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EDITORIAL

BAYRAM ŞAHİN *

We are pleased to present Volume 9, Issue 1, marking the beginning of a new publication year for our journal. As we proudly complete our 8th year, we would like to extend our sincere gratitude to our authors, reviewers, editorial board members, and readers who have contributed to the continuous growth of the journal.

Over the past year, we have taken important steps to strengthen our publication process and broaden the academic contribution of our journal. One of the most notable outcomes of these efforts has been the increase in the number of articles published, allowing us to offer a richer and more diverse selection of research to our readership while maintaining our commitment to academic quality and ethical publishing standards.

During this 8-year period, our journal has been included in numerous databases. This progress supports our goal of improving the visibility, accessibility, and impact of the research we publish. Being indexed in additional databases not only expands the reach of our authors' work but also contributes to the journal's ongoing advancement within the academic community.

We remain committed to further improving the quality of our journal, enhancing its international visibility, and providing an efficient and transparent review process. We thank everyone who has supported us on this journey and hope that this issue will be beneficial to researchers and practitioners alike.

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SUBMERSIONS OF CR-SUBMANIFOLDS WITH TOTALLY UMBILICAL FIBERS

AMINE YILMAZ *

Abstract. The purpose of this paper is to study totally umbilical submersions of a CR-submanifolds M of a Kaehler manifolds $\bar{M}$ onto an almost Hermitian manifolds N such that characteristic vector field is vertical or horizontal vector field. We first show the totally geodesicity of fibers of such submersions. In addition, inequality in terms of O'Neill tensor fields is presented and new results regarding the equality sitation are obtained. Finally, the geodesicity of the fibers by submersion with parallel mean curvature vector field was investigated in complex space form.

Keywords: Kaehler manifolds, Riemannian submersions, Submersions, CR-submanifolds, Complex space form.

MSC (2020): 53C15, 53C42.

1. Introduction

In 1966, O'Neill [18], introduced the concept of Riemannian submersion. In the seventies A. Bejancu [5] introduced the notion of a CR-submanifold as a natural generalization of both complex submanifolds and totally real submanifolds. In [14], S.Koboyashi observed the above similarity between the Riemannian submersion and CR-submanifold in terms of the distributions, thus he introduced the submersion of a CR-submanifold. It has been studied by many authors in different classes (see, [1], [2], [11], [13], [16], [17], [19], [22]). Also, the case of totally umbilic fibers was introduced by G.Baditoiu and S.Ianus in [3].

This paper focus on the geometry of the fibers, emphasing on the concept of totally umbilicity. The paper is organized in the following way: In section 2, we recall basic notions for later section. In section 3, we study the totally umbilical submersion of a CR-submanifolds M of a Kaehler manifolds $\bar{M}$ onto an almost Hermitian manifolds N. We show the totally geodesicity of fibers of such submersions. The relation between the bisectional curvatures of the fibers and the respective manifolds is given. In section 4, we prove that there exist no totally umbilical submersion of CR-submanifold with nonzero parallel mean curvature vector field in complex space form $c \neq 0$. In addition, the totally geodesic conditions of

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such submersions in the complex space form were examined and results related to the section curvatures were obtained.

2. Preliminaries

In this section, we give some definitions from [4], [5], [12], [18] and [23] for later sections. Let $\bar{M}$ be a Riemannian manifold with an almost complex structure J and Hermitian metric $g_{\bar{M}}$ satisfying

$$J^2 = -I, \qquad g_{\bar{M}}(JX, JY) = g_{\bar{M}}(X, Y). \tag{2.1}$$
$$(\bar{\nabla}_X J)Y = \bar{\nabla}_X JY - J\bar{\nabla}_X Y$$

for any $X, Y \in T\bar{M}$, where $\Gamma(T\bar{M})$ is the Lie algebra of vector fields in $\bar{M}$, then the $(\bar{M}, g_{\bar{M}})$ is called an almost Hermitian manifold. If the almost complex structure J also satisfies

$$(\bar{\nabla}_X J)Y = 0 \tag{2.2}$$

for every $X, Y \in T\bar{M}$, then $\bar{M}$ is said to be a Kaehler manifold [23].
In [18], B.O.Neill defined the concept of Riemannian submersions as follows: A surjective map $F : (M^m, g) \to (N^n, g)$ respectively m and n dimensional between Riemannian manifolds is called a *Riemannian submersion* if:
(S1) F has maximal rank, and
(S2) F_*, restricted to $(kerF_*)^{\perp}$, is a linear isometry.
Respectively, invariant tensors T and A are defined by

$$T_E F = \nu\nabla_{\nu E}\mathcal{H}F + \mathcal{H}\nabla_{\nu E}\nu F, \tag{2.3}$$

$$A_E F = \nu\nabla_{\mathcal{H}E}\mathcal{H}F + \mathcal{H}\nabla_{\mathcal{H}E}\nu F. \tag{2.4}$$

It is known that the tensors T and A satisfy that

$$T_{\mathcal{P}}\mathcal{Q} = T_{\mathcal{Q}}\mathcal{P}, \tag{2.5}$$

$$A_{\mathcal{R}}\mathcal{W} = -A_{\mathcal{W}}\mathcal{R} = \frac{1}{2}\nu[\mathcal{R}, \mathcal{W}] \tag{2.6}$$

for any $\mathcal{P}, \mathcal{Q} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{R}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$. From (2.3) and (2.4) equalities, we have

$$\nabla_{\mathcal{P}}\mathcal{Q} = T_{\mathcal{P}}\mathcal{Q} + \nu\nabla_{\mathcal{P}}\mathcal{Q}, \tag{2.7}$$

$$\nabla_{\mathcal{P}}\mathcal{R} = T_{\mathcal{P}}\mathcal{R} + \mathcal{H}\nabla_{\mathcal{P}}\mathcal{R}, \tag{2.8}$$

$$\nabla_{\mathcal{R}}\mathcal{P} = A_{\mathcal{R}}\mathcal{P} + \nu\nabla_{\mathcal{R}}\mathcal{P}, \tag{2.9}$$

$$\nabla_{\mathcal{R}}\mathcal{W} = \mathcal{H}\nabla_{\mathcal{R}}\mathcal{W} + A_{\mathcal{R}}\mathcal{W}. \tag{2.10}$$

For given a submanifold, M, of a Kaehler manifolds $(\bar{M}, J, g_{\bar{M}})$, in [7], Bejancu introduced the concept of CR-submanifolds as follows: D is invariant, $JD \subset D$ and $D^{\perp}$ is anti-invariant, $JD^{\perp} \subset TM^{\perp}$, where $TM^{\perp}$ is the normal space of M.
On the other hand, for given a CR submanifold M, an almost Hermitian manifold N, in [14], Kobayashi introduced the submersion of a CR-submanifold as follows: the submersion $F : M \to N$ such that (i) $D^{\perp}$ is the kernel of F_*, i.e., $F_* D^{\perp} = \{0\}$,
(ii) J interchanges $D^{\perp}$ and $TM^{\perp}$, i.e., $JD^{\perp} = TM^{\perp}$,
(iii) $F_* : D_p \to T_{F(p)}N$ is a complex isometry.

For a vector field $\mathcal{P}$ on M, we get

$$\mathcal{P} = H\mathcal{P} + V\mathcal{P}, \tag{2.11}$$

where H and V denote the horizontal and vertical part of $\mathcal{P}$.
Since the fibers of a Riemann submersion are submanifolds of the definition manifold, the concepts defined for submanifolds are also defined for fibers. Let $F : M \to N$ be a Riemannian submersion. Besse [8] defined the Riemannian curvatures of the M manifold, the N manifold and fibers as follows, R, R^* and $\bar{R}$, respectively

$$R(\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W}) = \bar{R}(\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W}) - g(T_{\mathcal{P}}\mathcal{W}, T_{\mathcal{Q}}\mathcal{R}) + g(T_{\mathcal{Q}}\mathcal{W}, T_{\mathcal{P}}\mathcal{R}), \tag{2.12}$$

$$R(\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{U}) = g((\nabla_{\mathcal{P}}T)(\mathcal{Q},\mathcal{R}),\mathcal{U}) - g((\nabla_{\mathcal{Q}}T)(\mathcal{P},\mathcal{R}),\mathcal{U}), \tag{2.13}$$

$$\begin{aligned} R(\mathcal{U},\mathcal{L},\mathcal{K},\mathcal{T}) &= R^*(\mathcal{U},\mathcal{L},\mathcal{K},\mathcal{T}) + 2g(A_{\mathcal{U}}\mathcal{L}, A_{\mathcal{K}}\mathcal{T}) \\ &\quad - g(A_{\mathcal{L}}\mathcal{K}, A_{\mathcal{U}}\mathcal{T}) + g(A_{\mathcal{U}}\mathcal{K}, A_{\mathcal{L}}\mathcal{T}), \end{aligned} \tag{2.14}$$

$\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{U},\mathcal{L},\mathcal{K},\mathcal{T} \in \mathcal{X}^{\mathcal{H}}(M)$.

In [20], the Gauss and Codazzi equations for submersions are given as follows

$$g(R(\mathcal{M},\mathcal{N})\mathcal{H},\mathcal{Y}) = g(\bar{R}(\mathcal{M},\mathcal{N})\mathcal{H},\mathcal{Y}) - g(T_{\mathcal{M}}\mathcal{Y}, T_{\mathcal{N}}\mathcal{H}) + g(T_{\mathcal{N}}\mathcal{Y}, T_{\mathcal{M}}\mathcal{H}), \tag{2.15}$$

$$g(R(\mathcal{M},\mathcal{N})\mathcal{H},\mathcal{Z}) = g((\nabla_{\mathcal{M}}T)_{\mathcal{N}}\mathcal{H},\mathcal{Z}) - g((\nabla_{\mathcal{N}}T)_{\mathcal{M}}\mathcal{H}),\mathcal{Z}), \tag{2.16}$$

$\mathcal{M},\mathcal{N},\mathcal{H},\mathcal{Y} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{Z} \in \mathcal{X}^{\mathcal{H}}(M)$

Definition 2.1. [23] *Let c be constant holomorphic sectional curvature. In this case the curvature tensor of $\bar{M}(c)$ is given by*

$$\begin{aligned} R(\mathcal{K},\mathcal{L})\mathcal{M} &= \frac{c}{4}[g(\mathcal{K},\mathcal{M})\mathcal{L} - g(\mathcal{L},\mathcal{M})\mathcal{K} + g(J\mathcal{K},\mathcal{M})J\mathcal{L} \\ &\quad - g(J\mathcal{L},\mathcal{M})J\mathcal{K} + 2g(J\mathcal{K},\mathcal{L})J\mathcal{M}] \end{aligned} \tag{2.17}$$

$\mathcal{K},\mathcal{L},\mathcal{M} \in \bar{M}$.

Example 2.1. *For any $\theta \in (0, \frac{\pi}{2})$, let $F : R^4 \to R^2$ be a map defined by $F(x_1, x_2, x_3, x_4) = (x_1 \cos\theta + x_3 \sin\theta, x_2 \sin\theta + x_4 \cos\theta)$. Then, by direct calculations, we obtain the Jacobian matrix of F as:*

$$F_* = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

The rank of F_* is equal to 2. Thus, the map F is a submersion. After some computations, we obtain

$$ker F_* = \nu = span\{V_1 = (\sin\theta, 0, -\cos\theta, 0), V_2 = (0, \cos\theta, 0, -\sin\theta)\}$$

$$(ker F_*)^{\perp} = \mathcal{H} = span\{H_1 = (\cos\theta, 0, \sin\theta, 0), H_2 = (0, \sin\theta, 0, \cos\theta)\}.$$

3. Submersions of CR-submanifolds with totally umbilical fibers in Kaehler Manifolds

In this section, we investigate the integrability of horizontal distribution of the totally umbilical submersion of a CR-submanifold and study totally geodesicness. We obtain relations between the curvatures of the fibers and the respective manifolds.

Let (M, g_1), (N, g_2) be Riemannian manifolds and $F : (M, g_1) \to (N, g_2)$ a Riemannian submersion. Any fiber of a Riemannian submersion F is called totally umbilical if

$$T_{\mathcal{U}}\mathcal{V} = g_1(\mathcal{U}, \mathcal{V})H, \tag{3.18}$$

for $\mathcal{U}, \mathcal{V} \in \mathcal{X}^{\nu}(M)$, where H is the mean curvature vector field of the fiber [18].
Now, we assume that $F : (M, g_M) \to (B, g_B)$ be a totally umbilical submersion of a CR submanifold M of a Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold B.

Theorem 3.1. *Let F be a submersion with totally umbilical fibers. Then the fibers of the submersion F are totally geodesic, if vertical distribution ν is parallel.*

Proof. By using equation (2.7), we get

$$\nabla_P \mathcal{Q} = T_P \mathcal{Q} + \hat{\nabla}_P \mathcal{Q}$$

for the vector fields $P, \mathcal{Q} \in \mathcal{X}^{\nu}(M)$.
Since the vertical distribution ν is parallel, we obtain $T_P\mathcal{Q} = g(P, \mathcal{Q})H = 0$. Hence, the fibers of the submersion F are totally geodesic. □

Theorem 3.2. *Let F be a submersion with totally umbilical fibers. Then we have the horizontal distribution $\mathcal{H}$ is integrable if and onl if*

$$A_{\mathcal{R}} J\mathcal{W} = A_{\mathcal{W}} J\mathcal{R}$$

for the vector fields $\mathcal{R}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$.

Proof. By (2.2) and (2.10), we obtain

$$\mathcal{H}\nabla_{\mathcal{R}} J\mathcal{W} + A_{\mathcal{R}} J\mathcal{W} = J\nabla_{\mathcal{R}}\mathcal{W}.$$

For the vertical vector field $\mathcal{Z}$, we have

$$g(A_{\mathcal{R}} J\mathcal{W} - A_{\mathcal{W}} J\mathcal{R}, \mathcal{Z}) = g(J[\mathcal{R}, \mathcal{W}], \mathcal{Z}).$$

Thus the proof the theorem completes. □

Corollary 3.1. *Let F be a submersion with totally umbilical fibers. Then M is locally Riemannian product if and only if $A_{\mathcal{Z}} J\mathcal{W} = A_{\mathcal{W}} J\mathcal{Z}$ and fibres are totally geodesic.*

Proof. M is called locally Riemannian product if the horizontal distributions $\mathcal{H}$ and vertical distribution ν are parallel. In this case, $\nabla_{\mathcal{Z}}\mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$ or equivalently $\nabla_{\mathcal{K}}\mathcal{L} \in \mathcal{X}^{\nu}(M)$ for all $\mathcal{Z}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$ and $\mathcal{K}, \mathcal{L} \in \mathcal{X}^{\nu}(M)$.

By using (2.7), we obtain

$$\nabla_{\mathcal{K}}\mathcal{L} = T_{\mathcal{K}}\mathcal{L} + \hat{\nabla}_{\mathcal{K}}\mathcal{L}.$$

Since fibres are totally geodesic, from Theorem 3.1 vertical distribution ν is parallel.
On the other hand by (2.10) we get

$$\nabla_{\mathcal{Z}}\mathcal{W} = \mathcal{H}\nabla_{\mathcal{Z}}\mathcal{W} + A_{\mathcal{Z}}\mathcal{W}.$$

Since the horizontal distribution $\mathcal{H}$ is integrable for $\mathcal{Z}, \mathcal{W} \in \mathcal{X}^{\mathcal{H}}(M)$, we have $[\mathcal{Z}, \mathcal{W}] \in \mathcal{X}^{\mathcal{H}}(M)$. Therefore $\nu[\mathcal{Z}, \mathcal{W}] = 0$. Then from (2.6), we have $A_{\mathcal{Z}}\mathcal{W} = 0$. Thus horizontal distribution is parallel. □

Theorem 3.3. *Let F be a submersion with totally umbilical fibers. The following assertions are equivalent:*

(i) the horizontal distribution $\mathcal{H}$ is integrable and $A_{J\mathcal{Z}}J\mathcal{W}=0$ for any $\mathcal{Z},\mathcal{W}\in\mathcal{X}^{\mathcal{H}}(M)$,

(ii) the horizontal distribution $\mathcal{H}$ is totally geodesic in M.

Proof. $(i)\Rightarrow(ii)$ We have

$$A_{J\mathcal{Z}}J\mathcal{W}=0.$$

By using (2.2), (2.10) and (2.11), we get

$$\begin{aligned}A_{J\mathcal{Z}}J\mathcal{W} &= \nabla_{J\mathcal{Z}}J\mathcal{W}-\mathcal{H}\nabla_{J\mathcal{Z}}J\mathcal{W}\\ &= J(\mathcal{H}\nabla_{J\mathcal{Z}}\mathcal{W}+A_{J\mathcal{Z}}\mathcal{W})-\mathcal{H}\nabla_{J\mathcal{Z}}J\mathcal{W}\\ &= J\mathcal{H}\nabla_{J\mathcal{Z}}\mathcal{W}+HA_{J\mathcal{Z}}\mathcal{W}+VA_{J\mathcal{Z}}\mathcal{W}-\mathcal{H}\nabla_{J\mathcal{Z}}J\mathcal{W}.\end{aligned}$$

By comparing vertical parts, we get $A_{J\mathcal{Z}}J\mathcal{W}=VA_{J\mathcal{Z}}\mathcal{W}=0$, that is, $A_{J\mathcal{Z}}\mathcal{W}=0$. Since the horizontal distribution $\mathcal{H}$ is integrable, from Theorem 3.2, we get

$$A_{J\mathcal{Z}}\mathcal{W}=A_{\mathcal{W}}J\mathcal{Z}=0.$$

By using (2.2) and (2.10), we obtain

$$\begin{aligned}g(A_{\mathcal{W}}J\mathcal{Z},\mathcal{K}) &= g(\nabla_{\mathcal{W}}J\mathcal{Z}-\mathcal{H}\nabla_{\mathcal{W}}J\mathcal{Z},\mathcal{K})\\ &= g(\nabla_{\mathcal{W}}J\mathcal{Z},\mathcal{K})\\ &= g(J\nabla_{\mathcal{W}}\mathcal{Z},\mathcal{K})=0\end{aligned}$$

for any $\mathcal{K}\in\mathcal{X}^{\nu}(M)$. Hence, the horizontal distribution $\mathcal{H}$ is totally geodesic in M. □

Theorem 3.4. *Let F be a submersion with totally umbilical fibers. If the bisectional curvatures $\bar{K}$ and K of $\bar{M}$ and M, respectively, then for orthonormal vertical vector $\mathcal{P}$ and $\mathcal{Q}$ we have*

$$K(\mathcal{P},\mathcal{Q})=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^2.$$

Proof. Since F is a totally umbilical submersion then by using (2.15), we have

$$\begin{aligned}g(R(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W}) &= g(\bar{R}(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W})-g(\mathcal{P},\mathcal{W})g(\mathcal{Q},\mathcal{R})\|H\|^2\\ &\quad+g(\mathcal{Q},\mathcal{W})g(\mathcal{P},\mathcal{R})\|H\|^2\end{aligned}\tag{3.19}$$

for the vertical vector fields $\mathcal{P},\mathcal{Q},\mathcal{R},\mathcal{W}\in\mathcal{X}^{\nu}(M)$. Taking $\mathcal{W}=\mathcal{P}$, $\mathcal{R}=\mathcal{Q}$ in (3.19), we have

$$\begin{aligned}g(R(\mathcal{P},\mathcal{Q})\mathcal{Q},\mathcal{P}) &= g(\bar{R}(\mathcal{P},\mathcal{Q})\mathcal{Q},\mathcal{P})-g(\mathcal{P},\mathcal{P})g(\mathcal{Q},\mathcal{Q})\|H\|^2\\ &\quad+g(\mathcal{Q},\mathcal{P})g(\mathcal{P},\mathcal{Q})\|H\|^2.\end{aligned}$$

Since orthonormal vectors $\mathcal{P}$ and $\mathcal{Q}$, we get

$$K(\mathcal{P},\mathcal{Q})=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^2.$$

□

4. Submersions of CR-submanifolds with totally umbilical fibers in Complex Space Forms

In this section, we study a totally umbilical submersion of a CR submanifold in complex space form. Conditions of such submersions with mean curvature vector field was investigated and results related to the section curvatures were obtained.
Now, we assume that $F^{'} : (M, g_M) \to (B, g_B)$ be a totally umbilical submersion of a CR submanifold M of a complex space form $\bar{M}(c)$ onto an almost Hermitian manifold B.

Lemma 4.1. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then $H \in \mathcal{X}^{\nu}(M)$.*

Proof. For $\mathcal{Z} \in \mathcal{X}^{\nu}(M)$ and $\mathcal{U} \in \mathcal{X}^{\mathcal{H}}(M)$, from (2.2) and (2.8) we obtain

$$T_{\mathcal{Z}} J\mathcal{U} + \mathcal{H}\nabla_{\mathcal{Z}} J\mathcal{U} = J(T_{\mathcal{Z}}\mathcal{U} + \mathcal{H}\nabla_{\mathcal{Z}}\mathcal{U}).$$

For any $\mathcal{W} \in \mathcal{X}^{\nu}(M)$ we have

$$g(T_{\mathcal{Z}} J\mathcal{U}, \mathcal{W}) = g(T_{\mathcal{Z}}\mathcal{U}, \mathcal{W}).$$

$$g(T_{\mathcal{Z}} J\mathcal{U}, \mathcal{W}) = -g(T_{\mathcal{Z}}\mathcal{W}, \mathcal{U}).$$

Since F is totally umbilical submersion, we get

$$g(\mathcal{Z}, J\mathcal{U})g(H, \mathcal{W}) = -g(\mathcal{Z}, \mathcal{W})g(H, \mathcal{U})$$

which means $H \in \mathcal{X}^{\nu}(M)$. □

Theorem 4.1. *There is no $F^{'}$ totally umbilical submersion, $c \neq 0$ with nonzero parallel mean curvature vector field H.*

Proof. For $Y \in \mathcal{X}^{\nu}(M)$ from Lemma 4.1, (2.7) and (3.18) we get

$$\begin{aligned} \nabla_Y H &= T_Y H + \nu\nabla_Y H \\ &= -g(H, H)Y + \nu\nabla_Y H \end{aligned}$$

On the other hand by (2.7) we have

$$R(Y, \mathcal{Z})H = T_Y T_{\mathcal{Z}} H + \nabla_Y \nabla_{\mathcal{Z}} H - \nabla_Y T_{\mathcal{Z}} H - T_{\mathcal{Z}} T_Y H - \nabla_{\mathcal{Z}} \nabla_Y H + \nabla_{\mathcal{Z}} T_Y H - \nabla_{[Y,\mathcal{Z}]} H$$

for any vertical field $\mathcal{Z}$. Since $F^{'}$ is totally umbilical submersion and H is parallel along the fibers, we get

$$R(Y, \mathcal{Z})H = 0. \tag{4.20}$$

Hence, since $\bar{M}(c)$ is complex space form, for $\mathcal{W} \in \mathcal{X}^{\nu}(M)$, we have

$$g(R(Y, \mathcal{Z})H, \mathcal{W}) = \frac{c}{4}[g(Y, H)g(\mathcal{Z}, \mathcal{W}) - g(\mathcal{Z}, H)g(Y, \mathcal{W})]. \tag{4.21}$$

Taking $\mathcal{Z} = \mathcal{W}$ in (4.21), we obtain

$$g(R(Y, \mathcal{Z})H, \mathcal{W}) = \frac{c}{4}[g(Y, H)g(\mathcal{Z}, \mathcal{Z})]$$

for any orthonormal vectors Y and $\mathcal{Z}$. From (4.20) and (4.21) we get

$$g(R(Y, \mathcal{Z})H, \mathcal{W}) = \frac{c}{4}[g(Y, H)g(\mathcal{Z}, \mathcal{Z})] = 0.$$

For $g(Y, H) \neq 0$ and $g(\mathcal{Z}, \mathcal{Z}) \neq 0$, we have $c = 0$. □

Theorem 4.2. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then the mean curvature vector H of M is parallel if and only if the fibers of submersion $F^{'}$ are totally geodesic.*

Proof. For any $\mathcal{P},\mathcal{Q},\mathcal{R}\in\mathcal{X}^{\nu}(M)$ and $\mathcal{W}\in\mathcal{X}^{\mathcal{H}}(M)$, by using the equation (2.17), we obtain

$$\begin{aligned} g(R(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W}) &= \frac{c}{4}[g(\mathcal{P},\mathcal{R})g(\mathcal{Q},\mathcal{W})-g(\mathcal{Q},\mathcal{R})g(\mathcal{P},\mathcal{W})+g(J\mathcal{P},\mathcal{R})g(J\mathcal{Q},\mathcal{W}) \\ &\quad -g(J\mathcal{Q},\mathcal{R})g(J\mathcal{P},\mathcal{W})+2g(J\mathcal{P},\mathcal{Q})g(J\mathcal{R},\mathcal{W})] \\ &= 0. \end{aligned}$$

On the other hand by (2.16) we have

$$g(R(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W})=g((\nabla_{\mathcal{P}}T)_{\mathcal{Q}}\mathcal{R},\mathcal{W})-g((\nabla_{\mathcal{Q}}T)_{\mathcal{P}}\mathcal{R},\mathcal{W}). \tag{4.22}$$

From [15], since $F^{'}$ is totally umbilical submersion, we have

$$\begin{aligned} (\nabla_{\mathcal{P}}T)_{\mathcal{Q}}\mathcal{R} &= \nabla_{\mathcal{P}}T_{\mathcal{Q}}\mathcal{R}-T_{\nabla_{\mathcal{P}}\mathcal{Q}}\mathcal{R}-T_{\mathcal{Q}}\nabla_{\mathcal{P}}\mathcal{R} \\ &= \nabla_{\mathcal{P}}\{g(\mathcal{Q},\mathcal{R})H\}-g(\nabla_{\mathcal{P}}\mathcal{Q},\mathcal{R})H-T_{\mathcal{Q}}\{T_{\mathcal{P}}\mathcal{R}+\bar{\nabla}_{\mathcal{P}}\mathcal{R}\} \\ &= g(\mathcal{Q},\mathcal{R})\nabla_{P}H-g(\nabla_{\mathcal{P}}\mathcal{Q},\mathcal{R})H-g(\mathcal{P},\mathcal{R})g(H,H)\mathcal{Q}. \end{aligned} \tag{4.23}$$

Similarly we may also prove that

$$(\nabla_{\mathcal{Q}}T)_{\mathcal{P}}\mathcal{R}=g(\mathcal{P},\mathcal{R})\nabla_{\mathcal{Q}}H-g(\nabla_{\mathcal{Q}}\mathcal{P},\mathcal{R})H-g(\mathcal{Q},\mathcal{R})g(H,H)\mathcal{P}. \tag{4.24}$$

Substituting (4.23) and (4.24) equation into (4.22), we have

$$g(\mathcal{Q},\mathcal{R})g(\nabla_{\mathcal{P}}H,\mathcal{W})=g(\mathcal{P},\mathcal{R})g(\nabla_{\mathcal{Q}}H,\mathcal{W}). \tag{4.25}$$

By using (2.7) and taking $\mathcal{Q}=\mathcal{R}$ in (4.25) we find

$$g(\mathcal{Q},\mathcal{Q})g(T_{\mathcal{P}}H,\mathcal{W})=0$$

for any orthonormal vectors $\mathcal{P}$ and $\mathcal{Q}$. We obtain $T_{\mathcal{P}}H=0$. Thus the proof the theorem completes. □

Theorem 4.3. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then for any vertical vector fields $\mathcal{P},\mathcal{Q}$, we have*

$$\frac{c}{4}=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^{2}.$$

Proof. From (2.12) and (2.17) we get

$$\begin{aligned} &\frac{c}{4}[g(\mathcal{P},\mathcal{R})g(\mathcal{Q},\mathcal{W})-g(\mathcal{Q},\mathcal{R})g(\mathcal{P},\mathcal{W})] \\ &=g(\bar{R}(\mathcal{P},\mathcal{Q})\mathcal{R},\mathcal{W})-g(\mathcal{P},\mathcal{W})g(\mathcal{Q},\mathcal{R})g(H,H)+g(\mathcal{Q},\mathcal{W})g(\mathcal{P},\mathcal{R})g(H,H) \end{aligned} \tag{4.26}$$

for any vertical vector fields $\mathcal{R},\mathcal{W}$. Taking $\mathcal{R}=\mathcal{Q}$ and $\mathcal{P}=\mathcal{W}$ in (4.26) equation, we obtain

$$\frac{c}{4}=\bar{K}(\mathcal{P},\mathcal{Q})-\|H\|^{2}$$

for any orthonormal vectors $\mathcal{P}$ and $\mathcal{Q}$. □

As an immediate consequence of Theorem 4.3, we have the following.

Corollary 4.1. *Let $F^{'}$ be a submersion with totally umbilical fibers. Then for any vertical vector fields $\mathcal{P},\mathcal{Q}$, we have*

$$c\leq 4\bar{K}(\mathcal{P},\mathcal{Q}).$$

The equality holds if and only if the fibers of submersion $F^{'}$ are totally geodesic.

Corollary 4.2. *Let $F^{'}$ be a totally umbilical submersion of a CR submanifold of a non-negatively complex space form. Then curvatures of fibers are positively.*

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INNOVATIVE AGGREGATION METHODS: SEMI-LINDELÖF PERFECT FUNCTIONS AND SEMI-PERFECT FUNCTIONS IN BITOPOLOGICAL SPACES, TOGETHER WITH THEIR UTILIZATION

ALI A. ATOOM AND MOHAMMAD A. BANI ABDELRAHMAN *

Abstract. Due to the significance of topological spaces in analysis and particular fields, numerous scholars use distinct frameworks to broaden topological space, encompassing the concept of topology. The mathematical description of perfect functions in bitopological spaces emerged as one of most profoundly prominent improvements. Precisely a consequence, we investigate different collections of operator strategies for creating paired semi perfect functions in this work. Accordance relates to the connections between specific kinds of pair semi perfect functions and related traditional topologies Function analysis allows us to explore properties and applications of classical topological concepts with regard for misalignment. The present study suggests and examines several new classes of perfect functions, such as pairwise semi-perfect functions and pairwise semi-Lindelöf perfect functions, within the framework of fundamental topological spaces. Through the use of cases and other instances, the characteristics they have and how they connect to various jobs are examined. It covers the computation of the Cartesian combination among finite intersection.

Keywords: Pairwise compact spaces, Pairwise semi-Lindelofness, Perfect functions, Pairwise semi normal, Applications in bitopological spaces.

MSC (2020): 54C10, 54D20.

1. Introduction

The characteristics of typical issues include large amounts of information with different degrees of complexity. Afterwards, it is necessary to establish innovative mathematical methods to cope with them. There have been some theorized extending topological configurations. Due to the topological space's significance in analysis and different uses, such as artificial intelligence, economics, physics, and statistics. Topological spaces provide a framework to examine continuity, convergence, and compactness, which are fundamental across various fields see ([3, 12, 13, 19, 22, 23, 29, 30, 37, 41]). Kelly [21] introduced the notion of bitopological spaces in 1963. Thereafter, many topologists have focused on applying well-known topological concepts to bitopological spaces. In 1969, Fletcher [16] further introduced the notion of pairwise compactness for bitopological spaces. Later, this concept was extended

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to pairwise Lindelöf spaces by Reilly [36] in 1973. In 1980s, Dorsett and Hanna [14, 15, 20] studied the concept of semi-compactness. The concept of semi-open sets and semi-continuity was introduced by Levine [26], and the notions were extended to bitopological spaces in 1977 by Maheshwari and Prasad [27]. The interrelations between different open sets in topological spaces were studied by Khedr [24] in 1992. The concepts of pairwise semi-Lindelöf and pairwise semi-Lindelöf bitopological spaces were defined by Balasubramanian [9] in 2009. A set $\mathscr{X}$ equipped with two topologies ξ_1 and ξ_2 is called a bitopological space and denoted by $(\mathscr{X}, \xi_1, \xi_2)$. Subsequently, the notions of semi perfect functions in topological spaces were defined by Atoom et al. [5]. A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is called pairwise perfect if ϱ is pairwise continuous, pairwise closed, and for each $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is pairwise compact. They also defined the notion of pairwise semi-perfect functions in bitopological spaces, provided some characterizations of pairwise semi-perfect functions and pairwise M-semi perfect functions, and studied the images and inverse images of certain bitopological properties under these functions. Let $(\mathscr{X}, \xi)$ be a topological space, and let $\mathscr{D}$ be a subset of $\mathscr{X}$. We denote the closure of $\mathscr{D}$ (resp., the interior of $\mathscr{D}$) by $\mathrm{Cl}(\mathscr{D})$ (resp., $\mathrm{Int}(\mathscr{D})$). A subset $\mathscr{D}$ of $(\mathscr{X}, \xi)$ is called semi-open [26] if $\mathscr{D} \subseteq Cl(Int(\mathscr{D}))$. The complement of a semi-open set is called semi-closed. The family of all semi-open sets in $(\mathscr{X}, \xi)$ is denoted by $\mathrm{S}(\xi)$. A $\mathscr{D}$ space $(\mathscr{X}, \xi)$ is said to be semi-compact [28] (resp., semi-Lindelöf) if every cover of $\mathscr{X}$ by semi-open sets has a finite (resp., countable) subcover. Let $\mathscr{D}$ be a subset of a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$. The closure of $\mathscr{D}$ and the interior of $\mathscr{D}$ with respect to ξ_i are denoted by $Cl_i(\mathscr{D})$ and $\mathrm{Int}_i(\mathscr{D})$, respectively. Also, the topologies on $\mathscr{D}$ inherited from ξ_1 and ξ_2 will be denoted by $\xi_1\mathscr{D}$ and $\xi_2\mathscr{D}$, respectively. The intersection of all semi-closed sets containing $\mathscr{D}$ is called the semi-closure [15] of $\mathscr{D}$, denoted by $\mathrm{sCl}(\mathscr{D})$. The semi-closure of $\mathscr{D}$ with respect to $\mathrm{S}(\xi_i)$ is denoted by $s\ Cl_i(\mathscr{D})$. A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is said to be pairwise continuous (resp., pairwise open, pairwise closed, a pairwise homeomorphism) if $\varrho_1 : (\mathscr{X}, \xi_1) \to (\mathscr{Y}, \mu_1)$ and $\varrho_2 : (\mathscr{X}, \xi_2) \to (\mathscr{Y}, \mu_2)$ are continuous (resp., open, closed, homeomorphism) functions. In recent years, numerous authors have examined the relationships between other topological and analytical concepts. Compactness, pairwise semi-compactness, and pairwise semi-Lindelöfness have been refined to develop new notions within the structure of pairwise perfect functions and pairwise Lindelöf perfect functions in topological spaces, including product properties for generalized pairwise Lindelöf spaces [39], Difference Compactness in Bitopological Spaces [4] pairwise weakly regular-Lindelöf spaces [25], Omega Closed Functions in Bitopological Spaces [8], mappings on pairwise para-Lindelöf bitopological spaces [11], some results on nearly pairwise compact spaces [32], KC-bitopological spaces [35], notes on p_1-Lindelöf spaces which are not contra second-countable spaces in bitopology [2], Study the structure of difference Lindelöf topological spaces and their properties [6], two forms of pairwise Lindelöfness and some results related to hereditary class in a bigeneralized topological spaces [1], some results on pairwise locally compact bitopological spaces [40], pairwise locally compact and compactification in double topological spaces [31], Pairwise α-perfect functions [7], and Semi-Compact and Semi-Lindelöf Spaces via Neutrosophic Crisp Set Theory [38]. Additionally, the concept of providing stronger and weaker forms of functions is becoming increasingly prevalent in research. The structure of the paper consists of several sections. In Section 2, we introduces the essential ideas and terms that are crucial to acknowledging semi-perfect functions in bitopological spaces. In Section 3 several kinds of pairwise semi-perfect functions(briefly; $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$) and their relationships are covered. We further explain and offer an alternative viewpoint on the many $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function forms in Section 4, also we investigate and establish the idea of $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions. Furthermore we look into the characteristics of pairwise semi-Lindelöf perfect functions(briefly; $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$). In Section 5, we examine a wide range of applications and discuss how they could lead to changes in multi-variable systems and enhance efficiency across various study areas.

2. Preliminaries

This section introduces the essential ideas and terms that are crucial to acknowledging semi-perfect functions in bitopological spaces.

Definition 2.1. *[33] In a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$, a collection of subsets $\mathscr{D}$ is considered $\xi_1\xi_2$-semi-open($\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$) if $\mathscr{D} \subset S(\xi_1) \cup S(\xi_2)$. Additionally, if $\mathscr{D} \cap S(\xi_1) \neq \emptyset$ and $\mathscr{D} \cap S(\xi_2) \neq \emptyset$, then $\mathscr{D}$ is further classified as pairwise semi-open ($\widetilde{P_w}\widetilde{S_e}\widetilde{O}$).*

Definition 2.2. *[33] A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ if every $\widetilde{P_w}\widetilde{S_e}$ cover of $\mathscr{X}$ has a finite subcover. Similarly, it is considered pairwise M-semi compact(briefly; $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$) if every $\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{X}$ has a finite subcover. It is obvious that every $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$) space is also $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, but the opposite is not true.*

Definition 2.3. *[10] Consider $(\mathscr{X}, \xi_1, \xi_2)$ be a bitopological space. A subset $\mathscr{D}$ of $\mathscr{X}$ is defined as a semi open set ($\widetilde{S_e}\widetilde{O}$) if $\mathscr{D} \subset cl_\kappa(int_\kappa \mathscr{D})$. Additionally, $\mathscr{D}$ is considered semi closed if $int_\kappa(cl_\kappa(\mathscr{D})) \subset \mathscr{D}$, for $\kappa = 1, 2$.*

Definition 2.4. *[10] A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is termed pairwise semi continuous(brf. $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$) if $\varrho^{-1}(V)$ is a $\widetilde{S_e}\widetilde{O}$ (or $\widetilde{S_e}\widetilde{C}$) set for every $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ (or $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$) set V in $\mathscr{Y}$.*

Definition 2.5. *[10] A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is referred to as $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ if both $\varrho_1 : (\mathscr{X}, \xi_1) \to (\mathscr{Y}, \mu_1)$ and $\varrho_2 : (\mathscr{X}, \xi_2) \to (\mathscr{Y}, \mu_2)$ are $\widetilde{S_e}\widetilde{C}$ functions. That is if ϱ_1 is $\widetilde{S_e}\widetilde{C}$ in $S(\xi_1)$, then $\varrho(F_1)$ is $\widetilde{S_e}\widetilde{C}$ in $S(\mu_1)$. Similarly, if F_2 is $\widetilde{S_e}\widetilde{C}$ in $S(\xi_2)$, then $\varrho(F_2)$ is $\widetilde{S_e}\widetilde{C}$ in $S(\mu_2)$.*

Theorem 2.1. *[33] A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is pairwise semi-compact ($\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$) iff every ξ_κ-semi-closed subset of $\mathscr{X}$ is semi-compact ($\widetilde{S_e}\widetilde{C_{pt}}$) with respect to the topology ξ_ι, where $\kappa, \iota \in \{1, 2\}$ and $\kappa \neq \iota$.*

Definition 2.6. *[34] If $\check{U}$ and $\check{N}$ are $\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$ covers of the bitopological space $(\mathscr{X}, \xi_1, \xi_2)$, then $\check{U}$ is said to be a refinement of $\check{N}$ if each $\mathscr{U}_\kappa \in \check{U} \cap S(\xi_\kappa)$ is contained in some $\mathscr{V}_\kappa \in \check{N} \cap S(\xi_\kappa)$, $\kappa = 1, 2$.*

Definition 2.7. *[34] A collection $\mathscr{D}$ of subsets of a space $(\mathscr{X}, \xi)$ is considered locally finite in $(\mathscr{X}, S(\xi))$ if for any $x \in \mathscr{X}$, there's a $\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ such that $x \in \mathscr{U}$ and $\mathscr{U}$ intersects at most finitely many elements of $\mathscr{D}$.*

Definition 2.8. *[34] In a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$. Let $\mathscr{U}$ and $\mathscr{N}$ be $\xi_1\xi_2$-$\widetilde{S_e}\widetilde{O}$ covers. The cover $\mathscr{U}$ is considered a refinement of $\mathscr{N}$ if any $\mathscr{U} \in \mathscr{U} \cap S(\xi_\kappa)$, there exists a $\mathscr{V} \in \mathscr{N} \cap S(\xi_\kappa)$, for $\kappa = 1, 2$.*

Definition 2.9. *[34] A collection $\mathscr{D}$ of subsets of a space $(\mathscr{X}, \xi)$ is considered locally finite in $(\mathscr{X}, S(\xi))$ if, for any $x \in \mathscr{X}$, there exists a $\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ such that $\mathscr{X} \in \mathscr{U}$ and $\mathscr{U}$ intersects with at most finitely many elements of $\mathscr{D}$.*

Definition 2.10. *[34] In a bitopological space $(\mathscr{X}, \xi_1, \xi_2)$, the collection $S(\xi_1)$ is considered semi-regular(brf. $\widetilde{S_e}\widetilde{R_{eg}}$) with respect to $S(\xi_2)$ if, for every $x \in \mathscr{X}$ and every ξ_1-$\widetilde{S_e}\widetilde{C}$ set f such that $\mathscr{X} \notin f$, there exists a ξ_1-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ and a ξ_2-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}$ such that $\mathscr{X} \in \mathscr{U}$, $f \subseteq \mathscr{V}$, and $\mathscr{U} \cap \mathscr{V} = \emptyset$. The space $(\mathscr{X}, \xi_1, \xi_2)$ is referred to as $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ if both sets of $\widetilde{S_e}\widetilde{O}$ sets are $\widetilde{S_e}\widetilde{R_{eg}}$ with respect to each other.*

Definition 2.11. *[7] A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is said to be $\widetilde{P_w}\widetilde{L}\widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{C}$, $\widetilde{P_w}\widetilde{C_{nt}}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is pairwise Lindelöf $\widetilde{P_w}\widetilde{L}$.*

3. A Taxonomy Of Semi Perfect Functions

This section explores the various types of perfect functions and the relationships between them.

Definition 3.1. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and any $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Theorem 3.1. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function, for every a pairwise semi Lindelöf ($\widetilde{P_w}\widetilde{S_e}\widetilde{L}$) $\mathscr{X} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{X})$ represents $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$.*

Theorem 3.2. *Every $\widetilde{P_w}\widetilde{P}$ function is also $\widetilde{P_w}\widetilde{L}\widetilde{P}$; however, the opposite is not always true.*

Proof. while every $\widetilde{P_w}\widetilde{C_{pt}}$ space is $\widetilde{P_w}\widetilde{L}$, it follows that every $\widetilde{P_w}\widetilde{P}$ function results in a $\widetilde{L}\widetilde{P}$. □

As we will show in an example, the reverse isn't always the case:

Example 3.1. *Consider $(\mathbb{R}, \xi_s, \xi_{dis})$ be a bitopological space, where ξ_s is the standard topology on $\mathbb{R}$ and ξ_{dis} is the discrete topology. Consider $U_\iota = (\iota, \iota+2), \iota \in \mathbb{Z}$ as an open cover of $\mathbb{R}$ under ξ_s that covers $\mathbb{R}$, that is,*

$$\mathbb{R} = \bigcup_{\iota \in \mathbb{Z}} (\iota, \iota + 2).$$

Since the cover is countable and there exists a countable subcover, it follows that $(\mathbb{R}, \xi_s)$ is Lindelöf.

Now, consider $(\mathbb{R}, \xi_{dis})$. In ξ_{dis}, every singleton set $\{x\}$ is open. Consider the open cover $\check{U} = \{\{x\} \mid x \in \mathbb{R}\}$ of $\mathbb{R}$ under ξ_{dis}. Since $\check{U}$ contains all points of $\mathbb{R}$, it is an open cover. However, no finite subcover of $\check{U}$ can cover $\mathbb{R}$, because each singleton only covers one point. Thus, $(\mathbb{R}, \xi_{dis})$ is not compact.

Theorem 3.3. *Every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function is also $\widetilde{P_w}\widetilde{L}$. however, but reverse does not necessarily true.*

Proof. Since every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ space is $\widetilde{P_w}\widetilde{L}$ space, a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function is $\widetilde{P_w}\widetilde{L}\widetilde{P}$ function. □

However, as the following example demonstrates, that the opposite not necessarily true:

Example 3.2. *Consider $(\mathbb{R}, \xi_s, \xi)$ be a bitopological space, where ξ_s is the standard topology on $\mathbb{R}$ and ξ is the Sorgenfrey topology with basic open sets $[a, b)$. The space $(\mathbb{R}, \xi_s)$ is Lindelöf. Meaning that for any open cover $\mathscr{U} = \{U_\alpha\}$, there exists a countable subcover. To illustrate this, consider an arbitrary open cover:*

$$\mathscr{U} = \{(n, n+2) \mid n \in \mathbb{Z}\}$$

has the countable subcover $\{(n, n+2) \mid n \in \mathbb{Z}\}$ itself, since $\mathbb{Z}$ is countable.

Now, consider the ξ-semi open cover:

$$\mathscr{C} = \{[x, x+1) \mid x \in \mathbb{R}\}.$$

This covers $\mathbb{R}$ because $\forall y \in \mathbb{R},\ y \in [y, y+1)$. So, $\mathscr{C}$ has no countable subcover.

To examine it; Suppose for contradiction there exists a countable subcover:

$$\mathscr{T} = \{[x_m, x_m+1) \mid m \in \mathbb{N}\}.$$

Let $Y = \sup\{x_m \mid m \in \mathbb{N}\}$. Two cases: If $Y = +\infty$, this contradicts $\mathbb{N}$ being countable. If $Y < +\infty$, then $Y + 1 \notin \bigcup_{m \in \mathbb{N}} [x_m, x_m + 1)$.

Both cases yield contradictions. Thus, $(\mathbb{R}, \xi)$ is not semi-Lindelöf.

Theorem 3.4. *Every $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function but the converse need not to be true.*

Proof. Since every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ is also $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$, so $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function. $\square$

Example 3.3. *Consider the bitopological space $(\mathbb{S}, \mu_s, \mu_2)$, where $\mathbb{S} = \mathbb{R}$, μ_s is the standard topology on $\mathbb{S}$ and μ_2 is the discrete topology on $\mathbb{S}$. A set $D \subseteq \mathbb{S}$ is semi-open in μ_s if $D \subseteq \mathrm{cl}(\mathrm{int}(D))$. For example, intervals $[a, b)$ are semi-open in μ_s. The space $(\mathbb{S}, \mu)$ is semi-Lindelöf because any semi open cover $\mathscr{V} = \{V_\beta \mid \beta \in B\}$ has a countable subcover $\{V_{\beta_i} \mid i \in \mathbb{N}\}$. Now, consider the μ_2-semi open cover*

$$\mathscr{C} = \{(\ell, \ell+1) \cup \{\ell+1\} \mid \ell \in \mathbb{Z}\}.$$

Each set is semi open because $(\ell, \ell+1)$ is open in μ_s, $\{\ell+1\}$ is open in μ_2 and $(\ell, \ell+1) \subseteq \mathrm{cl}(\mathrm{int}((\ell, \ell+1) \cup \{\ell+1\}))$.

Hence, $\mathscr{C}$ has no finite subcover.

To examine it, Suppose for contradiction there exists a finite subcover:

$$\{(\ell_k, \ell_k+1) \cup \{\ell_k+1\} \mid k = 1, \dots, n\}.$$

Let $M = \max\{\ell_k\}$. Then,

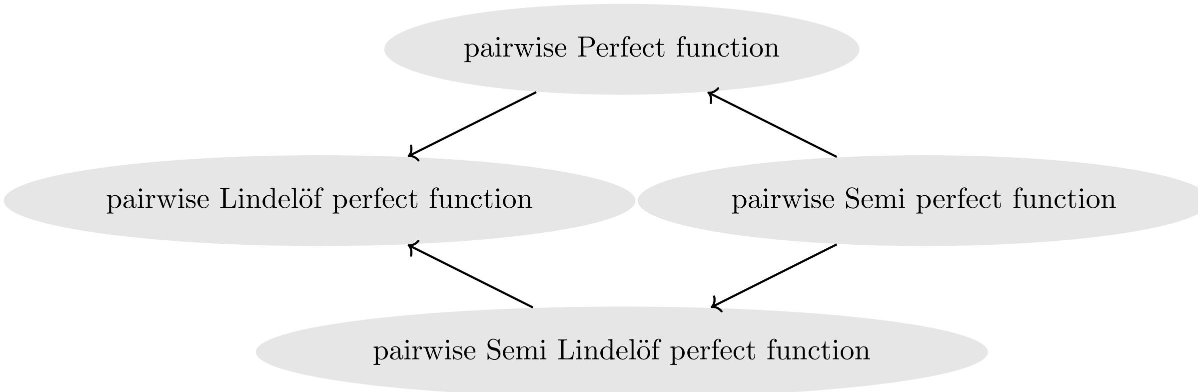
$$M + 1 \notin \bigcup_{k=1}^{n} (\ell_k, \ell_k+1) \cup \{\ell_k+1\}$$

contradicting $\mathscr{C}$ being a cover.

Theorem 3.5. *Every $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is a $\widetilde{P_w}\widetilde{P}$ function. However, the reverse is not necessarily true.*

Proof. Since every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ space is $\widetilde{P_w}\widetilde{C_{pt}}$ space, it follows that every $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function is also pairwise perfect. $\square$

We have the following diagram in which the converses of each implication need not be true as shown by the examples stated above. Moreover, the examples show that the pairwise semi perfect function are independent.



4. Essential Findings on Pairwise Semi-Lindelöf Perfect Functions

Definition 4.1. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$, if it is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Definition 4.2. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$.*

Definition 4.3. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is considered $\widetilde{P_w}\widetilde{S_e}Z^* - \widetilde{P}$, if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, and for every $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$.*

Theorem 4.1. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a$\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, then for every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ subset $\mathscr{Z} \subseteq \mathscr{Y}$, the inverse image $\varrho^{-1}\mathscr{Z}$ is also $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Proof. Let's consider $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$ be a pairwise open cover of $(\mathscr{X}, \xi_1, \xi_2)$, since ϱ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, then $\forall y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$. Thus,this implies there exist finite subsets Λ_y, Λ_y^* of Λ such that

$$\varrho^{-1}(y) \subseteq \bigcup_{\alpha \in \Lambda_y} \{\mathscr{V}_\alpha : \alpha \in \Lambda_y\} \cup \bigcup_{\alpha \in \Lambda_y^*} \{W_\alpha : \alpha \in \Lambda_y^*\},$$

where $\{\mathscr{V}_\alpha : \alpha \in \Lambda_y\}$ is semi open in $S(\xi_1)$, and $\{W_\alpha : \alpha \in \Lambda_y\}$ is semi open in $S(\xi_2)$.

Let $O_y = \mathscr{Y} - \varrho(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_1)$ containing y, and $O_y^* = \mathscr{Y} - \varrho(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y^*} W_\alpha)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_2)$ containing y, where $\varrho^{-1}(O_y) \subseteq \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha$ and $\varrho^{-1}(O_y^*) \subseteq \bigcup_{\alpha \in \Lambda_y^*} W_\alpha$. Define

$$\mathscr{O} = \{O_y : y \in \mathscr{Y}\} \cup \{O_y^* : y \in \mathscr{Y}\}$$

as a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover of $\mathscr{Y}$. $\mathscr{O}$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover of $\mathscr{Z}$.

Since $\mathscr{Z}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, we have $\mathscr{Z} \subseteq \bigcup_{i=1}^n O_{y_i} \cup \bigcup_{j=1}^m O_{y_j}^*$.

Thus,

$$\varrho^{-1}(\mathscr{Z}) \subseteq \bigcup_{i=1}^{n} \varrho^{-1}(O_{y_i}) \cup \bigcup_{j=1}^{m} \varrho^{-1}(O_{y_j}^*).$$

This is a subset of the union of a finite number of $\check{U}$, that is, $\varrho^{-1}(\mathscr{Z})$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.

□

Corollary 4.1. *A $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ space remains invariant under the inverse of a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function.*

Using a parallel reasoning, we can conclude these accompanying claims:

Theorem 4.2. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function, then for every $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ subset $\mathscr{Z} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{Z})$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$.*

Proof. Theorem 4.1 provides the justification for this claim. □

Corollary 4.2. *$\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$ space remains invariant under the inverse of a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function.*

Theorem 4.3. *given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$ function, then for every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ subset $\mathscr{Z} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{Z})$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$.*

Corollary 4.3. *A $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ space is remains invariant under the inverse of a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$.*

Theorem 4.4. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function, for every $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ $\mathscr{Z} \subseteq \mathscr{Y}$, $\varrho^{-1}(\mathscr{Z})$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$.*

Remark 4.1. *The composition of two $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions does not always results in a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions.*

Example 4.1. *Consider the bitopological spaces: $(\mathscr{X}, \xi_1, \xi_2)$ where $\mathscr{X} = \{a, b, c\}$, $\xi_1 = \{\mathscr{X}, \emptyset, \{a\}, \{a, b\}\}$, and $\xi_2 = \{\mathscr{X}, \emptyset, \{c\}\}$. $(\mathscr{Y}, \mu_1, \mu_2)$ where $\mathscr{Y} = \{1, 2, 3\}$, $\mu_1 = \{\mathscr{Y}, \emptyset, \{1\}, \{2\}, \{1, 2\}\}$, and $\mu_2 = \{\mathscr{Y}, \emptyset, \{3\}\}$. $(\mathscr{Z}, \delta_1, \delta_2)$ where $\mathscr{Z} = \{x, y, z\}$, $\delta_1 = \{\mathscr{Z}, \emptyset, \{x\}\}$, and $\delta_2 = \{\mathscr{Z}, \emptyset, \{y, z\}\}$.*

Let

$$\varrho : \mathscr{X} \to \mathscr{Y}, \quad \varrho(a) = 1,\ \varrho(b) = 2,\ \varrho(c) = 3$$

$$\psi : \mathscr{Y} \to \mathscr{Z}, \quad \psi(1) = x,\ \psi(2) = y,\ \psi(3) = z$$

ϱ and ψ are $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.
A function is pairwise semi continuous if the inverse of $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ set is semi-open in ξ_1 or ξ_2.
For ϱ: $\varrho^{-1}(\{1, 2\}) = \{a, b\}$ is ξ_1-open (hence semi-open), but not ξ_2-semi open. $\varrho^{-1}(\{3\}) = \{c\}$ is ξ_2-open (hence semi open), but not ξ_1-semi open. Thus, ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.
For ψ: $\psi^{-1}(\{x, y\}) = \{1, 2\}$ is μ_1-open (hence semi-open), but not μ_2-semi open. Thus, ψ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.

Now, Consider the $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ set $V = \{x, y\}$ in $\mathscr{Z}$:

$$(\psi \circ \varrho)^{-1}(V) = \varrho^{-1}(\psi^{-1}(V)) = \varrho^{-1}(\{1, 2\}) = \{a, b\}.$$

Check semi-openness in ξ_1 and ξ_2; In ξ_1: $\{a, b\}$ is open (hence semi open). In ξ_2: $\{a, b\}$ is not semi open because $\mathrm{cl}_{\xi_2}(\mathrm{int}_{\xi_2}(\{a, b\})) = \mathrm{cl}_{\xi_2}(\emptyset) = \emptyset \not\supseteq \{a, b\}$. Therefore, $(\psi \circ \varrho)^{-1}(V)$ is not semi-open in ξ_2, violating Definition 2.4. Thus, $\psi \circ \varrho$ is not $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$.

Remark 4.2. *the composition of two $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ functions does not necessarily yield another$\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function.*

Theorem 4.5. *the composition of two $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ functions does not necessarily yield another $\widetilde{P_w}Z^* - \widetilde{L}\widetilde{C_{pt}}$ function.*

Proof. Let $(\mathscr{X}, \xi_1, \xi_2)$ where $\mathscr{X} = \{a, b, c\}.(\mathscr{Y}, \mu_1, \mu_2)$ where $\mathscr{Y} = \{1, 2, 3\}.(\mathscr{Z}, \delta_1, \delta_2)$ where $\mathscr{Z} = \{x, y, z\}$.
Consider $\xi_1 = \{\mathscr{X}, \emptyset, \{a\}, \{a, b\}\}$. $\xi_2 = \{\mathscr{X}, \emptyset, \{c\}\}$. $\mu_2 = \{\mathscr{Y}, \emptyset, \{3\}\}$.
Define $\varrho(a) = 1,\ \varrho(b) = 2,\ \varrho(c) = 3$. $\psi(1) = x,\ \psi(2) = y,\ \psi(3) = z$
For ϱ: $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$: $\varrho^{-1}(\{1, 2\}) = \{a, b\}$ is ξ_1-semi-open. $\varrho^{-1}(\{3\}) = \{c\}$ is ξ_2-semi-open.
$\widetilde{P_w}\widetilde{S_e}\widetilde{C}$: $\varrho(\{a, b\}) = \{1, 2\}$ is μ_1-semi-closed. For ψ: $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$: $\psi^{-1}(\{x, y\}) = \{1, 2\}$ is μ_1-semi-open.
Now, consider $V = \{x, y\} \in \delta_1$-$\widetilde{S_e}\widetilde{O}$, then,

$$(\psi \circ \varrho)^{-1}(V) = \varrho^{-1}(\psi^{-1}(V)) = \{a, b\}.$$

Hence, composition fails $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ in ξ_2 since the topology mismatch between μ_2 and δ_2. □

Theorem 4.6. *given that ϱ:($\mathscr{X}$, ξ_1, ξ_2) $\to$ ($\mathscr{Y}$, μ_1, μ_2) is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and ω:($\mathscr{Y}$, μ_1, μ_2) $\to$ ($\mathscr{Z}$, δ_1, δ_2) is a pairwise perfect function, $\omega \circ \varrho$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function.*

Proof. Consider $\mathscr{H}$ be any δ_1-open set in $\mathscr{Z}$. Since ω is a pairwise perfect function, then $\omega^{-1}(\mathscr{H})$ is a μ_1-open set in $\mathscr{Y}$. Since ϱ is a pairwise semi perfect function, then $\varrho^{-1}(\omega^{-1}(\mathscr{H}))$ is a ξ_1-$\widetilde{S_e}\widetilde{O}$ set in $\mathscr{X}$.

Similarly, let $\mathscr{B}$ be any δ_2-open set in $\mathscr{Z}$. Then, $\omega \circ \varrho$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function. □

Using a parallel reasoning, we can conclude these accompanying claims:

Theorem 4.7. *given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function and ω:($\mathscr{Y}$, μ_1, μ_2) $\to$ ($\mathscr{Z}$, δ_1) is pairwise perfect function, $\omega \circ \varrho$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function.*

Corollary 4.4. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function and $\omega : (\mathscr{Y}, \mu_1, \mu_2) \to (\mathscr{Z}, \delta_1, \delta_2)$ is a pairwise perfect function, then $\omega \circ \varrho$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function.*

Proposition 4.1. *If the composition $\omega \circ \varrho$ of the $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ and the pairwise continuous function $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, then the function $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. consider $\mathscr{D}$ be a μ_1-closed set in $\mathscr{Y}$. Then $\varrho^{-1}(\mathscr{D})$ is ξ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{X}$. Since ϱ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ function and $\omega \circ \varrho$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$, then $\omega(\varrho(\varrho^{-1}(\mathscr{D})))$ is δ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{Z}$, i.e., $\omega(\mathscr{D})$ is δ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{Z}$.

Similarly, we can show that if $\mathscr{B}$ is a μ_2-closed set in $\mathscr{Y}$, then $\omega(\mathscr{B})$ is δ_2-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{Z}$. Thus, ω is a $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ function. □

Theorem 4.8. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ function, and pairwise continuous $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$, if the composition $\omega \circ \varrho$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$, then ω is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$.*

Proof. For every $z \in \mathscr{Z}$, $\omega^{-1}(z) = \varrho((\omega \circ \varrho)^{-1}(z))$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, because $\omega \circ \varrho$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$. Since ω is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ by the previous proposition, we get that ω is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$. □

By applying similar reasoning, we can derive the following related claims:

Theorem 4.9. *Given that the $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ function, $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$, and pairwise continuous $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$. If $\omega \circ \varrho$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$, then $\omega : (\mathscr{Y}, \mu_1, \mu_2) \xrightarrow{onto} (\mathscr{Z}, \delta_1, \delta_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$.*

Theorem 4.10. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$ function, for any $\mathscr{B} \subset \mathscr{Y}$, the restriction $\varrho_{\mathscr{B}} : \varrho^{-1}(\mathscr{B}) \to \mathscr{B}$ is also $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. Given that $\mathscr{B} \subset \mathscr{Y}$. let the function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{\text{onto}} (\mathscr{Y}, \mu_1, \mu_2)$, and consider $\mathscr{D}$ be a ξ_1-$\widetilde{S_e}\widetilde{C}$ set. Then,

$$\varrho_{\mathscr{B}}(\mathscr{D} \cap \varrho^{-1}(\mathscr{B})) = \varrho(\mathscr{D}) \cap \mathscr{B}$$

which is μ_1-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{B}$. As well, we can show that if $\mathscr{D}$ is a ξ_2-$\widetilde{S_e}\widetilde{C}$ set, then

$$\varrho_{\mathscr{B}}(\mathscr{D} \cap \varrho^{-1}(\mathscr{B})) = \varrho(\mathscr{D}) \cap \mathscr{B}$$

is μ_2-$\widetilde{S_e}\widetilde{C}$ in $\mathscr{B}$. Hence, $\varrho_{\mathscr{B}} : \varrho^{-1}(\mathscr{B}) \to \mathscr{B}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$. □

Theorem 4.11. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, for every $\mathscr{B} \subset \mathscr{Y}$, the restriction $\varrho_{\mathscr{B}} : \varrho^{-1}(\mathscr{B}) \to \mathscr{B}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$.*

Proof. This result is an immediate consequence of Theorem 4.10. □

Definition 4.4. *A space $(\{X\}, \xi_1, \xi_2)$ is said to $\widetilde{P_w}\widetilde{S_e}-$Hausdorff if, for each two distinct points x and y, there exist a $\widetilde{S_e} - \xi_1$-neighbourhood $\mathscr{U}$ of x and a $\widetilde{S_e} - \xi_2$-neighbourhood $\mathscr{V}$ of y such that $\mathscr{U} \cap \mathscr{V} = \emptyset$.*

Theorem 4.12. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$, where $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}-$Hausdorff, and $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, then ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. If $\mathscr{D}$ is a ξ_1-$\widetilde{S_e}\widetilde{C}$ subset of $(\mathscr{X}, \xi_1, \xi_2)$, it is also ξ_2-$\widetilde{S_e}\widetilde{C_{pt}}$ due to the $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ of $(\mathscr{X}, \xi_1, \xi_2)$. Since ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$, $\varrho(\mathscr{D})$ is a μ_2-$\widetilde{S_e}\widetilde{C_{pt}}$ subset of $(\mathscr{Y}, \mu_1, \mu_2)$.

Moreover, since $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}-$ Hausdorff, it follows that $\varrho(\mathscr{D})$ is a μ_1-$\widetilde{S_e}\widetilde{C}$ set. As well, if $\mathscr{B}$ is a ξ_2-$\widetilde{S_e}\widetilde{C}$ subset of $\mathscr{X}$, then $\varrho(\mathscr{B})$ is a μ_2-$\widetilde{S_e}\widetilde{C}$ subset of $(\mathscr{Y}, \mu_1, \mu_2)$.

□

Using similar reasoning, we can deduce the following related assertions:

Theorem 4.13. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$, where $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$, and $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}-$ Hausdorff, then ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Theorem 4.14. *If $\varrho : (\mathscr{X}, \xi_1, \xi_2) \xrightarrow{onto} (\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ function, where $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}$- Hausdorff, and $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Definition 4.5. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is termed a $\widetilde{P_w}\widetilde{S_e}$-homeomorphism if ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$ (or $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$), and ϱ is a bijection.*

Theorem 4.15. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a bijective function that is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{nt}}$. If $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space and $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, then ϱ is a $\widetilde{P_w}\widetilde{S_e}$-homeomorphism.*

Proof. It's enough to show that ϱ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$. Consider f be a ξ_κ-closed proper subset of $\mathscr{X}$, and hence f is a proper ξ_ι-$\widetilde{S_e}\widetilde{C_{pt}}$ set, where $\kappa \neq \iota$, and $\kappa, \iota = 1, 2$. By using Theorem 2.1, we know that $\varrho(f)$ is μ_ι-$\widetilde{S_e}\widetilde{C_{pt}}$. But since $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space, $\varrho(f)$ is μ_κ-$\widetilde{S_e}$ closed. Therefore, ϱ is a $\widetilde{P_w}\widetilde{S_e}$-homeomorphism function.

□

Definition 4.6. *A function $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is termed $\widetilde{P_w}\widetilde{S_e}$-strongly function (or $\widetilde{P_w}\widetilde{S_e}$-weakly function) if for any $\widetilde{P_w}\widetilde{O}$ cover $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$, there exists a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover $\check{V} = \{\mathscr{V}_\gamma : \gamma \in \Gamma\}$ of $\mathscr{Y}$ such that each $\varrho^{-1}(\mathscr{V}_\gamma)$ has a finite subcover of $\check{U}$.*

Theorem 4.16. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}$-strongly onto function. $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ if $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.*

Proof. Consider $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$ is a pairwise open cover of $(\mathscr{X}, \xi_1, \xi_2)$. Since ϱ is a $\widetilde{P_w}\widetilde{S_e}$-strongly function, there exists a $\widetilde{P_w}\widetilde{S_e}\widetilde{O_p}$ cover $\check{V} = \{\mathscr{V}_\gamma : \gamma \in \Gamma\}$ of $(\mathscr{Y}, \mu_1, \mu_2)$, such that each $\varrho^{-1}(\mathscr{V}_\gamma)$ has a finite subcover of $\check{U}$. But $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$, so there exists a finite subset $\Gamma_1 \subseteq \Gamma$, such that $\mathscr{Y} = \bigcup_{\gamma \in \Gamma_1} \mathscr{V}_\gamma$. therefore, $\mathscr{X} = \bigcup_{\gamma \in \Gamma_1} \varrho^{-1}(\mathscr{V}_\gamma)$. Thus, each $\varrho^{-1}(\mathscr{V}_\gamma)$ has a finite subcover of $\check{U}$, implying that $\mathscr{X}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$.

□

Definition 4.7. *A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is called $\widetilde{P_w}Z^* - \widetilde{S_e}$-paracompact if each $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{X}$ has a pairwise locally finite $\xi_1\xi_2$-open refinement.*

Definition 4.8. *A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is called $\widetilde{P_w}\widetilde{S_e}$-paracompact if each $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{X}$ has a $\widetilde{P_w}$-locally finite pairwise open refinement.*

Theorem 4.17. *Given that $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-paracompact. Then $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}$-paracompact.*

Proof. Consider $\check{U} = \{\mathscr{U}_\alpha : \alpha \in \Lambda\}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $(\mathscr{X}, \xi_1, \xi_2)$. Since ϱ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, then for all $y \in \mathscr{Y}$, $\varrho^{-1}(y)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$. There exists a finite subset Λ_y, Λ_y^* of Λ such that $\varrho^{-1}(y) \subseteq \bigcup_{\alpha \in \Lambda_y} \{\mathscr{V}_\alpha : \alpha \in \Lambda_y\} \cup \bigcup_{\alpha \in \Lambda_y^*} \{W_\alpha : \alpha \in \Lambda_y^*\}$, where $\{\mathscr{V}_\alpha : \alpha \in \Lambda_y\}$ is $\widetilde{S_e}\widetilde{O}$ in $S(\xi_1)$, and $\{W_\alpha : \alpha \in \Lambda_y^*\}$ is $\widetilde{S_e}\widetilde{O}$ in $S(\xi_2)$. Let $O_y = \mathscr{Y} - \varrho\left(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha\right)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_1)$ containing y, and $O_y^* = \mathscr{Y} - \varrho\left(\mathscr{X} - \bigcup_{\alpha \in \Lambda_y^*} W_\alpha\right)$ be a $\widetilde{S_e}\widetilde{O}$ set in $S(\mu_2)$ containing y, where $\varrho^{-1}(O_y) \subseteq \bigcup_{\alpha \in \Lambda_y} \mathscr{V}_\alpha$, and $\varrho^{-1}(O_y^*) \subseteq \bigcup_{\alpha \in \Lambda_y^*} W_\alpha$. Let $\check{O} = \{O_y : y \in \mathscr{Y}\} \cup \{O_y^* : y \in Y\}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{O}$ cover of $\mathscr{Y}$. Since $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}$-paracompact, $\check{O}$ has a $\widetilde{P_w}$-locally finite open refinement. Say $\check{H} = \{\mathscr{H}_\mathscr{B} : \mathscr{B} \in \Gamma_1\} \cup \{\mathscr{H}_\mathscr{B}^* : \mathscr{B} \in \Gamma_2\}$, where $\{H_\mathscr{B} : \mathscr{B} \in \Gamma_1\}$ is μ_1-locally finite paracompact of O_y, and $\{\mathscr{H}_\mathscr{B}^* : \mathscr{B} \in \Gamma_2\}$ is μ_2-locally finite paracompact of O_y^*, $\Gamma = \Gamma_1 \cup \Gamma_2$. Let $G_1 = \{\varrho^{-1}(\mathscr{H}_\mathscr{B}) \cap \mathscr{V}_{\alpha_\kappa} : \kappa = 1, 2, \ldots, n, \mathscr{B} \in \Gamma_1, \alpha \in \Lambda_y\}$ be a ξ_1-$\widetilde{S_e}$ open locally finite refinement of $\{\mathscr{V}_\alpha : \alpha \in \Lambda\}$, and let $G_2 = \{\varrho^{-1}(\mathscr{H}_\mathscr{B}^*) \cap W_{\alpha_\kappa} : \kappa = 1, 2, \ldots, n, \mathscr{B} \in \Gamma_2, \alpha \in \Lambda_\mathscr{Y}^*\}$ be a ξ_2-$\widetilde{S_e}\widetilde{O}$-locally finite refinement of $\{W_\alpha : \alpha \in \Lambda\}$. Let $\check{G} = G_1 \cup G_2$. Then $\check{G}$ is a $\widetilde{P_w}\widetilde{S_e}$-locally finite semi open refinement of $\check{U}$, so $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-paracompact space.

□

Corollary 4.5. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}$- paracompact,then $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}$-paracompact.*

Corollary 4.6. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{L}$ perfect function, and $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}Z^* - \widetilde{S_e}$-paracompact, then $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}$-paracompact.*

Theorem 4.18. *The $\widetilde{P_w}\widetilde{S_e}$-Hausdorff property is preserved under $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$.*

Proof. Consider $(\mathscr{X}, \xi_1, \xi_2)$ be a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space, $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $y_1 \neq y_2$ in $(\mathscr{Y}, \mu_1, \mu_2)$. Then $\varrho^{-1}(y_1)$ and $\varrho^{-1}(y_2)$ are disjoint and $\widetilde{P_w}\widetilde{S_e}\widetilde{C_{pt}}$ subsets of $(\mathscr{X}, \xi_1, \xi_2)$. Since $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space, there exists a ξ_1-$\widetilde{S_e}\widetilde{O}$ $\mathscr{U}$ of $\mathscr{X}$ and a ξ_2-$\widetilde{S_e}\widetilde{O}$ $\mathscr{V}$ such that $\varrho^{-1}(y_1) \subseteq \mathscr{U}$, $\varrho^{-1}(y_2) \subseteq \mathscr{V}$, and $\mathscr{U} \cap \mathscr{V} = \emptyset$. Let the sets $\mathscr{Y} - \varrho(\mathscr{X} - \mathscr{U})$ be a μ_1-$\widetilde{S_e}\widetilde{O}$ set in $(\mathscr{Y}, \mu_1, \mu_2)$ containing y_1, and $\mathscr{Y} - \varrho(\mathscr{X} - \mathscr{V})$ be a μ_2-$\widetilde{S_e}\widetilde{O}$ set in $(\mathscr{Y}, \mu_1, \mu_2)$ containing y_2, such that

$$\begin{aligned} &\mathscr{Y} - \varrho(\mathscr{X} - \mathscr{U}) \cap \mathscr{Y} - \varrho(\mathscr{X} - \mathscr{V}) \\ &= \mathscr{Y} - [\varrho(\mathscr{X} - \mathscr{U}) \cup \varrho(\mathscr{X} - \mathscr{V})] \\ &= \mathscr{Y} - \varrho(\mathscr{X} - (\mathscr{U} \cap \mathscr{V})) \\ &= \mathscr{Y} - \varrho(\mathscr{X}) \\ &= \emptyset. \end{aligned}$$

Therefore, $(\mathscr{Y}, \mu_1, \mu_2)$ is a $\widetilde{P_w}\widetilde{S_e}$-Hausdorff space.

