right].
 \end{aligned} \tag{3.18}$$

The relations (3.14), (3.17) and (3.18) imply that

$$\begin{aligned}
 &(\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\
 &= \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
 &\quad + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) \\
 &\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}] \\
 &\quad - \left\{ \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^\gamma \sigma_{22}^\gamma - (\sigma_{12}^\gamma)^2 \right] \right. \\
 &\quad \left. - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - (\sigma_{12}^{*\gamma})^2 \right] \right\} - \left\{ \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] \right. \\
 &\quad \left. - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^\gamma \sigma_{44}^\gamma - (\sigma_{34}^\gamma)^2 \right] - \frac{1}{2} \sum_{\gamma=d+1}^q \left[\sigma_{33}^{*\gamma} \sigma_{44}^{*\gamma} - (\sigma_{34}^{*\gamma})^2 \right] \right\} \\
 &\quad + 2\hat{K}^0(\pi_1) + 2\hat{K}^0(\pi_2) - 2\hat{\tau}^0.
 \end{aligned}$$

$$\begin{aligned}
&\geq \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_q}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
&\quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] \\
&\quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} \left\{ \left[\sigma_{ii}^\gamma \sigma_{jj}^\gamma - \sigma_{11}^\gamma \sigma_{22}^\gamma - \sigma_{33}^\gamma \sigma_{44}^\gamma \right] \right. \\
&\quad \left. + \left[\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - \sigma_{11}^{*\gamma} \sigma_{22}^{*\gamma} - \sigma_{33}^{*\gamma} \sigma_{44}^{*\gamma} \right] \right\} \\
&\quad + 2\hat{K}^0(\pi_1) + 2\hat{K}^0(\pi_2) - 2\hat{\tau}^0,
\end{aligned}$$

where we have used the known relation $\sigma + \sigma^* = 2\sigma^0$.

Now, we apply Lemma 3.1 for σ and σ^* to obtain the following inequality:

$$\begin{aligned}
&(\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\
&\geq \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
&\quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_1) + \Theta(\pi_1)] \\
&\quad - \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [1 + \Psi(\pi_2) + \Theta(\pi_2)] \\
&\quad - \frac{d^2(d-3)}{4(d-2)} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + 2(\hat{K}^0(\pi_1) + \hat{K}^0(\pi_2) - \hat{\tau}^0).
\end{aligned}$$

From this relation, we can easily derive our desired inequality (3.13). □

The equality occasion of inequality (3.13) is shaped by the following:

Theorem 3.2. *Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case, the equality holds for (3.13) identically if and only if we have*

- (i) $\sigma_{11}^\gamma + \sigma_{22}^\gamma = \sigma_{33}^\gamma + \sigma_{44}^\gamma = \sigma_{55}^\gamma = \cdots = \sigma_{dd}^\gamma$.
- (ii) $\sigma_{11}^{*\gamma} + \sigma_{22}^{*\gamma} = \sigma_{33}^{*\gamma} + \sigma_{44}^{*\gamma} = \sigma_{55}^{*\gamma} = \cdots = \sigma_{dd}^{*\gamma}$.
- (iii) $\sigma_{ij}^\gamma = \sigma_{ij}^{*\gamma} = 0$, $1 \leq i < j \leq d$, $i \neq j$, $(i, j) \neq (1, 2), (2, 1), (3, 4), (4, 3)$,

for $\gamma \in \{d+1, d+2, \dots, q\}$.

Corollary 3.1. *Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional totally real statistical submanifold. In that case, we have*

$$\begin{aligned} & (\tau - K(\pi_1) - K(\pi_2)) - (\tau^0 - K^0(\pi_1) - K^0(\pi_2)) \\ & \geq (d(d-2) - 2) \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) - \frac{d^2(d-3)}{4(d-2)} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] \\ & \quad + 2(\hat{K}^0(\pi_1) + \hat{K}^0(\pi_2) - \hat{\tau}^0). \end{aligned} \quad (3.19)$$

The equality case of (3.19) can be discussed as Theorem 3.2.

Next, we derive Euler inequality. Before that, we provide the following

Proposition 3.1. *Suppose that (M, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional totally real statistical submanifold.. In that case, we have*

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 & \geq \frac{8}{d^2} \left\{ 2\text{Ric}^0(X, X) - \text{Ric}(X, X) \right. \\ & \quad + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & \quad \left. + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{j=2}^d \hat{K}^0(X \wedge \varsigma_j) \right\}, \end{aligned} \quad (3.20)$$

where $\{X = \varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ is an orthonormal basis of $T_x N$.

