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Contents

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ON SUPRA e^*-OPEN SETS AND SUPRA e^*-CONTINUOUS FUNCTIONS

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Abstract. In the present study, we introduced a novel type of generalized supra open sets called supra e^*-open sets via supra δ-closure operator which we define. Through this new concept, we defined and studied supra e^*-continuous functions, supra e^*-open functions and supra e^*-closed functions. Also, we investigated relationships between supra e^*-continuous functions and different generalized types of supra continuity.

Keywords: Supra regular open set, Supra δ-closure operator, Supra e^*-open set, Supra e^*-continuous function, Supra e^*-open function.

2020 Mathematics Subject Classification: 54A10, 54A20, 54C10, 54C08.

1. Introduction

In the last decades, numerous investigators have worked on generalized types of open sets such as preopen [17], semi-open [16], α-open [16], b-open [9], β-open [1], e-open [12], e^*-open [13]. Some studies conducted with the help of generalized open sets are as follows: The class of somewhere dense sets [3] are contained all α-open, preopen, semi-open, β-open and b-open sets except for the empty set. Also, the concept of ST_1-space is defined in the same paper and its various features are investigated. Al-Shami and Noiri continued to study more properties of somewhere dense sets in [4].

On the other hand, Mashhour et al. defined the notion of supra open sets [18] in supra topological spaces in 1983. Later, many researchers introduced and studied generalizations of supra open sets. In 2008, Devi et al. [11] explored a kind of sets and functions called

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supra α-open sets and supra α-continuous functions, subsequently. In 2010, Noiri and Sayed investigated supra pre-open sets [19] and supra b-open sets [20]. In 2013, Vidyarani [22] introduced supra regular open sets. In 2013, Jafari and Tahiliani [15] worked on supra β-open sets and supra β-continuous functions. In 2017, Al-Shami [2] studied supra semi-continuous functions and supra semi-open functions through supra semi-open sets.

The other studies referring to recent contributions in supra topology and their applications can be listed as follows: M. E. El-Shafei et al. [14] introduced strong supra regularly ordered spaces, strong supra normally ordered spaces and strong supra T_i-ordered spaces $(i = 0, 1, 2, 3, 4)$ on supra topological ordered spaces and investigate the main properties of them. B. A. Asaad et al. [10] studied the notion of an operator γ on a supra topological space and then this notion is utilized to analyze supra γ-open sets. T. M. Al-Shami and I. Alshammari [5] found new rough-approximation operators inspired by an abstract structure called supra topology. T. M. Al-Shami et al. [6] introduced new forms of limit points of a set and separation axioms on supra topological spaces via supra α-open sets (resp. supra β-open sets [7]) T. M. Al-Shami et al. [8] defined three types of supra compactness and three types of supra Lindelöfness using supra topological spaces via supra pre-open sets.

In this paper, first of all, we introduced the concept of supra δ-closure operator via supra regular open sets. Then, we defined supra e^*-openness with the help of this operator. Later, we developed the ideas of supra e^*-continuous functions and supra e^*-open functions via supra e^*-open sets. We also obtained several characterizations of supra e^*-continuity and revealed some of its basic features.

2. Preliminaries

In the whole of this study, unless explicitly stated topological spaces $(\Psi, \top)$ and $(\Phi, \perp)$ (or simply Ψ and Φ) always mean on which no separation axioms are supposed. Then the closure and interior of E are expressed by $cl(E)$ and $int(E)$, subsequently. The collection of all open (resp. closed) sets of Ψ are expressed by $O(\Psi)$(resp. $C(\Psi)$). Also, (Ψ, μ) and (Φ, η) represent supra topological spaces.

A point $x \in \Psi$ is referred to as δ-cluster point [21] of E if $int(cl(O)) \cap E \neq \emptyset$ for every open neighborhood O of x. The set of all δ-cluster points of O is called the δ-closure [21] of E and is expressed by $cl_\delta(E)$. If $E = cl_\delta(E)$, then E is called δ-closed [21] and the complementary of a δ-closed set is called δ-open [21]. The set $int_\delta(E) := \{x | (\exists O \in O(\Psi, x))(int(cl(O)) \subseteq E)\}$ is called the δ-interior of E.

A subclass $\mu \subseteq 2^\Psi$ is referred to as a supra topology on Ψ [18] if Ψ is an element of μ and μ is closed under arbitrary union. (Ψ, μ) is referred to as a supra topological space (briefly, supra space) [18]. The members of μ are called supra open sets (briefly, s.o.) [18]. The complementary of supra open set is referred to as a supra closed set [18]. The intersection (resp. union) of all supra closed (resp. supra open) sets of Ψ containing (resp. contained in) E is called the supra closure [18] (resp. supra interior [18]) of E and is expressed by $cl^\mu(E)$ (resp. $int^\mu(E)$).

Definition 2.1. *Let (Ψ, μ) be a supra topological space. A subset E of Ψ is referred to as:*
(ι_1) *supra regular open* [22] *(briefly, s.r.o.) if $E = int^\mu(cl^\mu(E))$.*
(ι_2) *supra α-open* [11] *(briefly, s.α.o.) if $E \subseteq int^\mu(cl^\mu(int^\mu(E)))$.*
(ι_3) *supra semi-open* [3] *(briefly, s.s.o.) if $E \subseteq cl^\mu(int^\mu(E))$.*
(ι_4) *supra preopen* [19] *(briefly, s.p.o.) if $E \subseteq int^\mu(cl^\mu(E))$.*
(ι_5) *supra b-open* [20] *(briefly, s.b.o.) if $E \subseteq int^\mu(cl^\mu(E)) \cup cl^\mu(int^\mu(E))$.*
(ι_6) *supra β-open* [15] *(briefly, s.β.o.) if $E \subseteq cl^\mu(int^\mu(cl^\mu(E)))$.*

The complementary of a supra regular open (resp. supra α-open, supra semi-open, supra preopen, supra b-open, supra β-open) set is called supra regular closed [22] (resp. supra α-closed [11], supra semi-closed [3], supra preclosed [19], supra b-closed [20], supra β-closed [15]).

The collection of all supra open (resp. supra regular open, supra α-open, supra semi-open, supra preopen, supra b-open, supra β-open, supra closed, supra regular closed, supra α-closed, supra semi-closed, supra preclosed, supra b-closed, supra β-closed) sets in (Ψ, μ) is expressed by $\mu(\Psi)$ (resp. $R\mu(\Psi)$, $\alpha\mu(\Psi)$, $S\mu(\Psi)$, $P\mu(\Psi)$, $b\mu(\Psi)$, $\beta\mu(\Psi)$, $\mu^c(\Psi)$, $R\mu^c(\Psi)$, $\alpha\mu^c(\Psi)$, $S\mu^c(\Psi)$, $P\mu^c(\Psi)$, $b\mu^c(\Psi)$, $\beta\mu^c(\Psi)$).

Definition 2.2. [18] *Let Ψ be a topological space. If $\top \subseteq \mu$, then μ is called a supra topology associated with $\top$.*

Definition 2.3. *Let Ψ and Φ be two topological spaces and μ be a supra topology associated with $\top$. A function $\Delta : \Psi \to \Phi$ is called supra continuous (resp. supra α-continuous* [11]*, supra precontinuous* [19]*, supra semi-continuous* [3]*, supra b-continuous* [20]*, supra β-continuous* [15]*) if for all open set L of Φ, $\Delta^{-1}[L]$ is s.o. (resp. s.α.o., s.p.o., s.s.o., s.b.o., s.β.o.) in Ψ.*

Definition 2.4. *A function $\Delta : \Psi \to \Phi$ is referred to as e*-continuous* [13] *if for every open set E of Φ, $\Delta^{-1}[E]$ is e*-open in Ψ.*

3. Supra δ-closure Operator and Supra e^*-open Sets

In this part of the study, we define supra e^*-open sets via supra δ-closure operator and investigate some of its basic features.

Definition 3.1. *Let μ be a supra topology on Ψ, then the supra δ-closure of $E \subseteq \Psi$ is expressed as follows:*

$$cl^{\mu}_{\delta}(E) := \bigcap\{G|(E \subseteq G)(G \in R\mu^c(\Psi))\}.$$

and the supra δ-interior of $A \subseteq \Psi$ is expressed as follows:

$$int^{\mu}_{\delta}(E) := \bigcup\{V|(V \subseteq E)(V \in R\mu(\Psi))\}.$$

Definition 3.2. *Let (Ψ, μ) be a supra topological space and $E \subseteq \Psi$. The set E is called a supra δ-closed (briefly, s.δ.c.) set if $E = cl^{\mu}_{\delta}(E)$. The complementary of a supra δ-closed set is referred to as supra δ-open (briefly, s.δ.o.).*

Theorem 3.1.

(ι_1) *Every s.r.o. set is a s.δ.o. set.*

(ι_2) *Every s.δ.o. set is a s.o. set.*

Proof. The proofs are clear from Definition 3.2. □

Remark 3.1. *In the subsequent example demonstrated that a s.o. set need not be a s.δ.o. set.*

Example 3.1. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$. Define a supra topology $\mu = \{\Psi, \emptyset, \{\eth_1\}, \{\eth_1, \eth_3\}, \{\eth_2, \eth_3\}\}$ on Ψ. Then the set $\{\eth_1, \eth_3\}$ is a s.o. set, however, it is not s.δ.o.*

Question: Is there any s.δ.o. set which is not s.r.o.?

Definition 3.3. *Let (Ψ, μ) be a supra topological space and $E \subseteq \Psi$. The set E is called a supra e*-open (briefly, s.e*.o.) set if $E \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(E)))$. The complementary of a supra e*-open set is referred to as supra e*-closed. The collection of all supra e*-open (resp. supra e*-closed) set is expressed by $e^*\mu(\Psi)(e^*\mu^c(\Psi))$.*

Theorem 3.2. *Every s.β.o. set is a s.e*.o. set.*

Proof. Let $E \in \beta\mu(\Psi)$. Thus, $E \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}(E)))$. On the other hand, we have always $cl^{\mu}(E) \subseteq cl^{\mu}_{\delta}(E)$, then we get that $E \in e^{*}\mu(\Psi)$. □

Remark 3.2. *A s.e^*.o. set need not be s.β.o. as indicated by the subsequent example.*

Example 3.2. *Consider the supra topology in Example 3.1. Then the set $\{\eth_2\}$ is a s.e^*.o. set, however, it is not s.β.o.*

Remark 3.3. *From the above discussions and Theorem 3.2, we obtain the following diagram. However, the opposites of these implications don't hold always correct. Also, counterexamples of the other implications are shown in* [3], [11], [15] *and* [19].

$$\begin{array}{ccccccccccccc} & & & & & & & \text{s.}p\text{.o.} & & & & & \\ & & & & & & \nearrow & & \searrow & & & & \\ \text{s.}r\text{.o.} & \rightarrow & \text{s.}\delta\text{.o.} & \rightarrow & \text{s.o.} & \rightarrow & \text{s.}\alpha\text{.o.} & & \text{s.}b\text{.o.} & \rightarrow & \text{s.}\beta\text{.o.} & \rightarrow & \text{s.}e^{*}\text{.o.} \\ & & & & & & \searrow & & \nearrow & & & & \\ & & & & & & & \text{s.}s\text{.o.} & & & & & \end{array}$$

Theorem 3.3. *Let (Ψ, μ) be a supra topological space, then the following properties hold:*
(ι_1) *If $\mathcal{A} \subseteq e^{*}\mu(\Psi)$, then $\cup\mathcal{A} \in e^{*}\mu(\Psi)$.*
(ι_2) *The intersection of two s.e^*.o. sets is not necessarily s.e^*.o.*
(ι_3) *$\Psi \in e^{*}\mu(\Psi)$.*

Proof. (ι_1) : Let $\mathcal{A}$ be a collection of s.e^*.o. sets in Ψ.

$$\begin{aligned} E \in \mathcal{A} \quad &\Rightarrow \quad \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(E))) \subseteq \cup\mathcal{A} \\ &\Rightarrow \quad E \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(\cup\mathcal{A}))) \\ &\Rightarrow \quad \cup\mathcal{A} \subseteq cl^{\mu}(int^{\mu}(cl^{\mu}_{\delta}(\cup\mathcal{A}))). \end{aligned}$$

(ι_2) : Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and let $\mu = \{\emptyset, \{\eth_1\}, \{\eth_1, \eth_2\}, \{\eth_2, \eth_3\}, \Psi\}$ be a supra topological space on Ψ. Although the subsets $\{\eth_1, \eth_3\}$ and $\{\eth_2, \eth_3\}$ are s.e^*.o. in Ψ, which is the intersection set $\{\eth_3\}$ is not s.e^*.o. in Ψ.
(ι_3) : Obvious. □

Theorem 3.4. *Let (Ψ, μ) be a supra topological space, then the following properties hold:*
(ι_1) *If $\mathcal{A} \subseteq e^{*}\mu^{c}(\Psi)$, then $\cap\mathcal{A} \in e^{*}\mu^{c}(\Psi)$.*
(ι_2) *The union of two s.e^*.c. sets is not necessarily s.e^*.c.*

Proof. It is obvious from the proof of Theorem 3.3. □

Definition 3.4. *The supra e^*-closure (resp. supra e^*-interior) of a set E is the intersection (resp. union) of the supra e^*-closed (resp. supra e^*-open) sets including (resp. included in) E, which is expressed by $cl^{\mu}_{e^*}(E)$ (resp. $int^{\mu}_{e^*}(E)$).*

Remark 3.4. *It is obvious from the above definition that $int^{\mu}_{e^*}(E) \in e^*\mu(\Psi)$ and $cl^{\mu}_{e^*}(E) \in e^*\mu^c(\Psi)$.*

Theorem 3.5. *The following properties hold for the supra e^*-interior and supra e^*-closure of subsets F and G of a space Ψ.*
(ι_1) *$int^{\mu}_{e^*}(F) \subseteq F$ and $F \subseteq cl^{\mu}_{e^*}(F)$.*
(ι_2) *$int^{\mu}_{e^*}(F) = A$ iff F is a s.e*.o. set and $cl^{\mu}_{e^*}(F) = A$ iff A is a s.e*.c. set.*
(ι_3) *$int^{\mu}_{e^*}(\Psi \setminus F) = \Psi \setminus cl^{\mu}_{e^*}(F)$ and $cl^{\mu}_{e^*}(\Psi \setminus F) = \Psi \setminus int^{\mu}_{e^*}(F)$.*
(ι_4) *If $F \subseteq G$, then $int^{\mu}_{e^*}(F) \subseteq int^{\mu}_{e^*}(G)$ and $cl^{\mu}_{e^*}(F) \subseteq cl^{\mu}_{e^*}(G)$.*

Proof. Straightforward. □

Theorem 3.6. *Let A and B be any subsets of a space Ψ, then the following properties hold:*
(ι_1) *$int^{\mu}_{e^*}(A) \cup int^{\mu}_{e^*}(B) \subseteq int^{\mu}_{e^*}(A \cup B)$.*
(ι_2) *$cl^{\mu}_{e^*}(A \cap B) \subseteq cl^{\mu}_{e^*}(A) \cap cl^{\mu}_{e^*}(B)$.*

Proof. Straightforward. □

Remark 3.5. *The inclusions in (ι_1) and (ι_2) in Theorem 3.6 can not replaced by equalities by as can be seen from the following examples.*

Example 3.3. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and $\mu = \{\emptyset, \Psi, \{\eth_1\}, \{\eth_1, \eth_2\}, \{\eth_2, \eth_3\}\}$ be a supra topology on Ψ. Where, if $A = \{\eth_2\}$ and $B = \{\eth_3\}$, then $int^{\mu}_{e^*}(A) = int^{\mu}_{e^*}(B) = \emptyset$ and $int^{\mu}_{e^*}(A \cup B) = int^{\mu}_{e^*}(\{\eth_2, \eth_3\}) = \{\eth_2, \eth_3\}$.*

Example 3.4. *Let μ be the same supra topology on Ψ as given in the above example. If $C = \{\eth_1, \eth_2\}$ and $D = \{\eth_1, \eth_3\}$, then $cl^{\mu}_{e^*}(C) = cl^{\mu}_{e^*}(D) = \Psi$ and $cl^{\mu}_{e^*}(C \cap D) = cl^{\mu}_{e^*}(\{\eth_1\}) = \{\eth_1\}$.*

4. Supra e^*-continuous Functions

In this part of the study, we define a novel form of continuous functions called supra e^*-continuous. Also, we obtain several characterizations and investigate some of its fundamental properties.