□

Remark 4.3. *The $\widetilde{P_w}\widetilde{S_e}$-Hausdorff property preserved by the inverse images of $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ functions.*

Remark 4.4. *The $\widetilde{P_w}\widetilde{S_e}$-Hausdorff property is preserved under $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$.*

Remark 4.5. *The $\widetilde{P_w}\widetilde{S_e}$-Hausdorff property is preserved by the inverse images of $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$ functions*

Remark 4.6. *The $\widetilde{P_w}\widetilde{S_e}$-Hausdorff property is preserved by the inverse images of $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$ functions*

Lemma 4.1. *Consider $\mathscr{X}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space, and $\mathscr{D}$ be a ξ_κ-$\widetilde{S_e}\widetilde{C_{pt}}$ subset of $\mathscr{X}$, $\kappa = 1, 2$. Then for each ξ_κ-$\widetilde{S_e}$ neighbourhood $\mathscr{U}$ of $\mathscr{D}$, there exists a ξ_κ-$\widetilde{S_e}\widetilde{O}$ set W such that $\mathscr{D} \subset W \subset s\, Cl_{\xi_\iota}(W) \subset \mathscr{U}$, where $\kappa, \iota = 1, 2$ and $\kappa \neq \iota$.*

Proof. For every $d \in \mathscr{D}$, there exists a ξ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}(d)$ such that

$$\mathrm{sCl}_{\xi_\iota}\big(\mathscr{V}(d)\big) \subseteq \mathscr{U},$$

so,

$$\mathscr{D} \subseteq \bigcup_{t=1}^{n} \mathscr{V}(d_t) \subseteq \mathrm{sCl}_{\xi_\iota}\left(\bigcup_{t=1}^{n} \mathscr{V}(d_t)\right).$$

Let $W = \bigcup_{t=1}^{n} \mathscr{V}(d_t)$. Then, W is ξ_κ-$\widetilde{S_e}\widetilde{O}$. But,

$$\mathrm{sCl}_{\xi_\iota}(W) = \mathrm{sCl}_{\xi_\iota}\left(\bigcup_{t=1}^{n} \mathscr{V}(d_t)\right) = \bigcup_{t=1}^{n} \mathrm{sCl}_{\xi_\iota}\big(\mathscr{V}(d_t)\big) \subseteq \mathscr{U}.$$

Hence,

$$\mathscr{D} \subseteq W \subseteq \mathrm{sCl}_{\xi_\iota}(W) \subseteq \mathscr{U}, \quad \kappa, \iota \in \{1, 2\},\ \kappa \neq \iota.$$

□

Lemma 4.2. *Consider $\mathscr{X}$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space, and $\mathscr{D}$ be a ξ_κ-$\widetilde{S_e}\widetilde{L}$ subset of $\mathscr{X}$, $\kappa = 1, 2$. Then for each ξ_κ-$\widetilde{S_e}$ neighbourhood $\mathscr{U}$ of $\mathscr{D}$, there exists a ξ_κ-$\widetilde{S_e}\widetilde{O}$ set W such that $\mathscr{D} \subset W \subset s\ Cl_{\xi_\iota}(W) \subset \mathscr{U}$, where $\kappa, \iota = 1, 2$ and $\kappa \neq \iota$.*

Theorem 4.19. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$, then $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$.*

Proof. Given μ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}$, $y \in \mathscr{Y}$, $\kappa, \iota = 1, 2$, $\varrho^{-1}(y) \in \varrho^{-1}(\mathscr{V})$ in $\mathscr{X}$. Since $\mathscr{X}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$, there exists ξ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ (by using Lemma 4.1), such that $\varrho^{-1}(y) \subseteq W \subseteq \mathrm{sCl}_{\xi_\iota}(W) \subseteq \varrho^{-1}(\mathscr{V})$, $\kappa \neq \iota$. Since ϱ is ξ_κ-$\widetilde{S_e}\widetilde{C_{nt}}$, then there exists μ_κ-$\widetilde{S_e}$ neighborhood W of y, such that $\varrho^{-1}(y) \subseteq W$, but $W \subset \varrho(\mathrm{s\ Cl}_{\xi_\iota}\mathscr{U}) \subset \mathscr{V}$. Since $\varrho(\mathrm{s\ Cl}_{\xi_\iota}\mathscr{U})$ is μ_ι-$\widetilde{S_e}$ closed, $y \in W \subset \mathrm{s\ Cl}_{\mu_\iota}(W) \subset \varrho(\mathrm{s\ Cl}_{\xi_\iota}\mathscr{U}) \subset \mathscr{V}$.
Hence, $\mathscr{Y}$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$.

□

Corollary 4.7. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a$\widetilde{P_w}\widetilde{S_e}\widetilde{L}\widetilde{P}$ function, and $(\mathscr{X}, \xi_1, \xi_2)$ is a $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$, then $(\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$.*

Remark 4.7. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under the inverse image of $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$.*

Remark 4.8. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$.*

Corollary 4.8. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under the inverse image of $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{P}$.*

Corollary 4.9. *The $\widetilde{P_w}\widetilde{S_e}\widetilde{R_{eg}}$ space is preserved under the inverse image of $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}\widetilde{P}$.*

Definition 4.9. *A bitopological space $(\mathscr{X}, \xi_1, \xi_2)$ is called $\widetilde{P_w}\widetilde{S_e}$-normal if for each ξ_κ-$\widetilde{S_e}\widetilde{C}$ set $\mathscr{D}$ and ξ_ι-$\widetilde{S_e}\widetilde{C}$ set $\mathscr{B}$, there exist ξ_ι-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{U}$ and ξ_κ-$\widetilde{S_e}\widetilde{O}$ set $\mathscr{V}$, such that $\mathscr{D} \subset \mathscr{U}$, $\mathscr{B} \subset \mathscr{V}$, and $\mathscr{U} \cap \mathscr{V} = \emptyset$, where $\kappa, \iota = 1, 2$ and $\kappa \neq \iota$.*

Theorem 4.20. *Consider $\varrho : (\mathscr{X}, \xi_1, \xi_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ be a $\widetilde{P_w}\widetilde{S_e}\widetilde{P}$ function, and if $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}\widetilde{S_e}$-normal, then $(\mathscr{Y}, \mu_1, \mu_2)$ is also $\widetilde{P_w}\widetilde{S_e}$-normal.*

Proof. It follows by applying Lemma 4.1 and theorem 4.19. □

Theorem 4.21. *Consider $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be any bitopological spaces. If $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then the projection function: $\vartheta : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. If $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then $(\mathscr{X}, \xi_1)$ is $Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, and $(\mathscr{X}, \xi_2)$ is $Z^* - \widetilde{S_e}\widetilde{C_{pt}}$. Thus, the projection functions $\vartheta_1 : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1) \to (\mathscr{Y}, \mu_1)$ and $\vartheta_2 : (\mathscr{X} \times \mathscr{Y}, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_2)$ are semi closed. Therefore, ϑ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.

□

Theorem 4.22. *Consider $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be any bitopological spaces. If $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$, then the projection function: $\vartheta : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_1, \mu_2)$ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$.*

Proof. Given that $(\mathscr{X}, \xi_1, \xi_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$, then $(\mathscr{X}, \xi_1)$ is $Z^* - \widetilde{S_e}\widetilde{L}$, and $(\mathscr{X}, \xi_2)$ is $Z^* - \widetilde{S_e}\widetilde{L}$. Thus, the projection functions $\vartheta_1 : (\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1) \to (\mathscr{Y}, \mu_1)$ and $\vartheta_2 : (\mathscr{X} \times \mathscr{Y}, \xi_2 \times \mu_2) \to (\mathscr{Y}, \mu_2)$ are semi closed. Therefore, ϑ is $\widetilde{P_w}\widetilde{S_e}\widetilde{C}$. □

Corollary 4.10. *Let $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$ spaces, then $(\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{C_{pt}}$.*

Corollary 4.11. *Let $(\mathscr{X}, \xi_1, \xi_2)$ and $(\mathscr{Y}, \mu_1, \mu_2)$ be $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$ spaces, then $(\mathscr{X} \times \mathscr{Y}, \xi_1 \times \mu_1, \xi_2 \times \mu_2)$ is $\widetilde{P_w}Z^* - \widetilde{S_e}\widetilde{L}$.*

5. Prospective Applications

This section investigates the future possibilities of new aggregation approaches, specifically the applications of semi-Lindelöf perfect functions and semi-perfect functions within bitopological spaces. We next look at how these notions might be used to promote both theoretical research and practical applications in a variety of fields.

In neutrosophic topology, where any point can have degrees of truth, indeterminacy, and falsehood, pairwise perfect functions could provide a glimpse of how different topologies interact when some truth-values are preserved across mappings. This could find use in fuzzy systems, decision-making, or uncertainty modeling.

Having numerous topology types in mathematical modeling of economics and decision theory might be effective in representing several utility functions or preference structures, particularly in multicriteria decision-making. When it comes to optimization or equilibrium models, a pairwise perfect function can be thought of as a mapping that precisely maintains certain fundamental characteristics across various assessment criteria.

Additionally might enhance AI models. Artificial intelligence models may learn more quickly and achieve superior performance in tasks such as language processing and reinforcement learning if they simultaneously use pairwise semi perfect functions in bitopological spaces. Moreover, Semi-Lindelöf perfect functions may increase prediction models' accuracy by allowing better handling of complicated data topologies.

As well the semi-Lindelöf perfect function and semi perfect function in bitopological spaces may lead to the development of novel quantum algorithms that take use of quantum systems dual nature.

6. Conclusion

The paper investigated the relationships between pairwise semi-Lindelöf perfect functions, pairwise perfect spaces, pairwise Lindelöf perfect functions, and pairwise semi Lindelöf perfect functions in the bitopological spaces from which those functions come from. Using the conceptual framework of pairwise semi perfect functions provided currently, the research illustrated the requirements for connecting pairwise semi sets and continuous closed functions. We assessed the relationships between these ideas and declared them with many different functions. The other objective of the examination was to highlight specific details of the intricate features of pairwise semi-perfect functions as well as some peculiarities in the Cartesian multiplication of these functions under specific conditions. Additionally, some illustrative scenarios and the main features of these notions were thoroughly investigated. We outlined their essential traits in their entirety and clarified what was needed for getting them into parity. The research also highlighted these functions characteristics and offered numerous examples of them. These responsibilities will serve as a basis for additional research into the prospective outcomes of each of these roles. Further research could explore additional iterations of these features. Pairwise semi-perfect functions can be defined and studied in various contexts, such as: (1) Neutrosophic semi-perfect function in bitopological spaces; (2) Pairwise semi-perfect functions and domain theory; (3) Pairwise fuzzy semi-perfect functions; and (4) Finding applications for our newly discovered pairwise semi-perfect function results in Artificial Intelligence, Multi-Dimensional Optimization, Quantum Computing, Risk Management,and broad continuity.

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SOME IMPORTANT MAPS IN INTERVAL DIGITAL SIGNAL PROCESSING

HALISE LEVENT *

Abstract. In recent years there has been an increasing attention to intervals since an interval supplies necessary constraints on the uncertainties that arise from real world problems. Especially, in signal processing, when a signal value in a time t is completely unknown, intervals are used to process such signals, Mathematically, a signal is defined as a function from a subset of $\mathbb{R}$ into $\mathbb{C}$. The notion of interval signal is a function from a subset of $\mathbb{R}$ into $\mathbb{I}_{\mathbb{R}}$ which is the set of all intervals. This paper presents some elemental maps for the mathematical functions of interval digital signal processing. Finally, the interval circular convolution map is introduced to provide a suitable way for processing of interval signals.

Keywords: Interval-valued map, interval signal, discrete-interval sequence, discrete-time systems, circular convolution operator

MSC (2020): 06F99, 33B99, 46C05, 47H04, 54C60.

1. Introduction

In recent years, interval analysis has been used in many application areas such as global obtimization, economics, physics, control design, signal processing, robotics. The uncertainties that arise in real world problems have been taken into a suitable range thanks to the interval concept, and so the solution of the problem has been made more comfortable. In 1962 by R. Moore [7], the notion of interval has been widely handled to bound uncertainties arising from a system. In [7] is denoted a real interval x by $x = [\underline{x}, \overline{x}]$ where $\underline{x}$ and $\overline{x}$ are the left and right endpoints of x, respectively. Further, the set of all real intervals is denoted by $\mathbb{I}_{\mathbb{R}}$. Algebraic structures on this set have studied in [1, 2]. Nowadays interval analysis tools are widely used by many authors in signal processing, [9, 3, 11].

Mathematically, a signal is a map an independent variable in any time. Sometimes, the value of a signal at any time t may not be determined exactly. We need the concept of interval-valued data and interval-based signal processing to resolve such uncertainties in process, [4, 10, 14]. In [11], a *discrete-interval sequence* X is a map from $\mathbb{Z}$ into $\mathbb{I}_{\mathbb{R}}$ and it is represented $X = (X_n)_{n\in\mathbb{Z}}$. Additionally, the concept of discrete-time ınterval signal and continuous-time interval signal are presented in [13, 5]. In these works, some new mathematical techniques

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are developed to obtain some approximate estimations on interval digital signal processing. Especially, in [11], some important tools for interval digital signal processing are presented. These tools are: casualty, stability, additivity and homogeneity. Further, the notion of interval convolution is advanced to represent uncertainty systems and signals.

This study provides convenience to the designers dealing with uncertainties digital signal processing thanks to new definitions and results. Firstly, linear interval systems, memoryless interval systems, casual interval systems and BIBO-stable interval systems are presented by using the ınterval sequence $\mathbb{I}_{l_1}$ which denotes the set of all discrete-interval sequence $X = (X_n)_{n\in\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} \|X_n\| < \infty.$$

Thus, many concepts in interval digital processing are based on mathematical foundations by using interval analysis and functional analysis techniques. Finally, the notion of interval circular convolution is denoted. This definition is interval version of circular convolution in classical signal processing and it is one of the most important operations of digital signal processing.

2. Preliminaries

An interval x is the compact-convex subset of real numbers given by

$$x = [\underline{x}, \overline{x}] = \{t \in \mathbb{R} : \underline{x} \leq t \leq \overline{x}]$$

If x contains a single real number then x is called degenerate interval and we write that $\underline{x} = \overline{x}$.

If $x = [\underline{x}, \overline{x}]$ and $y = \left[\underline{y}, \overline{y}\right]$ are intervals then

$$x + y = [\underline{x}, \overline{x}] + \left[\underline{y}, \overline{y}\right] = \left[\underline{x} + \underline{y}, \overline{x} + \overline{y}\right], \tag{2.1}$$

$$x - y = [\underline{x}, \overline{x}] - \left[\underline{y}, \overline{y}\right] = \left[\underline{x} - \overline{y}, \overline{x} - \underline{y}\right]$$

where $y = \left[-\overline{y}, -\underline{y}\right]$,

$$\alpha x = \alpha\,[\underline{x}, \overline{x}] = \begin{cases} [\alpha\underline{x}, \alpha\overline{x}]\,, & \alpha \geq 0 \\ [\alpha\overline{x}, \alpha\underline{x}]\,, & \alpha < 0 \end{cases}, \tag{2.2}$$

and

$$x \cdot y = [\underline{x}, \overline{x}] \cdot \left[\underline{y}, \overline{y}\right] = [\min M, \max M] \tag{2.3}$$

where $M = \{\underline{xy}, \underline{x}\overline{y}, \overline{x}\underline{y}, \overline{xy}\}$.

The details of interval arithmetic in this article can be founded in [6, 8].

The set of all intervals is denoted by $\mathbb{I}_{\mathbb{R}}$ and it is not a linear space since some elements have not an additive inverse. For example, the interval $[2, 3]$ has not an additive inverse, since

$$[2, 3] + (-1)[2, 3] = [-1, 1]$$

and the set $[-1, 1]$ is different from the unit element $\{0\}$. The set $\mathbb{I}_{\mathbb{R}}$ is a quasilinear space with the operations given by 2.1, 2.2 and the partial order relation "$\preceq$ ",

$$x \preceq y \Longleftrightarrow \underline{y} \leq \underline{x},\ \overline{x} \leq \overline{y}. \tag{2.4}$$

The quasilinear spaces are investigated in [1]. If an interval x has an additive inverse then is called regular, otherwise it is called singular. The set of all regular elements of $\mathbb{I}_{\mathbb{R}}$ is isometrically isomorphic to $\mathbb{R}$.

Further, the arithmetic interval has psedo-distributivity:

$$x(y+z) \preceq xy + xz$$

where $x, y, z \in \mathbb{I}_{\mathbb{R}}$ and "$\preceq$ " is the partial order relation given by 2.4.

Now, let us give some basic notions with respect to the discrete interval sequences.

A discrete interval sequence is defined as follows:

$$X = (X_n)_{n\in\mathbb{Z}} = (..., X_{-2}, X_{-1}, X_0, X_1, X_2, ...)$$

such that $X_n \in \mathbb{I}_{\mathbb{R}}$ for $n \in \mathbb{Z}$, [13]. The sequence X is called as discrete-time interval sıgnal in interval signal processing.

The set of all discrete interval sequences $X = (X_n)_{n\in\mathbb{Z}}$ such that

$$\sum_{n\in\mathbb{Z}} \|X_n\|^2_{\mathbb{I}_{\mathbb{R}}} < \infty$$

is denoted by $\mathbb{I}_{l_2}$ and it is a Hilbert quasilinear space, [13]. Further, the norm on $\mathbb{I}_{\mathbb{R}}$ is defined by

$$\|x\|_{\mathbb{I}_{\mathbb{R}}} = \|[\underline{x}, \overline{x}]\|_{\mathbb{I}_{\mathbb{R}}} = \max\{|\underline{x}|, |\overline{x}|\},$$

[1]. For example, $\|[-3, 1]\| = \max\{|-3|, |1|\} = 3$.

The set $\mathbb{I}_{l_2}$ is in rapport with the set of all finite energy discrete-time interval signals. The addition, scalar multiplication operations and the relation are follows:

$$\begin{aligned} X + Y &= (..., X_{-2}, X_{-1}, X_0, X_1, X_2, ...) + (..., Y_{-2}, Y_{-1}, Y_0, Y_1, Y_2, ...) \\ &= (..., X_{-2} + Y_{-2}, X_{-1} + Y_{-1}, X_0 + Y_0, X_1 + Y_1, X_2 + Y_2, ...), \end{aligned}$$

$$\begin{aligned} \lambda X &= \alpha(..., X_{-2}, X_{-1}, X_0, X_1, X_2, ...) \\ &= (..., \alpha X_{-2}, \alpha X_{-1}, \alpha X_0, \alpha X_1, \alpha X_2, ...) \end{aligned}$$

and the relation

$$(X_n)_{n\in\mathbb{Z}} \lesssim (Y_n)_{n\in\mathbb{Z}} \text{ iff } X_n \preceq Y_n \text{ for each } n \in \mathbb{Z}.$$

For example, if

$$X_n = \begin{cases} [-1, \cos(\frac{\pi}{2}n)] & , \text{ for } n \in \mathbb{Z}^+ \\ 0 & , \text{ otherwise} \end{cases}$$

then $X = (X_n)_{n\in\mathbb{Z}} \notin \mathbb{I}_{l_2}$. If

$$X_n = \begin{cases} [0, \frac{1}{n^2}] & , \text{ for } n \in \mathbb{Z}^+ \\ 0 & , \text{ otherwise} \end{cases}$$

then $Y = (Y_n)_{n\in\mathbb{Z}} \in \mathbb{I}_{l_2}$.

3. Discrete Interval Systems and Some Their Properties

Discrete interval systems are important operators on the space of discrete interval sequences. Some of the most commonly used systems in applications are: Linear, memoryless, casual, stable interval systems. These systems is both important in applications and amenable to analysis. In this section, these discrete interval systems are examined and the properties of some important maps between interval sequence spaces are investigated.

Mathematically, a discrete interval system F maps an input discrete interval sequence X into an output discrete interval sequence Y such that

$$Y = F(x).$$

Typically, the space of discrete interval sequences is $\mathbb{I}_{l_2}$ or a subspace of $\mathbb{I}_{l_2}$.

3.1. **Linear Interval System.** A discrete interval system F is called linear if for any discrete interval sequences X and Y,

$$F(X+Y) = F(X) + F(Y),$$

$$F(\lambda X) = \lambda F(X)$$

and

$$\text{if } X \lesssim Y \text{ then } F(X) \lesssim F(Y)$$

where λ is a real constant.

The *Kronecker delta sequence* is defined in [12] as follows:

$$\delta_n = \begin{cases} 1 & , \text{ for } n = 0 \\ 0 & , \text{ otherwise} \end{cases}, n \in \mathbb{Z}.$$

According to this definition we define *an interval Kronecker delta sequence* as follows:

$$\Delta_n = [\delta_n, \delta_n].$$

In other words,

$$\Delta_n = \begin{cases} [1,1] & , \text{ for } n = 0 \\ 0 & , \text{ otherwise} \end{cases}, n \in \mathbb{Z}.$$

For example, the product of the sequence Δ_n with a constant sequence $X = (..., [1,2], [1,2], [1,2], ...)$ is as follows:

$$\Delta_n.X = \begin{cases} [1,2] & , \text{ for } n = 0 \\ 0 & , \text{ otherwise} \end{cases}, n \in \mathbb{Z}.$$

An interval sequence H_n is called the *impulse response* of linear interval system F if input Δ_n produces H_n, i.e.; $H_n = F(\Delta_n)$.

3.2. **Memoryless Interval System.** A discrete interval system F is called a memoryless system if the output Y_n depends only on its inputs X_n.

Definition 3.1. *The projection via domain restriction operator* $1_U : \mathbb{I}_{l_2} \to \mathbb{I}_{l_2}$ *is defined by*

$$Y = (Y_n)_{n\in\mathbb{Z}} = 1_U(X_n)_{n\in\mathbb{Z}}$$

with

$$Y_n = \begin{cases} X_n & , \text{ for } n \in U \\ 0 & , \text{ otherwise} \end{cases},$$

where U *is a subset of* $\mathbb{Z}$.

According to this definition, F is a memoryless system if at a time index t

$$1_{\{t\}}X_n = 1_{\{t\}}X_n^{'}$$

then

$$1_{\{t\}}Y_n = 1_{\{t\}}Y_n^{'}$$

where X_n and $X_n^{'}$ are inputs, Y_n and $Y_n^{'}$ are outputs.

3.3. **Shift-Invariant Interval System.** A discrete interval system F is called a *shift-invariant system* when for any integer k and the inputs X and $X^{'}$, if

$$X_n = X_{n-k}$$

then

$$F(X_n) = F(X_{n-k}^{'}).$$

Namely, in a shift-invariant system, shifting in the time of the input has the same variation in the time of the output.

3.4. **Casual Interval System.** A discrete interval system F is called a *casual system* if the output of the system F at time n depends on the input only up to time n. Namely, in the casual interval system F if

$$1_{\{-\infty,\dots,k\}}X_n = 1_{\{-\infty,\dots,k\}}X_n^{'}$$

then

$$1_{\{-\infty,\dots,k\}}Y_n = 1_{\{-\infty,\dots,k\}}Y_n^{'}$$

where X_n and $X_n^{'}$ are inputs, Y_n and $Y_n^{'}$ are outputs, $1_{\{-\infty,\dots,k\}}$ is domain restriction operator.

3.5. **BIBO Stable Interval System.** A discrete interval system F is called a *BIBO stable* if for each bounded interval sequence converts to bounded interval sequence.

If F is a BIBO stable interval system and

$$\sup_{n\in\mathbb{Z}} \|X_n\|_{\mathbb{I}_\mathbb{R}} < \infty$$

then

$$\sup_{n\in\mathbb{Z}} \|Y_n\|_{\mathbb{I}_\mathbb{R}} < \infty$$

where $X_n = F(X_n), n \in \mathbb{Z}$.

Now let us give an example of how these operators are used in applications.

Consider the discrete interval system F such that for the interval sequence $X = (X_n)_{n\in\mathbb{Z}}$,

$$F(X_n) = \frac{1}{3}(X_{n-1} + X_n + X_{n+1}).$$

This system is called *averaging interval operator.*

Averaging interval operator is linear system:

$$\begin{aligned}
F(X_n + \tilde{X}_n) &= \frac{1}{3}((X_{n-1} + \tilde{X}_{n-1}) + (X_n + \tilde{X}_n) + (X_{n+1} + \tilde{X}_{n+1})) \\
&= \frac{1}{3}(X_{n-1} + X_n + X_{n+1}) + \frac{1}{3}(\tilde{X}_{n-1} + \tilde{X}_n + \tilde{X}_{n+1}) \\
&= F(X_n) + F(\tilde{X}_n)
\end{aligned}$$

and

$$\begin{aligned} F(\lambda X_n) &= \frac{1}{3}(\lambda X_{n-1} + \lambda X_n + \lambda X_{n+1}) \\ &= \lambda(\frac{1}{3}(X_{n-1} + X_n + X_{n+1})) \\ &= \lambda F(X_n). \end{aligned}$$

Further, assume that $X \lesssim \tilde{X}$. Since $X_n \subseteq \tilde{X}_n$ for each $n \in \mathbb{Z}$, we write that

$$\begin{aligned} F(X_n) &= \frac{1}{3}(X_{n-1} + X_n + X_{n+1}) \\ &\lesssim \frac{1}{3}(\tilde{X}_{n-1} + \tilde{X}_n + \tilde{X}_{n+1}) \\ &= F(\tilde{X}_n). \end{aligned}$$

This system is BIBO stable:

Suppose that $\sup\limits_{n\in\mathbb{Z}} \|X_n\|_{\mathbb{I}_\mathbb{R}} < \infty$. Then

$$\begin{aligned} \sup_{n\in\mathbb{Z}} \|F(X_n)\| &= \sup_{n\in\mathbb{Z}} \left\| \frac{1}{3}(X_{n-1} + X_n + X_{n+1}) \right\| \\ &= \frac{1}{3} \sup_{n\in\mathbb{Z}} \|X_{n-1} + X_n + X_{n+1}\| \\ &\leq \frac{1}{3} \sup_{n\in\mathbb{Z}} \{\|X_{n-1}\| + \|X_n\| + \|X_{n+1}\|\} \\ &= \sup_{n\in\mathbb{Z}} \|X_n\| < \infty. \end{aligned}$$

But this discrete interval system is neither memoryless nor casual.

The matrix of the averaging interval operator is as follows:

For $F(X_n) = Y_n, n \in \mathbb{Z}$

$$\begin{bmatrix} . \\ . \\ . \\ Y_{-1} \\ \boxed{Y_0} \\ Y_1 \\ . \\ . \\ . \end{bmatrix} = \frac{1}{3} \begin{bmatrix} & & & . & . & . & . & . & & & \\ & & & . & . & . & . & . & & & \\ & & & . & . & . & . & . & & & \\ . & . & . & 1 & 1 & 0 & 0 & 0 & . & . & . \\ . & . & . & 0 & 1 & \boxed{1} & 1 & 0 & . & . & . \\ . & . & . & 0 & 0 & 1 & 1 & 1 & . & . & . \\ & & & . & . & . & . & . & & & \\ & & & . & . & . & . & . & & & \\ & & & . & . & . & . & . & & & \end{bmatrix} \cdot \begin{bmatrix} . \\ . \\ . \\ X_{-1} \\ \boxed{X_0} \\ X_1 \\ . \\ . \\ . \end{bmatrix}$$

Further, it is shown that this matrix is Toeplitz matrix.

As is known that the convolution operator has an important situation in digital signal processing. For this purpose, the interval convolution operator is defined in [11]. Now let us present the concept of interval circular convolution operator used to define the discrete-time Fourier transform.

In real life, many computations are done on finite impuls. For this purpose, the first method is to set $X_i = [0, 0]$ for all i outside of $\{0, 1, ..., N - 1\}$. Later; the second method is as follows:

$$X = (..., X_{N-1}, X_0, X_1, ..., X_{N-1}, X_0, X_1, ...).$$

Namely, a finite-length interval sequence can be written as follows:

$$X_{(i+k)N} = X_i, k \in \mathbb{Z}.$$

Let F be a linear interval system and H_n be the impulse response of F. If H_n is an element of $\mathbb{I}_{l_1}$, then we obtain the ınterval convolution as usual,

$$\begin{aligned} Y_i &= (X * H)_i \\ &= \sum_{i\in\mathbb{Z}} X_i H_{i-k} \\ &= \sum_{i\in\mathbb{Z}} H_i X_{i-k}. \end{aligned}$$

Since X is N-periodic interval sequence, then Y is N-periodic:

$$\begin{aligned} Y_{i+N} &= \sum_{i\in\mathbb{Z}} H_i X_{i+N-k} \\ &= \sum_{i\in\mathbb{Z}} H_i X_{i-k} \\ &= Y_n. \end{aligned}$$

Let us write the impulse response H_n with period N as follows:

$$(H^N)_n = \sum_{i\in\mathbb{Z}} H_{n-iN}.$$

The last sum is convergent, since $H \in \mathbb{I}_{l_1}$.

Now we will write the interval convolution of a N- periodic interval input sequence X and a N- periodic interval impulse response H.

$$\begin{aligned} (H * X)_n &= \sum_{i\in\mathbb{Z}} H_i X_{n-i} \\ &= \sum_{k\in\mathbb{Z}} \sum_{i=kN}^{(k+1)N-1} H_i X_{n-i} \\ &= \sum_{k\in\mathbb{Z}} \sum_{l=0}^{N-1} H_{l+kN} X_{n-l-kN}. \end{aligned}$$

If we do the change of variable $i = l$.

$$\sum_{k\in\mathbb{Z}} \sum_{l=0}^{N-1} H_{l+kN} X_{n-l-kN} = \sum_{k\in\mathbb{Z}} \sum_{i=0}^{N-1} H_{i+kN} X_{n-i}.$$

Since $H \in \mathbb{I}_{l_1}$ we can write that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{i=0}^{N-1} H_{i+kN}X_{n-i} &= \sum_{i=0}^{N-1}\sum_{k\in\mathbb{Z}} H_{i+kN}X_{n-i} \\ &= \sum_{i=0}^{N-1} H_i^N X_{n-i} \\ &= \sum_{i=0}^{N-1} H_i^N X_{(n-i)\bmod N}.\end{aligned}$$

Thus, *the interval circular convolution of* N-periodic interval sequences H and X is defined as

$$\begin{aligned}(H \circledast X)_n &= \sum_{i=0}^{N-1} X_i H_{(n-i)\bmod N} \\ &= \sum_{i=0}^{N-1} X_{(n-i)\bmod N} H_i.\end{aligned}$$

4. Conclusion

In this study, linear interval systems, memoryless interval systems, casual interval systems and BIBO-stable interval systems are presented by using the interval sequences. Further, the interval circular convolution is denoted by using interval analysis and functional analysis techniques. This study provides convenience to the designers dealing with uncertainties digital signal processing thanks to new definitions and results. Later, other important operators in the classical digital processing field can be adapted with the help of intervals to contribute to interval signal processing.

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CHARACTERIZATION OF m-QUASI-EINSTEIN STRUCTURES IN LP-KENMOTSU MANIFOLDS

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Abstract. In this paper, we investigate m-quasi Einstein metrics on LP-Kenmotsu manifolds, a recently introduced class of Lorentzian paracontact metric manifolds. We derive the curvature identity associated with a closed m-quasi Einstein structure and classify LP-Kenmotsu manifolds admitting such metrics. It is shown that if the potential vector field is conformal, collinear with the unit timelike vector field, or a strict infinitesimal contact transformation, then the manifold is either Einstein or η-Einstein under suitable conditions. Furthermore, we prove that the scalar curvature of such manifolds is necessarily constant.
Keywords: LP-Kenmotsu manifolds, m-Quasi-Einstein structures, Einstein manifolds, Conformal vector field.
MSC (2020): 53C18; 53C21; 53C25; 53C50.

1. Introduction

The study of Einstein manifolds has long been central to both mathematics and physics, especially in Riemannian geometry and the theory of general relativity. These are Riemannian manifolds (M, g) whose Ricci tensor satisfies $\mathrm{Ric} = \lambda g$ for some scalar $\lambda \in \mathbb{R}$. To explore geometric phenomena beyond this classical setting, various generalizations have been proposed among them, *Ricci solitons* and *m-quasi Einstein metrics* have gained significant attention due to their connections with the Ricci flow, warped product spaces, and weighted volume measures. For further information on Ricci solitons and their generalizations, the reader may refer to ([1, 5, 7, 12, 14, 15, 20, 26], [28]-[35]) and the sources cited within those works.

An *m-quasi Einstein manifold* (M, g, β_V, λ) is a Riemannian manifold equipped with a vector field β_V (called the potential vector field) that satisfies the equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^\sharp \otimes \beta_V^\sharp = \lambda g, \tag{1.1}$$

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where $\mathcal{L}_{\beta_V} g$ denotes the Lie derivative of the metric g along β_V, $\beta_V^\sharp$ is the 1-form metrically dual to β_V, and $m > 0$ (possibly $m = \infty$) (For further details, see [17, 29]).

The expression

$$\operatorname{Ric} + \nabla^2 f - \frac{1}{m}\, df \otimes df = \operatorname{Ric}_f^m$$

is known as the *m-Bakry-Émery Ricci tensor*. It arises naturally in the study of *warped product manifolds* $(M \times N, \bar{g})$, where (M^n, g) and (N^m, h) are Riemannian manifolds, and the warped metric is given by $\bar{g} = g + e^{-\frac{2f}{m}} h$.

Furthermore, when $m = \infty$, equation (1.1) reduces to the classical Ricci soliton equation, and when $\beta_V = 0$, it coincides with the Einstein condition. Thus, the m-quasi Einstein framework unifies several important geometric structures, including Einstein manifolds, both gradient and non-gradient Ricci solitons, and certain warped product spaces governed by Bakry-Émery Ricci tensors.

The theory of m-quasi Einstein metrics has been extended to various geometric settings. Case et al. [9, 10] studied such metrics in the context of warped product manifolds, while Barros and Ribeiro Jr. [3, 4] investigated conditions for their existence on compact and homogeneous spaces. Limoncu [22] introduced a formulation involving vector fields, generalizing the m-Bakry-Émery Ricci tensor. In contact geometry, Ghosh [16] explored m-quasi Einstein structures on contact and K-contact manifolds, showing that collinearity between the potential vector field β_V and the Reeb vector field ξ leads to η-Einstein or Sasakian manifolds. Venkatesha et al. [36] extended the study to almost Kenmotsu and Kenmotsu $(\kappa, \mu)'$-manifolds. Recently, De et al. [13] investigated m-quasi Einstein metrics in the framework of paracontact geometry, focusing on K-paracontact and (k, μ)-paracontact metric manifolds. They showed that a K-paracontact metric manifold admitting an m-quasi Einstein metric must have constant scalar curvature, and provided a classification and explicit examples of such structures, highlighting the compatibility of paracontact geometry with generalized Einstein-type conditions. Parallel to these developments, another significant branch of geometric study has emerged in *Lorentzian almost paracontact geometry* a natural generalization of classical contact geometry to the pseudo-Riemannian setting. Among these, *LP-Kenmotsu manifolds* (Lorentzian para-Kenmotsu manifolds) form a distinct and rich subclass. Originally, the concept of *LP-Sasakian manifolds* was introduced by Matsumoto [23] in 1989, and later independently studied by Mihai and Rosca [25], who obtained foundational results concerning their curvature and structure. As a continuation of this line of inquiry, Haseeb and Prasad [18] introduced *LP-Kenmotsu manifolds* in 2021, which generalize Kenmotsu manifolds to the Lorentzian setting. These manifolds are endowed with a Lorentzian paracontact structure (ϕ, ξ, η, g), satisfying the conditions:

$$(\nabla_{\beta_X}\phi)\beta_Y = -g(\phi\beta_X, \beta_Y)\xi - \eta(\beta_Y)\phi\beta_X, \quad \nabla_{\beta_X}\xi = \beta_X - \eta(\beta_X)\xi,$$

$$d\eta = 0, \quad d\Phi = 2\eta \wedge \Phi, \quad \text{where } \Phi(\beta_X, \beta_Y) = g(\beta_X, \phi\beta_Y).$$

These conditions mimic those of Kenmotsu geometry but are framed in the *Lorentzian* context. The geometric behavior of LP-Kenmotsu manifolds shares several structural features with both Kenmotsu and para-Sasakian manifolds, while also exhibiting distinct curvature properties due to the indefinite metric. The interplay of curvature, contact-type structure, and Lorentzian signature makes LP-Kenmotsu manifolds promising candidates for exploring generalized Einstein-type structures.

Despite the geometric richness and recent introduction of LP-Kenmotsu manifolds, the study of m-quasi Einstein metrics within this framework remains largely unexplored. This research gap forms the central motivation for the present work. Inspired by previous developments in contact and Kenmotsu geometry, this paper aims to investigate the existence, structural properties, and geometric implications of m-quasi Einstein structures on LP-Kenmotsu manifolds.

The paper is organized as follows: Section 2 presents a detailed overview of the fundamental definitions and properties of LP-Kenmotsu manifolds. In Section 3, we construct explicit examples of LP-Kenmotsu manifolds admitting m-quasi Einstein structures. Section 4 is devoted to the development and analysis of the theory of m-quasi Einstein metrics in the context of LP-Kenmotsu geometry.

2. **Lorentzian Para-Kenmotsu Manifolds**

Let M be a smooth manifold of dimension n. This manifold is referred to as a *Lorentzian almost paracontact metric manifold* if it is equipped with a tensor field ϕ of type $(1,1)$, a vector field ξ of type $(1,0)$, a 1-form η, and a Lorentzian metric g, which together satisfy the following conditions [23]:

$$\phi^2 = I + \eta \otimes \xi, \quad \eta(\xi) = -1, \quad \eta \circ \phi = 0, \tag{2.2}$$

$$g(\phi\beta_X, \phi\beta_Y) = g(\beta_X, \beta_Y) + \eta(\beta_X)\eta(\beta_Y), \quad g(\beta_X, \xi) = \eta(\beta_X), \tag{2.3}$$

for all vector fields β_X, β_Y on M, where I is the identity transformation on the tangent bundle of M. The structure (ϕ, ξ, η, g) is known as a *Lorentzian almost paracontact structure.*

If the manifold additionally satisfies

$$(\nabla_{\beta_X}\phi)\beta_Y = -g(\phi\beta_X, \beta_Y)\xi - \eta(\beta_Y)\phi\beta_X, \tag{2.4}$$

for any vector fields β_X, β_Y, where ∇ is the Levi-Civita connection associated with g, then M is called a *Lorentzian para-Kenmotsu manifold* (briefly, LP-Kenmotsu manifold).

From Equation (2.4), the following relations are obtained:

$$\nabla_{\beta_X}\xi = -\beta_X - \eta(\beta_X)\xi, \tag{2.5}$$

$$(\nabla_{\beta_X}\eta)(\beta_Y) = -g(\beta_X, \beta_Y) - \eta(\beta_X)\eta(\beta_Y). \tag{2.6}$$

In addition, the LP-Kenmotsu manifold satisfies the following curvature properties [18]:

$$\eta(R(\beta_X, \beta_Y)\beta_Z) = g(\beta_Y, \beta_Z)\eta(\beta_X) - g(\beta_X, \beta_Z)\eta(\beta_Y), \tag{2.7}$$

$$R(\beta_X, \beta_Y)\xi = \eta(\beta_Y)\beta_X - \eta(\beta_X)\beta_Y, \tag{2.8}$$

$$R(\xi, \beta_X)\beta_Y = g(\beta_X, \beta_Y)\xi - \eta(\beta_Y)\beta_X, \tag{2.9}$$

$$R(\xi, \beta_X)\xi = \beta_X + \eta(\beta_X)\xi, \tag{2.10}$$

$$\mathrm{Ric}(\beta_X, \xi) = (n-1)\eta(\beta_X), \tag{2.11}$$

$$Q\xi = (n-1)\xi, \tag{2.12}$$

for all $\beta_X, \beta_Y, \beta_Z \in \chi(M)$, where R, Ric, and Q denote the Riemannian curvature tensor, the Ricci tensor, and the Ricci operator, respectively.

A Lorentzian para-Kenmotsu manifold M is said to be *Einstein* if the Ricci tensor Ric is proportional to the metric tensor g, that is, $Ric = fg$ for some smooth function f. According to Lemma 1 in [21], the Ricci operator Q in such a manifold satisfies:

$$(\nabla_{\beta_X} Q)\xi = QX - 2n\beta_X, \tag{2.13}$$

$$(\nabla_\xi Q)\beta_X = 2QX - 4n\beta_X. \tag{2.14}$$

3. Examples

Example 3.1. *Let $M = \mathbb{R}^3$ be a 3-dimensional manifold with global coordinates (x, y, z), and consider the Lorentzian metric*

$$g = dx^2 + e^{2x} dy^2 - dz^2.$$

Define a vector field $\xi = \frac{\partial}{\partial z}$, a 1-form $\eta = dz$, and a (1,1)-tensor field ϕ by:

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}, \quad \phi\left(\frac{\partial}{\partial z}\right) = 0.$$

Then (ϕ, ξ, η, g) defines an LP-Kenmotsu structure on M, satisfying

$$(\nabla_{\beta_X}\phi)\beta_Y = -g(\phi\beta_X, \beta_Y)\xi - \eta(\beta_Y)\phi\beta_X, \quad \nabla_{\beta_X}\xi = \beta_X - \eta(\beta_X)\xi.$$

Define the potential vector field $\beta_V = f\xi$, where $f(x, y, z) = z$, and take $m = 2$. Then the manifold M satisfies the m-quasi Einstein equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^\sharp \otimes \beta_V^\sharp = \lambda g,$$

for a suitable scalar function λ. Thus, M admits a non-trivial m-quasi Einstein metric in the LP-Kenmotsu setting.

Example 3.2. *Let $M = \mathbb{R}^5$ be a 5-dimensional manifold with coordinates (x, y, z, u, v), and define a Lorentzian metric*

$$g = dx^2 + dy^2 + e^{2x}(dz^2 + du^2) - dv^2.$$

Let $\xi = \frac{\partial}{\partial v}$, $\eta = dv$, and define the (1,1)-tensor field ϕ by:

$$\phi\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y}, \quad \phi\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x},$$

$$\phi\left(\frac{\partial}{\partial z}\right) = \frac{\partial}{\partial u}, \quad \phi\left(\frac{\partial}{\partial u}\right) = -\frac{\partial}{\partial z}, \quad \phi(\xi) = 0.$$

Then the structure (ϕ, ξ, η, g) defines an LP-Kenmotsu manifold. Let the potential vector field be $\beta_V = a\xi$ for some constant $a \neq 0$, and choose $m = 3$. It can be verified that the manifold M satisfies the m-quasi Einstein equation

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_{\beta_V} g - \frac{1}{m}\beta_V^\sharp \otimes \beta_V^\sharp = \lambda g,$$

for a suitable constant λ. Therefore, M admits a constant-potential m-quasi Einstein metric within the LP-Kenmotsu framework.

4. Main Results

In this section, we consider the LP-Kenmotsu metric satisfying the m-quasi-Einstein metric. Before establishing our main results, we prove the following:

Lemma 4.1. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an LP-Kenmotsu manifold admitting a closed m-quasi Einstein structure with potential vector field β_V. Then, the curvature tensor satisfies the following identity:*

$$R(\beta_X,\beta_Y)\beta_V=(\nabla_{\beta_Y}Q)\beta_X-(\nabla_{\beta_X}Q)\beta_Y+\frac{1}{m}\left\{\beta_V^\sharp(\beta_X)\,QY-\beta_V^\sharp(\beta_Y)\,QX\right\}$$
$$+\frac{\lambda}{m}\left\{\beta_V^\sharp(\beta_Y)\,\beta_X-\beta_V^\sharp(\beta_X)\,\beta_Y\right\}, \tag{4.15}$$

for all vector fields $\beta_X,\beta_Y\in\Gamma(TM)$, where Q is the Ricci operator corresponding to the Ricci tensor, and $\beta_V^\sharp$ denotes the 1-form metrically dual to β_V.

Proof. Since the manifold admits a closed m-quasi Einstein structure, the associated metric satisfies

$$\mathrm{Ric}+\frac{1}{2}\mathcal{L}_{\beta_V}g-\frac{1}{m}\beta_V^\sharp\otimes\beta_V^\sharp=\lambda g.$$

The closedness of $\beta_V^\sharp$ implies the local existence of a smooth function f such that $\beta_V=\nabla f$. Consequently, the Lie derivative term simplifies to $\mathcal{L}_{\beta_V}g=2\,\mathrm{Hess}_f$, though we proceed considering β_V as a general vector field with closed dual 1-form.

Rewriting the m-quasi Einstein equation gives:

$$\nabla_{\beta_X}\beta_V+QX=\lambda\beta_X+\frac{1}{m}\beta_V^\sharp(\beta_X)\,\beta_V. \tag{4.16}$$

We now apply this identity to compute the curvature operator using:

$$R(\beta_X,\beta_Y)\beta_V=\nabla_{\beta_X}\nabla_{\beta_Y}\beta_V-\nabla_{\beta_Y}\nabla_{\beta_X}\beta_V-\nabla_{[\beta_X,\beta_Y]}\beta_V.$$

Substituting the expression for $\nabla\beta_V$, and simplifying using the symmetry of Q and properties of the Levi-Civita connection, we obtain:

$$R(\beta_X,\beta_Y)\beta_V=(\nabla_{\beta_Y}Q)\beta_X-(\nabla_{\beta_X}Q)\beta_Y+\frac{1}{m}\left\{\beta_V^\sharp(\beta_X)\,QY-\beta_V^\sharp(\beta_Y)\,QX\right\}$$
$$+\frac{\lambda}{m}\left\{\beta_V^\sharp(\beta_Y)\,\beta_X-\beta_V^\sharp(\beta_X)\,\beta_Y\right\}.$$

Hence, the required identity holds. □

Definition 4.1. *Let M be an almost contact metric manifold. A vector field β_V on M is said to be an* infinitesimal contact transformation *if the Lie derivative of the contact 1-form η along β_V satisfies*

$$\mathcal{L}_{\beta_V}\eta=\alpha\eta,$$

for some smooth function α on M. In particular, if $\alpha=0$, i.e., $\mathcal{L}_{\beta_V}\eta=0$, then β_V is called a strict infinitesimal contact transformation.

Theorem 4.1. *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be an LP-Kenmotsu manifold admitting a closed m-quasi Einstein structure with $m\neq1$. Then, one of the following holds:*

(1) *The potential vector field β_V is pointwise collinear with the unit timelike vector field ξ, and M is η-Einstein;*

(2) β_V *is a strict infinitesimal contact transformation, i.e.,* $\mathcal{L}_{\beta_V}\eta = 0$*;*
(3) M *is Einstein.*

Proof. First, by replacing β_Y with ξ in equation (4.15) and using equations (2.13) and (2.14), we obtain

$$R(\beta_X, \xi)\beta_V = QX - 2n\beta_X + \frac{2n-\lambda}{m}\beta_V^\sharp(\beta_X)\,\xi + \frac{\eta(\beta_V)}{m}(\lambda\beta_X - QX). \tag{4.17}$$

Now, taking the inner product of both sides of the curvature identity (2.8) yields

$$g(R(\beta_X, \xi)\xi, \beta_V) = -g(\beta_X, \beta_V) - \eta(\beta_X)\eta(\beta_V).$$

Together with equation (4.17), the preceding result leads to

$$QX - 2n\beta_X + \frac{2n+m-\lambda}{m}\beta_V^\sharp(\beta_X)\xi + \frac{\eta(\beta_V)}{m}(\lambda\beta_X - m\beta_X - Q\beta_X) = 0. \tag{4.18}$$

Next, by taking the inner product with ξ and using equation (2.12), we get

$$\left(\frac{2n+m-\lambda}{m}\right)(g(\beta_X, \beta_V) - \eta(\beta_X)\eta(\beta_V)) = 0. \tag{4.19}$$

As (4.19) holds for all β_X, we are led to two possibilities:

(i) $\beta_V = \eta(\beta_V)\xi$, i.e., β_V is collinear with ξ;
(ii) $\lambda = (2n+m)$.

Case (i): If $\beta_V = \eta(\beta_V)\xi$, then differentiating this along any vector field β_X and applying equation (2.5), we obtain

$$\nabla_{\beta_X}\beta_V = \eta(\nabla_{\beta_X}\beta_V)\xi - \eta(\beta_V)(\beta_X + \eta(\beta_X)\xi).$$

Substitute into (4.16), we get:

$$QX - \lambda\beta_X - \frac{1}{m}\beta_V^\sharp(\beta_X)\beta_V = 2n\,\eta(\beta_X)\,\xi - \lambda\eta(\beta_X)\xi - \frac{1}{m}\beta_V^\sharp(\beta_X)\,\eta(\beta_V)\,\xi + \eta(\beta_V)(\beta_X + \eta(\beta_X)\xi).$$

It follows from aforesaid equation, using the condition $\phi\beta_V = 0$, that

$$\phi QX = [\lambda + \eta(\beta_V)]\phi\beta_X. \tag{4.20}$$

Substituting $\beta_X = \phi\beta_X$ into equation (4.20), and recalling that the Ricci operator Q and ϕ commute on M, we obtain

$$QX = (\lambda + \eta(\beta_V))\beta_X + (\lambda + \eta(\beta_V) - 2n)\,\eta(\beta_X)\xi. \tag{4.21}$$

Taking the trace of equation (4.21) over β_X, we immediately obtain $\lambda + \eta(\beta_V) = \frac{r}{2n} - 1$. Substituting this result back into equation (4.21), we arrive at

$$QX = \left(\frac{r}{2n} - 1\right)\beta_X - \left(\frac{r}{2n} - (2n+1)\right)\eta(\beta_X)\xi,$$

which shows that M is an η-Einstein manifold. This completes the proof of part (1).

Case (ii): If $\lambda = (2n+m)$, substitute this into (4.18) and simplify:

$$\left(1 - \frac{\eta(\beta_V)}{m}\right)[Q\beta_X - 2n\beta_X] = 0. \tag{4.22}$$

Subcase 1: If $\eta(\beta_V) = m$, compute the Lie derivative:

$$\mathcal{L}_{\beta_V}\eta(\beta_X) = \mathcal{L}_{\beta_V}g(\beta_X, \xi) - g(\beta_X, \mathcal{L}_{\beta_V}\xi).$$

Since $\mathcal{L}_{\beta_V} g(\beta_X, \xi) = 2(\eta(\beta_X)\eta(\beta_V) - g(\beta_X, \beta_V)) = 0$, and $\mathcal{L}_{\beta_V}\xi = 0$, it follows that:

$$\mathcal{L}_{\beta_V}\eta = 0,$$

so β_V is a strict infinitesimal contact transformation.

Subcase 2: If $\eta(\beta_V) \neq m$, then from (4.22):

$$QX = 2n\beta_X,$$

which implies that M is Einstein. This completes the proof. □

Ghosh [16] demonstrated that a K-contact manifold admitting a conformal m-quasi Einstein metric is necessarily a non-trivial η-Einstein manifold, and the potential vector field β_V in this case is Killing. In a related study, Venkatesha et al. [36] showed that a Kenmotsu manifold equipped with a conformal m-quasi Einstein metric is Einstein with constant scalar curvature $-2n(2n+1)$.

In contrast, the investigation of m-quasi Einstein metrics on LP-Kenmotsu manifolds becomes particularly intriguing when the potential vector field β_V is assumed to be conformal. In this setting, we establish the following result:

Theorem 4.2. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an LP-Kenmotsu manifold. If M admits an m-quasi Einstein metric such that the potential vector field β_V is conformal, then M is Einstein with constant scalar curvature $r = -2n(2n+1)$.*

Proof. Since β_V is a conformal vector field, by Definition 2.1, the Lie derivative of the metric satisfies

$$\mathcal{L}_{\beta_V} g = 2\nu g, \tag{4.23}$$

for some smooth function ν on M.

Substituting (4.23) into the defining equation of an m-quasi Einstein metric (equation (1.1)), we obtain:

$$\mathrm{Ric}(\beta_X, \beta_Y) = (\lambda - \nu) g(\beta_X, \beta_Y) + \frac{1}{m}\beta_V^\sharp(\beta_X)\beta_V^\sharp(\beta_Y), \tag{4.24}$$

for all $\beta_X, \beta_Y \in \Gamma(TM)$.

Taking covariant derivative of both sides of (4.24) with respect to any vector field β_Z, and using compatibility of ∇ with g, we get:

$$(\nabla_{\beta_Z}\mathrm{Ric})(\beta_X, \beta_Y) = -(\beta_Z\nu)\, g(\beta_X, \beta_Y) + \frac{1}{m}\left\{(\nabla_{\beta_Z}\beta_V^\sharp)(\beta_X)\,\beta_V^\sharp(\beta_Y) + \beta_V^\sharp(\beta_X)\,(\nabla_{\beta_Z}\beta_V^\sharp)(\beta_Y)\right\}.$$

Now taking the cyclic sum over $\beta_X, \beta_Y, \beta_Z$, and using the symmetry of g, we derive:

$$\begin{aligned}\mathcal{C}(\beta_X, \beta_Y, \beta_Z) :=& (\nabla_{\beta_X}\mathrm{Ric})(\beta_Y, \beta_Z) + (\nabla_{\beta_Y}\mathrm{Ric})(\beta_Z, \beta_X) + (\nabla_{\beta_Z}\mathrm{Ric})(\beta_X, \beta_Y)\\ =& -\sum_{\mathrm{cyc}}(\beta_X\nu)\, g(\beta_Y, \beta_Z)\\ &+ \frac{1}{m}\sum_{\mathrm{cyc}}\left\{(\nabla_{\beta_X}\beta_V^\sharp)(\beta_Y)\beta_V^\sharp(\beta_Z) + \beta_V^\sharp(\beta_Y)(\nabla_{\beta_X}\beta_V^\sharp)(\beta_Z)\right\}.\end{aligned} \tag{4.25}$$

Contracting (4.25) over $\beta_Y = \beta_Z$, and recalling that β_V is conformal and $\mathcal{L}_{\beta_V} g = 2\nu g$, one obtains:

$$\frac{2}{2n+3}(\beta_X r) + \beta_X\nu - \frac{2\nu}{m}\beta_V^\sharp(\beta_X) = 0,$$

where r denotes the scalar curvature. Substituting this back into (4.25) eliminates the $(\beta_X \nu)$ terms, leading to:

$$\mathcal{C}(\beta_X, \beta_Y, \beta_Z) = -\frac{2}{2n+3} \left\{ (\beta_X r)\, g(\beta_Y, \beta_Z) + (\beta_Y r)\, g(\beta_Z, \beta_X) + (\beta_Z r)\, g(\beta_X, \beta_Y) \right\}. \tag{4.26}$$

Now, set $\beta_Y = \beta_Z = \xi$, and apply (2.12), (2.13) we obtain

$$\beta_X(r) + 2\,\xi(r)\,\eta(\beta_X) = 0. \tag{4.27}$$

Taking the trace of the equation yields $\xi(r) = 2(r - 2n(2n+1))$. By this equation, one can find that $\mathcal{L}_\xi r = 2(r - 2n(2n+1))$. Since $\mathcal{L}_\xi$ and d commute, this implies $\mathcal{L}_\xi(dr) = 2\,dr$, which means:

$$\mathcal{L}_\xi(Dr) = 2\,Dr.$$

Now using $\nabla_\xi Dr = Dr + (\xi r)\xi$ and combining with (4.27), we get:

$$(\xi r)\xi + 2\,\xi(\xi r)\xi = 0.$$

Solving this differential equation gives $\xi(r) = 0$, and therefore, the scalar curvature is constant:

$$r = 2n(2n+1).$$

Finally, substituting the constant r back into (4.25) and making use of (2.12), (2.13) shows that $QX = 2n\beta_X$, implying that M is Einstein.

□

Conclusion

The theory of geometric flows and differential operators, such as the Ricci flow and the Laplacian, plays a pivotal role in understanding deep structures in geometry and physics. These tools have been instrumental in resolving major conjectures (e.g., the Poincaré and geometrization conjectures) and have found applications in diverse fields like medical imaging, fluid dynamics, and computer vision. In the spirit of this geometric framework, the present work investigates the existence and implications of m-quasi Einstein metrics on LP-Kenmotsu manifolds, which are a class of Lorentzian paracontact manifolds with rich geometric structure.

Beginning with the closed m-quasi Einstein condition, we derived a fundamental curvature identity linking the Ricci operator, the potential vector field, and the structure tensors of the LP-Kenmotsu manifold. We further explored particular cases where the potential vector field is aligned with the unit timelike vector field, conformal, or acts as a strict infinitesimal contact transformation. These geometrically significant cases not only illuminate the curvature behavior but also suggest strong constraints on the underlying manifold.

Our results demonstrate that LP-Kenmotsu manifolds admitting such metrics must be Einstein or η-Einstein, with constant scalar curvature—a feature that reflects the rigidity imposed by the m-quasi Einstein framework. This generalizes several classical results known in Kenmotsu and contact metric geometry to the Lorentzian paracontact setting. The study of such metrics opens new avenues in the analysis of pseudo-Riemannian structures, particularly where the Laplacian and curvature operators interact in meaningful ways, thus aligning well with broader geometric applications in both mathematics and applied sciences.

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ON STANDARD RIEMANNIAN SPACE FORMS AND THEIR $\Box$-BICONSERVATIVE HYPERSURFACES

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Abstract. According to a variational problem, the tensor of stress-energy, as specified by Hilbert (1924), is a bicovariant symmetric tensor with null divergence. This property is named the conservativeness of stress-energy tensor. In this literature, the stress-energy tensor associated to the bi-energy function with null divergence is said to be biconservative. In differential geometric point of view, a hypersurface $\xi : M^n \to \mathbb{M}^{n+1}(c)$ of a Riemannian space form is called biconservative if $\Delta^2\xi$ has null tangential component, where Δ is the Laplace operator on M^n. It is proved that such a hypersurface has constant mean curvature. We consider the hypersurfaces satisfying a progressive version of biconservativity condition. The $\Box$-biconservativity condition is obtained by substituting the Cheng-Yau operator $\Box$ instead of Δ. We prove that $\Box$-biconservative hypersurfaces of Riemannian $(n+1)$-space forms (with some additional conditions) have constant scalar curvature.

Keywords: Cheng-Yau operator , $\Box$-biconservative, scalar curvature.

MSC (2020): 53C40, 53C42, 58G25.

1. Introduction

The subject of biconservative submanifolds is an interesting research topic in physics and mathematics, which has been started by Eells and Sampson ([6]) and followed by some other researchers (see for instance [10, 15]). From the physical points of view, this subject deals with the bienergy functional and its critical points arisen form the tension field. In geometric context, the subject of biconservative submanifolds has received much attentions. In 2015, Turgay studied the H-hypersurfaces of Euclidean spaces with at most three distinct principal curvatures ([16]). Also, the biconservative surfaces in spherical and hyperbolical cylinders have been studied in [8]. Recently, some researchers have studied biconservative hypersurfaces of Riemannian space forms of dimension 4 or 5 ([9, 17]). In this manuscript we study the $(n+1)$-dimensional Riemannian space forms and their hypersurfaces satisfying the $\Box$-biconservativeness condition. The $\Box$-biconservativity condition is obtained by substituting the Cheng-Yau operator $\Box$ instead of Δ. The operator $\Box$ denotes the linearized operator

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arisen from the first variation of the second mean curvature vector field ([5]). In general, The operator L_k denotes the linearized operator arisen from the first variation of the $(k+1)$th mean curvature vector field. In fact, $L_0 = \Delta$ and $L_1 = \Box$ are special cases ([1, 2, 3, 4, 11, 12, 14]).

By definition, a hypersurface in a Riemannian space form is called $\Box$-biconservative if its first and second mean curvatures H_1 and H_2 satisfy the equation

$$\mathcal{N}_2(\nabla H_2) - c\mathcal{N}_1(\nabla H_1) = \frac{3}{4}n(n-1)H_2\nabla H_2, \tag{1.1}$$

where $\mathcal{N}_1$ and $\mathcal{N}_2$ are the first and second Newton transformations. We show that this condition gives the constancy of scalar curvature in some cases.

2. Prerequisites

Some preliminary concepts from [1, 2, 12, 13, 19] are the opening discussions. The main study will be done on the hypersurfaces of standard Riemannian space forms of dimension $(n+1)$ and curvature c denoted by $\mathbb{M}^{n+1}(c)$ for $c = 0, \pm 1$. Clearly, $\mathbb{M}^{n+1}(c)$ is $\mathbb{E}^{n+1}$ when $c = 0$, $\mathbb{S}^{n+1}$ when $c = 1$ and $\mathbb{H}^{n+1}$ when $c = -1$. The sphere of radius $\rho > 0$ is defined by

$$\mathbb{S}^{n+1}(\rho) = \{z \in \mathbb{E}^{n+2} | \langle z, z\rangle = \rho^2\}.$$

For convenience, we put $\mathbb{S}^{n+1} := \mathbb{S}^{n+1}(1)$. Similarly, $\mathbb{M}^{n+1}(-1) = \mathbb{H}^{n+1}$ is the hyperbolic space of dimension $(n+1)$ and radius (-1). In general, hyperbolic m-space of radius $(-\rho)$, as a hyperquadric in the Lorentz-Minkowski space $\mathbb{L}^{m+1} = \mathbb{E}_1^{m+1}$, is defined by $\mathbb{H}^m(-\rho) = \{z \in \mathbb{L}^{m+1} | \langle z, z\rangle = -\rho^2, z_1 > 0\}$.

From now on, we consider a Riemannian hypersurface defined by an isometric immersion ξ from M^n into $\mathbb{M}^{n+1}(c)$ (for $c = \pm 1, 0$). The main tool of our work is the shape operator S of M^n associated to a chosen (local) orthonormal tangent basis $\mathcal{P} = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ and a unit normal vector field $\mathbf{n}$ on M^n. The eigenvalues of S are denoted by the real functions $\kappa_1, \ldots, \kappa_n$ on M^n. Using them, the jth mean curvature is defined by $\binom{n}{j}H_j = s_j$, where

$$s_j := \sum_{1\le i_1<\ldots<i_j\le n} \kappa_{i_1}\ldots\kappa_{i_j}$$

is the jth elementary symmetric function (for instance, see [1] and [2]).

The hypersurface M^n is named *j-minimal* if $H_{j+1} \equiv 0$. In special cases, H_1 is the ordinary mean curvature and $H_2 := (n-1)n(1-R)$, where R is the normalized scalar curvature function.

The Newton maps on M^n are defined as

$$\mathcal{N}_0 = I,\ \mathcal{N}_1 = -s_1 I + S,\ \mathcal{N}_2 = s_2 I - s_1 S + S^2,$$

where I denotes the identity map on the tangent bundle of M.

The Cheng-Yau operator $\Box : \mathcal{C}^\infty(M^n) \to \mathcal{C}^\infty(M^n)$ is defined by rule

$$\Box(f) = trace(\mathcal{N}_1 \circ \nabla^2 f),$$

where $\langle \nabla^2 f(\mathrm{V}), \mathrm{W}\rangle = Hess^f(\mathrm{V},\mathrm{W})$ for $f \in \mathcal{C}^\infty(M^n)$ and $\mathrm{V},\mathrm{W} \in \chi(M^n)$. In other words, $\Box(f)$ is given by $\Box(f) = \sum_{i=1}^n \mu_{i,1}(e_ie_if - \nabla_{e_i}e_if)$. So, we get

$$\Box\mathbf{x} = \alpha_n[H_2\mathbf{n} - cH_1\mathbf{x}],$$

and

$$\begin{aligned}\Box^2\mathbf{x} &= 2\alpha_n[\mathcal{N}_2(\nabla H_2) - c\mathcal{N}_1(\nabla H_1)] - \frac{3}{2}\alpha_n^2 H_2\nabla H_2\\ &+ \left[\alpha_n\Box(H_2) - \alpha_n(tr(S^2\circ\mathcal{N}_1) - \alpha_n H_1)H_2\right]\mathbf{n}\\ &+ c\left[\alpha_n\Box(H_1) - \alpha_n^2H_2^2 + \alpha_n^2 cH_1^2\right]\mathbf{x},\end{aligned} \tag{2.2}$$

where $\alpha_n = n(n-1)$. By definition, M^n is called $\Box$-*biconservative* if $\mathbf{x}$ satisfies $(\Box^2\mathbf{x})^\top = 0$ (i.e the condition (1.1)).

According to (local) orthonormal tangent basis $\{\mathbf{w}_m\}_{m=1}^{n+1}$ and its co-frame $\{\omega_m\}_{1\le m\le n+1}$ on $\mathbb{M}^{n+1}(c)$, where $\mathbf{w}_{n+1}$ is positively normal to M^n, by a Cartan Lemma, we have

$$\omega_{n+1,i} = \sum_{j=1}^n h_{ij}\omega_j, \tag{2.3}$$

where $h_{ij} = h_{ji}$ for $i, j = 1, \ldots, n$. Therefore, the structure equations of $\mathbb{M}^{n+1}(c)$ (as may be seen in [19]) are

$$d\omega_m = \sum_{k=1}^{n+1}\omega_{mk}\wedge\omega_k,\ \ \omega_{ij}+\omega_{ji} = 0,$$

$$d\omega_{ij} = \sum_{k=1}^{n+1}\omega_{ik}\wedge\omega_{kj} - \frac{1}{2}\sum_{k,l=1}^n R_{ijkl}\omega_k\wedge\omega_l.$$

Of course, we have $\omega_{n+1} = 0$ and then $d\omega_{n+1} = \sum\limits_{k=1}^n \omega_{n+1,k}\wedge\omega_k = 0$ on M. Also, we have the Gauss equation $R_{ijkl} = (h_{ik}h_{jl} - h_{il}h_{jk})$, where R_{ijkl} stand for the components of the tensor of Riemannian curvature on M^n. Finally, we have

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{2.4}$$

where h_{ijk} is the covariant derivative of h_{ij}. Thus, by exterior differentiation of (2.3), we obtain the Codazzi equation $h_{ijk} = h_{ikj}$. One can choose $w_1, \ldots, w_n$ such that $h_{ij} = \kappa_i\delta_{ij}$. On the other hand, the Levi-Civita connection of M^n satisfies $\nabla_{w_i}w_j = \sum_k \omega_{jk}(w_i)w_k$, and then $w_i(\kappa_j) = \omega_{ij}(w_j)(\kappa_i - \kappa_j)$ and $\omega_{ij}(w_l)(\kappa_i-\kappa_j) = \omega_{il}(w_j)(\kappa_i - \kappa_l)$ whenever i, j, l are distinct.

3. Examples

We see several samples of □-biconservative hypersurfaces in $\mathbb{S}^{n+1}$ and $\mathbb{H}^{n+1}$ with constant first and second mean curvatures (see [2, 14]).

Example 3.1. Suppose that $0 < \varrho < 1$ and $\Sigma_0 = \mathbb{S}^n(\varrho) \subset \mathbb{S}^{n+1}$ defined as

$$\Sigma_0 = \{\mathbf{y} = (y_1, \ldots, y_{n+2}) \in \mathbb{S}^{n+1} | y_1^2 + \ldots + y_{n+1}^2 = \varrho^2, y_{n+2} = \sqrt{1-\varrho^2}\}.$$

Its Gauss map is $\mathbf{n}(\mathbf{y}) = \frac{-\sqrt{1-\varrho^2}}{\varrho}(y_1, \dots, y_{n+1}, 0) + \frac{\varrho}{\sqrt{1-\varrho^2}}(0, \dots, 0, y_{n+2})$. It has a constant principal curvature $\kappa = \frac{\sqrt{1-\varrho^2}}{\varrho}$ of multiplicity n. One can see that Σ_0 is $\Box$-biconservative and its mean and scalar curvatures are constant.

Example 3.2. Let $1 \leq k \leq n-1$, $0 < \varrho < 1$ and $\Sigma_1 = \mathbb{S}^k(\varrho) \times \mathbb{S}^{n-k}(\sqrt{1-\varrho^2}) \subset \mathbb{S}^{n+1}$ defined as

$$\Sigma_1 = \{\mathbf{y} = (y_1, \dots, y_{n+2}) \in \mathbb{S}^{n+1} | y_1^2 + \dots + y_{k+1}^2 = \varrho^2, y_{k+2}^2 + \dots + y_{n+2}^2 = 1 - \varrho^2\},$$

whose Gauss map is $\mathbf{n}(\mathbf{y}) = \frac{-\sqrt{1-\varrho^2}}{\varrho}(y_1, \dots, y_{k+1}, 0, \dots, 0) + \frac{\varrho}{\sqrt{1-\varrho^2}}(0, \dots, 0, y_{k+2}, \dots, y_{n+2})$. Its principal curvatures are $\kappa_1 = \frac{\sqrt{1-\varrho^2}}{\varrho}$ of multiplicity k and $\kappa_2 = \frac{-\varrho}{\sqrt{1-\varrho^2}}$ of multiplicity $n-k$. One can see that Σ_1 is $\Box$-biconservative and its mean and scalar curvatures are constant.

Example 3.3. Let $1 \leq k \leq n-1$, $\varrho > 0$ and $\Sigma_2 = \mathbb{H}^k(-\sqrt{\varrho^2+1}) \times \mathbb{S}^{n-k}(\varrho) \subset \mathbb{H}^{n+1}$ defined as

$$\Sigma_2 = \{\mathbf{y} = (y_1, \dots, y_{n+2}) \in \mathbb{H}^{n+1} | -y_1^2 + y_2^2 + \dots + y_{k+1}^2 = -1 - \varrho^2, y_{k+2}^2 + \dots + y_{n+2}^2 = \varrho^2\},$$

with the Gauss map $\mathbf{n}(\mathbf{y}) = \frac{\varrho}{\sqrt{1+\varrho^2}}(y_1, \dots, y_{k+1}, 0, \dots, 0) + \frac{\sqrt{1+\varrho^2}}{\varrho}(0, \dots, 0, y_{k+2}, \dots, y_{n+2})$ and two distinct constant principal curvatures $\kappa_1 = \frac{-\varrho}{\sqrt{1+\varrho^2}}$ of multiplicity k and $\kappa_2 = \frac{-\sqrt{1+\varrho^2}}{\varrho}$ of multiplicity $n-k$. So, Σ_2 is $\Box$-biconservative.

Example 3.4. Assume that $\varrho > 0$ and $\Sigma_3 = \mathbb{S}^n(\varrho) \subset \mathbb{H}^{n+1}$ is defined by

$$\Sigma_3 = \{\mathbf{y} = (y_1, \dots, y_{n+2}) \in \mathbb{H}^{n+1} | y_1 = \sqrt{1+\varrho^2}, y_2^2 + \dots + y_{n+2}^2 = \varrho^2\}.$$

Its Gauss map is $\mathbf{n}(y) = \frac{\varrho}{\sqrt{1+\varrho^2}}(y_1, 0, \dots, 0) + \frac{\sqrt{1+\varrho^2}}{\varrho}(0, y_2, \dots, y_{n+2})$ and it has one constant principal curvature $\kappa = \frac{-\sqrt{1+\varrho^2}}{\varrho}$ of multiplicity n. So, Σ_3 is $\Box$-biconservative.

Example 3.5. Suppose that Σ_4 is the product $\mathbb{H}^n(-\sqrt{\varrho^2+1}) \subset \mathbb{H}^{n+1}$, where $\varrho \geq 0$. It can be represented as

$$\Sigma_4 = \{\mathbf{y} = (y_1, \dots, y_{n+2}) \in \mathbb{H}^{n+1} | -y_1^2 + y_2^2 + \dots + y_{n+1}^2 = -1 - \varrho^2, y_{n+2} = \varrho\}$$

with the Gauss map $\mathbf{n}(\mathbf{y}) = \frac{\varrho}{\sqrt{1+\varrho^2}}(y_1, \dots, y_{n+1}, 0) + \frac{\sqrt{1+\varrho^2}}{\varrho}(0, \dots, 0, y_{n+2})$, only one constant principal curvature $\kappa = \frac{-\varrho}{\sqrt{1+\varrho^2}}$ of multiplicity n. Hence, Σ_4 is $\Box$-biconservative.

Example 3.6. For each unit constant vector (point) $\mathbf{q} \in \mathbb{E}^{n+2}$ and each real number $-1 < \eta < 1$, we consider the totally umbilical hypersurface

$$\Pi_{\mathbf{q},\eta} := \{\mathbf{z} \in \mathbb{S}^{n+1} : \langle \mathbf{z}, \mathbf{q} \rangle = \eta\} = \mathbb{S}^n(\sqrt{1-\eta^2}).$$

Its Gauss map is $\mathbf{n}(\mathbf{z}) = \frac{1}{\sqrt{1-\eta^2}}(\mathbf{q} - \eta \mathbf{z})$ and its shape operator is $\mathrm{S} = \frac{\eta}{\sqrt{1-\eta^2}} I$. In particular, its first two mean curvatures are $H_1 = \frac{\eta}{\sqrt{1-\eta^2}}$, $H_2 = \frac{\eta}{1-\eta^2}$. So, Σ_η is $\Box$-biconservative.