Proof. Let $\{\varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ and $\{\varsigma_{d+1}, \dots, \varsigma_q\}$ be the orthonormal frames of $T_x N$ and $T_x^\perp N$, respectively. In that instance, equations (2.5), (2.11), and (2.12) are used to determine the scalar curvature:

$$\begin{aligned} \tau &= 2\tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & \quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(\varsigma_i \wedge \varsigma_j) \\ & \quad - \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{1 \leq i < j \leq d} [\{\sigma_{ii}^\gamma \sigma_{jj}^\gamma - (\sigma_{ij}^\gamma)^2\} + \{\sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma} - (\sigma_{ij}^{*\gamma})^2\}], \end{aligned}$$

where $\hat{K}^0$ is the sectional curvature of N in accordance with Levi-Civita connection.

Using $\sigma + \sigma^* = 2\sigma^0$, we get

$$\begin{aligned} \tau &= 2\tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) + \text{tr}^2(\phi) - \text{tr}(\phi^*)] \\ & \quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(\varsigma_i \wedge \varsigma_j) \\ & \quad - \frac{d^2}{2} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + \frac{1}{2} [\|\sigma\|^2 + \|\sigma^*\|^2]. \end{aligned} \quad (3.21)$$

To introduce an inequality in (3.21), we compute $\|\sigma\|^2$ and $\|\sigma^*\|^2$ as follows:

$$\begin{aligned}
\|\sigma\|^2 &= \sum_{\gamma=d+1}^q \sum_{i,j=1}^d (\sigma_{ij}^\gamma)^2 \\
&= \sum_{\gamma=d+1}^q \{(\sigma_{11}^\gamma)^2 + (\sigma_{12}^\gamma)^2 + \cdots + (\sigma_{1d}^\gamma)^2 + (\sigma_{21}^\gamma)^2 + (\sigma_{22}^\gamma)^2 \\
&\quad + \cdots + (\sigma_{2d}^\gamma)^2 + \cdots + (\sigma_{d1}^\gamma)^2 + (\sigma_{d2}^\gamma)^2 + \cdots + (\sigma_{dd}^\gamma)^2\} \\
&= \frac{1}{2} \sum_{\gamma=d+1}^q \{(\sigma_{11}^\gamma + \sigma_{22}^\gamma + \cdots + \sigma_{dd}^\gamma)^2 + (\sigma_{11}^\gamma - \sigma_{22}^\gamma - \cdots - \sigma_{dd}^\gamma)^2\} \\
&\quad + 2 \sum_{\gamma=d+1}^q \left\{ \sum_{j=2}^d (\sigma_{1j}^\gamma)^2 + \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma) \right\} \\
&\geq \frac{1}{2} \sum_{\gamma=d+1}^q (\sigma_{11}^\gamma + \sigma_{22}^\gamma + \cdots + \sigma_{dd}^\gamma)^2 + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma) \\
&= \frac{d^2}{2} \|\Upsilon\|^2 + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma).
\end{aligned}$$

Similarly,

$$\|\sigma^*\|^2 \geq \frac{d^2}{2} \|\Upsilon\|^2 + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^{*\gamma})^2 - \sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma}).$$

Combining the two inequalities yields

$$\begin{aligned}
\|\sigma\|^2 + \|\sigma^*\|^2 &\geq \frac{d^2}{2} [\|\Upsilon\|^2 + \|\Upsilon\|^2] + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^\gamma)^2 - \sigma_{ii}^\gamma \sigma_{jj}^\gamma) \\
&\quad + 2 \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} ((\sigma_{ij}^{*\gamma})^2 - \sigma_{ii}^{*\gamma} \sigma_{jj}^{*\gamma}) \\
&= \frac{d^2}{2} [\|\Upsilon\|^2 + \|\Upsilon\|^2] + \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} \{(\sigma_{ij}^\gamma + \sigma_{ij}^{*\gamma})^2 \\
&\quad - (\sigma_{ii}^\gamma + \sigma_{ii}^{*\gamma})(\sigma_{jj}^\gamma + \sigma_{jj}^{*\gamma}) \\
&\quad + (\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - \sigma_{ij}^\gamma \sigma_{ij}^{*\gamma}) \\
&\quad + (\sigma_{ii}^{*\gamma} \sigma_{jj}^\gamma - \sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma)\}.
\end{aligned}$$