Definition 4.1. *Let Ψ and Φ be two topological spaces. Let μ be an associated supra topology with $\top$. A function $\Delta : \Psi \to \Phi$ is called supra e^*-continuous if for each open set V of Φ, $\Delta^{-1}[V]$ is supra e^*-open in Ψ.*

Theorem 4.1. *Every continuous function is a supra e^*-continuous function.*

Proof. Let $V \in O(\Phi)$ and μ be an associated supra topology with $\top$.

$$\left.\begin{array}{r} \left.\begin{array}{r} V \in O(\Phi) \\ \Delta \text{ is continuous} \end{array}\right\} \Rightarrow \ \Delta^{-1}[V] \in O(\Psi) \\ O(\Psi) \subseteq \mu(\Psi) \end{array}\right\} \Rightarrow \Delta^{-1}[V] \in \mu(\Psi) \subseteq e^*\mu(\Psi).$$

That means Δ is supra e^*-continuous. □

Remark 4.1. *A supra e^*-continuous function need not be neither continuous nor supra β-continuous as shown by the following examples.*

Example 4.1. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and $\top = \{\emptyset, \Psi, \{\eth_3\}\}$ be a topology on Ψ and the supra topology μ is expressed as $\mu = \{\emptyset, \Psi, \{\eth_1\}, \{\eth_1, \eth_2\}\}$. Let $\Delta : (\Psi, \top) \to (\Psi, \top)$ be a function expressed as $\Delta := \{(\eth_1, \eth_1), (\eth_2, \eth_3), (\eth_3, \eth_2)\}$. The pre-image of the open set $\{\eth_3\}$ is $\{\eth_2\}$. In that case $\{\eth_2\} \in e^*\mu(\Psi)$ and $\{\eth_2\} \notin O(\Psi)$. Thus, Δ is supra e^*-continuous, however, it is not continuous.*

Example 4.2. *Let $\Psi = \{\eth_1, \eth_2, \eth_3\}$ and $\top = \{\emptyset, \Psi, \{\eth_2\}\}$ be a topology on Ψ. Consider the supra topology in Example 3.1 on Ψ. Then the identity function $\Delta : (\Psi, \top) \to (\Psi, \top)$ is supra e^*-continuous. However, it is not supra β-continuous.*

Remark 4.2. *From Remark 3.3 and Examples 4.1 and 4.2, we have the following diagram. However, the opposites of the requirements are not always true. Also, counterexamples of the other requirements are shown in* [15], [19] *and* [20].

$$\begin{array}{ccccccc} & & & & \text{supra pre-cont.} & & \\ & & & \nearrow & \downarrow & & \\ \text{supra cont.} & \to & \text{supra } \alpha\text{-cont.} & & \text{supra } b\text{-cont.} & \to & \text{supra } \beta\text{-cont.} \\ \uparrow & & & \searrow & \uparrow & & \downarrow \\ \text{cont.} & & & & \text{supra semi-cont.} & & \text{supra } e^*\text{-cont.} \end{array}$$

Theorem 4.2. *Let $\Delta : \Psi \to \Phi$ be a function and μ be an associated supra topology with $\top$. Then the following properties are equivalent:*

(ι_1) *Δ is s.e^*.c.;*

(ι_2) *If for each closed set F of Φ is $\Delta^{-1}[F]$ supra e^*-closed in Ψ;*

(ι_3) $cl^{\mu}_{e^*}\left(\Delta^{-1}[L]\right) \subseteq \Delta^{-1}[cl(L)]$ *for each* $L \subseteq \Phi$*;*

(ι_4) $\Delta\left[cl^{\mu}_{e^*}(E)\right] \subseteq cl(\Delta[E])$ *for each* $E \subseteq \Psi$*;*

(ι_5) $\Delta^{-1}[int(L)] \subseteq int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right)$ *for each* $L \subseteq \Phi$.

Proof. $(\iota_1) \Rightarrow (\iota_2)$: Let F be a closed set in Φ.

$$\left.\begin{array}{r} F \in C(\Phi) \Rightarrow \Phi \setminus F \in O(\Phi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}[\Phi \setminus F] = \Psi \setminus \Delta^{-1}[F] \in e^*\mu(\Psi) \Rightarrow \Delta^{-1}[F] \in e^*\mu^c(\Psi).$$

$(\iota_2) \Rightarrow (\iota_3)$: Let L be any subset of Φ.

$$\left.\begin{array}{r} L \subseteq \Phi \Rightarrow cl(L) \in C(\Phi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}[cl(L)] \in e^*\mu^c(\Psi) \Rightarrow cl^{\mu}_{e^*}\left(\Delta^{-1}[L]\right) \subseteq cl^{\mu}_{e^*}\left(\Delta^{-1}[cl(L)]\right) =$$

$\Delta^{-1}[cl(L)]$.

$(\iota_3) \Rightarrow (\iota_4)$: Let E be any subset of Ψ.

$$\left.\begin{array}{r} E \subseteq \Psi \Rightarrow \Delta[E] \subseteq \Phi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow cl^{\mu}_{e^*}(E) \subseteq cl^{\mu}_{e^*}\left(\Delta^{-1}[\Delta[E]]\right) \subseteq \Delta^{-1}[cl(\Delta[E])]$$

$$\Rightarrow \Delta\left[cl^{\mu}_{e^*}(E)\right] \subseteq \Delta[\Delta^{-1}[cl(\Delta[E])]] \subseteq cl(\Delta[E]).$$

$(\iota_4) \Rightarrow (\iota_5)$: Let L be any subset of Φ.

$$\left.\begin{array}{r} L \subseteq \Phi \Rightarrow \Psi \setminus \Delta^{-1}[L] \subseteq \Psi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta\left[cl^{\mu}_{e^*}\left(\Psi \setminus \Delta^{-1}[L]\right)\right] \subseteq cl\left(\Delta\left[\Psi \setminus \Delta^{-1}[L]\right]\right)$$

$$\Rightarrow \Delta\left[\Psi \setminus int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right)\right] \subseteq cl(\Phi \setminus L) = \Phi \setminus int(L)$$

$$\Rightarrow \Psi \setminus int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right) \subseteq \Delta^{-1}[\Phi \setminus int(L)]$$

$$\Rightarrow \Delta^{-1}[int(L)] \subseteq int^{\mu}_{e^*}\left(\Delta^{-1}[L]\right).$$

$(\iota_5) \Rightarrow (\iota_1)$: Let O be an open set in Φ.

$$\left.\begin{array}{r} O \in O(\Phi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}[O] \subseteq \Delta^{-1}[int(O)] \subseteq int^{\mu}_{e^*}(\Delta^{-1}[O])$$

$$\Rightarrow \Delta^{-1}[O] \in e^*\mu(\Psi)$$

Thus, Δ is supra e^*-continuous. □

Theorem 4.3. *If $\Delta : \Psi \to \Phi$ is supra e^*-continuous and $\Gamma : \Phi \to \zeta$ is continuous, then the composition $\Gamma \circ \Delta : \Psi \to \zeta$ is supra e^*-continuous.*

Proof. Let $V \in O(\zeta)$.

$$\left.\begin{array}{r} \left.\begin{array}{l} V \in O(\zeta) \\ \Gamma \text{ is cont.} \end{array}\right\} \Rightarrow \quad \Gamma^{-1}[V] \in O(\Phi) \\ \Delta \text{ is supra } e^*\text{-cont.} \end{array}\right\} \Rightarrow \Delta^{-1}[\Gamma^{-1}[V]] = (\Gamma \circ \Delta)^{-1}[V] \in e^*\mu(\Psi)$$

Thus, $\Gamma \circ \Delta$ is supra e^*-continuous. □

Theorem 4.4. *Let $\Delta : \Psi \to \Phi$ be a function and μ and η be the associated supra topologies with $\top$ and $\perp$, subsequently. Afterwards Δ is supra e^*-continuous if one of the following holds:*

(ι_1) $\Delta^{-1}\left[int^{\mu}_{e^*}(L)\right] \subseteq int\left(\Delta^{-1}[L]\right)$ for each $L \subseteq \Phi$.

(ι_2) $cl\left(\Delta^{-1}[L]\right) \subseteq \Delta^{-1}\left[cl^{\mu}_{e^*}(L)\right]$ for each $L \subseteq \Phi$.

(ι_3) $\Delta[cl(E)] \subseteq cl^{\mu}_{e^*}(\Delta[E])$ for each $E \subseteq \Psi$.

Proof. (ι_1) : Let $L \in O(\Phi)$.

$$\left.\begin{array}{r} L \in O(\Phi) \Rightarrow L \subseteq \Phi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta^{-1}\left[int^{\mu}_{e^*}(L)\right] \subseteq int\left(\Delta^{-1}[L]\right) \Rightarrow \Delta^{-1}[L] \subseteq int\left(\Delta^{-1}[L]\right)$$

$$\left.\begin{array}{r} \Rightarrow \Delta^{-1}[L] \in O(\Psi) \\ O(\Psi) \subseteq e^*\mu(\Psi) \end{array}\right\} \Rightarrow \Delta^{-1}[L] \in e^*\mu(\Psi)$$

Thus, Δ is supra e^*-continuous.

(ι_2) : Let $L \in O(\Phi)$.

$$\left.\begin{array}{r} L \in O(\Phi) \Rightarrow \Phi \setminus L \subseteq \Phi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow cl\left(\Delta^{-1}[\Phi \setminus L]\right) \subseteq \Delta^{-1}\left[cl^{\mu}_{e^*}(\Phi \setminus L)\right]$$

$$\Rightarrow \Psi \setminus int\left(\Delta^{-1}[L]\right) \subseteq \Psi \setminus \Delta^{-1}\left[int^{\mu}_{e^*}(L)\right]$$

$$\Rightarrow \Delta^{-1}\left[int^{\mu}_{e^*}(L)\right] \subseteq int\left(\Delta^{-1}[L]\right)$$

This condition is the same as (ι_1). Thus, Δ is supra e^*-continuous.

(ι_3) : Let $E \in O(\Phi)$.

$$\left.\begin{array}{r} E \in O(\Phi) \Rightarrow \Delta^{-1}[E] \subseteq \Psi \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta[cl(\Delta^{-1}[E])] \subseteq cl^{\mu}_{e^*}(\Delta[\Delta^{-1}[E]])$$

$$\Rightarrow \Delta[cl(\Delta^{-1}[E])] \subseteq cl^{\mu}_{e^*}(E)$$

$$\Rightarrow \Delta^{-1}[\Delta[cl(\Delta^{-1}[E])]] \subseteq \Delta^{-1}[cl^{\mu}_{e^*}(E)]$$

$$\Rightarrow cl(\Delta^{-1}[E]) \subseteq \Delta^{-1}[cl^{\mu}_{e^*}(E)]$$

This condition is the same as (ι_2). Thus, Δ is supra e^*-continuous. □

5. Supra e^*-open Functions and Supra e^*-closed Functions

Definition 5.1. *A function $\Delta : \Psi \to \Phi$ is called supra e^*-open (resp. supra e^*-closed) if for each open (resp. closed) set F of Ψ, $\Delta[F]$ is s.e*.o. (resp. s.e*.c.) in Φ.*

Theorem 5.1. *A function $\Delta : \Psi \to \Phi$ is supra e^*-open iff $\Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A])$ for each set A in Ψ.*

Proof. Necessity. Let $A \subseteq \Psi$ and suppose that Δ is supra e^*-open.

$$\left.\begin{array}{r} A \subseteq \Psi \Rightarrow A \supseteq int(A) \in O(\Psi) \\ \Delta \text{ is supra } e^*\text{-open} \end{array}\right\} \Rightarrow \left.\begin{array}{r} \Delta[A] \supseteq \Delta[int(A)] \in e^*\mu(\Phi) \\ int^{\mu}_{e^*}(\Delta[A]) = \cup\{B|(B \subseteq \Delta[A])(B \in e^*\mu(\Psi))\} \end{array}\right\} \Rightarrow$$

$\Rightarrow \Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A])$.

Sufficiency. Suppose that $\Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A])$ for each set $A \in \Psi$.

$$\left.\begin{array}{r} A \in O(\Psi) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow \Delta[A] \subseteq \Delta[int(A)] \subseteq int^{\mu}_{e^*}(\Delta[A]) \Rightarrow \Delta[A] \in e^*\mu(\Phi)$$

Hence, Δ is supra e^*-open. □

Theorem 5.2. *A function $\Delta : \Psi \to \Phi$ is supra e^*-closed iff $cl^{\mu}_{e^*}(\Delta[A]) \subseteq \Delta[cl(A)]$ for all set A in Ψ.*

Proof. It is obvious from the Theorem 5.1. □

Theorem 5.3. *Let $\Delta : \Psi \to \Phi$ and $\Gamma : \Phi \to \zeta$ be two functions. Then the following properties hold:*

(ι_1) *Whenever $\Gamma \circ \Delta$ is supra e^*-open and Δ is continuous surjective, afterwards Γ is supra e^*-open.*

(ι_2) *Whenever $\Gamma \circ \Delta$ is open and Γ is e^*-continuous injective, afterwards Δ is supra e^*-open.*

Proof. (ι_1) : Let $U \in O(\Phi)$.

$$\left.\begin{array}{r} U \in O(\Phi) \\ \Delta \text{ is continuous} \end{array}\right\} \Rightarrow \left.\begin{array}{r} \Delta^{-1}[U] \in O(\Psi) \\ \Gamma \circ \Delta \text{ is supra } e^*\text{-open} \end{array}\right\} \Rightarrow$$

$\Rightarrow (\Gamma \circ \Delta)[\Delta^{-1}[V]] = \Gamma[\Delta[\Delta^{-1}[V]]] \overset{\Delta \text{ is surj.}}{=} \Gamma[V] \in e^*\mu(\zeta)$

Hence, Γ is supra e^*-open.

(ι_2) : Let $U \in O(\Psi)$.

$$\left.\begin{array}{r} U \in O(\Psi) \\ \Gamma \circ \Delta \text{ is open} \end{array}\right\} \Rightarrow \left.\begin{array}{r} (\Gamma \circ \Delta)[U] \in O(\zeta) \\ \Gamma \text{ is } e^*\text{-continuous} \end{array}\right\} \Rightarrow$$

$\Rightarrow \Gamma^{-1}[(\Gamma \circ \Delta)[V]] = \Gamma^{-1}[\Gamma[\Delta[V]]] \overset{\Gamma \text{ is inj.}}{=} \Delta[V] \in e^*\mu(\Phi)$

Hence, Δ is supra e^*-open. □

Theorem 5.4. *Let $\Delta : \Psi \to \Phi$ be a bijection. Then the following functions are equivalent:*

(ι_1) *Δ is a s.e*.o.;*

(ι_2) *Δ is a s.e*.c.;*

(ι_3) *Δ^{-1} is a s.e*.c.*

Proof. $(\iota_1) \Rightarrow (\iota_2)$: Obvious.

$(\iota_2) \Rightarrow (\iota_3)$: Let $F \in C(\Psi)$.

$$\left.\begin{array}{r} F \in C(\Psi) \\ \Delta \text{ is supra } e^*\text{-closed} \end{array}\right\} \Rightarrow \Delta[F] \overset{\Delta \text{ is bij.}}{=} (\Delta^{-1})^{-1}[F] \in e^*\mu^c(\Phi)$$

By Theorem 4.2(ι_2) Δ^{-1} is supra e^*-continuous.

$(\iota_3) \Rightarrow (\iota_1)$: Let $F \in O(\Psi)$.

$$\left.\begin{array}{r} F \in O(\Psi) \\ \Delta^{-1} \text{ is supra } e^*\text{-continuous} \end{array}\right\} \Rightarrow (\Delta^{-1})^{-1}[F] \overset{\Delta \text{ is bij.}}{=} \Delta[F] \in e^*\mu(\Phi)$$

Hence, Δ is supra e^*-open. □

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References

[1] Abd El-Monsef M. E., El-Deeb S. N., & Mahmoud, R. A. (1983). β-open sets and β-continuous mappings. Bull. Fac. Sci. Assiut Univ., 12, 77-90.

[2] Al-Shami, T. M. (2017). On supra semi-open sets and some applications on topological spaces. J. Adv. Stud. Topol., 8(2), 144-153.

[3] Al-Shami, T. M. (2017). Somewhere dense sets and ST_1-spaces. Punjab Univ. J. Math., 49(2), 101-111.