Example 3.7. Let $\mathbf{w} \in \mathbb{L}^{n+2}$ be a nonzero constant vector such that $\langle \mathbf{w}, \mathbf{w} \rangle \in \{0, \pm 1\}$. The subset

$$\Upsilon_\sigma := \{\mathbf{q} \in \mathbb{H}^{n+1} : \langle \mathbf{q}, \mathbf{w} \rangle = \sigma\}$$

is a totally umbilical hypersurface of $\mathbb{H}^{n+1}$ if $\sigma^2 + \langle \mathbf{w}, \mathbf{w} \rangle > 0$. In fact

$$\Upsilon_\sigma = \begin{cases} \mathbb{S}^n(\sqrt{\sigma^2 - 1}) \subset \mathbb{E}^{n+2} & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = -1, |\sigma| > 1) \\ \mathbb{E}^n & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = 0, \sigma \neq 0) \\ \mathbb{H}^n(-\sqrt{1 + \sigma^2}) & (\text{if } \langle \mathbf{w}, \mathbf{w} \rangle = 1). \end{cases}$$

Its first and second mean curvatures are constant given by $H_1 = \frac{-\sigma}{\sqrt{\sigma^2 + \langle \mathbf{w}, \mathbf{w} \rangle}}$ and $H_2 = \frac{\sigma^2}{\sigma^2 + \langle \mathbf{w}, \mathbf{w} \rangle}$. So, Υ_σ is □-biconservative.

4. Main results

We study the hypersurfaces of standard Riemannian space form $\mathbb{M}^{n+1}(c)$ for $c = 0, \pm 1$ that satisfies the condition (1.1). Some similar studies may be found for the ordinary biconservativeness condition in [7, 16, 18]. Let $\xi : M^n \to \mathbb{M}^{n+1}(c)$ be a □-biconservative hypersurface which has at most two distinct principal curvatures. We show that such a hypersurface in $\mathbb{M}^{n+1}(c)$ under some additional conditions has a constant scalar curvature. First, we see a lemma which has a similar proof to the Lemma A in [15].

Lemma 4.1. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ is a hypersurface with principal curvatures of constant multiplicities. Then, the distribution generated by principal directions is completely integrable. Also, each principal curvature of multiplicity greater than 1 is fixed on each integral submanifold of its distribution.

Theorem 4.1. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ be a □-biconservative hypersurfaces having exactly one principal curvature of multiplicity n. Then, its scalar curvature is constant.

Proof. We define $\mathcal{U} := \{p \in M^n : \nabla H_2^2(p) \neq 0\}$ and show that it is empty. We assume that the set $\mathcal{U}$ is non-empty and then we derive a contradiction. In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = 1, \ldots, n$. Also, we have

$$\mu_{i;2} = \frac{1}{2}(n-2)(n-1)\lambda^2, \; H_2 = \lambda^2. \tag{4.5}$$

By condition (1.1) and constancy of H_1, we have $\mathcal{N}_2(\nabla H_2) = \frac{3}{4}n(n-1)H_2 \nabla H_2$, which, by expressing the vector field ∇H_2 in the orthonormal basis $\nabla H_2 = \sum_{i=1}^n \langle \nabla H_2, \mathbf{w}_i \rangle \mathbf{w}_i$, gives $\langle \nabla H_2, \mathbf{w}_i \rangle (\mu_{i;2} - \frac{3}{4}n(n-1)H_2) = 0$ for every i.

If $\langle \nabla H_2, \mathbf{w}_i \rangle \neq 0$ for an i, then we obtain $\mu_{i;2} = \frac{3}{4}n(n-1)H_2$, which using equalities (4.5), gives that $H_2 = 0$ on $\mathcal{U}$. So $\mathcal{U} = \emptyset$. This is a contradiction which gives the constancy of H_2 on M^n. □

Theorem 4.2. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ is a hypersurface of a Riemannian space form satisfying the condition (1.1). If it has constant mean curvature and its shape operator has only two eigenvalues η and λ of multiplicities 1 and $n-1$ (respectively), then its scalar curvature is constant.

Proof. We define $\mathcal{U} := \{p \in M^n : \nabla H_2^2(p) \neq 0\}$ and show that it is empty. We assume that the set $\mathcal{U}$ is non-empty and then we derive a contradiction.

In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = 1, \ldots, n$. Also, we have

In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \lambda \mathbf{w}_i$ for $i = 1, \ldots, n-1$ and $\mathrm{S}\mathbf{w}_n = \eta \mathbf{w}_n$. Also, we have

$$\begin{aligned} &\mu_{1,2} = \ldots = \mu_{n-1,2} = (n-2)\lambda\eta + \binom{n-2}{2}\lambda^2, \ \mu_{n,2} = \binom{n-1}{2}\lambda^2, \\ &nH_1 = \eta + (n-1)\lambda, \ \binom{n}{2}H_2 = \binom{n-1}{2}\lambda^2 + (n-1)\lambda\eta, \\ &\binom{n}{3}H_3 = \binom{n-1}{3}\lambda^3 + \binom{n-1}{2}\lambda^2\eta. \end{aligned} \tag{4.6}$$

By expressing the vector field ∇H_2 in the orthonormal basis $\nabla H_2 = \sum_{i=1}^n \langle \nabla H_2, \mathbf{w}_i \rangle \mathbf{w}_i$, from equality (1.1) we have

$$\langle \nabla H_2, \mathbf{w}_i \rangle (\mu_{i;2} - \frac{3(n-1)n}{4} H_2) = 0$$

for every i. On the other hand, we have $\langle \nabla H_2, \mathbf{w}_i \rangle \neq 0$ on $\mathcal{V}$ for some i. Therefore, we have

$$\mu_{i,2} = \frac{3}{4} n(n-1) H_2, \tag{4.7}$$

for some i. Two different situations can arise.

Situation 1. $\langle \nabla H_2, \mathbf{w}_i \rangle \neq 0$, for at least one $i \leq n-1$. Equality (4.7) (applying (4.6)) gives

$$(n-9)(n-2)\lambda^2 = 4(n+l)\lambda\eta.$$

If $\lambda \neq 0$, then we have $\eta = -\frac{(n+3)(n-2)}{2(n+1)}\lambda$, which gives $\lambda = \frac{2(n+1)n}{n^2-n+4} H_1$ and $H_2 = -\frac{8n(n-2)(n+1)}{(n^2-n+4)^2} H_1^2$. If $\lambda = 0$ then $H_2 = 0$. In each case, the constancy of H_2 is guaranteed.

Situation 2. $\langle \nabla H_2, \mathbf{w}_n \rangle \neq 0$ and $\langle \nabla H_2, \mathbf{w}_i \rangle = 0$, for each $i \leq n-1$. The equality (4.7) (using identities (4.6)) gives $\lambda = 0$ or $\eta = \frac{2-n}{6}\lambda$.

Assuming $\lambda \neq 0$, we get $\lambda = \frac{6n}{5n-4} H_1$ and $H_2 = \frac{24n(n-2)}{(5n-4)^2} H_1^2$. If $\lambda = 0$ then we get $H_2 = 0$. Again, in each case, the constancy of H_2 is guaranteed.

Therefore, we get $\mathcal{U} = \emptyset$, which is a contradiction. So H_2 is constant on M^n and then the scalar curvature of M^n is constant. □

Theorem 4.3. Suppose that $\xi : M^n \to \mathbb{M}^{n+1}(c)$ is C-biconservative hypersurface of Riemannian space form and suppose that η and λ are its only principal curvatures of multiplicities ℓ and $n - \ell$ (respectively), where $2 \leq \ell \leq n-2$. Then, it has constant scalar curvature if its mean curvature is constant.

Proof. We define $\mathcal{U} := \{p \in M^n : \nabla H_2^2(p) \neq 0\}$ and show that it is empty. We assume that the set $\mathcal{U}$ is non-empty and then we derive a contradiction. In the orthonormal tangent frame field on $\mathcal{U}$ containing principal directions, the shape operator satisfies $\mathrm{S}\mathbf{w}_i = \eta \mathbf{w}_i$ for

$i=1,\dots,\ell$ and $\mathrm{S}\mathbf{w}_i=\lambda\mathbf{w}_i$ for $i=\ell+1,\dots,n$. Also, we have

$$\begin{aligned} &nH_1=\ell\lambda+(n-\ell)\eta,\ n(n-1)H_2=(\ell-1)\ell\lambda^2+(n-\ell)(n-\ell-1)\eta^2+2(n-\ell)\ell\lambda\eta,\\ &\mu_{i;2}=\sum_{s=0}^{2}\binom{\ell-1}{s}\binom{n-\ell}{2-s}\lambda^{2-s}\eta^s,\ (i=1,\dots,\ell),\\ &\mu_{i;2}=\sum_{s=0}^{2}\binom{\ell}{s}\binom{n-\ell-1}{2-s}\lambda^{2-s}\eta^s,\ (i=\ell+1,\dots,n).\end{aligned} \tag{4.8}$$

By expressing the vector field ∇H_2 in the orthonormal basis $\nabla H_2=\sum_{i=1}^{n}\langle\nabla H_2,\mathbf{w}_i\rangle\mathbf{w}_i$, from equality (1.1) we obtain

$$\langle\nabla H_2,\mathbf{w}_i\rangle(\mu_{i,2}-\frac{3}{4}n(n-1)H_2)=0,$$

for $i=1,\dots,n$. On the other hand, we have $\langle\nabla H_2,\mathbf{w}_i\rangle\neq 0$ on $\mathcal{U}$ for at least one i and so

$$\mu_{i,2}=\frac{3}{4}n(n-1)H_2. \tag{4.9}$$

Two different situations can arise.

Situation 1. $\langle\nabla H_2,\mathbf{w}_i\rangle\neq 0$, for at least one $i\in\{1,\dots,\ell\}$. Equality (4.9) (using (4.8)) gives $(\ell-1)(\ell+4)\eta^2+(n-\ell-1)(n-\ell)\lambda^2+2(n-\ell)(\ell+2)\eta\lambda=0$. Hence, $\lambda=d_0\eta$, where

$$d_0=-\left(\frac{n-\ell+2}{n-\ell-1}\pm\frac{\sqrt{n(\ell+3)(n-\ell)+(n-\ell)(5n-5\ell-4)}}{(n-\ell-1)(n-\ell)}\right).$$

Hence, we get $\eta=\frac{n}{n-(n-\ell)(1-d_0)}H_1$ and $\lambda=\frac{nd_0}{n-(n-\ell)(1-d_0)}H_1$, which give $H_2=d_1H_1^2$ for a constant d_1. Hence, the constancy of H_2 is guaranteed.

Situation 2. Suppose that for each $i\leq\ell$ we have $\langle\nabla H_2,\mathbf{w}_i\rangle=0$ and for at least one $i\geq\ell+1$, $\langle\nabla H_2,\mathbf{w}_i\rangle\neq 0$. From equality (4.9), we get

$$[3n(n-1-2\ell)+\ell(5+\ell)]\eta^2+2\ell(n+2-\ell)\lambda\eta-[2n(n-3-2\ell)+\ell(9-\ell)]\lambda^2=0,$$

which gives $\ell\eta(6\eta+(n-2)\eta)=0$. If $\eta=0$ then $H_2=0$. Otherwise, we have $\lambda=-\frac{n-2}{6}\eta$, which gives $\eta=\frac{6n}{(6-n+\ell)n-4(n-\ell)}H_1$ and $\lambda=-\frac{n(n-2)}{(6-n+\ell)n-4(n-\ell)}H_1$ and then $H_2=d_2H_1^2$ for a constant d_2. So, in each case, the constancy of H_2 is guaranteed.

Therefore, $\mathcal{U}=\emptyset$, then H_2 and the scalar curvature are constant on M^n. □

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STUDY ON PARA-SASAKIAN MANIFOLDS ADMITTING η-EINSTEIN SOLITON

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AND MEGHLAL MALLIK

Abstract. In this paper we explore the properties of para-Sasakian manifold with the help of Schouten-van Kampen connection. We examine para-Sasakian manifold admitting η-Einstein soliton with respect to this connection. Moreover, we investigate η-Einstein soliton on para-Sasakian manifolds admitting cyclic parallel Ricci tensor with respect to Schouten-van Kampen connection. Additionally, we investigate η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi, F_1).\overline{S} = 0$, $\overline{K}(\xi, F_1).\overline{R} = 0$, for all smooth vector fields F_1, where $\overline{K}$, $\overline{S}$ and $\overline{R}$ are conharmonic curvature tensor, Ricci curvature tensor and Riemannian curvature tensor with respect to Schouten-van Kampen connection, respectively.

Keywords: Para-Sasakian manifold, Schouten-van Kampen connection, Einstein soliton, η-Einstein soliton.

MSC (2020): 53C15, 53C25.

1. Introduction

The concept of Schouten-van Kampen connection (abbreviated as SVK- connection) was introduced in the early twentieth century to study non-holomorphic manifolds [21, 26]. Later, in 2006, Bejancu [4] studied Schouten-van Kampen connection on Foliated manifolds. Recently, Biswas and Baisya [5] investigated some properties of pseudo symmetric Sasakian manifolds with respect to SVK-connection. Most recently, this connection has been introduced on para-Sasakian manifold by Sundriyal and Upreti [23]. They studied projective curvature tensor, concircular curvature tensor and Nijenhuis curvature tensor for the para-Sasakian manifold with respect to this connection. SVK-connection ($\overline{\nabla}$) for an almost contact metric manifold M of dimension n containing an almost contact metric structure (ϕ, ξ, η, g) composed of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, is defined by

$$\overline{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi, \tag{1.1}$$

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for all $X,\ Y\in\Gamma(M)$, where $\Gamma(M)$ is the set of all vector fields on M and ∇ being the Levi-Civita connection on M.

In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds was introduced by Sato and Matsumoto [20]. Later, Adati and Matsumoto investigated some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold was studied by many authors. For instance, we see [2, 15, 17, 19, 22] and their references.

The concept of Ricci flow was introduced by Hamilton [9] in the early 1980s. That the Ricci flow is an excellent tool by which the structure of a manifold can be simplified was observed by Hamilton. The differential equation of Ricci flow is

$$\frac{\partial g}{\partial t}=-2S,\quad g(0)=g_0, \tag{1.2}$$

where g is a Riemannian metric, S is Ricci curvature tensor and t is time. The solitons for Ricci flow is identified with the self similar solutions of the above differential equation, where the metrices at different times differ by a diffeomorphism of the manifold. A Ricci soliton is usually represented by a triple $\left(g,\mathcal{V}^{/},\lambda\right)$, where $\mathcal{V}^{/}$ is a vector field and λ is a constant satisfying

$$L_{\mathcal{V}^{/}}g+2S+2\lambda g=0, \tag{1.3}$$

where S is Ricci curvature tensor and $L_{\mathcal{V}^{/}}g$ denotes the Lie derivative of g along the vector field $\mathcal{V}^{/}$. A Ricci soliton is said to be shrinking, steady, expanding according as $\lambda<0,\lambda=0,\lambda>0$, respectively. The vector field $\mathcal{V}^{/}$ is called potential vector field and if it is a gradient of a continuously differentiable function, then the Ricci soliton $\left(g,\mathcal{V}^{/},\lambda\right)$ is called a gradient Ricci soliton and the associated function is said to be the potential function. Ricci soliton was further investigated by many researchers. For instance, we see [3, 14, 16, 18, 24, 25] and their references.

Catino and Mazzieri [7] in 2016 first introduced the notion of Einstein soliton as a generalization of Ricci soliton. An almost contact manifold M with structure (ϕ,ξ,η,g) is said to have an Einstein soliton $(g,\mathcal{V},\lambda)$ if

$$L_{\mathcal{V}}g+2S+(2\lambda-r)g=0, \tag{1.4}$$

holds, where r being the scalar curvature. The Einstein soliton $(g,\mathcal{V},\lambda)$ is said to be shrinking, steady, expanding according as $\lambda<0,\lambda=0,\lambda>0$, respectively. Einstein soliton creates some self-similar solutions of the Einstein flow equation given by

$$\frac{\partial g}{\partial t}=-2S+rg.$$

Again as a generalization of Einstein soliton, the η-Einstein soliton on manifold M was introduced by Blaga [6] and it is given by

$$L_{\mathcal{V}}g+2S+(2\lambda-r)g+2\beta\eta\otimes\eta=0, \tag{1.5}$$

where, β is some constant. When $\beta=0$ the notion of η-Einstein soliton simply reduces to the notion of Einstein soliton. And when $\beta\neq 0$, the data $(g,\mathcal{V},\lambda,\beta)$ is called proper η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda<0$, steady if $\lambda=0$, and expanding if $\lambda > 0$. This soliton was further studied by Mandal and Mallik [11, 12, 13] under various curvature conditions.

The conharmonic curvature tensor K [8, 10] of type (1,3) on an n-dimensional Riemannian manifold M is given by

$$\begin{aligned} K(F_1, F_2)F_3 &= R(F_1, F_2)F_3 - \frac{1}{n-2}\left[S(F_2, F_3)F_1 - S(F_1, F_3)F_2\right] \\ &\quad -\frac{1}{n-2}\left[g(F_2, F_3)QF_1 - g(F_1, F_3)QF_2\right], \end{aligned} \tag{1.6}$$

for all F_1, F_2, $F_3 \in \Gamma(M)$, where R is the Riemannian curvature tensor of type $(1,3)$ and r is the scalar curvature of M.

Definition 1.1. [13] *A para-Sasakian manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S(Y, Z) = l_1 g(Y, Z) + l_2 \eta(Y)\eta(Z),$$

for all Y, $Z \in \Gamma(M)$, where l_1, l_2 are scalars.

Definition 1.2. [13] *A para-Sasakian manifold M is called a generalized η-Einstein manifold if its Ricci tensor is of the form*

$$S(Y, Z) = k_1 g(Y, Z) + k_2 \eta(Y)\eta(Z) + k_3 g(Y, \phi Z),$$

for all Y, $Z \in \Gamma(M)$, where k_1, k_2 and k_3 are scalars.

Definition 1.3. *A para-Sasakian manifold M is said to be ξ-conharmonically flat if and only if*

$$K(X, Y)\xi = 0,$$

for all X, $Y \in \Gamma(M)$.

This paper is structured as follows:

First two sections of the paper have been kept for introduction and preliminaries. In **Section-3**, we study some properties of para-Sasakian manifold with respect to SVK-connection. In **Section-4**, we introduce η-Einstein soliton on para-Sasakian manifolds with respect to SVK-connection. In **Section-5**, we study η-Einstein soliton on para-Sasakian manifold admitting cyclic parallel Ricci tensor with respect to SVK-connection. **Section-6** deals with η-Einstein soliton on ξ-conharmonically flat para-Sasakian manifolds with respect to SVK-connection. **Section-7** deals with η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi, F_1).\overline{S} = 0$. Lastly, **Section-8** contains η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi, F_1).\overline{R} = 0$.

2. Preliminaries

An n-dimensional differentiable manifold M with structure (ϕ, ξ, η), where η is a 1-form, ξ is the structure vector field, ϕ is a $(1,1)$-tensor field satisfying [20]

$$\phi^2(F_1) = F_1 - \eta(F_1)\xi, \eta(\xi) = 1, \tag{2.7}$$

$$\phi(\xi) = 0, \eta \circ \phi = 0, \tag{2.8}$$

for all vector fields F_1 on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [27]

$$g(\phi F_1, \phi F_2) = -g(F_1, F_2) + \eta(F_1)\eta(F_2), \tag{2.9}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure (ϕ, ξ, η, g). From (2.9) one can deduce that

$$\begin{aligned} g(F_1, \phi F_2) &= -g(\phi F_1, F_2), && (2.10)\\ g(F_1, \xi) &= \eta(\xi). && (2.11) \end{aligned}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [27] if

$$g(F_1, \phi F_2) = d\eta(F_1, F_2),$$

for all vector fields F_1, F_2 on M, where

$$d\eta(F_1, F_2) = \frac{1}{2}\left\{F_1\eta(F_2) - F_2\eta(F_1) - \eta([F_1, F_2])\right\}.$$

The manifold M is called a para-Sasakian manifold if

$$(\nabla_{F_1}\varphi) F_2 = -g(F_1, F_2)\xi + \eta(F_2) F_1, \tag{2.12}$$

for any smooth vector fields F_1, F_2 on M.

In a para-Sasakian manifold the following relations also hold [27]:

$$\begin{aligned} (\nabla_{F_1}\eta) F_2 &= g(F_1, \phi F_2), \nabla_{F_1}\xi = -\phi F_1, && (2.13)\\ \eta(R(F_1, F_2) F_3) &= g(F_1, F_3)\eta(F_2) - g(F_2, F_3)\eta(F_1), && (2.14)\\ R(F_1, F_2)\xi &= \eta(F_1) F_2 - \eta(F_2) F_1, && (2.15)\\ R(\xi, F_1)F_2 &= -g(F_1, F_2)\xi + \eta(F_2) F_1, && (2.16)\\ R(F_1, \xi)F_2 &= g(F_1, F_2)\xi - \eta(F_2) F_1, && (2.17)\\ R(\xi, F_1)\xi &= F_1 - \eta(F_1)\xi, && (2.18)\\ S(F_1, \xi) &= -(n-1)\eta(F_1), && (2.19)\\ S(\xi, \xi) &= -(n-1), Q\xi = -(n-1)\xi, && (2.20)\\ S(\phi F_1, \phi F_2) &= S(F_1, F_2) + (n-1)\eta(F_1)\eta(F_2). && (2.21) \end{aligned}$$

for any smooth vector fields F_1, F_2, F_3 on M.

Example 2.1. *Let us consider* 3*-dimensional manifold*

$$M^3 = \left\{(x, y, z) \in R^3\right\},$$

where (x, y, z) *are the standard coordinates in* R^3*. We choose the linearly independent vector fields*

$$E_1 = e^x \frac{\partial}{\partial y}, E_2 = e^x\left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right), E_3 = -\frac{\partial}{\partial x}.$$

Let g *be the pseudo Riemannian metric defined by* $g(E_i, E_j) = 0$, $if\ i \neq j\ for\ i, j = 1, 2, 3,$ *and* $g(E_1, E_1) = -1, g(E_2, E_2) = -1, g(E_3, E_3) = 1.$

Let η *be the* 1*-form defined by* $\eta(X) = g(X, E_3)$ *for any* $X \in \Gamma(M^3)$. *Let* ϕ *be the* $(1,1)$ *tensor field defined by*

$$\phi E_1 = E_1, \phi E_2 = E_2, \phi E_3 = 0, \tag{2.22}$$

$$where, \quad trace(\phi) = \sum_{i=1}^{3} g(E_i, \phi E_i) = -2. \tag{2.23}$$

Let X, Y, $Z \in \Gamma(M^3)$ *be given by*

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3, \\ Y &= y_1E_1 + y_2E_2 + y_3E_3, \\ Z &= z_1E_1 + z_2E_2 + z_3E_3. \end{aligned}$$

Then, we have

$$\begin{aligned} g(X,Y) &= x_1y_1 + x_2y_2 + x_3y_3, \\ \eta(X) &= x_3, \\ g(\phi X, \phi Y) &= x_1y_1 + x_2y_2. \end{aligned}$$

Using the linearity of g *and* ϕ, $\eta(E_3) = 1, \phi^2 X = X - \eta(X) E_3$ *and* $g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y)$ *for all* X, $Y \in \Gamma(M)$. *We have*

$$\begin{aligned} [E_1, E_2] &= 0, [E_1, E_3] = -E_1, [E_2, E_3] = E_2, \\ [E_2, E_1] &= 0, [E_3, E_1] = E_1, [E_3, E_2] = -E_2. \end{aligned}$$

Let the Levi-Civita connection with respect to g *be* ∇, *then using Koszul formula we get the following*

$$\begin{pmatrix} \nabla_{E_1}E_1 & \nabla_{E_1}E_2 & \nabla_{E_1}E_3 \\ \nabla_{E_2}E_1 & \nabla_{E_2}E_2 & \nabla_{E_2}E_3 \\ \nabla_{E_3}E_1 & \nabla_{E_3}E_2 & \nabla_{E_3}E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above results we see that the structure (ϕ, ξ, η, g) *satisfies*

$$(\nabla_X \varphi) Y = -g(X,Y)\xi + \eta(Y) X,$$

for all X, $Y \in \Gamma(M^3)$, *where* $\eta(\xi) = \eta(E_3) = 1$. *Hence* $M^3(\phi, \xi, \eta, g)$ *is a para-Sasakian manifold.*

The components of Riemannian curvature tensor of M^3 *are given by*

$$\begin{pmatrix} R(E_1,E_2)E_2 & R(E_1,E_3)E_3 & R(E_1,E_2)E_3 \\ R(E_2,E_1)E_1 & R(E_2,E_3)E_3 & R(E_2,E_3)E_1 \\ R(E_3,E_1)E_1 & R(E_3,E_2)E_2 & R(E_3,E_1)E_2 \end{pmatrix} = \begin{pmatrix} -E_1 & -E_1 & 0 \\ E_2 & E_2 & 0 \\ E_3 & E_3 & 0 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 *are given by*

$$S(E_1, E_1) = S(E_3, E_3) = 0, S(E_2, E_2) = 2. \tag{2.24}$$

Therefore the scalar curvature of M^3 *is*

$$r = \sum_{i=1}^{3} S(E_i, E_i) = 2. \tag{2.25}$$

3. Schouten-van Kampen connection on para-Sasakian manifolds

In this section we get the relation between SVK-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SVK-connection. We also establish here the first Bianchi identity with respect to SVK-connection on M.

In view of (1.1), (2.13) and (2.11), we get the expression for SVK-connection in a para-Sasakian manifold M as

$$\overline{\nabla}_{F_1}F_2 = \nabla_{F_1}F_2 + g(F_1, \phi F_2)\xi + \eta(F_2)\phi F_1, \tag{3.26}$$

with torsion tensor

$$\overline{T}(F_1, F_2) = 2g(F_1, \phi F_2)\xi + \eta(F_2)\phi F_1 - \eta(F_1)\phi F_2.$$

On para-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\overline{\nabla}_{F_1}\xi = 0, \left(\overline{\nabla}_{F_1}\eta\right)F_2 = g(\phi F_1, F_2), \tag{3.27}$$

$$\left(\overline{\nabla}_{F_1}g\right)(F_2, F_3) = g(\phi F_1, F_2)\eta(F_3) + g(\phi F_2, F_3)\eta(F_1). \tag{3.28}$$

for all $F_1, F_2 \in \Gamma(M)$.

Proposition 3.1. *The SVK-connection on a para-Sasakian manifold is non metric compatible connection.*

Proposition 3.2. *The SVK-connection on a para-Sasakian manifold is non symmetric connection.*

Proposition 3.3. *The structure vector field of a para-Sasakian manifold is parallel with respect to SVK-connection.*

Let $\overline{R}$ be the Riemannian curvature tensor with respect to SVK-connection on a para-Sasakian manifold defined as

$$\overline{R}(F_1, F_2)F_3 = \overline{\nabla}_{F_1}\overline{\nabla}_{F_2}F_3 - \overline{\nabla}_{F_2}\overline{\nabla}_{F_1}F_3 - \overline{\nabla}_{[F_1,F_2]}F_3. \tag{3.29}$$

In reference of (2.12), (2.13) and (3.26) we have

$$\begin{aligned}
\overline{\nabla}_{F_1}\overline{\nabla}_{F_2}F_3 = {} & \nabla_{F_1}\nabla_{F_2}F_3 + g(\nabla_{F_1}F_2, \phi F_3)\xi - g(F_1, F_3)\eta(F_2)\xi + g(F_1, F_2)\eta(F_3)\xi \\
& + g(F_2, \phi\nabla_{F_1}F_3)\xi - g(F_2, \phi F_3)\phi F_1 + g(F_1, \phi F_3)\phi F_2 \\
& + \eta(\nabla_{F_1}F_3)\phi F_2 - g(F_1, F_2)\eta(F_3)\xi + \eta(F_2)\eta(F_3)F_1 \\
& + \eta(F_3)\phi\nabla_{F_1}F_2 + g(F_1, \phi\nabla_{F_2}F_3)\xi + g(F_1, F_2)\eta(F_3)\xi \\
& - \eta(F_1)\eta(F_2)\eta(F_3)\xi + \eta(\nabla_{F_1}F_3)\phi F_2 + g(F_2, \phi F_3)\phi F_1,
\end{aligned} \tag{3.30}$$

$$\begin{aligned}
\overline{\nabla}_{[F_1,F_2]}F_3 = {} & \nabla_{[F_1,F_2]}F_3 + g(\nabla_{F_1}F_2, \phi F_3)\xi - g(\nabla_{F_2}F_1, \phi F_3)\xi \\
& + \eta(F_3)\phi\nabla_{F_1}F_2 - \eta(F_3)\phi\nabla_{F_2}F_1.
\end{aligned} \tag{3.31}$$

Interchanging F_1 and F_2 in (3.30) and using it along with (3.30) and (3.31) in (3.29), we get

$$\begin{aligned}\overline{R}(F_1,F_2)F_3 &= R(F_1,F_2)F_3+g(F_2,F_3)\eta(F_1)\xi-g(F_1,F_3)\eta(F_2)\xi\\&+g(F_1,\phi F_3)\phi F_2-g(F_2,\phi F_3)\phi F_1\\&+\eta(F_2)\eta(F_3)F_1-\eta(F_1)\eta(F_3)F_2. \qquad (3.32)\end{aligned}$$

Writing the equation (3.32) by cyclic permutations of F_1, F_2 and F_3 and using the fact that $R(F_1,F_2)F_3+R(F_2,F_3)F_1+R(F_3,F_1)F_2=0$, we have

$$\overline{R}(F_1,F_2)F_3+\overline{R}(F_2,F_3)F_1+\overline{R}(F_3,F_1)F_2=0,$$

for all F_1, F_2, $F_3\in\Gamma(M)$.

Taking inner product of (3.32) with a vector field U and contracting over F_1 and U, we get

$$\overline{S}(F_2,F_3)=S(F_2,F_3)+(n-1)\eta(F_2)\eta(F_3)-\psi g(F_2,\phi F_3), \qquad (3.33)$$

where $\overline{S}$ denotes Ricci curvature tensor with respect to $\overline{\nabla}$ and $\psi=trace(\phi)$.

Proposition 3.4. *The SVK-connection on para-Sasakian manifold satisfies the first Bianchi identity.*

Lemma 3.1. *Let M be an n-dimensional para-Sasakian manifold admitting SVK-connection, then*

$$\begin{aligned}\overline{R}(F_1,F_2)\xi &= 0,\overline{R}(\xi,F_2)F_3=0,\overline{R}(F_1,\xi)F_3=0, &(3.34)\\ \overline{S}(F_1,\xi) &= 0=\overline{S}(\xi,F_2), &(3.35)\\ \overline{Q}F_1 &= QF_1+(n-1)\eta(F_1)\xi+\phi F_1\psi,\overline{Q}\xi=0, &(3.36)\\ \overline{r} &= r+(n-1)-\psi^2, &(3.37)\end{aligned}$$

for all F_1, F_2, $F_3\in\Gamma(M)$, where $\overline{R}$, $\overline{Q}$ and $\overline{r}$ denote Riemannian curvature tensor, Ricci operator and scalar curvature tensor with respect to $\overline{\nabla}$, respectively.

Remark 3.1. *Eigen value of Ricci operator with respect to SVK-connection corresponding to the eigen vector ξ is zero.*

4. η-Einstein soliton on para-Sasakian manifold with respect to SVK-connection

Writing equation (1.5) with respect to SVK-connection and expanding $\overline{L}_{F_4}$, we have

$$\begin{aligned}0 &= \left(\overline{L}_{F_4}g\right)(F_1,F_2)+2\overline{S}(F_1,F_2)+[2\lambda-\overline{r}]\,g(F_1,F_2)+2\beta\eta(F_1)\eta(F_2)\\&= g(\overline{\nabla}_{F_1}F_4,F_2)+g(F_1,\overline{\nabla}_{F_2}F_4)\\&\quad+2\overline{S}(F_1,F_2)+[2\lambda-\overline{r}]\,g(F_1,F_2)+2\beta\eta(F_1)\eta(F_2). \qquad (4.38)\end{aligned}$$

Using (3.33) and (3.37) in (4.38), we get

$$\begin{aligned}0 &= \left(L_{F_4}g\right)(F_1,F_2)+2S(F_1,F_2)+(2\lambda-r)g(F_1,F_2)+2\beta\eta(F_1)\eta(F_2)\\&\quad+g(F_1,\phi F_4)\eta(F_2)+g(F_2,\phi F_4)\eta(F_1)-\left[n-1-\psi^2\right]g(F_1,F_2)\\&\quad-2\psi g(F_1,\phi F_2)+2(n-1)\eta(F_1)\eta(F_2), \qquad (4.39)\end{aligned}$$

for all F_1, F_2, F_3 and $F_4 \in \Gamma(M)$.

Therefore we get the following theorem:

Theorem 4.1. *η-Einstein soliton on para-Sasakian manifold with respect to SVK-connection is invariant if and only if*

$$\begin{aligned} 0 &= g(F_1, \phi F_4)\eta(F_2) + g(F_2, \phi F_4)\eta(F_1) - 2\psi g(F_1, \phi F_2) \\ &\quad - \left[n - 1 - \psi^2\right] g(F_1, F_2) + 2(n-1)\eta(F_1)\eta(F_2), \end{aligned}$$

holds for all F_1, F_2, F_3 and $F_4 \in \Gamma(M)$, where $\psi = trace(\phi)$.

Setting $F_4 = \xi$ and using (3.27) in (4.38), we get

$$\overline{S}(F_1, F_2) = -\left[\lambda - \frac{\overline{r}}{2}\right] g(F_1, F_2) - \beta\eta(F_1)\eta(F_2). \tag{4.40}$$

Equation (4.40) gives

$$\overline{Q}F_1 = -\left[\lambda - \frac{\overline{r}}{2}\right] F_1 - \beta\eta(F_1)\xi. \tag{4.41}$$

Setting $F_2 = \xi$ in (4.40), we have

$$\overline{S}(F_1, \xi) = -\left[\lambda - \frac{\overline{r}}{2} + \beta\right] \eta(F_1). \tag{4.42}$$

Using (3.33) and (3.37) in (4.40), we obtain

$$\begin{aligned} S(F_1, F_2) &= -\left[\lambda - \frac{1}{2}\left\{r + (n-1) - \psi^2\right\}\right] g(F_1, F_2) \\ &\quad -(\beta + n - 1)\eta(F_1)\eta(F_2) + \psi g(F_1, \phi F_2). \end{aligned} \tag{4.43}$$

Hence we obtain the following theorem:

Theorem 4.2. *If an n-dimensional para-Sasakian manifold M contains η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection, then M is generalized η-Einstein manifold.*

Contracting (4.43) over F_1 and F_2, we get

$$r = \frac{2(n\lambda + \beta)}{n-2} - n + 1 + \psi^2. \tag{4.44}$$

Corollary 4.1. *If an n-dimensional para-Sasakian manifold M contains η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection, then the scalar curvature of M is given by equation (4.44).*

5. η-Einstein soliton on para-Sasakian manifold admitting cyclic parallel Ricci tensor with respect to SVK-connection

Suppose the para-Sasakian manifold M equipped with cyclic parallel Ricci tensor with respect to SVK-connection, then $\overline{S}$ satisfies

$$\left(\overline{\nabla}_{F_1}\overline{S}\right)(F_2, F_3) + \left(\overline{\nabla}_{F_2}\overline{S}\right)(F_3, F_1) + \left(\overline{\nabla}_{F_3}\overline{S}\right)(F_1, F_2) = 0, \tag{5.45}$$

for all F_1, F_2, $F_3 \in \Gamma(M)$.

Now, assuming $\overline{r}$ constant we get from (4.40) that

$$\begin{aligned}
\left(\overline{\nabla}_{F_1}\overline{S}\right)(F_2,F_3) &= -\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_1,F_2)\eta(F_3)+g(\phi F_2,F_3)\eta(F_1)\right] \\
&\quad -\beta\left[g(\phi F_1,F_2)\eta(F_3)+g(\phi F_1,F_3)\eta(F_2)\right], \qquad (5.46)\\
\left(\overline{\nabla}_{F_2}\overline{S}\right)(F_3,F_1) &= -\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_2,F_3)\eta(F_1)+g(\phi F_3,F_1)\eta(F_2)\right] \\
&\quad -\beta\left[g(\phi F_2,F_3)\eta(F_1)+g(\phi F_2,F_1)\eta(F_3)\right], \qquad (5.47)\\
\left(\overline{\nabla}_{F_3}\overline{S}\right)(F_1,F_2) &= -\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_3,F_1)\eta(F_2)+g(\phi F_1,F_2)\eta(F_3)\right] \\
&\quad -\beta\left[g(\phi F_3,F_1)\eta(F_2)+g(\phi F_3,F_2)\eta(F_1)\right]. \qquad (5.48)
\end{aligned}$$

Using (5.46), (5.47) and (5.48) in (5.45), we get

$$0=\left(\lambda-\frac{\overline{r}}{2}\right)\left[g(\phi F_1,F_2)\eta(F_3)+g(\phi F_2,F_3)\eta(F_1)+g(\phi F_3,F_1)\eta(F_2)\right]. \qquad (5.49)$$

Contracting (5.49) over F_1 and F_2, we get

$$\left(\lambda-\frac{\overline{r}}{2}\right)\psi=0. \qquad (5.50)$$

If $\psi\neq 0$, then (3.37) and (5.50) give

$$\lambda=\frac{1}{2}\left[r+(n-1)-\psi^2\right]. \qquad (5.51)$$

From (4.44) and (5.51), we get

$$\beta=-\left[r+(n-1)-\psi^2\right]. \qquad (5.52)$$

This implies the following theorem:

Theorem 5.1. *If an n-dimensional para-Sasakian manifold containing η-Einstein soliton (g,ξ,λ,β) admits cyclic parallel Ricci tensor with respect to SVK-connection, then the soliton constants are given by equations (5.51) and (5.52).*

6. η-Einstein soliton on ξ-conharmonically flat para-Sasakian manifold with respect to SVK-connection

The conharmonic curvature tensor with respect to SVK-connection is given by

$$\begin{aligned}
\overline{K}(F_1,F_2)F_3 &= \overline{R}(F_1,F_2)F_3-\frac{1}{n-2}\left[\overline{S}(F_2,F_3)F_1-\overline{S}(F_1,F_3)F_2\right] \\
&\quad -\frac{1}{n-2}\left[g(F_2,F_3)\overline{Q}F_1-g(F_1,F_3)\overline{Q}F_2\right], \qquad (6.53)
\end{aligned}$$

for all F_1, F_2, $F_3\in\Gamma(M)$.

The condition must be satisfied by $\overline{K}$ is

$$\overline{K}(F_1,F_2)\xi=0. \qquad (6.54)$$

In view of (6.53) and (6.54), we get

$$\begin{aligned}\overline{R}(F_1,F_2)\xi &= \frac{1}{n-2}\left[\overline{S}(F_2,\xi)F_1-\overline{S}(F_1,\xi)F_2\right] \\ &+\frac{1}{n-2}\left[\eta\left(F_2\right)\overline{Q}F_1-\eta\left(F_1\right)\overline{Q}F_2\right].\end{aligned} \tag{6.55}$$

In reference to (4.41), (4.42), (3.34) and (6.55), we have

$$(2\lambda-\overline{r}+\beta)\left[\eta\left(F_2\right)F_1-\eta\left(F_1\right)F_2\right]=0.$$

Since $\eta\left(F_2\right)F_1\neq\eta\left(F_1\right)F_2$, we have

$$2\lambda+\beta=\overline{r}. \tag{6.56}$$

In view of (3.37), (4.44) and (6.56), we have

$$\frac{2(\lambda+2)-n}{n-4}=\frac{\beta-2}{2}=\frac{r-\left(\psi^2-1\right)}{n-2}, \tag{6.57}$$

where $\psi=trace(\phi)$.

Hence we have the following theorem:

Theorem 6.1. *If a ξ-conharmonically flat para-Sasakian manifold of dimension n admits an η-Einstein soliton (g,ξ,λ,β) with respect to SVK-connection, then the soliton constants are given by (6.57).*

Setting $\beta=0$ in (6.57), we get

$$\lambda=\frac{1}{2}\left[r+(n-1)-\psi^2\right].$$

Corollary 6.1. *If an n-dimensional ξ-conharmonically flat para-Sasakian manifold admits an Einstein soliton (g,ξ,λ) with respect to SVK-connection, then the soliton is*

$$\begin{aligned} i)\ \textit{shrinking if } r &< -(n-1)+\psi^2, \\ ii)\ \textit{steady if } r &= -(n-1)+\psi^2, \\ iii)\ \textit{expanding if } r &> -(n-1)+\psi^2,\end{aligned}$$

where $\psi=trace(\phi)$.

7. η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi,F_1).\overline{S}=0$

Setting $F_1=\xi$ in (6.53), we get

$$\begin{aligned}\overline{K}(\xi,F_1)F_2 &= -\frac{1}{n-2}\left[\overline{S}(F_1,F_2)\xi-\overline{S}(\xi,F_2)F_1\right] \\ &-\frac{1}{n-2}\left[g(F_1,F_2)\overline{Q}\xi-\eta\left(F_2\right)\overline{Q}F_1\right].\end{aligned} \tag{7.58}$$

Using (4.40). (4.41) and (4.42) in (7.58), we get

$$\overline{K}(\xi,F_1)F_2=\left[\frac{2\lambda-\overline{r}+\beta}{n-2}\right]\left[g(F_1,F_2)\xi-\eta\left(F_2\right)F_1\right]. \tag{7.59}$$

The condition that must be satisfied by $\overline{S}$ is

$$\overline{S}(\overline{K}(\xi,F_1)F_2,F_3)+\overline{S}(F_2,\overline{K}(\xi,F_1)F_3)=0, \tag{7.60}$$

for all $F_1, F_2, F_3 \in \Gamma(M)$.

Replacing the value of $\overline{S}$ from (4.40) and using (7.59) in (7.60), we get

$$\begin{aligned} 0 &= (2\lambda - \overline{r} + \beta)^2 \left[g(F_1, F_3)\eta(F_2) + g(F_1, F_2)\eta(F_3)\right] \\ &\quad +2(2\lambda - \overline{r} + \beta)\beta\eta(F_1)\eta(F_2)\eta(F_3). \end{aligned} \tag{7.61}$$

Setting $F_1 = F_2 = \xi$ in (7.61), we have

$$(2\lambda - \overline{r} + \beta)\left[2(\lambda + \beta) - \overline{r}\right] = 0. \tag{7.62}$$

In view of (3.37) and (7.62), we have

$$2\lambda + \beta = r + (n-1) - \psi^2, \tag{7.63}$$

$$or,$$

$$2(\lambda + \beta) = r + (n-1) - \psi^2, \tag{7.64}$$

where $\psi = trace(\phi)$.

From (4.44) and (7.63), we have

$$\lambda = \frac{1}{2}\left[\frac{n-4}{n-2}\right]\left(r + n - 1 - \psi^2\right), \beta = \left[\frac{2\left(r + n - 1 - \psi^2\right)}{n-2}\right]. \tag{7.65}$$

Again, in reference to (4.44) and (7.64), we get

$$\lambda = \frac{1}{2}\left[\frac{n-3}{n-1}\right]\left(r + n - 1 - \psi^2\right), \beta = \frac{1}{n-1}\left(r + n - 1 - \psi^2\right). \tag{7.66}$$

Hence we have the following theorem:

Theorem 7.1. *Let M be an n-dimensional para-Sasakian manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection. If M satisfies*

$$\overline{K}(\xi, F_1).\overline{S} = 0,$$

then the soliton scalars are given by (7.65) or (7.66).

8. η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{K}(\xi, F_1).\overline{R} = 0$

The condition must be satisfied by $\overline{R}$ is

$$\left(\overline{K}(\xi, F_1)\overline{R}\right)(F_2, F_3)F_4 = 0, \tag{8.67}$$

for all F_1, F_2, F_3 and $F_4 \in \Gamma(M)$.

Equation (8.67) gives

$$\begin{aligned} 0 &= \overline{K}(\xi, F_1)\overline{R}(F_2, F_3)F_4 - \overline{R}(\overline{K}(\xi, F_1)F_2, F_3)F_4 \\ &\quad -\overline{R}(F_2, \overline{K}(\xi, F_1)F_3)F_4 - \overline{R}(F_2, F_3)\overline{K}(\xi, F_1)F_4. \end{aligned} \tag{8.68}$$

In reference to (3.32), (7.58) and (8.68), we have

$$\begin{aligned} 0 &= \left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right]\left[g(F_1, \overline{R}(F_2, F_3)F_4)\xi + \overline{R}(F_1, F_3)F_4)\eta(F_2)\right] \\ &\quad + \left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right]\left[\overline{R}(F_2, F_1)F_4)\eta(F_3) + \overline{R}(F_2, F_3)F_1)\eta(F_4)\right]. \end{aligned} \tag{8.69}$$

Setting $F_4 = \xi$ and using (3.34) in (8.69), we get

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] \overline{R}(F_2, F_3)F_1 = 0. \tag{8.70}$$

Taking inner product of (8.70) with a vector field W and contracting over F_2 and W, we get

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] \overline{S}(F_1, F_3) = 0. \tag{8.71}$$

By the help of (4.40) and (8.71), we obtain

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] \left[\left(\lambda - \frac{\overline{r}}{2}\right) g(F_1, F_3) + \beta\eta(F_1)\,\eta(F_3)\right] = 0. \tag{8.72}$$

Replacing F_3 by ξ in and using (3.37) in (8.72), we have

$$\left[\frac{2\lambda - \overline{r} + \beta}{n-2}\right] [2(\lambda + \beta) - \overline{r}] = 0. \tag{8.73}$$

Using (3.37) in (8.73), we have

$$\lambda + \beta = \frac{1}{2}\left[r + (n-1) - \psi^2\right], \tag{8.74}$$

or,

$$2\lambda + \beta = \left[r + (n-1) - \psi^2\right], \tag{8.75}$$

where $\psi = trace(\phi)$.

In view of (4.44) and (8.74), we have

$$\frac{2\lambda - (n-3)}{2(n-3)} = \frac{\beta - 1}{2} = \frac{r - \psi^2}{2(n-1)}. \tag{8.76}$$

Again, in reference to (4.44) and (8.75), we get

$$\frac{2(\lambda + 2) - n}{n-4} = \frac{\beta - 2}{2} = \frac{r - \left(\psi^2 - 1\right)}{n-2}. \tag{8.77}$$

Hence we have the following theorem:

Theorem 8.1. *Let M be an n-dimensional para-Sasakian manifold admitting η-Einstein soliton (g, ξ, λ, β) with respect to SVK-connection. If M satisfies*

$$\overline{K}(\xi, F_1).\overline{R} = 0,$$

then the soliton scalars are given by (8.76) or (8.77).

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ON THE ZAGREB INDICES OF THE ZERO-DIVISOR GRAPH OF $\mathbb{Z}_n$

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Abstract. In this paper, we compute the first and second Zagreb indices, first and second multiplicative Zagreb indices, first and second Zagreb coindices, first and second multiplicative Zagreb coindices of the Beck's zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$, where $n = p^k, pq, pqr$ and p, q, r are distinct primes. These indices are important in chemical graph theory and network analysis, providing information about the structural characteristics of graph.
Keywords: Zero-divisor graph, Commutative ring, Zagreb indices, Vertex degree.
MSC (2020): 05C25, 05C07, 05C30, 05A18.

1. Introduction

The zero-divisor graph is a graphical presentation of an algebraic structure and has emerged as a powerful tool for investigating its structural properties of the structure. In 1988, while studying coloring of a ring, Beck [3] took all the elements of a ring as the vertices of the graph, connecting two vertices by an edge if and only if the product of corresponding elements in the ring is zero. After modification of Beck's definition, Anderson and Livingston [1] introduced the zero-divisor graph by removing all the non zero-divisors from the vertex set, denoted by $\Gamma(R)$. Subsequently, extensive research has been conducted on zero-divisor graphs within both commutative and non-commutative context, encompassing thorough analyses of their spectral and non-spectral graph characteristics.
Graph invariants are structural properties of the graph that gives quantitative measures its characteristic. The Zagreb indices are a set of topological indices in graph theory, used to characterize molecular graphs, particularly in the field of chemical graph theory. These indices can record information of degree-based information of the molecular graphs and have been been successfully employed in modeling a host of physico-chemical properties of organic compounds; including boiling points, stability, strain energy.

The first Zagreb index M_1 was first introduced by Gutman et al. [8] in 1972 is the sum of the squares of the degrees of all vertices in the graph

$$M_1(G) = \sum_{v \in V(G)} d(v)^2.$$

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It can be also expressed as a sum over edges of G,

$$M_1(G) = \sum_{(u,v)\in E(G)} [d(u) + d(v)].$$

The second Zagreb index $M_2(G)$ is the sum of the product of the degrees of pairs of adjacent vertices in the graph.

$$M_2(G) = \sum_{(u,v)\in E(G)} d(u)d(v).$$

In [4, 9, 11], the principal characteristics of $M_1(G)$ and $M_2(G)$ are explained in detail. The first and second Zagreb indices are strongly related with the total π-electrons energy of conjugated hydrocarbons. Zagreb indices provides information about the topological features of a graph which are used in quantitative structure activity relationships (QSAR). They offers various chemical applications to predict several properties and activities of molecules. Different molecular structures can have different Zagreb indices, which can be helpful in comparing and classifying compounds based on their structural properties.

Eliasi et al. [6] and Todeschini et al. [13, 14] have introduced modified version of conventional Zagreb indices using multiplication operations, are

$$\prod\nolimits_1(G) = \prod_{v\in V(G)} d(v)^2,$$
$$\prod\nolimits_2(G) = \prod_{(u,v)\in E(G)} d(u)d(v).$$

Došlić [5] introduced these calculations involving summations over edges of complements of graph G and referred as Zagreb coindices. The first Zagreb coindex of graph G is

$$\overline{M_1}(G) = \sum_{(u,v)\notin E(G)} [d(u) + d(v)].$$

The second Zagreb coindex is defined in [5] as

$$\overline{M_2}(G) = \sum_{(u,v)\notin E(G)} [d(u)d(v)].$$

In 2013, Xu et al. [15] introduced the concept of multiplicative Zagreb coindices.

$$\begin{aligned}\overline{\prod}_1(G) &= \prod_{(u,v)\notin E(G)} [d(u) + d(v)],\\ \overline{\prod}_2(G) &= \prod_{(u,v)\notin E(G)} d(u)d(v).\end{aligned}$$

In this article, we use Beck's definition of the zero-divisor graph to compute the first Zagreb index, second Zagreb index, first multiplicative Zagreb index, second multiplicative Zagreb index, first Zagreb coindex, second Zagreb coindex, first multiplicative Zagreb coindex and second multiplicative Zagreb coindex of the zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$ for $n = p^k,\ n = pq,\ n = pqr$, where $p,\ q,\ r$ are distinct primes.

2. Preliminaries

We use the partition of vertex set of zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$ given in [10, Remark 2.2].

Remark 2.1. (Partition of zero-divisor graph) *Let $\mathbb{Z}_{p^k}$ be the ring of integers modulo p^k. Then we can write $\Gamma_0(\mathbb{Z}_{p^k}) = \bigcup_{i=1}^{k+1} A_i$, where A_i are disjoint subsets of $\Gamma_0(\mathbb{Z}_{p^k})$, for*

$1 \leq i \leq k+1$, where $A_1,\ A_2, \cdots, A_{k+1}$ for $1 \leq i \leq k+1$, which are defined as follows

$$\begin{aligned}
A_1 &= (p^{k-1})\backslash\{0\},\\
A_2 &= (p^{k-2})\backslash\{\ A_1 \cup \{0\}\},\\
A_3 &= (p^{k-3})\backslash\{\ A_1 \cup A_2 \cup \{0\}\},\\
&\vdots\\
A_{k-1} &= (p)\backslash\{\{\textstyle\bigcup_{i=1}^{k-2} A_i\} \cup \{0\}\},\\
A_k &= \{unit\ vertices\},\\
A_{k+1} &= \{0\}
\end{aligned}$$

Also, we can write $A_i = \{k_i p^{k-i} : k_i = 1, 2, \ldots, (p^i - 1);\ p \nmid k_i\}$, for $1 \leq i \leq k-1$. Thus for any $1 \leq i \leq k$, $|A_i| = p^i - p^{(i-1)} = \phi(p^i)$ and $|A_{k+1}| = 1$.

Lemma 2.1. [10] *Let A_i be subsets of $V(\Gamma_0(\mathbb{Z}_{p^k}))$ which are defined in the above Remark 2.1, for $1 \leq i \leq k$; and let s, t be two integers with $1 \leq s \leq t \leq k$, then $\sum_{i=s}^{t} |A_i| = p^t - p^{s-1}$.*

The following lemma gives the adjacency among the vertices in the partitioning sets $A_i,\ A_j$.

Lemma 2.2. [10] *Let p be a prime, $\Gamma_0(\mathbb{Z}_{p^k})$ be zero-divisor graph of $\mathbb{Z}_{p^k}$ and $x \in A_i$, $y \in A_j$, for $i, j \in \{1, 2, \cdots, k+1\}$. Then $d(x, y) = \begin{cases} 1, & \text{if } x \neq y, \quad i+j \leq k; \\ 2, & \text{if } x \neq y, \quad i+j > k. \end{cases}$*

First we state the results given in Mahale et al. [10], which we use frequently.

Lemma 2.3. [10] *Let A_i be subsets of $V(\Gamma_0(\mathbb{Z}_{p^k}))$, for $1 \leq i \leq k+1$, as in given Remark 2.1. Then the following assertions hold.*

(1) *A vertex in A_{k+1} is adjacent to every vertex in $\bigcup_{i=1}^{k} A_i$.*
(2) *Each vertex of A_i is adjacent to every vertex in $\bigcup_{j=1}^{k-i} A_j$.*
(3) *If k is odd, then any two vertices in $\bigcup_{i=1}^{\frac{k-1}{2}} A_i$ are adjacent; and any two vertices in $\bigcup_{i=\frac{k+1}{2}}^{k-1} A_i$ are non adjacent.*
(4) *If k is even, then any two vertices in $\bigcup_{i=1}^{\frac{k}{2}} A_i$ are adjacent; and any two vertices in $\bigcup_{i=\frac{k+2}{2}}^{k-1} A_i$ are non adjacent.*

Next result gives the partition of $\Gamma_0(\mathbb{Z}_{pq})$.

Lemma 2.4. [10] *Let p and q be distinct primes. Then $\Gamma_0(\mathbb{Z}_{pq}) = \dot{\bigcup}_{i=1}^{4} A_i$, where*

$$\begin{aligned}
A_1 &= \{pk_1\colon k_1 = 1, 2, \cdots, (q-1),\ p \nmid k_1\},\\
A_2 &= \{qk_2\colon k_2 = 1, 2, \cdots, (p-1),\ q \nmid k_2\},\\
A_3 &= \{unit\ verices\},\\
A_4 &= \{0\} \quad \text{are such that,}
\end{aligned}$$

(1) *Any vertex of A_4 is adjacent to every vertex of $\bigcup_{i=1}^{3} A_i$.*
(2) *A_1, A_2 and A_3 are independent sets.*
(3) *Each element of A_1 is adjacent to every element in A_2.*

The following result of Mahale et al. [10] gives the partition of vertex set of $\Gamma_0(\mathbb{Z}_{pqr})$, for the sake of completeness we provide the proof.

Lemma 2.5. [10] *Let p, q and r be distinct primes. Then* $\Gamma_0(\mathbb{Z}_{pqr}) = A_1 \dot{\cup} A_2 \dot{\cup} \cdots \dot{\cup} A_8$.

Proof. Here p, q and r are distinct primes. The vertex set of $\Gamma_0(\mathbb{Z}_{pqr})$ can be partitioned as

$$\begin{aligned}
A_1 &= \{pk_1 \colon k_1 = 1, 2, \cdots, (qr-1) \text{ with } q \nmid k_1 \text{ and } r \nmid k_1\},\\
A_2 &= \{qk_2 \colon k_2 = 1, 2, 3, \cdots, (pr-1),\ p \nmid k_2, r \nmid k_2\},\\
A_3 &= \{rk_3 \colon k_3 = 1, 2, 3, \cdots, (pq-1),\ p \nmid k_3, q \nmid k_3\},\\
A_4 &= \{qrk_4 \colon k_4 = 1, 2, 3, \cdots, (p-1)\},\\
A_5 &= \{prk_5 \colon k_5 = 1, 2, 3, \cdots, (q-1)\},\\
A_6 &= \{pqk_6 \colon k_6 = 1, 2, 3, \cdots, (r-1)\},\\
A_7 &= \{\textit{unit vertices}\}\\
A_8 &= \{0\}
\end{aligned}$$

Then the sets $A_1, A_2, \cdots, A_8$ are as desired. □

Lemma 2.6. [2] *Let G be a simple graph on n vertices and m edges. Then*

(1) $\overline{M_1}(G) = 2m(n-1) - M_1(G)$.
(2) $\overline{M_2}(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)$.

Lemma 2.7. *The number of edges of zero-divisor graph $\Gamma_0(\mathbb{Z}_{p^k})$ is*

$$E(\Gamma_0(\mathbb{Z}_{p^k})) = \begin{cases} \frac{1}{2}[kp^k + p^k - kp^{k-1} - p^{\frac{k-1}{2}}], & \textit{if } k \textit{ is odd;} \\ \frac{1}{2}[kp^k - kp^{k-1} - p^{\frac{k}{2}} + 2p^k - p^{k-1}], & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. In the view of Lemma 2.3, the number of vertices of zero-divisor graph $\Gamma_0(\mathbb{Z}_{p^k})$ is p^k and we have the degree of each vertex. Thus by Handshaking lemma[7], we get number of edges of $\Gamma_0(\mathbb{Z}_{p^k})$ as $\sum_{i=1, v\in A_i}^{k+1} d(v)|A_i| = 2|E|$.
Now we have two cases for k.
Case I) If k is odd. From (3) of Lemma 2.3, we get

$$\begin{aligned}
\sum_{\substack{i=1\\ v\in A_i}}^{k+1} d(v)|A_i| &= \sum_{i=1}^{\frac{k-1}{2}} d(v)|A_i| + \sum_{i=\frac{k+1}{2}}^{k-1} d(v)|A_i| + d(v)|A_k| + d(v)|A_{(k+1)}| \\
&= \sum_{i=1}^{\frac{k-1}{2}} (p^{k-i} - 1)(p^i - p^{i-1}) + \sum_{i=\frac{k+1}{2}}^{k-1} p^{(k-i)}(p^i - p^{i-1}) + 1.(p^k - p^{(k-1)}) \\
&\quad + (p^k - 1)
\end{aligned}$$

Therefore, $|E(\Gamma_0(\mathbb{Z}_{p^k}))| = \frac{1}{2}[kp^k + p^k - kp^{k-1} - p^{\frac{k-1}{2}}]$.
Case II) If k is even, then $|E(\Gamma_0(\mathbb{Z}_{p^k}))| = \frac{1}{2}[kp^k - kp^{k-1} - p^{\frac{k}{2}} + 2p^k - p^{k-1}]$. □

3. Zagreb indices

In this section, we compute the Zagreb indices for the zero-divisor graph $\Gamma_0(\mathbb{Z}_n)$, for $n = p^k$, $n = pq$, $n = pqr$ where p, q and r are distinct primes.

Theorem 3.1. *Let p be a prime. Then the first Zagreb index M_1 of $\Gamma_0(\mathbb{Z}_{p^k})$ is*

$$M_1(\Gamma_0(\mathbb{Z}_{p^k})) = \begin{cases} p^{2k} + p^{(2k-1)} + (k-3)p^k - kp^{(k-1)} + p^{(\frac{k-1}{2})}, & \textit{if } k \textit{ is odd;} \\ p^{2k} + p^{2k-1} + (k-2)p^k - (k+1)p^{k-1} + p^{\frac{k}{2}}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. Using the fact given in the Remark 2.1, the vertex set $V(\Gamma_0(\mathbb{Z}_{p^k}))$ is partitioned into $k+1$ sets $A_1, A_2, \ldots, A_{k+1}$ with $|A_i| = p^i - p^{i-1}$, for any $1 \le i \le k$ and $|A_{k+1}| = 1$. There are two cases for k.

Case I) If k is odd. In view of Lemma 2.3, we have following subcases.

Subcase 1) Consider the set A_i, for $1 \le i \le \frac{k-1}{2}$, the degree of each A_i is $(p^{k-i} - 1)$. Let $V_1 = \bigcup_{i=1}^{\frac{k-1}{2}} A_i$. Then

$$\begin{aligned}\sum_{v\in V_1} d(v)^2 = \sum_{i=1}^{\frac{k-1}{2}} \sum_{v\in A_i} d(v)^2 &= \sum_{i=1}^{\frac{k-1}{2}} (p^{k-i}-1)^2 |A_i| \\ &= \sum_{i=1}^{\frac{k-1}{2}} (p^{2(k-i)} - 2p^{k-i} + 1)(p^i - p^{i-1}).\end{aligned}$$

Now consider

$$\begin{aligned}\sum_{i=1}^{\frac{k-1}{2}} (p^{2(k-i)})(p^i - p^{i-1}) &= p^{2k}(1-p^{-1}) \left(\sum_{i=1}^{\frac{k-1}{2}} p^{-i} \right) \\ &= p^{2k}(1-p^{-1}) \left(\frac{p^{-1} - p^{-(\frac{k+1}{2})}}{(1-p^{-1})} \right) = p^{2k-1} - p^{\frac{3k-1}{2}}.\end{aligned}$$

Also, $2\sum_{i=1}^{\frac{k-1}{2}} p^{k-i}(p^i - p^{i-1}) = 2\sum_{i=1}^{\frac{k-1}{2}} (p^k - p^{k-1}) = (k-1)(p^k - p^{k-1})$

and $\sum_{i=1}^{\frac{k-1}{2}} (p^i - p^{i-1}) = p^{\frac{k-1}{2}} - 1$ (from Lemma 2.1).

Thus,

$$\sum_{v\in V_1} d(v)^2 = p^{2k-1} - p^{\frac{3k-1}{2}} + (k-1)(p^k - p^{k-1}) + (p^{\frac{k-1}{2}} - 1). \tag{3.1}$$

Subcase 2) Let $V_2 = \bigcup_{i=\frac{k+1}{2}}^{k-1} A_i$. Then degree of each vertex in A_i is $p^{\frac{k-1}{2}}, \ldots, p^3, p^2, p$ respectively.

$$\begin{aligned}\sum_{v\in V_2} d(v)^2 = \sum_{i=\frac{k+1}{2}}^{k-1} \sum_{v\in A_i} d(v)^2 &= \sum_{\frac{k+1}{2}}^{k-1} (p^{(k-i)})^2 |A_i| \\ &= \sum_{\frac{k+1}{2}}^{k-1} p^{(2k-2i)}(p^i - p^{(i-1)}) \\ &= p^{2k}(1-p^{-1}) \sum_{\frac{k+1}{2}}^{k-1} p^{-i} \\ &= p^{\frac{3k-1}{2}} - p^k.\end{aligned} \tag{3.2}$$

Subcase 3) If $v \in A_K$ then $d(v) = 1$ and $|A_k| = p^k - p^{k-1}$. Thus,

$$\sum d(v)^2 = \sum d(v)^2 |A_k| = p^k - p^{k-1}. \tag{3.3}$$

Subcase 4) If $v \in A_{k+1}$ then $d(v) = p^k - 1$ and $|A_{k+1}| = 1$. Thus,

$$\sum d(v)^2 = \sum d(v)^2 |A_{k+1}| = (p^k - 1)^2. \tag{3.4}$$

Hence, using Equations (3.1) to (3.4) we get

$$\sum_{i=1}^{k+1} d(v)^2 = p^{2k} + p^{(2k-1)} + (k-3)p^k - kp^{(k-1)} + p^{(\frac{k-1}{2})}.$$

Case II) If k is even. The techniques of Case (I) can be applied to obtain Case (II).

$$\sum_{i=1}^{k+1} d(v)^2 = p^{2k} + p^{2k-1} + (k-2)p^k - (k+1)p^{k-1} + p^{\frac{k}{2}}.$$

□

Theorem 3.2. *Let p be a prime. Then the second Zagreb index M_2 of $\Gamma_0(\mathbb{Z}_{p^k})$ is*

$$\begin{cases} \frac{1}{8p^2(p+1)}\Big((3k^2+2k+3)p^{(2k+3)} - (3k^2-14k+7)p^{(2k+2)} - (3k^2-6k+7)p^{(2k+1)} \\ +(3k^2-6k+3)p^{2k} - (k^2-9)p^{(k+3)} - (k^2-12k+23)p^{(k+2)} + (k^2-8k+11)p^{(k+1)} \\ -(k^2-4k+3)p^k - (4k+16)p^{\frac{3k+5}{2}} + (4k+4)p^{\frac{3k+1}{2}} + (4k-16)p^{\frac{k+3}{2}} - (4k+8)p^{\frac{3(k+1)}{2}} \\ -(4k-4)p^{\frac{k-1}{2}} - 4p^{\frac{3k-1}{2}} - 4p^{\frac{k+5}{2}} - 8p^{\frac{k+1}{2}} + 8p^2 + 8p\Big), \text{ if } k \text{ is odd;} \\ \frac{1}{8p^2(p+1)}\Big((3k^2+4k)p^{(2k+3)} - (3k^2-4k-4)p^{(2k+2)} - (3k^2+4k-4)p^{(2k+1)} + (3k^2-4k) \\ p^{2k} - (k^2-2k-20)p^{(k+3)} - (k^2-2k+28)p^{(k+2)} + (k^2-2k)p^{(k+1)} - (k^2-2k)p^k \\ -(4k+20)p^{\frac{3k+4}{2}} + (4k-4)p^{\frac{3k+2}{2}} + (4k-4)p^{\frac{k+4}{2}} - (4k-16)p^{\frac{3(k+2)}{2}} + 4p^{\frac{k+6}{2}} + 4kp^{\frac{3k}{2}} \\ -4kp^{\frac{k}{2}} + 8p^{\frac{k+2}{2}} - 8p^3 - 8p^2 + 8p\Big), \text{ if } k \text{ is even.} \end{cases}$$

Proof. In the view of Lemma 2.3. There are two cases for k.

Case I) If k is odd.