We put this inequality into (3.21), it follows that

$$\begin{aligned}
\tau &\geq 2\tau^0 + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\
&\quad + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(\varsigma_i \wedge \varsigma_j) \\
&\quad - \frac{d^2}{4} [\|\Upsilon\|^2 + \|\Upsilon^*\|^2] + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} \{(\sigma_{ij}^\gamma + \sigma_{ij}^{*\gamma})^2 \\
&\quad - (\sigma_{ii}^\gamma + \sigma_{ii}^{*\gamma})(\sigma_{jj}^\gamma + \sigma_{jj}^{*\gamma}) + (\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - \sigma_{ij}^\gamma \sigma_{ij}^{*\gamma}) + (\sigma_{ii}^{*\gamma} \sigma_{jj}^\gamma - \sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma)\}. \quad (3.22)
\end{aligned}$$

After rearranging (3.22), we arrive at

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 \geq & \frac{4}{d^2} \left\{ 2\tau^0 - \tau \right. \\ & + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}(\phi^*) + \text{tr}^2(\phi)] \\ & + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) - 4 \sum_{1 \leq i < j \leq d} \hat{K}^0(e_i \wedge e_j) \\ & + \frac{1}{2} \sum_{\gamma=d+1}^q \sum_{2 \leq i < j \leq d} [(\sigma_{ij}^\gamma + \sigma_{ij}^{*\gamma})^2 - (\sigma_{ii}^\gamma + \sigma_{ii}^{*\gamma})(\sigma_{jj}^\gamma + \sigma_{jj}^{*\gamma}) \\ & \left. + (\sigma_{ii}^\gamma \sigma_{jj}^{*\gamma} - \sigma_{ij}^\gamma \sigma_{ij}^{*\gamma}) + (\sigma_{ii}^{*\gamma} \sigma_{jj}^\gamma - \sigma_{ij}^{*\gamma} \sigma_{ij}^\gamma) \right\}. \end{aligned}$$

Thus, we have

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 \geq & \frac{8}{d^2} \left\{ 2\text{Ric}^0(X, X) - \text{Ric}(X, X) + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) \right. \\ & - \text{tr}(\phi^*) + \text{tr}^2(\phi)] + \left(-\frac{(1-\beta)\zeta_p + \beta\zeta_q}{4} \right) 2(d-1)\text{tr}(\phi) \\ & \left. - 4 \sum_{j=2}^d \hat{K}^0(X \wedge \varsigma_j) \right\}, \end{aligned}$$

where $\{X = \varsigma_1, \varsigma_2, \dots, \varsigma_d\}$ is an orthonormal basis of $T_x N$. □

Proposition 3.2. Suppose that (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case, the equality of (3.20) holds at $x \in N$ if and only if $2\sigma_{11} = d\Upsilon$, $2\sigma_{11}^* = d\Upsilon^*$ and $\sigma_{1j} = 0$, $\sigma_{1j}^* = 0$ for $j = 2, \dots, d$.

By using Proposition 3.1, we have

Theorem 3.3. Suppose (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold.

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 \geq & \frac{8}{d^3} \left\{ 2\tau^0 - \tau + \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) [d(d-2) - \text{tr}^2(\phi^*) \right. \\ & + \text{tr}(\phi)] + \left(-\frac{(1-\beta)\zeta_a + \beta\zeta_b}{4} \right) 2(d-1)\text{tr}(\phi) \\ & \left. - 4 \sum_{j=2}^d \hat{K}^0(X \wedge \varsigma_j) \right\}. \end{aligned} \quad (3.23)$$

Theorem 3.4. Suppose (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is an d -dimensional statistical submanifold. In that case the equality of 3.23 holds at $x \in N$ if and only if either (i) $\sigma = \sigma^* = 0$, if $d \geq 3$ or (ii) $\sigma = \mu \otimes \Upsilon$ and $\sigma^* = \mu \otimes \Upsilon^*$, if $d = 2$.