[4] Al-Shami, T. M., & Noiri, T. (2019). More notions and mappings via somewhere dense sets. Afr. Mat., 30, 1011-1024.

[5] Al-Shami, T. M., & Alshammari, I. (2023). Rough sets models inspired by supra-topology structures. Artif. Intell. Rev., 56, 6855–6883.

[6] Al-Shami, T. M., Asaad, B. A., & El-Bably, M. K. (2020). Weak types of limit points and separation axioms on supra topological spaces. Adv. Math. Sci. Journal, 9(10), 8017–8036.

[7] Al-Shami, T. M., Abo-Tabl, E. A., & Asaad, B. A. (2020). Investigation of limit points and separation axioms using supra β-open sets. Missouri J. Math. Sci., 32(2), 171-187.

[8] Al-Shami, T. M., Asaad, B. A., & El-Gayar, M. A. (2020). Various types of supra pre-compact and supra pre-lindelöf spaces. Missouri J. Math. Sci., 32(1), 1-20.

[9] Andrijević, D. (1996). On b-open sets. Mat. Vesnik, 48, 59-64.

[10] Asaad, B. A., Al-Shami, T. M., & Abo-Tabl, E. A. (2020). Applications of some operators on supra topological spaces. Demonstratio Math., 53, 292–308.

[11] Devi, R., Sampathkumar S., & Caldas, M. (2008). On supra α-open sets and s_α-continuous maps. Gen. Math., 16(2), 77-84.

[12] Ekici, E. (2008). On e-open sets, $\mathcal{DP}^*$-sets and $\mathcal{DPE}^*$-sets and decompositions of continuity. Arabian J. Sci. Eng., 33(2A), 269-282.

[13] Ekici, E. (2009). On e-open sets, On e^*-open sets and $(\mathcal{D},\mathcal{S})^*$-sets. Math. Morav., 13(1), 29-36.

[14] El-Shafei, M. E., Abo-Elhamayel, M., & Al-Shami, T. M. (2017). Strong separation axioms in supra topological ordered spaces. Math. Sci. Lett., 6(3), 271-277.

[15] Jafari, S., & Tahiliani, S. (2013). Supra β-open sets and supra β-continuity on topological spaces. Annales Univ. Sci. Budapest., 56, 1-9.

[16] Levine, N. (1963). Semi-open sets and semi-continuity in topological spaces. Amer. Math. Monthly, 70, 36-41.

[17] Mashhour, A. S., Abd El-Monsef, M. E., & El-Deeb, S. N. (1982). On precontinuous and weak precontinuous mappings. Proc. Math. Phys. Soc. Egypt, 53, 47-53.

[18] Mashhour, A. S., Allam, A. A., Mahmoud F. S., & Khedr, F. H. (1983). Semi-open sets and semi-continuity in topological spaces. Indian J. Pure and Appl. Math., 14(4), 502-510.

[19] Sayed, O. R. (2010). Supra pre-open sets and supra pre-continuous on topological spaces. Ser. Math. Inform., 20, 79–88.

[20] Sayed, O. R., & Noiri, T. (2010). On supra b-open sets and supra b-continuity on topological spaces. Eur. J. Pure and Appl. Math., 3, 295-302.

[21] Velicko, N. V. (1968). H-closed topological spaces. Amer. Math. Soc. Transl., 78, 103-118.

[22] Vidyarani, L., & Vigneshwaran, M. (2013). N-Homeomorphism and N^*-Homeomorphism in supra topological spaces. Int. J. Math. Stat. Invent., 2(1), 79-83.

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ON ROUGH $\mathcal{I}$-STATISTICAL CONVERGENCE OF COMPLEX UNCERTAIN SEQUENCES

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ABSTRACT. In this paper, we introduce the notion of rough $\mathcal{I}$-statistical convergence of complex uncertain sequences in four aspects of uncertainty, viz., almost surely, measure, mean, distribution as an extension of rough convergence, rough statistical convergence, and rough $\mathcal{I}$-convergence of complex uncertain sequences. Also, we explore the concept of rough $\mathcal{I}$-statistical convergence in p-distance, and rough $\mathcal{I}$-statistical convergence in metric of complex uncertain sequences. Overall, this study mainly presents a diagrammatic scenario of interrelationships among all rough $\mathcal{I}$-statistical convergence concepts of complex uncertain sequences and include some observations about the above convergence concepts.

Keywords: Uncertainty space, Complex uncertain variable, Complex uncertain sequence, Almost surely, Rough $\mathcal{I}$-statistical convergence.

2020 Mathematics Subject Classification: 60B10, 40A35, 40G15.

1. INTRODUCTION

The idea of the convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [12] and Steinhaus [30]. Later, it was studied by Fridy [13] and many other researchers. A sequence (x_m) is said to be statistically convergent to ℓ provided that for each $\varepsilon > 0$ such that

$$\lim_{m\to\infty} \frac{1}{m} |\{k \leq m : |x_k - \ell| \geq \varepsilon\}| = 0, \ m \in \mathbb{N}.$$

The concept of $\mathcal{I}$-convergence was introduced by Kostyrko et al. [20] as a generalization

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of statistical convergence. The idea of $\mathcal{I}$-convergence was further extended to $\mathcal{I}$-statistical convergence by Savas and Das [27]. Later on, more investigation in this direction can be found in the works of [11, 15, 28].

The idea of rough convergence was first introduced by Phu [23] in finite-dimensional normed spaces. A sequence (x_m) is said to be rough convergent to ℓ provided that for each $\varepsilon > 0$ $\exists$ $m_\varepsilon \in \mathbb{N}$ such that

$$|x_m - \ell| < r + \varepsilon \text{ for all } m \geq m_\varepsilon,$$

where r is a non-negative real number and called roughness degree. After that, Dündar and Çakan [10] introduced the notion of rough $\mathcal{I}$-convergence of sequence. The concept of rough $\mathcal{I}$-statistical convergence of sequences was introduced by Savaş et al. [29] in the year 2018.

On the other hand, in 2007, Liu [21] introduced a theory named uncertainty theory, including different types of convergence of uncertain sequences and identifying the relationships among various forms of convergence, such as convergence in measure, distribution, mean, and convergence a.s. Then the concept has been extended to the c.u.v.s by Peng [22]. After that, Chen et al. [2] subsequently studied the idea of convergence of c.u.s.s using c.u.v.s. In 2017, Tripathy and Nath [31] proposed the idea of statistical convergence of c.u.s.s in the context of uncertainty theory. After that, Debnath and Das [6,7] introduced the notion of rough convergence and rough statistical convergence of c.u.s.s, and this field has also seen a lot of exciting changes; for details, see [1, 3–5, 9, 14, 16, 17, 19, 24–26]. The concept of rough $\mathcal{I}$-convergence of complex uncertain sequences was recently introduced by Debnath and Halder [8].

Inspired by the above works, in this paper we introduce the notion of rough $\mathcal{I}$-statistical convergence of c.u.s.s in four aspects of uncertainty, viz., a.s., measure, mean, and distribution. We also explore the concepts of rough $\mathcal{I}$-statistical convergence in p-distance, and rough $\mathcal{I}$-statistical convergence in metric of c.u.s.s. Finally, we try to establish the relationship among all rough $\mathcal{I}$-statistical convergence concepts of c.u.s.s with an attached diagramatic section.

2. Definitions and Preliminaries

In this section, we provide some basic ideas and results on generalized convergence concepts and the theory of uncertainty that will be used throughout the article.

Definition 2.1. *[20] Consider a non-empty set S. An ideal on S is defined as a family of subsets $\mathcal{I}$ that satisfies the following conditions:*

(i) The empty set, ϕ, belongs to $\mathcal{I}$.

(ii) For any $U, V \in \mathcal{I}$, the union of U and V, denoted as $U \cup V$, is also in $\mathcal{I}$.

(iii) For any $U \in \mathcal{I}$ and any subset $V \subset U$, V is a member of $\mathcal{I}$.

An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \{\Phi\}$ and $S \notin \mathcal{I}$.

A non-trivial ideal $\mathcal{I}$ is called an admissible ideal in S if and only if $\{\{s\} : s \in S\} \subset \mathcal{I}$.

Example 2.1. (*i*) $\mathcal{I}_f$:= *The set of all finite subsets of $\mathbb{N}$ forms a non-trivial admissible ideal.*

(*ii*) $\mathcal{I}_d$:= *The set of all subsets of $\mathbb{N}$ whose natural density is zero forms a non-trivial admissible ideal.*

Definition 2.2. *[20] A sequence (x_m) is said to be $\mathcal{I}$-convergent to ℓ, if for every $\varepsilon > 0$, the set $\{m \in \mathbb{N} : |x_m - \ell| \geq \varepsilon\} \in \mathcal{I}$.*

The usual convergence of sequences is a special case of $\mathcal{I}$-convergence ($\mathcal{I}$=$\mathcal{I}_f$-the ideal of all finite subsets of $\mathbb{N}$). The statistical convergence of sequences is also a special case of $\mathcal{I}$-convergence. In this case, $\mathcal{I}$=$\mathcal{I}_d$ $= \left\{A \subseteq \mathbb{N} : \lim\limits_{m\to\infty} \frac{|A\cap\{1,2,\cdots,m\}|}{m} = 0\right\}$, where $|A|$ is the cardinality of the set A.

Definition 2.3. *[29] A sequence (x_m) is said to be rough $\mathcal{I}$-statistically convergent to $\ell \in \mathbb{R}$, if for every $\delta, \upsilon > 0$,*

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \leq m : |x_k - \ell| \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree. For $r = 0$, rough $\mathcal{I}$-statistical convergence coincides with $\mathcal{I}$-statistical convergence.

Definition 2.4. *[21] Let $\mathscr{P}$ be a σ-algebra on a non-empty set Υ. If the set function $\mathcal{X}$ on Υ satisfies the following axioms, it is referred to be an uncertain measure:*

- *The first axiom, which deals with normality, is $\mathcal{X}\{\Upsilon\} = 1$;*
- *The second, which deals with duality, is $\mathcal{X}\{\Xi\} + \mathcal{X}\{\Xi^c\} = 1$ for any $\Xi \in \mathscr{P}$;*
- *The third, which deals with subadditivity is for every countable sequence of $\{\Xi_m\} \in \mathscr{P}$,*

$$\mathcal{X}\{\bigcup_{m=1}^{\infty} \Xi_m\} \leq \sum_{m=1}^{\infty} \mathcal{X}\{\Xi_m\}.$$

An u.s. is denoted by the triplet $(\Upsilon, \mathscr{P}, \mathcal{X})$, and an event is denoted by each member Ξ in $\mathscr{P}$.

Definition 2.5. *[21] A c.u.v. is represented by a variable ζ in the uncertainty space $(\Upsilon, \mathscr{P}, \mathscr{X})$ if and only if both its real part ξ and imaginary part η are uncertain variables.*

Here, ξ and η correspond to the real and imaginary components of the complex variable $\zeta = \xi + i\eta$, respectively.

Definition 2.6. *[22] Let $\zeta = \xi + i\eta$ be a c.u.v., where ξ is the real part and η is the imaginary part of ζ. Then the complex uncertainty distribution of ζ is denoted by $\Psi : \mathbb{C} \to [0, 1]$ and is defined by $\Psi(z) = \mathcal{X}\{\xi \leq s, \eta \leq t\}$ for any complex number $z = s + it$.*

Definition 2.7. *[22] Let $\zeta = \xi + i\eta$ be a c.u.v. If the expected value of ξ and η i.e., $E[\xi]$ and $E[\eta]$ exists, then the expected value of ζ is defined by*

$$E[\zeta] = E[\xi] + iE[\eta].$$

Definition 2.8. *[25] Let ζ and ζ^* be two c.u.v.s. Then the p-distance between them is defined as*

$$d_p(\zeta, \zeta^*) = \left(E\left[\|\zeta - \zeta^*\|^p\right]\right)^{\frac{1}{p+1}}, p > 0.$$

Definition 2.9. *[26] A c.u.s. sequence (ζ_m) is considered statistically convergent in p-distance to ζ if*

$$\lim_{m\to\infty} \frac{1}{m} \left|\left\{k \leq m : \left(E\left[\|\zeta_k - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \geq \varepsilon\right\}\right| = 0 \text{ for every } \varepsilon > 0.$$

Definition 2.10. *[4] Let ζ and ζ^* be two c.u.v.s, then the metric between them is defined as follows*

$$D(\zeta, \zeta^*) = \inf\{t : \mathcal{X}\{\|\zeta - \zeta^*\| \leq t\} = 1\}.$$

Definition 2.11. *[4] If the condition $\lim_{m\to\infty} D(\zeta_m, \zeta) = 0$ is hold for a c.u.s. (ζ_m), then (ζ_m) is called convergent in metric to ζ.*

Definition 2.12. *[8] A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-convergent a.s. to ζ if for every small positive value δ, and for any event Ξ where $\mathcal{X}\{\Xi\} = 1$ we have the following condition satisfied for every element $\varrho \in \Xi$:*

$$\{m \in \mathbb{N} : \|\zeta_m(\varrho) - \zeta(\varrho)\| \geq r + \delta\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 2.13. *[8] A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-convergent in measure to ζ if, for every given small positive values ε and δ, there exists a set satisfying the condition*

$$\{m \in \mathbb{N} : \mathcal{X}(\|\zeta_m - \zeta\| \geq \varepsilon) \geq r + \delta\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 2.14. *[8] Let $\Psi, \Psi_1, \Psi_2, \cdots$ denote the complex uncertainty distributions of c.u.v.s $\zeta, \zeta_1, \zeta_2, \cdots$, respectively. The c.u.s. (ζ_m) is called rough $\mathcal{I}$-convergent in distribution to ζ if, for every small positive values δ, there exists a set satisfying the condition:*

$$\{m \in \mathbb{N} : \|\Psi_m(z) - \Psi(z)\| \geq r + \delta\} \in \mathcal{I},$$

where r is called roughness degree and for all z at which $\Psi(z)$ is continuous.

Definition 2.15. *[8] A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-convergent in mean to ζ if, for every given small positive values δ, there exists a set satisfying the condition*

$$\{m \in \mathbb{N} : E\left[\|\zeta_m - \zeta\|\right] \geq r + \delta\} \in \mathcal{I},$$

where r is called roughness degree.

In this article, we assume that $\mathcal{I}$ to be a non-trivial admissible ideal of $\mathbb{N}$ and r as a non-negative real number .

3. Main Results

Definition 3.1. *A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent a.s. to ζ if, for every small positive value δ and υ, and for any event Ξ where $\mathcal{X}\{\Xi\} = 1$ we have the following condition satisfied for every element $\varrho \in \Xi$:*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : \|\zeta_k(\varrho) - \zeta(\varrho)\| \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree. If we take $r = 0$ we obtain the notion of $\mathcal{I}$-statistical convergence a.s. of c.u.s. which was introduced by Halder and Debnath [14].

Definition 3.2. *A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in measure to ζ if, for every given small positive values ε, δ and υ, there exists a set satisfying the condition*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 3.3. *A c.u.s. (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in mean to ζ if, for every given small positive values δ, and υ, there exists a set satisfying the condition*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : E\left[\|\zeta_k - \zeta\|\right] \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree.

Definition 3.4. *Let $\Psi, \Psi_1, \Psi_2, \cdots$ denote the complex uncertainty distributions of c.u.v.s $\zeta, \zeta_1, \zeta_2, \cdots$, respectively. The c.u.s. (ζ_m) is called rough $\mathcal{I}$-statistically convergent in distribution to ζ if, for every small positive values δ and υ, there exists a set satisfying the condition:*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : \|\Psi_k(z) - \Psi(z)\| \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree and for all z at which $\Psi(z)$ is continuous.

Theorem 3.1. *The c.u.s. (ζ_m) where $\zeta_m = \xi_m + i\eta_m$ is rough $\mathcal{I}$-statistically convergent in measure to $\zeta = \xi + i\eta$ if and only if the uncertain sequence (ξ_m) and (η_m) are rough $\mathcal{I}$-statistically convergent in measure to ξ and η, respectively.*

Proof. Omitted, since it can be established using standard technique. □

Theorem 3.2. *If a c.u.s. (ζ_m) is rough $\mathcal{I}$-statistically convergent in mean to ζ, then it is rough $\mathcal{I}$-statistically convergent in measure to ζ.*

Proof. The proof follows from the following Markov inequality. □

Remark 3.1. *However, the reverse of the above theorem does not hold in general.*

Example 3.1. *Consider the u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$ to be $\{\varrho_1, \varrho_2, \cdots\}$ with power set and $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\Phi\} = 0$ and*

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \frac{m}{(2m+1)}, & if \ \sup\limits_{\varrho_m \in \Xi} \frac{m}{(2m+1)} < \frac{1}{2} \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \frac{m}{(2m+1)}, & if \ \sup\limits_{\varrho_m \in \Xi^c} \frac{m}{(2m+1)} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots$$

Also, $\zeta_m(\varrho)$ (the c.u.v.s) are defined by

$$\zeta_m(\varrho) = \begin{cases} im, \ if \ \varrho = \varrho_m \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. Take $\mathcal{I}$=$\mathcal{I}_d$.