Subcase 1) As given in (3) of Lemma 2.3, the vertex set $V_1 = \bigcup_{i=1}^{\frac{k-1}{2}} A_i$ forms a complete subgraph of $\Gamma_0(\mathbb{Z}_{p^k})$ and degree of each vertex in the respective set is $(p^{(k-i)}-1)$. Thus in this case we get,

$$\begin{aligned} d_1 &= \sum_{i=1}^{\frac{k-1}{2}} d(v_i) \sum_{v_i,v_j\in V_1} d(v_j) = \sum_{i=1}^{\frac{k-1}{2}} d(v)|A_i|\left(\tfrac{1}{2}d(v)(|A_i|-1)\right) \\ &= \tfrac{1}{2}\sum_{i=1}^{\frac{k-1}{2}} (p^{(k-i)}-1)(p^i - p^{(i-1)})((p^{(k-i)}-1)(p^i - p^{(i-1)}-1)) \\ &= \tfrac{1}{2}\sum_{i=1}^{\frac{k-1}{2}} (p^k-p^i)(1-p^{(-1)})\left[(p^k-p^i)(1-p^{(-1)}) + 1 - p^{(k-i)}\right] \\ &= \tfrac{1}{2}(1-p^{(-1)})^2\sum_{i=1}^{\frac{k-1}{2}}(p^k-p^i)^2 + \tfrac{1}{2}(1-p^{(-1)})\sum_{i=1}^{\frac{k-1}{2}}(p^k-p^i) - \tfrac{1}{2}(1-p^{(-1)})\sum_{i=1}^{\frac{k-1}{2}}(p^k-p^i)p^{(k-i)} \\ &= (p-1)^2p^{(2k-2)}(\tfrac{k-1}{4}) + (p-1)p^{(k-1)}(1-p^{\frac{(k-1)}{2}}) + \frac{(p-1)(p^{(k-1)}-1)}{2(p+1)} \\ &\quad + \tfrac{1}{2}[(p-1)p^{(k-1)}(\tfrac{k-1}{2}) + (1-p^{\frac{k-1}{2}}) + p^{(2k-1)} - p^{(\frac{3k-1}{2})} - p^{(k-1)}(p-1)(\tfrac{k-1}{2})]. \end{aligned} \tag{3.5}$$

Subcase 2) In this case, we take the adjacency in the complete graph $\bigcup_{i=1}^{\frac{k-1}{2}} A_i$, from vertex set $V_1 = \bigcup_{i=1}^{\frac{k-3}{2}} A_i$ to the vertex set $V_2 = \bigcup_{j=i+1}^{\frac{k-1}{2}} A_j$. Hence,

$$
\begin{aligned}
d_2 &= \sum_{i=1}^{\frac{k-1}{2}} \sum_{v_i\in V_1, v_j\in V_2} d(v_i)d(v_j) = \sum_{i=1}^{\frac{k-3}{2}} d(v)|A_i| \left(\sum_{j=i+1}^{\frac{k-1}{2}} d(v)|A_j| \right) \\
&= \sum_{i=1}^{\frac{k-3}{2}} (p^{k-i}-1)(p^i - p^{i-1}) \left(\sum_{j=i+1}^{\frac{k-1}{2}} (p^{k-j}-1)(p^j - p^{j-1}) \right) \\
&= (1-p^{-1})^2 \sum_{i=1}^{\frac{k-3}{2}} (p^k - p^i) \left(\sum_{j=i+1}^{\frac{k-1}{2}} (p^k - p^j) \right) \\
&= (1-p^{-1})^2 \sum_{i=1}^{\frac{k-3}{2}} (p^k - p^i) \left(\frac{p^k[(k-1)-2i]}{2} - \frac{p^{\frac{k+1}{2}} - p^{i+1}}{(p-1)} \right) \\
&= (1-p^{-1})^2 \left(\frac{p^{2k}(k-1)(k-3)}{4} - \frac{p^k(k-3)(k-1)}{8} - \frac{p^{\frac{3k+1}{2}}(k-3)}{2(p-1)} \right. \\
&\quad + \frac{p^{k+1}p(p^{\frac{(k-3)}{2}}-1)}{(p-1)^2} - \frac{p^k(k-1)p(p^{\frac{k-3}{2}})}{2(p-1)} + \frac{2p-(k-1)p^{\frac{k-1}{2}} + (k-3)p^{\frac{k+1}{2}}}{2(p-1)^2} \\
&\quad \left. + \frac{p^{\frac{k+1}{2}}p(p^{\frac{k-3}{2}}-1)}{(p-1)^2} - \frac{(p^k-p^3)}{(p-1)(p^2-1)} \right).
\end{aligned} \tag{3.6}
$$

Subcase 3) By using (2) of Lemma 2.3, we get the adjacency of the sets from vertex set $V_1 = \bigcup_{i=1}^{\frac{k-1}{2}} A_i$ to the vertex set $V_4 = \bigcup_{j=\frac{k+1}{2}}^{k-i} A_j$ and degree of each vertex of the set V_4 is $p^{(k-j)}$ respectively.

$$
\begin{aligned}
d_3 &= \sum_{i=1}^{\frac{k-1}{2}} \sum_{v\in A_i} d(v)|A_i| \left(\sum_{v\in A_j} \sum_{j=\frac{k+1}{2}}^{k-i} d(v)|A_j| \right) \\
&= \sum_{i=1}^{\frac{k-1}{2}} (p^{k-i}-1)(p^i - p^{i-1}) \left(\sum_{j=\frac{k+1}{2}}^{k-i} (p^{k-j})(p^j - p^{j-1}) \right) \\
&= \sum_{i=1}^{\frac{k-1}{2}} (1-p^{-1})(p^k - p^i) \left(\sum_{j=\frac{k+1}{2}}^{k-i} (p^k - p^{k-1}) \right) \\
&= (1-p^{-1}) \sum_{i=1}^{\frac{k-1}{2}} (p^k - p^i) \left((p^k - p^{k-1})(\frac{k+1-2i}{2}) \right) \\
&= (1-p^{-1})(p^k - p^{k-1}) \sum_{i=1}^{\frac{k-1}{2}} (p^k - p^i) \left(\frac{k+1-2i}{2} \right) \\
&= \left(\frac{(p-1)^2 p^k}{2p^2} \right) \left((k+1)p^k(\frac{k-1}{2}) - \frac{p^k(k^2-1)}{4} - (k+1)\frac{p(1-p^{\frac{k-1}{2}})}{1-p} \right. \\
&\quad \left. + (\frac{2p-(k+1)p^{\frac{k+1}{2}} + (k-1)p^{\frac{k+3}{2}}}{(p-1)^2}) \right) \\
&= \left(\frac{(p-1)^2 p^{k-1}}{2} \right) \left(\frac{p^{k-1}(k^2-1)}{4} + \frac{(k+1)(1-p^{\frac{k-1}{2}})}{p-1} + \frac{2-(k+1)p^{\frac{k-1}{2}} + (k-1)p^{\frac{k+1}{2}}}{(p-1)^2} \right) \\
&= (\frac{p^{k-2}}{8}) \left[(k^2-1)(p^{k+2} - 2p^{k+1} + p^k) + 4(k+1)p^2 - 4(k-1)p - 8p^{\frac{k+3}{2}} \right]
\end{aligned} \tag{3.7}
$$

Subcase 4) By using (1) of Lemma 2.3, a vertex in A_{k+1} is adjacent to every vertex in $\bigcup_{i=1}^{k} A_i$.

$$\begin{aligned} d_4 &= \sum_{i=1}^{k}\sum_{v\in A_i} d(v_{k+1})|A_{k+1}|d(v)|A_i| \\ &= \left(\sum_{i=1}^{\frac{k-1}{2}} d(v_{k+1})d(v)|A_i|\right) + \left(\sum_{i=\frac{k+1}{2}}^{k-1} d(v_{k+1})d(v)|A_i|\right) + \Big((p^k-1)d(v)|A_k|\Big) \\ &= \left(\sum_{i=1}^{\frac{k-1}{2}}(p^k-1)(p^{k-i}-1)(p^i-p^{i-1})\right) + \left(\sum_{i=\frac{k+1}{2}}^{k-1}(p^k-1)(p^{k-i})(p^i-p^{i-1})\right) \\ &\quad +(p^k-1)(p^k-p^{k-1}) \\ &= (p^k-1)(1-p^{-1})\left(p^k(\frac{k-1}{2}) - (\frac{p-p^{\frac{k+1}{2}}}{1-p})\right) + (p^k-1)(p^k-p^{k-1})(\frac{k-1}{2}) \\ &\quad +(p^k-1)p^k(1-p^{-1}). \end{aligned} \tag{3.8}$$

Hence, using Equations (3.5) to (3.8) we get

$$\begin{aligned} M_2(\Gamma_0(R)) &= \frac{1}{8p^2(p+1)}\Big((3k^2+2k+3)p^{2k+3} - (3k^2-14k+7)p^{2k+2} - (3k^2-6k+7) \\ &\quad p^{2k+1} + (3k^2-6k+3)p^{2k} - (k^2-9)p^{k+3} - (k^2-12k+23)p^{k+2} + (k^2-8k \\ &\quad +11)p^{k+1} - (k^2-4k+3)p^k - (4k+16)p^{\frac{3k+5}{2}} + (4k+4)p^{\frac{3k+1}{2}} + (4k-16)p^{\frac{k+3}{2}} \\ &\quad -(4k+8)p^{\frac{3(k+1)}{2}} - (4k-4)p^{\frac{k-1}{2}} - 4p^{\frac{3k-1}{2}} - 4p^{\frac{k+5}{2}} - 8p^{\frac{k+1}{2}} + 8p^2 + 8p\Big). \end{aligned}$$

Case II) If k is even: The techniques of Case (I) can be applied to obtain Case (II).

$$\begin{aligned} M_2(\Gamma_0(R)) &= \frac{1}{8p^2(p+1)}\Big((3k^2+4k)p^{2k+3} - (3k^2-4k-4)p^{2k+2} - (3k^2+4k-4)p^{2k+1} \\ &\quad +(3k^2-4k)p^{2k} - (k^2-2k-20)p^{k+3} - (k^2-2k+28)p^{k+2} + (k^2-2k)p^{k+1} \\ &\quad -(k^2-2k)p^k - (4k+20)p^{\frac{3k+4}{2}} + (4k-4)p^{\frac{3k+2}{2}} + (4k-4)p^{\frac{k+4}{2}} \\ &\quad -(4k-16)p^{\frac{3(k+2)}{2}} + 4p^{\frac{k+6}{2}} + 4kp^{\frac{3k}{2}} - 4kp^{\frac{k}{2}} + 8p^{\frac{k+2}{2}} - 8p^3 - 8p^2 + 8p\Big). \end{aligned}$$

□

Theorem 3.3. *Let p be a prime. Then the first Zagreb coindex $\overline{M_1}(\Gamma_0(\mathbb{Z}_{p^k}))$ is given by*

$$\begin{cases} kp^{2k} - (k+1)p^{2k-1} - 2(k-1)p^k + 2kp^{k-1} - p^{\frac{3k-1}{2}}, & \textit{if } k \textit{ is odd;} \\ (k+1)p^{2k} - (k+2)p^{2k-1} - 2kp^k + 2(k+1)p^{k-1} - p^{\frac{3k}{2}}, & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. We have the vertices in $\Gamma_0(\mathbb{Z}_{p^k})$ is p^k and the number of edges are $\frac{1}{2}(kp^k+p^k-kp^{k-1}-p^{\frac{k-1}{2}})$, if k is odd and $\frac{1}{2}[kp^k-kp^{k-1}-p^{\frac{k}{2}}+2p^k-p^{k-1}]$, if k is even. By using (1) of Lemma 2.6, the first Zagreb coindex is

If k is odd:

$$\begin{aligned} \overline{M_1}(\Gamma_0(\mathbb{Z}_{p^k})) &= 2m(n-1) - M_1(\Gamma_0(\mathbb{Z}_{p^k})) \\ &= 2(\tfrac{1}{2}[kp^k+p^k-kp^{k-1}-p^{\frac{k-1}{2}}])(p^k-1) - [p^{(2k)}+p^{(2k-1)}+(k-3)p^k \\ &\quad -kp^{(k-1)}+p^{(\frac{k-1}{2})}] \\ \overline{M_1}(\Gamma_0(\mathbb{Z}_{p^k})) &= kp^{2k} - (k+1)p^{2k-1} - 2(k-1)p^k + 2kp^{k-1} - p^{\frac{3k-1}{2}}. \end{aligned}$$

If k is even, then $\overline{M}_1(\Gamma_0(\mathbb{Z}_{p^k})) = (k+1)p^{2k} - (k+2)p^{2k-1} - 2kp^k + 2(k+1)p^{k-1} - p^{\frac{3k}{2}}$. □

Theorem 3.4. *Let p be a prime. Then the second Zagreb coindex $\overline{M_2}(\Gamma_0(\mathbb{Z}_{p^k}))$ is given by*

$$\begin{cases} \frac{1}{8p^2(p+1)}[4(kp^{k+1}+p^{k+1}-kp^k-p^{\frac{k+1}{2}})^2(p+1)-(3k^2+2k+7)p^{2k+3}+(3k^2-14k-1) \\ p^{2k+2}+(3k^2-6k+3)p^{2k+1}-(3k^2-6k+3)p^{2k}+(k^2-4k+3)p^{k+3}+(k^2-12k+35) \\ p^{k+2}-(k^2-12k+11)p^{k+1}+(k^2-4k+3)p^k+4(k+4)p^{\frac{3k+5}{2}}-4(k+1)p^{\frac{3k+1}{2}}-4(k-3) \\ p^{\frac{k+3}{2}}+4(k+2)p^{\frac{3(k+1)}{2}}+4(k-1)p^{\frac{k-1}{2}}+4p^{\frac{3k-1}{2}}+8p^{\frac{k+1}{2}}-8p^2-8p], \text{ if } k \text{ is odd;} \\ \frac{1}{8p^2(p+1)}[4((k+2)p^{k+1}-(k+1)p^k-p^{\frac{k+1}{2}})^2(p+1)-(k^2+2k)p^{k+1}+(3k^2-4k-12) \\ p^{2k+2}+(3k^2+4k-8)p^{2k+1}-(3k^2+4k)p^{2k}+(k^2-2k+8)p^{k+3}+(k^2-2k+40)p^{k+2} \\ -(3k^2-4k-4)p^{2k+3}+(k^2-2k)p^k+4(k+5)p^{\frac{3k+4}{2}}-4(k-1)p^{\frac{3k+2}{2}}-4kp^{\frac{k+4}{2}}+12p^2 \\ +4(k-4)p^{\frac{3(k+2)}{2}}-8p^{\frac{k+6}{2}}-4kp^{\frac{3k}{2}}+4kp^{\frac{k}{2}}-8p^{\frac{k+2}{2}}+12p^3-8p], \text{ if } k \text{ is even.} \end{cases}$$

Proof. Using (2) of Lemma 2.6 we get result. ∎

Theorem 3.5. *Let p and q be distinct primes with $p<q$. Then Zagreb indices of $\Gamma_0(\mathbb{Z}_{pq})$ are given by*

(1) $M_1(\Gamma_0(\mathbb{Z}_{pq}))=(pq-1)(pq+p+q)-(p^2+q^2)+2$.
(2) $M_2(\Gamma_0(\mathbb{Z}_{pq}))=4(pq)^2-(p+q)(3pq-2)-pq-1$.
(3) $\prod_1(\Gamma_0(\mathbb{Z}_{pq}))=p^{2(q-1)}q^{2(p-1)}(pq-1)^2$.
(4) $\prod_2(\Gamma_0(\mathbb{Z}_{pq}))=p^{p(q-1)}q^{q(p-1)}(pq-1)^{(pq-1)}$.
(5) $\overline{M_1}(\Gamma_0(\mathbb{Z}_{pq}))=3(pq)^2-3(p^2q+pq^2+pq-p-q)+p^2+q^2-2$.
(6) $\overline{M_2}(\Gamma_0(\mathbb{Z}_{pq}))=\frac{1}{2}[7(pq)^2-11(p^2q+pq^2)+11pq+5(p^2+q^2)-3(p+q)]$.
(7) $\overline{\prod}_1(\Gamma_0(\mathbb{Z}_{pq}))=(p+1)^{(p-1)(q-1)^2}\times(2p)^{\frac{(q-1)(q-2)}{2}}\times(q+1)^{(p-1)^2(q-1)}\times(2q)^{\frac{(p-1)(p-2)}{2}}$
$\times 2^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}$.
(8) $\overline{\prod}_2(\Gamma_0(\mathbb{Z}_{pq}))=p^{(q-1)(pq-p-1)}q^{(p-1)(pq-q-1)}$.

Proof. The vertex set $V(\Gamma_0(\mathbb{Z}_{pq}))$ is partitioned in given 2.4 as $A_1,\ A_2,\ A_3,\ A_4$. The degree of each vertex in the set $A_1,\ldots,A_4$ is $p,\ q,\ 1,\ pq-1$ and the cardinality of the these sets are $(q-1),\ (p-1),\ (p-1)(q-1),\ 1$ respectively.
The first Zagreb index $M_1(\Gamma_0(\mathbb{Z}_{pq}))$ is

$$\begin{aligned} M_1(\Gamma_0(\mathbb{Z}_{pq})) &= \sum_{u\in A_1}(p)^2+\sum_{u\in A_2}(q)^2+\sum_{u\in A_3}(1)^2+\sum_{u\in A_4}(pq-1)^2 \\ &= (q-1)p^2+(p-1)q^2+(p-1)(q-1)+(pq-1)^2 \\ &= (pq-1)(pq+p+q)-(p^2+q^2)+2. \end{aligned}$$

The adjacency of the vertices within the respective sets is provided in the Lemma 2.4. The second Zagreb index $M_2(\Gamma_0(\mathbb{Z}_{pq}))$ is

$$\begin{aligned} M_2(\Gamma_0(\mathbb{Z}_{pq})) &= \sum_{u\in A_1,v\in A_2}pq+\sum_{u\in A_1,v\in A_4}p(pq-1)+\sum_{u\in A_2,v\in A_4}q(pq-1)+\sum_{u\in A_3,v\in A_4}(pq-1)1 \\ &= pq(q-1)(p-1)+p(pq-1)(q-1)+q(pq-1)(p-1)+(pq-1)(p-1)(q-1) \\ &= 4(pq)^2-(p+q)(3pq-2)-pq-1. \end{aligned}$$

The first multiplicative Zagreb index of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\begin{aligned} \prod\nolimits_1(\Gamma_0(\mathbb{Z}_{pq})) &= \prod_{v\in A_1}(p)^2\times\prod_{v\in A_2}(q)^2\times\prod_{v\in A_3}(1)^2\times\prod_{v\in A_4}(pq-1)^2 \\ &= (p^2)^{(q-1)}(q^2)^{(p-1)}(1^2)^{(p-1)(q-1)}[(pq-1)^2]^1 \\ &= p^{2(q-1)}q^{2(p-1)}(pq-1)^2. \end{aligned}$$

The second multiplicative Zagreb indices of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\begin{aligned}\prod\nolimits_2(\Gamma_0(\mathbb{Z}_{pq})) &= \prod_{u\in A_1,v\in A_2}(p)(q)\times\prod_{u\in A_1,v\in A_4}(p)(pq-1)\times\prod_{u\in A_2,v\in A_4}(q)(pq-1)\\ &\quad\times\prod_{u\in A_3,v\in A_4}(1)(pq-1)\\ &= (pq)^{(q-1)(p-1)}\times[p(pq-1)]^{(q-1)}\times[q(pq-1)]^{(p-1)}\times(pq-1)^{(p-1)(q-1)}\\ &= p^{p(q-1)}q^{q(p-1)}(pq-1)^{(pq-1)}.\end{aligned}$$

The number of edges of $\Gamma_0(\mathbb{Z}_{pq})$ are $2pq-p-q$. Using (1) of Lemma 2.6, the first Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\begin{aligned}\overline{M_1}(\Gamma_0(\mathbb{Z}_{pq})) &= 2m(n-1)-M_1(\Gamma_0(\mathbb{Z}_{pq}))\\ &= 2(2pq-p-q)(pq-1)-[(pq-1)(pq+p+q)-(p^2+q^2)+2]\\ &= 3(pq)^2-3(p^2q+pq^2+pq-p-q)+p^2+q^2-2.\end{aligned}$$

Using (2) of Lemma 2.6, the second Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pq})$ is

$$\overline{M_2}(\Gamma_0(\mathbb{Z}_{pq})) = \tfrac{1}{2}[7(pq)^2-11(p^2q+pq^2)+11pq+5(p^2+q^2)-3(p+q)].$$

The second multiplicative Zagreb coindex of $(\Gamma_0(\mathbb{Z}_{pq}))$is

$$\begin{aligned}\overline{\prod}_1(\Gamma_0(\mathbb{Z}_{pq})) &= \prod_{u\in A_1,v\in A_3}(p+1)\times\prod_{u,v\in A_1}(p+p)\times\prod_{u,v\in A_2}(q+q)\times\prod_{u\in A_2,v\in A_3}(q+1)\\ &\quad\times\prod_{u,v\in A_3}(1+1)\\ &= (p+1)^{(q-1)(p-1)(q-1)}\times(2p)^{\frac{(q-1)(q-2)}{2}}\times(2q)^{\frac{(p-1)(p-2)}{2}}\\ &\quad\times(q+1)^{(p-1)(p-1)(q-1)}\times 2^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\ &= (p+1)^{(p-1)(q-1)^2}\times(2p)^{\frac{(q-1)(q-2)}{2}}\times(q+1)^{(p-1)^2(q-1)}\\ &\quad\times(2q)^{\frac{(p-1)(p-2)}{2}}\times 2^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}.\end{aligned}$$

The second multiplicative Zagreb coindex of $(\Gamma_0(\mathbb{Z}_{pq}))$ is

$$\begin{aligned}\overline{\prod}_2(\Gamma_0(\mathbb{Z}_{pq})) &= p^{(q-1)(p-1)(q-1)}\times(p^2)^{\frac{(q-1)(q-2)}{2}}\times q^{(p-1)(p-1)(q-1)}\times(q^2)^{\frac{(p-1)(p-2)}{2}}\\ &\quad\times 1^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\ &= p^{(q-1)(p-1)(q-1)}\times p^{(q-1)(q-2)}\times q^{(p-1)(p-1)(q-1)}\times q^{(p-1)(p-2)}\\ &= p^{(q-1)(pq-p-1)}q^{(p-1)(pq-q-1)}.\end{aligned}$$

□

Theorem 3.6. *Let p,q and r be distinct primes with $p<q<r$. Then Zagreb indices of $\Gamma_0(R)$ where R is $\mathbb{Z}_{pqr}$ are given by*

(1) $M_1(\Gamma_0(R)) = (pqr)^2+pqr(p+q+r+pq+qr+pr-1)+(p^2+q^2+r^2)-pr(1+p+r+pr)-pq(1+p+q+pq)-qr(1+q+r+qr)+(p+q+r)$.

(2) $M_2(\Gamma_0(R)) = 13(p^2q^2r^2)-9(p^2q^2r+p^2r^2q+q^2r^2p)+6(p^2qr+q^2pr+r^2pq)-11pqr+4(pq+pr+qr)-2(p+q+r)+1$.

(3) $\prod_1(\Gamma_0(R)) = p^{(2qr-2)}q^{(2pr-2)}r^{(2pq-2)}(pqr-1)^2$.

(4) $\prod_2(\Gamma_0(R)) = p^{[3pqr-2(pq+pr)+p]}\times q^{[3pqr-2(pq+qr)+q]}\times r^{[3pqr-(pr+qr)+r]}\times(pqr-1)^{(pqr-1)}$.

(5) $\overline{M_1}(\Gamma_0(R)) = 7(pqr)^2-5(p^2q^2r+p^2r^2q+q^2r^2p)+(p^2qr+q^2pr+r^2pq)-9pqr+5(pq+pr+qr)-3(p+q+r)+(p^2q^2+p^2r^2+q^2r^2)-(p^2+q^2+r^2)+(p^2q+p^2r+q^2p+q^2r+r^2p+r^2q)+2$.

(6) $\overline{M_2}(\Gamma_0(R)) = \frac{1}{2}[5p^2q^2r^2-15(p^2q^2r+p^2r^2q+q^2r^2p)+9(p^2q^2+p^2r^2+q^2r^2)+19(p^2qr+q^2pr+r^2pq)-17pqr-5(pq+pr+qr)-(p+q+r)+7(p^2q+p^2r+q^2p+q^2r+r^2p+r^2q)+(p^2+q^2+r^2)]$.

(7) $\overline{\prod}_1(\Gamma_0(R)) = (p+q)^{(p-1)(q-1)(r-1)^2} \times (p+r)^{(p-1)(q-1)^2(r-1)} \times (q+r)^{(p-1)^2(q-1)(r-1)}$
$\times\, p^{\frac{1}{2}[(qr)^2-5qr+2q+2r]} \times q^{\frac{1}{2}[(pr)^2-5pr+2p+2r]} \times r^{\frac{1}{2}[(pq)^2-5pq+2p+2q]}$
$\times (1+p)^{(p-1)[(q-1)^2+(r-1)^2+(q-1)^2(r-1)^2]} \times (1+q)^{(q-1)[(p-1)^2+(r-1)^2+(p-1)^2(r-1)^2]}$
$\times (1+r)^{(r-1)[(p-1)^2+(q-1)^2+(p-1)^2(q-1)^2]} \times 2^{(p-1)^2(q-1)^2(r-1)^2} \times 2^{(p-1)^2(q-1)^2}$
$\times\, 2^{(p-1)^2(r-1)^2+(q-1)^2(r-1)^2-(p-1)(q-1)(r-1)}$
$\times\, 2^{(1-p)(q-p+1)+(1-r)(p-r+1)+(1-q)(r-q+1)}.$

(8) $\overline{\prod}_2(\Gamma_0(R)) = p^{[q^2r^2p+2(pq+pr)-(p+qr)-4pqr+1]} \times q^{[p^2r^2q+2(pq+qr)-(q+pr)-4pqr+1]}$
$\times\, r^{[p^2q^2r+2(pr+qr)-(r+pq)-4pqr+1]}$

Proof. The vertex set of $\Gamma_0(\mathbb{Z}_{pqr})$ can be partitioned into $\bigcup_{i=1}^{8} A_i$ as given in the proof of Lemma 2.5. The degree of each vertex in the sets $A_1,\ A_2,\ \ldots,\ A_8$ is $p,\ q,\ r,\ qr,\ pr,\ pq,\ 1,\ (pqr-1)$ respectively.

The total number of vertices in sets $A_1,\ A_2,\ A_3,\ \cdots,\ A_8$ is $(q-1)(r-1),\ (p-1)(r-1),\ (p-1)(q-1),\ (p-1),\ (q-1),\ (r-1),\ (p-1)(q-1)(r-1),\ 1$ respectively. Then we have the result.

The first Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned} M_1(\Gamma_0(R)) &= \sum_{u\in A_1}(p)^2 + \sum_{u\in A_2}(q)^2 + \sum_{u\in A_3}(r)^2 + \sum_{u\in A_4}(qr)^2 + \sum_{u\in A_5}(pr)^2 + \sum_{u\in A_6}(pq)^2 \\ &\quad + \sum_{u\in A_7}(1)^2 + \sum_{u\in A_8}(pqr-1)^2 \\ &= (q-1)(r-1)p^2 + (p-1)(r-1)q^2 + (p-1)(q-1)r^2 + (p-1)(qr)^2 \\ &\quad +(q-1)(pr)^2 + (r-1)(pq)^2 + (p-1)(q-1)(r-1) + (pqr-1)^2 \\ &= (pqr)^2 + pqr(p+q+r+pq+qr+pr-1) + (p^2+q^2+r^2) - pr(1+p+r \\ &\quad +pr) - pq(1+p+q+pq) - qr(1+q+r+qr) + (p+q+r). \end{aligned}$$

By the adjacency between the vertices as given in the Lemma 2.5, the second Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned} M_2(\Gamma_0(R)) &= \sum_{u\in A_1, v\in A_4} p(qr) + \sum_{u\in A_1, v\in A_8} p(pqr-1) + \sum_{u\in A_2, v\in A_5} q(pq) + \sum_{u\in A_2, v\in A_8} q(pqr-1) \\ &\quad + \sum_{u\in A_3, v\in A_6} r(pq) + \sum_{u\in A_3, v\in A_8} r(pqr-1) + \sum_{u\in A_4, v\in A_8} qr(pqr-1) \\ &\quad + \sum_{u\in A_4, v\in A_5} qr(pr) + \sum_{u\in A_4, v\in A_6} qr(pq) + \sum_{u\in A_5, v\in A_6} pr(pq) + \sum_{u\in A_5, v\in A_8} pr(pqr-1) \\ &\quad + \sum_{u\in A_6, v\in A_8} pq(pqr-1) + \sum_{u\in A_7, v\in A_8} 1(pqr-1) \\ &= (pqr)(q-1)(r-1)(p-1) + p(pqr-1)(q-1)(r-1) + pqr(p-1) \\ &\qquad (q-1)(r-1) + q(pqr-1)(p-1)(r-1) + pqr(p-1)(q-1)(r-1) \\ &\quad +r(pqr-1)(p-1)(q-1) + qr(pqr-1)(p-1) + (qrpr)(p-1)(q-1) \\ &\quad +(qrpq)(p-1)(r-1) + (prpq)(q-1)(r-1) + pr(pqr-1)(q-1) \\ &\quad +pq(pqr-1)(r-1) + (pqr-1)(p-1)(q-1)(r-1)1q1 \\ &= 13(p^2q^2r^2) - 9(p^2q^2r + p^2r^2q + q^2r^2p) + 6(p^2qr + q^2pr + r^2pq) - 11pqr \\ &\quad +4(pq+pr+qr) - 2(p+q+r) + 1. \end{aligned}$$

The first multiplicative Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\prod\nolimits_1(\Gamma_0(R)) &= \prod_{v\in A_1}(p)^2\times\prod_{v\in A_2}(q)^2\times\prod_{v\in A_3}(r)^2\times\prod_{v\in A_4}(qr)^2\times\prod_{v\in A_5}(pr)^2\times\prod_{v\in A_6}(pq)^2\\ &\quad\times\prod_{v\in A_7}(1)^2\times\prod_{v\in A_8}(pqr-1)^2\\ &= (p)^{2(q-1)(r-1)}\times(q)^{2(p-1)(r-1)}\times(r)^{2(p-1)(q-1)}\times(qr)^{2)(p-1)}\times(pr)^{2(q-1)}\\ &\quad\times(pq)^{2(r-1)}\times 1\times(pqr-1)^2\\ &= p^{(2qr-2)}q^{(2pr-2)}r^{(2pq-2)}(pqr-1)^2.\end{aligned}$$

The second multiplicative Zagreb index of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\prod\nolimits_2(\Gamma_0(R)) &= \prod_{u\in A_1,v\in A_4}p(qr)\times\prod_{u\in A_1,v\in A_8}p(pqr-1)\times\prod_{u\in A_2,v\in A_5}q(pr)\times\prod_{u\in A_2 v\in A_8}q(pqr-1)\\ &\quad\times\prod_{u\in A_3,v\in A_6}r(pq)\times\prod_{u\in A_3,v\in A_8}r(pqr-1)\times\prod_{u\in A_4,v\in A_5}qr(pr)\times\prod_{u\in A_4,v\in A_6}qr(pq)\\ &\quad\times\prod_{u\in A_4,v\in A_8}qr(pqr-1)\times\prod_{u\in A_5 v\in A_6}pr(pq)\times\prod_{u\in A_5,v\in A_8}pr(pqr-1)\\ &\quad\times\prod_{u\in A_6,v\in A_8}pq(pqr-1)\times\prod_{u\in A_7,v\in A_8}1(pqr-1)\\ &= (pqr)^{(q-1)(r-1)(p-1)}\times[p(pqr-1)]^{(q-1)(r-1)}\times(pqr)^{(p-1)(r-1)(q-1)}\\ &\quad\times[q(pqr-1)]^{(p-1)(r-1)}\times(pqr)^{(p-1)(q-1)(r-1)}\times[r(pqr-1)]^{(p-1)(q-1)}\\ &\quad\times(pqr^2)^{(p-1)(q-1)}\times(pq^2r)^{(p-1)(r-1)}\times[qr(pqr-1)]^{(p-1)}\times(p^2qr)^{(q-1)(r-1)}\\ &\quad\times[pr(pqr-1)]^{(q-1)}\times[pq(pqr-1)]^{(r-1)}\times(pqr-1)^{(p-1)(q-1)(r-1)}\\ &= (pqr)^{3pqr-2(pq+qr+pr)+(p+q+r)}\times p^{2qr-q-r}\times q^{2pr-p-r}\times r^{2pq-p-q}\\ &\quad\times(pqr-1)^{(pqr-1)}\\ &= p^{[3pqr-2(pq+pr)+p]}\times q^{[3pqr-2(pq+qr)+q]}\times r^{[3pqr-(pr+qr)+r]}\times(pqr-1)^{(pqr-1)}.\end{aligned}$$

The first Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\overline{M_1}(\Gamma_0(R)) &= 2m(n-1)-M_1(R)\\ &= 2[4pqr-2(pq+pr+qr)+(p+q+r)-1](pqr-1)-[(pqr)^2+pqr(p+q\\ &\quad+r+pq+qr+pr-1)-pr(1+p+r+pr)-pq(1+p+q+pq)-qr(1\\ &\quad+q+r+qr)+(p+q+r)+(p^2+q^2+r^2)]\\ &= 7(pqr)^2-5(p^2q^2r+p^2r^2q+q^2r^2p)+(p^2qr+q^2pr+r^2pq)-9pqr+5(pq\\ &\quad+pr+qr)-3(p+q+r)+(p^2q^2+p^2r^2+q^2r^2)-(p^2+q^2+r^2)+(p^2q\\ &\quad+p^2r+q^2p+q^2r+r^2p+r^2q)+2.\end{aligned}$$

By using (2) of Lemma 2.6, the second Zagreb coindex $M_2(\Gamma_0(\mathbb{Z}_{pqr}))$ is given by

$$\begin{aligned}\overline{M_2}(\Gamma_0(R)) &= \tfrac{1}{2}[5p^2q^2r^2-15(p^2q^2r+p^2r^2q+q^2r^2p)+9(p^2q^2+p^2r^2+q^2r^2)+19(p^2qr\\ &\quad+q^2pr+r^2pq)-17pqr-5(pq+pr+qr)-(p+q+r)+7(p^2q+p^2r+q^2p\\ &\quad+q^2r+r^2p+r^2q)+(p^2+q^2+r^2)].\end{aligned}$$

The first multiplicative Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}\overline{\prod}_1(\Gamma_0(R)) &= \prod_{u\in A_1,v\in A_2}(p+q)\times\prod_{u\in A_1,v\in A_3}(p+r)\times\prod_{u\in A_1,v\in A_5}(p+pr)\times\prod_{u\in A_1,v\in A_6}(p+pq)\\ &\quad\times\prod_{u\in A_1,v\in A_7}(p+1)\times\prod_{u,v\in A_1}(p+p)\times\prod_{u\in A_2,v\in A_3}(q+r)\times\prod_{u\in A_2,v\in A_4}(q+qr)\\ &\quad\times\prod_{u\in A_2,v\in A_6}(q+pq)\times\prod_{u\in A_2,v\in A_7}(q+1)\times\prod_{u,v\in A_2}(q+q)\times\prod_{u\in A_3,v\in A_4}(r+qr)\\ &\quad\times\prod_{u\in A_3,v\in A_5}(r+pr)\times\prod_{u\in A_3,v\in A_7}(r+1)\times\prod_{u,v\in A_3}(r+r)\times\prod_{u\in A_4,v\in A_7}(qr+1)\\ &\quad\times\prod_{u,v\in A_4}(qr+qr)\times\prod_{u\in A_5,v\in A_7}(pr+1)\times\prod_{u,v\in A_5}(pr+pr)\times\prod_{u\in A_6,v\in A_7}(pq+1)\end{aligned}$$

$$
\begin{aligned}
&\times \prod_{u,v\in A_6}(pq+pq)\times \prod_{u,v\in A_7}(1+1)\\
&=(p+q)^{(q-1)(r-1)(p-1)(r-1)}\times(p+r)^{(q-1)(r-1)(p-1)(q-1)}\times(p+pr)^{(q-1)(r-1)(q-1)}\\
&\times(p+pq)^{(q-1)(r-1)(r-1)}\times(p+1)^{(q-1)(r-1)(p-1)(q-1)(r-1)}\times(2p)^{\frac{(q-1)(r-1)[(q-1)(r-1)-1]}{2}}\\
&\times(q+r)^{(p-1)(r-1)(p-1)(q-1)}\times(q+qr)^{(p-1)(r-1)(p-1)}\times(q+pq)^{(p-1)(r-1)(r-1)}\\
&\times(q+1)^{(p-1)(r-1)(p-1)(q-1)(r-1)}\times(2q)^{\frac{(p-1)(r-1)[(p-1)(r-1)-1]}{2}}\times(r+qr)^{(p-1)(q-1)(p-1)}\\
&\times(r+pr)^{(p-1)(q-1)(q-1)}\times(r+1)^{(p-1)(q-1)(p-1)(q-1)(r-1)}\times(2r)^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\
&\times(qr+1)^{(p-1)(p-1)(q-1)(r-1)}\times(2qr)^{\frac{(p-1)(p-2)}{2}}\times(pr+1)^{(q-1)(p-1)(q-1)(r-1)}\\
&\times(2pr)^{\frac{(q-1)(q-2)}{2}}\times(pq+1)^{(r-1)(p-1)(q-1)(r-1)}\times(2pr)^{\frac{(r-1)(r-2)}{2}}\\
&\times 2^{\frac{(p-1)(q-1)(r-1)[(p-1)(q-1)(r-1)-1]}{2}}\\
&=(p+q)^{(p-1)(q-1)(r-1)^2}\times(p+r)^{(p-1)(q-1)^2(r-1)}\times(q+r)^{(p-1)^2(q-1)(r-1)}\\
&\times p^{\frac{1}{2}[(qr)^2-5qr+2q+2r]}\times q^{\frac{1}{2}[(pr)^2-5pr+2p+2r]}\times r^{\frac{1}{2}[(pq)^2-5pq+2p+2q]}\\
&\times(1+p)^{(p-1)[(q-1)^2+(r-1)^2+(q-1)^2(r-1)^2]}\times(1+q)^{(q-1)[(p-1)^2+(r-1)^2+(p-1)^2(r-1)^2]}\\
&\times(1+r)^{(r-1)[(p-1)^2+(q-1)^2+(p-1)^2(q-1)^2]}\times 2^{(p-1)^2(q-1)^2(r-1)^2}\times 2^{(p-1)^2(q-1)^2}\\
&\times 2^{(p-1)^2(r-1)^2+(q-1)^2(r-1)^2-(p-1)(q-1)(r-1)}\times 2^{(1-p)(q-p+1)+(1-r)(p-r+1)+(1-q)(r-q+1)}.
\end{aligned}
$$

The second multiplicative Zagreb coindex of $\Gamma_0(\mathbb{Z}_{pqr})$ is

$$
\begin{aligned}
\overline{\prod}_2(\Gamma_0(R)) &= (pq)^{(q-1)(r-1)(p-1)(r-1)}\times(pr)^{(q-1)(r-1)(p-1)(q-1)}\times(p^2r)^{(q-1)(r-1)(q-1)}\\
&\times(p^2q)^{(q-1)(r-1)(r-1)}\times(p)^{(q-1)(r-1)(p-1)(q-1)(r-1)}\times(p^2)^{\frac{(q-1)(r-1)[(q-1)(r-1)-1]}{2}}\\
&\times(qr)^{(p-1)(r-1)(p-1)(q-1)}\times(q^2r)^{(p-1)(r-1)(p-1)}\times(pq^2)^{(p-1)(r-1)(r-1)}\\
&\times(q)^{(p-1)(r-1)(p-1)(q-1)(r-1)}\times(q^2)^{\frac{(p-1)(r-1)[(p-1)(r-1)-1]}{2}}\times(qr^2)^{(p-1)(q-1)(p-1)}\\
&\times(pr^2)^{(p-1)(q-1)(q-1)}\times(r)^{(p-1)(q-1)(p-1)(q-1)(r-1)}\times(r^2)^{\frac{(p-1)(q-1)[(p-1)(q-1)-1]}{2}}\\
&\times(qr)^{(p-1)(p-1)(q-1)(r-1)}\times[(qr)^2]^{\frac{(p-1)(p-2)}{2}}\times(pr)^{(q-1)(p-1)(q-1)(r-1)}\\
&\times[(pr)^2]^{\frac{(q-1)(q-2)}{2}}\times(pq)^{(r-1)(p-1)(q-1)(r-1)}\times[(pq)^2]^{\frac{(r-1)(r-2)}{2}}\\
&\times(1)^{\frac{(p-1)(q-1)(r-1)[(p-1)(q-1)(r-1)-1]}{2}}\\
&= p^{[q^2r^2p+2(pq+pr)-(p+qr)-4pqr+1]}\times q^{[p^2r^2q+2(pq+qr)-(q+pr)-4pqr+1]}\\
&\times r^{[p^2q^2r+2(pr+qr)-(r+pq)-4pqr+1]}.
\end{aligned}
$$

□

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ON BIPOLAR FUZZY IMPLICATIVE IDEALS IN SHEFFER STROKE BG-ALGEBRAS

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Abstract. This paper presents a comprehensive investigation into the algebraic integration of bipolar fuzzy logic and Sheffer stroke BG-algebras, introducing the concept of bipolar fuzzy Sheffer stroke BG-algebras. By embedding bipolar fuzzy sets—distinguished by their dual positive and negative membership degrees—into the Sheffer stroke BG-algebraic framework, we systematically examine the structure and properties of bipolar fuzzy subalgebras and various classes of SBG-ideals. Central to our approach is the analysis of level sets associated with bipolar fuzzy subsets, through which we establish rigorous correspondences between these fuzzy constructs and classical subalgebras and ideals. Our results reveal that the level sets of bipolar fuzzy SBG-subalgebras and SBG-ideals naturally inherit the underlying algebraic structure, subject to well-defined conditions. This theoretical advancement not only enhances the algebraic modeling of systems characterized by uncertainty and duality but also lays a solid foundation for further applications in logical inference, information processing, and computational intelligence within bipolar fuzzy contexts.
Keywords: Sheffer stroke (BG-algebra), BG-ideal, fuzzy SBG-ideal.
MSC (2020): 06F05, 03G25, 03G10.

1. Introduction

In [2], published in 2008 by C. B. Kim and H. S. Kim, the concept of BG-algebras was introduced as a generalization of the notion of B-algebras (as defined in [4]). BG-algebras, generalize Boolean algebras by introducing advanced operations and properties, thereby enabling the analysis of complex logical and scientific systems beyond the scope of classical Boolean logic. Rooted in the foundations of Boolean algebra, BG-algebras provide a versatile framework that has proven valuable in areas such as mathematical logic and fuzzy logic, where nuanced forms of inference must be represented.

A central operation in logic, the Sheffer stroke (NAND), was introduced by Henry Maurice Sheffer [10]. This operation is functionally complete, meaning any logical expression or axiom

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can be constructed using only the Sheffer stroke [3]. This property not only simplifies the formulation of logical systems but also underscores the Sheffer stroke's fundamental role in both algebraic and logical contexts. As a result, the Sheffer stroke has inspired extensive research, including studies on its application to basic algebras and their congruences [6], MLT-algebras [9], ortholattices [1], new state operators and their filters in basic algebras [7], Riečan and Bosbach state operators on MTL-algebras [8], and BG-algebras [5, 11].

Sheffer stroke BG-algebras are of particular interest due to their ability to unify various logical operations within a single structure, thereby broadening the analytical scope for logical deduction and facilitating the exploration of relationships among diverse algebraic frameworks. These algebras have significant implications for both theoretical research and practical applications, especially in computational models relevant to quantum computing and fuzzy logic.

Fuzzy set theory, pioneered by Zadeh [13], marked a significant advancement in modeling uncertainty and vagueness, providing a robust framework for approximate reasoning and decision-making. Building on this, bipolar fuzzy sets—introduced by Zhang [14]—extend the concept by simultaneously representing positive and negative information, thus reflecting real-world situations where data may be contradictory or dual in nature.

This paper contributes to the field by investigating bipolar fuzzy Sheffer stroke BG-algebras, a novel algebraic system that synthesizes bipolar fuzzy logic with Sheffer stroke operations. We aim to generalize classical notions of subalgebras and ideals to the bipolar fuzzy context, with a particular focus on their level sets. These level sets provide a means to connect membership functions with algebraic properties, offering deeper insight into the structure of fuzzy subsets.

The motivation for this research lies in the potential of algebraic models equipped with bipolar fuzzy logic to address contradictions and uncertainties prevalent in decision-making, data analysis, and computational intelligence. By establishing rigorous correspondences between bipolar fuzzy subalgebras and ideals and their level sets, this study lays the groundwork for future applications and extensions within fuzzy algebraic systems.

The list of acronyms is provided in Table 1.1.

Table 1.1. List of acronyms

Acronyms	Representation
SBG-algebra	Sheffer stroke BG-algebra
BFISBG-ideal	bipolar fuzzy implicative SBG-ideal
BVFISBG-ideal	bipolar-valued fuzzy implicative SBG-ideal

2. Preliminaries

In this section, basic definitions and notions about Sheffer stroke BG-algebras and bipolar fuzzy structures concept.

Definition 2.1. [10] *Let $(H,|)$ be a groupoid. The operation $|$ is said to be a Sheffer stroke operation if it satisfies the following conditions:*

$$(S1)\ \omega \mid \zeta = \zeta \mid \omega$$
$$(S2)\ (\omega \mid \omega) \mid (\omega \mid \zeta) = \omega$$
$$(S3)\ \omega \mid ((\zeta \mid \upsilon) \mid (\zeta \mid \upsilon)) = ((\omega \mid \zeta) \mid (\omega \mid \zeta)) \mid \upsilon$$
$$(S4)\ (\omega \mid ((\omega \mid \omega) \mid (\zeta \mid \zeta))) \mid (\omega \mid ((\omega \mid \omega) \mid (\zeta \mid \zeta))) = \omega.$$

Definition 2.2. [5] *A Sheffer stroke BG-algebra (briefly, SBG-algebra) is a structure $(A,|)$ of type (2) such that 0 is the fixed element in A and the following conditions are satisfied for all $x,y,z \in A$*

$$(SBG-1)\ (\omega \mid (\omega \mid \omega)) \mid (x \mid (\omega \mid \omega)) = 0$$
$$(SBG-2)\ (0 \mid (\zeta \mid \zeta)) \mid (\omega \mid (\zeta \mid \zeta)) \mid (\omega \mid (\zeta \mid \zeta)) = \omega \mid \omega.$$

To enhance the readability of this manuscript about Sheffer stroke BG-algebras, we will consistently use the following notation:

$$\omega \mid (\zeta \mid \zeta) = \omega^{\zeta},$$

$$\omega \mid (\zeta \mid (\omega \mid \omega)) = \omega_{\zeta}.$$

Proposition 2.1. [5] *Let $(A,|)$ be an SBG-algebra. Then the binary relation $\omega \leq \zeta$ if and only if $\zeta^{\omega} \mid y^{x} = 0$ is a partial order on A.*

Definition 2.3. [5] *Let $(A,|)$ be an SBG-algebra. A nonempty subset G of A is called an SBG-subalgebra of A if $\omega^{\zeta} \mid \omega^{\zeta} \in G$ for all $\omega, \zeta \in G$.*

Definition 2.4. [5] *Let $(A,|)$ be an SBG-algebra. A nonempty subset I of a A is called an SBG-ideal of A if for all $\omega, \zeta \in I$*

(1) $0 \in I$,
(2) $\omega^{\zeta} \mid \omega^{\zeta} \in I$ *and* $\zeta \in I \Rightarrow \omega \in I$.

Definition 2.5 ([14])**.** *Let X be a nonempty set. A* bipolar fuzzy set *B on X is defined as*

$$B = \{(\omega, \Psi^{-}(\omega), \Psi^{+}(\omega)) \mid \omega \in X\},$$

where $\Psi^{+} : X \to [0,1]$ and $\Psi^{-} : X \to [-1,0]$ are two functions. The value $\Psi^{+}(\omega)$ represents the degree to which the element x satisfies the property associated with the bipolar fuzzy set B (the positive membership degree), while $\Psi^{-}(\omega)$ represents the degree to which x satisfies an implicit counter-property of B (the negative membership degree).

If $\Psi^{+}(\omega) \neq 0$ and $\Psi^{-}(\omega) = 0$, then x is interpreted as satisfying only the positive aspect of B. Conversely, if $\Psi^{+}(\omega) = 0$ and $\Psi^{-}(\omega) \neq 0$, then x is viewed as not satisfying the main property of B but partially satisfying its counter-property. It is also possible that both $\Psi^{+}(\omega) = 0$ and $\Psi^{-}(\omega) = 0$, which indicates an overlap or neutrality between the property and its counter-property at ω.

For simplicity, we denote the bipolar fuzzy set B by the pair of functions $\Psi = (\Psi^{+}, \Psi^{-})$.

Lemma 2.1 ([12])**.** *Let $\alpha, \beta, \gamma \in \mathbb{R}$. Then the following identities hold:*

(1) $\alpha - \min\{\beta, \gamma\} = \max\{\alpha - \beta, \alpha - \gamma\}$,
(2) $\alpha - \max\{\beta, \gamma\} = \min\{\alpha - \beta, \alpha - \gamma\}$.

3. Bipolar fuzzy implicative SBG-ideals

This section is devoted to the study of bipolar fuzzy implicative SBG-ideals in SBG-algebras. We begin by formally defining the notion of a bipolar fuzzy implicative SBG-ideal and investigate its fundamental properties and characterizations. Several key results are established, including necessary and sufficient conditions for a bipolar-valued fuzzy set to be a BVFISBG-ideal in terms of its level sets. The relationships between various classes of bipolar fuzzy ideals—such as sub-implicative, completely closed, and p-ideals—are explored, and their connections with implicative SBG-ideals are clarified. Moreover, closure properties under infimum and translation operators are examined. The section also provides illustrative theorems and propositions that characterize these ideals within specific algebraic settings, such as medial and implicative SBG-algebras, thereby enriching the structural understanding of bipolar fuzzy ideals in this context.

Definition 3.1. *A bipolar fuzzy set $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ on an SBG-algebra $\mathcal{L}$ is called a bipolar fuzzy implicative SBG-ideal of $\mathcal{L}$ if*

$$(\forall \omega, \zeta \in \mathcal{L}) \left(\begin{array}{c} \Psi^-(0) \leq \Psi^-(\omega) \leq \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\} \\ \Psi^+(0) \geq \Psi^+(\omega) \geq \min\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\} \end{array} \right). \tag{3.1}$$

Proposition 3.1. *Every BFISBG-ideal of an SBG-algebra $\mathcal{L}$ is also a bipolar fuzzy SBG-ideal of $\mathcal{L}$.*

Proof. Let Ψ be a BFISBG-ideal of an SBG-algebra $\mathcal{L}$. Then, the following inequalities hold for all $\omega \in \mathcal{L}$:

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{and} \quad \Psi^+(0) \geq \Psi^+(\omega).$$

Also,

$$\begin{array}{rcl} \Psi^-(\omega) & \leq & \max\{\Psi^-(\zeta), \Psi^-((\omega_\omega \mid \omega_\zeta)^\zeta \mid (\omega_\omega \mid \omega_\zeta)^\zeta)\} \\ & = & \max\{\Psi^-(\zeta), \Psi^-((\omega^0 \mid \omega^0)^\zeta \mid (\omega^0 \mid \omega^0)^\zeta)\} \\ & = & \max\{\Psi^-(\zeta), \Psi^-(\omega^\zeta \mid \omega^\zeta)\}, \end{array}$$

and

$$\begin{array}{rcl} \Psi^+(\omega) & \geq & \min\{\Psi^+(\zeta), \Psi^+((\omega_\omega \mid \omega_\zeta)^\zeta \mid (\omega_\omega \mid \omega_\zeta)^\zeta)\} \\ & = & \min\{\Psi^+(\zeta), \Psi^+((\omega^0 \mid \omega^0)^\zeta \mid (\omega^0 \mid \omega^0)^\zeta)\} \\ & = & \min\{\Psi^+(\zeta), \Psi^+(\omega^\zeta \mid \omega^\zeta)\}. \end{array}$$

Therefore, Ψ is a bipolar fuzzy SPG-ideal of $\mathcal{L}$. ☐

Theorem 3.1. *Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar-valued fuzzy set in $\mathcal{L}$. Then Ψ is a BVFISBG-ideal of $(\mathcal{L}, |)$ if and only if, for all $(\mathfrak{p}, \mathfrak{q}) \in [-1, 0] \times [0, 1]$, the negative $\mathfrak{p}$-cut and the positive $\mathfrak{q}$-cut of Ψ (whenever they are nonempty) are implicative SBG-ideals of $\mathcal{L}$.*

Proof. Assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a BVFISBG-ideal of $(\mathcal{L}, |)$, and that for all $(\mathfrak{p}, \mathfrak{q}) \in [-1, 0] \times [0, 1]$, the negative $\mathfrak{p}$-cut $\mathcal{L}(\Psi^-, \mathfrak{p})$ and the positive $\mathfrak{q}$-cut $U(\Psi^+, \mathfrak{q})$ are nonempty.

Let $\omega, \alpha \in \mathcal{L}$ be such that $(\omega, \alpha) \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$, i.e.,

$$\Psi^-(\omega) \leq \mathfrak{p} \quad \text{and} \quad \Psi^+(\alpha) \geq \mathfrak{q}.$$

Since $\Psi^-(0) \leq \Psi^-(\omega) \leq \mathfrak{p}$ and $\Psi^+(0) \geq \Psi^+(\alpha) \geq \mathfrak{q}$, it follows that $(0, 0) \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$.

Now, let $\omega, \zeta, \upsilon, \alpha, \beta, \gamma \in \mathcal{L}$ satisfy

$$(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \in L(\Psi^-, \mathfrak{p}),$$

$$(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma) \in U(\Psi^+, \mathfrak{q}),$$

and $(\upsilon, \gamma) \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$. Then we have

$$\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \leq s, \quad \Psi^-(\upsilon) \leq \mathfrak{p},$$

$$\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \geq \mathfrak{q}, \quad \Psi^+(\gamma) \geq \mathfrak{q}.$$

From this, it follows that

$$\Psi^-(\omega) \leq \max\{\Psi^-(\omega^\zeta \mid \omega^\zeta), \Psi^-(\upsilon)\} \leq \mathfrak{p},$$

and

$$\Psi^+(\alpha) \geq \min\{\Psi^+(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma), \Psi^+(c)\} \geq \mathfrak{q},$$

which shows that $\omega, \alpha \in L(\Psi^-, \mathfrak{p}) \times U(\Psi^+, \mathfrak{q})$. Hence, $L(\Psi^-, \mathfrak{p})$ and $U(\Psi^+, \mathfrak{q})$ are implicative SBG-ideals of $\mathcal{L}$.

Conversely, assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a bipolar-valued fuzzy set in $\mathcal{L}$ such that its negative $\mathfrak{p}$-cuts and positive $\mathfrak{q}$-cuts are implicative SBG-ideals of $\mathcal{L}$ whenever they are nonempty.

Suppose there exists $\alpha \in \mathcal{L}$ such that $\Psi^-(0) > \Psi^-(\alpha)$. Then $\alpha \in L(\Psi^-, \Psi^-(\alpha))$, but $0 \notin L(\Psi^-, \Psi^-(\alpha))$, contradicting the fact that $L(\Psi^-, \Psi^-(\alpha))$ is an implicative SBG-ideal. Hence,

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{for all } \omega \in \mathcal{L}.$$

Similarly, suppose there exists $\omega \in \mathcal{L}$ such that $\Psi^+(0) < \Psi^+(\omega)$. Then $\omega \in U(\Psi^+, \Psi^+(\omega))$, but $0 \notin U(\Psi^+, \Psi^+(\omega))$, a contradiction. Therefore,

$$\Psi^+(0) \geq \Psi^+(\alpha) \quad \text{for all } \alpha \in \mathcal{L}.$$

Now, suppose there exist $\alpha, \beta, \gamma, \omega, \zeta, \upsilon \in \mathcal{L}$ such that

$$\Psi^-(\alpha) > \max\left\{\Psi^-(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma), \Psi^-(\gamma)\right\}$$

or

$$\Psi^+(\omega) < \min\left\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\right\}.$$

Define

$$\mathfrak{p} = \max\left\{\Psi^-(((\alpha_\beta \mid \alpha_\beta)^\gamma) \mid (\alpha_\beta \mid \alpha_\beta)^\gamma), \Psi^-(\gamma)\right\},$$

$$\mathfrak{q} = \min\left\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\right\}.$$

Since $a \notin L(\Psi^-, s)$ or $x \notin U(\Psi^+, t)$, this contradicts the assumption that these cuts are implicative SBG-ideals.

Therefore,

$$\Psi^-(\omega) \leq \max\left\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\right\}$$

and

$$\Psi^+(\omega) \geq \min\left\{\Psi^+\left(x^y \mid x^y\right), \Psi^+(\upsilon)\right\}$$

for all $\omega, \zeta, \upsilon \in \mathcal{L}$. Hence, Ψ is a BVFISBG-ideal of $\mathcal{L}$. $\square$

Theorem 3.2. *A bipolar-valued fuzzy set $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ in $\mathcal{L}$ is a BVFISBG-ideal of $(\mathcal{L}, |)$ if and only if the fuzzy sets Ψ^-_γ and Ψ^+ are fuzzy implicative SBG-ideals of $(\mathcal{L}, |)$, where $\Psi^-_\gamma : \mathcal{L} \to [0,1]$ is defined by $\Psi^-_\gamma(\omega) = 1 - \Psi^-(\omega)$ for all $\omega \in \mathcal{L}$.*

Proof. Assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a BVFISBG-ideal of $(\mathcal{L}, |)$. Then Ψ satisfies the following: For every $\omega, \zeta \in \mathcal{L}$,

$$\begin{aligned} \Psi^-_\gamma(0) &= 1 - \Psi^-(0) \\ &\geq 1 - \Psi^-(\omega) \\ &= \Psi^-_\gamma(\omega), \end{aligned}$$

$$\begin{aligned} \Psi^-_\gamma(\omega) &= 1 - \Psi^-(\omega) \\ &\geq 1 - \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\} \\ &= \min\{1 - \Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), 1 - \Psi^-(\upsilon)\} \\ &= \min\{\Psi^-_c(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_c(\upsilon)\}. \end{aligned}$$

Thus, Ψ^-_γ satisfies the conditions of a fuzzy implicative SBG-ideal. Since Ψ^+ is already assumed to be one, the claim holds.

Conversely, Conversely, suppose that Ψ^-_γ and Ψ^+ are fuzzy implicative SBG-ideals of $(\mathcal{L}, |)$. Then for $\omega, \zeta, \upsilon \in \mathcal{L}$ we have $\Psi^-(0) \leq \Psi^-(\omega)$ and

$$\begin{aligned} 1 - \Psi^-(\omega) &= \Psi^-_\gamma(\omega) \\ &\geq \min\{\Psi^-_\gamma(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_\gamma(\upsilon)\} \\ &= \min\{1 - \Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), 1 - \Psi^-(\upsilon)\} \\ &= 1 - \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\}, \\ \Psi^-(\omega) &\leq \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\}. \end{aligned}$$

Hence $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a BVFISBG-ideal of $\mathcal{L}$. □

Theorem 3.3. *Let $\mathcal{F}$ be a nonempty subset of $\mathcal{L}$. Define the bipolar-valued fuzzy set $\Psi_\mathcal{F} = (L, \Psi^-_\mathcal{F}, \Psi^+_\mathcal{F})$ in $\mathcal{L}$ as follows:*

$$\Psi^-_\mathcal{F} : \mathcal{L} \to [-1, 0], \alpha \mapsto \begin{cases} \mathfrak{p}^- & \text{if } \alpha \in \mathcal{F}, \\ \mathfrak{q}^- & \text{otherwise,} \end{cases} \quad \Psi^+_\mathcal{F} : \mathcal{L} \to [0, 1], \omega \mapsto \begin{cases} \mathfrak{p}^+ & \text{if } \omega \in \mathcal{F}, \\ \mathfrak{q}^+ & \text{otherwise,} \end{cases}$$

where $\mathfrak{p}^- < \mathfrak{q}^-$ in $[-1, 0]$ and $\mathfrak{p}^+ > \mathfrak{q}^+$ in $[0, 1]$. Then $\Psi_\mathcal{F} = (\mathcal{L}, \Psi^-_\mathcal{F}, \Psi^+_\mathcal{F})$ is a BVFISBG-ideal of $(\mathcal{L}, |)$ if and only if $\mathcal{F}$ is an implicative SBG-ideal of $\mathcal{L}$.

Proof. Assume that $\Psi_\mathcal{F} = (\mathcal{L}, \Psi^-_\mathcal{F}, \Psi^+_\mathcal{F})$ is a BVFISBG-ideal of $(\mathcal{L}, |)$. Let $\omega, \zeta, \upsilon \in \mathcal{L}$ be such that $\omega, \zeta \in \mathcal{F}$. Then, by the definition of $\Psi^-_\mathcal{F}$ and $\Psi^+_\mathcal{F}$, we have $\Psi^-_\mathcal{F}(\omega) = \mathfrak{p}^-$ and $\Psi^+_\mathcal{F}(\omega) = \mathfrak{p}^+$. Since $\Psi_\mathcal{F}$ is a BVFISBG-ideal, it satisfies the conditions $\Psi^-_\mathcal{F}(0) \leq \Psi^-_\mathcal{F}(\omega)$ and $\Psi^+_\mathcal{F}(0) \geq \Psi^+_\mathcal{F}(\omega)$, which gives $\Psi^-_\mathcal{F}(0) \leq \mathfrak{p}^-$ and $\Psi^+_\mathcal{F}(0) \geq \mathfrak{p}^+$. However, by the definition of $\Psi^-_\mathcal{F}$ and $\Psi^+_\mathcal{F}$, we also know that $\Psi^-_\mathcal{F}(0)$ can only be $\mathfrak{p}^-$ or $\mathfrak{q}^-$, and $\Psi^+_\mathcal{F}(0)$ can only be $\mathfrak{p}^+$ or $\mathfrak{q}^+$. But since $^- < \mathfrak{q}^-$ and $\mathfrak{p}^+ > \mathfrak{q}^+$, the inequalities above can hold only if $\Psi^-_\mathcal{F}(0) = \mathfrak{p}^-$, and $\Psi^+_\mathcal{F}(0) = \mathfrak{p}^+$, which implies that $0 \in \mathcal{F}$. Also $\Psi^-(\omega) \leq \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\zeta)\} = \mathfrak{p}^-$, $\Psi^+(\omega) \geq \min\{\Psi^+(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon)), \Psi^+(\zeta)\} = s^+$, and so $\Psi^-(\omega) = \mathfrak{p}^-$ and $\Psi^+(\omega) = \mathfrak{p}^+$. Therefore, $\omega \in \mathcal{F}$. Hence, $\mathcal{F}$ is closed under the implicative SBG-ideal condition, and since $0 \in \mathcal{F}$, we conclude that $\mathcal{F}$ is an implicative SBG-ideal of $\mathcal{L}$.

Conversely, assume that $\mathcal{F}$ is an implicative SBG-ideal of $(\mathcal{L}, |)$. Let $\omega, \zeta, \upsilon \in \mathcal{L}$. If $\omega \in \mathcal{F}$, then $0 \in \mathcal{F}$, and thus $\Psi^-_{\mathcal{F}}(0) = \mathfrak{p}^- = \Psi^-_{\mathcal{F}}(\omega)$, and $\Psi^+_{\mathcal{F}}(0) = \mathfrak{p}^+ = \Psi^+_{\mathcal{F}}(\omega)$. If $0 \notin \mathcal{F}$, then $\Psi^-_{\mathcal{F}}(0) = \mathfrak{q}^- > \mathfrak{p}^- = \Psi^-_{\mathcal{F}}(\omega)$, and $\Psi^+_{\mathcal{F}}(0) = \mathfrak{q}^+ < s^+ = \Psi^+_{\mathcal{F}}(\omega)$, which still satisfies the required inequality conditions for a BVFISBG-ideal.

Furthermore, suppose $\alpha = (((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) \in \mathcal{F}$ and $\upsilon \in \mathcal{F}$. Then, since $\mathcal{F}$ is an implicative SBG-ideal, we have $\omega \in \mathcal{F}$. Hence,

$$\Psi^-_{\mathcal{F}}(\omega) = \mathfrak{p}^- = \max\{\Psi^-_{\mathcal{F}}(\eta), \Psi^-_{\mathcal{F}}(\upsilon)\}, \quad \Psi^+_{\mathcal{F}}(\omega) = \mathfrak{p}^+ = \min\{\Psi^+_{\mathcal{F}}(\eta), \Psi^+_{\mathcal{F}}(\upsilon)\}.$$

If either $\eta \notin \mathcal{F}$ or $\upsilon \notin \mathcal{F}$, then at least one of $\Psi^-_{\mathcal{F}}(\eta)$ or $\Psi^-_{F}(\upsilon)$ equals t^-, and so

$$\Psi^-_{\mathcal{F}}(\omega) \leq \mathfrak{q}^- = \max\{\Psi^-_{\mathcal{F}}(\eta), \Psi^-_{\mathcal{F}}(\upsilon)\}, \quad \Psi^+_{\mathcal{F}}(\omega) \geq \mathfrak{q}^+ = \min\{\Psi^+_{\mathcal{F}}(\eta), \Psi^+_{\mathcal{F}}(\upsilon)\},$$

which still satisfies the defining conditions.

Thus, $\Psi_{\mathcal{F}} = (\mathcal{L}, \Psi^-_{F}, \Psi^+_{\mathcal{F}})$ is a BVFISBG-ideal of $\mathcal{L}$. □

Proposition 3.2. *Let $\{\Psi_{\mathfrak{k}} = (\Psi^-_{\mathfrak{k}}, \Psi^+_{\mathfrak{k}}) : \mathfrak{k} \in \Delta\}$ be a family of bipolar-valued fuzzy implicative SBG-ideals of a SBG-algebra $\mathcal{L}$. Then the infimum*

$$\bigwedge_{\mathfrak{k}\in\Delta} \Psi_{\mathfrak{k}} = \left(\bigvee_{\mathfrak{k}\in\Delta} \Psi^-_{\mathfrak{k}}, \bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}} \right)$$

is also a BVFISBG-ideal of $\mathcal{L}$.

Proof. Let $\Psi_{\mathfrak{k}} = \{(\Psi^+_{\mathfrak{k}}, \Psi^-_{\mathfrak{k}}) : \mathfrak{k} \in \Delta\}$ be a family of bipolar fuzzy implicative SBG-ideals of a SBG-algebra $\mathcal{L}$. Let $\omega, \zeta \in \mathcal{L}$. Then we have

$$(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}})(0) = \inf_{\mathfrak{k}\in\Delta}\{\Psi^+_{\mathfrak{k}}(0)\} \geq \inf_{\mathfrak{k}\in\Delta}\{\Psi^+_{\mathfrak{k}}(\omega)\} = (\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}})(\omega),$$

$$(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^-_{\mathfrak{k}})(0) = \sup_{\mathfrak{k}\in\Delta}\{\Psi^-_{\mathfrak{k}}(0)\} \leq \sup_{\mathfrak{k}\in\Delta}\{\Psi^-_{\mathfrak{k}}(\omega)\} = (\bigwedge_{i\in\Delta} \Psi^-_{\mathfrak{k}})(\omega).$$

Let $\omega, \zeta \in \mathcal{L}$. Then we have

$$\begin{aligned}
(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}})(\omega) &= \inf_{\mathfrak{k}\in\Delta}\{\Psi^+_{\mathfrak{k}}(\omega)\} \\
&\geq \inf_{\mathfrak{k}\in\Delta}\{\min\{\Psi^+_{\mathfrak{k}}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+_{\mathfrak{k}}(\upsilon)\}\} \\
&= \min\{\inf_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \inf_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}}(\upsilon)\} \\
&= \min\{(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}})(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), (\bigwedge_{\mathfrak{k}\in\Delta} \Psi^+_{\mathfrak{k}})(\upsilon)\},
\end{aligned}$$

and

$$\begin{aligned}
(\bigwedge_{i\in\Delta} \Psi^-_{\mathfrak{k}})(\omega) &= \sup_{\mathfrak{k}\in\Delta}\{\Psi^-_{\mathfrak{k}}(\omega)\} \\
&\leq \sup_{\mathfrak{k}\in\Delta}\{\max\{\Psi^-_{\mathfrak{k}}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_{\mathfrak{k}}(\upsilon)\}\} \\
&= \max\{\sup_{\mathfrak{k}\in\Delta} \Psi^-_{\mathfrak{k}}(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \sup_{\mathfrak{k}\in\Delta} \Psi^-_{\mathfrak{k}}(\upsilon)\} \\
&= \max\{(\bigwedge_{\mathfrak{k}\in\Delta} \Psi^-_{\mathfrak{k}})(((\omega_\zeta \mid \omega_\zeta)^\upsilon) \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), (\bigwedge_{i\in\Delta} \Psi^-_{\mathfrak{k}})(\upsilon)\}.
\end{aligned}$$

Hence $\bigwedge_{\mathfrak{k}\in\Delta} \Psi_{\mathfrak{k}}$ is a BFISBG-ideal of a SBG-algebra $\mathcal{L}$. □

Definition 3.2. *A bipolar fuzzy subset $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ of a SBG-algebra $\mathcal{L}$ is called a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$ if it satisfies the following conditions:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L})\left(\begin{array}{l}\Psi^-(0) \leq \Psi^-(\omega),\\ \Psi^+(0) \geq \Psi^+(\omega),\\ \Psi^-((\zeta|\zeta^\omega)|(\zeta|\zeta^\omega)) \leq \max\{\Psi^-(((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon \mid ((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon), \Psi^-(\upsilon)\},\\ \Psi^+((\zeta|\zeta^\omega)|(\zeta|\zeta^\omega)) \geq \min\{\Psi^+(((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon \mid ((\omega|\omega^\zeta)|(\omega|\omega^\zeta))^\upsilon), \Psi^+(\upsilon)\},\end{array}\right). \tag{3.2}$$

Proposition 3.3. *Let $\mathcal{L}$ be a SBG-algebra. Then every bipolar fuzzy subimplicative SBG-ideal of $\mathcal{L}$ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. Then $\Psi^-(0) \leq \Psi^-(\omega)$, $\Psi^+(0) \geq \Psi^+(\omega)$,

$$\begin{aligned}\Psi^-(\omega) &= \Psi^-(\omega^0 \mid \omega^0)\\ &= \Psi^-((\omega|\omega^\omega)|(\omega|\omega^\omega))\\ &\leq \max\{\Psi^-(((((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon) \mid ((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon)), \Psi^-(\upsilon)\}\\ &= \max\{\Psi^-((\omega^0 \mid \omega^0)^\upsilon \mid (\omega^0 \mid \omega^0)^\upsilon), \Psi^-(\upsilon)\}\\ &= \max\{\Psi^-(\omega^\upsilon \mid \omega^\upsilon), \Psi^-(\upsilon)\},\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &= \Psi^+(\omega^0 \mid \omega^0)\\ &= \Psi^+((\omega|\omega^\omega)|(\omega|\omega^\omega))\\ &\geq \min\{\Psi^+(((((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon) \mid ((\omega|\omega^\omega)|(\omega|\omega^\omega))^\upsilon)), \Psi^+(\upsilon)\}\\ &= \min\{\Psi^+((\omega^0 \mid \omega^0)^\upsilon \mid (\omega^0 \mid \omega^0)^\upsilon), \Psi^+(\upsilon)\}\\ &= \min\{\Psi^+(\omega^\upsilon \mid \omega^\upsilon), \Psi^+(\upsilon)\}.\end{aligned}$$

Therefore, $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. □

Theorem 3.4. *Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of a SBG-algebra $\mathcal{L}$. Then $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$ if and only if*

$$\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \leq \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))$$

and

$$\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \geq \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)).$$

Proof. Assume that $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. Then we have

$$\begin{aligned}&\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))\\ &\leq \max\left\{\Psi^-(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^0 \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^0), \Psi^-(0)\right\}\\ &= \max\left\{\Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^-(0)\right\}\\ &= \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)).\end{aligned}$$

Similarly,

$$\begin{aligned}
&\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \\
&\geq \min\left\{\Psi^+(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^0 \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^0), \Psi^+(0)\right\} \\
&= \min\left\{\Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^+(0)\right\} \\
&= \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)).
\end{aligned}$$

Conversely, assume that the given inequalities hold and that Ψ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Then we have

$$\Psi^-(0) \leq \Psi^-(\omega), \quad \Psi^+(0) \geq \Psi^+(\omega) \quad \text{for all } \omega \in \mathcal{L}.$$

Now, from the assumption, we obtain

$$\begin{aligned}
\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &\leq \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\
&\leq \max\left\{\Psi^-(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon), \Psi^-(\upsilon)\right\},
\end{aligned}$$

and similarly,

$$\begin{aligned}
\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &\geq \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\
&\geq \min\left\{\Psi^+(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon), \Psi^+(\upsilon)\right\}.
\end{aligned}$$

Hence, Ψ satisfies the condition for being a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Theorem 3.5. *Let $\mathcal{L}$ be an implicative SBG-algebra. Then every bipolar fuzzy SBG-ideal of $\mathcal{L}$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Then it satisfies $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$ for all $\omega \in \mathcal{L}$. By the properties of bipolar fuzzy SBG-ideals, we have

$$\begin{aligned}
&\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \\
&\quad \leq \max\{\Psi^-(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon), \Psi^-(\upsilon)\} \\
&\quad = \max\{\Psi^-(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon, \Psi^-(\upsilon)\},
\end{aligned}$$

$$\begin{aligned}
&\Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \\
&\quad \geq \min\{\Psi^+(((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon \mid ((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega))^\upsilon), \Psi^+(\upsilon)\} \\
&\quad = \min\{\Psi^+(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^\upsilon, \Psi^+(\upsilon)\}.
\end{aligned}$$

Thereby, Ψ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Lemma 3.1. *In an SBG-algebra $\mathcal{L}$, the following identity holds for all $\omega, \zeta \in \mathcal{L}$:*

$$((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \mid \zeta^\omega = \omega \mid \omega^\zeta.$$

Theorem 3.6. *Every bipolar fuzzy SBG-ideal of a medial SBG-algebra $\mathcal{L}$ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of a medial SBG-algebra $\mathcal{L}$. Then by definition, for all $\omega \in \mathcal{L}$ we have $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$. Note that in a medial SBG-algebra, the following identity holds:

$$(\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega}) = \omega.$$

Therefore,

$$\Psi^{-}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) = \Psi^{-}(\omega),$$

and

$$\Psi^{+}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) = \Psi^{+}(\omega).$$

Next, we observe that

$$\begin{aligned}\Psi^{-}(\omega) &\leq \max\{\Psi^{-}(\omega^{\upsilon} \mid \omega^{\upsilon}),\ \Psi^{-}(\upsilon)\}\\ &= \max\left\{\Psi^{-}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}),\ \Psi^{-}(\upsilon)\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Psi^{+}(\omega) &\geq \min\{\Psi^{+}(\omega^{\upsilon} \mid \omega^{\upsilon}),\ \Psi^{+}(\upsilon)\}\\ &= \min\left\{\Psi^{+}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}),\ \Psi^{+}(\upsilon)\right\}.\end{aligned}$$

Thus, the conditions for Ψ to be a bipolar fuzzy sub-implicative SBG-ideal are satisfied. Hence, every bipolar fuzzy SBG-ideal of a medial SBG-algebra $\mathcal{L}$ is indeed a bipolar fuzzy sub-implicative SBG-ideal. □

Theorem 3.7. *Let $\mathcal{L}$ be an SBG-algebra satisfying the following condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \begin{pmatrix} \Psi^{-}(\zeta^{\upsilon} \mid \zeta^{\upsilon}) \leq \Psi^{-}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}) \\ \Psi^{+}(\zeta^{\upsilon} \mid \zeta^{\upsilon}) \geq \Psi^{+}(((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon} \mid ((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta}))^{\upsilon}) \end{pmatrix}. \tag{3.3}$$

Then every bipolar fuzzy SBG-ideal of $\mathcal{L}$ is also a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.

Proof. We are given that the following inequality holds:

$$\begin{aligned}\Psi^{-}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) &\leq \Psi^{-}((((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}) \mid (((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}))\\ &= \Psi^{-}((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Psi^{+}((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) &\geq \Psi^{+}((((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}) \mid (((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})) \mid \zeta^{\omega}))\\ &= \Psi^{+}((\omega \mid \omega^{\zeta}) \mid (\omega \mid \omega^{\zeta})).\end{aligned}$$

Hence, Ψ is a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Theorem 3.8. *Every medial SBG-algebra is an implicative SBG-algebra.*

Theorem 3.9. *Let $\Psi = (\mathcal{L}, \Psi^{-}, \Psi^{+})$ be a bipolar fuzzy SBG-ideal of a SBG-algebra $\mathcal{L}$. Then Ψ is a BFISBG-ideal of $\mathcal{L}$ if and only if the following condition holds:*

$$(\forall \omega, \zeta \in \mathcal{L}) \quad \begin{cases} \Psi^{-}(\omega) \leq \Psi^{-}(\omega_{\zeta} \mid \omega_{\zeta}), \\ \Psi^{+}(\omega) \geq \Psi^{+}(\omega_{\zeta} \mid \omega_{\zeta}). \end{cases} \tag{3.4}$$

Proof. Let Ψ be a BFISBG-ideal of $\mathcal{L}$. Then

$$\begin{aligned}\Psi^-(\omega) &\leq \max\{\Psi^-(0), \Psi^-((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0)\} \\ &= \max\{\Psi^-(0), \Psi^-(\omega_\zeta \mid \omega_\zeta)\} \\ &= \Psi^-(\omega_\zeta \mid \omega_\zeta),\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &\geq \min\{\Psi^+(0), \Psi^+((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0)\} \\ &= \min\{\Psi^+(0), \Psi^+(\omega_\zeta \mid \omega_\zeta)\} \\ &= \Psi^+(\omega_\zeta \mid \omega_\zeta),\end{aligned}$$

for all $\omega, \zeta \in \mathcal{L}$.