Example 3.1. Consider the Golden-like statistical manifold $(\mathbb{R}_d^{2d+q}, \hat{\nabla}, g, \phi)$ with standard coordinates and the semi-Riemannian metric g with the signature $(\underbrace{-, \dots, -}_{d\text{-times}}, \underbrace{+, \dots, +}_{(d+q)\text{-times}})$. The tensor field ϕ type $(1, 1)$ on $\mathbb{R}_d^{2d+q}$ is the following:

$$\phi = \frac{1}{2} \begin{bmatrix} \lambda_{ij} & \sqrt{5}\lambda_{ij} & 0 \\ \sqrt{5}\lambda_{ij} & \lambda_{ij} & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

where β is the Golden mean. In that case, ϕ is golden structure on $\mathbb{R}_d^{2d+q}$. Furthermore, the conjugate tensor field ϕ^* is defined as

$$\phi^* = \frac{1}{2} \begin{bmatrix} \lambda_{ij} & -\sqrt{5}\lambda_{ij} & 0 \\ -\sqrt{5}\lambda_{ij} & \lambda_{ij} & 0 \\ 0 & 0 & \beta \end{bmatrix},$$

Next, let N be any d -dimensional statistical submanifold of the Golden-like statistical manifold $(\mathbb{R}_d^{2d+q}, \hat{\nabla}, \mu, \phi)$. In that case, the inequality (3.23) is satisfied. In particular, the statistical submanifold $N = \mathbb{R}_1^3$ of $\mathbb{R}_2^5$ attains equality in (3.23) because N is a totally geodesic submanifold of $\mathbb{R}_2^5$ in accordance with the Levi-Civita connection, that is, $\sigma = \sigma^* = 0$.

To justify this claim explicitly, take $N = \mathbb{R}_1^3$ to be embedded in $F = \mathbb{R}_2^5$ as the standard linear (affine) subspace determined by fixing the last two ambient coordinates to be constant. On $F = \mathbb{R}_2^5$ with the standard flat statistical structure, the dual connections $\hat{\nabla}$ and $\hat{\nabla}^*$ both coincide with the ordinary directional derivative D of $\mathbb{R}_2^5$ (in flat coordinates the connection coefficients vanish identically, and $\hat{\nabla} = \hat{\nabla}^* = \hat{\nabla}^0 = D$). For tangent vector fields X_1, Y_1 on N , extended to F so that their components along the two fixed (normal) directions vanish identically, the directional derivative $D_{X_1}Y_1$ again has vanishing components in those directions, i.e. $D_{X_1}Y_1$ is tangent to N at every point. By the Gauss formula

$$\hat{\nabla}_{X_1}Y_1 = \nabla_{X_1}Y_1 + \sigma(X_1, Y_1), \quad \hat{\nabla}_{X_1}^*Y_1 = \nabla_{X_1}^*Y_1 + \sigma^*(X_1, Y_1),$$

the normal-valued terms $\sigma(X_1, Y_1)$ and $\sigma^*(X_1, Y_1)$ are exactly the components of $\hat{\nabla}_{X_1}Y_1 = D_{X_1}Y_1$ and $\hat{\nabla}_{X_1}^*Y_1 = D_{X_1}^*Y_1$ transverse to N . Since $D_{X_1}Y_1$ has been shown to be tangent to N , these transverse components vanish for all X_1, Y_1 tangent to N , and hence $\sigma \equiv 0$ and $\sigma^* \equiv 0$ on N . This confirms that $N = \mathbb{R}_1^3$ is totally geodesic in $F = \mathbb{R}_2^5$ with respect to both dual connections, as claimed.

Corollary 3.2. Suppose (F, μ, ϕ) is an q -dimensional Golden-like statistical manifold and N is a d -dimensional totally real statistical submanifold.. In that case, we have

$$\begin{aligned} \|\Upsilon\|^2 + \|\Upsilon^*\|^2 &\geq \frac{8}{d^3} \left\{ 2\tau^0 - \tau + d(d-2) \left(-\frac{(1-\beta)\zeta_a - \beta\zeta_b}{2\sqrt{5}} \right) \right. \\ &\quad \left. - 4 \sum_{j=2}^d \hat{K}^0(X \wedge e_j) \right\}. \end{aligned} \quad (3.24)$$

The equality case of (3.24) can be discussed as Theorem 3.4

4. CONCLUSION

This topic focuses on Euler and Chen type inequalities in submanifold geometry, as well as intrinsic and extrinsic curvature in statistical manifolds with Golden structures, which fills a niche that is unique. This study fills a gap in the literature by extending statistical

manifold results to geometric settings and analyzing the situation of statistical manifolds with golden structures. This study provides formal characterizations of various Pinching theorems with trivial normal connections, thereby strengthening the theoretical value of the optimal bounds. We turn up the results relative to Chen-invariant, expanding the range of ambient spaces where sharp bounds are obtained. The goal is to understand the relationship between intrinsic scalar curvature and extrinsic type invariants, as well as identify examples of equality or rigidity.