For every $\varepsilon, \delta, \upsilon > 0$ and $r \geq 0$ we have,

$$\begin{aligned} &\left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \\ &\quad = \left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{k \leq m : \mathcal{X}\left(\varrho : \|\zeta_k(\varrho) - \zeta(\varrho)\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \\ &\quad = \left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\{k \leq m : \mathcal{X}\{\varrho_k\} \geq r + \delta\}\right| \geq \upsilon\right\} \\ &\quad = \left\{m \in \mathbb{N} : \tfrac{1}{m} \left|\left\{k \leq m : \tfrac{k}{2k+1} \geq r + \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}. \end{aligned}$$

Thus the sequence (ζ_m) is rough $\mathcal{I}$-statistically convergent in measure to ζ for $r = \frac{1}{2}$.

However, for each m, we have the complex uncertainty distributions of uncertain variable

$\|\zeta_m - \zeta\|$ *is*

$$\Psi_m(t) = \begin{cases} 0, & \text{if } t < 0 \\ 1 - \frac{m}{2m+1}, & \text{if } 0 \le t < m \\ 1, & \text{if } t \ge m \end{cases} \quad \text{for } m = 1, 2, 3, \cdots.$$

Now $E\left[\|\zeta_m - \zeta\|\right] = \int_0^{+\infty} (1 - \Psi_m(t))\, dt = \int_0^m \frac{m}{2m+1} dt = \frac{m^2}{2m+1}$.

Consequently, for any given δ *and* υ *both greater than zero, and* $r = \frac{1}{2}$,

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \le m : E\left[\|\zeta_k - \zeta\|\right] \ge r + \delta\right\}\right| \ge \upsilon\right\}$$
$$= \left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \le m : \frac{k^2}{2k+1} \ge r + \delta\right\}\right| \ge \upsilon\right\} \notin \mathcal{I}.$$

Hence the sequence (ζ_m) *is not rough* $\mathcal{I}$*-statistically convergent in mean to* ζ *for* $r = \frac{1}{2}$.

Theorem 3.3. *Let* (ξ_m) *and* (η_m) *be the real and imaginary part of a c.u.s.* (ζ_m) *are considered to be rough* $\mathcal{I}$*-statistical convergence in measure to* ξ *and* η *respectively. then* (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in distribution to* $\zeta = \xi + i\eta$.

Proof. Let $z = s + it$ be a continuous point of the complex uncertainty distribution Ψ. For any $\alpha > s$ and $\beta > t$, we can express

$$\{\xi_m \le s, \eta_m \le t\} = \{\xi_m \le s, \eta_m \le t, \xi \le \alpha, \eta \le \beta\} \cup \{\xi_m \le s, \eta_m \le t, \xi > \alpha, \eta > \beta\}$$
$$\cup \{\xi_m \le s, \eta_m \le t, \xi \le \alpha, \eta > \beta\} \cup \{\xi_m \le s, \eta_m \le t, \xi > \alpha, \eta \le \beta\}$$
$$\subset \{\xi \le \alpha, \eta \le \beta\} \cup \{|\xi_m - \xi| \ge \alpha - s\} \cup \{|\eta_m - \eta| \ge \beta - t\}.$$

By the subadditivity axiom, we can conclude that:

$$\Psi_m(z) = \Psi_m(s + it) \le \Psi(\alpha + i\beta) + \mathcal{X}\{|\xi_m - \xi| \ge \alpha - s\} + \mathcal{X}\{|\eta_m - \eta| \ge \beta - t\}.$$

Since (ξ_m) and (η_m) are rough $\mathcal{I}$-statistically convergent in measure to ξ and η respectively, then it follows that for any given δ, υ and $r \ge 0$, we can conclude that:

$$\Psi_m(z) = \Psi_m(s + it) \le \Psi(\alpha + i\beta) + \mathcal{X}\{|\xi_m - \xi| \ge \alpha - s\} + \mathcal{X}\{|\eta_m - \eta| \ge \beta - t\}.$$

Since (ξ_m) and (η_m) are rough $\mathcal{I}$-statistically convergent in measure to ξ and η respectively, then it follows that for any given δ, υ and $r \ge 0$, we can conclude that:

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \le m : \mathcal{X}\left(|\xi_k - \xi| \ge \alpha - s\right) \ge r + \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}$$
$$\text{and } \left\{m \in \mathbb{N} : \frac{1}{m} \left|\left\{k \le m : \mathcal{X}\left(|\eta_k - \eta| \ge \beta - t\right) \ge r + \delta\right\}\right| \ge \upsilon\right\} \in \mathcal{I}.$$

Then for any $\alpha > s, \beta > t$ and letting $\alpha + i\beta \to s + it$, we have

$$\|\Psi_m(z) - \Psi(z)\| \le \mathcal{X}\left\{|\xi_m - \xi| \ge \alpha - s\right\} + \mathcal{X}\left\{|\eta_m - \eta| \ge \beta - t\right\}.$$

Then for every $\delta > 0$ and $r \ge 0$,

$$\{k \le m : \|\Psi_k(z) - \Psi(z)\| \ge r + \delta\}$$
$$\subseteq \{k \le m : \mathcal{X}\left\{|\xi_k - \xi| \ge \alpha - s\right\} \ge r + \delta\}$$
$$\cup \{k \le m : \mathcal{X}\left\{|\eta_k - \eta| \ge \beta - t\right\} \ge r + \delta\}.$$

$$\Rightarrow \frac{1}{m}\left|\{k \leq m : \|\Psi_k(z) - \Psi(z)\| \geq r + \delta\}\right|$$
$$\leq \frac{1}{m}\left|\{k \leq m : \mathcal{X}\{|\xi_k - \xi| \geq \alpha - s\} \geq r + \delta\}\right|$$
$$+ \{k \leq m : \mathcal{X}\{|\eta_k - \eta| \geq \beta - t\} \geq r + \delta\}.$$

For every $\upsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \leq m : \|\Psi_k(z) - \Psi(z)\| \geq r + \delta\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \leq m : \mathcal{X}\{|\xi_k - \xi| \geq \alpha - s\} \geq r + \delta\}\right| \geq \upsilon\right\}$$
$$\cup \{m \in \mathbb{N} : \{k \leq m : \mathcal{X}\{|\eta_k - \eta| \geq \beta - t\} \geq r + \delta\} \geq \upsilon\} \in \mathcal{I}.$$

Hence the c.u.s. (ζ_m) is rough $\mathcal{I}$-statistically convergent in distribution to ζ. □

Remark 3.2. *However, the reverse of the above theorem does not hold in general.*

Example 3.2. *Consider the u.s.* $(\Upsilon, \mathscr{P}, \mathcal{X})$ *to be* $\{\varrho_1, \varrho_2\}$ *with* $\mathcal{X}(\varrho_1) = \mathcal{X}(\varrho_2) = \frac{1}{2}$. *We define a c.u.v. as*

$$\zeta(\varrho) = \begin{cases} i, & if\ \varrho = \varrho_1, \\ -i, & if\ \varrho = \varrho_2. \end{cases}$$

We also define $\zeta_m = -\zeta$ *for* $m = 1, 2, \cdots$ *and take* $\mathcal{I}=\mathcal{I}_d$.

Then the sequence (ζ_m) *and* ζ *have the same distribution as:*

$$\Psi_m(z) = \Psi_m(s + it) = \begin{cases} 0, & if\ s < 0, -\infty < t < +\infty, \\ 0, & if\ s \geq 0, t < -1, \\ \frac{1}{2}, & if\ s \geq 0, -1 \leq t < 1, \\ 1, & if\ s \geq 0, t \geq 1. \end{cases}$$

So the sequence (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in distribution to* ζ.

However, for a given $\varepsilon, \delta, \upsilon > 0$ *and* $r \geq 0$, *we have*

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \leq m : \mathcal{X}\left(\|\zeta_k - \zeta\| \geq \varepsilon\right) \geq r + \delta\}\right| \geq \upsilon\right\} \notin \mathcal{I}.$$

Thus the sequence (ζ_m) *is not rough* $\mathcal{I}$*-statistically convergent in measure to* ζ *for* $r = 0.1$.

Definition 3.5. *A c.u.s.* (ζ_m) *is said to be rough* $\mathcal{I}$*-statistically convergent in p-distance to* ζ *if for every* $\delta, \upsilon > 0$ *such that*

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{k \leq m : \left(E\left[\|\zeta_k - \zeta\|^p\right]\right)^{\frac{1}{p+1}} \geq r + \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r *is called roughness degree.*

Theorem 3.4. *Let* $\zeta, \zeta_1, \zeta_2, \cdots$ *be c.u.v.s defined on u.s.* $(\Upsilon, \mathscr{P}, \mathcal{X})$. *Then* (ζ_m) *is considered to be rough* $\mathcal{I}$*-statistically convergent in measure to* ζ *if it is rough* $\mathcal{I}$*-statistically convergent in p-distance to* ζ.

Proof. Let the c.u.s. (ζ_m) be rough $\mathcal{I}$-statistically convergent in p-distance to ζ, then for every choice of δ and υ greater than zero, we obtain

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{k \leq m : (E[\|\zeta_k - \zeta\|^p])^{\frac{1}{p+1}} \geq r + \delta\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Then for any given $\varepsilon, p > 0$, we have

$\mathcal{X}(\|\zeta_m - \zeta\| \geq \varepsilon) \leq \dfrac{E[\|\zeta_m - \zeta\|^p]}{\varepsilon^p}$ (Using Markov Inequality).

So for every $\delta > 0$ and $r \geq 0$,

$$\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta\| \geq \varepsilon\} \geq r + \delta\}$$
$$\subseteq \left\{k \leq m : (E[\|\zeta_k - \zeta\|^p])^{\frac{1}{p+1}} \geq r' + \delta'\right\}, \text{ where } r' + \delta' = [(r+\delta)\cdot\varepsilon^p]^{\frac{1}{p+1}}.$$

For every $\upsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m}\left|\{k \leq m : \mathcal{X}(\|\zeta_k - \zeta\| \geq \varepsilon) \geq r + \delta\}\right| \geq \upsilon\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m}\left|\left\{k \leq m : (E[\|\zeta_k - \zeta\|^p])^{\frac{1}{p+1}} \geq r' + \delta'\right\}\right| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_m) is rough $\mathcal{I}$-statistically convergent in measure to ζ. □

Remark 3.3. *However, the reverse of the above theorem does not hold in general.*

Example 3.3. *Let* $\mathbb{N} = \bigcup_{j=1}^{\infty} D_j$, *where* $D_j = \{2^{j-1}j^* : 2$ *does not divide* $j^*, j^* \in \mathbb{N}\}$ *be the decomposition of* $\mathbb{N}$ *such that each* D_j *is infinite and* $D_j \cap D_{j^*} = \Phi$, *for* $j \neq j^*$. *Let* $\mathcal{I}$ *be the class of all subsets of* $\mathbb{N}$ *that can intersect only finite number of* $D_j{}'s$. *Then* $\mathcal{I}$ *is a non-trivial admissible ideal of* $\mathbb{N}$ *(see for details in [20]).*

Now we consider the u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$ *to be* $\{\varrho_1, \varrho_2, \cdots\}$ *with power set and* $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\Phi\} = 0$ *and*

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \beta_m, & if \ \sup\limits_{\varrho_m \in \Xi} \beta_m < \frac{1}{2} \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \beta_m, & if \ \sup\limits_{\varrho_m \in \Xi^c} \beta_m < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

where $\beta_m = \frac{1}{j+1}$, $if\ m \in D_j\ for\ m = 1, 2, 3, \cdots$.

Also, the c.u.v.s are defined by

$$\zeta_m(\varrho) = \begin{cases} i(m+1), & if\ \varrho = \varrho_m \\ 0, & otherwise \end{cases} \qquad for\ m = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$.

It can be shown that, the sequence (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in measure to* $\zeta \equiv 0$ *but it is not rough* $\mathcal{I}$*-statistically convergent in* p*-distance to* $\zeta \equiv 0$.

Theorem 3.5. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be c.u.v.s defined on u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$. Then (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in distribution to ζ if it is rough $\mathcal{I}$-statistically convergent in p-distance to ζ.*

Proof. If the sequence (ζ_k) exhibits rough $\mathcal{I}$-statistically convergent in p-distance to ζ, then, according to theorems 3.4 and 3.3, it also demonstrates rough $\mathcal{I}$-statistically convergent in distribution to the same limit ζ. □

Remark 3.4. *However, the reverse of the above theorem does not hold in general.*

Example 3.4. *In example 3.3, the complex uncertainty distributions of (ζ_m) are*

$$\Psi_m(z) = \Psi_m(s+it) = \begin{cases} 0, & \textit{if } s < 0, t < \infty \\ 0, & \textit{if } s \geq 0, t < 0 \\ 1 - \beta_m, & \textit{if } s \geq 0, 0 \leq t < (m+1) \\ 1, & \textit{if } s \geq 0, t \geq (m+1) \end{cases} \quad \textit{for } m = 1, 2, 3, \cdots$$

and the complex uncertainty distributions of ζ is

$$\Psi(z) = \Psi(s+it) = \begin{cases} 0, & \textit{if } s < 0, t < \infty \\ 0, & \textit{if } s \geq 0, t < 0 \\ 1, & \textit{if } s \geq 0, t \geq 0. \end{cases}$$

It can be shown that the c.u.s. (ζ_m) is rough $\mathcal{I}$-statistically convergent in distribution to $\zeta \equiv 0$ but it is not rough $\mathcal{I}$-statistically convergent in p-distance to $\zeta \equiv 0$.

Definition 3.6. *A c.u.s. (ζ_m) is said to be rough $\mathcal{I}$-statistically convergent in metric to ζ if for every $\delta, \upsilon > 0$ such that*

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : D\left(\zeta_k, \zeta\right) \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where r is called roughness degree.

Theorem 3.6. *Let $\zeta, \zeta_1, \zeta_2, \cdots$ be c.u.v.s defined on u.s. $(\Upsilon, \mathscr{P}, \mathcal{X})$. Then (ζ_m) is considered to be rough $\mathcal{I}$-statistically convergent in mean to ζ if it is rough $\mathcal{I}$-statistically convergent in metric to ζ.*

Proof. Let the c.u.s. (ζ_m) be rough $\mathcal{I}$-statistically convergent in metric to ζ, then for every $\delta, \upsilon > 0$ and $r \geq 0$ we have,

$$\left\{m \in \mathbb{N} : \frac{1}{m} \left|\{k \leq m : D\left(\zeta_k, \zeta\right) \geq r + \delta\}\right| \geq \upsilon\right\} \in \mathcal{I},$$

where $D(\zeta_m, \zeta) = \inf\{t : \mathcal{X}\{\|\zeta_m - \zeta\| \leq t\} = 1\}$.

Let $D(\zeta_m, \zeta) = q$ and $\Psi_m(t)$ represent the complex uncertainty distributions of the uncertain variable $\|\zeta_m - \zeta\|$. Then, we have $D(\zeta_m, \zeta) = \inf\{t : \Psi_m(t) = 1\}$.