Conversely, let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a bipolar fuzzy SBG-ideal of $\mathcal{L}$ satisfying the inequality (3.4). Then it is evident that $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$ for all $\omega \in \mathcal{L}$. Since

$$\begin{aligned}\Psi^-(\omega) &\leq \Psi^-(\omega_\zeta \mid \omega_\zeta) \\ &\leq \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\},\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &\geq \Psi^+(\omega_\zeta \mid \omega_\zeta) \\ &\geq \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\},\end{aligned}$$

for all $\omega, \zeta, \upsilon \in \mathcal{L}$. Therefore, we conclude that Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Theorem 3.10. *Let $\mathcal{L}$ be a medial SBG-algebra satisfying the following condition:*

$$(\forall \omega, \zeta \in \mathcal{L}) \left(\begin{array}{l} \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \leq \Psi^-(\omega_\zeta \mid \omega_\zeta) \\ \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \geq \Psi^+(\omega_\zeta \mid \omega_\zeta) \end{array} \right). \tag{3.5}$$

Then every bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$ is a bipolar fuzzy implicative ideal of $\mathcal{L}$.

Proof. Let Ψ be a bipolar fuzzy sub-implicative SBG-ideal of a medial SBG-algebra $\mathcal{L}$ satisfying the inequality (3.5). Then we obtain that

$$\begin{aligned}\Psi^-(\omega) &= \Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \\ &\leq \max\{\Psi^-(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0 \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0), \Psi^-(0)\} \\ &= \max\{\Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^-(0)\} \\ &= \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\ &\leq \Psi^-(\omega_\zeta \mid \omega_\zeta),\end{aligned}$$

and

$$\begin{aligned}\Psi^+(\omega) &= \Psi^+((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) \\ &\geq \min\{\Psi^+(((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0 \mid ((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta))^0), \Psi^+(0)\} \\ &= \min\{\Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)), \Psi^+(0)\} \\ &= \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\ &\geq \Psi^+(\omega_\zeta \mid \omega_\zeta).\end{aligned}$$

Thus, Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Theorem 3.11. *Let $\mathcal{L}$ be an implicative SBG-algebra. Then every BFISBG-ideal of $\mathcal{L}$ is also a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Proof. Let $\Psi = (\mathcal{L}, \Psi^-, \Psi^+)$ be a BFISBG-ideal of an implicative SBG-algebra $\mathcal{L}$. By definition, Ψ is also a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Therefore, we have $\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$ for all $\omega \in \mathcal{L}$. Hence, for all $\omega, \zeta, \upsilon \in \mathcal{L}$, we obtain

$$\begin{aligned}\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &= \Psi^-((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\ &\leq \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\},\end{aligned}$$

and

$$\begin{aligned}\Psi^-((\zeta \mid \zeta^\omega) \mid (\zeta \mid \zeta^\omega)) &= \Psi^+((\omega \mid \omega^\zeta) \mid (\omega \mid \omega^\zeta)) \\ &\geq \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\},\end{aligned}$$

Thus, Ψ satisfies the defining conditions of a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$. □

Corollary 3.1. *Let $\mathcal{L}$ be a medial SBG-algebra. Then every BFISBG-ideal of $\mathcal{L}$ is also a bipolar fuzzy sub-implicative SBG-ideal of $\mathcal{L}$.*

Definition 3.3. *A bipolar fuzzy SBG-ideal $\Psi = (\mathcal{L}, \Psi^+, \Psi^-)$ of a SBG-algebra $\mathcal{L}$ is said to be* bipolar fuzzy closed *if it satisfies the following condition:*

$$(\forall \omega \in \mathcal{L}) \begin{pmatrix} \Psi^-(0^\omega \mid 0^\omega) \leq \Psi^-(\omega) \\ \Psi^+(0^\omega \mid 0^\omega) \geq \Psi^+(\omega) \end{pmatrix}. \tag{3.6}$$

Definition 3.4. *Let $\Psi = (\mathcal{L}, \Psi^+, \Psi^-)$ be a bipolar fuzzy SBG-ideal of a SBG-algebra $\mathcal{L}$. Then Ψ is called a* bipolar fuzzy completely closed SBG-ideal *of $\mathcal{L}$ if it satisfies the following condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \begin{pmatrix} \Psi^-(\omega^\zeta \mid \omega^\zeta) \leq \max\{\Psi^-(\omega), \Psi^-(\zeta)\}, \\ \Psi^+(\omega^\zeta \mid \omega^\zeta) \geq \min\{\Psi^+(\omega), \Psi^+(\zeta)\}, \end{pmatrix}. \tag{3.7}$$

Theorem 3.12. *Let $\mathcal{L}$ be a SBG-algebra satisfying the following condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \quad (((\omega^\zeta \mid \omega^\zeta) \mid \omega^\upsilon) \mid ((\omega^\zeta \mid \omega^\zeta) \mid \omega^\upsilon)) \mid \upsilon^\zeta = 0 \mid 0. \tag{3.8}$$

Then $\mathcal{L}$ is implicative if and only if every bipolar fuzzy closed SBG-ideal of $\mathcal{L}$ is a BFISBG-ideal of $\mathcal{L}$.

Proof. Let $\mathcal{L}$ be a SBG-algebra satisfying (3.8). Assume that $\mathcal{L}$ is implicative and let $\Psi = (\mathcal{L}, \Psi^+, \Psi^-)$ be a bipolar fuzzy closed SBG-ideal of $\mathcal{L}$. Then Ψ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Thus, for all $\omega \in \mathcal{L}$, we have

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{and} \quad \Psi^+(0) \geq \Psi^+(\omega).$$

Also, for all $\omega, \zeta, \upsilon \in \mathcal{L}$, we have

$$\begin{aligned}\Psi^-(\omega) &\leq \max\left\{\Psi^-(\upsilon), \Psi^-(\omega^\upsilon \mid \omega^\upsilon)\right\} \\ &= \max\left\{\Psi^-(\upsilon), \Psi^-((((\omega \mid \omega)^\omega \mid x^y)^\upsilon \mid \upsilon) \mid (((\omega \mid \omega)^\omega \mid x^y)^\upsilon \mid \upsilon)) \right. \\ &= \max\left\{\Psi^-(\upsilon), \Psi^-(((\omega \mid \zeta^\omega) \mid (\omega \mid \zeta^\omega))^\upsilon \mid ((\omega \mid \zeta^\omega) \mid (\omega \mid \zeta^x))^\upsilon)\right\}.\end{aligned}$$

Similarly,

$$\begin{aligned}\Psi^+(\omega) &\geq \min\left\{\Psi^+(\upsilon), \Psi^+(\omega^\upsilon \mid \omega^\upsilon)\right\} \\ &= \min\left\{\Psi^+(\upsilon), \Psi^+((((\omega \mid \omega)^\omega \mid x^y)^\upsilon \mid \upsilon) \mid (((\omega \mid \omega)^\omega \mid \omega^\zeta)^\upsilon \mid \upsilon)) \mid \right. \\ &= \min\left\{\Psi^+(\upsilon), \Psi^+(((x \mid \zeta^\omega) \mid (\omega \mid \zeta^\omega))^\upsilon \mid ((\omega \mid \zeta^\omega) \mid (\omega \mid \zeta^\omega))^\upsilon)\right\}.\end{aligned}$$

This implies that Ψ is a BFISBG-ideal of $\mathcal{L}$.

Conversely, suppose that every bipolar fuzzy closed SBG-ideal of $\mathcal{L}$ is a BFISBG-ideal. Then, from equation (3.8), we have

$$\upsilon^{\zeta} = \omega^{\upsilon} \mid (\omega^{\zeta} \mid \omega^{\zeta}).$$

Using identities (S1) and (S3), we also have

$$\upsilon^{\zeta} = ((\omega \mid \omega^{\upsilon}) \mid (\omega \mid \omega^{\upsilon}))^{\zeta}.$$

Then, applying (S2) and Lemma 2.1(3), it follows that

$$\upsilon = (\omega \mid \omega^{\upsilon}) \mid (\omega \mid \omega^{\upsilon}).$$

Now, we derive

$$\begin{aligned}
(\omega \mid \omega^{\zeta}) &= (((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) \mid ((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})))^{y} \\
&= ((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) \mid (y \mid (((\zeta \mid \zeta) \mid \zeta^{\omega}) \mid ((\zeta \mid \zeta) \mid \zeta^{\omega}))) \\
&= ((\zeta \mid \zeta^{\omega}) \mid (\zeta \mid \zeta^{\omega})) \mid y^{y} \\
&= ((\zeta^{\zeta} \mid (\zeta \mid \zeta)^{\zeta}) \mid (\zeta^{\zeta} \mid (\zeta \mid \zeta)^{\zeta})) \mid \zeta^{\omega} \\
&= \zeta \mid \zeta^{\omega},
\end{aligned}$$

for all $\omega, \zeta \in \mathcal{L}$, which means that $\mathcal{L}$ is implicative. □

Proposition 3.4. *Let A be an implicative SBG-algebra satisfying equation (3.8). Then every bipolar fuzzy completely closed SBG-ideal of A is also a BFISBG-ideal.*

Proof. Let Ψ be a bipolar fuzzy completely closed SBG-ideal of an implicative SBG-algebra $\mathcal{L}$. By definition, Ψ is a bipolar fuzzy SBG-ideal of $\mathcal{L}$. Note that

$$\Psi^{-}(0^{\zeta} \mid 0^{\zeta}) \leq \max\{\Psi^{-}(0), \Psi^{-}(\zeta)\} = \Psi^{-}(\zeta)$$

and

$$\Psi^{+}(0^{\zeta} \mid 0^{\zeta}) \geq \min\{\Psi^{+}(0), \Psi^{+}(\zeta)\} = \Psi^{+}(\zeta).$$

These inequalities imply that Ψ is a bipolar fuzzy closed SBG-ideal of $\mathcal{L}$. Hence, Ψ is a BFISBG-ideal of $\mathcal{L}$. □

Corollary 3.2. *Let $\mathcal{L}$ be a medial SBG-algebra satisfying equation (3.8). Then every bipolar fuzzy completely closed SBG-ideal of $\mathcal{L}$ is a BFISBG-ideal.*

Definition 3.5. *A bipolar fuzzy set Ψ on an SBG-algebra $\mathcal{L}$ is said to be a bipolar fuzzy p-ideal of $\mathcal{L}$ if it satisfies the condition:*

$$(\forall \omega, \zeta, \upsilon \in \mathcal{L}) \begin{pmatrix} \Psi^{-}(0) \leq \Psi^{-}(\omega) \\ \Psi^{+}(0) \geq \Psi^{+}(\omega) \\ \Psi^{-}(\omega) \leq \max\{\Psi^{-}(((\omega^{\upsilon} \mid \omega^{\upsilon}) \mid \zeta^{\upsilon}) \mid ((\omega^{\upsilon} \mid \omega^{\upsilon}) \mid \zeta^{\upsilon})), \Psi^{-}(\zeta)\} \\ \Psi^{+}(\omega) \geq \min\{\Psi^{+}(((\omega^{\upsilon} \mid \omega^{\upsilon}) \mid \zeta^{\upsilon}) \mid ((\omega^{\upsilon} \mid \omega^{\upsilon}) \mid |\zeta^{\upsilon})), \Psi^{+}(\zeta)\} \end{pmatrix}. \tag{3.9}$$

Definition 3.6. *Let $\mathcal{L}$ be an SBG-algebra. The subset*

$$\mathcal{A}^{+} = \{x \in A : 0^{\omega} \mid 0^{\omega} = 0\}$$

is called the BCA-part *of $\mathcal{L}$.*

Theorem 3.13. *Let $\mathcal{A} = \mathcal{A}^+$ be an SBG-algebra. Then every bipolar fuzzy p-ideal of $\mathcal{L}$ is a BFISBG-ideal of $\mathcal{L}$.*

Proof. Let Ψ be a bipolar fuzzy p-ideal of $\mathcal{L}$. Then

$$\begin{array}{rcl}
\Psi^-(\omega) & \leq & \max\{\Psi^-(((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega)) \mid ((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega))), \Psi^-(0)\} \\
& = & \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0), \Psi^-(0)\} \\
& = & \max\{\Psi^-(\omega_\zeta \mid \omega_\zeta), \Psi^-(0)\} \\
& = & \Psi^-(\omega_\zeta \mid \omega_\zeta),
\end{array}$$

and

$$\begin{array}{rcl}
\Psi^+(\omega) & \geq & \min\{\Psi^+(((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega)) \mid ((\omega_\zeta \mid \omega_\zeta) \mid (0 \mid \zeta^\omega))), \Psi^+(0)\} \\
& = & \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^0 \mid (\omega_\zeta \mid \omega_\zeta)^0), \Psi^+(0)\} \\
& = & \min\{\Psi^+(\omega_\zeta \mid \omega_\zeta), \Psi^+(0)\} \\
\omega & = & \Psi^+(\omega_\zeta \mid \omega_\zeta).
\end{array}$$

Hence Ψ is a BFISBG-ideal of $\mathcal{L}$. $\square$

Theorem 3.14. *Let $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ be a BFISBG-ideal of $\mathcal{A}$. Then, for every $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$, the bipolar fuzzy (θ, ϑ)-translation*

$$\Psi^{T_2}_{(\theta,\vartheta)} = (A, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$$

is also a BFISBG-ideal of $\mathcal{A}$.

Proof. Assume that $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$. Then for all $\omega \in \mathcal{A}$ and $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$, we have

$$\Psi^-(0) \leq \Psi^-(\omega) \quad \text{and} \quad \Psi^+(0) \geq \Psi^+(\omega).$$

It follows that

$$\Psi^-_{(\theta,T_2)}(0) = \Psi^-(0) - \theta \leq \Psi^-(\omega) - \theta = \Psi^-_{(\theta,T_2)}(\omega),$$
$$\Psi^+_{(\vartheta,T_2)}(0) = \Psi^+(0) - \vartheta \geq \Psi^+(\omega) - \vartheta = \Psi^+_{(\vartheta,T_2)}(\omega).$$

Now let $\omega, \zeta, \upsilon \in \mathcal{A}$. Since Ψ is a BFISBG-ideal, we have

$$\Psi^-(\omega) \leq \max\left\{\Psi^-(\mathfrak{q}), \Psi^-(\upsilon)\right\}, \quad \Psi^+(\omega) \geq \min\left\{\Psi^+(\mathfrak{q}), \Psi^+(\upsilon)\right\},$$

where

$$\mathfrak{q} = (\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon.$$

Subtracting θ and ϑ respectively, we obtain

$$\Psi^-_{(\theta,T_2)}(\omega) = \Psi^-(\omega) - \theta \leq \max\left\{\Psi^-(\mathfrak{q}) - \theta, \Psi^-(\upsilon) - \theta\right\} = \max\left\{\Psi^-_{(\theta,T_2)}(\mathfrak{q}), \Psi^-_{(\theta,T_2)}(\upsilon)\right\},$$

$$\Psi^+_{(\vartheta,T_2)}(\omega) = \Psi^+(\omega) - \vartheta \geq \min\left\{\Psi^+(\mathfrak{q}) - \vartheta, \Psi^+(\upsilon) - \vartheta\right\} = \min\left\{\Psi^+_{(\vartheta,T_2)}(\mathfrak{q}), \Psi^+_{(\vartheta,T_2)}(\upsilon)\right\}.$$

Therefore, the (θ, ϑ)-translation $f^{T_2}_{(\theta,\vartheta)} = (\mathcal{A}, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$ satisfies the conditions of a BFISBG-ideal of $\mathcal{A}$. $\square$

Theorem 3.15. *Let $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ be a bipolar fuzzy set on an SBG-algebra $\mathcal{A}$. If there exists a pair $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$ such that the (θ, ϑ)-translation*

$$\Psi^{T_2}_{(\theta,\vartheta)} = (\mathcal{A}, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$$

is a BFISBG-ideal of $\mathcal{A}$, then Ψ itself is a BFISBG-ideal of $\mathcal{A}$.

Proof. Suppose there exist $(\theta, \vartheta) \in [\pm, 0] \times [0, \mp]$ such that the (θ, ϑ)-translation

$$\Psi^{T_2}_{(\theta,\vartheta)} = (\mathcal{A}, \Psi^-_{(\theta,T_2)}, \Psi^+_{(\vartheta,T_2)})$$

is a BFISBG-ideal of $\mathcal{A}$. Then, for all $\omega, \zeta, \upsilon \in \mathcal{A}$, we have
$\Psi^-(0) \leq \Psi^-(\omega)$ and $\Psi^+(0) \geq \Psi^+(\omega)$.

$$\Psi^-(0) - \theta = \Psi^-_{(\theta,T_2)}(0) \leq \Psi^-_{(\theta,T_2)}(\omega) = \Psi^-(\omega) - \theta,$$

$$\Psi^+(0) - \vartheta = \Psi^+_{(\vartheta,T_2)}(0) \geq \Psi^+_{(\vartheta,T_2)}(\omega) = \Psi^+(\omega) - \vartheta.$$

Now let $x, y, z \in A$. Then

$$\begin{array}{rcl}
\Psi^-(\omega) - \theta & = & \Psi^-_{(\theta,T_2)}(\omega) \\
& \leq & \max\{\Psi^-_{(\theta,T_2)}((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-_{(\theta,T_2)}(\zeta)\} \\
& = & \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) - \theta, \Psi^-(\zeta) - \theta\} \\
& = & \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\zeta)\} - \theta,
\end{array}$$

$$\begin{array}{rcl}
\Psi^+(\omega) - \vartheta & = & \Psi^+_{(\vartheta,T_2)}(\omega) \\
& \geq & \min\{\Psi^+_{(\vartheta,T_2)}((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+_{(\vartheta,T_2)}(\upsilon)\} \\
& = & \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon) - \vartheta, \Psi^+(\upsilon) - \vartheta\} \\
& = & \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\} - \vartheta,
\end{array}$$

Hence $\Psi^-(\omega) \leq \max\{\Psi^-((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^-(\upsilon)\}$ and $\Psi^+(\omega) \geq \min\{\Psi^+((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon), \Psi^+(\upsilon)\}$. Hence $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$. □

Theorem 3.16. *Let $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ be a bipolar fuzzy set in $\mathcal{A}$. Then, $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$ if and only if for every $(\mathfrak{q}^-, \mathfrak{q}^+) \in [-1, 0] \times [0, 1]$, the sets $N_U(\Psi, \mathfrak{q}^-)$ and $P_L(\Psi, \mathfrak{q}^+)$ are implicative SBG-ideals of $\mathcal{A}$, whenever these sets are nonempty.*

Proof. Assume that $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$. Let $(\mathfrak{q}^-, \mathfrak{q}^+) \in [-1, 0] \times [0, 1]$ be such that the level sets $N_U(\Psi, \mathfrak{q}^-)$ and $P_L(\Psi, \mathfrak{q}^+)$ are nonempty. Consider any $\omega \in \mathcal{A}$ with $\omega \in N_U(\Psi, \mathfrak{q}^-)$, which implies that $\Psi^-(\omega) \geq \mathfrak{q}^-$. Since $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$, it follows that

$$\begin{array}{rcl}
\overline{\Psi^-}(0) & \leq & \overline{\Psi^-}(\omega) \\
1 - \Psi^+(0) & \leq & 1 - \Psi^+(\omega) \\
\Psi^+(0) & \geq & \Psi^+(\omega) \\
& \geq & \mathfrak{q}^-.
\end{array}$$

Hence, we have $0 \in N_U(\Psi, \mathfrak{q}^-)$.

Next, let $\omega, \zeta, \upsilon \in \mathcal{A}$ be such that

$$\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \upsilon \in N_U(\Psi, \mathfrak{q}^-).$$

Then

$$\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \geq t^-, \quad \text{and} \quad \Psi^-(\upsilon) \geq \mathfrak{q}^-.$$

Since $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$, it follows that

$$\begin{aligned}
\overline{\Psi^-}(\omega) &\leq \max\left\{\overline{\Psi^-}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, \overline{\Psi^-}(\upsilon)\right\},\\
1 - \Psi^-(\omega) &\leq \max\left\{1 - \Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, 1 - \Psi^-(\upsilon)\right\},\\
1 - \Psi^-(\omega) &\leq 1 - \min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, \Psi^-(\upsilon)\right\},\\
\Psi^-(\omega) &\geq \min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big),\, \Psi^-(\upsilon)\right\} \geq \mathfrak{q}^-.
\end{aligned}$$

Hence, $\omega \in N_U(\Psi, \mathfrak{q}^-)$. Therefore, $N_U(\Psi, \mathfrak{q}^-)$ is an implicative SBG-ideal of $\mathcal{A}$.

Let $\omega \in \mathcal{A}$ be such that $\omega \in P_L(\Psi, \mathfrak{q}^+)$. Then $\Psi^+(\omega) \geq \mathfrak{q}^+$. Since $\Psi = (\mathcal{A}, \Psi^-, \Psi^+)$ is a BFISBG-ideal of $\mathcal{A}$, we have

$$\begin{aligned}
\overline{\Psi^+}(0) &\geq \overline{\Psi^+}(\omega),\\
1 - \Psi^+(0) &\geq 1 - \Psi^+(\omega),\\
\Psi^+(0) &\leq \Psi^+(\omega) \leq \mathfrak{q}^+.
\end{aligned}$$

Hence, $0 \in P_L(\Psi, \mathfrak{q}^+)$.

Next, let $\omega, \zeta, \upsilon \in \mathcal{A}$ be such that $\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \upsilon \in P_L(f, t^+)$. Then $\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \geq \mathfrak{q}^+$ and $\Psi^+(\upsilon) \geq \mathfrak{q}^+$. Since $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$, we have

$$\begin{aligned}
\overline{\Psi^+}(\omega) &\leq \max\{\overline{\Psi^+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \overline{\Psi^+}(\upsilon)\}\\
1 - \Psi^+(\omega) &\leq \max\{1 - \Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), 1 - \Psi^+(\upsilon)\}\\
1 - \Psi^+(\omega) &\leq 1 - \min\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\}\\
\Psi^+(\omega) &\geq \min\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\}\\
&\geq \mathfrak{q}^+.
\end{aligned}$$

Hence, $\omega \in P_L(\Psi, \mathfrak{q}^+)$. Therefore, $P_L(\Psi, \mathfrak{q}^+)$ is an implicative SBG-ideal of A.

Conversely, suppose that for all $(\mathfrak{q}^-, \mathfrak{q}^+) \in [-1, 0] \times [0, 1]$, the sets $N_U(\Psi, \mathfrak{q}^-)$ and $P_L(\Psi, \mathfrak{q}^+)$ are implicative SBG-ideals of $\mathcal{A}$, provided they are nonempty.

Let $\omega \in \mathcal{A}$. Since $\Psi^-(\omega) \in [-1, 0]$, choose $\mathfrak{q}^- = \Psi^-(\omega)$. Then

$$\Psi^-(0) \geq \Psi^-(\omega) = \mathfrak{q}^-,$$

which implies $0 \in N_U(\Psi, \mathfrak{q}^-) \neq \emptyset$. By assumption, $N_U(\Psi, \mathfrak{q}^-)$ is an implicative SBG-ideal of $\mathcal{A}$, and therefore $0 \in N_U(\Psi, \mathfrak{q}^-)$ holds. Consequently,

$$\Psi^-(0) \geq \mathfrak{q}^- = \Psi^-(\omega),$$

and so

$$\overline{\Psi^-}(0) = -1 - \Psi^-(0) \leq -1 - \Psi^-(\omega) = \overline{\Psi^-}(\omega).$$

Now let $x, y, z \in A$. Note that

$$\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \Psi^-(\upsilon) \in [-1, 0].$$

Choose

$$\mathfrak{q}^- = \min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^-(\upsilon)\right\}.$$

Then

$$\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \geq t^-,$$

and

$$\Psi^{-}(\upsilon) \geq \mathfrak{q}^{-},$$

so

$$\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \upsilon \in N_U(\Psi, \mathfrak{q}^-) \neq \emptyset.$$

By assumption, $N_U(\Psi, \mathfrak{q}^-)$ is an implicative SBG-ideal of $\mathcal{A}$, and hence $\omega \in N_U(\Psi, \mathfrak{q}^-)$, which implies

$$\Psi^-(\omega) \geq \mathfrak{q}^- = \min\left\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^-(\upsilon)\right\}.$$

By Lemma 2.1 (1), we have

$$\begin{aligned}
\overline{\Psi^-}(\omega) &= -1 - \Psi^-(\omega) \\
&\leq -1 - \min\{\Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^-(\upsilon)\} \\
&= \max\{-1 - \Psi^-\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), -1 - \Psi^-(\upsilon)\} \\
&= \max\{\overline{\Psi^-}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \overline{\Psi^-}(\upsilon)\}.
\end{aligned}$$

Let $\omega, \zeta \in \mathcal{A}$. Since $\Psi^+(\omega) \in [0,1]$, choose $\mathfrak{q}^+ = \Psi^+(\omega)$. Then, as

$$\Psi^+(0) \leq \Psi^+(\omega) = \mathfrak{q}^+,$$

it follows that $\omega \in P_L(\Psi, \mathfrak{q}^+) \neq \emptyset$. By assumption, $P_L(f, t^+)$ is an implicative SBG-ideal of $\mathcal{A}$, hence $0 \in P_L(\Psi, \mathfrak{q}^+)$.

Therefore,

$$\Psi^+(0) \geq \mathfrak{q}^+ = \Psi^+(\omega),$$

and thus

$$\overline{\Psi^+}(0) = 1 - \Psi^+(0) \geq 1 - \Psi^+(\omega) = \overline{\Psi^+}(\omega).$$

Let $\omega, \zeta, \upsilon \in \mathcal{A}$. Note that

$$\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \Psi^+(\upsilon) \in [0,1].$$

Choose

$$t^+ = \max\left\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\right\}.$$

Then,

$$\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big) \leq \mathfrak{q}^+$$

and

$$\Psi^+(\upsilon) \leq t^+,$$

which implies

$$\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \quad \upsilon \in P_L(\Psi, \mathfrak{q}^+) \neq \emptyset.$$

By assumption, $P_L(\Psi, \mathfrak{q}^+)$ is an implicative SBG-ideal of $\mathcal{A}$, and therefore $x \in P_L(\Psi, \mathfrak{q}^+)$, which yields

$$\Psi^+(\omega) \geq \mathfrak{q}^+ = \max\left\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\right\}.$$

By Lemma 2.1 (1), we have

$$\begin{aligned}
\overline{\Psi^+}(\omega) &= 1-\Psi^+(\omega)\\
&\leq 1-\max\{\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \Psi^+(\upsilon)\}\\
&= \min\{1-\Psi^+\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), 1-\Psi^+(\upsilon)\}\\
&= \min\{\overline{\Psi^+}\big((\omega_\zeta \mid \omega_\zeta)^\upsilon \mid (\omega_\zeta \mid \omega_\zeta)^\upsilon\big), \overline{\Psi^+}(\upsilon)\}.
\end{aligned}$$

Hence $\overline{\Psi} = (\mathcal{A}, \overline{\Psi^-}, \overline{\Psi^+})$ is a BFISBG-ideal of $\mathcal{A}$. □

4. Conclusion

In this paper, we have systematically developed the theory of bipolar fuzzy Sheffer stroke BG-algebras, focusing on the interplay between bipolar fuzzy sets and algebraic structures defined by the Sheffer stroke operation. By introducing and analyzing the notions of bipolar fuzzy subalgebras and various classes of SBG-ideals, we have established clear criteria under which the level sets of bipolar fuzzy subsets correspond to conventional algebraic substructures and ideals. This approach provides a rigorous algebraic foundation for representing and manipulating dual (positive and negative) information within SBG-algebras.

Our results demonstrate that the framework of bipolar fuzzy sets not only extends the expressive power of fuzzy algebraic systems, but also enables a more nuanced treatment of uncertainty and contradiction in logical and computational contexts. The equivalence conditions and closure properties we have proven for bipolar fuzzy (implicative, sub-implicative, completely closed, and p-) SBG-ideals contribute to a deeper understanding of the algebraic behavior of these fuzzy structures.

The integration of bipolar fuzzy logic with Sheffer stroke BG-algebras, as presented here, paves the way for further research into the algebraic modeling of complex systems characterized by ambiguity, duality, or inconsistent information. Future investigations may include the study of categorical aspects, the extension to other algebraic systems, or the application of these results to areas such as soft computing, knowledge representation, and decision support systems.

In summary, this work highlights the foundational role of level set techniques in relating bipolar fuzzy membership functions to algebraic properties, thereby enriching both the theoretical landscape and the potential applications of fuzzy algebraic systems within mathematics and computer science.

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Hsu-UNIFIED STRUCTURE MANIFOLDS COUPLED WITH A GENERALIZED WINTGEN TYPE INEQUALITY

MOHD DANISH SIDDIQI * AND MOHD NOMAN ALI

Abstract. The goal of this research note is that, Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection has been investigated. The Riemannian curvature tensor, Ricci curvature tensor, and scalar curvature of the Hsu-unified structure manifold with a semi-symmetric non-metric S-connection are transformed into their respective formulations. Mainly, we derive the generalized Wintgen inequalities for submanifolds in Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection. In addition, we discuss the Wintgen inequality for totally umbilical submanifolds of Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection. Also, we deduce the same inequality for the almost complex manifold, almost tangent manifold, almost product manifold and GF-manifold, and π-structure ambient manifolds. Finally, we deduced a universal lower bound of the norms of specific functions involving scalar curvature, normalized scalar curvature, and mean curvature to establish topological obstructions for submanifolds in Hsu-unified structure manifolds with a semi-symmetric non-metric S-connection.

Keywords: Hsu-unified structure manifold, semi-symmetric non-metric S-connection, Submanifolds, Generalized Wintgen inequalities, Totally umbilical submanifolds.

MSC (2020): 53C15, 53C20, 53C25, 53C44.

1. Introduction

In the year 1924, Friedmann and Schouten [9] opened a new door for researchers by working on differentiable manifolds with semi-symmetric linear connection ∇. Later on, such connections were given geometrical meaning by Bartolotti [3] and Hayden [14] produced the discourse of semi-symmetric metric connection if $\nabla \mathrm{g} = 0$. After a lengthy break, Prvanonic [28] started the research of a semi-symmetric connection ∇ satisfying $\nabla \mathrm{g} \neq 0$ under the name pseudo-metric semi-symmetric connection, and Andonie [2] followed. A semi-symmetric non-metric connection was introduced and explored by Agashe and Chafle [1] in 1992. Yano [39] improved the semi-symmetric non-metric connection study on Riemannian manifolds in a systematic manner, and other geometers looked into it further (see [15], [13], [29], [30], [37] for details). Moreover, a quarter symmetric or pseudo-metric connection satisfies the specific

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form of torsion mentioned below:

$$\mathcal{T}(\mho_1, \mho_2) = \eta(\mho_1)\varphi\mho_2 - \eta(\mho_2)\varphi\mho_1, \tag{1.1}$$

where η is a 1-form and φ is a (1,1) tensor field. A quarter symmetric or pseudo-metric connection can be further classified as either metric or non-metric, as we have mentioned above. If in the quarter-symmetric condition, we have $\varphi\mho = \mho$ (the identity transformation), the connection becomes a semi-symmetric connection. Therefore, in the present study, we have focused on this specific case with the condition of non-metric $\nabla \mathsf{g} \neq 0$.

The Wintgen inequality is a sharp geometric inequality for surfaces in 4-dimensional Euclidean space $\mathbb{E}^4$ that involves major extrinsic invariants like normal curvature, square mean curvature, and shape operator, as well as intrinsic invariants like sectional curvatures, Ricci curvatures, and scalar curvatures.

P. Wintgen [38] derived an inequality for any surface $\mathbb{S}^2$ in $\mathbb{E}^4$ by the following way

$$||\mathbb{H}||^2 \geq \Omega + |\Omega_{Nor}|,$$

where the normal curvature is represented by Ω_{Nor} and the Gaussian curvature by Ω. To indicate the squared mean curvature, use $||\mathbb{H}||^2$.

In 1983, Guadalupe and Rodriguez [11] developed an extension for arbitrary codimension m in real space forms $\overline{M}^{m+2}(c)$ by

$$||\mathbb{H}||^2 + c \geq \Omega + |\Omega_{Nor}|. \tag{1.2}$$

In real space form $N^{n+2}(c)$ with $n \geq 2$, De Smet, Dillen, Verstraelen, and Vrancken conjectured the generalized Wintgen inequality for submanifolds [8]. Ge and Tang [10] also established this conjecture, which is also known as the DDVV conjecture for real space form. Several geometers have recently proven DDVV inequality for various classes of submanifolds in various ambient manifolds (see [21, 7, 31, 32, 33, 34]). Recently, Siddiqi et al. [32] established a generalized Wintgen inequality for quaternionic bi-slant submanifolds and QR-submanifolds (with minimal codimension) in quaternion space forms.

In 1960 C. Hsu [16] constructed a new structure on differentiable manifold of smooth class, in terms of any tensor field φ of type $(0,2)$-type and a vector field $\mathcal{U}$, named Hsu-unified structure manifold ($\mathcal{HUS}$-manifolds). Then Singh also discussed hypersurfaces of an Hsu-unified structure [35]. Nivas et al. ([22], [23]) are also explored some properties of $\mathcal{HUS}$-manifolds endowed with a non-metric connection. After that, other authors also studied $\mathcal{HUS}$-manifolds endowed with a induced non-metric connection (see [4], [19]).

On the other hand, in this research paper, we have derived the generalized Wintgen inequality for totally umbilical submanifolds of geometric structures such as almost complex, tangent, product, GF-manifolds, providing a broad framework unified under Hsu-unified structure with a semi-symmetric non-metric S-connection. In addition, exploring some topological obstructions through integral inequalities involving Betti numbers that connect the scalar curvature (an intrinsic property), the normal scalar curvature (an extrinsic property), and the squared mean curvature (also an extrinsic property), with a semi-symmetric non-metric S-connection.

The purpose of this monograph is to further investigate submanifolds in of type $(0,2)$-type and a vector field $\mathcal{U}$ endowed with a semi-symmetric non-metric S-connection ($SSNMs$) by means of generalized Wintgen inequalities.

2. Preliminaries

Let $\mathcal{H}^n$ be C^∞ class, $(n = 2p)$-even dimensional differentiable manifold. Let φ be a vector valued real linear function of differentiable class C^∞ satisfies

$$\varphi^2\mho_1 = \alpha^r I(\mho_1) \tag{2.3}$$

for any vector field $\mho_1 \in \chi(\mathcal{H}^n)$.
In addition, a Riemannian metric g, such that

$$\mathsf{g}(\varphi\mho_1, \varphi\mho_2) = \alpha^r \mathsf{g}(\mho_1, \mho_2), \tag{2.4}$$

where α is a real or complex number and $0 \leq r \leq n$ and I indicates the identity operator. Then $(\mathcal{H}^n, \varphi, \mathsf{g}, r, \alpha)$ is said to be a $\mathcal{HUS}$-manifolds [16].

Let Φ in $\mathcal{HUS}$-manifolds $\mathcal{H}^n$ is a $(0,2)$-type symmetric tensor field expressed as

$$\Phi(\mho_1, \mho_2) = \mathsf{g}(\varphi\mho_1, \mho_2) = \Phi(\mho_1, \mho_2) = \mathsf{g}(\mho_1, \mho_2), \tag{2.5}$$

and we have

$$\Phi(\varphi\mho_1, \mho_2) = \alpha^r \mathsf{g}(\mho_1, \mho_2). \tag{2.6}$$

Also, an $\mathcal{HUS}$-manifolds $(\mathcal{H}^n, \varphi, \mathsf{g}, r, \alpha)$ is satisfied [16]

$$\nabla_{\mho_1}\zeta = \varphi\mho_1. \tag{2.7}$$

Different structures manifolds are presented by equation (2.3) and (2.4) according to the values of r and α [16, 4, 27].

Value of r	Value of α	Manifolds type
$r = \pm 1$	$\alpha = -1 \quad or \quad \alpha = \pm i$	almost complex manifold
$r \neq 0$	$\alpha = 0$	almost tangent manifold
$r = 0$	$\alpha = +1$	almost product manifold
$r = 2$	$\alpha \neq 0$	GF-manifold [27] and π-structure

3. Semi-symmetric non-metric s-connection

An affine metric connection $\nabla^\sharp$ obeys

$$(\nabla^\sharp{}_{\mho_1}\varphi)\mho_2 = \eta(\mho_2)\mho_1 - \mathsf{g}(\mho_1, \mho_2)\zeta, \tag{3.8}$$

is said to be s-connection [25], where ζ is a 1-form and η is a vector field associated with ζ.

A s-connection $\nabla^\sharp$ is said to be a $SSNMs$-connection [24] if and only if

$$(\nabla^\sharp{}_{\mho_1}\varphi)\mho_2 = \nabla_{\mho_1}\mho_2 - \eta(\mho_2)\mho_1 - \mathsf{g}(\mho_1, \mho_2)\zeta. \tag{3.9}$$

Moreover,

$$(\nabla^{\sharp}{}_{\mho_1}\varphi)(\mho_2,\mho_3) = 2\eta(\mho_2)\mathsf{g}(\mho_1,\mho_3) + 2\eta(\mho_3)\mathsf{g}(\mho_1,\mho_2), \tag{3.10}$$

adopting (3.9), the torsion tensor $\mathcal{T}$ of $\mathcal{H}^n$ with respect to the connection $\nabla^{\sharp}$, is represented by

$$\mathcal{T}(\mho_1,\mho_2) = \eta(\mho_1)\mho_2 - \eta(\mho_2)\mho_1, \tag{3.11}$$

where

$$\mathcal{T}(\mho_1,\mho_2) = \nabla^{\sharp}_{\mho_1}\mho_2 - \nabla^{\sharp}_{\mho_2}\mho_1 - [\mho_1,\mho_2].$$

We know that $\mathsf{g}(\mho_1,\zeta) = \eta(\mho_1)$. In view of (3.10) as

$$\nabla^{\sharp}_{\mho_1}\mho_2 = \nabla_{\mho_1}\mho_2 + \mathcal{T}(\mho_1,\mho_2). \tag{3.12}$$

Then

$$\mathcal{T}(\mho_1,\mho_2) = -\eta(\mho_1)\mho_2 - \mathsf{g}(\mho_1,\mho_2)\zeta. \tag{3.13}$$

Let us explain

$$\mathcal{T}(\mho_1,\mho_2,\mho_3) = \mathsf{g}(\mathcal{T}(\mho_1,\mho_2),\mho_3). \tag{3.14}$$

In light of (3.13) and (3.14) we obtain

$$(\nabla^{\sharp}{}_{\mho_1}\eta) = (\nabla_{\mho_1}\eta)\mho_2 + \eta(\mho_1)\eta(\mho_2) + \mathsf{g}(\mho_1,\mho_2)\eta(\zeta). \tag{3.15}$$

4. H*su*-unified structure manifold with a SSNMs-connection

Analogous to the definition of curvature tensor of $\mathcal{H}^n$ with respect to the Levi-Civita connection ∇, we express the curvature tensor of $\mathcal{H}^n$ with a new SSNMs-connection $\nabla^{\sharp}$, is given as

$$\mathcal{R}^{\sharp}(\mho_1,\mho_2,\mho 3) = \nabla^{\sharp}_{\mho_1}\nabla^{\sharp}_{\mho_2}\mho_3 - \nabla^{\sharp}_{\mho_2}\nabla^{\sharp}_{\mho_1}\mho_3 - \nabla^{\sharp}_{[\mho_1,\mho_2]}\mho 3. \tag{4.16}$$

Therefore, adopting (2.7), (3.15) and 3.14), the curvature tensor $\mathcal{R}^{\sharp}$ with respect to the SSNMs-connection $\nabla^{\sharp}$ is given by

$$\begin{aligned}\mathcal{R}^{\sharp}(\mho_1,\mho_2,\mho 3) = \mathcal{R}(\mho_1,\mho_2,\mho 3) &- [(\nabla_{\mho_1}\eta)\mho_2 + \eta(\mho_1)\eta(\mho_3) + \mathsf{g}(\mho_1,\mho_3)\eta(\zeta)]\mho_2 \\ &+ [(\nabla_{\mho_2}\eta)\mho_3 + \eta(\mho_2)\eta(\mho_3) + \mathsf{g}(\mho_2,\mho_3)\eta(\zeta)]\mho_1 \\ &- [\nabla_{\mho_1}\zeta - \eta(\mho_1)\zeta]\mathsf{g}(\mho_2,\mho_3) + +\mathsf{g}(\mho_1,\mho_3)[\nabla_{\mho_2}\zeta - \eta(\mho_2)\zeta)],\end{aligned} \tag{4.17}$$

where $\mathcal{R}$ is curvature tensor of $\mathcal{H}^n$ with respect to the Riemannian connection ∇.

Let $\mathcal{H}^n$ be an n-dimensional $\mathcal{HUS}$-manifolds. Then, with regard to the SSNMs-connection $\nabla^{\sharp}$, the scalar curvature R_{scal} and Ricci tensor $\mathcal{S}_{ric}$ of the manifold $\mathcal{H}^n$ are given as

$$\mathcal{S}^{\sharp}_{ric}(\mho_1,\mho_2) = \sum_{i=1}^{n} \varepsilon_i \mathsf{g}(\mathcal{R}^{\sharp}(I_i,\mho_1,\mho_2), I_i) \tag{4.18}$$

and

$$R^{\sharp}_{scal} = \sum_{i=1}^{n} \varepsilon_i \mathcal{S}^{\sharp}_{ric}(I_i, I_i), \tag{4.19}$$

where $\{I_1, I_2 \cdots, I_n\}$ is an orthonormal frame with $\mathsf{g}(I_i, I_i) = \varepsilon_i$.

Now, we gain the next result:

Theorem 4.1. *If $(\mathcal{H}^n, \varphi, g, r, \alpha)$ be a n-dimensional $\mathcal{HUS}$-manifolds, the Ricci tensor $\mathcal{S}^{\sharp}_{ric}$ and the scalar curvature $R^{\sharp}_{scal}$ endowed with a SSNMs-connection $\nabla^{\sharp}$ are given by*

$$\mathcal{S}^{\sharp}_{ric}(\mho_1, \mho_2) = \mathcal{S}_{ric}(\mho_1, \mho_2) + (n-1)B(\mho_1, \mho_2) \tag{4.20}$$
$$+\alpha^r g(\mho_1, \mho_2) + g(\varphi\mho_1, \mho_2) - \eta(\mho_1)\eta(\mho_2),$$

where $B(\mho_1, \mho_2) = (\nabla_{\mho_1}\eta)\mho_2 + \eta(\mho_1)\eta(\mho_2) + g(\mho_1, \mho_2)\eta(\zeta)$.

$$R^{\sharp}_{scal} = R_{scal} + \alpha^r(n-1)(n+2). \tag{4.21}$$

where $\mathcal{S}_{ric}$ and R_{scal} denotes the Ricci tensor and scalar curvature with respect to the Levi-Civita connection ∇, respectively.

Proof. In the light of (4.17), (4.18) and (4.19), easily we obtain (4.20) and (4.21). □

5. Wintgen inequalities for submanifolds in Hsu-unified structure manifold with a $SSNMs$-connection

The generalized Wintgen inequalities for submanifolds in a $\mathcal{HUS}$-manifolds with a $SSNMs$-connection are derived in this section.

Let $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional $\mathcal{HUS}$-manifolds with a $SSNMs$-connection and induced metric g. Let $\nabla^{\sharp}$ and $\nabla^{\perp}$ represent the induced connections on the tangent bundle $T\mathcal{H}'$ and $T\mathcal{H}'^{\perp}$ of $\mathcal{H}'$, respectively and indicate the second fundamental form of $\mathcal{H}'$ by $\hbar$, for all $\mho_1, \mho_2 \in \Gamma(T\mathcal{H}')$ and $\mathbb{N} \in \Gamma(T^{\perp}\mathcal{H}')$.

Recall the Gauss and Weingarten formulas by

$$\nabla^{\sharp}_{\mho_1}\mho_2 = \nabla_{\mho_1}\mho_2 + \hbar(\mho_1, \mho_2), \tag{5.22}$$

and

$$\nabla^{\sharp}_{\mho_1}\mathbb{N} = -\Lambda_{\mathbb{N}}\mho_1 + \nabla^{\perp}_{\mho_1}\mathbb{N}, \tag{5.23}$$

where the shape operator of $\mathcal{H}'$ with respect to $\mathcal{N}$ is notated using $\Lambda_{\mathbb{N}}$. The equation that follows is widely recognized.

$$\mathsf{g}(\Lambda_{\mathbb{N}}\mho_1, \mho_2) = \mathsf{g}(\hbar(\mho_1, \mho_2), \mathbb{N}), \qquad \forall\ \mho_1, \mho_2 \in \Gamma(T\mathcal{H}'), \qquad \mathbb{N} \in \Gamma(T^{\perp}\mathcal{H}'). \tag{5.24}$$

Now, let the induced SSNMs-connection $\nabla^{\sharp}$ on $\mathcal{H}'$ indicated by $\nabla^{\sharp^{\mathcal{H}'}}$ and the induced Levi-Civita connection ∇ on $\mathcal{H}$ identified by $\nabla^{\mathcal{H}}$. Next, by the equation of Gauss

$$\begin{aligned}\mathcal{R}^{\sharp}(\mho_1, \mho_2, \mho_3, \mho_4) &= \mathcal{R}(\mho_1, \mho_2, \mho_3, \mho_4) - \mathsf{g}(\hbar(\mho_1, \mho_4), \hbar(\mho_3, \mho_3)) \\ &\quad + \mathsf{g}(\hbar(\mho_1, \mho_3), \hbar(\mho_2, \mho_4))\end{aligned} \tag{5.25}$$

for all $\mho_1, \mho_2, \mho_3, \mho_4 \in T\mathcal{H}'$ and $\hbar$ is the second fundamental form of $\mathcal{H}'$ in $\mathcal{H}$ with respect to the $\nabla^{\sharp}$.

Let $\{I_1, \ldots, I_m\}$ and $\{I_{m+1}, \ldots, I_n\}$ indicated a local orthonormal tangent frame of the tangent bundle $T\mathcal{H}'$ of $\mathcal{H}'$ and a local orthonormal normal frame of the normal bundle $T^{\perp}\mathcal{H}'$ of $\mathcal{H}'$ in $\mathcal{H}$.

The mean curvature vector Π of $\mathcal{H}'$ should be defined through

$$\Pi = \sum_{i=1}^{m} \frac{1}{m} \hbar(I_i, I_i) \tag{5.26}$$

as well as the squared norm of the second fundamental form by

$$||\hbar||^2 = \sum_{i,j=1}^{m} \mathtt{g}\big(\hbar(I_i, I_j), \hbar(I_i, I_j)\big)^2. \tag{5.27}$$

At $p \in \mathcal{H}'$, we express the scalar curvature R_{scal} as

$$R_{scal} = \sum_{1 \le i < j \le m} \mathcal{R}(I_i, I_j, I_j, I_i) \tag{5.28}$$

and specify the normalized scalar curvature Ω of $\mathcal{H}'$ by

$$\Omega = \frac{2R_{scal}}{m(m-1)} = \frac{2}{m(m-1)} \sum_{1 \le i < j \le m} \mathbb{K}(I_i \wedge I_j) \tag{5.29}$$

where $\mathbb{K}$ is the sectional curvature function on $\mathcal{H}'$.

The following expression defines the scalar normal curvature $\mathbb{K}_{\mathbb{N}}$ in terms of the second fundamental form's components.[21]

$$\mathbb{K}_{\mathbb{N}} = \sum_{1 \le i < j \le m} \sum_{1 \le r < s \le n} (\sum_{k=1}^{m} \hbar^r_{jk} \hbar^s_{ik} - \hbar^r_{ik} \hbar^s_{jk})^2. \tag{5.30}$$

Additionally, the normalized scalar normal curvature has the following relationship.

$$\Omega_{Nor} = \frac{2}{m(m-1)} \sqrt{\mathbb{K}_{\mathbb{N}}}. \tag{5.31}$$

Now, we demonstrate the generalized Wintgen inequality for submanifolds of Hsu-unified structure manifold which possess an SSNMs connection.

Theorem 5.1. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \le \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m}. \tag{5.32}$$

Proof. Assume that $\{I_1, \ldots, I_m\}$ and $\{I_{m+1}, \ldots, I_n\}$ denotes the local orthonormal tangent frame and local orthonormal normal frame on $\mathcal{H}'$ respectively. Then, from (5.25) and Gauss equation, we have

$$\begin{aligned} \sum_{1 \le i < j \le m} \mathcal{R}^\sharp(I_i, I_j, I_j, I_i) \quad &= \quad R_{scal} + \alpha^r(m-1)(m+2) \\ &+ \quad \sum_{r=m+1}^{n} \sum_{1 \le i < j \le m} \Big[h^r_{ii} h^r_{jj} - (h^r_{ij})^2\Big]. \end{aligned} \tag{5.33}$$

Also

$$2\mathcal{R}^\sharp_{scal} = \sum_{1 \le i < j \le m} \mathcal{R}^\sharp(I_i, I_j, I_j, I_i). \tag{5.34}$$

Using (5.33) and (5.34), we obtain

$$2\mathcal{R}^{\sharp}_{scal} = R_{scal} + \alpha^r(m-1)(m+2) + \sum_{r=m+1}^{n} \sum_{1\le i<j\le m} \Big[h^r_{ii}h^r_{jj} - (h^r_{ij})^2\Big]. \tag{5.35}$$

We also note that

$$m^2||\mathbb{H}||^2 = \sum_{r=m+1}^{n} \Big(\sum_{i=1}^{m} h^r_{ii}\Big)^2 = \frac{1}{m-1}\sum_{r=m+1}^{n} \sum_{1\le i<j\le m} (h^r_{ii} - h^r_{jj})^2 + \frac{2m}{m-1}\sum_{r=m+1}^{n} \sum_{1\le i<j\le m} h^r_{ii}h^r_{jj}. \tag{5.36}$$

In [20], Lu proved the Normal Scalar Curvature Conjecture and the Böttcher-Wenzel Conjecture. He also develops a new Bochner formula and uses the results to establish new pinching theorems for minimal submanifolds in spheres.
Therefore, from (page 6, Lemma 1, [20]) it is known

$$\sum_{r=m+1}^{n} \sum_{1\le i<j\le m} (\hbar^r_{ii} - \hbar^r_{jj})^2 + 2m \sum_{r=m+1}^{n} \sum_{1\le i<j\le m} (\hbar^r_{ij})^2 \ge 2m\Big[\sum_{m+1\le r<s\le n} \sum_{1\le i<j\le m} (\sum_{k=1}^{m} (\hbar^r_{jk}\hbar^s_{ik} - \hbar^r_{ik}\hbar^s_{jk}))^2\Big]^{\frac{1}{2}} \tag{5.37}$$

Thanks to (5.36), (5.37) and (5.30), we have

$$m^2||\mathbb{H}||^2 - m^2\Omega_{Nor} \ge \frac{2m}{m-1}\sum_{r=m+1}^{n} \sum_{1\le i<j\le m} [\hbar^r_{ii}\hbar^r_{jj} - (\hbar^r_{ij})^2]. \tag{5.38}$$

Hence, taking view of (5.31), (5.35) and (5.38), we find

$$\Omega_{Nor} - ||\mathbb{H}||^2 \le \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m} - \Omega$$

whereby proving the inequality (5.32). □

Theorem 5.2. *If $\mathcal{H}'$ be m-dimensional submanifold of a n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a SSNMs-connection. The equality holds if and only if the shape operators of $\mathcal{H}'$ in $\mathcal{H}$ take the following forms with the suitable orthonormal frames*

$$\{b_1 \cdots b_m\} \quad and \quad \{b_{m+1} \cdots b_n, b_{n+1} = \zeta\}$$

$$\Lambda_{b_m+1} = \begin{pmatrix} \theta_1 & \delta & 0 & \dots & 0 \\ \delta & \theta_1 & 0 & \dots & 0 \\ 0 & 0 & \theta_1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_1 \end{pmatrix}, \tag{5.39}$$

$$\Lambda_{b_m+2} = \begin{pmatrix} \theta_2+\delta & 0 & 0 & \dots & 0 \\ 0 & \theta_2-\delta & 0 & \dots & 0 \\ 0 & 0 & \theta_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_2 \end{pmatrix}, \tag{5.40}$$

$$\Lambda_{b_m+3} = \begin{pmatrix} \theta_3 & 0 & 0 & \dots & 0 \\ 0 & \theta_3 & 0 & \dots & 0 \\ 0 & 0 & \mu_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \theta_3 \end{pmatrix}, \quad \Lambda_{b_m+4} = \cdots = \Lambda_{b_{2m}} = \Lambda_{b_{2m+1}} = 0, \tag{5.41}$$

where $\theta_1, \theta_2, \theta_3$, and δ are real numbers.

Example 5.1. [6] *Totally umbilical submanifolds of real space forms, trivially realize the equality in (1.2). For a classification of totally umbilical submanifolds in real space forms.*

Example 5.2. *If M^2 is a surface in a real space form $N^4(c)$ with an ellipse of curvature a circle, then the equality is realized in (1.2)* [8].

Example 5.3. *A special case of the surfaces in Example 5.2 are the superminimal (i.e. minimal and ellipse of curvature a circle) surfaces in R^4. Also a cylinder on a superminimal surface in R^4 satisfies equality in (1.2)* [8].

As a consequence of Theorem 5.1 and the above table, we give the following results:

Corollary 5.1. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional almost complex manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} - \frac{(m+2)}{m}. \tag{5.42}$$

Or

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)i}{m}. \tag{5.43}$$

Corollary 5.2. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional almost tangent manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)}. \tag{5.44}$$

Corollary 5.3. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional almost product manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m}. \tag{5.45}$$

Corollary 5.4. *If $\mathcal{H}'$ be m-dimensional submanifold of n-dimensional GF-manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad ||\mathbb{H}||^2 + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^2(m+2)}{m}. \tag{5.46}$$

Remark 5.1. *In fact for particular values of r and α we can turn up the similar equality condition for almost complex manifold, almost tangent manifold, almost product manifold and GF-manifold and π-structure manifolds, respectively.*

Again, in view of Theorem 5.1, we can turn up the following inequalities for the totally umbilical submanifolds as well.

Theorem 5.3. *If $\mathcal{H}'$ be m-dimensional totally umbilical submanifold of n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\Omega_{Nor} + \Omega \quad \leq \quad + \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m}. \tag{5.47}$$

Moreover, we gain the same inequality for the various manifolds in the following table:

Value of r and α	Manifolds type	Type of Inequality
$r = \pm 1$, $\alpha = -1$ or $\alpha = \pm i$	almost complex manifold	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)i}{m}$
$r \neq 0, \alpha = 0,$	almost tangent manifold	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)}$
$r = 0$, $\alpha = +1$	almost product manifold	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m}$
$r = 2$, $\alpha \neq 0$	GF-manifold [27] and π-structure	$\Omega_{Nor} + \Omega \leq \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^2(m+2)}{m}$

6. Some Implications

In this section, we use universal lower bounds of the norms of specific functions involving scalar curvature, normalized scalar curvature, and mean curvature to establish topological obstructions for be m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection. The Betti numbers are used to express these barriers.

By extending Wintgen's inequality in the previously given form, Guadalupe and Rodriguez [12] created an inequality for compact surfaces in a simple way. This inequality establishes a connection between the integral of $c + \mathbb{H}^2 - |\Omega_{Nor}|$ and the Euler-Poincar´e characteristic of the surface.

In (Theorem 1, [26]), Onti et al. used universal lower limits on the $L^{m/2}$-norms of various functions involving Ω, $\mathbb{H}$, and $\Omega^{\perp}$ to deduce several topological obstacles for compact m-dimensional submanifolds. The Betti numbers are used to describe these barriers. $\mathcal{H}^{m+n}(c)$ for compact submanifolds into space, where $c \geq 0$. It is assumed that every manifold examined in this work is oriented, connected without limits, and has an i-th Betti number $B_i(\mathcal{H}^m; \mathbb{F})$ over an arbitrary coefficient field $\mathbb{F}$.

Onti et al. [26], apply for any $1 \leq k \leq n-1$, but it generally fails for $k = n$, where the involved norm vanishes precisely for Wintgen ideal submanifolds. They demonstrated this by providing a method of constructing new compact 3-dimensional minimal Wintgen ideal submanifolds in even-dimensional spheres. Specifically, they proved that such submanifolds exist in $\mathbb{S}^6$ with arbitrarily large first Betti number.

Remark 6.1. *Let m -diemnsional Riemannian manifold M^m denote by $\lambda_1 \leq \cdots \leq \lambda_m$ the eigenvalues of the normalized Ricci tensor at each point. In view of above facts Onti et al.*

[26] *proved an inequality for given integers $m \geq 3$ and $n \geq 1$, there exists for every $\lambda \in [0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if M^m is a compact subamnifolds of a Riemannian manifold $M^{m+n}(c)$, then*

$$\int_{M^m} (\mathbb{H}^2 + c - \lambda\Omega_{Nor} - \Omega)^{\frac{m}{2}} \geq \varepsilon_\lambda(m,n) \sum_{i=1}^{m-1} B_i(M^m;\mathbb{F}) \tag{6.48}$$

for any coefficient field $\mathbb{F}$.

Moreover, Let M be homeomorphic to the n-sphere $\mathbb{S}^n$. If $f : M \longrightarrow \mathbb{S}^n$ is a minimal immersion with trivial normal bundle, then f is totally geodesic.

Proof. The following outcome is obtained by combining (5.32), (6.48), Theorem 5.1, and the aforementioned Remark 6.1. □

Theorem 6.1. *Given integers $m \geq 3$ and $n \geq 1$, there exists for every $\lambda \in [0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection. Then*

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m,n) \sum_{i=1}^{m-1} B_i(M^m;\mathbb{F}). \tag{6.49}$$

In particular, $\mathcal{HUS}$-manifolds $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or or it is an Eells-Kuiper manifold if

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{\alpha^r(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m,n). \tag{6.50}$$

Proof. Using (6.48) and (5.32), we easily get the (6.50). □

In particular, tangent manifolds $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold

Corollary 6.1. *Given integers $m \geq 3$ and $n \geq 1$, there exists for every $\lambda \in [0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional almost complex manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m,n) \sum_{i=1}^{m-1} B_i(M^m;\mathbb{F}). \tag{6.51}$$

In particular, almost complex manifold $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or or it is an Eells-Kuiper manifold, then we have

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} + \frac{(m+2)}{m} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m,n). \tag{6.52}$$

Corollary 6.2. *Given integers $m \geq 3$ and $n \geq 1$, there exists for every $\lambda \in [0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional almost tangent manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\int_{M^m} \left[\|\mathbb{H}\|^2 - \lambda\Omega_{Nor} - \frac{2\mathcal{R}_{scal}}{m(m-1)} \right]^{\frac{m}{2}} \geq \varepsilon_\lambda(m,n) \sum_{i=1}^{m-1} B_i(M^m;\mathbb{F}). \tag{6.53}$$

If an almost tangent manifold $\mathcal{H}$ with a $SSNMs$-connection s homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold, then

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n). \tag{6.54}$$

Corollary 6.3. *Given integers $m\geq 3$ and $n\geq 1$, there exists for every $\lambda\in[0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional almost product manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}-\frac{(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n)\sum_{i=1}^{m-1}B_i(M^m;\mathbb{F}). \tag{6.55}$$

If almost product manifold $\mathcal{H}$ with a $SSNMs$-connection is homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold, then

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}-\frac{(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n). \tag{6.56}$$

Corollary 6.4. *Given integers $m\geq 3$ and $n\geq 1$, there exists for every $\lambda\in[0,1)$ a positive constant $\varepsilon_\lambda(m,n)$, depending only on m and n, such that if m-dimensional submanifold $\mathcal{H}'$ of an n-dimensional GF-manifold $\mathcal{H}$ with a $SSNMs$-connection . Then*

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}-\frac{\alpha^2(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n)\sum_{i=1}^{m-1}B_i(M^m;\mathbb{F}). \tag{6.57}$$

In particular, GF-manifold $\mathcal{H}^m$ is homeomorphic to $\mathbb{S}^m$ or it is an Eells-Kuiper manifold, then we have

$$\int_{M^m}\left[\|\mathbb{H}\|^2-\lambda\Omega_{Nor}-\frac{2\mathcal{R}_{scal}}{m(m-1)}\frac{\alpha^2(m+2)}{m}\right]^{\frac{m}{2}}\geq\varepsilon_\lambda(m,n). \tag{6.58}$$

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DOMINATION AND INDEPENDENCE NUMBERS OF EXACT ZERO-DIVISOR GRAPH OF THE RING $\mathbb{Z}_n$

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Abstract. This study examines the structural properties of exact zero-divisor graph associated with commutative rings possessing a non-zero identity, emphasizing rings that are not integral domains. By focusing on key graph-theoretic parameters, such as domination and independence numbers, we provide a detailed analysis of their behavior in specific rings, including $\mathbb{Z}_{p^2}$, $\mathbb{Z}_{p^k}$, and $\mathbb{Z}_{pq}$. These findings reveal significant relationships between these parameters and the fundamental algebraic properties of rings, enhancing the comprehension of the interaction between graph theory and ring theory.

Keywords: Exact zero-divisor graph, Domination number, Independence number.

MSC (2020): 13A15, 05C69.

1. Introduction

Exploring algebraic structures through their corresponding graphs has been a dynamic field of mathematical research, providing valuable insights into the relationship between algebra and graph theory. The zero-divisor graph of a commutative ring, first introduced by Beck [4], and later refined by Anderson and Livingston [1], exemplifies this interplay by Anderson et al. in [2]. In their definition, the vertex set comprises the nonzero zero-divisors of the ring, with adjacency determined by the zero-product condition. This framework has catalyzed numerous investigations into properties of such graphs, including connectivity, chromatic number, and domination number.

Graph theory, as a field, has a rich history and has developed into a powerful tool for analyzing relationships in various mathematical structures. Foundational text by Haynes et al. [6] has laid the groundwork for understanding concepts such as domination, independence, connectivity, and coloring in graphs. These graph-theoretic concepts have profound implications when applied to algebraic structures like rings, providing a means to explore their structural intricacies through visual and quantitative analysis.

In recent years, the concept of exact zero-divisors, introduced by Henriques and Sega [5], has provided a nuanced perspective on zero-divisor graphs. An exact zero-divisor is defined

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via principal annihilators with symmetric properties, thereby enabling the construction of the exact zero-divisor graph $E\Gamma(R)$ for a commutative ring R. The graph $E\Gamma(R)$ differs from the traditional zero-divisor graph $\Gamma(R)$ in its focus on a refined subset of zero-divisors, leading to richer structural properties and new avenues for research. As in [7], Lalchandani introduced the exact zero-divisor graph, particularly for the ring $\mathbb{Z}_n$, highlighting key differences between $\Gamma(\mathbb{Z}_n)$ and $E\Gamma(\mathbb{Z}_n)$. Later developments, such as in [8], explored further extensions, including properties of exact zero-divisor graphs for reduced rings, providing foundational results.These studies highlight the intricate interplay between the algebraic properties of a ring and the graph-theoretic properties of its associated graphs, a theme central to this paper.

The ring $\mathbb{Z}_n$, comprising integers modulo n, is a foundational object in algebra, with its arithmetic properties influencing a wide range of mathematical and applied studies [3]. When n is not a prime power, $\mathbb{Z}_n$ admits nontrivial zero-divisors, making it a fertile ground for exploring zero-divisor graphs. Visweswaran and Lalchandani in [9] have shown that $E\Gamma(\mathbb{Z}_n)$ offers distinct structural properties compared to $\Gamma(\mathbb{Z}_n)$, particularly in terms of connectivity and bipartite decompositions. These insights form the backdrop for examining fundamental graph-theoretic parameters, such as domination and independence, in $E\Gamma(\mathbb{Z}_n)$.

Domination and independence are fundamental concepts in graph theory, with applications spanning optimization, network design, and social dynamics. The domination number, denoted $\gamma(G)$, quantifies the minimum size of a set of vertices such that every vertex in the graph is either in the set or adjacent to a vertex in the set [6]. On the other hand, the independence number $\alpha(G)$ captures the maximum size of a subset of vertices such that no two vertices in the subset are adjacent [10]. Both these invariants reveal critical information about the structure and behavior of a graph. For $E\Gamma(\mathbb{Z}_n)$, understanding these parameters sheds light on the intricate relationships among exact zero-divisors of $\mathbb{Z}_n$.

This article aims to advance the understanding of domination and independence numbers in the exact zero-divisor graph of the ring $\mathbb{Z}_n$. By building on foundational results, we develop methods to compute these invariants for various classes of $\mathbb{Z}_n$. Theoretical results are illustrated with concrete examples to elucidate the differences between $E\Gamma(\mathbb{Z}_n)$ and $\Gamma(\mathbb{Z}_n)$.

The organization of the paper is as follows. In Section 2, we review the definitions and fundamental preliminaries of exact zero-divisors and their associated graphs. Section 3 presents methods to compute the domination number for $E\Gamma(\mathbb{Z}_n)$ under various conditions on n. Also, we explore the independence number of $E\Gamma(\mathbb{Z}_n)$ for different values of n. We conclude with discussions on the implications of our findings and potential avenues for future research.

By integrating graph-theoretic techniques with algebraic insights, this work aims to contribute to the broader understanding of the structural properties of graphs derived from rings. The results presented here have potential implications for both pure mathematics and applications involving modular arithmetic and network analysis. Throughout the article, all rings considered are commutative rings with non-zero identity.

2. Preliminaries

In this section, we introduce relevant terminology along with illustrative examples for this paper. For further details, readers are encouraged to consult references [3] and [10]. We begin with the following key definitions:

Definition 2.1. *A zero-divisor in a commutative ring R with identity is a nonzero element $a \in R$ such that there exists a nonzero $b \in R$ with $ab = 0$.*

Definition 2.2. *The zero-divisor graph $\Gamma(R)$ of a commutative ring R is the graph where the vertex set consists of all nonzero zero-divisors of R, and two vertices a, b are adjacent if and only if $ab = 0$.*

Definition 2.3. [7] *An exact zero-divisor $a \in R$ is a zero-divisor for which there exists a non zero element $b \in R$ such that the annihilator $Ann(a) = \{r \in R \mid ra = 0\}$ is a principal ideal generated by b, and $Ann(b)$ is the principal ideal generated by a.*

Definition 2.4. [7] *The exact zero-divisor graph $E\Gamma(R)$ of a commutative ring R is the graph where the vertex set consists of all nonzero exact zero-divisors of R, and two vertices a, b are adjacent if and only if $Ann(a) = bR$ and $Ann(b) = aR$.*

Definition 2.5. *The domination number $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set, which is a subset $D \subseteq V(G)$ such that every vertex in G is either in D or adjacent to a vertex in D.*

Definition 2.6. *The independence number $\alpha(G)$ of a graph G is the maximum cardinality of an independent set, which is a subset $I \subseteq V(G)$ such that no two vertices in I are adjacent.*

We now compute the domination and independence numbers of $E\Gamma(R)$ for the rings $\mathbb{Z}_6$, $\mathbb{Z}_{27}$, and $\mathbb{Z}_{49}$. For each ring, the graphs are constructed using exact zero-divisors, and the results are summarized.

Example 2.1. *Let $R = \mathbb{Z}_{2\cdot 3} = \mathbb{Z}_6$. Then the nonzero exact zero-divisors in $\mathbb{Z}_6$ are found to be $\{2, 3, 4\}$. The exact zero-divisor graph $E\Gamma(\mathbb{Z}_6)$ is as in Figure 1.*

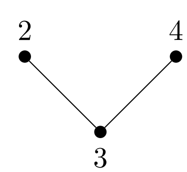

FIGURE 1. Exact Zero-Divisor Graph $E\Gamma(\mathbb{Z}_6)$

From the graph:

- ***Domination Number**: $\gamma(E\Gamma(\mathbb{Z}_6)) = 1$, with $\{3\}$ as a dominating set.*
- ***Independence Number**: $\alpha(E\Gamma(\mathbb{Z}_6)) = 2$, with $\{2, 4\}$ as an independent set.*

Example 2.2. *Let $R = \mathbb{Z}_{27}$. Then the nonzero exact zero-divisors in $\mathbb{Z}_{27}$ are $\{3, 6, 9, 12, 15, 18, 21, 24\}$. The exact zero-divisor graph $E\Gamma(\mathbb{Z}_{27})$ is in Figure 2.*

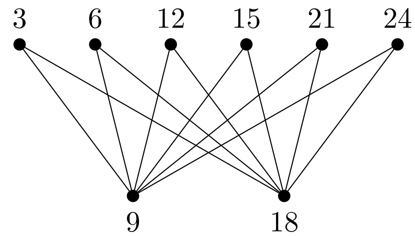

FIGURE 2. Exact Zero-Divisor Graph $E\Gamma(\mathbb{Z}_{27})$

From the graph:

- ***Domination Number**: $\gamma(E\Gamma(\mathbb{Z}_{27})) = 2$, with $\{9, 18\}$ as a dominating set.*
- ***Independence Number**: $\alpha(E\Gamma(\mathbb{Z}_{27})) = 6$, with $\{3, 6, 12, 15, 21, 24\}$ as an independent set.*

Example 2.3. *Let $R = \mathbb{Z}_{7^2} = \mathbb{Z}_{49}$. Then the nonzero exact zero-divisors in $\mathbb{Z}_{49}$ are $\{7, 14, 21, 28, 35, 42\}$. The exact zero-divisor graph $E\Gamma(\mathbb{Z}_{49})$ is in Figure 3.*

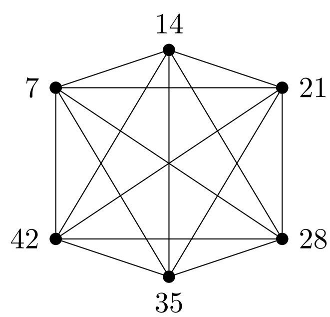

FIGURE 3. Exact Zero-Divisor Graph $E\Gamma(\mathbb{Z}_{49})$

From the graph:

- ***Domination Number**: $\gamma(E\Gamma(\mathbb{Z}_{49})) = 1$, with $\{7\}$ as a dominating set.*
- ***Independence Number**: $\alpha(E\Gamma(\mathbb{Z}_{49})) = 1$, with $\{7\}$ as an independent set.*

3. THE DOMINATION AND INDEPENDENCE NUMBERS OF $E\Gamma(\mathbb{Z}_n)$

Theorem 3.1. *Let $R = \mathbb{Z}_{p^2}$, where p is prime number. Then the exact zero-divisor graph $E\Gamma(R)$ of the ring R is complete graph K_{p-1} with $p-1$ vertices and the domination number of $E\Gamma(R)$ is 1.*

Proof. We know that the exact zero-divisor graph $E\Gamma(R)$ of the ring R is complete graph K_{p-1} with $p-1$ vertices for $R = \mathbb{Z}_{p^2}$ [8]. Consider any single vertex $v_1 \in V(E\Gamma(R))$, where $R = \mathbb{Z}_{p^2}$, since $E\Gamma(R)$ is complete graph with $p-1$ vertices, v_1 is adjacent to every other vertex in $V \setminus \{v_1\}$. Therefore, v_1 is adjacent to all the remaining $p-2$ vertices. Hence, the set $\{v_1\}$ is a dominating set because every vertex in $V \setminus \{v_1\}$ is adjacent to v_1.