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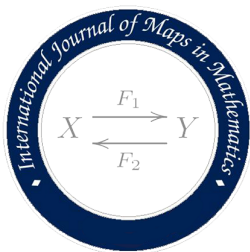
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RIGIDITY OF YAMABE SOLITONS ON $N(k)$ -CONTACT METRIC MANIFOLDS

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Abstract. We study Yamabe solitons on $N(k)$ -contact metric manifolds under D -homothetic and infinitesimal D -conformal deformations. We obtain rigidity results for Yamabe solitons under non-trivial D -homothetic deformation and show that an infinitesimal D -conformal deformation reduces to a homothetic deformation with constant factor. Consequently, the associated deformation vector field is affine, and no genuinely nontrivial Yamabe solitons arise under these natural geometric constraints.

Keywords: Infinitesimal D -conformal deformation, Yamabe soliton, $N(k)$ -manifold, D -homothetic deformation.

MSC (2020): 53C15, 53D10

1. INTRODUCTION

The study of conformal geometric evolution equations relies heavily on Yamabe solitons, which are self-similar solutions of the Yamabe flow. Yamabe solitons are more delicate and flexible in their classification because they rely solely on scalar curvature, in contrast to Ricci solitons. The notion of Yamabe flow was introduced by Hamilton [8] as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian metrics on a Riemannian manifold (M^n, g) , $n \geq 3$. The Yamabe flow is an evolution equation for metrics on a Riemannian manifold, given by

$$\frac{\partial}{\partial t} g = -rg,$$

where r is the scalar curvature corresponding to the metric g . In dimension $n = 2$, the Yamabe flow is equivalent to the Ricci flow. However, in dimension $n > 2$, the Yamabe and Ricci flows do not agree, since the Yamabe flow preserves the conformal class of the metric, whereas the Ricci flow does not in general. A Yamabe soliton on a Riemannian manifold (M, g) is a triplet (g, V, ρ) satisfying

$$\mathcal{L}_V g = 2(r - \rho)g, \tag{1.1}$$

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where $\mathcal{L}_V$ denotes the Lie derivative of g in the direction of the vector field V and ρ is a constant. The Yamabe soliton is said to be shrinking, steady, or expanding according as $\rho < 0$, $\rho = 0$, or $\rho > 0$, respectively.

If ρ is allowed to be a smooth function on M , then the metric satisfying (1.1) is called an almost Yamabe soliton. The rigidity of Ricci solitons and Ricci nearly solitons on K-contact and Sasakian manifolds has been studied in detail in the subject of contact metric geometry, especially in the context of symmetry assumptions like projective vector fields or D -homothetic invariance. Beyond the Sasakian context, however, very little is known about Yamabe solitons. A natural setting for extending these rigidity results is the class of $N(k)$ -contact metric manifolds. The study of curvature conditions involving the Reeb vector field naturally gives rise to these manifolds, which include Sasakian manifolds as the specific case ($k = 1$).

In [6], Daskalopoulos and Sesum showed that a compact gradient Yamabe soliton has constant scalar curvature. In [10], Sharma studied Yamabe solitons in contact Riemannian geometry, and proved that a 3-dimensional Sasakian manifold whose metric is Sasakian has constant scalar curvature and the soliton vector field V is Killing. Nagaraja and Sharma [7] established that a complete K -contact Ricci almost soliton with projective potential vector field is compact Sasakian and shrinking and that divergence is invariant under D -homothetic deformation. The purpose of this paper is to establish a unified rigidity theory for Yamabe solitons on $N(k)$ -contact metric manifolds under D -homothetic deformation.

2. PRELIMINARIES

A $(2n + 1)$ -dimensional differentiable manifold M is said to admit an almost contact structure if it admits a tensor field ϕ of type $(1, 1)$, a vector field ξ , and a 1-form η satisfying [1, 3]

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \quad (2.2)$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M \times \mathbb{R}$ defined by $J(X, f \frac{d}{dt}) = (\phi X - f\xi, \eta(X) \frac{d}{dt})$ is integrable, where X is tangent to M , t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$.