Now for any positive number ℓ,

$$E[\|\zeta_m - \zeta\|] = \int_0^{+\infty} (1 - \Psi_m(t))\, dt = \int_0^{q+\ell} (1 - \Psi_m(t))\, dt + \int_{q+\ell}^{+\infty} (1 - \Psi_m(t))\, dt$$
$$= \int_0^{q+\ell} (1 - \Psi_m(t))\, dt < 1 \cdot (q + \ell) = q + \ell$$

$$\Rightarrow E[\|\zeta_m - \zeta\|] \leq q \Rightarrow E[\|\zeta_m - \zeta\|] \leq D(\zeta_m, \zeta).$$

So for every $\delta > 0$ and $r \geq 0$,

$$\{k \leq m : E[\|\zeta_k - \zeta\|] \geq r + \delta\} \subseteq \{k \leq m : D(\zeta_k, \zeta) \geq r + \delta\}$$
$$\Rightarrow \frac{1}{m} |\{k \leq m : E[\|\zeta_k - \zeta\|] \geq r + \delta\}| \leq \frac{1}{m} |\{k \leq m : D(\zeta_k, \zeta) \geq r + \delta\}|.$$

Then for every $\upsilon > 0$,

$$\left\{m \in \mathbb{N} : \frac{1}{m} |\{k \leq m : E[\|\zeta_k - \zeta\|] \geq r + \delta\}| \geq \upsilon\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m} |\{k \leq m : D(\zeta_k, \zeta) \geq r + \delta\}| \geq \upsilon\right\} \in \mathcal{I}.$$

Hence the sequence (ζ_m) is rough $\mathcal{I}$-statistically convergent in mean to ζ. □

Remark 3.5. *However, the reverse of the above theorem does not hold in general.*

Example 3.5. *Consider the u.s.* $(\Upsilon, \mathscr{P}, \mathcal{X})$ *to be* $\{\varrho_1, \varrho_2, \cdots\}$ *with power set and* $\mathcal{X}\{\Upsilon\} = 1$, $\mathcal{X}\{\Phi\} = 0$ *and*

$$\mathcal{X}\{\Xi\} = \begin{cases} \sup\limits_{\varrho_m \in \Xi} \frac{m\beta_m}{2m+1}, & if \ \sup\limits_{\varrho_m \in \Xi} \frac{m\beta_m}{2m+1} < \frac{1}{2} \\ 1 - \sup\limits_{\varrho_m \in \Xi^c} \frac{m\beta_m}{2m+1}, & if \ \sup\limits_{\varrho_m \in \Xi^c} \frac{m\beta_m}{2m+1} < \frac{1}{2} \\ \frac{1}{2}, & otherwise \end{cases}$$

$$where\ \beta_m = \begin{cases} 1, \ if \ m = k^2, k \in \mathbb{N} \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots.$$

Also, the c.u.v.s are defined by

$$\zeta_m(\varrho) = \begin{cases} i(m+1), \ if \ \varrho = \varrho_m \\ 0, \quad otherwise \end{cases} \quad for \ m = 1, 2, 3, \cdots$$

and $\zeta \equiv 0$. *Take* $\mathcal{I} = \mathcal{I}_d$.

The complex uncertainty distributions associated with the uncertain variable $\|\zeta_m - \zeta\|$ *is*

$$\Psi_m(t) = \begin{cases} 0, & if\, t < 0 \\ 1 - \frac{m\beta_m}{2m+1}, & if\, 0 \leq t < (m+1) \\ 1, & if\, t \geq (m+1) \end{cases} \quad for \ m = 1, 2, 3, \cdots.$$

Now $E\left[\|\zeta_m-\zeta\|\right]=\int_0^{+\infty}\left(1-\Psi_m(t)\right)dt=\int_0^{(m+1)}\frac{m\beta_m}{2m+1}dt=\frac{m(m+1)\beta_m}{2m+1}$.

Then for every $\delta,\upsilon>0$ *and* $r\geq 0$, *we have*

$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:E\left[\|\zeta_k-\zeta\|\right]\geq r+\delta\right\}\right|\geq\upsilon\right\}$$
$$=\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:\frac{k(k+1)\beta_k}{2k+1}\geq r+\delta\right\}\right|\geq\upsilon\right\}\in\mathcal{I}.$$

Again the metric between complex uncertain veriables ζ_m *and* ζ *is given by*

$$D\left(\zeta_m,\zeta\right)=\inf\left\{t:\mathcal{X}\left\{\|\zeta_m-\zeta\|\leq t\right\}=1\right\}=\inf\left\{t:\Psi_m(t)=1\right\}=m+1.$$

Thus for every $\delta,\upsilon>0$ *and* $r\geq 0$,

$$\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:D\left(\zeta_k,\zeta\right)\geq r+\delta\right\}\right|\geq\upsilon\right\}$$
$$=\left\{m\in\mathbb{N}:\frac{1}{m}\left|\left\{k\leq m:(k+1)\geq r+\delta\right\}\right|\geq\upsilon\right\}\notin\mathcal{I}.$$

Hence the c.u.s. (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in mean to* $\zeta\equiv 0$ *but it is not rough* $\mathcal{I}$*-statistically convergent in metric to* $\zeta\equiv 0$.

Theorem 3.7. *Let* $\zeta,\zeta_1,\zeta_2,\cdots$ *be c.u.v.s defined on u.s.* $(\Upsilon,\mathscr{P},\mathcal{X})$. *If* (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in metric to* ζ, *then it is rough* $\mathcal{I}$*-statistically convergent in measure to* ζ.

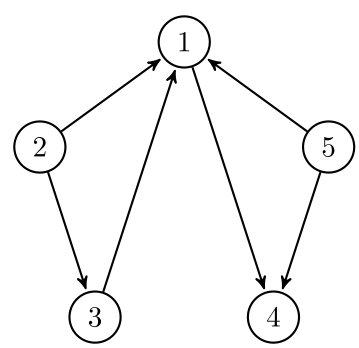
Proof. Let (ζ_m) be rough $\mathcal{I}$-statistically convergent in metric to ζ, then it is rough $\mathcal{I}$-statistically convergent in measure to ζ by theorem 3.6 and 3.2. □

Remark 3.6. *However, the reverse of the above theorem does not hold in general.*

Example 3.6. *From example 3.5, it can be shown that the c.u.s.* (ζ_m) *is rough* $\mathcal{I}$*-statistically convergent in measure to* $\zeta\equiv 0$ *but it is not rough* $\mathcal{I}$*-statistically convergent in metric to* $\zeta\equiv 0$.

4. Diagramatic representation among all convergence concepts

1. rough $\mathcal{I}$-statistically convergence in measure
2. rough $\mathcal{I}$-statistically convergence in metric
3. rough $\mathcal{I}$-statistically convergence in mean
4. rough $\mathcal{I}$-statistically convergence in distribution
5. rough $\mathcal{I}$-statistically convergence in p-distance



5. Conclusion

This paper has mainly discussed some rough $\mathcal{I}$-statistical convergence concepts of c.u.s.s, such as rough $\mathcal{I}$-statistical convergence in measure, mean, distribution, a.s., and established the relationships among them. Also, we initiate the notion of rough $\mathcal{I}$-statistical convergence in p-distance, and rough $\mathcal{I}$-statistical convergence in metric of c.u.s.s and include some interesting examples related to the notion. Furthermore, this paper is a more generalized form of rough $\mathcal{I}$-convergence of c.u.s.s, which was introduced by Debnath and Halder [8], which is a very recent and a new approach in complex uncertainty theory. In this paper, we try to establish relationships among all rough $\mathcal{I}$-statistical convergence concepts of c.u.s.s. However, we observe that certain concepts are unrelated to each other. It may attract future researchers in this direction.

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References

[1] Baliarsingh, P. (2021). On statistical deferred A-convergence of uncertain sequences. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 29(4), 499-515.

[2] Chen, X., Ning, Y., & Wang, X. (2016). Convergence of complex uncertain sequences. J. Intell. Fuzzy Syst., 30(6), 3357-3366.

[3] Choudhury, C., & Debnath, S. (2022). On rough $\mathcal{I}$-statistical convergence of sequences in gradual normed linear spaces. Mat. Vesnik, 74(3), 218-228.

[4] Das, B. (2022). Convergence of complex uncertain triple sequence via metric operator, p-distance and complete convergence. Facta Univ. Ser. Math. Inform., 37(2), 377-396.

[5] Das, B., & Tripathy, B. C. (2023). On λ^2-statistical convergence of complex uncertain sequences. Asian-Eur. J. Math., 16(5), Article number-2350083.

[6] Debnath, S., & Das, B. (2023). On rough statistical convergence of complex uncertain sequences. New Math. Nat. Comput., 19(1), 1-17.

[7] Debnath, S., & Das, B. (2023). On rough convergence of complex uncertain sequences. J. Uncertain Syst., 14(4), Article number-2150021.

[8] Debnath, S., & Halder, A. (2024). On rough $\mathcal{I}$-convergence of complex uncertain sequences. Boletim da Sociedade Paranaense de Matemática. (Accepted).

[9] Dowari, P. J., & Tripathy, B. C. (2023). Lacunary statistical convergence of sequences of complex uncertain variables. Bol. Soc. Paran. Mat., 41, 1-10.

[10] Dündar, E., & Çakan, C. (2019). Rough $\mathcal{I}$-convergence. Gulf J. Math., 2(1), 45-51.

[11] Dündar, E., & Ulusu, U. (2023). On rough $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy sequence for functions defined on amenable semigroups. Universal J. Math. Appl., 6(2), 86-90.

[12] Fast, H. (1951). Sur la convergence statistique. Colloq. Math., 2(3-4), 241-244.

[13] Fridy, J. A. (1985). On statistical convergence. Analysis, 5, 301-313.

[14] Halder, A., & Debnath, S. (2023). On $\mathcal{I}$-statistical convergence almost surely of complex uncertain sequences. Adv. Math. Sci. Appl., 32(2), 431-445.

[15] Hazarika, B. (2014). Ideal convergence in locally solid Riesz spaces. Filomat, 28(4), 797–809.

[16] Khan, V.A., Hazarika, B., Khan, I.A., & Rahman, Z. (2022). A study on $\mathcal{I}$-deferred strongly Cesàro summable and μ-deferred $\mathcal{I}$-statistically convergence for complex uncertain sequences. Filomat, 36(20), 7001–7020.

[17] Khan, V.A., Khan, I.A., & Hazarika, B. (2022). On μ-deferred $\mathcal{I}_2$-statistical convergence of double sequence of complex uncertain variables. Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat, 116, Article: 121.

[18] Kişi, Ö., & Gürdal, M. (2022). Rough statistical convergence of complex uncertain triple sequence. Acta Math. Univ. Comen., 91(4), 365–376.

[19] Kişi, Ö., & Gürdal, M. (2023). On I_2 and I_2^*-convergence in almost surely of complex uncertain double sequences. Probl. Anal. Issues Anal., 12(30) (2), 51-67.

[20] Kostyrko, P., Mačaj, M., & Sleziak, M. (2000/2001). $\mathcal{I}$-convergence. Real Anal. Exchange, 26, 669-686.

[21] Liu, B. (2015). Uncertainty Theory. (4^{th} edition), Springer-Verlag, Berlin.

[22] Peng, Z. (2012). Complex uncertain variable. Doctoral Dissertation, Tsinghua University.

[23] Phu, H. X. (2001). Rough convergence in normed linear spaces. Numer. Funct. Anal. Optim., 22(1-2), 199-222.

[24] Raj, K., Sharma, S., & Mursaleen, M. (2022). Almost λ-statistical convergence of complex uncertain sequences. Internat. J. Uncertain. Fuzziness Knowledge-Based Systems, 30(5), 795-811.

[25] Roy, S., Tripathy, B. C., & Saha, S. (2016). Some results on $p-$distance and sequence of complex uncertain variables. Commun. Korean. Math. Soc., 35(3), 907-916.

[26] Saha, S., Tripathy, B. C. & Roy, S. (2021). Relationships between statistical convergence concepts of complex uncertain sequences. Appl. Sci., 23, 137-144.

[27] Savaş, E., & Das, P. 2011, A generalized statistical convergence via ideals. A Math. Lett., 24, 826-830.

[28] Savaş, E., & Das, P. (2014). On $\mathcal{I}$-statistically pre-Cauchy sequences. Taiwanese J. Math., 18(1), 115-126.

[29] Savaş, E., Debnath, S., & Rakshit, D. (2018). On $\mathcal{I}$-statistically rough convergence. Publ. Inst. Math., 105(119), 145-150.

[30] Steinhaus, H. (1951). Sur la convergence ordinaire et la convergence asymptotique. Colloq. Math., 2(1), 73-74.

[31] Tripathy, B. C., & Nath, P. K. (2017). Statistical convergence of complex uncertain sequences. New Math. Nat. Comput., 13(3), 359-374.

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CHARACTERIZATION OF $\mathcal{W}_6$-CURVATURE TENSOR ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract. The aim of this study is to explore the characteristics of n-dimensional Lorentzian para-Kenmotsu {briefly, $(LPK)_n$} manifolds with $\mathcal{W}_6$-curvature tensor. Firstly, we explore $(LPK)_n$ manifold with the condition $'\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta) = 0$ and find that it is an Einstein manifold. Next, we consider the conditions of Φ-$\mathcal{W}_6$-symmetric, $\mathcal{W}_6$-semisymmetric, and Φ-$\mathcal{W}_6$-flat on the $(LPK)_n$ manifold. Moreover, an example has been constrcuted to verify the results. Lastly, we explain the condition $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$ on $(LPK)_n$ manifold that establishes ω-Einstein manifold.

Keywords: Lorentzian para-Kenmotsu manifold, Scalar curvature, $\mathcal{W}_6$-curvature tensor, Einstein manifold.

2020 Mathematics Subject Classification: 53C25, 53C50.

1. Introduction

In 1989 [14], B.B. Sinha and K.L. Sai Prasad have defined para-Kenmotsu manifolds. They investigated the significant properties of para-Kenmotsu manifolds. Later on, para-Kenmotsu manifolds drew attention of several authors to study the characteristics of such manifolds. Lorentzian para-Kenmotsu manifolds were initiated in 2018 by A. Haseeb and R. Prasad [3]. R. Sari et al. have explained slant manifolds of a Lorentzian Kenmotsu manifold [11]. Mobin Ahmad studied semi-invariant $\zeta^{\perp}$-submanifolds of Lorentzian para-Sasakian manifolds in

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2019 [1]. Moreover, Abhishek Singh et al., in 2024, explored some results on β-Kenmotsu manifolds with a non-symmetric non-metric connection [12, 13]. In 2022, Shashikant Pandey et al. have described certain results of Ricci-soliton on 3-dimensional Lorentzian para α-Sasakian manifolds [5]. For invariant submanifolds of Lorentzian para-Kenmotsu manifold to be totally geodesic, Atceken [2] gave the necessary and sufficient conditions.

G.P. Pokhariyal gave the concept of $\mathcal{W}_6$-curvature tensor with the support of Weyl curvature tensor in 1982 [6, 7], and is described as

$$\mathcal{W}_6(\mathtt{A},\mathtt{B})\mathtt{C} = \mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} + \frac{1}{n-1}[g(\mathtt{A},\mathtt{B})Q\mathtt{C} - \mathcal{S}(\mathtt{B},\mathtt{C})\mathtt{A}], \tag{1.1}$$

and

$$`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) = \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) + \frac{1}{n-1}[g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\mathtt{T}) - g(\mathtt{A},\mathtt{T})\mathcal{S}(\mathtt{B},\mathtt{C}), \tag{1.2}$$

$\forall\ \mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T} \in \chi(M^n)$, where, $\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) = g(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\mathtt{T})$, $\mathcal{R}$ and Q denote Riemann curvature tensor and Ricci operator, respectively.

This article has been organized in the following manner: Section-1 contains introduction, where corresponding concepts and their brief histories are given. Section-2 covers preliminaries, containing some basic results, which have been used extensively throughout this manuscript. Section-3 describes the Lorentzian para-Kenmotsu manifold with the condition $`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta) = 0$. Section-4 studies the nature of $\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0$ on Lorentzian para-Kenmotsu manifold with the construction of an example. Section-5 and section-6 examine the behavior of $\mathcal{W}_6$-semisymmetric, and Φ-$\mathcal{W}_6$-flat on $(LPK)_n$ manifold, respectively. In section-7, we see that an $(LPK)_n$ manifold with the condition $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$ gives an ω-Einstein manifold.

2. Preliminaries

We assume that M^n is a Lorentzian metric manifold, meaning there by, it is equipped with an structure (Φ,ζ,ω,g), where Φ is a (1, 1)-type tensor field, ζ is a vector field, ω is a one-form on M^n, and g is a Lorentzian metric tensor holding the subsequent results [8, 9, 10]:

$$\left.\begin{aligned} &\Phi^2(\mathtt{A}) = \mathtt{A} + \omega(\mathtt{A})\zeta, \quad g(\mathtt{A},\zeta) = \omega(\mathtt{A}), \quad \omega(\zeta) = -1, \\ &\Phi\zeta = 0, \quad \omega(\Phi\mathtt{A}) = 0, \quad g(\Phi\mathtt{A},\Phi\mathtt{B}) = g(\mathtt{A},\mathtt{B}) + \omega(\mathtt{A})\omega(\mathtt{B}), \end{aligned}\right\} \tag{2.3}$$

$\forall$ vector fields $\mathtt{A}$, $\mathtt{B}$ on M^n. Thus, $M^n(\Phi,\zeta,\omega,g)$ is called a Lorentzian almost paracontact manifold.