Thus, by the definition of the domination number, we conclude that

$$\gamma(E\Gamma(\mathbb{Z}_{p^2})) = 1.$$

$\square$

Theorem 3.2. *Let $R = \mathbb{Z}_{pq}$, where p and q are distinct prime numbers such that $p < q$. Then the exact zero-divisor graph $E\Gamma(R)$ is isomorphic to the complete bipartite graph $K_{p-1,q-1}$, and its domination number is given by*

$$\gamma(E\Gamma(R)) = \begin{cases} 1, & \text{if } p = 2, \\ 2, & \text{if } p > 2. \end{cases}$$

Proof. The structure of $E\Gamma(\mathbb{Z}_{pq})$ aligns with the complete bipartite graph $K_{p-1,q-1}$, where the vertex sets correspond to the nonzero zero-divisors from the respective prime factors of pq [8]. When $p = 2$, the graph reduces to $K_{1,q-1}$, wherein the single vertex from the smaller partition is adjacent to all others. Hence, a single vertex suffices to dominate the graph, and we have:

$$\gamma(E\Gamma(\mathbb{Z}_{2q})) = 1.$$

For $p > 2$, each partite set has more than one vertex, and no single vertex can dominate both partitions. Selecting one vertex from each suffices to dominate the entire graph. Thus:

$$\gamma(E\Gamma(\mathbb{Z}_{pq})) = 2.$$

□

Theorem 3.3. *Let $R = \mathbb{Z}_{p^k}$, where p is a prime and $k \geq 3$ is odd. Then the domination number of the exact zero-divisor graph $E\Gamma(R)$ is*

$$\gamma(E\Gamma(\mathbb{Z}_{p^k})) = k - 1.$$

Proof. When $k \geq 3$ is odd, the graph $E\Gamma(\mathbb{Z}_{p^k})$ consists of $\frac{k-1}{2}$ disconnected components, each isomorphic to a complete bipartite graph [8]. These components arise from annihilating pairs (p^i, p^{k-i}) for $i = 1, 2, \ldots, \frac{k-1}{2}$. In each component K_{m_i,n_i}, the partite sets have sizes $m_i = \varphi(p^i)$ and $n_i = \varphi(p^{k-i})$, both strictly greater than one. Thus, the domination number of each such bipartite component is exactly 2, as no single vertex can dominate both partitions. Since the components are disjoint, their domination numbers add. Therefore,

$$\gamma(E\Gamma(\mathbb{Z}_{p^k})) = 2 \cdot \frac{k-1}{2} = k - 1.$$

□

Theorem 3.4. *Let $R = \mathbb{Z}_{p^k}$, where p is a prime and $k \geq 4$ is even. Then the domination number of the exact zero-divisor graph $E\Gamma(R)$ is*

$$\gamma(E\Gamma(\mathbb{Z}_{p^k})) = k - 1.$$

Proof. For even $k \geq 4$, the graph $E\Gamma(\mathbb{Z}_{p^k})$ consists of $\frac{k}{2}$ disconnected components [8]. Among these, the first $\frac{k}{2} - 1$ components are complete bipartite graphs, each arising from annihilating pairs (p^i, p^{k-i}), where $i = 1, 2, \ldots, \frac{k}{2} - 1$. Each such component contributes 2 to the domination number. The remaining component, corresponding to $i = \frac{k}{2}$, involves zero-divisors with annihilators $\langle p^{k/2} \rangle$, forming a complete graph. In this case, a single vertex dominates the entire component. Combining all contributions, we obtain:

$$\gamma(E\Gamma(\mathbb{Z}_{p^k})) = 2 \cdot \left(\frac{k}{2} - 1 \right) + 1 = k - 1.$$

□

Theorem 3.5. *Let $R = \mathbb{Z}_{p^2}$, where p is a prime number, and let $E\Gamma(R)$ denote the exact zero-divisor graph of R. Then the independence number of $E\Gamma(R)$ is given by:*

$$\alpha(E\Gamma(\mathbb{Z}_{p^2})) = 1.$$

Proof. We want to show that the independence number of the exact zero-divisor graph $E\Gamma(R)$ for the ring $R = \mathbb{Z}_{p^2}$ is 1. To do this, we begin by understanding the structure of the graph $E\Gamma(R)$. It is known that $E\Gamma(\mathbb{Z}_{p^2})$ is a complete graph with $p-1$ vertices [8].

In a complete graph, each vertex is connected to every other vertex. Consider any vertex v_1 in $E\Gamma(\mathbb{Z}_{p^2})$. Since the graph is complete, v_1 is adjacent to all other vertices. Therefore, the only independent set that can include v_1 is the set $\{v_1\}$, which contains just one vertex. This reasoning holds for any vertex in the graph. Hence, we conclude that the independence number of $E\Gamma(\mathbb{Z}_{p^2})$ is 1.

$$\alpha(E\Gamma(\mathbb{Z}_{p^2})) = 1$$

□

Theorem 3.6. *Let $R = \mathbb{Z}_{pq}$, where p and q are distinct prime numbers with $p < q$, and let $E\Gamma(R)$ denote the exact zero-divisor graph of R. Then the independence number of $E\Gamma(R)$ is given by:*

$$\alpha(E\Gamma(\mathbb{Z}_{pq})) = \phi(q) = q - 1.$$

Proof. To prove that the independence number of the exact zero-divisor graph $E\Gamma(R)$ for the ring $R = \mathbb{Z}_{pq}$, where p and q are distinct prime numbers, is $q-1$, we start by considering the structure of the graph. The graph $E\Gamma(\mathbb{Z}_{pq})$ is a complete bipartite graph, specifically $K_{p-1,q-1}$ [8]. This means that the set of $p-1$ vertices corresponds to the zero divisors in $\mathbb{Z}_p$, and the set of $q-1$ vertices corresponds to the zero divisors in $\mathbb{Z}_q$. In this graph, every vertex in the first set is connected to every vertex in the second set, with no edges within either set.

Since $E\Gamma(\mathbb{Z}_{pq})$ is a complete bipartite graph, the largest independent set must come from one of the two sets, as there are no edges within either set. We can choose all the vertices from set B, which has $q-1$ vertices. This is the larger of the two sets since $q > p$. Therefore, the largest independent set in $E\Gamma(\mathbb{Z}_{pq})$ contains $q-1$ vertices.

Thus, we conclude that the independence number of $E\Gamma(\mathbb{Z}_{pq})$ is $q-1$.

$$\alpha(E\Gamma(\mathbb{Z}_{pq})) = q - 1$$

□

Theorem 3.7. *Let $R = \mathbb{Z}_{p^k}$, where p is a prime number, and let $E\Gamma(\mathbb{Z}_{p^k}), k \geq 3$ denote the exact zero-divisor graph of R. Then the independence number of $E\Gamma(\mathbb{Z}_{p^k})$ is given by:*

$$\alpha(E\Gamma(\mathbb{Z}_{p^k})) = \begin{cases} \sum\limits_{i=\frac{k+1}{2}}^{k-1} \phi(p^i), & \text{if } k \text{ is odd,} \\ \sum\limits_{i=\frac{k}{2}+1}^{k-1} \phi(p^i) + 1, & \text{if } k \text{ is even.} \end{cases}$$

Proof. Let $R = \mathbb{Z}_{p^k}$, where $k \geq 3$. The vertices of $E\Gamma(\mathbb{Z}_{p^k})$ are all nonzero exact zero-divisors, given by $V = \{p^i \cdot m \mid m \in \mathbb{Z}^*_{p^k},\ 1 \leq i \leq k-1\}$, where $\mathbb{Z}^*_{p^k}$ consists of all elements in $\mathbb{Z}_{p^k}$ (integers modulo p^k) that are coprime to p. By Theorem 3.2 of [8], $E\Gamma(\mathbb{Z}_{p^k})$ is a disjoint union of $\lfloor \frac{k}{2} \rfloor$ components.

For $i = 1, 2, \ldots, \lfloor \frac{k}{2} \rfloor - 1$, each component corresponds to the annihilator pair $(\langle p^i \rangle, \langle p^{k-i} \rangle)$, forming a complete bipartite graph K_{X_i,Y_i} with $|X_i| = \phi(p^i)$ and $|Y_i| = \phi(p^{k-i})$. For $i = \frac{k}{2}$ (if k is even), the vertices form a complete graph $K_{\phi(p^{\frac{k}{2}})}$.

The independence number α of a graph is the size of its largest independent set. For K_{X_i,Y_i}, $\alpha(K_{X_i,Y_i}) = \max(|X_i|, |Y_i|) = \phi(p^{k-i})$, since $\phi(p^i) \leq \phi(p^{k-i})$. For $K_{\phi(p^{\frac{k}{2}})}$, $\alpha(K_{\phi(p^{\frac{k}{2}})}) = 1$.

Case 1: k is odd; When $k = 2m+1$, $E\Gamma(\mathbb{Z}_{p^k})$ has m bipartite components. The independent set in each component corresponds to Y_i, where $i = \frac{k+1}{2}, \frac{k+3}{2}, \ldots, k-1$. Thus:

$$\alpha(E\Gamma(\mathbb{Z}_{p^k})) = \phi(p^{\frac{k+1}{2}}) + \phi(p^{\frac{k+3}{2}}) + \cdots + \phi(p^{k-1}) = \sum_{i=\frac{k+1}{2}}^{k-1} \phi(p^i).$$

Case 2: k is even; When $k = 2m$, $E\Gamma(\mathbb{Z}_{p^k})$ has $m-1$ bipartite components and one complete graph. The independent set in each bipartite component is Y_i for $i = \frac{k}{2}+1, \frac{k}{2}+2, \ldots, k-1$. The independent set in $K_{\phi(p^{\frac{k}{2}})}$ contributes 1. Therefore:

$$\alpha(E\Gamma(\mathbb{Z}_{p^k})) = \phi(p^{\frac{k}{2}+1}) + \phi(p^{\frac{k}{2}+2}) + \cdots + \phi(p^{k-1}) + 1 = \sum_{i=\frac{k}{2}+1}^{k-1} \phi(p^i) + 1.$$

Therefore, the independence number of the exact zero-divisor graph $E\Gamma(\mathbb{Z}_{p^k})$ is given by:

$$\alpha(E\Gamma(\mathbb{Z}_{p^k})) = \begin{cases} \displaystyle\sum_{i=\frac{k+1}{2}}^{k-1} \phi(p^i), & \text{if } k \text{ is odd,} \\ \displaystyle\sum_{i=\frac{k}{2}+1}^{k-1} \phi(p^i) + 1, & \text{if } k \text{ is even.} \end{cases}$$

□

4. Domination and Independence Numbers in $\Gamma(\mathbb{Z}_n)$ vs $E\Gamma(\mathbb{Z}_n)$

Table 4.1 presents a clear comparison of the domination and independence numbers of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ and the exact zero-divisor graph $E\Gamma(\mathbb{Z}_n)$ for selected values of n. This helps to highlight how these graph invariants vary between the two graph types.

n	$\gamma(\Gamma(\mathbb{Z}_n))$	$\gamma(E\Gamma(\mathbb{Z}_n))$	$\alpha(\Gamma(\mathbb{Z}_n))$	$\alpha(E\Gamma(\mathbb{Z}_n))$
16	1	3	4	5
27	1	2	6	6
32	1	4	14	12

Table 4.1. Comparison of domination and independence numbers for $\Gamma(\mathbb{Z}_n)$ and $E\Gamma(\mathbb{Z}_n)$

5. Conclusion

In this paper, we analyzed the properties of the exact zero-divisor graph $E\Gamma(R)$ of various commutative rings R of the form $\mathbb{Z}_n$, where n is a prime as well as composite number. This study explored the structure of exact zero-divisor graphs $E\Gamma(R)$ for rings of integers modulo n, focusing on their domination and independence numbers. The domination and independence numbers are tied to Euler's totient function, reflecting the deeper interplay between algebra and graph theory. These results reveal how the algebraic properties of rings influence the structure of their zero-divisor graphs, providing insights into both mathematical theory and practical applications in combinatorics and algebra. The domination and independence numbers computed for exact zero-divisor graphs are not only theoretically significant but also practically meaningful in systems where modular arithmetic and graph structure intersect. We highlight some applications below with examples and mathematical justification as following illustrations:

5.1. **Optimal Monitoring in Ring-Structured Networks:** In communication networks or fault-monitoring systems modeled by rings such as $\mathbb{Z}_n$, each exact zero-divisor can represent a node, and edges in $E\Gamma(\mathbb{Z}_n)$ capture mutual influence via annihilation. The domination number then corresponds to the smallest number of control points required to observe the entire system. As shown in Example 2.3, the graph $E\Gamma(\mathbb{Z}_{49})$ consists of six vertices with one dominating vertex namely 7. Since this vertex is adjacent to all others, selecting it ensures full coverage of the network. Thus, the domination number $\gamma(E\Gamma(\mathbb{Z}_{49})) = 1$ signifies that a single monitoring unit suffices. This principle applies to minimal sensor placement in systems with modular timing or synchronization constraints.

5.2. **Construction of Non-Interfering Code-words in Modular Coding Schemes:** In error-correcting code design, independent sets in graphs are often used to identify elements that behave distinctly under encoding operations. In $E\Gamma(\mathbb{Z}_n)$, an independent set corresponds to exact zero-divisors that do not annihilate each other, ensuring they operate without algebraic interference. From Example 2.2, the independent set $\{3, 6, 12, 15, 21, 24\}$ in $E\Gamma(\mathbb{Z}_{27})$ has size six, and no two elements within this set are adjacent. This makes them suitable for representing code-words with minimal overlap in structure. When mapped to vectors over $\mathbb{Z}_{27}$, these elements enable the design of modular codes with high Hamming distance and clear signal distinction which is essential in digital communication over noisy channels, especially where the underlying operations are defined over finite rings.

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CERTAIN GEOMETRIC PROPERTIES OF THE COTANGENT BUNDLE ENDOWED WITH THE HORIZONTALLY DEFORMED SASAKI METRIC

ABDERRAHIM ZAGANE *

Abstract. This study presents a new family of Riemannian metrics on the cotangent bundle of a Riemannian manifold, referred to as horizontally deformed Sasaki metric. These metrics extend the classical Sasaki metric by incorporating a horizontal deformation. The paper conducts a detailed examination of the geometric structure of the cotangent bundle endowed with the horizontally deformed Sasaki metric, focusing on the Levi-Civita connection and various curvature tensors. Additionally, the geometry of a naturally arising hypersurface within the cotangent bundle is investigated through the derivation of its induced Levi-Civita connection and associated curvature formulas. The findings broaden the existing class of Riemannian metrics defined on cotangent bundles and provide deeper insight into the geometric framework of Riemannian manifolds and their cotangent structures. These contributions enhance the understanding of invariant metric structures on cotangent bundles and offer a foundation for further research in differential geometry.

Keywords: Cotangent bundle, horizontally deformed Sasaki metric, Riemannian curvature tensor, hyperspace.

MSC (2020): 53C20, 53C05, 53B05.

1. Introduction

One of the earliest contributions in the literature that treats the cotangent bundle of a manifold as a Riemannian manifold is due to Patterson and Walker [10], who introduced a Riemannian metric on the cotangent bundle derived from an affine symmetric connection on the base manifold, now known as the Riemannian extension of the connection. This construction was subsequently generalized by Sekizawa [12], who developed a classification of natural transformations mapping affine connections on manifolds to metrics on their cotangent bundles. His work produced a two-parameter family of metrics termed natural Riemann extensions, which have since become the focus of extensive study.

This foundational work has inspired further research into alternative Riemannian metrics on cotangent bundles (see [2, 7, 9]), with the Sasaki metric [11] emerging as one of the most widely studied. In this context, Ağca introduced the notion of g-natural metrics on cotangent

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bundles, drawing from similar constructions previously developed for tangent bundles [1]. For related works, see [3, 4, 5, 8]. In addition, deformations of the Sasaki metric have been investigated in the setting of anti-para-Kähler and standard Kähler manifolds (see [6, 14, 15, 16, 18]).

In this work, we propose a new deformation, termed the horizontally deformed Sasaki metric, which acts on the horizontal distribution of the cotangent bundle. This construction broadens the class of invariant Riemannian metrics on cotangent bundles and sheds new light on the geometric interplay between a manifold and its cotangent bundle.

Motivated by fundamental questions concerning the influence of such deformations on curvature behavior, connections, and induced geometric structures, we undertake a systematic investigation of this metric. These inquiries bear relevance to both theoretical and applied aspects of differential geometry.

The structure of the paper is as follows:

Section 2 reviews fundamental results on the cotangent bundle.

Section 3 establishes the foundational properties of the horizontally deformed Sasaki metric and derives explicit formulas for the associated Levi-Civita connection.

Section 4 examines all types of curvature in the cotangent bundle under this metric, highlighting the geometric effects of the horizontal deformation in comparison with the classical Sasaki metric.

Section 5 explores the geometry of hypersurfaces within the cotangent bundle, deriving their induced Levi-Civita connections. These results generalize previous work and lay the groundwork for further study of submanifolds in the context of deformed Riemannian structures.

2. Basic information and preliminary results

Consider an m-dimensional Riemannian manifold (M^m, g), with its associated cotangent bundle T^*M and the canonical projection map $\pi : T^*M \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a corresponding local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)$ on T^*M, where the index $\bar{i}$ is defined by $\bar{i} = i + 1$. Here p_i denote the components of a covector $p \in T^*_x M$, expressed relative to the canonical coframe $\{dx^i\}$ for each point $x \in U$.

Let $C^\infty(M)$ be the ring of smooth real-valued functions on M and $\Im^r_s(M)$ (respectively, $\Im^r_s(T^*M)$) denote the $C^\infty(M)$-module of smooth tensor fields of type (r,s) on the manifold M (respectively, on its cotangent bundle T^*M). The Levi-Civita connection associated with the Riemannian metric g is denoted by ∇, and its corresponding Christoffel symbols are written as Γ^k_{ij}.

The connection ∇ naturally defines a splitting of the tangent bundle of the cotangent bundle T^*M into a direct sum:

$$TT^*M = VT^*M \oplus HT^*M, \tag{2.1}$$

where VT^*M and HT^*M represent the vertical and horizontal distributions, respectively. These distributions are given explicitly by

$$\begin{array}{rcl} V_{(x,p)}T^*M &=& \ker(d_{(x,p)}\pi) = \{\omega_i \partial_{\bar{i}}|_{(x,p)} \mid \omega_i \in \mathbb{R}\}, \\ H_{(x,p)}T^*M &=& \{X^i \partial_i|_{(x,p)} + X^i p_a \Gamma^a_{hi} \partial_{\bar{h}}|_{(x,p)} \mid X^i \in \mathbb{R}\} \end{array}$$

for each point $(x,p) \in T^*M$, V and H represent the projections onto the vertical and horizontal sub-bundles, induced by the connection ∇ respectively, and $\partial_i = \frac{\partial}{\partial x^i}$ and $\partial_{\bar{i}} = \frac{\partial}{\partial p_i}$ represent the coordinate vector fields.

The mapping

$$w \mapsto {}^V w = w_i \partial_{\bar{i}}|_{(x,p)}$$

defines an isomorphism between $T^*_x M$ and $V_{(x,p)}T^*M$. Similarly, the mapping

$$v \mapsto {}^H v = v^i \partial_i|_{(x,p)} + v^i p_a \Gamma^a_{hi} \partial_{\bar{h}}|_{(x,p)}$$

establishes an isomorphism between the vector spaces $T_x M$ and $H_{(x,p)}T^*M$. Consequently, any tangent vector $W \in T_{(x,p)}T^*M$ can be uniquely expressed as

$$W = {}^H v + {}^V w, \tag{2.2}$$

where $v \in T_x M$ and $w \in T^*_x M$ are uniquely determined.

Let $Z = Z^i \partial_i$ and $\omega = \omega_i dx^i$ be the local representations of a vector field $Z \in \Im^1_0(M)$ and a covector field $\omega \in \Im^0_1(M)$, respectively, in a local chart $(U, x^i)_{i=\overline{1,m}}$. The horizontal lifts of Z, denoted by ${}^H Z$ in $\Im^1_0(T^*M)$, as well as the vertical lift of ω, denoted by ${}^V\omega$ in $\Im^1_0(T^*M)$, are defined as follows:

$$\begin{array}{rcll} {}^H Z &=& Z^i \partial_i + p_h \Gamma^h_{ij} Z^j \partial_{\bar{i}}, & (2.3) \\ {}^V \omega &=& \omega_i \partial_{\bar{i}}, & (2.4) \end{array}$$

In particular, the vertical distribution ${}^V p$ on T^*M is defined by

$${}^V p = p_i {}^V(dx^i) = p_i \partial_{\bar{i}},$$

which is also known as the canonical or Liouville vector field on T^*M.

The Lie bracket operation between vertical and horizontal vector fields on the cotangent bundle T^*M can be explicitly described by the formulas

$$\begin{cases} [{}^V\omega, {}^V\eta] = 0, \\ [{}^H Z, {}^V\eta] = {}^V(\nabla_Z \eta), \\ [{}^H Z, {}^H Y] = {}^H[Z,Y] + {}^V(p\mathrm{R}(Z,Y)), \end{cases} \tag{2.5}$$

for each vector fields $Y, Z \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$[13], such that $p\mathrm{R}(Z,Y) = p_a \mathrm{R}^a_{ijk} Z^i Y^j dx^k$, where R^a_{ijk} are local components of R on (M^m, g).

Given a Riemannian manifold (M^m, g) , the map

$$\begin{array}{cccc} \sharp : & \Im^0_1(M) & \to & \Im^1_0(M) \\ & \theta & \mapsto & \sharp\theta \end{array}$$

defined by

$$g(\sharp\theta, Z) = \theta(Z)$$

for each vector field $Z \in \Im_0^1(M)$ is $C^\infty(M)$-linear isomorphism.
Locally, for each $\theta = \theta_i dx^i \in \Im_1^0(M)$, we have

$$\sharp\theta = g^{ij}\theta_i\partial_j,$$

where (g^{ij}) is the inverse of the matrix (g_{ij}).

For each point $x \in M$, the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space T_x^*M by

$$g^{-1}(\omega,\eta) = g(\sharp\omega,\sharp\eta) = g^{ij}\omega_i\eta_j,$$

from this we write $\sharp\omega = g^{-1}\circ\omega$.

Lemma 2.1. *Given a Riemannian manifold (M^m, g). Then, the following results are satisfied:*

$$\begin{aligned}
\nabla_X\sharp\omega &= \sharp(\nabla_X\omega), &(2.6)\\
Xg^{-1}(\omega,\eta) &= g^{-1}(\nabla_X\omega,\eta) + g^{-1}(\omega,\nabla_X\eta), &(2.7)\\
\sharp(\omega\mathrm{R}(X,Y)) &= -\mathrm{R}(X,Y)\sharp\omega, &(2.8)\\
\omega(\nabla_X F) &= \nabla_X(\omega F) - (\nabla_X\omega)F, &(2.9)\\
\omega(\nabla_X\mathrm{R})(Y,Z) &= \nabla_X(\omega\mathrm{R}(Y,Z)) - (\nabla_X\omega)\mathrm{R}(Y,Z) - \omega\mathrm{R}(\nabla_X Y, Z)\\
&\quad -\omega\mathrm{R}(Y,\nabla_X Z), &(2.10)\\
\sharp(\omega(\nabla_X\mathrm{R})(Y,Z)) &= -(\nabla_X\mathrm{R})(Y,Z)\sharp\omega. &(2.11)
\end{aligned}$$

for each vector fields $X, Y, Z \in \Im_0^1(M)$, covector fields $\omega,\eta \in \Im_1^0(M)$ and $F \in \Im_1^1(M)$, where ∇ is the Levi-Civita connection of (M^m, g) and R is its curvature tensor.

3. The horizontally deformed Sasaki metric

Definition 3.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M, we introduce a new Riemannian metric g^f on T^*M defined as follows:*

$$\begin{aligned}
g^f({}^HZ, {}^HY)_{(x,p)} &= f(r)g_x(Z,Y), &(3.12)\\
g^f({}^HZ, {}^V\eta)_{(x,p)} &= 0\\
g^f({}^V\omega, {}^V\eta)_{(x,p)} &= g_x^{-1}(\omega,\eta), &(3.13)
\end{aligned}$$

for each $(x,p) \in T^*M$, vector fields $Y, Z \in \Im_0^1(M)$ and covector fields $\omega,\eta \in \Im_1^0(M)$, where $f : [0,+\infty[\to]0,+\infty[$ is a smooth real-valued function and $r = g^{-1}(p,p)$. We refer to this metric as the horizontally deformed Sasaki metric.
For $f = 1$, g^f coincides with the Sasaki metric introduced in [11].

Lemma 3.1. [17] *Let (M^m, g) be a Riemannian manifold, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth real-valued function. Then, the following results are satisfied:*

(1) ${}^HZ(f(r)) = 0$,
(2) ${}^V\omega(f(r)) = 2f'(r)g^{-1}(\omega,p)$,
(3) ${}^HZg(X,Y) = Zg(X,Y) = g(\nabla_Z X, Y) + g(X,\nabla_Z Y)$,
(4) ${}^HZg^{-1}(\eta,\mu) = Zg^{-1}(\eta,\mu) = g^{-1}(\nabla_Z\eta,\mu) + g^{-1}(\eta,\nabla_Z\mu)$,
(5) ${}^HZg^{-1}(\eta,p) = g^{-1}(\nabla_Z\eta,p)$,

(6) ${}^{V}\omega g^{-1}(\eta,p)=g^{-1}(\eta,\omega)$,
(7) ${}^{V}\omega g(Y,Z)=0$,
(8) ${}^{V}\omega g^{-1}(\eta,\mu)=0$,

for each vector fields $X,Y,Z\in\Im^1_0(M)$ *and covector fields* $\omega,\eta,\mu\in\Im^0_1(M)$, *where* $r=g^{-1}(p,p)$.

Proof. Lemma 3.1 is a direct consequence of the expressions given in equations (2.3) and (2.4). □

Lemma 3.2. *Let* (M^m,g) *be a Riemannian manifold, and consider its cotangent bundle* T^*M *endowed with the horizontally deformed Sasaki metric* g^f. *Then, the following relations hold:*

$$\begin{aligned}
(1)\ {}^{H}Z\left(g^f({}^{H}X,{}^{H}Y)\right) &= g^f({}^{H}(\nabla_Z X),{}^{H}Y)+g^f({}^{H}X,{}^{H}(\nabla_Z Y)),\\
(2)\ {}^{V}\mu\left(g^f({}^{H}X,{}^{H}Y)\right) &= 2f'(r)g(X,Y)g^{-1}(\mu,p),\\
(3)\ {}^{H}X\left(g^f({}^{V}\eta,{}^{V}\mu)\right) &= g^f({}^{V}(\nabla_X\eta),{}^{V}\mu)+g^f({}^{V}\eta,{}^{V}(\nabla_X\mu)),
\end{aligned}$$

for each vector fields $X,Y,Z\in\Im^1_0(M)$ *and covector fields* $\eta,\mu\in\Im^0_1(M)$.

Proof. Lemma 3.2 is an immediate consequence of Lemma 3.1. □

We now define the Levi-Civita connection $\widetilde{\nabla}$ associated with the horizontally deformed Sasaki metric g^f on T^*M, derived through application of the Koszul formula

$$\begin{aligned}
2g^f(\widetilde{\nabla}_P Q,W) &= Pg^f(Q,W)+Qg^f(W,P)-Wg^f(P,Q)+g^f(W,[P,Q])\\
&\quad +g^f(Q,[W,P])-g^f(P,[Q,W]),
\end{aligned}\tag{3.14}$$

for each vector fields $P,Q,W\in\Im^1_0(T^*M)$. The following theorem gives all the different forms of the Levi-Civita connection.

Theorem 3.1. *Let* (M^m,g) *be a Riemannian manifold, and consider its cotangent bundle* T^*M *endowed with the horizontally deformed Sasaki metric* g^f. *The associated Levi-Civita connection* $\widetilde{\nabla}$ *with respect to this metric satisfies the following properties:*

$$\begin{aligned}
1.\ \widetilde{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y)-f'(r)g(X,Y){}^{V}p+\frac{1}{2}{}^{V}(p\mathrm{R}(X,Y)),\\
2.\ \widetilde{\nabla}_{{}^{H}X}{}^{V}\eta &= {}^{V}(\nabla_X\eta)+\frac{f'(r)}{f(r)}g^{-1}(\eta,p){}^{H}X+\frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p,\sharp\eta)X),\\
3.\ \widetilde{\nabla}_{{}^{V}\omega}{}^{H}Y &= \frac{f'(r)}{f(r)}g^{-1}(\omega,p){}^{H}Y+\frac{1}{2f(r)}{}^{H}(\mathrm{R}(\sharp p,\sharp\omega)Y),\\
4.\ \widetilde{\nabla}_{{}^{V}\omega}{}^{V}\eta &= 0,
\end{aligned}$$

for each vector fields $X,Y\in\Im^1_0(M)$ *and covector fields* $\omega,\eta\in\Im^0_1(M)$.

Proof. The proof relies on formula (2.5), together with Lemmas 2.1, 3.1, and 3.2, in addition to Koszul's formula (3.14). Through straightforward computations, we derive the following:

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{{}^HX}{}^HY, {}^HZ) &= {}^HX\big(g^f({}^HY, {}^HZ)\big) + {}^HY\big(g^f({}^HZ, {}^HX)\big) - {}^HZ\big(g^f({}^HX, {}^HY)\big) \\
&\quad + g^f({}^HZ, [{}^HX, {}^HY]) + g^f({}^HY, [{}^HZ, {}^HX]) - g^f({}^HX, [{}^HY, {}^HZ]) \\
&= g^f({}^H(\nabla_X Y), {}^HZ) + g^f({}^HY, {}^H(\nabla_X Z)) + g^f({}^H(\nabla_Y Z), {}^HX) \\
&\quad + g^f({}^HZ, {}^H(\nabla_Y X)) - g^f({}^H(\nabla_Z X), {}^HY) - g^f({}^HX, {}^H(\nabla_Z Y)) \\
&\quad + g^f({}^HZ, {}^H[X, Y]) + g^f({}^HY, {}^H[Z, X]) - g^f({}^HX, {}^H[Y, Z]) \\
&= 2g^f({}^H(\nabla_X Y), {}^HZ),
\end{aligned}$$

and

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{{}^HX}{}^HY, {}^V\mu) &= {}^HX\big(g^f({}^HY, {}^V\mu)\big) + {}^HY\big(g^f({}^V\mu, {}^HX)\big) - {}^V\mu\big(g^f({}^HX, {}^HY)\big) \\
&\quad + g^f({}^V\mu, [{}^HX, {}^HY]) + g^f({}^HY, [{}^V\mu, {}^HX]) - g^f({}^HX, [{}^HY, {}^V\mu]) \\
&= -2f'(r)g(X, Y)g^{-1}(\mu, p) - g^f({}^V\mu, {}^V(p\mathrm{R}(X, Y))) \\
&= -g^f\big(2f'(r)g(X, Y){}^Vp + {}^V(p\mathrm{R}(X, Y)), {}^V\mu\big),
\end{aligned}$$

from which we find the expression of $\widetilde{\nabla}_{{}^HX}{}^HY$.
Analogous computations to those presented above yield

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{{}^HX}{}^V\eta, {}^HZ) &= {}^HX\big(g^f({}^V\eta, {}^HZ)\big) + {}^V\eta\big(g^f({}^HZ, {}^HX)\big) - {}^HZ\big(g^f({}^HX, {}^V\eta)\big) \\
&\quad + g^f({}^HZ, [{}^HX, {}^V\eta]) + g^f({}^V\eta, [{}^HZ, {}^HX]) - g^f({}^HX, [{}^V\eta, {}^HZ]) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g^f({}^V\eta, {}^V(p\mathrm{R}(Z, X))) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g^{-1}(\eta, p\mathrm{R}(Z, X)) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g(\sharp\eta, \sharp(p\mathrm{R}(Z, X))) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) - g(\sharp\eta, \mathrm{R}(Z, X)\sharp p) \\
&= 2f'(r)g(X, Z)g^{-1}(\eta, p) + g(\mathrm{R}(\sharp p, \sharp\eta)X, Z) \\
&= \frac{2f'(r)}{f(r)}g^{-1}(\eta, p)g^f({}^HX, {}^HZ) + \frac{1}{f(r)}g^f({}^H(\mathrm{R}(\sharp p, \sharp\eta)X), {}^HZ).
\end{aligned}$$

Also it follows that

$$\begin{aligned}
2g^f(\widetilde{\nabla}_{{}^HX}{}^V\eta, {}^V\mu) &= {}^HX\big(g^f({}^V\eta, {}^V\mu)\big) + {}^V\eta\big(g^f({}^V\mu, {}^HX)\big) - {}^V\mu\big(g^f({}^HX, {}^V\eta)\big) \\
&\quad + g^f({}^V\mu, [{}^HX, {}^V\eta]) + g^f({}^V\eta, [{}^V\mu, {}^HX]) - g^f({}^HX, [{}^V\eta, {}^V\mu]) \\
&= g^f({}^V(\nabla_X\eta), {}^V\mu) + g^f({}^V\eta, {}^V(\nabla_X\mu)) + g^f({}^V\mu, {}^V(\nabla_X\eta)) - g^f({}^V\eta, {}^V(\nabla_X\mu)) \\
&= 2g^f({}^V(\nabla_X\eta), {}^V\mu).
\end{aligned}$$

from which we find the expression of $\widetilde{\nabla}_{{}^HX}{}^V\eta$.
The remaining formulas can be derived through similar calculations. □

Lemma 3.3. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. Then, the following relations hold:*

$$\begin{array}{rcl}
(1)\ \widetilde{\nabla}_{^HX}{}^Vp &=& \dfrac{rf'(r)}{f(r)}{}^HX,\\
(2)\ \widetilde{\nabla}_{^Vp}{}^HX &=& \dfrac{rf'(r)}{f(r)}{}^HX,\\
(3)\ \widetilde{\nabla}_{^V\omega}{}^Vp &=& {}^V\omega,\\
(4)\ \widetilde{\nabla}_{^Vp}{}^V\omega &=& 0,\\
(5)\ \widetilde{\nabla}_{^Vp}{}^Vp &=& {}^Vp,
\end{array}$$

for each vector field $X \in \Im^1_0(M)$ and covector field $\omega \in \Im^0_1(M)$.

Proof. In the proof, we use the formulas (2.3) and (2.4), as well as the Theorem 3.1. Through simple calculations, we obtain:

$$\begin{array}{rcl}
(1)\ \widetilde{\nabla}_{^HX}{}^Vp &=& \widetilde{\nabla}_{^HX}(p_k{}^V(dx^k)) = {}^HX(p_k){}^V(dx^k) + p_k\widetilde{\nabla}_{^HX}{}^V(dx^k)\\
&=& X^i\partial_i(p_k){}^V(dx^k) + p_h\Gamma^h_{ij}X^j\partial_{\bar{i}}(p_k){}^V(dx^k) + p_k{}^V(\nabla_X dx^k)\\
&& +p_k\dfrac{f'(r)}{f(r)}g^{-1}(dx^k,p){}^HX + \dfrac{p_k}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp(dx^k))X)\\
&=& X(p_k){}^V(dx^k) + p_h\Gamma^h_{kj}X^j{}^V(dx^k) + p_k{}^V(\nabla_X dx^k)\\
&& +\dfrac{f'(r)}{f(r)}g^{-1}(p,p){}^HX + \dfrac{p_k}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp p)X)\\
&=& {}^V(\nabla_X(p_k dx^k)) - {}^V(\nabla_X p) + \dfrac{f'(r)}{f(r)}g^{-1}(p,p){}^HX\\
&=& \dfrac{rf'(r)}{f(r)}{}^HX.
\end{array}$$

$$\begin{array}{rcl}
(2)\ \widetilde{\nabla}_{^Vp}{}^HX &=& \widetilde{\nabla}_{p_k{}^V(dx^k)}{}^HX = p_k\widetilde{\nabla}_{^V(dx^k)}{}^HX\\
&=& p_k\dfrac{f'(r)}{f(r)}g^{-1}(dx^k,p){}^HX + \dfrac{p_k}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp(dx^k))X)\\
&=& \dfrac{f'(r)}{f(r)}g^{-1}(p,p){}^HX + \dfrac{1}{2f(r)}{}^H(\mathrm{R}(\sharp p, \sharp p)X)\\
&=& \dfrac{rf'(r)}{f(r)}{}^HX.\\
(3)\ \widetilde{\nabla}_{^V\omega}{}^Vp &=& \widetilde{\nabla}_{^V\omega}(p_k{}^V(dx^k)) = {}^V\omega(p_k){}^V(dx^k) + p_k\widetilde{\nabla}_{^V\omega}{}^V(dx^k)\\
&=& \omega_k{}^V(dx^k)\\
&=& {}^V\omega.\\
(4)\ \widetilde{\nabla}_{^Vp}{}^V\omega &=& \widetilde{\nabla}_{p_k{}^V(dx^k)}{}^V\omega = p_k\widetilde{\nabla}_{^V(dx^k)}{}^V\omega = 0.\\
(5)\ \widetilde{\nabla}_{^Vp}{}^Vp &=& \widetilde{\nabla}_{p_h{}^V(dx^h)}(p_k{}^V(dx^k)) = p_h{}^V(dx^h)(p_k){}^V(dx^k) + p_hp_k\widetilde{\nabla}_{^V(dx^h)}{}^V(dx^k)\\
&=& p_k{}^V(dx^k)\\
&=& {}^Vp.
\end{array}$$

$\square$

Definition 3.2. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. For any $(1,1)$-tensor field F on M, we can define associated horizontal and vertical vector fields ${}^HF$ and ${}^VF$ of F to T^*M as follows:*

$$\begin{array}{rcl} {}^HF : T^*M & \to & TT^*M \\ (x,p) & \mapsto & {}^HF(x,p) = {}^H(F_x(\sharp p)), \\ {}^VF : T^*M & \to & TT^*M \\ (x,p) & \mapsto & {}^VF(x,p) = {}^V(pF_x). \end{array}$$

Locally we have

$$ {}^H(F(\sharp p)) = (\sharp p)^i\, {}^H(F(\partial_i)), \tag{3.15}$$

$$ {}^V(pF) = p_i\, {}^V(dx^iF), \tag{3.16}$$

where $p = p_j dx^j$ and $\sharp p = (\sharp p)^i \partial_i = g^{ij} p_j \partial_i$.

From Theorem 3.1, we obtain

Proposition 3.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. For any $(1,1)$-tensor field F on M. Then, the following relations hold:*

$$\begin{array}{rcl} \widetilde{\nabla}_{{}^HX}{}^H(F(\sharp p)) & = & {}^H((\nabla_X F)(\sharp p)) - f'(r) g(X, F(\sharp p))\, {}^Vp + \dfrac{1}{2}\, {}^V(p\mathrm{R}(X, F(\sharp p))), \\ \widetilde{\nabla}_{{}^HX}{}^V(pF) & = & {}^V(p(\nabla_X F)) + \dfrac{f'(r)}{f(r)} g^{-1}(pF, p)\, {}^HX + \dfrac{1}{2f(r)}\, {}^H(\mathrm{R}(\sharp p, \sharp(pF))X), \\ \widetilde{\nabla}_{{}^V\omega}{}^H(F(\sharp p)) & = & {}^H(F(\sharp\omega)) + \dfrac{f'(r)}{f(r)} g^{-1}(\omega, p)\, {}^H(F(\sharp p)) + \dfrac{1}{2f(r)}\, {}^H(\mathrm{R}(\sharp p, \sharp\omega)F(\sharp p)), \\ \widetilde{\nabla}_{{}^V\omega}{}^V(pF) & = & {}^V(\omega F), \end{array}$$

for each vector field $X \in \Im^1_0(M)$ and covector field $\omega \in \Im^0_1(M)$, where $\sharp(pF) = g^{-1} \circ (pF)$.

4. The Riemannian curvature tensors of the horizontally deformed Sasaki metric

We now consider the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ associated with the cotangent bundle T^*M equipped with the horizontally deformed Sasaki metric g^f. The theorem below presents the various forms of the curvature tensor in explicit form.

Theorem 4.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated Riemannian* *curvature tensor* $\widetilde{\mathrm{R}}$ *with respect to this metric satisfies the following properties:*

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^HX,{}^HY){}^HZ &= {}^H(\mathrm{R}(X,Y)Z)+\frac{1}{2f(r)}{}^H(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Z)\\
&\quad+\frac{1}{4f(r)}\big({}^H(\mathrm{R}(\sharp p,\mathrm{R}(X,Z)\sharp p)Y)-{}^H(\mathrm{R}(\sharp p,\mathrm{R}(Y,Z)\sharp p)X)\big)\\
&\quad+\frac{r(f'(r))^2}{f(r)}\big(g(X,Z){}^HY-g(Y,Z){}^HX\big)\\
&\quad-\frac{1}{2}{}^V\big(p(\nabla_Z\mathrm{R})(X,Y)\big), \qquad (4.17)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^HX,{}^V\eta){}^HZ &= \frac{1}{2f(r)}{}^H((\nabla_X\mathrm{R})(\sharp p,\sharp\eta)Z)+\frac{1}{4f(r)}{}^V(p\mathrm{R}(X,\mathrm{R}(\sharp p,\sharp\eta)Z))\\
&\quad-\frac{1}{2}{}^V(\eta\mathrm{R}(X,Z))+\frac{f'(r)}{2f(r)}g^{-1}(\eta,p){}^V(p\mathrm{R}(X,Z))\\
&\quad+f'(r)g(X,Z){}^V\eta+\frac{f'(r)}{2f(r)}g(\mathrm{R}(\sharp p,\sharp\eta)X,Z){}^Vp\\
&\quad+\big(2f''(r)-\frac{(f'(r))^2}{f(r)}\big)g^{-1}(\eta,p)g(X,Z){}^Vp, \qquad (4.18)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^HX,{}^HY){}^V\mu &= \frac{1}{2f(r)}\big({}^H((\nabla_X\mathrm{R})(\sharp p,\sharp\mu)Y)-{}^H((\nabla_Y\mathrm{R})(\sharp p,\sharp\mu)X)\big)\\
&\quad+\frac{1}{4f(r)}\big({}^V(p\mathrm{R}(X,\mathrm{R}(\sharp p,\sharp\mu)Y))-{}^V(p\mathrm{R}(Y,\mathrm{R}(\sharp p,\sharp\mu)X))\big)\\
&\quad+\frac{f'(r)}{f(r)}\big(g^{-1}(\mu,p){}^V(p\mathrm{R}(X,Y))+g(\mathrm{R}(\sharp p,\sharp\mu)X,Y){}^Vp\big)\\
&\quad-{}^V(\mu\mathrm{R}(X,Y)), \qquad (4.19)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^HX,{}^V\eta){}^V\mu &= \Big(\big((\frac{f'(r)}{f(r)})^2-2\frac{f''(r)}{f(r)}\big)g^{-1}(\eta,p)g^{-1}(\mu,p)-\frac{f'(r)}{f(r)}g^{-1}(\eta,\mu)\Big){}^HX\\
&\quad+\frac{f'(r)}{2(f(r))^2}\big(g^{-1}(\eta,p){}^H(\mathrm{R}(\sharp p,\sharp\mu)X)-g^{-1}(\mu,p){}^H(\mathrm{R}(\sharp p,\sharp\eta)X)\big)\\
&\quad-\frac{1}{2f(r)}{}^H(\mathrm{R}(\sharp\eta,\sharp\mu)X)-\frac{1}{4(f(r))^2}{}^H(\mathrm{R}(\sharp p,\sharp\eta)\mathrm{R}(\sharp p,\sharp\mu)X), \qquad (4.20)
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^V\omega,{}^V\eta){}^HZ &= \frac{1}{4(f(r))^2}\big({}^H(\mathrm{R}(\sharp p,\sharp\omega)\mathrm{R}(\sharp p,\sharp\eta)Z)-{}^H(\mathrm{R}(\sharp p,\sharp\eta)\mathrm{R}(\sharp p,\sharp\omega)Z)\big)\\
&\quad+\frac{f'(r)}{(f(r))^2}\big(g^{-1}(\eta,p){}^H(\mathrm{R}(\sharp p,\sharp\omega)Z)-g^{-1}(\omega,p){}^H(\mathrm{R}(\sharp p,\sharp\eta)Z)\big)\\
&\quad+\frac{1}{f(r)}{}^H(\mathrm{R}(\sharp\omega,\sharp\eta)Z), \qquad (4.21)
\end{aligned}$$

$$\widetilde{\mathrm{R}}({}^V\omega,{}^V\eta){}^V\mu = 0, \qquad (4.22)$$

for each vector fields $X,Y,Z\in\Im^1_0(M)$ *and covector fields* $\omega,\eta,\mu\in\Im^0_1(M)$.

Proof. The proof makes use of Lemmas 2.1, 3.1, and 3.2, together with Theorem 3.1 and Proposition 3.1.

(1) $\widetilde{\mathrm{R}}({}^HX, {}^HY){}^HZ = \widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^HY}{}^HZ - \widetilde{\nabla}_{{}^HY}\widetilde{\nabla}_{{}^HX}{}^HZ - \widetilde{\nabla}_{[{}^HX,{}^HY]}{}^HZ$.

Let $F : TM \to TM$ be the bundle endomorphism given by $pF = p\mathrm{R}(Y, Z)$.
Using (2.8) and direct calculations, we have:

$$\begin{aligned}
\widetilde{\nabla}_{{}^HX}\widetilde{\nabla}_{{}^HY}{}^HZ &= \widetilde{\nabla}_{{}^HX}\big({}^H(\nabla_Y Z) - f'(r)g(Y,Z)^Vp + \frac{1}{2}{}^V(pF)\big)\\
&= {}^H(\nabla_X\nabla_Y Z) - f'(r)g(X,\nabla_Y Z)^Vp + \frac{1}{2}{}^V(p\mathrm{R}(X,\nabla_Y Z))\\
&\quad - f'(r)g(\nabla_X Y, Z)^Vp - f'(r)g(Y,\nabla_X Z)^Vp\\
&\quad - \frac{r(f'(r))^2}{f(r)}g(Y,Z)^HX + \frac{1}{2}{}^V(\nabla_X(p\mathrm{R}(Y,Z)))\\
&\quad - \frac{1}{2}{}^V((\nabla_X p)\mathrm{R}(Y,Z)) - \frac{1}{4f(r)}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(Y,Z)\sharp p)X). \qquad (4.23)
\end{aligned}$$

From which, with permutation of X by Y, we get:

$$\begin{aligned}
\widetilde{\nabla}_{{}^HY}\widetilde{\nabla}_{{}^HX}{}^HZ &= {}^H(\nabla_Y\nabla_X Z) - f'(r)g(Y,\nabla_X Z)^Vp + \frac{1}{2}{}^V(p\mathrm{R}(Y,\nabla_X Z))\\
&\quad - f'(r)g(\nabla_Y X, Z)^Vp - f'(r)g(X,\nabla_Y Z)^Vp\\
&\quad - \frac{r(f'(r))^2}{f(r)}g(X,Z)^HY + \frac{1}{2}{}^V(\nabla_Y(p\mathrm{R}(X,Z)))\\
&\quad - \frac{1}{2}{}^V((\nabla_Y p)\mathrm{R}(X,Z)) - \frac{1}{4f(r)}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(X,Z)\sharp p)Y). \qquad (4.24)
\end{aligned}$$

Also, we find:

$$\begin{aligned}
\widetilde{\nabla}_{[{}^HX,{}^HY]}{}^HZ &= \widetilde{\nabla}_{{}^H[X,Y]}{}^HZ + \widetilde{\nabla}_{{}^V(p\mathrm{R}(X,Y))}{}^HZ\\
&= {}^H(\nabla_{[X,Y]}Z) - f'(r)g([X,Y],Z)^Vp + \frac{1}{2}{}^V(p\mathrm{R}([X,Y],Z))\\
&\quad - \frac{1}{2f(r)}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(X,Y)\sharp p)Z). \qquad (4.25)
\end{aligned}$$

From the formulas (4.23), (4.24), (4.25) and using (2.10) we get:

$$\begin{aligned}
\widetilde{\mathrm{R}}({}^HX, {}^HY){}^HZ &= {}^H(\mathrm{R}(X,Y)Z) + \frac{1}{2f(r)}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(X,Y)\sharp p)Z)\\
&\quad + \frac{1}{4f(r)}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(X,Z)\sharp p)Y) - \frac{1}{4f(r)}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(Y,Z)\sharp p)X)\\
&\quad + \frac{r(f'(r))^2}{f(r)}\big(g(X,Z)^HY - g(Y,Z)^HX\big)\\
&\quad + \frac{1}{2}{}^V\big(p(\nabla_X\mathrm{R})(Y,Z)\big) - \frac{1}{2}{}^V\big(p(\nabla_Y\mathrm{R})(X,Z)\big).
\end{aligned}$$

Using (2.11) and the second Bianchi identity, we obtain the formula (4.17).
The other formulas (4.18)-(4.22) are obtained by a similar calculation. □

Proposition 4.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. Then, the following assertions holds:*
(i) If M is flat and $f = const$ then, T^*M is flat.
(ii) Conversely, if T^*M is flat then, M is flat.

Proof.
(i) This follows directly from Theorem 4.1, which implies that $\widetilde{\mathrm{R}}=0$.
(ii) Suppose that $\widetilde{\mathrm{R}}=0$. Evaluating the Riemannian curvature tensor $\widetilde{\mathrm{R}}$ at a point $(x,0)\in T^*M$, and using equation (4.19), we deduce that $\mathrm{R}=0$. □

Corollary 4.1. *Let (M^m,g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. If $f\neq const$, then T^*M is never flat.*

Let $\{e_i\}_{i=\overline{1,m}}$ (resp. $\{\omega^i\}_{i=\overline{1,m}}$) be a local orthonormal frame (resp. coframe on M), then

$$\{\frac{1}{\sqrt{f(r)}}{}^He_i,{}^V\omega^i\}_{i=\overline{1,m}} \tag{4.26}$$

is a local orthonormal frame on (T^*M,g^f). Moreover, we have

$$\sharp\omega^a=e_a \tag{4.27}$$

Indeed

$$\begin{aligned}\sharp\omega^a &= \sum_{i=1}^m g(\sharp\omega^a,e_i)e_i=\sum_{i,h,k=1}^m g_{hk}(\sharp\omega^a)^h e_i^k e_i=\sum_{i,h,k,j=1}^m g_{hk}g^{jh}\omega_j^a e_i^k e_i\\ &= \sum_{i,k,j=1}^m \delta_k^j\omega_j^a e_i^k e_i=\sum_{i,j=1}^m \omega_j^a e_i^j e_i=\sum_{i=1}^m\omega^a(e_i)e_i=\sum_{i=1}^m\delta_i^a e_i=e_a.\end{aligned}$$

Theorem 4.2. *Let (M^m,g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated Ricci curvature $\widetilde{\mathrm{Ric}}$ with respect to this metric satisfies the following properties:*

$$\begin{aligned}1.\ \widetilde{\mathrm{Ric}}({}^HX,{}^HY) &= \mathrm{Ric}(X,Y)-\frac{1}{2f(r)}\sum_{i=1}^m g(\mathrm{R}(e_i,X)\sharp p,\mathrm{R}(e_i,Y)\sharp p)\\ &\quad -\big(2rf''(r)+mf'(r)+(m-2)r\frac{(f'(r))^2}{f(r)}\big)g(X,Y),\end{aligned} \tag{4.28}$$

$$2.\ \widetilde{\mathrm{Ric}}({}^HX,{}^V\eta) = \frac{1}{2f(r)}\sum_{i=1}^m g((\nabla_{e_i}\mathrm{R})(\sharp p,\sharp\eta)X,e_i), \tag{4.29}$$

$$\begin{aligned}3.\ \widetilde{\mathrm{Ric}}({}^V\omega,{}^V\eta) &= \frac{1}{4(f(r))^2}\sum_{i=1}^m g(\mathrm{R}(\sharp p,\tilde{\omega})e_i,\mathrm{R}(\sharp p,\sharp\eta)e_i)-\frac{mf'(r)}{f(r)}g^{-1}(\omega,\eta)\\ &\quad +\big(m(\frac{f'(r)}{f(r)})^2-2m\frac{f''(r)}{f(r)}\big)g^{-1}(\omega,p)g^{-1}(\eta,p),\end{aligned} \tag{4.30}$$

for each vector fields $X,Y\in\Im_0^1(M)$ and covector fields $\omega,\eta\in\Im_1^0(M)$, where Ric *represents the Ricci curvature associated with the Riemannian manifold (M^m,g).*

Proof. Using (4.26), we have

$$\begin{aligned}\widetilde{\mathrm{Ric}}({}^HX,{}^HY) &= \sum_{i=1}^m g^f(\widetilde{\mathrm{R}}(\frac{1}{\sqrt{f(r)}}{}^He_i,{}^HX){}^HY,\frac{1}{\sqrt{f(r)}}{}^He_i)+\sum_{i=1}^m g^f(\widetilde{\mathrm{R}}({}^V\omega^i,{}^HX){}^HY,{}^V\omega^i)\\ &= \frac{1}{f(r)}\sum_{i=1}^m g^f(\widetilde{\mathrm{R}}({}^He_i,{}^HX){}^HY,{}^He_i)-\sum_{i=1}^m g^f(\widetilde{\mathrm{R}}({}^HX,{}^V\omega^i){}^HY,{}^V\omega^i)\end{aligned}$$

Using (4.17), (4.18) and (4.27), and performing direct calculations, we find

$$\begin{aligned}\widetilde{\mathrm{Ric}}({}^{H}X,{}^{H}Y) &= \mathrm{Ric}(X,Y)-\frac{3}{4f(r)}\sum_{i=1}^{m} g(\mathrm{R}(e_i,X)\sharp p,\mathrm{R}(e_i,Y)\sharp p)\\ &\quad+\frac{1}{4f(r)}\sum_{i=1}^{m} g(\mathrm{R}(e_i,\sharp p)X,\mathrm{R}(e_i,\sharp p)Y)\\ &\quad-\big(2rf''(r)+mf'(r)+(m-2)r\frac{(f'(r))^2}{f(r)}\big)g(X,Y).\end{aligned}$$

To simplify this expression, we utilize

$$\sum_{i=1}^{m} g(\mathrm{R}(e_i,X)\sharp p,\mathrm{R}(e_i,Y)\sharp p)=\sum_{i=1}^{m} g(\mathrm{R}(e_i,\sharp p)X,\mathrm{R}(e_i,\sharp p)Y).$$

This completes the proof.
The remaining identities, namely (4.29) and (4.30), can be derived through analogous computations. □

We proceed to evaluate the sectional curvature of the cotangent bundle (T^*M, g^f). The sectional curvature $\widetilde{\mathrm{K}}(V,W)$ corresponding to the plane P spanned by two linearly independent tangent vectors $V, W \in TT^*M$ is given by the expression:

$$\widetilde{\mathrm{K}}(V,W)=\frac{g^f(\widetilde{\mathrm{R}}(V,W)W,V)}{g^f(V,V)g^f(W,W)-g^f(V,W)^2}, \tag{4.31}$$

where $\widetilde{\mathrm{R}}$ denotes the Riemann curvature tensor associated with the Levi-Civita connection of the metric g^f.

We examine three specific cases of sectional curvature on (T^*M, g^f)

- $\widetilde{\mathrm{K}}({}^{H}X,{}^{H}Y)$ for the horizontal plane spanned by $\{{}^{H}X,{}^{H}Y\}$.
- $\widetilde{\mathrm{K}}({}^{H}X,{}^{V}\eta)$ for the mixed horizontal-vertical plane spanned by $\{{}^{H}X,{}^{V}\eta\}$.
- $\widetilde{\mathrm{K}}({}^{V}\omega,{}^{V}\eta)$ or the vertical plane spanned by $\{{}^{V}\omega,{}^{V}\eta\}$.

where X, Y are orthonormal vector fields and ω, η are orthonormal covector fields on the base manifold M.

Theorem 4.3. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated sectional curvature $\widetilde{\mathrm{K}}$ with respect to this metric satisfies the following properties:*

$$\begin{aligned}\widetilde{\mathrm{K}}({}^{H}X,{}^{H}Y) &= \frac{1}{f(r)}\mathrm{K}(X,Y)-\frac{3}{4(f(r))^2}|\mathrm{R}(X,Y)\sharp p|^2-r\big(\frac{f'(r)}{f(r)}\big)^2,\\ \widetilde{\mathrm{K}}({}^{H}X,{}^{V}\eta) &= \frac{1}{4(f(r))^2}|\mathrm{R}(\sharp p,\sharp\eta)X|^2+\big(\big(\frac{f'(r)}{f(r)}\big)^2-2\frac{f''(r)}{f(r)}\big)g^{-1}(\eta,p)^2-\frac{f'(r)}{f(r)},\\ \widetilde{\mathrm{K}}({}^{V}\omega,{}^{V}\eta) &= 0,\end{aligned}$$

where K represents the sectional curvature associated with the Riemannian manifold (M^m, g).

Proof. Using Theorem 4.1,(4.31) and direct calculations we have,

$$
\begin{aligned}
1.\ \widetilde{\mathrm{K}}({}^{H}X,{}^{H}Y) &= \frac{g^f(\widetilde{\mathrm{R}}({}^{H}X,{}^{H}Y)^{H}Y,{}^{H}X)}{g^f({}^{H}X,{}^{H}X)g^f({}^{H}Y,{}^{H}Y)-g^f({}^{H}X,{}^{H}Y)^2}\\
&= \frac{1}{(f(r))^2}\Big(f(r)g(\mathrm{R}(X,Y)Y,X)+\frac{1}{2}g(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Y,X)\\
&\quad +\frac{1}{4}g(\mathrm{R}(\sharp p,\mathrm{R}(X,Y)\sharp p)Y,X)-r(f'(r))^2\Big)\\
&= \frac{1}{f(r)}\mathrm{K}(X,Y)-\frac{3}{4(f(r))^2}|\mathrm{R}(X,Y)\sharp p|^2-r\big(\frac{f'(r)}{f(r)}\big)^2.
\end{aligned}
$$

$$
\begin{aligned}
2.\ \widetilde{\mathrm{K}}({}^{H}X,{}^{V}\eta) &= \frac{g^f(\widetilde{\mathrm{R}}({}^{H}X,{}^{V}\eta)^{V}\eta,{}^{H}X)}{g^f({}^{H}X,{}^{H}X)g^f({}^{V}\eta,{}^{V}\eta)-g^f({}^{H}X,{}^{V}\eta)^2}\\
&= \frac{1}{f(r)}\Big(\big(\frac{(f'(r))^2}{f(r)}-2f''(r)\big)g^{-1}(\eta,p)^2-\frac{f'(r)}{f(r)}g^{-1}(\eta,\eta)\\
&\quad +\frac{1}{4f(r)}|\mathrm{R}(\sharp p,\sharp\eta)X|^2\Big)\\
&= \frac{1}{4(f(r))^2}|\mathrm{R}(\sharp p,\sharp\eta)X|^2+\big(\big(\frac{f'(r)}{f(r)}\big)^2-2\frac{f''(r)}{f(r)}\big)g^{-1}(\eta,p)^2-\frac{f'(r)}{f(r)}.\\
3.\ \widetilde{\mathrm{K}}({}^{V}\omega,{}^{V}\eta) &= \frac{g^f(\widetilde{\mathrm{R}}({}^{V}\omega,{}^{V}\eta)^{V}\eta,{}^{V}\omega)}{g^f({}^{V}\omega,{}^{V}\omega)g^f({}^{V}\eta,{}^{V}\eta)-g^f({}^{V}\omega,{}^{V}\eta)^2}=0.
\end{aligned}
$$

□

Theorem 4.4. *Let (M^m,g) be a Riemannian manifold, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. The associated scalar curvature $\widetilde{\sigma}$ with respect to this metric satisfies the following:*

$$
\begin{aligned}
\widetilde{\sigma} &= \frac{1}{f(r)}\sigma-\frac{1}{4(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)\sharp p|^2-4mr\frac{f''(r)}{f(r)}-2m^2\frac{f'(r)}{f(r)}\\
&\quad -(m^2-3m)r\big(\frac{f'(r)}{f(r)}\big)^2. \qquad (4.32)
\end{aligned}
$$

where σ represents the scalar curvature associated with the Riemannian manifold (M^m,g).

Proof. Let $\{\frac{1}{\sqrt{f(r)}}{}^{H}e_i,{}^{V}\omega^i\}_{i=1,\dots,m}$ be a local orthonormal frame on (T^*M,g^f) defined by (4.26). We will compute the scalar curvature of (T^*M,g^f), as follows:

$$
\widetilde{\sigma} = \frac{1}{f(r)}\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{H}e_j,{}^{H}e_j)+\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{V}\omega^j,{}^{V}\omega^j). \qquad (4.33)
$$

Using (4.28), we have

$$
\begin{aligned}
\frac{1}{f(r)}\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^{H}e_j,{}^{H}e_j) &= \frac{1}{f(r)}\sigma-\frac{1}{2(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)\sharp p|^2\\
&\quad -2mr\frac{f''(r)}{f(r)}-m^2\frac{f'(r)}{f(r)}-m(m-2)r\big(\frac{f'(r)}{f(r)}\big)^2. \qquad (4.34)
\end{aligned}
$$

Using (4.30), we have

$$\sum_{j=1}^{m}\widetilde{\mathrm{Ric}}({}^V\omega^j,{}^V\omega^j) = \frac{1}{4(f(r))^2}\sum_{i,j=1}^{m}|\mathrm{R}(\sharp p,\sharp\omega^j)e_i|^2 -2mr\frac{f''(r)}{f(r)}-m^2\frac{f'(r)}{f(r)}+mr\Big(\frac{f'(r)}{f(r)}\Big)^2. \tag{4.35}$$

On the other hand, we have

$$\sum_{i,j=1}^{m}|\mathrm{R}(\sharp p,e_j)e_i|^2=\sum_{i,j=1}^{m}|\mathrm{R}(e_i,e_j)\sharp p|^2. \tag{4.36}$$

Substituting (4.34), (4.35) and (4.36) into (4.33), we find (4.32). □

From Theorem 4.4, we obtain

Proposition 4.2. *Let (M^m,g) be a Riemannian manifold of constant sectional curvature κ, and consider its cotangent bundle T^*M endowed with the horizontally deformed Sasaki metric g^f. Then scalar curvature $\widetilde{\sigma}$ of (T^*M,g^f) is given by:*

$$\widetilde{\sigma}=\frac{m(m-1)\kappa}{f(r)}-\frac{\kappa^2(m-1)r}{2(f(r))^2}-4mr\frac{f''(r)}{f(r)}-2m^2\frac{f'(r)}{f(r)}-(m^2-3m)r\Big(\frac{f'(r)}{f(r)}\Big)^2.$$

5. Horizontally deformed Sasaki metric on hypersurface

Let us denote by T^*M_0 the cotangent bundle of a manifold M, excluding the zero section. Consider a smooth real-valued function $f:[0,+\infty[\to]0,+\infty[$ satisfying $f'(r)\neq 1$, where $r=g^{-1}(p,p)$.

We define a hypersurface of T^*M_0 by

$$T^*_{f(r)}M=\big\{(x,p)\in T^*M_0,\ f(r)=r\big\}.$$

To describe this hypersurface, consider the smooth function $\phi:T^*M_0\to\mathbb{R}$ defined by

$$\phi(x,p)=f(r)-r,$$

Then the hypersurface $T^*_{f(r)}M$ is given as the zero level set of ϕ:

$$T^*_{f(r)}M=\{(x,p)\in T^*M_0,\quad \phi(x,p)=0\}.$$

The gradient gradϕ computed with respect to the horizontally deformed Sasaki metric g^f, provides a normal vector field to the hypersurface $T^*_{f(r)}M$. From Lemma 3.1, we obtain:

$$\begin{aligned} g^f({}^HZ,\mathrm{grad}\phi) &= {}^HZ(\phi)={}^HZ(f(r)-r)=0,\\ g^f({}^V\omega,\mathrm{grad}\phi) &= {}^V\omega(\phi)={}^V\omega(f(r)-r)=2(f'(r)-1)g^{-1}(\omega,p)=2(f'(r)-1)g^f({}^V\omega,{}^Vp).\end{aligned}$$

for each vector field $Z\in\Im^1_0(M)$ and covector field $\omega\in\Im^0_1(M)$. Thus, we conclude

$$\mathrm{grad}\phi=2(f'(r)-1)^Vp.$$

Accordingly, a unit normal vector field $\mathcal{N}$ to $T^*_{f(r)}M$ is expressed by

$$\mathcal{N}=\frac{\mathrm{grad}\phi}{|\mathrm{grad}\phi|_{g^f}}=\frac{{}^Vp}{\sqrt{g^f({}^Vp,{}^Vp)}}=\frac{1}{\sqrt{r}}{}^Vp.$$

We define the tangential lift ${}^T\omega$ of a covector $\omega \in T_x^*M$ at the point $(x,p) \in T^*_{f(r)}M$ with respect to the metric g^f as the orthogonal projection of the vertical lift ${}^V\omega$ onto the tangent space of the hypersurface, that is:

$$ {}^T\omega = {}^V\omega - g^f_{(x,p)}({}^V\omega, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} = {}^V\omega - \frac{1}{r}g_x^{-1}(\omega,p)^Vp. $$

Using the decomposition given by Equation (2.2), any tangent vector $W \in T_{(x,p)}T^*M$ at a point (x,p) can be expressed in terms of horizontal and vertical lifts. That is, there exist $v \in T_xM$ and $w \in T_x^*M$, such that

$$ \begin{aligned} W &= {}^Hv + {}^Vw \\ &= {}^Hv + {}^Tw + g^f_{(x,p)}({}^Vw, \mathcal{N}_{(x,p)})\mathcal{N}_{(x,p)} \\ &= {}^Hv + {}^Tw + \frac{1}{r}g_x^{-1}(w,p)^Vp, \end{aligned} $$

where $(x,p) \in T^*_{f(r)}M$.This expression leads directly to the orthogonal decomposition:

$$ T_{(x,p)}T^*M = T_{(x,p)}T^*_{f(r)}M \oplus span\{\mathcal{N}_{(x,p)}\}, \tag{5.37} $$

which implies that the tangent space to the hypersurface $T_{(x,p)}T^*M$ at (x,p) is expressed by:

$$ T_{(x,p)}T^*_{f(r)}M = \{{}^Hv + {}^Tw \,/\, v, \in T_xM, \omega \in p^\perp\}, $$

where

$$ p^\perp = \{w \in T_x^*M,\ g^{-1}(w,p) = 0\} \subset T_x^*M. $$

denotes the orthogonal complement of p with respect to the inverse metric g^{-1}.

Given a covector field ω on M, its tangential lift to the hypersurface $T_{(x,p)}T^*M$ is expressed by:

$$ {}^T\omega_{(x,p)} = \left({}^V\omega - g^f({}^V\omega, \mathcal{N})\mathcal{N}\right)_{(x,p)} = {}^V\omega_{(x,p)} - \frac{1}{r}g_x^{-1}(\omega_x,p)^Vp. $$

To simplify notation, we define the projected covector

$$ \overline{\omega} = \omega - \frac{1}{r}g^{-1}(\omega,p)p, $$

so that ${}^T\omega = {}^V\overline{\omega}$.

Now, consider the hypersurface $T^*_{f(r)}M$ endowed with the induced metric from the horizontally deformed Sasaki metric g^f on T^*M. With this induced metric, also denoted by g^f, $T^*_{f(r)}M$ becomes a Riemannian submanifold. Its Levi-Civita connection $\widehat{\nabla}$ is described by the Gauss formula:

$$ \widehat{\nabla}_P Q = \widetilde{\nabla}_P Q - g^f(\widetilde{\nabla}_P Q, \mathcal{N})\mathcal{N}, \tag{5.38} $$

for all vector fields P and Q tangent to $T^*_{f(r)}M$, where $\widetilde{\nabla}$ is the Levi-Civita connection on (T^*M, g^f).

The following theorem provides explicit formulas for $\widehat{\nabla}$ in terms of the horizontal and tangential lifts.

Theorem 5.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle hypersurface $(T^*_{f(r)}M, g^f)$. Then the associated Levi-Civita connection $\widehat{\nabla}$ satisfies the following identities:*

$$\begin{aligned}
\widehat{\nabla}_{^HX}{}^HY &= {}^H(\nabla_XY) + \frac{1}{2}{}^T(p\mathrm{R}(X,Y)), \\
\widehat{\nabla}_{^HX}{}^T\eta &= {}^T(\nabla_X\eta) + \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\eta)X), \\
\widehat{\nabla}_{^T\omega}{}^HY &= \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\omega)Y), \\
\widehat{\nabla}_{^T\omega}{}^T\eta &= \frac{-1}{r}g^{-1}(\eta, p)^T\omega,
\end{aligned}$$

for each vector fields $X, Y \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$.

Proof. The proof relies on Theorem 3.1, Lemma 3.3, and the Gauss formula (5.38):

• From Theorem 3.1, we compute:

$$\begin{aligned}
\widehat{\nabla}_{^HX}{}^HY &= \widetilde{\nabla}_{^HX}{}^HY - g^f(\widetilde{\nabla}_{^HX}{}^HY, \mathcal{N})\mathcal{N} \\
&= {}^H(\nabla_XY) - f'(r)g(X,Y)^Vp + \frac{1}{2}{}^V(p\mathrm{R}(X,Y)) + f'(r)g(X,Y)g^f({}^Vp, \mathcal{N})\mathcal{N} \\
&\quad -\frac{1}{2}g^f({}^V(p\mathrm{R}(X,Y)), \mathcal{N})\mathcal{N}.
\end{aligned}$$

Noting that

$$g^f({}^Vp, \mathcal{N}) = \sqrt{r} \quad \textit{and} \quad g^f({}^V(p\mathrm{R}(X,Y)), \mathcal{N}) = \frac{1}{\sqrt{r}}g^{-1}(p\mathrm{R}(X,Y), p),$$

we obtain:

$$\widehat{\nabla}_{^HX}{}^HY = {}^H(\nabla_XY) + \frac{1}{2}{}^T(p\mathrm{R}(X,Y)).$$

• We apply the Gauss formula:

$$\widehat{\nabla}_{^HX}{}^T\eta = \widetilde{\nabla}_{^HX}{}^T\eta - g^f(\widetilde{\nabla}_{^HX}{}^T\eta, \mathcal{N})\mathcal{N},$$

From Theorem 3.1 and Lemma 3.3, we obtain

$$\widetilde{\nabla}_{^HX}{}^T\eta = {}^T(\nabla_X\eta) + \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\eta)X),$$

and since both terms are tangent to $T^*_{f(r)}M$, the scalar product with $\mathcal{N}$ vanishes:

$$g^f(\widetilde{\nabla}_{^HX}{}^T\eta, \mathcal{N})\mathcal{N} = 0.$$

Hence:

$$\widehat{\nabla}_{^HX}{}^T\eta = {}^T(\nabla_X\eta) + \frac{1}{2r}{}^H(\mathrm{R}(\sharp p, \sharp\eta)X).$$

• The remaining formulas for $\widehat{\nabla}_{^T\omega}{}^HY$ and $\widehat{\nabla}_{^T\omega}{}^T\eta$ follow analogously, by direct computation from the formulas for $\widetilde{\nabla}$ and applying the orthogonal projection via the Gauss formula. □

We now proceed to compute the Riemannian curvature tensor of the hypersurface $(T^*_{f(r)}M, g^f)$. Let $\widehat{\mathrm{R}}$ denote the Riemannian curvature tensor associated with the Levi-Civita connection $\widehat{\nabla}$ of $(T^*_{f(r)}M, g^f)$. By applying the Gauss equation for hypersurfaces, we obtain the following expression for $\widehat{\mathrm{R}}(P,Q)W$:

$$\widehat{\mathrm{R}}(P,Q)W = {}^t(\widetilde{\mathrm{R}}(P,Q)W) - B(P,W).A_{\mathcal{N}}Q + B(Q,W).A_{\mathcal{N}}P, \tag{5.39}$$

for all vector fields P, Q and W vector fields on $T^*_{f(r)}M$. Here

- $\widetilde{\mathrm{R}}$ denotes the curvature tensor of the ambient manifold (T^*M, g^f),
- ${}^t(\widetilde{\mathrm{R}}(P,Q)W)$ is the tangential component of $\widetilde{\mathrm{R}}(P,Q)W$, taken with respect to the orthogonal decomposition given in Equation (5.37),
- $A_{\mathcal{N}}$ is the shape operator (or Weingarten operator) associated with the unit normal vector field $\mathcal{N}$,
- B denotes the second fundamental form of the hypersurface $T^*_{f(r)}M$ with respect to $\mathcal{N}$.

The shape operator $A_{\mathcal{N}}$ is defined via the Weingarten formula:

$$\widetilde{\nabla}_P\mathcal{N} = -A_{\mathcal{N}}P + \nabla^{\mathcal{N}}_P\mathcal{N},$$

where $\nabla^{\mathcal{N}}$ denotes the normal connection on $T^*_{f(r)}M$. Consequently, we have

$$A_{\mathcal{N}}P = -{}^t(\widetilde{\nabla}_P\mathcal{N}). \tag{5.40}$$

The second fundamental form B is given by the Gauss formula:

$$\widetilde{\nabla}_PQ = \widehat{\nabla}_PQ + B(P,Q)\mathcal{N}.$$

from which we deduce the explicit expression:

$$B(P,Q) = g^f(\widetilde{\nabla}_PQ, \mathcal{N}). \tag{5.41}$$

Lemma 5.1. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle hypersurface $(T^*_{f(r)}M, g^f)$. Then, the following relation holds:*

$$\begin{array}{rcl} A_{\mathcal{N}}{}^HX &=& \dfrac{-f'(r)}{\sqrt{r}}{}^HX, \\ A_{\mathcal{N}}{}^T\omega &=& \dfrac{-1}{\sqrt{r}}{}^T\omega, \\ B({}^HX, {}^HY) &=& -\sqrt{r}f'(r)g(X,Y), \\ B({}^HX, {}^T\eta) &=& B({}^T\omega, {}^HY) = 0, \\ B({}^T\omega, {}^T\eta) &=& \dfrac{-1}{\sqrt{r}}g^{-1}(\overline{\omega}, \overline{\eta}), \end{array}$$

for each vector fields $X, Y \in \Im^1_0(M)$ and covector fields $\omega, \eta \in \Im^0_1(M)$.