Let g be a compatible Riemannian metric with the almost contact structure (ϕ, ξ, η) , that is,

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad (2.3)$$

for all vector fields X, Y on M .

Then M becomes an almost contact metric manifold equipped with the almost contact metric structure (ϕ, ξ, η, g) . In an almost contact metric manifold the following hold.

$$\eta(X) = g(X, \xi), \quad g(\phi X, Y) = -g(X, \phi Y), \quad (2.4)$$

for all vector fields X, Y on M . An almost contact metric structure (ϕ, ξ, η, g) is said to be a contact metric structure if

$$d\eta(X, Y) = g(X, \phi Y), \quad (2.5)$$

for all vector fields X, Y on M . A normal contact metric manifold is called a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \quad (2.6)$$

for all $X, Y \in TM$, where ∇ denotes the Levi-Civita connection of the Riemannian metric g . A contact metric manifold (M, ϕ, ξ, η, g) for which ξ is a Killing vector field is said to be a K -contact manifold. Every Sasakian manifold is K -contact, but the converse is not true. However, a 3-dimensional K -contact manifold is Sasakian [9]. Blair, Koufogiorgos and Papantoniou [5] introduced the (k, μ) -nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ) -nullity distribution $N(k, \mu)$ of a contact metric manifold M is defined by

$$N(k, \mu) : p \longmapsto N_p(k, \mu) = \{W \in T_p M : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\},$$

for all $X, Y \in TM$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M with $\xi \in N(k, \mu)$ is called a (k, μ) -contact metric manifold. If $\mu = 0$, the (k, μ) -nullity distribution reduces to the k -nullity distribution [11].

The k -nullity distribution $N(k)$ of a Riemannian manifold is defined by [11]

$$N(k) : p \longmapsto N_p(k) = \{Z \in T_p M : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

where k is a constant. If the characteristic vector field $\xi \in N(k)$, then the contact metric manifold is called an $N(k)$ -contact metric manifold [4]. If $k = 1$, then the manifold is Sasakian, and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [2].

However, for an $N(k)$ -contact metric manifold M of dimension $(2n + 1)$, we have [4]

$$(\nabla_X \phi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX), \quad (2.7)$$

$$(\nabla_X \eta)(Y) = -g(\phi X + \phi hX, Y), \quad (2.8)$$

$$\nabla_X \xi = -\phi X - \phi hX, \quad (2.9)$$

where $h = \frac{1}{2}\mathcal{L}_\xi \phi$. Moreover, $h^2 = (k - 1)\phi^2$. In an $N(k)$ -contact metric manifold the following hold.

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y], \quad (2.10)$$

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X], \quad (2.11)$$

$$\text{Ric}(Y, \xi) = 2nk\eta(Y), \quad (2.12)$$

$$r = 2n(2n - 2 + k), \quad (2.13)$$

where r denotes the scalar curvature and Ric is the Ricci tensor.

A contact metric manifold (M, ϕ, ξ, η, g) is said to be Einstein if its Ricci tensor Ric satisfies $\text{Ric}(X, Y) = \lambda g(X, Y)$, where λ is a constant. It is said to be η -Einstein if

$\text{Ric}(X, Y) = \alpha g(X, Y) + \beta \eta(X)\eta(Y)$, where α and β are smooth functions on M .

Definition 2.1. A vector field V on a Riemannian manifold (M, g) is called an affine vector field if its local flow preserves the Levi-Civita connection ∇ . Equivalently,

$$\mathcal{L}_V \nabla = 0. \quad (2.14)$$

Definition 2.2. A D -homothetic deformation of a contact metric structure is given by [12]

$$\bar{\eta} = a\eta, \quad \bar{\xi} = \frac{1}{a}\xi, \quad \bar{\phi} = \phi, \quad \bar{g} = ag + a(a-1)\eta \otimes \eta, \quad (2.15)$$

where a is a positive constant. If (M, ϕ, ξ, η, g) is an almost contact metric manifold with contact form η , then $(M, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is also an almost contact metric manifold.

3. RIGIDITY UNDER D -HOMOTHEIC DEFORMATION

In this section we study Yamabe solitons on $N(k)$ -contact metric manifolds under D -homothetic deformation. The quantities with bar denote those with respect to D -homothetically deformed space.