Definition 2.1. *A Lorentzian almost paracontact manifold $M^n(\Phi,\zeta,\omega,g)$ is called a Lorentzian para-Kenmotsu manifold if*

$$(\nabla_{\mathtt{A}}\Phi)\mathtt{B} = -g(\Phi\mathtt{A},\mathtt{B})\zeta - \omega(\mathtt{B})\Phi\mathtt{A},$$

$\forall$ vector fields $\mathtt{A}$, $\mathtt{B} \in \chi(M^n)$. Here, ∇ and $\chi(M^n)$ represent Levi-Civita connection, and a collection of differentiable vector fields on M^n, respectively.

We assume that $M^n(\Phi,\zeta,\omega,g)$ is an $(LPK)_n$ manifold. The succeeding results hold for $M^n(\Phi,\zeta,\omega,g)$:

$$\nabla_{\mathtt{A}}\zeta = -\mathtt{A} - \omega(\mathtt{A})\zeta, \qquad g(\Phi\mathtt{A},\mathtt{B}) = g(\mathtt{A},\Phi\mathtt{B}), \tag{2.4}$$

$$(\nabla_{\mathtt{A}}\omega)\mathtt{B} = -g(\mathtt{A},\mathtt{B}) - \omega(\mathtt{A})\omega(\mathtt{B}), \tag{2.5}$$

$$\omega(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}) = \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta) = g(\mathtt{B},\mathtt{C})\omega(\mathtt{A}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B}), \tag{2.6}$$

$$\left.\begin{aligned} \mathcal{R}(\mathtt{A},\zeta)\mathtt{B} &= \omega(\mathtt{B})\mathtt{A} - g(\mathtt{A},\mathtt{B})\zeta, \\ \mathcal{R}(\zeta,\mathtt{A})\zeta &= \mathtt{A} + \omega(\mathtt{A})\zeta, \\ \mathcal{R}(\mathtt{A},\mathtt{B})\zeta &= \omega(\mathtt{B})\mathtt{A} - \omega(\mathtt{A})\mathtt{B}, \end{aligned}\right\} \tag{2.7}$$

$$\mathcal{K}(\zeta,\mathtt{A},\mathtt{B},\mathtt{C}) = g(\mathtt{A},\mathtt{B})\ \omega(\mathtt{C}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B}), \tag{2.8}$$

$$\left.\begin{aligned} \mathcal{S}(\mathtt{A},\zeta) = (n-1)\omega(\mathtt{A}),\ \ \mathcal{S}(\zeta,\zeta) = -(n-1), \\ (\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{C},\zeta) = \mathcal{S}(\mathtt{E},\mathtt{C}) - (n-1)g(\mathtt{E},\mathtt{C}), \end{aligned}\right\} \tag{2.9}$$

$$div\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} = (\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C}) - (\nabla_{\mathtt{B}}\mathcal{S})(\mathtt{A},\mathtt{C}), \qquad . \tag{2.10}$$

here, $\mathcal{S}$ denotes Ricci tensor of $M^n(\Phi,\zeta,\omega,g)$.

Particularly, setting $\mathtt{A} = \zeta$, $\mathtt{B} = \zeta$, and $\mathtt{C} = \zeta$, respectively, in 1.1 on an $(LPK)_n$ manifold, it yields

$$\mathcal{W}_6(\zeta,\mathtt{B})\mathtt{C} = g(\mathtt{B},\mathtt{C})\zeta - \omega(\mathtt{C})\mathtt{B} + \frac{1}{n-1}[\omega(\mathtt{B})Q\mathtt{C} - \mathcal{S}(\mathtt{B},\mathtt{C})\zeta], \tag{2.11}$$

$$\mathcal{W}_6(\mathtt{A},\zeta)\mathtt{C} = -g(\mathtt{A},\mathtt{C})\zeta + \frac{1}{n-1}\omega(\mathtt{A})Q\mathtt{C}, \tag{2.12}$$

$$\mathcal{W}_6(\mathtt{A},\mathtt{B})\zeta = g(\mathtt{A},\mathtt{B})\zeta - \omega(\mathtt{A})\mathtt{B}. \tag{2.13}$$

Definition 2.2. *An $(LPK)_n$ manifold is called an ω-Einstein manifold if its Ricci tensor satisfies the following relation*

$$\mathcal{S}(\mathtt{A},\mathtt{B}) = \alpha g(\mathtt{A},\mathtt{B}) + \beta\omega(\mathtt{A})\omega(\mathtt{B}),$$

here, α, and β are scalar functions on M^n. In case of $\beta = 0$, manifold becomes Einstein manifold [4].

3. $`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$ on a Lorentzian para-Kenmotsu Manifolds

In this part, we discuss the condition $`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$ on $(LPK)_n$ manifolds M^n. We begin with the subsequent theorem:

Theorem 3.1. *An n-dimensional Lorentzian para-Kenmotsu manifold is an Einstein manifold if,* $`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$.

Proof. $`\mathcal{W}_6$-curvature tensor is defined by 1.2

$$`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T})=\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T})+\frac{1}{n-1}[g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\mathtt{T})-g(\mathtt{A},\mathtt{T})\mathcal{S}(\mathtt{B},\mathtt{C})].$$

$\forall$ $\mathtt{A}$, $\mathtt{B}$, $\mathtt{C}$, $\mathtt{T}\in\chi(M^n)$.

Putting $\mathtt{T}=\zeta$ into the above equation, we have

$$`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)+\frac{1}{n-1}[g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\zeta)-g(\mathtt{A},\zeta)\mathcal{S}(\mathtt{B},\mathtt{C})].$$

Applying $`\mathcal{W}_6(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=0$ in the above relation, we get

$$\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)=\frac{1}{n-1}[g(\mathtt{A},\zeta)\mathcal{S}(\mathtt{B},\mathtt{C})-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\zeta)]. \tag{3.14}$$

Using 2.3, 2.6, and 2.9, the relation 3.14 yields

$$\frac{1}{n-1}\mathcal{S}(\mathtt{B},\mathtt{C})\omega(\mathtt{A})=g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})+g(\mathtt{A},\mathtt{B})\omega(\mathtt{C}). \tag{3.15}$$

Applying $\mathtt{A}=\zeta$ into 3.15, it yields

$$\frac{1}{n-1}\mathcal{S}(\mathtt{B},\mathtt{C})\omega(\zeta)=g(\mathtt{B},\mathtt{C})\omega(\zeta)-g(\zeta,\mathtt{C})\omega(\mathtt{B})+g(\zeta,\mathtt{B})\omega(\mathtt{C}). \tag{3.16}$$

Using 2.3, on simplification, the relation 3.16 provides

$$\mathcal{S}(\mathtt{B},\mathtt{C})=(n-1)g(\mathtt{B},\mathtt{C}). \tag{3.17}$$

This completes the proof. □

4. nature of Φ-$\mathcal{W}_6$-symmetric on $(LPK)_n$ manifolds

We begin this part with the definition of Φ-$\mathcal{W}_6$-symmetric Lorentzian para-Kenmotsu manifold:

Definition 4.1. *A Lorentzian para-Kenmotsu manifold is said to be a Φ-$\mathcal{W}_6$-symmetric Lorentzian para-Kenmotsu manifold, if it satisfies the relation*

$$\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C})=0,$$

for every $\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{E}$ *on* M^n.

Theorem 4.1. *A Φ-W_6-symmetric Lorentzian para-Kenmotsu manifold is an Einstein manifold.*

Proof. Covariant differentiation of relation 1.1 along $\mathtt{E}$ yields

$$(\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C} = (\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C} - \frac{1}{n-1}[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A} - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C}]. \tag{4.18}$$

Operating Φ^2 on both sides of the equation 4.18 and using 2.3, it gives

$$\begin{aligned}\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) &= (\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C} + \omega((\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C})\zeta \\ &\quad - \frac{1}{n-1}[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A} + (\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\zeta \\ &\quad - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C} - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{C},\zeta)\zeta].\end{aligned} \tag{4.19}$$

Using 2.9 with condition $\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0$ into the relation 4.19, it yields

$$\begin{aligned}0 &= (\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C} + \omega((\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C})\zeta \\ &\quad - \frac{1}{n-1}[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A} + (\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\zeta \\ &\quad - g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C} - g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{E},\mathtt{C})\zeta + (n-1)g(\mathtt{A}.\mathtt{B})g(\mathtt{E},\mathtt{C})\zeta].\end{aligned} \tag{4.20}$$

Differentiating covariantly 2.6 along $\mathtt{E}$, it gives

$$\begin{aligned}&(\nabla_{\mathtt{E}}g)(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\zeta) + g(\nabla_{\mathtt{E}}\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\zeta) \\ &\quad + g(\mathcal{R}(\nabla_{\mathtt{E}}\mathtt{A},\mathtt{B})\mathtt{C},\zeta) + g(\mathcal{R}(\mathtt{A},\nabla_{\mathtt{E}}\mathtt{B})\mathtt{C},\zeta) \\ &\quad + g(\mathcal{R}(\mathtt{A},\mathtt{B})\nabla_{\mathtt{E}}\mathtt{C},\zeta) + g(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\nabla_{\mathtt{E}}\zeta) \\ &= (\nabla_{\mathtt{E}}g)(\mathtt{B},\mathtt{C})\,g(\mathtt{A},\zeta) + g(\nabla_{\mathtt{E}}\mathtt{B},\mathtt{C})\,g(\mathtt{A},\zeta) \\ &\quad + g(\mathtt{B},\nabla_{\mathtt{E}}\mathtt{C})\,g(\mathtt{A},\zeta) + g(\mathtt{B},\mathtt{C})\,(\nabla_{\mathtt{E}}g)(\mathtt{A},\zeta) \\ &\quad + g(\mathtt{B},\mathtt{C})\,g(\nabla_{\mathtt{E}}\mathtt{A},\zeta) + g(\mathtt{B},\mathtt{C})\,g(\mathtt{A},\nabla_{\mathtt{E}}\zeta) \\ &\quad - (\nabla_{\mathtt{E}}g)(\mathtt{A},\mathtt{C})\,g(\mathtt{B},\zeta) - g(\nabla_{\mathtt{E}}\mathtt{A},\mathtt{C})\,g(\mathtt{B},\zeta) \\ &\quad - g(\mathtt{A},\nabla_{\mathtt{E}}\mathtt{C})\,g(\mathtt{B},\zeta) - g(\mathtt{A},\mathtt{C})\,(\nabla_{\mathtt{E}}g)(\mathtt{B},\zeta) \\ &\quad - g(\mathtt{A},\mathtt{C})\,g(\nabla_{\mathtt{E}}\mathtt{B},\zeta) - g(\mathtt{A},\mathtt{C})\,g(\mathtt{B},\nabla_{\mathtt{E}}\zeta).\end{aligned} \tag{4.21}$$

Applying 2.3 and 2.4 into 4.21, it gives

$$\omega((\nabla_{\mathtt{E}}\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}) = g(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C},\mathtt{E}) + g(\mathtt{A},\mathtt{C})g(\mathtt{B},\mathtt{E}) - g(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{E}). \tag{4.22}$$

Relations 4.20 and 4.22, provide

$$\begin{aligned}&(\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C}+\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{E})\zeta+g(\mathtt{A},\mathtt{C})\,g(\mathtt{B},\mathtt{E})\,\zeta-g(\mathtt{B},\mathtt{C})\,g(\mathtt{A},\mathtt{E})\,\zeta\\&-\frac{1}{n-1}\Big[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\mathtt{A}+(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\,\omega(\mathtt{A})\,\zeta-g(\mathtt{A},\mathtt{B})(\nabla_{\mathtt{E}}Q)\mathtt{C}\\&-g(\mathtt{A},\mathtt{B})\,\mathcal{S}(\mathtt{E},\mathtt{C})\,\zeta+(n-1)\,g(\mathtt{A},\mathtt{B})\,g(\mathtt{E},\mathtt{C})\,\zeta\Big]=0.\end{aligned} \tag{4.23}$$

Innerproduct of 4.23 along $\mathtt{F}$ is given by

$$\begin{aligned}&g((\nabla_{\mathtt{E}}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C},\mathtt{F})+\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{E})\omega(\mathtt{F})+g(\mathtt{A},\mathtt{C})g(\mathtt{B},\mathtt{E})\omega(\mathtt{F})-g(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{E})\omega(\mathtt{F})\\&-\frac{1}{n-1}\Big[(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{F})+(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{F})\\&-g(\mathtt{A},\mathtt{B})g((\nabla_{\mathtt{E}}Q)\mathtt{C},\mathtt{F})-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{E},\mathtt{C})\omega(\mathtt{F})+(n-1)g(\mathtt{A},\mathtt{B})g(\mathtt{E},\mathtt{C})\omega(\mathtt{F})\Big]=0.\end{aligned} \tag{4.24}$$

Contracting 4.24 along $\mathtt{E}$ and $\mathtt{F}$, we have

$$\begin{aligned}&\sum_{i=1}^{n}\epsilon_i g((\nabla_{\mathcal{E}_i}\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C},\mathcal{E}_i)+\sum_{i=1}^{n}\epsilon_i\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathcal{E}_i)g(\zeta,\mathcal{E}_i)\\&+\sum_{i=1}^{n}\epsilon_i g(\mathtt{A},\mathtt{C})g(\mathtt{B},\mathcal{E}_i)g(\zeta,\mathcal{E}_i)-\sum_{i=1}^{n}\epsilon_i g(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathcal{E}_i)g(\zeta,\mathcal{E}_i)\\&-\frac{1}{n-1}\sum_{i=1}^{n}\epsilon_i\Big[(\nabla_{\mathcal{E}_i}\mathcal{S})(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathcal{E}_i)+(\nabla_{\mathcal{E}_i}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})g(\zeta,\mathcal{E}_i)\\&-g(\mathtt{A},\mathtt{B})g((\nabla_{\mathcal{E}_i}Q)\mathtt{C},\mathcal{E}_i)-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathcal{E}_i,\mathtt{C})g(\zeta,\mathcal{E}_i)\\&+(n-1)g(\mathtt{A},\mathtt{B})g(\mathcal{E}_i,\mathtt{C})g(\zeta,\mathcal{E}_i)\Big]=0.\end{aligned}$$

where, $\epsilon_i=g(\mathcal{E}_i,\mathcal{E}_i)$ and $\{\mathcal{E}_1,\mathcal{E}_2....\mathcal{E}_{n-1},\zeta\}$ are orthonormal base field on $(LPK)_n$ manifold. Using relations 2.3, 2.6, 2.8, and 2.9 into the above relation, it gives

$$\begin{aligned}&(div\mathcal{R})(\mathtt{A},\mathtt{B})\mathtt{C}+\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\zeta)+g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})-g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\\&-\frac{1}{n-1}\Big[(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C})+(\nabla_{\zeta}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\\&-g(\mathtt{A},\mathtt{B})\frac{\mathtt{C}(r)}{2}-g(\mathtt{A},\mathtt{B})\mathcal{S}(\mathtt{C},\zeta)+(n-1)g(\mathtt{A},\mathtt{B})g(\mathtt{C},\zeta)\Big]=0.\end{aligned} \tag{4.25}$$

where, $divQ(\mathtt{C}){=}\frac{\mathtt{C}(r)}{2}$.