Proof. The result is obtained directly by applying Theorem 3.1, Lemma 3.3, together with formulas (5.40) and (5.41). □

Theorem 5.2. *Let (M^m, g) be a Riemannian manifold, and consider its cotangent bundle hypersurface $(T^*_{f(r)}M, g^f)$. Then, the following identities hold:*

$$\begin{array}{rcl} 1.\ \widehat{\mathrm{R}}({}^HX, {}^HY){}^HZ &=& {}^H(\mathrm{R}(X,Y)Z) + \dfrac{1}{2r}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(X,Y)\sharp p)Z) + \dfrac{1}{4r}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(X,Z)\sharp p)Y) \\ && -\dfrac{1}{4r}{}^H(\mathrm{R}(\sharp p, \mathrm{R}(Y,Z)\sharp p)X) - \dfrac{1}{2}{}^T\big(p(\nabla_Z R)(X,Y)\big). \end{array}$$

$$
\begin{aligned}
2.\ \widehat{\mathrm{R}}({}^HX, {}^T\eta)^HZ &= \frac{1}{2r}{}^H((\nabla_X R)(\sharp p, \sharp\eta)Z) + \frac{1}{4r}{}^T(p\mathrm{R}(X, \mathrm{R}(\sharp p, \sharp\eta)Z)) - \frac{1}{2}{}^T(\overline{\eta}\mathrm{R}(X, Z)).\\
3.\ \widehat{\mathrm{R}}({}^HX, {}^HY)^T\mu &= \frac{1}{2r}{}^H((\nabla_X R)(\sharp p, \sharp\mu)Y) - \frac{1}{2r}{}^H((\nabla_Y R)(\sharp p, \sharp\mu)X) - {}^T(\overline{\mu}\mathrm{R}(X, Y))\\
&\quad + \frac{1}{4r}{}^T(p\mathrm{R}(X, \mathrm{R}(\sharp p, \sharp\mu)Y)) - \frac{1}{4r}{}^T(p\mathrm{R}(Y, \mathrm{R}(\sharp p, \sharp\mu)X)).\\
4.\ \widehat{\mathrm{R}}({}^HX, {}^T\eta)^T\mu &= -\frac{1}{2r}{}^H(\mathrm{R}(\sharp\overline{\eta}, \sharp\overline{\mu})X) - \frac{1}{4r^2}{}^H(\mathrm{R}(\sharp p, \sharp\eta)\mathrm{R}(\sharp p, \sharp\mu)X).\\
5.\ \widehat{\mathrm{R}}({}^T\omega, {}^T\eta)^HZ &= \frac{1}{4r^2}{}^H(\mathrm{R}(\sharp p, \sharp\omega)\mathrm{R}(\sharp p, \sharp\eta)Z) - \frac{1}{4r^2}{}^H(\mathrm{R}(\sharp p, \sharp\eta)\mathrm{R}(\sharp p, \sharp\omega)Z)\\
&\quad + \frac{1}{r}{}^H(\mathrm{R}(\sharp\overline{\omega}, \sharp\overline{\eta})Z).\\
6.\ \widehat{\mathrm{R}}({}^T\omega, {}^T\eta)^T\mu &= \frac{1}{r}g(\overline{\eta}, \overline{\mu})^T\omega - \frac{1}{r}g(\overline{\omega}, \overline{\mu})^T\eta.
\end{aligned}
$$

for each $X, Y, Z \in \Im^1_0(M)$ *and* $\omega, \eta, \mu \in \Im^0_1(M)$, *where* $\overline{\omega} = \omega - \frac{1}{r}g^{-1}(\omega, p)p$ *and* $\sharp\overline{\omega} = \sharp\omega - \frac{1}{r}g^{-1}(\omega, p)\sharp p$.

Proof. The proof follows directly from Theorem 4.1, Gauss's equation (5.39) and Lemma 3.2. □

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FS-GKM: A CLUSTERING-DRIVEN FEATURE SELECTION FRAMEWORK FOR ENHANCED SUPERVISED LEARNING PERFORMANCE

DUYGU SELIN TURAN *

Abstract. Feature selection is a fundamental process in machine learning that involves identifying and selecting the most informative features from a dataset to be used in model training. This step is crucial, as it can significantly enhance model performance, mitigate overfitting, reduce computational complexity, and improve the interpretability of the resulting models. By eliminating irrelevant or redundant features, feature selection facilitates the development of more efficient and accurate predictive models. Therefore, the development of new feature selection algorithms is of great importance. In this study, a novel feature selection algorithm—Feature Selection using Global k-Means (FS-GKM)—is proposed. In this study, a novel feature selection algorithm—Feature Selection using Global k-Means (FS-GKM)—is proposed. The method leverages the global k-means clustering algorithm to group features based on their similarity. Subsequently, the algorithm assesses the discriminative power of features by analyzing class density distributions within each cluster. This clustering-based evaluation enables the identification of features that contribute most significantly to class separability. To validate the effectiveness of the FS-GKM method, a comprehensive experimental analysis was conducted using 13 benchmark datasets. Each dataset was subjected to dimensionality reduction through 13 different feature selection techniques, and the resulting feature subsets were evaluated using four distinct classifiers. The proposed FS-GKM algorithm achieved superior performance in 40 out of 52 comparative cases, demonstrating its robustness and effectiveness across diverse scenarios.

Keywords: Feature selection, Classification, Clustering, Preprocessing.

MSC (2020): 68T05, 68T09, 68T10.

1. Introduction

Feature selection plays a vital role in machine learning and data processing by significantly improving the efficiency, scalability, and precision of predictive models, particularly when dealing with large-scale or high-dimensional datasets. By identifying and retaining the most informative features while eliminating irrelevant or redundant ones, feature selection reduces model complexity and computational burden, which is especially critical in text

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mining and vector space modeling applications [26]. This reduction not only improves classification accuracy but also mitigates overfitting, shortens training time, and enhances model interpretability. From a broader perspective, feature selection can be viewed as a search or optimization problem, which becomes increasingly challenging as data dimensionality and volume grow, as highlighted in big data bioinformatics studies [30]. Moreover, in complex real-world systems such as industrial control systems and network intrusion detection, effective feature selection is essential to handle high-dimensional traffic data, remove redundancy, and improve detection efficiency and robustness [7]. Consequently, feature selection is not merely a preprocessing step but a fundamental component for building robust, generalizable, and computationally efficient models capable of handling large and complex datasets.

Several approaches have been developed for feature selection, including filter methods, wrapper methods, and embedded methods. Filter methods evaluate each feature individually, typically using statistical measures, but often fail to consider interactions between features. Wrapper methods assess subsets of features based on model performance and tend to yield higher accuracy; however, they are computationally expensive. Embedded methods perform feature selection during the model training process itself—examples include LASSO and decision trees—offering a good balance between computational efficiency and predictive performance. Nevertheless, each technique comes with its own limitations, such as the overfitting risk in wrapper methods or limited flexibility in filter-based approaches [16].

A variety of feature selection techniques have been proposed in the literature. RELIEF ranks features based on their ability to distinguish between instances of different classes, taking feature interactions into account. While effective for complex datasets, it can be computationally intensive due to repeated nearest-neighbor searches [15]. ANOVA (Analysis of Variance) assesses the importance of features by comparing the means across different classes. It ranks features based on the variance ratio, making it suitable for identifying features that contribute significantly to class separation [25].

The FARNeMF (Feature Adaptive Robust Non-Negative Matrix Factorization) algorithm is designed for feature selection and dimensionality reduction, particularly in high-dimensional datasets. By accounting for noise and outliers, FARNeMF ensures a more robust selection of features. It extends Non-Negative Matrix Factorization (NMF) to enhance the efficiency of feature extraction [12]. Principal Component Analysis (PCA) is another widely used technique for dimensionality reduction in high-dimensional datasets. It transforms data into a set of uncorrelated principal components, simplifying the dataset while retaining most of its variance. Although PCA does not directly select features, it effectively reduces redundancy and enhances computational efficiency while preserving critical information [1].

The Laplacian Score is a filter-based method that evaluates feature importance based on its ability to preserve locality, independent of a learning algorithm. It can be applied in both supervised and unsupervised settings, and has demonstrated superior performance compared to data variance and the Fisher score in several experiments [11]. Recursive Feature Elimination (RFE) improves model performance by iteratively removing the least important features based on model-assigned importance scores. This process continues until only the most informative features are retained, making RFE particularly useful for high-dimensional classification and regression tasks [17].

GBNRS is a novel feature selection approach based on rough set theory, specifically designed to handle continuous data without the need for membership functions or parameter tuning. By incorporating an adaptive neighborhood radius selection mechanism, GBNRS improves attribute reduction and outperforms the state-of-the-art NRS algorithm in classification accuracy [31]. GRRS (Granular-Rectangular Rough Set) offers another innovative approach that avoids distance metrics altogether. Instead, it divides space into granular-rectangular regions and constructs the neighborhood radius based on hierarchical spatial relationships. This method enhances accuracy and efficiency, especially in cases where traditional distance measures are ineffective [32].

To identify a small subset of genes from large-scale gene expression data collected via DNA microarrays, Guyon et al. proposed a gene selection method based on Support Vector Machines (SVM) in combination with RFE. Their approach aims to improve classification performance while also identifying genes with biological relevance to cancer. Experimental results show that their method outperforms traditional correlation-based techniques [10].

The Generalized Fisher Score extends the traditional Fisher score by jointly selecting features that maximize a lower bound of the Fisher criterion. This results in a mixed-integer programming problem, which is solved using a cutting plane algorithm that alternates between multivariate ridge regression and projected gradient descent. The method demonstrates improved performance over both the standard Fisher score and other state-of-the-art approaches [8].

ILFS (Infinite Latent Feature Selection) introduces a robust probabilistic, graph-based algorithm that ranks features by modeling all possible subsets as paths on a graph. By treating feature relevance as a latent variable in a PLSA-inspired generative model, it avoids the combinatorial explosion of subset evaluation. Extensive benchmarking shows that ILFS outperforms eleven state-of-the-art methods across diverse datasets [21].

The method proposed in [29] addresses high-dimensional generalized linear models with Lipschitz loss functions by establishing a non-asymptotic oracle inequality for empirical risk minimization under a LASSO penalty. This penalty is applied to the model coefficients, normalized by their empirical norm. The method is demonstrated through logistic regression, density estimation, classification with hinge loss, and least squares regression tasks. Infinite Feature Selection (Inf-FS) models feature subsets as paths in a graph, where edges encode pairwise feature relationships. It employs matrix power series and Markov chain principles to evaluate paths of arbitrary lengths, ranking features based on relevance and redundancy. An unsupervised variant is also proposed, and experiments show that Inf-FS outperforms 18 state-of-the-art methods across a variety of benchmark datasets [22].

This paper introduces the FS-GKM method, which is based on the Global k-Means clustering algorithm. The primary objective of FS-GKM is to identify the most discriminative features within a dataset. In this approach, the instances in the dataset are kept fixed, and each feature is individually evaluated by clustering its values using the global k-means algorithm. This algorithm addresses the sensitivity of standard k-means to the initial selection of cluster centers. The number of clusters k is set equal to the number of classes in the dataset. After clustering, the class distributions within the resulting clusters are analyzed. A feature is retained if any cluster contains a number of instances from a specific class exceeding a predefined threshold; otherwise, it is discarded.

The remainder of this paper is organized as follows: Section 2 reviews related work in the field. Section 3 provides a detailed description of the proposed FS-GKM method. Section 4 presents the datasets used in the experiments, outlines the four classification algorithms and performance metrics employed, and discusses the experimental results. Finally, Section 5 concludes the paper with a summary of findings and suggestions for future research.

2. Related Work

Categories of feature selection methods are presented in this section.

2.1. Filter Methods.

Filter methods evaluate the importance of features based on statistical measures independently of any machine learning model. These methods are computationally efficient and scalable, making them ideal for large datasets. For instance, ANOVA (Analysis of Variance) is a commonly used filter method that assesses the relationship between categorical target variables and numerical features by analyzing variance. Other examples include correlation coefficients, which measure linear relationships, and chi-square tests, which evaluate the independence of categorical features. While filter methods are fast and easy to implement, they often overlook interactions between features, limiting their effectiveness in complex datasets. Research suggests that filter methods are best suited for preliminary feature selection before applying more advanced techniques [9].

2.2. Wrapper Methods.

Wrapper methods take a distinct approach by using a machine learning model to assess the performance of various feature subsets. These methods are iterative and can be computationally costly, as they involve training and evaluating models multiple times. A popular example is RFE (Recursive Feature Elimination), which systematically eliminates the least important features based on model coefficients or feature importance scores. Other wrapper methods include forward selection, which adds features one at a time, and backward elimination, which removes them sequentially. Although wrapper methods often lead to better model performance by accounting for feature interactions, they are not suitable for very large datasets due to their computational demands. Research indicates that wrapper methods are most effective when sufficient computational resources are available and optimizing model performance is a priority [14].

2.3. Embedded Methods.

Embedded methods integrate feature selection directly into the model training process, offering greater efficiency compared to wrapper methods. These techniques are specific to the model being used and often leverage regularization to penalize less important features. For example, Lasso Regression (L1 Regularization) shrinks the coefficients of less relevant features to zero, effectively performing feature selection. Similarly, tree-based models such as decision trees and random forests naturally rank features based on their importance in splitting nodes. Embedded methods offer a good balance between computational efficiency and model performance, as they account for feature interactions during training. However, their reliance on particular algorithms can limit their adaptability to other models. Research indicates that embedded methods are especially beneficial for high-dimensional datasets, where computational efficiency is a key consideration [27].

2.4. **Unsupervised Methods.** Unsupervised methods are applied when there is no target variable, making them ideal for exploratory data analysis and dimensionality reduction. These techniques focus on transforming or selecting features based on their inherent properties rather than their relationship with a target. A key example is PCA (Principal Component Analysis), which reduces dimensionality by generating new, uncorrelated variables (principal components) that capture the maximum variance in the data. Other unsupervised methods include Independent Component Analysis (ICA) and clustering-based feature selection. While unsupervised methods effectively reduce noise and redundancy, they may not always align with the features' predictive power in supervised tasks. Research suggests that PCA is especially useful for preprocessing high-dimensional data before applying supervised learning methods [13].

2.5. **Hybrid Methods.** Hybrid methods combine the advantages of both filter and wrapper techniques to strike a balance between computational efficiency and model performance. These approaches typically use filter methods to initially reduce the feature space, then apply wrapper methods for further refinement. For instance, genetic algorithms can be guided by filter-based scores to identify the optimal feature subset. Another strategy is ensemble feature selection, where results from multiple methods are combined to enhance robustness. Hybrid methods are particularly valuable for large datasets, where pure wrapper methods would be too computationally expensive. However, they can be complex to implement and require careful parameter tuning. Research indicates that hybrid methods hold great potential for feature selection, particularly in fields like bioinformatics and text mining [24].

2.6. **Domain Knowledge-Based Methods.** Domain knowledge-based methods rely on the insights of experts to identify features known to be significant in a particular field. These approaches often involve feature engineering, where new features are constructed based on domain-specific knowledge. For instance, in healthcare, factors like patient age, medical history, and laboratory test results might be chosen based on clinical expertise. While these methods can produce highly interpretable and contextually relevant features, they can also be subjective and may not easily transfer across different domains. Furthermore, they necessitate collaboration between domain specialists and data scientists, which can be resource-intensive. Research suggests that domain knowledge-based methods are most effective when integrated with data-driven techniques, ensuring both practical relevance and statistical robustness [18].

2.7. **Stability-Based Methods.** Stability-based methods focus on identifying features that remain consistently important across different subsets of data or models. These techniques aim to reduce variability in feature selection results, which can be influenced by random fluctuations in the data. One example is stability selection, which uses subsampling and regularization to identify features consistently selected in multiple iterations. Another approach is bootstrap aggregating (bagging), which combines feature importance scores from multiple bootstrap samples. Stability-based methods are especially beneficial for enhancing the reliability of feature selection in high-dimensional datasets. However, they can be computationally demanding and may require careful parameter tuning. Research suggests that stability-based methods are a robust feature selection approach, particularly when reproducibility is crucial [20].

3. The Proposed Algorithm

In this section, a new clustering-based approach for feature selection is proposed. The core idea of the proposed algorithm is to cluster the features using the global k-means algorithm and select those that effectively distinguish classes based on class densities within the clusters.

3.1. **The Global k-Means Algorithm.** Various algorithms have been developed to address the clustering problem, with the incremental k-means algorithm representing a notable improvement over the traditional k-means method [19]. Unlike classical k-means, which is highly sensitive to the initial choice of cluster centers and often converges to local minima, the incremental k-means algorithm incrementally constructs clusters by systematically evaluating candidate centers. This stepwise approach improves stability and increases the likelihood of reaching a globally optimal solution. Moreover, its ability to dynamically update clusters as new data points are introduced makes it particularly suitable for large-scale and evolving datasets. The pseudocode for the incremental k-means algorithm—given a training set $X^1, \ldots, X^k$ and a predefined number of clusters k—is presented in 1 [28]. The pseudocode of the global k-means algorithm is presented in Algorithm 1.

Algorithm 1 Global k-means

Input: Data set $x_1, \ldots, x_m$, Number of clusters k
Output: Cluster centers $x^1, \ldots, x^k$
1: Initialize the first cluster center as $c^1 \leftarrow \frac{1}{m}\sum_{i=1}^{m} x_i$;
2: Set $t \leftarrow 1$;
3: **while** termination condition $t > k$ is not met **do**
4:   Set $p \leftarrow 1$;
5:   **while** $p \leq m$ **do**
6:     Run the k-means algorithm with initial centers $(c^1, \ldots, c^t, x_p)$;
7:     **if** the clustering result improves upon the previous one **then**
8:       Update cluster centers: $x^i \leftarrow c^i$ for $1 \leq i \leq t-1$ and $x^t \leftarrow c^{t+1}$;
9:     **end if**
10:     $p \leftarrow p + 1$;
11:   **end while**
12:   Update $c^i \leftarrow x^i$ for $1 \leq i \leq t$ and increment $t \leftarrow t + 1$;
13: **end while**
14: **Return** final cluster centers $x^1, \ldots, x^k$;

The global k-means algorithm aims to obtain an optimal clustering configuration at each iteration. It starts by calculating the mean of all data points to establish the first cluster center, initializing $t = 1$. In each subsequent iteration, t is incremented, and every data point is evaluated as a potential candidate for the tth cluster center. During Step 7, the algorithm seeks to minimize the total distance between data points and their assigned centers, retaining the configuration that yields the lowest overall error for the current t-cluster setup. This iterative process continues until the desired number of clusters is reached, at which point the algorithm converges to a final clustering solution.

3.2. **Feature Selection Using the Global k-Means Algorithm.** The FS-GKM algorithm is introduced in this section. It consists of two main stages. In the first stage, each feature is clustered, and the class distributions within the resulting clusters are analyzed. In the second stage, the discriminative power of each feature is evaluated against a predefined threshold. Features exceeding this threshold are deemed informative and included in the selected feature set; those falling below are discarded. The pseudocode of the FS-GKM algorithm is presented in Algorithm 2.

Algorithm 2 Feature Selection Using the Global k-Means Algorithm

```
Input: Data set D, feature set A = {a_1, a_2, ..., a_m}, threshold r, number of clusters k
Output: Selected feature set B
Initialize B ← ∅
for i ← 1 to m do
  Cluster the ith feature into k clusters using Algorithm 1
  for j ← 1 to k do
    for s ← 1 to k do
      Count the number of data points in cluster j that belong to class s, denote this as
      count
      if count ≥ r then
        B ← B ∪ {a_i}
      end if
    end for
  end for
end for
Return the selected feature set B
```

The algorithm takes as input an $m \times n$ data set D, the feature set A, a threshold r used to evaluate class distribution within clusters, and the number of clusters k. Initially, the selected feature set B is empty. Each feature is clustered into k clusters, where k corresponds to the number of classes in the data set. The class distribution within each cluster is then analyzed. If, for any cluster, the number of data points belonging to a class exceeds the threshold r, the corresponding feature is added to the selected set B.

3.3. **Complexity of the FS-GKM Algorithm.** The time complexity of the global k-means algorithm is generally higher than that of standard k-means, typically estimated as $O(n \times k^2 \times m)$, where n denotes the number of data points, k is the number of clusters, m represents the number of features per data point, and t is the number of iterations per k-means run. This increased complexity arises from the algorithm's iterative procedure of adding one cluster center at a time and exhaustively evaluating each data point as a potential new centroid. At each step, a full k-means clustering is performed to optimize cluster assignments, significantly increasing computational cost compared to standard k-means. Although the global k-means offers advantages such as reduced sensitivity to initial centroid placement and a higher likelihood of reaching a global optimum rather than local minima, it does so at the expense of considerably higher computational demands.

In the FS-GKM algorithm, the global k-means is executed m times, where m is the number of features. However, in each run, only a single feature is considered (i.e., dimensionality is 1). Therefore, the time complexity of FS-GKM can be approximated as $O(n \times m \times t \times k^4)$.

TABLE 4.1. Brief description of the datasets

Dataset	Number of Instances	Number of Features	Number of Classes
iris	150	4	3
hepatitis	155	19	2
wine	178	12	3
heart1	270	13	2
iono	351	34	2
horse	368	23	2
wdbc	569	30	2
credit	690	15	2
german	1000	19	2
digits	1797	64	10
wifiLocalization	2000	8	4
madelon	2000	500	2
htru2	17898	8	2

4. EXPERIMENTS

In this section, we present a comparison of the proposed algorithm with other widely used or state-of-the-art feature selection algorithms on 13 benchmark datasets. The datasets were selected from the UCI Machine Learning Repository [6]. A description of these datasets is provided in Table 4.1. Thirteen feature selection algorithms were used for comparison: PCA [1], RFE [17], ANOVA [25], FARNeMF [12], GBNRS [31], CFS [10], Fisher [8], ILFS [21], LASSO [29], Laplacian [29], Inf-FS [22], GRRS-N, and GRRS-1 [32]. We utilized the results obtained by these algorithms as presented in [32]. The FS-GKM algorithm, CFS, Fisher, ILFS, Laplacian, LASSO, Inf-FS, RFE, PCA, and ANOVA were implemented in Python using the Feature Selection Code Library (FSLib) and scikit-learn. The FARNeMF algorithm was run with the neighborhood radius varying from 0.01 to 0.5 in increments of 0.01. The GBNRS algorithm was executed ten times, and the optimal result was retained. The GRRS algorithm was executed with the neighborhood radius varying from 1 to 5 in increments of 1. For the proposed algorithm under test, the threshold value r was considered with increments of 10, ranging from 50 to 90, and the optimal result was retained. For the global k-Means algorithm in FS-GKM, the number of clusters k was set equal to the number of classes in each dataset.

4.1. Classifiers and Performance Measure.

The FS-GKM algorithm was evaluated using four classifiers: SVM, CART, KNN, and MLP. Support Vector Machines (SVM) is a supervised learning algorithm primarily used for classification tasks, aiming to identify the optimal hyperplane that maximizes the margin between data points of different classes in high-dimensional spaces. A notable advantage of SVM is its ability to handle non-linear data efficiently through the kernel trick, which transforms the data into a higher-dimensional space to facilitate separation. However, SVM can be computationally demanding, especially with large datasets, which may limit its suitability for time-sensitive applications [4].

Classification and Regression Trees (CART) is a decision tree algorithm that partitions data into subsets by evaluating feature values, constructing a tree-like structure applicable to both classification and regression. Its main strengths lie in simplicity and interpretability, as the resulting tree can be easily visualized and understood, making the decision-making process transparent. However, CART models are prone to overfitting when the tree grows too deep and may struggle with noisy data or imbalanced class distributions [2].

TABLE 4.2. Accuracy results for all threshold values on the credit dataset

Threshold Value	Accuracy (SVM)	Accuracy (CART)	Accuracy (MLP)
50	0.84	0.77	0.77
60	0.86	**0.86**	**0.80**
70	0.86	0.81	0.77
80	**0.87**	0.84	0.77
90	0.84	0.83	0.78

Multilayer Perceptron (MLP) is a type of artificial neural network composed of multiple layers of interconnected neurons, trained using the backpropagation algorithm. It excels at modeling complex patterns and relationships in data, making it particularly effective for large datasets. However, MLPs require substantial computational resources during training, are susceptible to overfitting, and demand careful tuning of hyperparameters, which can be complex and time-consuming [23].

K-Nearest Neighbors (KNN) is a simple, instance-based algorithm commonly used for both classification and regression. It classifies a data point based on the majority class among its k closest neighbors in the feature space. The main advantage of KNN lies in its simplicity and effectiveness in low-dimensional spaces with minimal training. However, KNN can be computationally intensive during prediction, especially with large datasets, due to the need to compute distances to all training instances [5].

Accuracy, a widely used metric for evaluating machine learning algorithms, was employed to compare the algorithms in this study. It is calculated as the ratio of correctly predicted instances to the total number of instances, as shown in equation 4.1. In this context, True Positive (TP) refers to instances that are correctly predicted as positive, False Positive (FP) refers to instances that are incorrectly predicted as positive, True Negative (TN) refers to instances that are correctly predicted as negative, and False Negative (FN) refers to instances that are incorrectly predicted as negative. TP and TN indicate correct predictions, while FP and FN represent incorrect predictions [3].

$$acc = \frac{\text{TP+TN}}{\text{TP+FP+FN+TN}} \times 100 \tag{4.1}$$

Higher accuracy indicates better model performance and greater reliability.

4.2. **Selection of the Threshold Value for FS-kM and the Number of Nearest Neighbors for KNN.** The FS-GKM algorithm is sensitive to the input threshold value r. Depending on this threshold, the selected feature set B varies. Therefore, FS-GKM was executed for threshold values $r = 50, 60, 70, 80$, and 90. For each threshold value, a feature set B is obtained, resulting in five different B sets. Classification is then performed using all these sets, and the best result among them is considered the final accuracy. The best result among all threshold values was considered the final accuracy. The accuracy results for all threshold values on the credit dataset are given in Table 4.2. The best results are highlighted in bold.

This procedure was repeated for all datasets, and the highest accuracy values were obtained at a threshold of $r = 50$ for the German, Heart1, HTRU2, Madelon, WDBC, and Iris datasets; at $r = 60$ for the Horse and Wine datasets; at $r = 70$ for the Iono, WiFiLocalization, and

TABLE 4.3. Accuracy results of the KNN algorithm for different k values on the credit dataset

Number of Nearest Neighbors k	Threshold Value r				
	50	60	70	80	90
1	0.67	0.65	0.66	0.71	0.74
2	0.67	0.71	0.65	0.66	0.72
3	0.70	0.70	0.68	0.73	0.77
4	0.74	0.74	0.71	0.73	0.75
5	0.72	0.77	0.71	0.75	0.79
6	0.74	0.75	0.72	0.74	0.79
7	0.71	0.73	0.75	0.76	0.79
8	0.72	0.74	0.73	0.78	0.79
9	0.70	0.72	0.72	0.77	**0.80**
10	0.71	0.73	0.73	0.77	0.79

Digits datasets; and at $r = 80$ for the Hepatitis and Credit datasets. The classifiers were run on the datasets obtained from the features selected using the specified threshold values.

The performance of the k-Nearest Neighbors (KNN) algorithm depends heavily on the choice of k, the number of nearest neighbors used for classification or regression. A small k can make the model sensitive to noise and outliers, resulting in overfitting, while a large k smooths the decision boundary and may cause underfitting. Selecting an appropriate k balances bias and variance. In this study, k was tested for values from 1 to 10, and the best accuracy was retained, highlighted in bold in Table 4.3.

4.3. **Computational Results.** Computational experiments are carried out on a laptop with Intel(R) Core(TM) i7-8550U CPU 1.80 GHz and RAM 8 GB. To illustrate the performance of the FS-GKM algorithm, the performance of the algorithm is compared with other 13 feature selection algorithms presented in Section IV using three classifiers in Subsection 4.1. Accuracy as a performance measure in Subsection 4.1 is used for comparison. The results are given in Tables 4.4, 4.5, 4.6, 4.7, 4.9, 4.10, 4.11 and 4.12. In the tables, we report the accuracy values and the best results obtained by the algorithms are highlighted using bold font.

In [32], the first 10 feature selection algorithms listed in Tables 4.4, 4.5, 4.6, 4.7 were applied to the 10 datasets given in the same table. The SVM, MLP, Cart, and KNN classifiers were applied on the reduced datasets. The classification accuracy values from [32] are listed in Tables 4.4, 4.5, 4.6, 4.7.

The SVM classifier under consideration five-fold cross-validation was then executed on the new datasets, consisting of the features selected by the feature selection methods for each dataset, and the accuracy values are shown in the Table 4.4. The accuracy values in the table show that, with the exception of the two datasets named Horse and Madelon, the FS-GKM algorithm outperformed all others in the remaining datasets.

The Cart classifier under consideration five-fold cross-validation was executed on the reduced datasets, consisting of the features selected by the feature selection methods for each dataset, and the accuracy values are shown in the Table 4.5. With the exception of the Hepatitis and Madelon datasets, the best accuracy value was obtained by running the CART classifier on the reduced datasets using the FS-GKM algorithm.

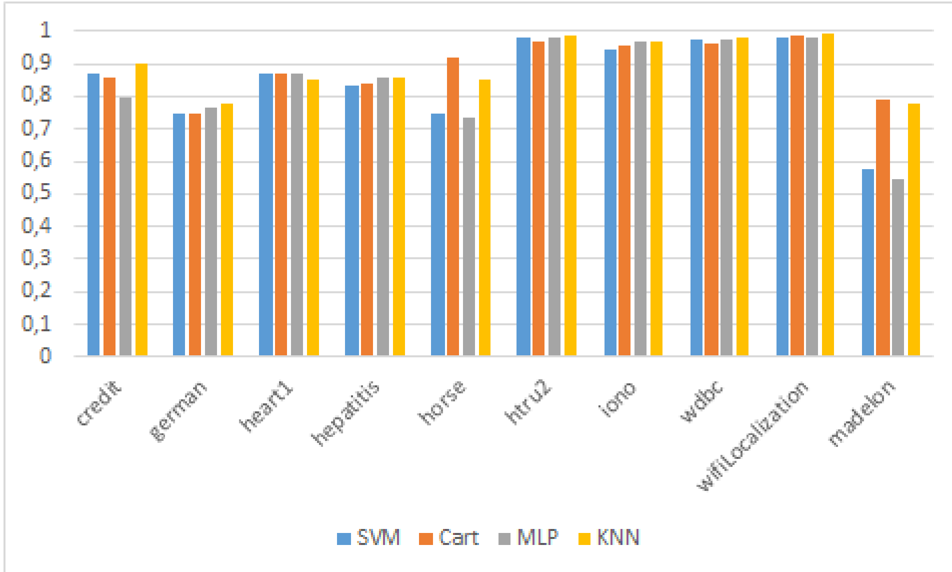

FIGURE 1. Comparison of classifiers for the FS-GKM

The MLP classifier was evaluated using five-fold cross-validation on the reduced datasets, which were composed of features selected by the feature selection methods for each dataset. The corresponding accuracy values are presented in Table 4.6. When the data in the table is examined, the FS-GKM algorithm has overperformed in 70% of the problems.

The KNN classifier was assessed using five-fold cross-validation on the reduced datasets, which consisted of features selected by the feature selection methods for each dataset. The resulting accuracy values are shown in Table 4.7. Upon reviewing the data in the table, it is evident that the FS-GKM algorithm outperformed the others in 80% of the cases.

The graph of accuracy values with four different classifiers, obtained by classifying the reduced datasets using the FS-GKM algorithm, is shown in Figure 1. According to the graph, KNN achieved the highest accuracy in 80% of the cases.

Since the proposed algorithm generates different reduced datasets for different threshold values, it is sensitive to the threshold. Figure 2 visualizes the percentage decrease in the number of features in the dataset for various threshold values. Upon examining the graph, it is evident that as the threshold value increases, the number of features decreases.

In addition to the feature selection methods in [32], the FS-GKM algorithm was compared with the ANOVA, PCA, and RFE feature selection methods, using SVM, MLP, and CART classifiers, on the digits, wine, iris, and wdbc datasets from the Table 4.1. For the ANOVA, RFE, and PCA feature selection algorithms, the number of features to be selected was specified as input. To determine this, the FS-GKM algorithm was first run on the datasets with threshold values of 50, 60, 70, 80, and 90. Based on the result with the highest accuracy, the number of features selected by the FS-GKM algorithm was then used as input for the ANOVA, RFE, and PCA algorithms. The number of features of the original datasets and the number of features selected after the FS-GKM are presented in Table 4.8.

The SVM, MLP, and CART classifiers were trained and tested on the obtained reduced datasets using a 33% percentage split, and accuracy values were obtained. The results

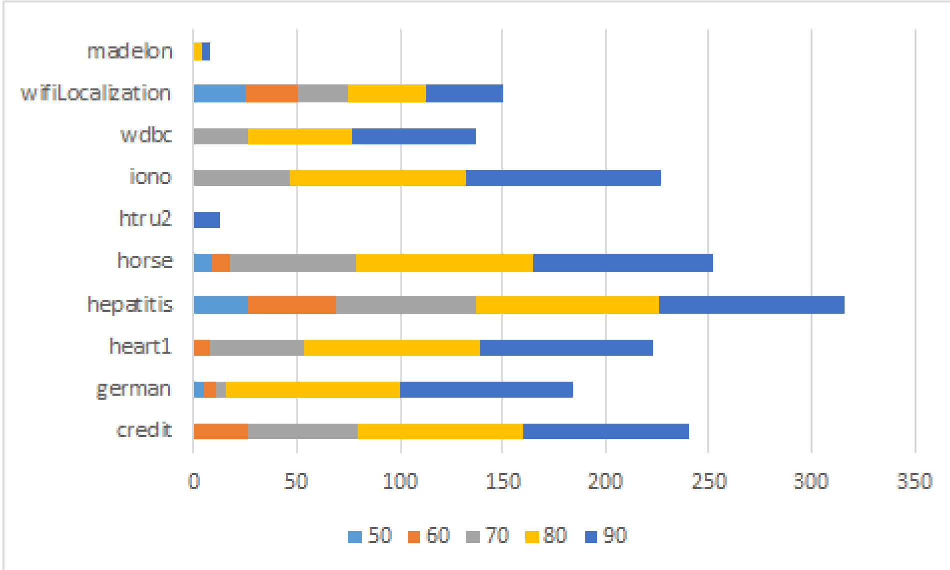

FIGURE 2. Comparison of the number of features for different threshold values

obtained in this way are presented in Tables 4.9, 4.10, 4.11 and 4.12, with the best accuracy values highlighted in bold.

According to Tables 4.9, 4.10, 4.11, and 4.12, using the reduced datasets obtained with the FS-GKM algorithm, the best accuracy was achieved in 1 out of 3 cases for the digits dataset, in 2 out of 3 cases for the wdbc dataset, and in all cases for the wine and iris datasets.

5. CONCLUSION

In this paper, a novel feature selection algorithm, FS-GKM, is introduced. The algorithm partitions the instances in the dataset into clusters based on the number of classes, using the global k-means clustering algorithm. If the class distributions within the resulting clusters exceed a predefined threshold, the feature is considered discriminative, and these features are included in the reduced feature set. Since the proposed algorithm is sensitive to the threshold, it is run with different threshold values, and the best results are retained. To illustrate the effectiveness of the method, 13 benchmark datasets were reduced using 14 different feature selection algorithms, including FS-GKM. Then, SVM, KNN, CART, and MLP classifiers were trained and tested on both the original and reduced datasets. Based on the accuracy values obtained from the training and testing results, it was observed that the datasets reduced with the FS-GKM algorithm showed better performance in approximately 77% of the cases.

Although the effectiveness of the method has been demonstrated through computational experiments, it suffers from sensitivity to the threshold value. As future work, it is planned to explore whether better results can be achieved by using different clustering methods, and to develop an approach for determining the threshold value provided as input to the method.

TABLE 4.4. Accuracy Results of Feature Selection Algorithms Using SVM

Dataset	original	FARNeMF	GBNRS	GRRS-N	GRRS-1	CFS	Fisher	ILFS	Laplacian	LASSO	Inf-FS	FS-GKM
credit	0.855 ± 0.022	0.855 ± 0.021	0.855 ± 0.021	0.855 ± 0.022	0.855 ± 0.022	0.659 ± 0.020	0.629 ± 0.022	0.855 ± 0.022	0.651 ± 0.022	0.855 ± 0.022	0.629 ± 0.022	**0.868 ± 0.037**
german	0.708 ±0.008	0.716 ± 0.017	0.713 ± 0.017	0.721 ± 0.003	0.709 ± 0.008	0.700 ± 0.006	0.727 ± 0.011	0.730 ± 0.005	0.700 ± 0.010	0.702 ± 0.009	0.727 ± 0.011	**0.750 ± 0.033**
heart1	0.742 ± 0.121	0.742 ±0.090	0.734 ± 0.116	0.792 ± 0.069	0.792 ± 0.076	0.786 ± 0.081	0.793 ± 0.078	0.745 ± 0.073	0.742 ± 0.128	0.789 ± 0.078	0.793 ± 0.078	**0.870 ± 0.025**
hepatitis	0.794 ± 0.017	0.793 ± 0.011	0.793 ± 0.011	0.793 ± 0.011	0.793 ± 0.017	0.794 ± 0.017	0.794 ± 0.017	0.793 ± 0.017	0.794 ± 0.017	0.794 ± 0.017	0.794 ± 0.017	**0.834 ± 0.094**
horse	0.809 ± 0.028	0.809 ± 0.028	0.809 ± 0.028	0.809 ± 0.028	**0.810 ± 0.028**	0.649 ± 0.028	0.622 ± 0.057	0.665 ± 0.028	0.613 ± 0.028	0.657 ± 0.053	0.622 ± 0.057	0.750 ± 0.146
htru2	0.974 ± 0.003	0.973 ± 0.003	0.974 ± 0.002	0.976 ± 0.002	0.976 ± 0.002	0.947 ± 0.002	0.975 ± 0.002	0.947 ± 0.003	0.908 ± 0.003	0.976 ± 0.002	0.975 ± 0.002	**0.982 ± 0.002**
iono	0.866 ± 0.015	0.889 ± 0.033	0.854 ± 0.020	0.889 ± 0.030	0.889 ± 0.044	0.900 ± 0.025	0.903 ± 0.047	0.858 ± 0.045	0.843 ± 0.038	0.761 ± 0.042	0.903 ± 0.047	**0.943 ± 0.046**
wdbc	0.952 ± 0.017	0.957 ± 0.020	0.963 ± 0.019	0.963 ± 0.020	0.956 ± 0.006	0.942 ± 0.015	0.926 ± 0.017	0.947 ± 0.017	0.937 ± 0.018	0.909 ± 0.017	0.926 ± 0.017	**0.974 ± 0.013**
wifiLocalization	0.972 ± 0.004	0.974 ± 0.008	0.974 ± 0.008	0.976 ± 0.010	0.976 ± 0.010	0.972 ± 0.020	0.972 ± 0.011	0.972 ± 0.004	0.976 ± 0.010	0.975 ± 0.010	0.972 ± 0.011	**0.983 ± 0.007**
madelon	0.598 ± 0.021	0.616 ± 0.022	0.620 ± 0.026	0.616 ± 0.021	0.610 ± 0.021	0.590 ± 0.026	**0.625 ± 0.015**	0.620 ± 0.010	0.612 ± 0.023	0.615 ± 0.017	0.615 ± 0.017	0.578 ± 0.026

TABLE 4.5. Accuracy Results of Feature Selection Algorithms Using Cart

Dataset	original	FARNeMF	GBNRS	GRRS-N	GRRS-1	CFS	Fisher	ILFS	Laplacian	LASSO	Inf-FS	FS-GKM
credit	0.785 ± 0.018	0.807 ± 0.018	0.816 ± 0.026	0.829 ± 0.018	0.838 ± 0.025	0.744 ± 0.041	0.613 ± 0.035	0.794 ± 0.030	0.701 ± 0.020	0.755 ± 0.022	0.607 ± 0.016	**0.860 ± 0.125**
german	0.684 ±0.043	0.689 ± 0.032	0.680 ± 0.030	0.684 ± 0.019	0.713 ± 0.024	0.697 ± 0.006	0.705 ± 0.013	0.706 ± 0.020	0.656 ± 0.023	0.710 ± 0.019	0.706 ± 0.019	**0.745 ± 0.047**
heart1	0.664 ± 0.084	0.711 ±0.079	0.731 ± 0.058	0.714 ± 0.060	0.731 ± 0.063	0.742 ± 0.080	0.667± 0.030	0.745 ± 0.073	0.671 ± 0.076	0.789 ± 0.079	0.670 ± 0.078	**0.870 ± 0.091**
hepatitis	0.720 ± 0.042	0.811 ± 0.066	0.830 ± 0.097	0.839 ± 0.028	0.832 ± 0.062	**0.845 ± 0.053**	0.768 ± 0.035	0.819 ± 0.062	0.799 ± 0.077	0.794 ± 0.053	0.768 ± 0.035	0.839 ± 0.094
horse	0.793 ± 0.027	0.784 ± 0.051	0.817 ± 0.013	0.820 ± 0.038	0.837 ± 0.044	0.698 ± 0.045	0.614 ± 0.057	0.684 ± 0.049	0.668 ± 0.065	0.636 ± 0.014	0.625 ± 0.062	**0.917 ± 0.172**
htru2	0.967 ± 0.002	0.968 ± 0.003	0.967 ± 0.003	0.967 ± 0.002	0.969 ± 0.002	0.937 ± 0.003	0.963 ± 0.002	0.957 ± 0.003	0.924 ± 0.005	0.967 ± 0.003	0.969 ± 0.002	**0.970 ± 0.002**
iono	0.886 ± 0.038	0.928 ± 0.037	0.891 ± 0.015	0.931 ± 0.030	0.929 ± 0.017	0.920 ± 0.032	0.926 ± 0.039	0.901 ± 0.038	0.897 ± 0.040	0.857 ± 0.037	0.923 ± 0.039	**0.957 ± 0.059**
wdbc	0.928 ± 0.019	0.949 ± 0.015	0.943 ± 0.009	0.953 ± 0.018	0.961 ± 0.023	0.947 ± 0.019	0.940 ± 0.015	0.928 ± 0.026	0.921 ± 0.030	0.912 ± 0.016	0.940 ± 0.010	**0.965 ± 0.027**
wifiLocalization	0.960 ± 0.024	0.963 ± 0.022	0.964 ± 0.021	0.966 ± 0.014	0.969 ± 0.011	0.957 ± 0.028	0.957 ± 0.029	0.965 ± 0.023	0.964 ± 0.015	0.967 ± 0.011	0.955 ± 0.027	**0.988 ± 0.022**
madelon	0.756 ± 0.016	0.785 ± 0.012	0.590 ± 0.026	**0.820 ± 0.034**	0.799 ± 0.023	0.744 ± 0.017	0.761 ± 0.024	0.758 ± 0.016	0.794 ± 0.015	0.752 ± 0.028	0.796 ± 0.024	0.793 ± 0.018

Table 4.6. Accuracy Results of Feature Selection Algorithms Using MLP

Dataset	original	FARNeMF	GBNRS	GRRS-N	GRRS-1	CFS	Fisher	ILFS	Laplacian	LASSO	Inf-FS	FS-GKM
credit	0.866 ± 0.023	**0.872 ± 0.026**	0.865 ± 0.023	0.861 ± 0.019	0.869 ± 0.025	0.750 ± 0.046	0.641 ± 0.025	0.857 ± 0.018	0.762 ± 0.029	0.865 ± 0.022	0.629 ± 0.021	0.798 ± 0.033
german	0.742 ±0.013	0.755 ± 0.021	0.748 ± 0.013	0.749 ± 0.021	0.760 ± 0.023	0.702 ± 0.016	0.752 ± 0.028	0.744 ± 0.005	0.725 ± 0.022	0.724 ± 0.014	0.750 ± 0.027	**0.765 ± 0.037**
heart1	0.756 ± 0.099	0.756 ±0.120	0.762 ± 0.101	0.792 ± 0.069	0.792 ± 0.076	0.762 ± 0.108	0.789± 0.078	0.745 ± 0.061	0.745 ± 0.068	0.789 ± 0.078	0.793 ± 0.078	**0.870 ± 0.040**
hepatitis	0.825 ± 0.048	0.839 ± 0.058	0.813 ± 0.032	0.825 ± 0.039	0.852 ± 0.028	0.833 ± 0.035	0.813 ± 0.036	0.813 ± 0.053	0.820 ± 0.062	0.825 ± 0.035	0.800 ± 0.042	**0.859 ± 0.094**
horse	0.801 ± 0.043	0.817 ± 0.026	0.820 ± 0.018	0.826 ± 0.013	**0.829 ± 0.017**	0.684 ± 0.036	0.649 ± 0.075	0.677 ± 0.045	0.641 ± 0.026	0.649 ± 0.041	0.654 ± 0.071	0.733 ± 0.113
htru2	0.979 ± 0.002	0.979 ± 0.002	0.979 ± 0.002	0.979 ± 0.001	0.980 ± 0.003	0.955 ± 0.001	0.978 ± 0.002	0.974 ± 0.002	0.928 ± 0.001	0.979 ± 0.003	0.978 ± 0.002	**0.981 ± 0.002**
iono	0.914 ± 0.028	0.917 ± 0.018	0.900 ± 0.019	0.897 ± 0.039	0.912 ± 0.029	0.918 ± 0.025	0.923 ± 0.025	0.903 ± 0.032	0.889 ± 0.034	0.872 ± 0.033	0.923 ± 0.032	**0.971 ± 0.043**
wdbc	0.972 ± 0.011	0.968 ± 0.022	0.971 ± 0.019	0.970 ± 0.021	0.972 ± 0.009	0.961 ± 0.012	0.945 ± 0.011	0.949 ± 0.007	0.958 ± 0.015	0.933 ± 0.017	0.942 ± 0.015	**0.976 ± 0.024**
wifiLocalization	0.976 ± 0.014	0.977± 0.013	0.980 ± 0.009	0.979 ± 0.003	0.980 ± 0.006	0.975 ± 0.016	0.973 ± 0.022	0.979 ± 0.010	0.979 ± 0.004	0.976 ± 0.014	0.975 ± 0.024	**0.981 ± 0.006**
madelon	0.559 ± 0.030	0.715 ± 0.012	0.630 ± 0.026	**0.769 ± 0.015**	0.737 ± 0.016	0.571 ± 0.029	0.715 ± 0.023	0.658 ± 0.012	0.757 ± 0.012	0.595 ± 0.020	0.755 ± 0.012	0.543 ± 0.015

TABLE 4.7. Accuracy Results of Feature Selection Algorithms Using KNN

Dataset	original	FARNeMF	GBNRS	GRRS-N	GRRS-1	CFS	Fisher	ILFS	Laplacian	LASSO	Inf-FS	FS-GKM
credit	0.871 ± 0.021	0.863 ± 0.022	0.856 ± 0.032	0.876 ± 0.031	0.860 ± 0.028	0.723 ± 0.015	0.634 ± 0.021	0.839 ± 0.020	0.737 ± 0.020	0.855 ± 0.032	0.634 ± 0.021	**0.898 ± 0.036**
german	0.712 ±0.014	0.741 ± 0.029	0.727 ± 0.016	0.734 ± 0.018	0.739 ± 0.029	0.657 ± 0.017	0.738 ± 0.049	0.732 ± 0.020	0.680 ± 0.020	0.686 ± 0.019	0.738 ± 0.049	**0.780 ± 0.037**
heart1	0.760 ± 0.092	0.758 ±0.097	0.762 ± 0.096	0.775 ± 0.062	0.782 ± 0.076	0.775 ± 0.080	0.738± 0.074	0.734 ± 0.092	0.748 ± 0.092	0.649 ± 0.097	0.738 ± 0.074	**0.852 ± 0.074**
hepatitis	0.799 ± 0.080	0.848 ± 0.037	0.857 ± 0.052	0.851 ± 0.041	0.858 ± 0.028	0.851 ± 0.017	0.781 ± 0.043	0.839 ± 0.048	0.831 ± 0.062	0.805 ± 0.080	0.781 ± 0.043	**0.859 ± 0.094**
horse	0.777 ± 0.056	0.801 ± 0.032	0.845 ± 0.034	**0.861 ± 0.019**	**0.861 ± 0.019**	0.657 ± 0.050	0.640 ± 0.047	0.649 ± 0.062	0.638 ± 0.048	0.640 ± 0.061	0.640 ± 0.047	0.850 ± 0.182
htru2	0.977 ± 0.002	0.977 ± 0.002	0.9977 ± 0.002	0.978 ± 0.001	0.979 ± 0.002	0.950 ± 0.001	0.977 ± 0.002	0.972 ± 0.001	0.929 ± 0.001	0.978 ± 0.001	0.977 ± 0.002	**0.986 ± 0.002**
iono	0.848 ± 0.045	0.911 ± 0.039	0.889 ± 0.057	0.928 ± 0.030	0.926 ± 0.028	0.903 ± 0.033	0.914 ± 0.043	0.903 ± 0.032	0.857 ± 0.050	0.871 ± 0.050	0.914 ± 0.043	**0.971 ± 0.054**
wdbc	0.966 ± 0.015	0.973 ± 0.020	0.966 ± 0.024	0.966 ± 0.022	0.970 ± 0.007	0.959 ± 0.011	0.950 ± 0.015	0.954 ± 0.013	0.956 ± 0.020	0.917 ± 0.017	0.950 ± 0.020	**0.979 ± 0.028**
wifiLocalization	0.975 ± 0.005	0.975± 0.003	0.976 ± 0.006	0.977 ± 0.006	0.977 ± 0.004	0.975 ± 0.030	0.975 ± 0.058	0.975 ± 0.027	0.977 ± 0.014	0.976 ± 0.027	0.975 ± 0.058	**0.993 ± 0.010**
madelon	0.574 ± 0.025	0.750 ± 0.018	0.579 ± 0.027	**0.900 ± 0.004**	0.891 ± 0.013	0.569 ± 0.035	0.737 ± 0.010	0.637 ± 0.022	0.840 ± 0.015	0.571 ± 0.032	0.824 ± 0.015	0.780 ± 0.033

Table 4.8. Comparison of the number of features

Datasets	The number of features of original dataset	The number of features of reduced dataset
iris	4	2
wine	19	7
wdbc	30	23
digits	64	30

Table 4.9. Accuracy results for the dataset digits

	Feature Selection Methods			
Classifiers	ANOVA	RFE	PCA	FS-GKM
SVM	**0.97**	0.96	**0.97**	0.84
MLP	0.97	**0.98**	**0.98**	**0.98**
Cart	0.84	**0.87**	0.84	0.76

Table 4.10. Accuracy results for the dataset wine

	Feature Selection Methods			
Classifiers	ANOVA	RFE	PCA	FS-GKM
SVM	**1**	**1**	0.96	**1**
MLP	0.13	0.70	0.89	**0.93**
Cart	0.94	0.91	0.91	**0.96**

Table 4.11. Accuracy results for the dataset iris

	Feature Selection Methods			
Classifiers	ANOVA	RFE	PCA	FS-GKM
SVM	**1**	0.98	**1**	**1**
MLP	**1**	**1**	**1**	**1**
Cart	**1**	0.98	**1**	**1**

Table 4.12. Accuracy results for the dataset wdbc

	Feature Selection Methods			
Classifiers	ANOVA	RFE	PCA	FS-GKM
SVM	**0.98**	0.97	0.96	**0.98**
MLP	0.95	**0.97**	0.96	0.95
Cart	**0.94**	0.93	0.93	**0.94**

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TORSE-FORMING VECTOR FIELDS AND NORMALIZED NULL HYPERSURFACES

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Abstract. The study focuses on normalized null hypersurfaces within semi-Riemannian manifolds that possess torse-forming vector fields. We demonstrate that these hypersurfaces exhibit a triple product structure. Additionally, we present obstruction results related to both topology and specific geometric criteria concerning the rigging and rigged vector field. We also provide a result on the splitting of the hypersurface, which is endowed with its rigged Riemannian structure. Furthermore, taking the potential vector field as the screen component of a torse-forming vector field, we investigate screen distribution with Ricci solitons.

Keywords: Normalized null hypersurface, Rigging vector field, Ricci solitons, Torse-forming vector field, Screen quasi-conformal.

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1. Introduction

A torse-forming vector field is defined as a vector field K on a semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ that fulfills the following equation:

$$\overline{\nabla}_U K = \Psi U + \mu(U) K,$$

for all vector field $U \in \Gamma(\overline{\Sigma})$, where Ψ is a smooth function on $\overline{\Sigma}$, μ is a one-form, and $\overline{\nabla}$ denotes the Levi-Civita connection of $(\overline{\Sigma}, \overline{g})$. The function Ψ is referred to as the conformal scalar, while the one-form μ is called the generating[24].

Torse-forming vector fields are of particular interest because of their use in submanifold theory, relativity, and cosmology. These vector fields are significant in a number of differential geometry and physics topics (see [7], [23]). K. Yano examined the definition and analysis of torse-forming vector fields for the first time in [30].

In addition to these facts, the notion of a Ricci soliton is initially observed by Hamilton's. After G. Perelman [28] used Ricci solitons to solve the Poincaré conjecture, Ricci flow and Ricci solitons gained interest. A Riemannian manifold $(\overline{\Sigma}, \overline{g})$ with a metric tensor $\overline{g}$ is referred

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to as a Ricci soliton if a smooth vector field V tangent to $\overline{\Sigma}$ exists and satisfies the equation described below:

$$L_V \overline{g} + \overline{R}ic = \Lambda \overline{g},$$

where $L_V \overline{g}$ is the Lie derivative of $\overline{g}$ with respect to V, $\overline{R}$ic represents the Ricci tensor of $(\overline{\Sigma}, \overline{g})$, and Λ is a constant. We call the vector field V the potential field. A Ricci soliton is called shrinking if $\Lambda > 0$, steady if $\Lambda = 0$, expanding if $\Lambda < 0$.

The geometry of Riemannian submanifolds and non-degenerate submanifolds of semi-Riemannian manifolds share a number of similarities, according to [27]. However, the geometry is quite different and difficult when the induced metric on a submanifold is degenerate, and the existing methods fail.

The authors began the general study of arbitrary null submanifolds of semi-Riemannian manifolds in [14],[13]. They avoid the anomaly resulting from the degeneracy of the induced metric by fixing geometric data consisting of a null section and a screen distribution (or, alternatively, a null section and a null transversal section) on a null hypersurface. Although this approach has been very successful, it has the disadvantage of depending on two different and arbitrary decisions, and it doesn't appear to be appropriate for examining the inherent geometry of the null hypersurfaces.

One of the novel methods for dealing with this issue, as presented in [16], is to choose a transversal vector field E defined on an open neighborhood of the null hypersurface Σ on semi-Riemannian manifolds $\overline{\Sigma}$. This is known as the rigging for Σ. Through this selection, a null section ξ (also known as a rigged vector field) is created, such that $\overline{g}(E, \xi) = 1$. This method has also been used in other studies [5, 19, 20, 22, 26, 21]

Many researchers have studied the geometry of null hypersurfaces in Lorentzian manifolds furnished with different kinds of vector fields, including parallel, conformal, closed conformal, homothetic, Killing vector fields, and affine conformal Killing (see [4, 2, 5, 8, 20, 29, 8, 6]). Nevertheless, the case with torse-forming vector fields has not been investigated thus far.

The purpose of this work is to study the geometry of normalized null hypersurfaces of semi-Riemannian manifolds admitting a torse-forming vector field, with a focus on the consequences of the existence of such a vector field on both the geometry of null hypersurfaces and the curvature of the ambient space, as well as exploring the conditions under which screen distributions can admit Ricci soliton structures that arise from screen component of a torse-forming vector field K.

The following is the structure of this paper. Section 2 includes the necessary preliminary information. We demonstrate in Section 3 that such hypersurfaces have a triple product structure (Theorem 3.1, Theorem 3.2, Theorem 3.5, and Theorem 3.6). Theorem 4.1 and Theorem 4.4 establish obstruction results involving both topology and prescribed geometric conditions on the rigging and rigged vector field, while Theorem 4.2 provides a splitting result on the hypersurface equipped with its rigged Riemannian structure. These results are found in Section 4. According to Theorem 5.1, we characterize null hypersurfaces in Section 5, whose screen distributions admit a Ricci soliton induced from the screen component of a torse-forming vector field K.

2. Rigging technique for null hypersurfaces

Let $(\overline{\Sigma}, \overline{g})$ be a $(n+2)$-dimensional semi-Riemannian manifold with index t, where $0 < to < n+2$, and (Σ, g) a null hypersurface of $(\overline{\Sigma}, \overline{g})$. A *rigging* for Σ is a vector field E defined on some open set of $\overline{\Sigma}$ containing Σ such that for each $p \in \Sigma$, $\Sigma_p \notin T_p\Sigma$ [16].

For a fixed rigging E for Σ, we set $\alpha = \overline{g}(E, \cdot)$, $\omega = i^*\alpha$, $\breve{g} = \overline{g} + \alpha \otimes \alpha$ and $\widetilde{g} = i^\star \breve{g}$, where $i : \Sigma \hookrightarrow \overline{\Sigma}$ is the canonical inclusion map. It is well known that $\widetilde{g}$ is a non degenerate metric on Σ. The *rigged vector field* on Σ is the unique null vector field ξ given by $\tilde{g}(\xi, .) = \alpha$ and it satisfies

$$\overline{g}(E, \xi) = 1. \tag{2.1}$$

A null hypersurface Σ furnished with a rigging E is said to be normalized and is denoted (Σ, E). A rigging E defines a screen distribution $\mathscr{S}(E)$ given by $\mathscr{S}(E) = T\Sigma \cap E^\perp = \ker \omega$. The null transverse vector field on Σ is

$$N = E - \frac{1}{2}\overline{g}(E, E)\xi, \tag{2.2}$$

which is the unique null vector field such that $\overline{g}(N, \xi) = 1$. Moreover, $T\overline{\Sigma}$ admits the following splitting

$$\begin{aligned} T\overline{\Sigma}|_\Sigma &= T\Sigma \oplus span(N) \\ &= \{\mathscr{S}(E) \oplus span(\xi)\} \oplus span(N). \end{aligned} \tag{2.3}$$

From the decomposition (2.3), the Gauss and Weingarten equations of Σ and $\mathscr{S}(E)$ are the following [13] :

$$\overline{\nabla}_U V = \nabla_U V + B(U, V)N, \qquad \overline{\nabla}_U N = -A_N U + \tau(U)N, \tag{2.4}$$

$$\nabla_U PY = \overset{\star}{\nabla}_U PY + C(U, PY)\xi, \qquad \nabla_U \xi = -\overset{\star}{A}_\xi U - \tau(U)\xi, \tag{2.5}$$

for any $U, V \in \Gamma(T\Sigma)$, where $\overline{\nabla}$ represents the Levi-Civita connection on $(\overline{\Sigma}, \overline{g})$, ∇ represents the connection on Σ that is induced from $\overline{\nabla}$ via the projection along the null transverse vector field N and $\overset{\star}{\nabla}$ indicates the connection on the screen distribution $\mathscr{S}(E)$ that is induced from ∇ via the projection morphism P of $\Gamma(T\Sigma)$ onto $\Gamma\big(\mathscr{S}(E)\big)$ in accordance with the decomposition (2.5). Now the $(0, 2)$ tensors B and C are the second fundamental forms on $T\Sigma$ and $\mathscr{S}(E)$ respectively, A_N and $\overset{\star}{A}_\xi$ are the shape operators on $T\Sigma$ with respect to the rigging E and the rigged vector field ξ respectively and τ a 1-form on $T\Sigma$ defined by

$$\tau(U) = \overline{g}(\overline{\nabla}_U N, \xi).$$

As $(\nabla_U g)(V, Z) = B(U, V)\omega(Z) + B(U, Z)\omega(V)$, $\forall U, V, Z \in \Gamma(T\Sigma)$, then the induced linear connection ∇ is not a metric connection. Also C is not symmetric as $C(U, V) - C(V, U) = g(\nabla_U V - \nabla_V U, N) = \omega([U, V]), \forall U, V \in \mathscr{S}(E)$.

Let denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Then the following are the Gauss-Codazzi equations [13].

$$
\begin{aligned}
\overline{g}(\overline{R}(U,V)Z,\xi) &= (\nabla_U B)(V,Z) - (\nabla_V B)(U,Z) \\
&\quad + \tau^N(U)B(V,Z) - \tau^N(V)B(U,Z), \\
\overline{g}(\overline{R}(U,V)Z,PT) &= \overline{g}(R(U,V)Z,PT) + B(U,Z)C^N(V,PT) \\
&\quad - B(V,Z)C^N(U,PT), \\
\overline{g}(\overline{R}(U,V)\xi,N) &= \overline{g}(R(U,V)\xi,N) = C^N(V,\overset{\star}{A}_\xi\, U) \\
&\quad - C^N(\overset{\star}{A}_\xi\, V,U) \\
&\quad - 2d\tau^N(U,V), \quad \forall U,V,Z,W \in \Gamma(TM|_{\mathscr{U}}). \qquad (2.6) \\
\overline{g}(\overline{R}(U,V)PZ,N) &= \overline{g}((\nabla_U A_N)V,PZ) - \overline{g}((\nabla_Y A_N)U,PZ) \\
&\quad + \tau^N(V)\overline{g}(A_N U,PZ) - \tau^N(U)\overline{g}(A_N V,PZ).
\end{aligned}
$$

for every U,V,Z and T in $\Gamma(T\Sigma)$.

A normalized null hypersurface (Σ,E) is said to be totally umbilical (resp. totally geodesic) if there exists a smooth function β on Σ such that at each $p\in\Sigma$ and $\forall U,V\in T_p\Sigma$,

$$B(p)(U,V) = \beta(p)g(U,V) \text{ (resp. } B \text{ vanishes identically on } \Sigma \text{).} \tag{2.7}$$

Also the screen distribution $\mathscr{S}(E)$ is totally umbilical (resp. totally geodesic) if for all $U,V\in\Gamma(T\Sigma)$

$$C(U,PV) = \lambda g(U,V) \text{ (resp. } A_N = 0). \tag{2.8}$$

The null mean curvature $\mathbf{H}_\xi$ and the screen mean curvature $\mathbf{H}_N$ are defined by :

$$\mathbf{H}_\xi = \frac{1}{n}\sum_{i=1}^{n} B(\overset{\circ}{E}_i,\overset{\circ}{E}_i), \quad \mathbf{H}_N = \frac{1}{n}\sum_{i=1}^{n} C(\overset{\circ}{E}_i,\overset{\circ}{E}_i),$$

where $\{\overset{\circ}{E}_1,...,\overset{\circ}{E}_n\}$ is an orthonormal basis of $\mathscr{S}(E)$.

With regard to ξ and N, we have seen that two shape operators are supplied for any null hypersurface: $\overset{\star}{A}_\xi$ and A_N. The authors of [3] simplify the analysis by taking into account the scenario in which these shape operators are linearly connected, in which case Σ is referred to as a null hypersurface with conformal screen distribution. Navarro et al. expanded on this idea in [25], defining the screen quasi-conformal hypersurfaces.

Definition 2.1. *We say that the normalized null hypersurface (Σ,E) (or the rigging E) has a quasi-conformal screen distribution if the shape operators A_N and $\overset{\star}{A}_\xi$ of Σ and $\mathscr{S}(E)$ satisfy*

$$A_N U = \kappa\overset{\star}{A}_\xi U + \sigma PU, \tag{2.9}$$

for any $U\in\Gamma(T\Sigma)$ and some functions κ and σ. Equivalently,

$$C(U,PV) = \kappa B(U,PV) + \sigma g(PU,PV)$$

for any $U,V\in\Gamma(T\Sigma)$. For $\sigma=0$, we simply say that (Σ,E) (or the rigging E) has a conformal screen distribution.

We conclude this part by providing an example of a null hypersurface equipped with a rigging vector field that meets the definition previously mentioned.

Example 2.1. *Let $\overline{\Sigma}$ be a Generalized Robertson-Walker (GRW) spacetime; that is the warped product $\overline{\Sigma} = I \times_\psi H$, where I (the base) is an open interval of the real line $\mathbb{R}$, (H, g_H) the fiber is a Riemannian manifold of dimensionn $n+1$ and $\psi > 0$ is a smooth warping function defined on I. It is then endowed with the Lorentzian metric $\overline{g} = -du^2 + \psi^2(t)g_H$, where u stands for the natural (global) parameter on $\mathbb{R}$. It is established that the vector field $E = \psi\partial t$ is a timelike closed conformal vector field with a closed conformal factor ψ'. Therefore, we can utilize it as a rigging for any null hypersurface Σ in GRW. Let ξ represent the associated rigged vector field. The related null transverse vector field is denoted as*
$N = E - \frac{1}{2}\overline{g}(E,E)\xi = E + \frac{1}{2}\psi^2\xi$. From this, it is worth nothing that [1, Eq.3.1] $\tau(U) = 0,\ PU \cdot \psi = 0,\ A_N U = \frac{1}{2}\psi^2 \overset{\star}{A}_\xi U - \psi' PU,\ A_N\xi = 0,\ \tau(\xi) = 0,\ \xi \cdot \psi = -\frac{\psi'}{\psi}, \forall U \in \Gamma(T\Sigma)$.

Example 2.2. *The null cone Σ^{n+1} of $\mathbb{R}^{n+1}_1 (n > 2)$ is given by the equation $-(u^0)^2 + \sum_{i=1}^{n+1}(u_i)^2 = 0, u_0 \neq 0$. It is known that Σ^{n+1} is a normalized null hypersurface of $\mathbb{R}^{n+2}_1$. The rigging vector field is given by $E = -\frac{1}{u_0}\frac{\partial}{\partial u_0}$. The corresponding rigged vector field is given by $\xi = \sum_{i=0}^{n+1} u_i \frac{\partial}{\partial u_i}$. The associated null transversal vector field is $N = \frac{1}{2(u^0)^2}\{-u^0\frac{\partial}{\partial u^0} + \sum_{a=1}^{n+1} u_i \frac{\partial}{\partial u_i}\}$. As ξ is the position vector field, we get: $\bar{\nabla}_U\xi = \nabla_U\xi = U, \forall U \in \Gamma(T\Sigma)$. Then, $-\overset{\star}{A}_\xi U - \tau(U)\xi = U$. Since $\overset{\star}{A}_\xi$ is $\mathscr{S}(E)$-valued, we obtain $\forall U \in \Gamma(\mathscr{S}(E))$ $\overset{\star}{A}_\xi U = -U$. Next, it is known that $A_N U = \frac{1}{2(u^0)^2} \overset{\star}{A}_\xi U, \forall U \in \Gamma(T\Sigma^{n+1})$. Which means that Σ^{n+1} is screen conformal.*

3. Torse-forming vector fields

In the entire context, K represents a fixed torse-forming vector field in a semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ with potential function Ψ, that is

$$\overline{\nabla}_U K = \Psi U + \mu(U)K, \tag{3.10}$$

$\forall U$ on $(\overline{\Sigma}, \overline{g})$, where Ψ is a smooth function on $\overline{\Sigma}$, μ is a one-form. The function Ψ is called the conformal scalar while the one-form μ is called the generating form [24]. When $\mu(K) = 0$, the vector field K is said to be a torqued vector field [9]. Let W be the dual vector field corresponding to the generating form μ, satisfying $\mu(U) = \overline{g}(U, W)$ for every vector field U on $(\overline{\Sigma}, \overline{g})$. If the one-form μ is identically zero, then K is known as a closed conformal vector field (see Example 2.1) . When Ψ is a non-zero constant and $\mu = 0$, K is known as a concurrent vector field (the position vector field in Example 2.2 is concurrent), and if $\Psi = 0$ with μ is a non-zero, K is described as a recurrent vector field.

Remark 3.1. *If K is null, then by Eq.(3.10) we get*

$$0 = U\overline{g}(K,K) = 2\overline{g}(\overline{\nabla}_U K, K) = 2\Psi\overline{g}(U,K),$$

$\forall U \in \Gamma(T\overline{\Sigma})$. *Thus, since Ψ must be identically zero, K is a recurrent vector field.*

Example 3.1. *Let $\overline{\Sigma} = (\mathbb{R}^4_1, \overline{g}_0)$ be the 4 dimension Minkowski space time, that is $\overline{g}_0 = -dt^2 + dx_1^2 + dx_2^2 + dx_3^2$, with (t, x_1, x_2, x_3) being coordinates in $\mathbb{R}^4$. We consider the conformal* *metric* $\overline{g} = e^{2t}\overline{g}_0$. *As* $(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ *is an orthogonal frame basis with respect to* $\overline{g}_0$, *then is worth noting that the vectors fields*

$$u_1 = e^{-t}\partial_1, \quad u_2 = e^{-t}\partial_2, \quad u_3 = e^{-t}\partial_3, \quad u_4 = e^{-t}\partial_t$$

is a frame basis with respect to $\bar{g}$. *Let* $\overline{\nabla}$ *denotes the Levi-Civita connection of the metric* $\overline{g}$ *and* ∇^0 *the Levi-Civita connection of* $\overline{g}_0$. *It is known that*

$$\overline{\nabla}_U V = \nabla^0_U V + U(f)V - V(f)U - \overline{g}_0(U,V)grad_{\overline{g}_0}(f)$$

$\forall U, V \in \Gamma(T\overline{\Sigma})$, *where* $f = t$. *Using this known result, we obtain*

$$\overline{\nabla}_{u_i} u_i = e^{-t}u_4, \quad \overline{\nabla}_{u_4} u_i = 0, \quad \overline{\nabla}_{u_i} u_j = 0 \ \textit{with} \ (i \neq j)$$
$$\overline{\nabla}_{u_i} u_4 = -e^{-t}u_i, \quad \overline{\nabla}_{u_4} u_4 = -2e^{-t}u_4, \qquad \textit{with} \ (1 \leq i, j \leq 3).$$

Any vector field U *on* $(\overline{\Sigma}, \overline{g})$ *can be split as* $U = au_1 + bu_2 + cu_3 + du_4$, *where* a, b, c, d *are smooth functions on* $\overline{\Sigma}$. *It is easy to show that* u_4 *is torse-forming vector field :*

$$\begin{aligned}
\overline{\nabla}_U u_4 &= a\overline{\nabla}_{u_1} u_4 + b\overline{\nabla}_{u_2} u_4 + c\overline{\nabla}_{u_3} u_4 + d\overline{\nabla}_{u_4} u_4 \\
&= -ae^{-t}u_1 - be^{-t}u_2 - ce^{-t}u_3 - 2de^{-t}u_4 \\
&= -e^{-t}(au_1 + bu_2 + cu_3 + du_4) - de^{-t}u_4 \\
&= -e^{-t}U + \mu(U)u_4.
\end{aligned}$$

Thus, $K = u_4$ *is a torse-forming vector field with conformal scalar* $\Psi = -e^{-t}$ *and generating form* $\mu(.) = \overline{g}(., W)$, *with* $W = e^{-t}u_4$.

In [20],[2], a starting point to describe the geometry of Σ through closed conformal vector fields is the following pointwise decomposition along Σ:

$$K = K^T + bN = K_{\mathscr{S}} + a\xi + bN, \tag{3.11}$$

where a and b are smooth functions on $\overline{\Sigma}$ defined by $a = \overline{g}(K, N)$, $b = \overline{g}(K, \xi)$ and $K_{\mathscr{S}} \in \Gamma(\mathscr{S}(E))$.

The reader may compare this with [22, Proposition 3.1].