Theorem 3.1. Let (M, ϕ, ξ, η, g) be an $N(k)$ -contact metric manifold and let $(M, \bar{g})$ be obtained from (M, g) by a non-trivial D -homothetic deformation. If (M, g, V, ρ) and $(M, \bar{g}, V, \bar{\rho})$ are both Yamabe solitons with the same soliton vector field V . Then the following assertions hold:

- (i) V is an infinitesimal contact transformation, and is strict if and only if $\bar{r} - \bar{\rho} = r - \rho$.
- (ii) V transforms η , ξ and ϕ according to

$$\mathcal{L}_V \eta = c\eta, \quad \mathcal{L}_V \xi = \sigma\xi, \quad \mathcal{L}_V \phi = \sigma\phi,$$

where

$$c = \frac{2a(\bar{r} - \bar{\rho}) - (a+1)(r - \rho)}{a-1}, \quad \sigma = \frac{2a(\bar{r} - \bar{\rho}) - (3a-1)(r - \rho)}{a-1}.$$

Proof. Let (M, g) be an $N(k)$ -contact metric manifold and let $(M, \bar{g})$ be obtained from (M, g) by a D -homothetic deformation. Suppose that (M, g, V, ρ) and $(M, \bar{g}, V, \bar{\rho})$ are Yamabe solitons. Then

$$\mathcal{L}_V g = 2(r - \rho)g, \quad (3.16)$$

$$\mathcal{L}_V \bar{g} = 2(\bar{r} - \bar{\rho})\bar{g}. \quad (3.17)$$

Taking the Lie derivative of the D -homothetically deformed metric (2.15) along V , we obtain

$$\mathcal{L}_V \bar{g} = a\mathcal{L}_V g + a(a-1)((\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta)). \quad (3.18)$$

Using (3.16) and (3.17), equation (3.18) becomes

$$2(\bar{r} - \bar{\rho})\bar{g} = 2a(r - \rho)g + a(a-1)[(\mathcal{L}_V \eta) \otimes \eta + \eta \otimes (\mathcal{L}_V \eta)]. \quad (3.19)$$

Evaluating (3.19) on $(X, \bar{\xi})$, and using $\bar{g}(X, \bar{\xi}) = \bar{\eta}(X)$, $\bar{\xi} = \frac{1}{a}\xi$, and $\bar{\eta} = a\eta$, we obtain

$$2(\bar{r} - \bar{\rho})a\eta(X) = 2(r - \rho)\eta(X) + (a-1)[(\mathcal{L}_V \eta)(X) + \eta(X)(\mathcal{L}_V \eta)(\xi)]. \quad (3.20)$$

Lie differentiating $\eta(\xi) = 1$ along V gives

$$(\mathcal{L}_V \eta)(\xi) = -\eta(\mathcal{L}_V \xi). \quad (3.21)$$

Also, Lie differentiating $g(\xi, \xi) = 1$ gives

$$(\mathcal{L}_V g)(\xi, \xi) + 2\eta(\mathcal{L}_V \xi) = 0. \quad (3.22)$$

Using (3.16), this yields

$$\eta(\mathcal{L}_V \xi) = -(r - \rho), \quad (3.23)$$

and hence

$$(\mathcal{L}_V \eta)(\xi) = r - \rho. \quad (3.24)$$

Substitution of (3.24) into (3.20) gives

$$(\mathcal{L}_V \eta)(X) = \frac{2a(\bar{r} - \bar{\rho}) - (a+1)(r - \rho)}{a-1} \eta(X). \quad (3.25)$$

Thus

$$\mathcal{L}_V \eta = c\eta, \quad c = \frac{2a(\bar{r} - \bar{\rho}) - (a+1)(r - \rho)}{a-1}. \quad (3.26)$$

Hence V is an infinitesimal contact transformation. Moreover, V is strict precisely when $c = 0$.

Next, using

$$(\mathcal{L}_V g)(X, \xi) = (\mathcal{L}_V \eta)(X) - g(X, \mathcal{L}_V \xi)$$

and (3.16), we obtain

$$\mathcal{L}_V \xi = \sigma \xi, \quad (3.27)$$

where

$$\sigma = \frac{2a(\bar{r} - \bar{\rho}) - (3a-1)(r - \rho)}{a-1}. \quad (3.28)$$

Finally, since $\mathcal{L}_V$ commutes with d , from $d\eta(X, Y) = g(X, \phi Y)$ we have

$$d(\mathcal{L}_V \eta)(X, Y) = (\mathcal{L}_V g)(X, \phi Y) + g(X, (\mathcal{L}_V \phi)Y). \quad (3.29)$$