Putting the value from 2.6, and 2.10 into 4.25, we have

$$\begin{aligned}&(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C})-(\nabla_{\mathtt{B}}\mathcal{S})(\mathtt{A},\mathtt{C})+g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})\\&+g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})-g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-\frac{1}{n-1}\Big[(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\mathtt{C})\\&+(\nabla_{\zeta}\mathcal{S})(\mathtt{B},\mathtt{C})\omega(\mathtt{A})-\frac{\mathtt{C}(r)}{2}g(\mathtt{A},\mathtt{B})\Big]=0.\end{aligned} \tag{4.26}$$

Putting $\mathtt{C}=\zeta$ into 4.26, we get

$$\frac{(n-2)}{(n-1)}(\nabla_{\mathtt{A}}\mathcal{S})(\mathtt{B},\zeta)-(\nabla_{\mathtt{B}}\mathcal{S})(\mathtt{A},\zeta)-\frac{1}{n-1}(\nabla_{\zeta}\mathcal{S})(\mathtt{B},\zeta)\omega(\mathtt{A})+\frac{1}{2(n-1)}\zeta(r)g(\mathtt{A},\mathtt{B})=0. \tag{4.27}$$

Using the relation 2.9 into 4.27, it gives

$$\begin{aligned}&\frac{(n-2)}{(n-1)}\big[\mathcal{S}(\mathtt{A},\mathtt{B})-(n-1)g(\mathtt{A},\mathtt{B})\big]-\big[\mathcal{S}(\mathtt{A},\mathtt{B})-(n-1)g(\mathtt{A},\mathtt{B})\big]\\&-\frac{1}{n-1}\big[\mathcal{S}(\mathtt{B},\zeta)-(n-1)\omega(\mathtt{B})\big]\omega(\mathtt{A})+\frac{1}{2(n-1)}\zeta(r)g(\mathtt{A},\mathtt{B})=0.\end{aligned} \tag{4.28}$$

After simplification, 4.28 yields

$$\mathcal{S}(\mathtt{A},\mathtt{B})=[(n-1)+\frac{\zeta(r)}{2}]g(\mathtt{A},\mathtt{B})]. \tag{4.29}$$

Further, contracting 4.24 along $\mathtt{A}$ and $\mathtt{F}$ and using 2.8, we have

$$(\nabla_{\mathtt{E}}\mathcal{S})(\mathtt{C},\mathtt{B})=-\mathcal{S}(\mathtt{E},\mathtt{C})\omega(\mathtt{B})+(n-1)g(\mathtt{E},\mathtt{C})\omega(\mathtt{B}). \tag{4.30}$$

Again, contracting the above equation along $\mathtt{B}$ and $\mathtt{C}$, we have

$$(\nabla_{\mathtt{E}}r)=-S(\mathtt{E},\zeta)+(n-1)\omega(\mathtt{E}). \tag{4.31}$$

Using 2.9, it yields that scalar curvature r is constant. Therefore, 4.29 concludes the following:

$$S(\mathtt{A},\mathtt{B})=(n-1)g(\mathtt{A},\mathtt{B}). \tag{4.32}$$

Hence, we establish that $\mathit{\Phi}$-$\mathcal{W}_6$-symmetric $(LPK)_n$ manifold is an Einstein manifold.

□

We consider an $(LPK)_n$ manifold of constant curvature, then

$$\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C}=k[g(\mathtt{B},\mathtt{C})\mathtt{A}-g(\mathtt{A},\mathtt{C})\mathtt{B}], \tag{4.33}$$

where, k is constant.

The relations 1.1 and 4.33, taken together, we have

$$\mathcal{W}_6(\mathtt{A},\mathtt{B})\mathtt{C} = k[g(\mathtt{A},\mathtt{B})\mathtt{C} - g(\mathtt{A},\mathtt{C})\mathtt{B}]. \tag{4.34}$$

Differentiating covariantly the relation 4.34 along $\mathtt{E}$ and operating Φ^2 on both sides, it yields

$$\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0. \tag{4.35}$$

This establishes the subsequent corollary:

Corollary 4.1. *The $(LPK)_n$ manifolds of constant curvature are Φ-$\mathcal{W}_6$-symmetric $(LPK)_n$ manifolds.*

Example 4.1. *Consider a differentiable manifold $M^4 = \{(u,v,w,t) \in \Re^4\colon$ u, v, w is non zero, t>0$\}$. Suppose that $\{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4\}$ are linearly independent vectors at every point of M^4. We define,*

$$\mathcal{E}_1 = e^{u+t}\frac{\partial}{\partial u}, \qquad \mathcal{E}_2 = e^{v+t}\frac{\partial}{\partial v}, \qquad \mathcal{E}_3 = e^{w+t}\frac{\partial}{\partial w}, \qquad \mathcal{E}_4 = \frac{\partial}{\partial t}.$$

Lorentzian metric g on M^4 is established in the following way:

$$g_{ij} = g(\mathcal{E}_i,\mathcal{E}_j) = \begin{cases} 0 & \text{if } i \neq j \\ -1 & \text{if } i = j = 4 \\ 1 & \text{or else.} \end{cases}$$

Assuming ω is one-form corresponding to g is defined by

$$\omega(\mathtt{A}) = g(\mathtt{A},\mathcal{E}_4),$$

$\forall\ \mathtt{A} \in \chi(M^4)$, *here $\chi(M^4)$ be collection of vector fields on M^4. We define Φ as $(1,1)$-tensor field as follows:*

$$\Phi(\mathcal{E}_1) = \mathcal{E}_1, \qquad \Phi(\mathcal{E}_2) = \mathcal{E}_2, \qquad \Phi(\mathcal{E}_3) = \mathcal{E}_3, \qquad \Phi(\mathcal{E}_4) = 0.$$

From linear characteristic of Φ and g, the following results can easily be proved:

$$\omega(\mathcal{E}_4) = -1, \qquad \Phi^2(\mathtt{A}) = \mathtt{A} + \omega(\mathtt{A})\mathcal{E}_4, \qquad g(\Phi\mathtt{A},\Phi\mathtt{B}) = g(\mathtt{A},\mathtt{B}) + \omega(\mathtt{A})\omega(\mathtt{B}),$$

$\forall\ \mathtt{A},\mathtt{B} \in \chi(M^4)$. *So, when $\mathcal{E}_4 = \zeta$, structure (Φ,ζ,ω,g) leading to Lorentzian paracontact structure as well as manifold M equipped with Lorentzian paracontact structure is said to be Lorentzian paracontact manifold of dimension-4.*

We represent $[\mathtt{A},\mathtt{B}]$ *as Lie-derivative of* $\mathtt{A}$, $\mathtt{B}$, *defined as* $[\mathtt{A},\mathtt{B}] = \mathtt{AB} - \mathtt{BA}$. *The non-zero constituents of Lie bracket are evaluated as below:*

$$[\mathcal{E}_1,\mathcal{E}_4] = -\mathcal{E}_1, \qquad [\mathcal{E}_2,\mathcal{E}_4] = -\mathcal{E}_2, \qquad [\mathcal{E}_3,\mathcal{E}_4] = -\mathcal{E}_3.$$

Let Riemannian connection with respect to g *be denoted by* ∇. *So, when* $\mathcal{E}_4 = \zeta$, *we have the subsequent results:*

$$\begin{array}{llll}
\nabla_{\mathcal{E}_1}\mathcal{E}_1 = -\mathcal{E}_4, & \nabla_{\mathcal{E}_1}\mathcal{E}_2 = 0, & \nabla_{\mathcal{E}_1}\mathcal{E}_3 = 0, & \nabla_{\mathcal{E}_1}\mathcal{E}_4 = -\mathcal{E}_1,\\
\nabla_{\mathcal{E}_2}\mathcal{E}_1 = 0, & \nabla_{\mathcal{E}_2}\mathcal{E}_2 = -\mathcal{E}_4, & \nabla_{\mathcal{E}_2}\mathcal{E}_3 = 0, & \nabla_{\mathcal{E}_2}\mathcal{E}_4 = -\mathcal{E}_2,\\
\nabla_{\mathcal{E}_3}\mathcal{E}_1 = 0, & \nabla_{\mathcal{E}_3}\mathcal{E}_2 = 0, & \nabla_{\mathcal{E}_3}\mathcal{E}_3 = -\mathcal{E}_4, & \nabla_{\mathcal{E}_3}\mathcal{E}_4 = -\mathcal{E}_3,\\
\nabla_{\mathcal{E}_4}\mathcal{E}_1 = 0, & \nabla_{\mathcal{E}_4}\mathcal{E}_2 = 0, & \nabla_{\mathcal{E}_4}\mathcal{E}_3 = 0, & \nabla_{\mathcal{E}_4}\mathcal{E}_4 = 0.
\end{array}$$

Assuming $\mathtt{A} \in \chi(M^4)$, *so* $\mathtt{A} = a_1\mathcal{E}_1 + a_2\mathcal{E}_2 + a_3\mathcal{E}_3 + a_4\mathcal{E}_4$, *here* $\{\mathcal{E}_1,\mathcal{E}_2,\mathcal{E}_3,\mathcal{E}_4\}$ *be the basis of* $\chi(M^4)$. *Above relations help verify* $\nabla_{\mathtt{A}}\mathcal{E}_4 = -\mathtt{A} - \omega(\mathtt{A})\mathcal{E}_4$ *for each* $\mathtt{A} \in \chi(M^4)$. *Hence,* M^4 *is a Lorentzian para-Kenmotsu manifold of dimension-4. From the above relations, the non-vanishing constituents of the curvature tensor are evaluated as subsequently,*

$$\begin{array}{lll}
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_1 = -\mathcal{E}_2, & \mathcal{R}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_1 = -\mathcal{E}_3, & \mathcal{R}(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_1 = -\mathcal{E}_4,\\
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_2 = \mathcal{E}_1, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_2 = -\mathcal{E}_3, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_2 = -\mathcal{E}_4,\\
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_3 = \mathcal{E}_1, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_3 = \mathcal{E}_2, & \mathcal{R}(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_3 = -\mathcal{E}_4,\\
\mathcal{R}(\mathcal{E}_1,\mathcal{E}_4)\mathcal{E}_4 = -\mathcal{E}_1, & \mathcal{R}(\mathcal{E}_2,\mathcal{E}_4)\mathcal{E}_4 = -\mathcal{E}_2, & \mathcal{R}(\mathcal{E}_3,\mathcal{E}_4)\mathcal{E}_4 = -\mathcal{E}_3.
\end{array}$$

It can easily be seen that $\quad \mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} = g(\mathtt{B},\mathtt{C})\mathtt{A} - g(\mathtt{A},\mathtt{C})\mathtt{B}$.
From definition of Ricci tensor $\mathcal{S}$ *on* M^4, *the subsequent result holds,*

$$\mathcal{S}(\mathtt{A},\mathtt{B}) = \Sigma_{i=1}^4 \varepsilon_i g(\mathcal{R}(\mathcal{E}_i,\mathtt{A})\mathtt{B},\mathcal{E}_i), \qquad \varepsilon_i = g(\mathcal{E}_i,\mathcal{E}_i).$$

Therefore, matrix representation of $\mathcal{S}$ *is mentioned by*

$$\mathcal{S} = \begin{bmatrix} 3 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & -3 \end{bmatrix}.$$

This gives, $\mathcal{S}(\mathtt{A},\mathtt{B}) = 3g(\mathtt{A},\mathtt{B})$ *and scalar curvature* $\kappa = \Sigma_{i=1}^4 \varepsilon_i \mathcal{S}(\mathcal{E}_i,\mathcal{E}_i) = 12$, *this implies that* $(LPK)_4$ *manifold has constant scalar curvature. Hence, relation* $\Phi^2((\nabla_{\mathtt{E}}\mathcal{W}_6)(\mathtt{A},\mathtt{B})\mathtt{C}) = 0$ *holds.*
Thus, the above example verifies the results of this section.

5. $\mathcal{W}_6$-semisymmetric Lorentzian para-Kenmotsu Manifolds

This part covers the behavior of $\mathcal{W}_6$, when $\mathcal{R}(\mathtt{A},\ \mathtt{B})$ operates on it in $(LPK)_n$ manifold. Now, we have the following theorem:

Theorem 5.1. *Let $(M^n,\ g)$ be an $(LPK)_n$ manifold. If $\mathcal{R}(\mathtt{A},\mathtt{B}).\mathcal{W}_6 = 0$. Then, M^n is an Einstein manifold, where $\mathcal{R}(\mathtt{A},\mathtt{B})$ is a Riemannian operator, and $\mathcal{W}_6$ is a curvature tensor.*

Proof. We assume that M^n is an $(LPK)_n$ manifold satisfying subsequent condition:

$$(\mathcal{R}(\mathtt{A},\mathtt{B}).\mathcal{W}_6)(\mathtt{E},\mathtt{F})\mathtt{T} = 0. \tag{5.36}$$

From relation 5.36, we have

$$\mathcal{R}(\mathtt{A},\mathtt{B}).\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T} \ = \ \mathcal{W}_6(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{E},\mathtt{F})\mathtt{T} + \mathcal{W}_6(\mathtt{E},\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{F})\mathtt{T} + \mathcal{W}_6(\mathtt{E},\mathtt{F})(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{T}). \tag{5.37}$$

Taking innerproduct of 5.37 along $\mathtt{C}$, we have

$$\mathcal{K}(\mathtt{A},\mathtt{B},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T},\mathtt{C}) = {}^{\backprime}\mathcal{W}_6(\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\mathtt{C}) + {}^{\backprime}\mathcal{W}_6(\mathtt{E},\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{F},\mathtt{T},\mathtt{C}) + {}^{\backprime}\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{T},\mathtt{C}). \tag{5.38}$$

Applying $\mathtt{A} = \mathtt{C} = \zeta$ into 5.38, it provides

$$\mathcal{K}(\zeta,\mathtt{B},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T},\zeta) = {}^{\backprime}\mathcal{W}_6(\mathcal{R}(\zeta,\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\zeta) + {}^{\backprime}\mathcal{W}_6(\mathtt{E},\mathcal{R}(\zeta,\mathtt{B})\mathtt{F},\mathtt{T},\zeta) + {}^{\backprime}\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\zeta,\mathtt{B})\mathtt{T},\zeta). \tag{5.39}$$

Evaluation of left hand side of 5.39 with relation 2.6, it yields

$$\begin{aligned} \mathcal{K}(\zeta,\mathtt{B},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{T},\zeta) = &-\mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{T},\mathtt{B}) \\ &- \frac{1}{n-1}[g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{T},\mathtt{B}) - g(\mathtt{E},\mathtt{B})\mathcal{S}(\mathtt{F},\mathtt{T})] \\ &- \omega(\mathtt{B})\omega(\mathtt{T})g(\mathtt{E},\mathtt{F}) - \omega(\mathtt{E})\omega(\mathtt{B})g(\mathtt{F},\mathtt{T}) \\ &+ g(\mathtt{E},\mathtt{T})\omega(\mathtt{F})\omega(\mathtt{B}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{B})\mathcal{S}(\mathtt{F},\mathtt{T}). \end{aligned} \tag{5.40}$$

Evaluation of first term of right hand side of 5.39 with the relation 2.6 in the following way:

$${}^{\backprime}\mathcal{W}_6(\mathcal{R}(\zeta,\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\zeta) = g(\mathtt{B},\mathtt{E}){}^{\backprime}\mathcal{W}_6(\zeta,\mathtt{F},\mathtt{T},\zeta) - \omega(\mathtt{E}){}^{\backprime}\mathcal{W}_6(\mathtt{B},\mathtt{F},\mathtt{T},\zeta).$$

Applying the definition of $\mathcal{W}_6$-curvature tensor, the above relation becomes

$$\begin{aligned}\mathcal{W}_6(\mathcal{R}(\zeta,\mathtt{B})\mathtt{E},\mathtt{F},\mathtt{T},\zeta) &= -g(\mathtt{B},\mathtt{E})g(\mathtt{F},\mathtt{T}) - g(\mathtt{B},\mathtt{E})\omega(\mathtt{F})\omega(\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{E})\omega(\mathtt{F})\omega(\mathtt{T}) + \frac{1}{n-1}g(\mathtt{B},\mathtt{E})\mathcal{S}(\mathtt{F},\mathtt{T}) - \omega(\mathtt{E})\omega(\mathtt{B})g(\mathtt{F},\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{F}) - g(\mathtt{B},\mathtt{F})\omega(\mathtt{E})\omega(\mathtt{T}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{B})\mathcal{S}(\mathtt{F},\mathtt{T}). \quad (5.41)\end{aligned}$$

Evaluation of middle term of right hand side of 5.39 with 2.6 into the following way:

$$`\mathcal{W}_6(\mathtt{E},\mathcal{R}(\zeta,\mathtt{B})\mathtt{F},\mathtt{T},\zeta) = `\mathcal{W}_6(\mathtt{E},\zeta,\mathtt{T},\zeta)g(\mathtt{B},\mathtt{F}) - \omega(\mathtt{F})`\mathcal{W}_6(\mathtt{E},\mathtt{B},\mathtt{T},\zeta).$$