Proposition 3.1. *Let* K *be a torse-forming vector field on* $(\overline{\Sigma}, \overline{g})$ *and* (Σ, E) *a normalized null hypersurface of* $\overline{\Sigma}$. *Then, for any* U *tangent to* Σ, *the following holds:*

$$\nabla^*_U K_{\mathscr{S}} = a \overset{\star}{A}_\xi (U) + bA_N(U) + \Psi PU + \mu(U)K_{\mathscr{S}}, \tag{3.12}$$
$$\Psi\omega(U) + a\mu(U) = C(U, K_{\mathscr{S}}) + U(a) - a\tau(U), \tag{3.13}$$
$$b\mu(U) = B(U, K_{\mathscr{S}}) + U(b) + b\tau(U). \tag{3.14}$$

Proof. Applying (3.11) along with the property that K represents torse-forming vector fields with the conformal factor Ψ, $\forall U \in \Gamma(T\Sigma)$, we obtain on one side

$$\begin{aligned}
\overline{\nabla}_U K &= \Psi U + \mu(U)K \\
&= \Psi PU + \mu(U)K_{\mathscr{S}} + (\Psi\omega(U) + a\mu(U))\xi + b\mu(U)N.
\end{aligned} \tag{3.15}$$

On other hand, we have

$$\begin{aligned}
\overline{\nabla}_U(K_{\mathscr{S}} + a\xi + bN) &= \overline{\nabla}_U K_{\mathscr{S}} + \overline{\nabla}_U(a\xi) + \overline{\nabla}_U(bN) \\
&= \nabla_U K_{\mathscr{S}} + B(U, K_{\mathscr{S}})N + U(a)\xi + a\overline{\nabla}_U\xi + U(b)N + b\overline{\nabla}_U N \\
&= \nabla^*_U K_{\mathscr{S}} + C(U, K_{\mathscr{S}})\xi + B(U, K_{\mathscr{S}})N + U(a)\xi + a(-\overset{\star}{A}_\xi(U) - \tau(U)\xi) \\
&\quad + U(b)N + b(-A_N(U) + \tau(U)N) \\
&= \nabla^*_U K_{\mathscr{S}} - a\overset{\star}{A}_\xi(U) - bA_N(U) + \{C(U, K_{\mathscr{S}}) + U(a) - a\tau(U)\}\xi \\
&\quad + \{B(U, K_{\mathscr{S}}) + U(b) + b\tau(U)\}N. \qquad (3.16)
\end{aligned}$$

Matching the screen, the radical, and the transversal components of Eq.(3.15), we derive Eq.(3.12), Eq.(3.13), and Eq.(3.14). ☐

As a direct result of Proposition 3.1, we can outline the scenarios in which K is completely contained within $span(\xi)$, $span(N)$, or $\mathscr{S}(\zeta)$.

Corollary 3.1. *Let K be a torse-forming vector field on $(\overline{\Sigma}, \overline{g})$ and (Σ, E) a normalized null hypersurface of $\overline{\Sigma}$. Given $U \in T\Sigma$, the following holds :*

(i) *If K is tangent to $span(\xi)$, then*

$$a\overset{\star}{A}_\xi(U) + \Psi PU = 0, \quad \Psi\omega(U) + a\mu(U) = C(U, K_{\mathscr{S}}) + U(a) - a\tau(U). \qquad (3.17)$$

(ii) *If K is tangent to $\mathscr{S}(E)$, then*

$$\nabla^*_U K = \Psi PU + \mu(U)K,$$
$$C(U, K) = \Psi\omega(U), \quad C(PU, K) = B(U, K) = 0.$$

(iii) *If K is tangent to $span(N)$, then*

$$bA_N(U) = 0, \quad b\mu(U) = U(b) + b\tau(U).$$

(iv) *If K is tangent to $span(\xi) \oplus span(N)$, then*

$$a\overset{\star}{A}_\xi(U) + bA_N(U) + \Psi PU = 0, \quad \Psi\omega(U) + a\mu(U) = U(a) - a\tau(U),$$
$$b\mu(U) = U(b) + b\tau(U).$$

Proof. If K is tangent to ξ, then $K = a\xi$, i.e. $K_{\mathscr{S}} = 0$ and $b = 0$. Using this in Eq.(3.12) and Eq.(3.13), we have $a\overset{\star}{A}_\xi(U) + \Psi PU = 0$ and $a\Psi\omega(U) + a\mu(U) = U(a) - a\tau(U)$, respectively. Which completes the proof of item (i). Next, if K is tangent to $\mathscr{S}(E)$, then $a = b = 0$. From this, Eq.(3.12), Eq.(3.13) and Eq.(3.14) leads to $\nabla^*_U K_{\mathscr{S}} = \Psi PU + \mu(U)K_{\mathscr{S}}$, $\Psi\omega(U) = C(U, K_{\mathscr{S}})$ and $B(U, K_{\mathscr{S}}) = 0$, respectively. Which gives the proof of item (ii). If K is tangent to N, then $a = 0$, $K_{\mathscr{S}} = 0$ and $K = bN$ with $b \neq 0$. Using this in Eq.(3.12), we obtain $bA_N(U) + \Psi PU = 0$. But, Eq.(3.13) gives $\Psi\omega(U) = 0$, so in particular for $U = \xi$, $\Psi = 0$. Thus $bA_N = 0$. Also, Eq.(3.14) leads to $b\mu(U) = U(b) + b\tau(U)$. Finally, if K is tangent to $span(\xi) \oplus span(N)$, then $K_{\mathscr{S}} = 0$, i.e., $K = a\xi + bN$. Thus, from Eq.(3.12), Eq.(3.13) and Eq.(3.14), we have $a\overset{\star}{A}_\xi(U) + bA_N(U) + \Psi PU = 0$, $\Psi\omega(U) + a\mu(U) = U(a) - a\tau(U) = 0$, $b\mu(U) = U(b) + b\tau(U)$. ☐

Based on the decomposition in (3.11) and Corollary 3.1, we obtain the subsequent Theorem :

Theorem 3.1. *Let (Σ, E) be a normalized null hypersurface of a semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ admitting a non zero torse-forming vector field K.*

(1) *If K belongs to $span(\xi)$, then Σ is totally umbilic.*
(2) *If K belongs to $span(N)$, then Σ is totally screen geodesic and locally a product manifold $c_\xi \times \overset{\circ}{\Sigma}$, where c_ξ is a null curve tangent to $span(\xi)$ and $\overset{\circ}{\Sigma}$ is a leaf of $\mathscr{S}(E)$.*

Proof. Since K belongs to $span(\xi)$, from Eq. (3.17) of Corollary 3.1), we get $a \overset{\star}{A}_\xi (U) + \Psi PU = 0$, $\forall U \in \mathfrak{X}(\Sigma)$, that is $\overset{\star}{A}_\xi (U) = -\frac{\Psi}{a} PU$ as $a \neq 0$, which implies that Σ is totally umbilic, which gives item (1). Next, as K belongs to $span(N)$, then from item (iii) of Corollary 3.1 it follows that $A_N(U) = 0$ for any $U \in \mathfrak{X}(\Sigma)$, since $b \neq 0$. Thus $\mathscr{S}(E)$ is totally geodesic, Moreover, from the first equation of (2.5), we see that $\mathscr{S}(E)$ is parallel with respect to ∇, and thus an integrable distribution on Σ. It follows that it is locally a product manifold $c_\xi \times \overset{\circ}{\Sigma}$ where c_ξ is a null curve tangent to $T\Sigma^\perp$ and $\overset{\circ}{\Sigma}$ is a leaf of $\mathscr{S}(E)$, which gives item (2) and completes the proof. $\square$

Corollary 3.2. *Let (Σ, E) be a normalized null hypersurface of a semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ admitting a non zero torse-forming vector field K. If K lies in $span(N)$, then $\overline{\Sigma}$ admits no proper totally screen umbilic null hypersurface.*

Corollary 3.3. *Let (Σ, E) be a normalized null hypersurface of a semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ admitting a non zero torse-forming vector field K. If K lies in $\mathscr{S}(E)$, then Σ can not be a proper totally umbilic or proper totally screen umbilic null hypersurface.*

Proof. Suppose that Σ is totally umbilic and K belongs to $\mathscr{S}(E)$, then item (ii) of Corollary (3.1) and Eq.(2.7) imply that $0 = B(X, K) = \beta \overline{g}(X, K)$, for any X tangent to $\mathscr{S}(E)$. Since $\mathscr{S}(E)$ is non-degenerate, we have $\beta = 0$, that is Σ is totally geodesic. Similarly, if $\mathscr{S}(E)$ is totally umbilic, we prove that $\lambda = 0$ from Eq.(2.8) and item (ii) of Corollary3.1. $\square$

Setting $U = \xi$ in (3.13)-(3.14) and using equation (2.9) we have the following.

Proposition 3.2. *Let $(\overline{\Sigma}, \overline{g})$ be a Lorentzian manifold equipped with a torse-forming vector field K with conformal factor Ψ and (Σ, E) a normalized null hypersurface of $\overline{\Sigma}$. Then we have the following partial differential equation of first order*

$$b\big(\tau(\xi) - \mu(\xi)\big) + \xi(b) = 0. \tag{3.18}$$

Moreover, if (Σ, E) is screen quasi-conformal, then

$$\Psi + a\big(\tau(\xi) + \mu(\xi)\big) - \xi(a) = 0, \tag{3.19}$$

where $a = \overline{g}(K, N)$, $b = \overline{g}(K, \xi)$ and μ is the generating form.

From Eqs.(3.18) and (3.19) the following Corollaries hold.

Corollary 3.4. *Let $(\overline{\Sigma}, \overline{g})$ be a Lorentzian manifold equipped with a timelike torse-forming vector field K with conformal factor Ψ and (Σ, E) a normalized null hypersurface. If b is constant along the integral curves of ξ, then $\tau(\xi) = \mu(\xi)$. Moreover, if (Σ, E) is screen quasi-conformal then Ψ satisfies the following relation $\Psi = \xi(a) - 2a\tau(\xi)$.*

Proof. As b is constant along the integral curves of ξ, we have from Eq.(3.18) that $b\big(\tau(\xi) - \mu(\xi)\big) = 0$, which implies that $\tau(\xi) = \mu(\xi)$. Indeed, $b = 0$ leads to $\overline{g}(K, K) = \overline{g}(K_{\mathscr{S}}, K_{\mathscr{S}})$ which is also not possible since $K_{\mathscr{S}}$ is a spacelike vector and K is a timelike. Setting $\tau(\xi) = \mu(\xi)$ in Eq.(3.19), the last claim follows. □

At this point, it is natural to ask under which circumstances we can choose the rigging vector field E in a way that the corresponding null hypersurface (Σ, E) is screen quasi conformal(see Defintion 2.9). Based on Example 2.1, it is worth noting that if the ambient space admits a closed conformal rigging vector field, then it induces a screen quasi-conformal structure in any null hyperfurface. As the next result shows, there is an analogous connection between torse forming vector fields and screen quasi conformal distributions, thus generalizing the above assertion.

Theorem 3.2. *Let (Σ, E) be a normalized null hypersurface of $(\overline{\Sigma}, \overline{g})$ and $K \in \Gamma(T\Sigma)$ a torse forming vector field with $\overline{g}(K, K) \neq 0$. If K is nowhere tangent to Σ then there exists a normalization E such that the corresponging screen distribution $\mathscr{S}(E)$ is screen quasi conformal. Moreover, Σ is locally a product manifold $c_\xi \times \overset{\circ}{\Sigma}$, where c_ξ is a null curve tangent to span(ξ) and $\overset{\circ}{\Sigma}$ is a leaf of $\mathscr{S}(E)$.*

Proof. Since K is nowhere tangent to Σ, we use it as a rigging E for Σ. Setting $K = E$, and using Eq.(2.2), we have that $a = \frac{1}{2}\overline{g}(E, E)$ and $b = 1$. Using item (iii) of Corollary 3.1, we have the following equations:

$$\begin{aligned}
&\frac{1}{2}\overline{g}(E, E) \overset{\star}{A}_\xi (U) + A_N(U) + \Psi PU = 0,\\
&\Psi\omega(U) + \frac{1}{2}\overline{g}(E, E)\mu(U) = +\frac{1}{2}U(\overline{g}(E, E)) - \frac{1}{2}\overline{g}(E, E)\tau(U),\\
&\mu(U) = \tau(U) = \overline{g}(\overline{\nabla}_U N, \xi).
\end{aligned} \tag{3.20}$$

Thus, $A_N = -\frac{1}{2}\overline{g}(E, E) \overset{\star}{A}_\xi - \Psi P$, and we have that Σ is screen quasi-conformal with $\kappa = -\frac{1}{2}\overline{g}(E, E)$ and $\sigma = -\Psi$. Then $\mathscr{S}(E)$ is integrable and from [25], Σ is locally a product manifold $c_\xi \times \overset{\circ}{\Sigma}$, where c_ξ is a null curve tangent to *span*(ξ) and $\overset{\circ}{\Sigma}$ is a leaf of $\mathscr{S}(E)$. Which completes the proof □

The following Lemmas provide some basic facts about torse-forming vector fields. To performed the previously mentioned action, we set $\theta = g(K, K)$, and let γ be the metrically one-form equivalent to K i.e $\gamma = \overline{g}(., K)$.

Lemma 3.1. *Given $U, V \in \mathfrak{X}(\overline{\Sigma})$, we have :*

(i) $d\gamma(U, V) = \mu(U)\gamma(V) - \mu(V)\gamma(U)$,

(ii) $(L_K\overline{g})(U, V) = 2\Psi\overline{g}(U, V) + \mu(U)\gamma(V) + \mu(V)\gamma(U)$,

(iii) $\overline{div}(K) = \Psi(n + 2) + \mu(K)$.

Proof. $\forall\ U, V \in \mathfrak{X}(\overline{\Sigma})$, we have

$$\begin{aligned}
d\gamma(U, V) &= \overline{g}(\overline{\nabla}_U K, V) - \overline{g}(U, \overline{\nabla}_V K)\\
&= \overline{g}(\Psi U + \mu(U)K, V) - \overline{g}(U, \Psi V + \mu(V)K)\\
&= \mu(U)\gamma(V) - \mu(V)\gamma(U),
\end{aligned} \tag{3.21}$$

also,

$$\begin{aligned}(L_K\overline{g})(U,V) &= \overline{g}(\overline{\nabla}_U K, V) + \overline{g}(U, \overline{\nabla}_V K) \\ &= \overline{g}(\Psi U + \mu(U)k, V) + \overline{g}(u, \Psi V + \mu(V)K) \\ &= 2\Psi\overline{g}(U,V) + \mu(U)\gamma(V) + \mu(V)\gamma(U). \end{aligned} \tag{3.22}$$

Which give item (i) and (ii). From Eqs.(3.21) and (3.22), we obtain item (iii).

Now, let $\{e_1, e_2, \cdots, e_{n+2}\}$ be an orthonormal frame basis of $(\overline{\Sigma}, \overline{g})$. We compute the divergence of K as :

$$\begin{aligned}\overline{div}(K) &= \sum_{i=1}^{n+2} \epsilon_i \overline{g}(\overline{\nabla}_{e_i} K, e_i) = \sum_{i=1}^{n+2} \epsilon_i \overline{g}(\Psi e_i + \mu(e_i)K, e_i) \\ &= \Psi \sum_{i=1}^{n+2} \epsilon_i \overline{g}(e_i, e_i) + \sum_{i=1}^{n+2} \epsilon_i \mu(e_i)\overline{g}(K, e_i) = (n+2)\Psi + \sum_{i=1}^{n+2} \epsilon_i \overline{g}(W, e_i)\overline{g}(K, e_i) \\ &= (n+2)\Psi + \overline{g}(W, K) = (n+2)\Psi + \mu(K).\end{aligned}$$

□

Lemma 3.2. *Suppose that $K \in \mathfrak{X}(\overline{\Sigma})$ is torse-forming vector field. Then for all $U, V \in \mathfrak{X}(\overline{\Sigma})$, we have*

(i) $\overline{\nabla}\theta = 2\Psi K + 2\theta W$.

(ii) $\bar{\Delta}\theta = 2K(\Psi) + 2\Psi((n+2)\Psi + 3\mu(K)) + 2\theta(2\mu(W) + \overline{div}(W))$.

(iii) *If $U \in \mathfrak{X}(\overline{\Sigma})$, with $U \perp K$, then* $2\theta U(\Psi) = \Psi U(\theta) + 2\theta d\mu(K, U)$.

(iv) $\overline{R}_{UV}K = (U(\Psi) - \Psi\mu(U))V - (V(\Psi) - \Psi\mu(V))U + (d\mu(U,V))K$.
$\overline{Ric}(U, K) = -(n+1)(\overline{g}(\overline{\nabla}\Psi, U) - \Psi\mu(U)) - d\mu(K, U)$.

Proof. $\forall U \in \mathfrak{X}(\overline{\Sigma})$ together with Eq.(3.10), give

$$\begin{aligned} g(\overline{\nabla}\theta, U) = U.\theta &= U\overline{g}(K, K) \\ &= 2\overline{g}(\overline{\nabla}_U K, K) \\ &= 2\overline{g}(\Psi U + \mu(U)K, K) \\ &= \overline{g}(2\Psi K, U) + 2\overline{g}(K,K)\overline{g}(W, U) \\ &= \overline{g}(2\Psi K + 2\theta W, U). \end{aligned} \tag{3.23}$$

The first item follows using the fact that $\overline{g}$ is non-degenerate.

By definition, we get

$$
\begin{aligned}
\bar{\Delta}\theta &= \overline{div}(\overline{\nabla}\theta) \\
&= \overline{div}(2\Psi K + 2\theta W) \\
&= 2K(\Psi) + 2\Psi\overline{div}K + 2W(\theta) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 2W\overline{g}(K,K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 4\overline{g}(\overline{\nabla}_W K, K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 4\overline{g}(\Psi W + \mu(W)K, K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + \mu(K)) + 4\Psi\overline{g}(W,K) + 4\mu(W)\overline{g}(K,K) + 2\theta\overline{div}W \\
&= 2K(\Psi) + 2\Psi((n+2)\Psi + 3\mu(K)) + 2\theta(2\mu(W) + \overline{div}(W)),
\end{aligned}
$$

which gives item (ii).

(iii) From the first item and for any $U \in \mathfrak{X}(\overline{\Sigma})$, if $U \perp K$, we have

$$
\begin{aligned}
\overline{g}(\overline{\nabla}\theta, U) &= U.\theta \\
&= \overline{g}(2\Psi K + 2\theta W, U) \\
&= 2\theta\overline{g}(W, U).
\end{aligned} \tag{3.24}
$$

Since $U \perp K$, this leads to $K(\theta) = 2\theta(\Psi + \mu(K)) = \overline{g}(\overline{\nabla}\theta, K)$, and $U(\theta) = 2\theta\mu(U)$. Now taking the derivative of $K(\theta)$ with respect to U gives

$$
\begin{aligned}
U(K(\theta)) &= 2\Psi U(\theta) + 2\mu(K)U(\theta) + 2\theta U(\Psi) + 2\theta U(\mu(K)) \\
&= 2\Psi U(\theta) + 2\mu(K)U(\theta) + 2\theta U(\Psi) + 2\theta(\overline{g}(\overline{\nabla}_U W, K) + \overline{g}(W, \overline{\nabla}_U K)) \\
&= 2\Psi U(\theta) + 2\mu(K)U(\theta) + 2\theta U(\Psi) + 2\theta\mu(U)\mu(K) \\
&+ 2\theta\Psi\mu(U) + 2\theta\overline{g}(\overline{\nabla}_U W, K) \\
&= 3\Psi U(\theta) + 2\theta U(\Psi) + 3\mu(K)U(\theta) + 2\theta\overline{g}(\overline{\nabla}_U W, K).
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
But, \quad U(K(\theta)) &= U\overline{g}(\overline{\nabla}\theta, K) \\
&= \overline{g}(\overline{\nabla}_U\overline{\nabla}\theta, K) + \overline{g}(\overline{\nabla}\theta, \overline{\nabla}_U K) \\
&= \overline{g}(\overline{\nabla}_K\overline{\nabla}\theta, U) + \overline{g}(2\Psi K + 2\theta W, \Psi U + \mu(U)K) \\
&= \overline{g}(\overline{\nabla}_K(2\Psi K + 2\theta W), U) + 2\Psi U(\theta) + U(\theta)\mu(K) \\
&= 2K(\theta)\mu(U) + 2\theta\overline{g}(\overline{\nabla}_K W, U) + 2\Psi U(\theta) + U(\theta)\mu(K) \\
&= 4\Psi U(\theta) + 3\mu(K)U(\theta) + 2\theta\overline{g}(\overline{\nabla}_K W, U).
\end{aligned} \tag{3.26}
$$

By equating Eq.(3.25) and Eq.(3.26), we obtain $2\theta U(\Psi) = \Psi U(\theta) + 2\theta d\mu(K, U)$, which gives item (iii).

Now, let $U, V \in \mathfrak{X}(\overline{\Sigma})$, we have

$$\begin{aligned}
\overline{R}_{UV}K &= \overline{\nabla}_U\overline{\nabla}_V K - \overline{\nabla}_V\overline{\nabla}_U K - \overline{\nabla}_{[U,V]}K \\
&= \overline{\nabla}_U(\Psi V + \mu(V)K) - \overline{\nabla}_V(\Psi U + \mu(U)K) - \Psi[U,V] - \mu([U,V])K \\
&= U(\Psi)V + \Psi\overline{\nabla}_U V + U(\mu(V))K + \mu(V)\overline{\nabla}_U K - V(\Psi)U - \Psi\overline{\nabla}_V U - V(\mu(U))K \\
&\quad - \mu(U)\overline{\nabla}_V K - \Psi[U,V] - \mu([U,V])K \\
&= U(\Psi)V + U(\mu(V))K + \mu(V)(\Psi U + \mu(U)K) - V(\Psi)U - V(\mu(U))K \\
&\quad - \mu(U)(\Psi V + \mu(V)K) - \mu([U,V])K \\
&= (U(\Psi) - \Psi\mu(U))V - (V(\Psi) - \Psi\mu(V))U - (d\mu(U,V))K,
\end{aligned}$$

which gives item (iv).

If $\{e_1, e_2, \cdots, e_{n+2}\}$ an orthonormal frame basis of $(\overline{\Sigma}, \overline{g})$, then using Eq.(3.10), we have

$$\begin{aligned}
\overline{Ric}(U,K) &= \sum_{i=1}^{n+2}\epsilon_i\overline{g}(\overline{R}(e_i,U)K, e_i) \\
&= \sum_{i=1}^{n+2}\epsilon_i\overline{g}((e_i(\Psi) - \Psi\mu(e_i))U - (U(\Psi) - \Psi\mu(U))e_i - (d\mu(e_i,U))K, e_i) \\
&= \sum_{i=1}^{n+2}\epsilon_i\overline{g}((\overline{g}(\overline{\nabla}\Psi, e_i)U - \Psi\overline{g}(W,e_i)U - U(\Psi)e_i + \Psi\mu(U)e_i - (d\mu(e_i,U))K, e_i) \\
&= \sum_{i=1}^{n+2}\epsilon_i(\overline{g}(\overline{\nabla}\Psi, e_i)\overline{g}(U,e_i) - \Psi\overline{g}(W,e_i)\overline{g}(U,e_i) - U(\Psi)\overline{g}(e_i,e_i) + \Psi\mu(U)\overline{g}(e_i,e_i) \\
&\quad - d\mu(e_i,U)\overline{g}(K,e_i) \\
&= \overline{g}(\overline{\nabla}\Psi, U) - \Psi\overline{g}(W,U) - (n+2)U(\Psi) + (n+2)\Psi\mu(U) - d\mu(\sum_{i=1}^{n+2}\epsilon_i\overline{g}(K,e_i)e_i, U) \\
&= \overline{g}(\overline{\nabla}\Psi, U) - \Psi\mu(U) - (n+2)\overline{g}(\overline{\nabla}\Psi, U) + (n+2)\Psi\mu(U) - d\mu(K,U) \\
&= -(n+1)(\overline{g}(\overline{\nabla}\Psi, U) - \Psi\mu(U)) - d\mu(K,U),
\end{aligned}$$

which gives item (iv) and completes the proof of the Lemma. □

Corollary 3.5. *From item (iv) of Lemma (3.2), for a torse-forming vector field K on an $(n+2)$ dimensional semi-Riemannian manifold, we have :*

$$\overline{Ric}(K,K) = (n+1)(\Psi\mu(K) - \overline{g}(\overline{\nabla}\Psi, K)). \tag{3.27}$$

Corollary 3.6. *Let $K \in \Gamma(\overline{\Sigma})$ be a torse-forming vector field and $U \in \Gamma(\overline{\Sigma})$ such that $\overline{g}(U,K) = \overline{g}(U,W) = 0$. Then $U \cdot \theta = 0$.*

Proof. The proof follows from Eq. (3.24). □

The subsequent lemma serves as a generalization of [5, Lemma 5. p. 8], where it was assumed that K is a closed conformal vector field.

Proposition 3.3. *Let $K \in \mathfrak{X}(\overline{\Sigma})$ be a torse-forming vector field on a Lorentzian manifold $(\overline{\Sigma}, \overline{g})$.*

(i) *If K is never null and it has constant length with $\overline{g}(K,K) \neq 0$, then γ is closed and $\Psi^2 = \theta\mu(W)$, where W is the dual vector of the generating form μ and $\theta = \overline{g}(K,K)$.*

(ii) *If K is never null torqued and has constant length, then either K is parallel or W is lightlike.*

(iii) *If K is torqued , then either K is a null recurrent or W is lightlike.*

Proof. We recall $\theta = \overline{g}(K,K)$. If θ is constant then from item (i) of Lemma 3.2 it worth noting that $\Psi K = -\theta W$. Thus, $d\gamma = 0$ as K and W are linearly related. Also setting $U = K$ in Eq.(3.23) together with the fact that θ is constant, we have $0 = K(\theta) = 2\Psi\theta + 2\theta\mu(K)$, which implies $\Psi = -\mu(K)$, that is, $\Psi^2 = \theta\mu(W)$ by applying μ to the first item of Lemma 3.2. Which proof item (i). Next, if K is torqued, i.e., $\mu(K) = 0$, then from item (i), we have $\theta\mu(W) = 0$ and $\Psi = 0$. From the later equation, we have for any $U \in \mathfrak{X}(\overline{\Sigma})$ $0 = U\overline{g}(K,K) = 2\overline{g}(\overline{\nabla}_U K, K) = 2\overline{g}(\mu(U)K, K) = 2\mu(U)\overline{g}(K,K)$, which implies $\mu(U) = 0$ as $\overline{g}(K,K) \neq 0$, which gives the proof of item (ii). (iii) If K is torqued, i.e, $\mu(K) = 0$, then it is worth noting that $0 = K \cdot \mu(K) = K \cdot \bar{g}(K,W) = \bar{g}(K,K)\bar{g}(W,W) = \theta\mu(W)$. That is either, K is null or W is null. From Remark 3.1, the claims follws. □

We remember the following from [9],[15].

Theorem 3.3. [9] *Every torqued vector field K associated with a twisted product manifold $I \times_f F$ is of the form $K = \Psi f \partial_t$, where t is an arc-length parameter of I, Ψ is a nonzero function on F, and f is the twisting function.*

We have the following classification of torqued vector fields on Einstein manifolds, which arises as a consequence of Theorem 3.3.

Theorem 3.4. [9] *Every torqued vector field K on an Einstein manifolds is of the form*

$$K = \varPhi Z, \tag{3.28}$$

where Z is a concircular vector field and $\varPhi$ is a function satisfying $Z(\varPhi) = 0$.

Conversely, every vector field of the form (3.28) is a torqued vector field on an Einstein manifold.

To end up this section, we assume that K is a torqued vector field tangent to the screen distribution and we have the following Lemmas :

Lemma 3.3. *Let K be a torse-forming vector field on $(\overline{\Sigma}, \overline{g})$. Assume that K is a torqued vector field. For any $U \in \mathfrak{X}(\overline{\Sigma})$ with $U \perp K$, we have:*

$$\Psi\mu(U) - U(\Psi) = d\mu(U,K) \quad \textit{and} \quad \overline{Ric}(U,K) = (n+2)(\Psi\mu(U) - U(\Psi)). \tag{3.29}$$

Proof. From item (iv) of Lemma 3.2 together with the fact that K is torqued and $\theta = \overline{g}(K,K)$, we have

$$\begin{aligned} \overline{R}_{UK}K &= (U(\Psi) - \Psi\mu(U))K - (K(\Psi) - \Psi\mu(K))U + (d\mu(U,K))K \\ &= (U(\Psi) - \Psi\mu(U))K - K(\Psi)U + (d\mu(U,K))K. \end{aligned}$$

This leads to $0 = \overline{g}(\overline{R}_{UK}K, K) = \theta(U(\Psi) - \Psi\mu(U)) + \theta d\mu(U,K)$, which implies $d\mu(U,K) = -U(\Psi) + \Psi\mu(U)$, as $\theta \neq 0$. Replace this just in the second point of item (iv) of Lemma 3.2 give the second equality. □

From the previous Lemma and item (iii) of Lemma 3.2, for any $U \in \mathfrak{X}(\overline{\Sigma})$ with $U \perp K$, it holds that Ψ and θ are related by

$$2U(\Psi) - \Psi\mu(U) = \Psi U(\log(\sqrt{\theta})). \tag{3.30}$$

We have the following Lemma.

Lemma 3.4. *From the above Lemma and since $\xi \perp K$ and $N \perp K$ with $\theta = \overline{g}(K, K)$, we get :*

$$\xi(\theta) = 2\theta\mu(\xi), \quad \text{and} \quad N(\theta) = 2\theta\mu(N).$$
$$\xi(\Psi) = \mu(\xi), \quad \text{and} \quad N(\Psi) = \mu(N).$$

Corollary 3.7. *If the dual vector field W of the generating form μ belongs to $\mathscr{S}(E)$ then :*

$$\xi(\theta) = N(\theta) = 0, \quad \text{and} \quad \xi(\Psi) = N(\Psi) = 0.$$

The main result of this section is the following Theorems.

Theorem 3.5. *Let (Σ, E) be a screen quasi-conformal normalized null hypersurface of a Lorentzian manfold $(\overline{\Sigma}, \overline{g})$, furnished with a nonzero torse-forming vector field K tangent to $\mathscr{S}(E)$ with conformal scalar Ψ and generating vector field W. If W is tangent to $span(\xi) \oplus span(N)$, then Σ is locally a triple product manifold $c_\xi \times c_K \times \Sigma'$, where c_ξ is a null curve tangent to $T\Sigma^\perp$, c_K is a curve of the line bundle $\langle K \rangle$ and Σ' is an $(n-1)$-dimensional leaf of the orthogonal complement of K_S in $\mathscr{S}(\zeta)$.*

Proof. Since Σ is screen quasi-constant then $\mathscr{S}(E)$ is integrable and from [25], Σ is locally a product manifold $c_\xi \times \overset{\circ}{\Sigma}$, where c_ξ is a null curve tangent to $span(\xi)$ and $\overset{\circ}{\Sigma}$ is a leaf of $\mathscr{S}(E)$. Also, as K belongs to $\mathscr{S}(E)$, Corollary 3.1 gives $\nabla^*_X K = \Psi PX + \mu(X)K, \quad \Psi = C(\xi, K), \quad B(X, K) = 0$. But, $\Psi = C(\xi, K) \overset{(2.9)}{=} \kappa B(\xi, K) + \sigma g(\xi, K) = 0$, and $\mu(X) = 0$ as $W \in span(\xi) \oplus span(N)$. That is, K is parallel on $\mathscr{S}(\zeta)$. Moreover, it worth noting that $g(K, \overset{\star}{\nabla}_X Y) = 0$, $\forall X, Y \in T\overset{\circ}{\Sigma}$ and orthogonal to K. Thus, if L is the complementary distribution of the line bundle $\langle K \rangle$ spanned by K in $\overset{\circ}{\Sigma}$, then L is parallel and thus, an integrable distribution over $\mathscr{S}(E)$. Therefore, by de-Ram's decomposition Theorem in [12], each leaf $\overset{\circ}{\Sigma}$ of $\mathscr{S}(E)$ is locally a product manifold $c_K \times \Sigma'$, where c_K is a curve of the line bundle $\langle K \rangle$ spanned by K and Σ' is an $(n-1)$-dimensional leaf of L. □

Theorem 3.6. *Let (Σ, E) be a screen integrable normalized null hypersurface of a Lorentzian manfold $(\overline{\Sigma}, \overline{g})$, admitting a nonzero torqued vector field K tangent to $\mathscr{S}(\zeta)$. Then Σ is locally a triple product manifold $c_\xi \times I \times_f \Sigma'$, where c_ξ is a null curve tangent to $T\Sigma^\perp$, f is a non zero smooth function on an open set I with arc length t. $\partial_t = e_1$ is a unit vector field satisfying $K = \sqrt{\theta} e_1$, and Σ' is a semi-Riemannian $(n-1)$-dimensional manifold, where $\theta = \overline{g}(K, K)$.*

Proof. As $\mathscr{S}(\zeta)$ is integrable, C is symmetric, hence, Σ is a product manifold $c_\xi \times \overset{\circ}{\Sigma}$, where c_ξ is a null curve tangent to $T\Sigma^\perp$ and $\overset{\circ}{\Sigma}$ is a leaf of $\mathscr{S}(E)$. Since K belongs to $\mathscr{S}(E)$, Corollary 3.1 gives

$$\nabla^*_X K = \Psi PX + \mu(X)K, \quad \Psi = C(\xi, K), \quad B(X, K) = 0.$$

Now let us extend e_1 to a quasi-orthonormal basis $\{e_1\}^{\perp} \cup \{e_1\} = \{e_1, e_2, \cdots, e_n\}$ of $\overset{\circ}{\Sigma}$. Thus, from Eq.(3.30), we have $2e_i(\Psi) - \Psi\mu(e_i) = \Psi e_i(\log(\sqrt{\theta}))$. On the other hands we have :

$$\nabla^*_{e_i} K = \Psi e_i + \mu(e_i)\sqrt{\theta} e_1,$$
$$\nabla^*_{e_i} K = e_i(\sqrt{\theta})e_1 + \sqrt{\theta}\nabla^*_{e_i} e_1,$$

which give

$$e_i(\log(\theta) = 2\mu(e_i), \; and \; \nabla^*_{e_i} e_1 = \frac{\Psi}{\sqrt{\theta}} e_i, \; \text{ with } i \neq 1$$

and since K is torqued, we get $0 = \mu(K) = \sqrt{\theta}\mu(e_1)$, which implies $\mu(e_1) = 0$.
Hence, by assuming $\Psi \neq 0$, we have :

$$\begin{aligned} 2e_i(\Psi) &= \Psi\mu(e_i) + \Psi e_i(\log(\sqrt{\theta})) \\ &= \Psi e_i(\log(\sqrt{\theta}) + \Psi e_i(\log(\sqrt{\theta}) \\ &= 2\Psi e_i(\log(\sqrt{\theta}) \\ \frac{e_i(\Psi)}{\Psi} &= e_i(\log(\sqrt{\theta})) \implies e_i(\log(\Psi) - \log(\sqrt{\theta})) = 0. \end{aligned}$$

This implies that, $\{e_1\}^{\perp}$ is an integrable distribution since $\nabla^*_{e_i} \in \{e_1\}^{\perp}$ whose leaves are totally umbilical hypersurface of $\overset{\circ}{\Sigma}$ with umbilicity factor $-\frac{\Psi}{\sqrt{\theta}}$. There for from [27, Proposition 35, p. 206], the leaf $\overset{\circ}{\Sigma}$ is a semi-Riemannian warped product $I \times_f \Sigma'$, where f is a function on the open interval I and Σ' is a Riemannian $(n-1)$-dimensional manifold, and θ is a positive function on $I \times \Sigma'$.
Since $e_1 = \partial_t$, we may set $\sqrt{\theta} = \Psi f(t)$ and admit that Ψ is define only on Σ'.
Thus it is comes that Σ is locally in the form $c_\xi \times I \times_{f(t)} \Sigma'$. □

4. Compact null hypersurfaces and Torse-Forming vector field

The question here is to know if it is always possible to select E such that it is torse forming, along with prescribed geometric properties for the null hypersurface.

Theorem 4.1. *Let (Σ, E) be a simply connected compact normalized null hypersurface of Lorentzian manifold $(\overline{\Sigma}, \overline{g})$ furnished with a torse-forming vector field E. Then the generating vector field W can not be a rigged for Σ.*

Proof. Assume there exists a compact null hypersurface in $\overline{\Sigma}$ with a torse-forming vector field E and a rigged generating vector field W. Then $\alpha = \gamma$ and $\mu(U) = 0$ for all $U \in \Gamma(T\Sigma.)$ Substituting this in Eq.(3.21), we have that $d\gamma(U, V) = d\alpha(U, V) = 0$, $\forall U, V \in \Gamma(T\Sigma)$, which implies that $\omega = i^*\alpha$ is a closed 1-form on Σ and since Σ is simply connected, there exists $f \in C^\infty(\Sigma)$ such that $\omega = df$. It follows that $\widetilde{\nabla} f = W$. However, Σ being compact, the function f has a critical point $x \in \Sigma$, where $W(x) = 0$ and this contradicts the fact $\widetilde{g}(\widetilde{\nabla} f, \widetilde{\nabla} f) = 1$. □

B.Y.Chen and K.Yano[11], introduced the notion of a Riemannian manifold of quasi-constant curvature as a Riemannian manifolds $(\overline{\Sigma}, \overline{g})$ endowed with the curvature tensor $\overline{R}$

satisfying the following equation:

$$\begin{aligned} \overline{g}(\overline{R}(U,V)Z,T) &= s\{\overline{g}(V,Z)\overline{g}(U,T)-\overline{g}(U,Z)\overline{g}(V,T)\} \\ &+ t\{\overline{g}(U,T)\theta(V)\theta(Z)-\overline{g}(U,Z)\theta(V)\theta(T) \\ &+ \overline{g}(V,Z)\theta(U)\theta(T)-\overline{g}(V,T)\theta(U)\theta(Z)\}, \end{aligned} \tag{4.31}$$

for any vector fields $U,V,Z,T\in\Gamma(T\overline{\Sigma})$, where s and t are smooth functions and

$$\theta(U)=\overline{g}(U,\zeta),$$

is $\overline{g}-$dual to a non vanishing smooth vector field ζ called the curvature vector field of $\overline{\Sigma}$. It is well known that if the curvature tensor $\overline{R}$ is of the form $(1,1)$, then $\overline{\Sigma}$ is conformally flat. If $t=0$, then $\overline{\Sigma}$ is a space of constant curvature.

Here, we suppose that the curvature vector field ζ never belongs to the tangent space of the null hypersurface Σ. In this case ζ can be taken as a rigging for Σ. This leads to the following definition.

Definition 4.1. [19] *A null hypersurface Σ of of Lorentzian manifolds of quasi-constant curvature such that the curvature vector field ζ is a rigging for Σ is said to be ζ-**rigging null hypersurface**.*

It is known that a totally umbilic null hypersurface with a closed normalization locally decomposes as a twisted product, [16, Theorem 5.3]. Here, we demonstrate that if it has a torse-forming rigging within an ambient of quasi-constant curvature, the local twisted product structure of the Riemann structure $(\Sigma,\tilde{g})$ is actually a warped product.

Theorem 4.2. *Let $(\overline{\Sigma}^{n+2},\overline{g})$ be a simply connected Lorentzian manifold of quasi-constant curvature (with $n\geq 2$) and Σ a ζ-rigging totally umbilic null hypersurface such that ζ is a torse forming vector field with the generating form W as a rigged for Σ. Then given $x\in\Sigma$, $(\Sigma,\widetilde{g})$ is locally isometric to a warped product $(\mathbb{R}\times\overset{\circ}{\Sigma},dr^2+f^2g_0)$, where $\overset{\circ}{\Sigma}$ is the leaf of $\mathscr{S}(\zeta)$ through x, and g_0 is a conformal metric to $g_{|_{\overset{\circ}{\Sigma}}}$. Moreover, the decomposition is global whenever Σ is simply connected and the rigged vector field ξ complete, .*

Proof. Utilizing [16, Theorem 5.3] as mentioned earlier, our sole focus will be to demonstrate the warped decomposition of $(\Sigma,\widetilde{g})$.

In [16, Theorem 4.8], it is demonstrated that for $X,Y\in T\Sigma$, the following is true,

$$R(X,Y)\xi-\widetilde{R}(X,Y)\xi=\overline{g}(\overline{R}(X,Y)\xi,N)\xi-\tau(X)\overset{\star}{A}_\xi(Y)+\tau(Y)\overset{\star}{A}_\xi(X).$$

As the generating vector field W is rigged for Σ, then from Eq.(3.20), the $1-$form τ vanishes along the null hypersurfaces. Using the Gauss-Codazzi equation we have $R(X,Y)\xi=\overline{R}(X,Y)\xi$. Finally $\overline{R}(X,Y)\xi=t\Big(\alpha(Y)X-\alpha(X)Y\Big)$ as $(\overline{\Sigma},\overline{g})$ has quasi-constant curvature (see (4.31)) and the above equality becomes $\widetilde{R}(X,Y)\xi=t\Big(\alpha(Y)X-\alpha(X)Y\Big)$, $\forall X,Y\in\Gamma(T\Sigma)$. Then, $\widetilde{Ric}(U,\xi)=0$ for all $\mathscr{S}(\zeta)-$valued vector field U. The conclusion is derived from the condition of mixed Ricci flat, as stated in [17, Theorem 1]. □

Theorem 4.3. *Let $(\overline{\Sigma},\overline{g})$ be a Lorentzian manifold of quasi-constant curvature with $t_{|_M}$ non-negative and Σ a ζ-rigging simply connected compact null hypersurface of $\overline{\Sigma}$. If ζ is a* *torse-forming vector field with conformal scalar Ψ and generating form μ, then Σ is totally geodesic in $\overline{\Sigma}$ and $\mathscr{S}(\zeta)$ is totally umbilic in $\overline{\Sigma}$ with umbilic factor σ.*

Proof. Since ζ is torse forming, from above Theorem 3.2, ζ is screen quasi-conformal with $A_N(U) = -\frac{1}{2}\overline{g}(\zeta,\zeta)\overset{\star}{A}_\xi(U) - \Psi PU$ i.e $\kappa = -\frac{1}{2}\overline{g}(\zeta,\zeta)$ and $\sigma = -\Psi$. From (2.6), we have

$$\overline{g}(\overline{R}(U,V)\xi, N) = C(V, \overset{*}{A}_\xi X) - C(U, \overset{*}{A}_\xi V) - 2d\tau(U,V), \tag{4.32}$$

for all $U, V \in \Gamma(T\Sigma)$. Given that $\overline{\Sigma}$ possesses quasi-constant curvature, the left side of (4.32) vanishes. In addition, since Σ is screen quasi-conformal, we obtain

$$C(V, \overset{*}{A}_\xi U) - C(U, \overset{*}{A}_\xi V) = \overline{g}(A_N V, \overset{*}{A}_\xi U) - \overline{g}(A_N U, \overset{*}{A}_\xi V) = 0,$$

implying that $d\tau = 0$, i.e., τ is closed. Given that Σ is simply connected, we can find a function f defined on Σ such that $\tau(U) = df(U)$ holds for every vector field $U \in \Gamma(T\Sigma)$. By introducing a new rigging defined as $\widehat{\zeta} = \gamma\zeta$, the associated rigged vector field is expressed as $\widehat{\xi} = \gamma^{-1}\xi$, leading to the relation $\tau(U) = \widehat{\tau}(U) + X(\log(\gamma^{-1}))$. Hence, by setting $\gamma = \exp(f)^{-1}$ in this expression, we conclude that $\widehat{\tau}(U) = 0$ for all $U \in \Gamma(T\Sigma)$.

Let $\widehat{\mathbf{H}}_\xi$ represent the mean curvature function and $\overset{*}{A}_{\widehat{\xi}}$ denote the screen shape operator associated with $\widehat{\zeta}$. It is a well-established fact that

$$\overline{Ric}(\widehat{\xi}) = \widehat{\xi}(n\widehat{\mathbf{H}}_\xi) + n\widehat{\tau}(\widehat{\xi})\widehat{\mathbf{H}}_\xi - \|\overset{*}{A}_{\widehat{\xi}}\|^2.$$

But $\overline{Ric}(\xi) = nt_{|\Sigma}$ and $\tau(\xi) = 0$, it follows that $\xi(\widehat{\mathbf{H}}_\xi) - |\overset{\star}{A}_\xi|^2 - nt_{|_\Sigma} = 0$. From the inequality $|\overset{\star}{A}_\xi|^2 \geq \frac{1}{n}\widehat{\mathbf{H}}^2_\xi$, we get $\xi(\widehat{\mathbf{H}}_\xi) - \frac{1}{n}\widehat{\mathbf{H}}^2_\xi - nt_{|M} \geq 0$, and since ξ is complete (Σ being compact), we obtain that $\widehat{\mathbf{H}}_\xi = 0$. From the equation $\xi(\widehat{\mathbf{H}}_\xi) - |\overset{\star}{A}_{\widehat{\xi}}|^2 - nt_{|_\Sigma} = 0$, it can be deduced that $|\overset{\star}{A}_{\widehat{\xi}}|^2 = -nt_{|_\Sigma}$, resulting in $\overset{\star}{A}_{\widehat{\xi}} = 0$ and $t_{|_\Sigma} = 0$ on Σ given that $t_{|_\Sigma}$ is non-negative. Since Σ is totally geodesic and considering that ζ possesses a quasi-conformal screen distribution, we can derive from Eq.(2.9) that $\mathscr{S}(\zeta)$ is totally umbilic with the umbilic factor σ. $\square$

Theorem 4.4. *Let $(\overline{\Sigma}^4, \overline{g})$ be a 4-dimensional Lorentzian manifold of quasi-constant curvature with $t_{|_\Sigma}$ non-negative and Σ a compact ζ-rigging null hypersurface. If Σ has finite fundamental group then the normalization ζ can not be torse-forming.*

Proof. Let Σ be defined as above. Assuming there exists a torse-forming normalization ζ, and given that Σ has a finite fundamental group, the first De Rham cohomology group $H^1(\Sigma, \mathbb{R})$ must be trivial. From Theorem 4.3, it can be deduced that Σ is totally geodesic. Additionally, since ζ is established as torse-forming, $\mathscr{S}(\zeta)$ is integrable and leads to a foliation on Σ. We demonstrate the leaves of $\mathscr{S}(\zeta)$ are totally geodesic in $(\Sigma, \widetilde{g})$. To support this, let's recall (from [16], Proposition 3.7) that for U and V in $\mathscr{S}(\zeta)$,

$$\widetilde{\nabla}_U V = \overset{\star}{\nabla}_U V - \widetilde{g}(\widetilde{\nabla}_U \xi, V)\xi,$$

but we also have

$$\widetilde{g}(\widetilde{\nabla}_U \xi, V) + \widetilde{g}(\widetilde{\nabla}_V \xi, U) = L_\xi \widetilde{g}(U,V) = -2B(U,V).$$

Given that $\mathscr{S}(\zeta)$ is integrable, we conclude that $\widetilde{g}(\widetilde{\nabla}_U\xi, V) = \widetilde{g}(\widetilde{\nabla}_V\xi, U)$. This leads to the result that $\widetilde{g}(\widetilde{\nabla}_U\xi, V) = -B(U, V)$, from which we derive that

$$\widetilde{\nabla}_U V = \overset{\star}{\nabla}_U V + B(U, V)\xi.$$

In other words, the second fundamental form of each leaf in $\mathscr{S}(\zeta)$ within the manifold $(M, \widetilde{g})$ is B, leading to the conclusion that all of these leaves are totally geodesic in $(\Sigma, \widetilde{g})$, since Σ is totally geodesic in the $(\overline{\Sigma}^4, \overline{g})$. Consequently, there exists a totally geodesic foliation of codimension one on the compact 3-manifold Σ, which implies that Σ must possess an infinite fundamental group (refer to [18]), resulting in a contradiction. □

5. $K_{\mathscr{S}}$ as a Ricci soliton potential on leaves of $\mathscr{S}(E)$

Consider a normalized null hypersurface (Σ, E) within a $(n+2)$-dimensional semi-Riemannian manifold $(\overline{\Sigma}, \overline{g})$ that possesses a torse-forming vector field K. In this context, we examine the screen distribution $\mathscr{S}(E)$, which is integrable on Σ, ensuring that the structure $(\overset{\circ}{\Sigma}, \overset{\circ}{g}, K_{\mathscr{S}}, \Lambda)$ forms a Ricci soliton. Here, $\overset{\circ}{\Sigma}$ denotes a leaf of $\mathscr{S}(E)$, Λ is constant, and $\overset{\circ}{g}= g_{|_{\mathscr{S}(E)}}$. In this scenario, we have :

$$L_{K_{\mathscr{S}}} \overset{\circ}{g} +2 \overset{\circ}{Ric}= 2\Lambda \overset{\circ}{g}, \tag{5.33}$$

where $L_{K_{\mathscr{S}}}$ is the Lie derivative operator, $\overset{\circ}{Ric}$ is the Ricci tensor of $\overset{\circ}{\Sigma}$ and Λ is a real constant.

Proposition 5.1. *$(\overset{\circ}{\Sigma}, \overset{\circ}{g}, K_{\mathscr{S}}, \Lambda)$ is a Ricci soliton if and only if :*

$$\overset{\circ}{Ric} (U, V) = (\Lambda - \Psi) \overset{\circ}{g} (U, V) - aB(U, V) - bC(U, V) - \frac{1}{2}\{\gamma(U)\mu(V) + \gamma(V)\mu(U)\}, \tag{5.34}$$

for any $U, V \in\overset{\circ}{\Sigma}$.

Proof. As $\overset{\star}{\nabla}$ is a $\overset{\circ}{g}$-metric connection, for any $U, V \in\overset{\circ}{\Sigma}$, from Eq.(3.12) we have :

$$\begin{aligned}
L_{K_{\mathscr{S}}} \overset{\circ}{g} (U, V) &=\overset{\circ}{g} (\nabla^*_U K_{\mathscr{S}}, V)+ \overset{\circ}{g} (U, \nabla^*_V K_{\mathscr{S}}) \\
&=\overset{\circ}{g} (a \overset{\star}{A}_\xi (U) + bA_N(U) + \Psi U + \mu(U)K_{\mathscr{S}}, V) \\
&+ \overset{\circ}{g} (U, a \overset{\star}{A}_\xi (V) + bA_N(V) + \Psi V + \mu(V)K_{\mathscr{S}}) \\
&= 2\Psi \overset{\circ}{g} (U, V) + 2aB(U, V) + 2bC(U, V) + \gamma(U)\mu(V) + \gamma(V)\mu(U).
\end{aligned}$$

From this together with Eq.(5.33), the claim follows. □

If $aB + bC = \eta g$, for some smooth function η, then Σ is screen quasi-conformal and Eq.(5.34) becomes :

$$\overset{\circ}{Ric} (U, V) = (\Lambda - \Psi - \eta) \overset{\circ}{g} (U, V) - \frac{1}{2}\{\gamma(U)\mu(Y) + \gamma(V)\mu(U)\}.$$

This leads to the following Corollary

Corollary 5.1. *Under the previous mentionned hypothesis, the scalar curvature $\overset{\circ}{Sc}$ of $\overset{\circ}{\Sigma}$ is :*

$$\overset{\circ}{Sc}= n(\Lambda - \Psi - \eta)- \overset{\circ}{g} (K_{\mathscr{S}}, W_{\mathscr{S}}). \tag{5.35}$$

Additionally, if the dual vector W of the torse-forming vector field belongs to $(span(\xi) \oplus span(N))$, then $\mu(X) = 0$ for any $X \in \overset{\circ}{\Sigma}$. From this, Eq.(5.35) becomes $\overset{\circ}{Sc} = n(\Lambda - \Psi - \sigma)$.

Theorem 5.1. *Let K be a torse-forming vector field on a Lorentzian manifold $(\overline{\Sigma}^{n+1}, \overline{g})$ where $n \geq 4$, and let (Σ, E) be a normalized null hypersurface of $\overline{\Sigma}$. Suppose that $(\overset{\circ}{M}, \overset{\circ}{g}, K_{\mathscr{S}}, \Lambda)$ is a Ricci soliton on Σ with $aB + bC = \eta \overset{\circ}{g}$, where η is smooth function, and the dual vector field W of the generating form μ is a rigging. Then the following are true :*

(i) *$\Psi + \eta$ is constant function, and given by $\Psi + \sigma = \Lambda$;*

(ii) *$\Sigma = c_\xi \times I \times_{f(t)} \Sigma'$, where c_ξ is a null curve tangent to $T\Sigma^\perp$, I is an open interval with arc length t, f is smooth function on Σ', and Σ' is an Einstein $(n-2)$-dimensional manifold,*

Proof. Since the screen distribution $\mathscr{S}(E)$ is integrable (given that $a \overset{\star}{A}_\xi + bA_N = \eta P$), Theorem 3.2 implies that Σ is locally structured as a product manifold, specifically $c_\xi \times \Sigma'$, where c_ξ represents a null curve tangent to $T\Sigma^\perp$ and $\overset{\circ}{\Sigma}$ is a Riemannian leaf of $\mathscr{S}(E)$. Furthermore, by utilizing Eq.(5.34) alongside the fact that W serves as a rigging for Σ and $aB + bC = \eta \overset{\circ}{g}$, we get $\overset{\circ}{Ric}(U, V) = (\Lambda - \Psi - \eta) \overset{\star}{g}(U, V)$, for all $U, V \in \mathscr{S}(E)$. Which means that $\overset{\circ}{M}$ is Einstein manifold. Since $dim\Sigma \geq 4$, it follows that $dim\ \overset{\circ}{\Sigma} \geq 3$, thus $\overset{\circ}{\Sigma}$ has constant curvature ([10]). Thus $\Lambda - \Psi - \eta$ is constant, hence $\Psi + \eta$ is constant. Moreover, by Eq.(3.12) with the fact that W is a rigging for Σ , we have $\nabla^*_U K_{\mathscr{S}} = (\Psi + \eta)U$. Now it is worth noting that

$$\begin{aligned}
\overset{\circ}{R}(U, K_{\mathscr{S}})K_{\mathscr{S}} &= \nabla^*_U {\nabla^*}_{K_{\mathscr{S}}} K_{\mathscr{S}} - {\nabla^*}_{K_{\mathscr{S}}} \nabla^*_U K_{\mathscr{S}} - {\nabla^*}_{[U, K_{\mathscr{S}}]} K_{\mathscr{S}} \\
&= \nabla^*_U (\Psi + \sigma) K_{\mathscr{S}} - {\nabla^*}_{K_{\mathscr{S}}} (\Psi + \sigma) U - (\Psi + \sigma)[U, K_{\mathscr{S}}] \\
&= U(\Psi + \eta) K_{\mathscr{S}} - K_{\mathscr{S}}(\Psi + \eta) U \\
&= 0.
\end{aligned}$$

This leads to

$$\overset{\circ}{Ric}(K_{\mathscr{S}}, K_{\mathscr{S}}) = \sum_{i=1}^{n-1} \overset{\circ}{g}(\overset{\circ}{R}(e_i, K_{\mathscr{S}})K_{\mathscr{S}}, e_i) = 0.$$

Using this with Eq.(5.34), we get $(\Lambda - \Psi - \eta) \overset{\circ}{g}(X, Y) = 0$, which gives $\Lambda = \Psi + \eta$ since $\overset{\circ}{g}$ is Riemannian. To show item (ii), we see that $K_{\mathscr{S}}$ is closed conformal vector field on $\overset{\circ}{\Sigma}$, and by Theorem 3.6 $\overset{\circ}{\Sigma}$ is a warped product manifold $I \times_{f(t)} \Sigma'$, where $f(t)$ is a positive smooth function. □

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A GENERALIZED TOPOLOGICAL CONVERGENCE OF FUNCTION'S SEQUENCES CONFINED BY WEIGHTS

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Abstract. In this article, an extended method based on weight functions for pointwise and uniform statistical convergence for function sequences in topological settings is presented. Several fundamental theorems are proven and new definitions are suggested in order to rigorously describe these sorts of generalized convergence. Illustrations and counterexamples that highlight the differences and advantages of the suggested approaches provide further support for the theoretical development. Furthermore, this study has also examined the topological implications of these new types of convergence with the presence of Dini's theorem.

Keywords: Sequence of function, Weighted statistical convergence, Weighted statistical limit of sequence of function, Weight function.
MSC (2020): 40A30, 54A20.

1. Introduction

The traditional concept of convergence for sequences frequently turns out to be inappropriate for some findings. To overcome this restriction, H. Fast proposed the concept of statistical convergence in 1951 [15] with the advancement of A. Zygmund's summability technique [30], providing a more adaptable substitute that allows for a limited number of terms differ from the limit. Independently, J.A. Schoenberg [24] explored the same, connecting statistical convergence with summability theory. The notion of natural density (also known as asymptotic density) for a set $Q \subseteq \mathbb{N}$, has been defined as,

$$\delta(Q) = \lim_{n\to\infty} \frac{|\{l \leq n : l \in Q\}|}{n}$$

if the limit exists.

A significant breakthrough was made in 2008 when G. Di Maio and L. Kočinac [21] expanded the idea to topological spaces, allowing for the analysis of convergence behavior beyond purely metric or numerical contexts. As evidenced by further research [4, 16, 18, 19],

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this cleared the path for more generalizations and applications, expanding the theory's use across a different branch of subjects.

Within the realm of function spaces, statistical convergence gained prominence with the work of A. Gökhan and M. Güngür [17] in 2002. They introduced pointwise statistical convergence and statistical Cauchy conditions for real-valued function sequences, demonstrating their equivalence under suitable conditions. Their results established a solid foundation for extending statistical convergence to function-level analysis. Later, Balcerzak, Dems, and Komisarski [3] contributed further by considering statistical and ideal convergence for function sequences in metric contexts.

Explorations into more generalized normed structures followed, with Sarabadan and Talebi [23], and subsequently Yegül and Dündar [27], examining statistical convergence within 2-normed spaces. These efforts revealed rich structural behavior of function sequences under alternative topological constraints and highlighted the versatility of statistical convergence as an analytical tool. Numerous work on statistical convergence can be found in [5, 6, 7, 8, 9, 11, 12, 14, 25, 28, 29].

In an effort to refine these convergence methods further, the concept of weighted statistical convergence emerged. By replacing the standard counting function with a suitable weight function g, a generalized weighted density was introduced in 2020 by Adem et. al. [1] that defined as,

$$\delta_g(Q) = \lim_{n\to\infty} \frac{|\{l \leq n : l \in Q\}|}{g(n)}.$$

A generalized work has been paved way in the field of Topology by Das et. al. in 2024 [10] using the weight function g that offers some important applications in the field of weighted statistical convergence in topological space.

In parallel to these developments, a classical result such as Dini's theorem which asserts the uniform convergence of monotonic, pointwise-converging sequences of continuous functions on compact domains have attracted renewed interest. Various researchers have investigated how this theorem might be extended under statistical convergence schemes on metric spaces and normed spaces.

In view of these developments, this current study presents a framework for examining uniform statistical convergence and pointwise statistical convergence of function sequences in topological spaces by the weights. Lastly, the work provides novel versions of Dini's theorem in the weighted context.

2. Preliminaries

Before delving into the new variant of statistical convergence, some prerequisite definitions, terms and notations are added for the reader's convenience. For common symbols, nomenclature and concepts we follow [13].

Definition 2.1. [1] *Let* $g : \mathbb{N} \longrightarrow [0, \infty)$ *be a function is called weight function if* $\lim_{n\to\infty} g(n) = \infty$ *and* $\lim_{n\to\infty} \frac{n}{g(n)} \neq 0$.

Definition 2.2. [2] *Let* $\langle f_n \rangle_{n\in\mathbb{N}}$ *be a sequence of functions where* $f_n, f : (X, \tau) \longrightarrow (Y, \sigma)$ *for all* $n \in \mathbb{N}$*. Then for an arbitrary* $a \in X$*,*

$$\diamondsuit^-_{f(a)}(f_n(a)) = \{n \in \mathbb{N} : \{f_n(a)\} \cap V = \emptyset \textit{ for at least one neighborhood } V \textit{ of } f(a)\}.$$

Definition 2.3. [10] *Let g be a weight function. A sequence $\{x_n : n \in \mathbb{N}\}$ is said to be g-statistically convergent to x_0 in a topological space (X, τ), if for every neighborhood U of x_0, $\delta_g(\{n \in \mathbb{N} : x_n \notin U\}) = 0$.*

Theorem 2.1. [22] *A pointwise convergent sequence $\{f_n\}$ of functions is also uniformly convergent f on A if the following conditions are satisfied:*

i) *A is compact set in a metric space (X, d);*
ii) *f and each f_n is continuous on A for $n \in \mathbb{N}$;*
iii) *$\{f_n\}$ is decreasing: $0 \le f_{n+1}(x) \le f_n(x)$ for all $n \in \mathbb{N}$ and all $x \in A$.*

3. On s_g-limit for sequence of function

Based on the operator approach of statistical convergence for sequence of function by the author of [2], this paper aims to propose a new variant of statistical convergence restricted with a weight function to control the rate of convergence under the topological settings.

Throughout the paper, a mapping $g : \mathbb{N} \to \mathbb{R}^+$ is considered as the weight function defined as $g(n) = \log(1 + n)$ and $g(n) = n^p$ for all $n \in \mathbb{N}$ and $0 < p \le 1$ with the existence of $\lim_{n\to\infty} \frac{n}{g(n)} \neq 0$. These weight functions are used for the development of examples.

Definition 3.1. *A sequence of function $(\varphi_m(x))_{m\in\mathbb{N}}$ where $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ is termed as pointwise weighted statistically convergent (or shortly pointwise s_g-convergent) to $\varphi(x)$, if $\delta_g(\diamondsuit^-_{\varphi(x)}(\varphi_m(x))) = 0$ for all $x \in X$. Mathematically indicated as Pt-$s_g - \lim_{m\to\infty} \varphi_m = \varphi$ or $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varphi$.*

Definition 3.2. *A sequence of function $(\varphi_m(x))_{m\in\mathbb{N}}$ where $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ is termed as uniform weighted statistically convergent (or shortly uniform s_g-convergent) to $\varphi(x)$, if $\delta_g(\bigcup_{x\in X} \diamondsuit^-_{\varphi(x)}(\varphi_m(x))) = 0$. Mathematically indicated as U-$s_g - \lim_{m\to\infty} \varphi_m = \varphi$ or $\varphi_m \overset{U-s_g-\lim}{\longrightarrow} \varphi$.*

Theorem 3.1. *Every uniform s_g-convergent sequence of function is pointwise s_g-convergent.*

Proof. Let $(\varphi_m)_{m\in\mathbb{N}}$ be uniform s_g-convergent sequence of function that converges to φ, i.e., $\delta_g(\bigcup_{x\in X}\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) = 0$. Now, it is obvious that $\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\} \subseteq \bigcup_{x\in X}\{m \in \mathbb{N} : \varphi(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}$ that gives $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) \le \delta_g(\bigcup_{x\in X}\{m \in \mathbb{N} : \varphi(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) = 0$. As a result, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ for at least one neighborhood W of $\varphi(x)\}) = 0$ for all $x \in X$. Thus, $\delta_g(\diamondsuit^-_{\varphi(x)}\varphi_m(x)) = 0$ for all $x \in X$. Hence $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varphi$. □

Example 3.1. *Any sequence of function that is pointwise s_g-convergent does not imply the sequence of function to be uniform s_g-convergent.*

Let us consider a function $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ *for all* $m \in \mathbb{N}$ *such that* $Y = \{p_1, p_2\}$ *and* $\sigma = \{\emptyset, \{p_1\}, Y\}$ *defined for all* $x \in X$ *and let the sequence of function* $(\varphi_m(x))$ *is defined by*

$$\varphi_m(x) = \begin{cases} p_2, & \text{if } m = l^l : l \in \mathbb{N} \\ p_1, & \text{otherwise} \end{cases}$$

Here the neighborhoods of $\varphi(x) = p_1$ *are* $W_1 = \{p_1\}$ *and* $W_2 = \{Y\}$. *For the neighborhood* W_1 *of* p_1, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_1 = \emptyset$ *for the neighborhood* $\{p_1\}$ *of* $p_1\}) = \delta_g(\{1, 4, 27, 256, \cdots\})$ $= 0$ *and for the neighborhood* W_2 *of* p_1, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_2 = \emptyset$ *for the neighborhood* Y *of* $p_1\}) = \delta_g(\emptyset) = 0$ *for all* $x \in X$.
Thus, for every $x \in X$, $\varphi_m(x) \overset{Pt-s_g-\lim}{\longrightarrow} \varphi(x)$.
But, $\delta_g(\bigcup_{x\in X}\{m \in \mathbb{N} : \varphi_m(x) \cap W = \emptyset$ *for at least one neighborhood* W *of* $\varphi(x)\}) \neq 0$.
Thus, $\varphi_m(x) \overset{U-s_g-\lim}{\not\longrightarrow} \varphi(x)$.

Example 3.2. *Limit of a pointwise* s_g*-convergent sequence of function may have more than one limit point.*
Let us consider a function $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ *such that* $Y = \{a_1, a_2, a_3\}$ *and* $\sigma = \{\emptyset, Y, \{a_1, a_2\}, \{a_3\}\}$. *Then for all* $m \in \mathbb{N}$ *the sequence of function* (φ_m) *is defined as,*

$$\varphi_m = \begin{cases} a_1, & \text{if } m = l^l, l \in \mathbb{N} \\ a_2, & \text{otherwise} \end{cases}$$

Here the open neighborhoods of a_2 *are* $W_1 = \{a_1, a_2\}$ *and* $W_2 = Y$. *For the neighborhood* W_1 *of* a_2, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_1 = \emptyset$ *for the neighborhood* $\{a_1, a_2\}$ *of* $a_2\}) = \delta_g(\{1, 4, 27, \cdots\}) = 0$ *and for the neighborhood* W_2 *of* a_2, $\delta_g(\{m \in \mathbb{N} : \varphi_m(x) \cap W_2 = \emptyset$ *for the neighborhood* Y *of* $a_2\}) = \delta_g(\emptyset) = 0$ *for all* $x \in X$.
Thus, for every neighborhood W *of* a_2, $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} a_2$. *But the neighborhoods of* a_2 *are same as the neighborhoods of* a_1. *So, for every neighborhood of* a_1, $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} a_1$.
Thus, limit of a pointwise s_g*-convergent sequence of function has more than one limit point.*

Theorem 3.2. *Every pointwise weighted statistically convergent sequence of function does not have more than one limit point in a Hausdorff co-domain space.*

Proof. Let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence of function that converges pointwise weighted statistically to two limit points $\varphi(b)$ and $\varpi(b)$ with $\varphi(b) \neq \varpi(b)$ for any specified $b \in X$.
But (Y, σ) is a Hausdorff space so, there exist two open neighborhoods $W_1, W_2 \subseteq \sigma$ of the limit points $\varphi(b)$ and $\varpi(b)$ such that $\varphi(b) \in W_1$, $\varpi(b) \in W_2$ and also $W_1 \cap W_2 = \emptyset$ that implies $W_2 \subseteq W_1^{\complement}$.
Since $W_2 \subseteq W_1^{\complement}$ so, for a specified $b \in X$,
$\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 \neq \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\} \subseteq \{m \in \mathbb{N} : \varphi_m(b) \cap W_1^{\complement} \neq \emptyset$ for at least one neighborhood W_1 of $\varphi(b)\}$.
i.e., $\delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 \neq \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\}) \leq \delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_1^{\complement} \neq \emptyset$ for at least one neighborhood W_1 of $\varphi(b)\}) = 0$.
i.e, $\delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 \neq \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\}) = 0$. But $\delta_g(\{m \in \mathbb{N} : \varphi_m(b) \cap W_2 = \emptyset$ for at least one neighborhood W_2 of $\varpi(b)\}) \neq 0$ which contradicts the fact that $\varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi(b)$.

As a result, $\varphi(b) = \varpi(b)$ for a specified $b \in X$.
Hence, limit of a pointwise s_g-convergent sequence of function is always unique in Hausdorff co-domain space. $\square$

Theorem 3.3. *If $\varphi_m, \varphi : (X, \tau) \longrightarrow (Y, \sigma)$ and $\varpi_m, \varpi : (Y, \sigma) \longrightarrow (Z, \eta)$ are functions such that $\varpi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi$, then $\varpi_m \circ \varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi \circ \varphi$.*

Proof. As, $\varpi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi$ so, $\delta_g(\diamondsuit^-_{\varpi(y)} \varpi_m(y)) = 0$ for all $y \in Y$.
Let $b \in X$ be arbitrary. Therefore, $\varphi(b) \in Y$ that results $\delta_g(\diamondsuit^-_{\varpi(\varphi(b))} \varpi_m(\varphi_m(b))) = 0$.
Thus, $\delta_g(\diamondsuit^-_{\varpi\circ\varphi(b)} \varpi_m \circ \varphi_m(b)) = 0$ for all $x \in X$. Hence $\varpi_m \circ \varphi_m \overset{Pt-s_g-\lim}{\longrightarrow} \varpi \circ \varphi$. $\square$

From the article [10], it is clear that every s_g-convergent sequence is an s-convergent. So, it can be possible to show that every pointwise s_g-convergent sequence of function is pointwise s-convergent. Also every uniform s_g-convergent sequence of function is uniform s-convergent. For this applications, the following diagram is depicted below.

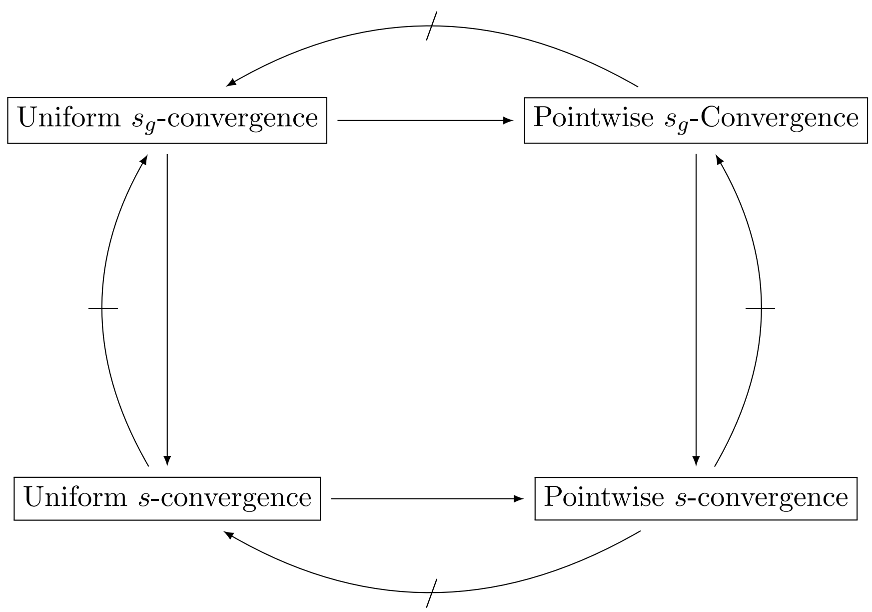

Figure 1. Relationship chart

Theorem 3.4. *A pointwise weighted statistical convergent sequence of function (φ_n) that converges to φ is also uniform weighted statistically convergent to φ, if the following conditions holds,*

i) *X is countable and (X, τ) is sequential compact.*
ii) *φ and every φ_n is continuous on X for all $n \in \mathbb{N}$.*
iii) *φ_n is monotonic for all $n \in \mathbb{N}$.*

Proof. Let $\varphi_n \overset{Pt-s_g-\lim}{\longrightarrow} \varphi$ and φ_n is a monotone increasing sequence. Therefore, $\delta_g(\diamondsuit^-_{\varphi(x)} \varphi_n(x)) = 0$ for all $x \in X$.
Let $\varpi_n(x) = \varphi_n(x) - \varphi(x)$ for all $x \in X$ therefore, $\varpi_n \overset{Pt-s_g-\lim}{\longrightarrow} 0$ as φ_n is increasing. Since

φ and φ_n are continuous, ϖ_n is also continuous for all $n \in \mathbb{N}$. Therefore, $\delta_g(\diamondsuit_0^- \varpi_n(x)) = 0$ for all $x \in X$.

If possible, let us assume $\delta_g(\bigcup_{x_m \in X} \diamondsuit_0^- \varpi_n(x_m)) \neq 0$. Then by sequential compactness $\{x_m\}$ has a convergent sub sequence $x_{m_k} \longrightarrow x_0 \in X$. Also, ϖ_n is continuous so, by continuity $\varpi_n(x_{m_k}) \longrightarrow \varpi_n(x_0)$. But, $\varpi_n(x_0) \xrightarrow{Pt-s_g-\lim} 0$. So, $\varpi_n(x_{m_k}) \xrightarrow{U-s_g-\lim} 0$ for all $k \in \mathbb{N}$ which contradicts the fact that $\delta_g(\bigcup_{x_m \in X} \diamondsuit_0^- \varpi_n(x_m)) \neq 0$.

Hence $\varphi_n \xrightarrow{U-s_g-\lim} \varphi$. □

4. Conclusion

In this work, we developed two new variants of statistical convergence (pointwise and uniform) for function's sequences that controlled with the weight function g to witness the pace of convergence under topological space and also established the relation among them. We showed that the pointwise weighted statistically convergent sequence of functions admits multiple limit points in a topological space while in a Hausdorff codomain space, the limit becomes unique. It was further shown that the composition of two different pointwise weighted statistical convergent sequence of function also result a pointwise statistical convergent. Lastly, a theorem has been developed to generalize the Dini's theorem with the weight function, that provides a broader framework for convergence of function's sequences.

This work may be further extended by the regulation of new parameter α that will effect the rate of convergence under topological settings.

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