Using $\mathcal{L}_V \eta = c\eta$ and (3.16), equation (3.29) becomes

$$c g(X, \phi Y) = 2(r - \rho)g(X, \phi Y) + g(X, (\mathcal{L}_V \phi)Y). \quad (3.30)$$

Therefore

$$\mathcal{L}_V \phi = (c - 2(r - \rho))\phi = \sigma \phi. \quad (3.31)$$

This proves (i) – (ii). □

4. RIGIDITY OF YAMABE SOLITONS UNDER INFINITESIMAL D -CONFORMAL DEFORMATIONS

We begin with the notion of an infinitesimal D -conformal transformation in the contact metric setting.

Definition 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a contact metric manifold. A vector field V is said to generate an infinitesimal D -conformal deformation if*

$$\mathcal{L}_V g = 2\alpha g + 2\beta \eta \otimes \eta, \quad (4.32)$$

for smooth functions α, β on M .

Remark 4.1. When α is constant and $\beta = \alpha(\alpha - 1)$, the above definition corresponds to the infinitesimal generator of a global D -homothetic deformation.

Theorem 4.1. Let $(M, \varphi, \xi, \eta, g)$ be an $N(k)$ -contact metric manifold admitting a Yamabe soliton (g, V, ρ) . If V generates an infinitesimal D -conformal deformation, then

$$\mathcal{L}_V g = 2(r - \rho)g,$$

where $r = 2n(2n - 2 + k)$ is constant. Consequently, V is a homothetic and affine vector field.

Proof. Since (M, g, V, ρ) is a Yamabe soliton, we have

$$\mathcal{L}_V g = 2(r - \rho)g. \quad (4.33)$$

On the other hand, since V generates an infinitesimal D -conformal deformation, we have

$$\mathcal{L}_V g = 2\alpha g + 2\beta \eta \otimes \eta. \quad (4.34)$$

Comparing (4.33) and (4.34), we obtain

$$(r - \rho)g = \alpha g + \beta \eta \otimes \eta. \quad (4.35)$$

Let $X \in \ker \eta$, so that $\eta(X) = 0$. Evaluating (4.35) on (X, X) gives

$$(r - \rho)g(X, X) = \alpha g(X, X).$$

Therefore,

$$\alpha = r - \rho. \quad (4.36)$$

Substituting (4.36) into (4.35), we obtain

$$\beta \eta \otimes \eta = 0,$$

and hence

$$\beta = 0. \quad (4.37)$$

Thus the infinitesimal D -conformal deformation reduces to the infinitesimal homothetic deformation

$$\mathcal{L}_V g = 2(r - \rho)g. \quad (4.38)$$

For an $N(k)$ -contact metric manifold, the scalar curvature is

$$r = 2n(2n - 2 + k), \quad (4.39)$$

and hence r is constant. Since ρ is also constant, $r - \rho$ is constant. Therefore V is a homothetic vector field with constant homothety factor. Consequently,

$$\mathcal{L}_V g = 2cg, \quad c = r - \rho = \text{constant}. \quad (4.40)$$

Thus, V is a homothetic vector field. Using the standard formula for the Lie derivative of the Levi-Civita connection, we have

$$\begin{aligned} 2g((\mathcal{L}_V \nabla)(X, Y), Z) &= (\nabla_X \mathcal{L}_V g)(Y, Z) + (\nabla_Y \mathcal{L}_V g)(X, Z) \\ &\quad - (\nabla_Z \mathcal{L}_V g)(X, Y). \end{aligned} \quad (4.41)$$

Since c is constant and $\nabla g = 0$, it follows that

$$\nabla_X(\mathcal{L}_V g) = \nabla_X(2cg) = 0.$$

Similarly,

$$\nabla_Y(\mathcal{L}_V g) = 0, \quad \nabla_Z(\mathcal{L}_V g) = 0.$$

Consequently,

$$\mathcal{L}_V \nabla = 0.$$

Hence, V is an affine vector field.

□

5. CONCLUDING REMARKS

The results demonstrate that Yamabe solitons on $N(k)$ -contact metric manifolds exhibit strong rigidity under D -homothetic and infinitesimal D -conformal deformations. In particular, an infinitesimal D -conformal deformation reduces to a homothetic deformation with constant factor, and the corresponding vector field is affine.

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