Now, from the definition of $\mathcal{W}_6$-curvature tensor, the above relation becomes

$$\begin{aligned}`\mathcal{W}_6(\mathtt{E},\mathcal{R}(\zeta,\mathtt{B})\mathtt{F},\mathtt{T},\zeta) &= g(\mathtt{B},\mathtt{F})g(\mathtt{E},\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{F})\omega(\mathtt{E})\omega(\mathtt{T}) - \omega(\mathtt{E})\omega(\mathtt{F})g(\mathtt{B},\mathtt{T}) + g(\mathtt{E},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{B})\\ &- g(\mathtt{E},\mathtt{B})\omega(\mathtt{F})\omega(\mathtt{T}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{F})\mathcal{S}(\mathtt{B},\mathtt{T}). \quad (5.42)\end{aligned}$$

Evaluation of the last term of 5.39 into the following way:

In view of relation 2.7, the last term of 5.39 becomes

$$`\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\zeta,\mathtt{B})\mathtt{T},\zeta) = g(\mathtt{B},\mathtt{T})`\mathcal{W}_6(\mathtt{E},\mathtt{F},\zeta,\zeta) - \omega(\mathtt{T})`\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathtt{B},\zeta),$$

Using the definition 1.2 with relation 2.7 and 2.9 into the above relation, we have

$$\begin{aligned}`\mathcal{W}_6(\mathtt{E},\mathtt{F},\mathcal{R}(\zeta,\mathtt{B})\mathtt{T},\zeta) &= -g(\mathtt{B},\mathtt{T})g(\mathtt{E},\mathtt{F}) - \omega(\mathtt{E})\omega(\mathtt{F})g(\mathtt{B},\mathtt{T})\\ &- \omega(\mathtt{T})\omega(\mathtt{E})g(\mathtt{F},\mathtt{B}) + g(\mathtt{E}.\mathtt{B})\omega(\mathtt{F})\omega(\mathtt{T})\\ &- \omega(\mathtt{T})\omega(\mathtt{E})g(\mathtt{F},\mathtt{B}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{T})\mathcal{S}(\mathtt{F},\mathtt{B}). \quad (5.43)\end{aligned}$$

Putting the values from 5.40, 5.41, 5.42, and 5.43 into 5.39, we have

$$\begin{aligned}\mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{T},\mathtt{B}) &+ \frac{1}{n-1}g(\mathtt{E},\mathtt{F})S(\mathtt{T},\mathtt{B}) - g(\mathtt{B},\mathtt{E})g(\mathtt{F},\mathtt{T})\\ &+ g(\mathtt{B},\mathtt{F})g(\mathtt{E},\mathtt{T}) + \frac{1}{n-1}S(\mathtt{B},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{F}) - g(\mathtt{B},\mathtt{T})g(\mathtt{E},\mathtt{F})\\ &- g(\mathtt{B},\mathtt{T})\omega(\mathtt{E})\omega(\mathtt{F}) - g(\mathtt{F},\mathtt{B})\omega(\mathtt{E})\omega(\mathtt{T}) + \frac{1}{n-1}\omega(\mathtt{E})\omega(\mathtt{T})S(\mathtt{F},\mathtt{B}) = 0. \quad (5.44)\end{aligned}$$

Contracting 5.44 along $\mathtt{E}$, and $\mathtt{B}$, on evaluation, it provides

$$\mathcal{S}(\mathtt{F},\mathtt{T}) = (n-1)g(\mathtt{F},\mathtt{T}). \quad (5.45)$$

This completes the proof. □

6. Φ-$\mathcal{W}_6$-flat Lorentzian para-Kenmotsu Manifolds

Theorem 6.1. *If an $(LPK)_n$ manifold is Φ-$\mathcal{W}_6$-flat, then it is an Einstein manifold.*

Proof. Let us consider that an $(LPK)_n$ manifold is Φ-$\mathcal{W}_6$-flat. Then,

$$`\mathcal{W}_6(\Phi \mathtt{A}, \Phi \mathtt{B}, \Phi \mathtt{C}, \Phi \mathtt{T}) = 0. \tag{6.46}$$

By definition of $\mathcal{W}_6$ curvature tensor 1.2

$$\mathcal{K}(\Phi \mathtt{A}, \Phi \mathtt{B}, \Phi \mathtt{C}, \Phi \mathtt{T}) + \frac{1}{n-1}[g(\Phi \mathtt{A}, \Phi \mathtt{B})\mathcal{S}(\Phi \mathtt{C}, \Phi \mathtt{T}) - \mathcal{S}(\Phi \mathtt{B}, \Phi \mathtt{C})g(\Phi \mathtt{A}, \Phi \mathtt{T})] = 0. \tag{6.47}$$

By definition of Riemann curvature tensor, we have

$$\mathcal{R}(\mathtt{A}, \mathtt{B})\Phi \mathtt{C} = \nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi \mathtt{C} - \nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\Phi \mathtt{C} - \nabla_{[\mathtt{A},\mathtt{B}]}\Phi \mathtt{C}.$$

Taking innerproduct of the above relation with respect to $\Phi \mathtt{T}$, it gives

$$g(\mathcal{R}(\mathtt{A}, \mathtt{B})\Phi \mathtt{C}, \Phi \mathtt{T}) = g(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi \mathtt{C}, \Phi \mathtt{T}) - g(\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\Phi \mathtt{C}, \Phi \mathtt{T}) - g(\nabla_{[\mathtt{A},\mathtt{B}]}\Phi \mathtt{C}, \Phi \mathtt{T}). \tag{6.48}$$

Evaluation of the term $\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi \mathtt{C}$ provides

$$\begin{aligned}
\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi \mathtt{C} &= -g(\nabla_{\mathtt{A}}\Phi \mathtt{B}, \mathtt{C})\zeta - g(\Phi \mathtt{B}, \nabla_{\mathtt{A}}\mathtt{C})\zeta \\
&+ g(\Phi \mathtt{B}, \mathtt{C})\mathtt{A} + g(\Phi \mathtt{B}, \mathtt{C})\omega(\mathtt{A})\zeta - (\nabla_{\mathtt{A}}\omega)(\mathtt{C})\Phi \mathtt{B} - \omega(\nabla_{\mathtt{A}}\mathtt{C})\Phi \mathtt{B} \\
&+ g(\Phi \mathtt{A}, \mathtt{B})\omega(\mathtt{C})\zeta + \omega(\mathtt{B})\omega(\mathtt{C})\Phi \mathtt{A} - \omega(\mathtt{C})\Phi(\nabla_{\mathtt{A}}\mathtt{B}) \\
&- g(\Phi \mathtt{A}, \nabla_{\mathtt{B}}\mathtt{C})\zeta - \omega(\nabla_{\mathtt{B}}\mathtt{C})\Phi \mathtt{A} + \Phi(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\mathtt{C}).
\end{aligned} \tag{6.49}$$

Taking innerproduct of 6.49 with $\Phi \mathtt{T}$, we have

$$\begin{aligned}
g(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi \mathtt{C}, \Phi \mathtt{T}) &= -g(\nabla_{\mathtt{A}}\Phi \mathtt{B}, \mathtt{C})g(\zeta, \Phi \mathtt{T}) - g(\Phi \mathtt{B}, \nabla_{\mathtt{A}}\mathtt{C})g(\zeta, \Phi \mathtt{T}) \\
&+ g(\Phi \mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi \mathtt{T}) + g(\Phi \mathtt{B}, \mathtt{C})\omega(\mathtt{A})g(\zeta, \Phi \mathtt{T}) - (\nabla_{\mathtt{A}}\omega)(\mathtt{C})g(\Phi \mathtt{B}, \Phi \mathtt{T}) \\
&- \omega(\nabla_{\mathtt{A}}\mathtt{C})g(\Phi \mathtt{B}, \Phi \mathtt{T}) + g(\Phi \mathtt{A}, \mathtt{B})\omega(\mathtt{C})g(\zeta, \Phi \mathtt{T}) + \omega(\mathtt{B})\omega(\mathtt{C})g(\Phi \mathtt{A}, \Phi \mathtt{T}) \\
&- \omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{A}}\mathtt{B}), \Phi \mathtt{T}) - g(\Phi \mathtt{A}, \nabla_{\mathtt{B}}\mathtt{C})g(\zeta, \Phi \mathtt{T}) \\
&- \omega(\nabla_{\mathtt{B}}\mathtt{C})g(\Phi \mathtt{A}, \Phi \mathtt{T}) + g(\Phi(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\mathtt{C}), \Phi \mathtt{T}).
\end{aligned} \tag{6.50}$$

Using 2.3 into 6.50, we have

$$\begin{aligned}
g(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\Phi \mathtt{C}, \Phi \mathtt{T}) &= g(\Phi \mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi \mathtt{T}) - (\nabla_{\mathtt{A}}\omega)(\mathtt{C})g(\Phi \mathtt{B}, \Phi \mathtt{T}) \\
&- \omega(\nabla_{\mathtt{A}}\mathtt{C})g(\Phi \mathtt{B}, \Phi \mathtt{T}) + \omega(\mathtt{B})\omega(\mathtt{C})g(\Phi \mathtt{A}, \Phi \mathtt{T}) - \omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{A}}\mathtt{B}), \Phi \mathtt{T}) \\
&- \omega(\nabla_{\mathtt{B}}\mathtt{C})g(\Phi \mathtt{A}, \Phi \mathtt{T}) + g(\Phi(\nabla_{\mathtt{A}}\nabla_{\mathtt{B}}\mathtt{C}), \Phi \mathtt{T}).
\end{aligned} \tag{6.51}$$

Applying $\mathtt{A} \leftrightarrow \mathtt{B}$ in 6.51, we have

$$g(\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\Phi\mathtt{C}, \Phi\mathtt{T}) = g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) - (\nabla_{\mathtt{B}}\omega)(\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) - \omega(\nabla_{\mathtt{B}}\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) + \omega(\mathtt{A})\omega(\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) - \omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{B}}\mathtt{A}), \Phi\mathtt{T}) - \omega(\nabla_{\mathtt{A}}\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) + g(\Phi(\nabla_{\mathtt{B}}\nabla_{\mathtt{A}}\mathtt{C}), \Phi\mathtt{T}). \tag{6.52}$$

Differentiating covariantly $\Phi\mathtt{C}$ along $[\mathtt{A}, \mathtt{B}]$, we find

$$\nabla_{[\mathtt{A},\mathtt{B}]}(\Phi\mathtt{C}) = -g(\Phi[\mathtt{A}, \mathtt{B}], \mathtt{C})\zeta - \omega(\mathtt{C})[\Phi(\nabla_{\mathtt{A}}\mathtt{B} - \nabla_{\mathtt{B}}\mathtt{A})] + \Phi(\nabla_{[\mathtt{A},\mathtt{B}]}\mathtt{C}). \tag{6.53}$$

Taking innerproduct of 6.53 with $\Phi\mathtt{T}$, we have

$$g(\nabla_{[\mathtt{A},\mathtt{B}]}(\Phi\mathtt{C}), \Phi\mathtt{T}) = -\omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{A}}\mathtt{B}), \Phi\mathtt{T}) + \omega(\mathtt{C})g(\Phi(\nabla_{\mathtt{B}}\mathtt{A}), \Phi\mathtt{T}) + g(\Phi(\nabla_{[\mathtt{A},\mathtt{B}]}\mathtt{C}), \Phi\mathtt{T}). \tag{6.54}$$

Putting values 6.51, 6.52 and 6.54 into relation 6.48, it yields

$$\mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) = g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) + (\nabla_{\mathtt{B}}\omega)(\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) - (\nabla_{\mathtt{A}}\omega)(\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) + \omega(\mathtt{B})\omega(\mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) - \omega(\mathtt{A})\omega(\mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}) + g(\Phi(\mathcal{R}(\mathtt{A}, \mathtt{B})\mathtt{C}, \Phi\mathtt{T}). \tag{6.55}$$

Applying the relation 2.4 into the last term of right hand side of 6.55, and then transposing to left hand side, we have

$$\mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) - \mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \Phi^2\mathtt{T}) = g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) + [(\nabla_{\mathtt{B}}\omega)(\mathtt{C}) + \omega(\mathtt{B})\omega(\mathtt{C})]g(\Phi\mathtt{A}, \Phi\mathtt{T}) - [(\nabla_{\mathtt{A}}\omega)(\mathtt{C}) + \omega(\mathtt{A})\omega(\mathtt{C})]g(\Phi\mathtt{B}, \Phi\mathtt{T}). \tag{6.56}$$

Using 2.3, and 2.5 into 6.56, we have

$$\mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) - \mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{T}) - \omega(\mathtt{T})\mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \zeta) = g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) - g(\mathtt{B}, \mathtt{C})g(\Phi\mathtt{A}, \Phi\mathtt{T}) + g(\mathtt{A}, \mathtt{C})g(\Phi\mathtt{B}, \Phi\mathtt{T}). \tag{6.57}$$

Using 2.3, and 2.6 into 6.57, we have

$$\mathcal{K}(\mathtt{A}, \mathtt{B}, \Phi\mathtt{C}, \Phi\mathtt{T}) - \mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{T}) = g(\Phi\mathtt{B}, \mathtt{C})g(\mathtt{A}, \Phi\mathtt{T}) - g(\Phi\mathtt{A}, \mathtt{C})g(\mathtt{B}, \Phi\mathtt{T}) - g(\mathtt{B}, \mathtt{C})g(\mathtt{A}, \mathtt{T}) + g(\mathtt{A}, \mathtt{C})g(\mathtt{B}, \mathtt{T}). \tag{6.58}$$

By Riemann curvature property, we have

$$\mathcal{K}(\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{T}) = \mathcal{K}(\mathtt{C}, \mathtt{T}, \mathtt{A}, \mathtt{B}). \tag{6.59}$$

Applying $X \leftrightarrow Z$, and $Y \leftrightarrow T$ into 6.58, we have

$$\mathcal{K}(\mathtt{C},\mathtt{T},\Phi\mathtt{A},\Phi\mathtt{B}) - \mathcal{K}(\mathtt{C},\mathtt{T},\mathtt{A},\mathtt{B}) = g(\Phi\mathtt{T},\mathtt{A})g(\mathtt{C},\Phi\mathtt{B}) - g(\Phi\mathtt{C},\mathtt{A})g(\mathtt{T},\Phi\mathtt{B}) - g(\mathtt{T},\mathtt{A})g(\mathtt{C},\mathtt{B}) + g(\mathtt{C},\mathtt{A})g(\mathtt{T},\mathtt{B}). \quad (6.60)$$

Subtracting 6.60 from 6.58, and using 6.59, we have

$$\mathcal{K}(\mathtt{A},\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T}) = \mathcal{K}(\mathtt{C},\mathtt{T},\Phi\mathtt{A},\Phi\mathtt{B}). \quad (6.61)$$

Applying $\mathtt{A} \to \Phi\mathtt{A}$, and $\mathtt{B} \to \Phi\mathtt{B}$ into 6.61, we have

$$\mathcal{K}(\Phi\mathtt{A},\Phi\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T}) = \mathcal{K}(\mathtt{C},\mathtt{T},\Phi^2\mathtt{A},\Phi^2\mathtt{B}). \quad (6.62)$$

Applying 2.3, and 6.59 into 6.62, on simplification, we have

$$\mathcal{K}(\Phi\mathtt{A},\Phi\mathtt{B},\Phi\mathtt{C},\Phi\mathtt{T}) = \mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) + g(\mathtt{A},\mathtt{T})\omega(\mathtt{B})\omega(\mathtt{C}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})\omega(\mathtt{T}) + g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{T}) - g(\mathtt{B},\mathtt{T})\omega(\mathtt{A})\omega(\mathtt{C}). \quad (6.63)$$

Putting value from relation 6.63 into 6.47, we have

$$\mathcal{K}(\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{T}) - g(\mathtt{A},\mathtt{C})\omega(\mathtt{B})\omega(\mathtt{T}) + g(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{T}) - g(\mathtt{B},\mathtt{T})\omega(\mathtt{A})\omega(\mathtt{C}) + \frac{1}{n-1}[\mathcal{S}(\mathtt{C},\mathtt{T})g(\mathtt{A},\mathtt{B}) + (n-1)g(\mathtt{A},\mathtt{B})\omega(\mathtt{C})\omega(\mathtt{T}) + \mathcal{S}(\mathtt{C},\mathtt{T})\omega(\mathtt{A})\omega(\mathtt{B}) - \mathcal{S}(\mathtt{B},\mathtt{C})g(\mathtt{A},\mathtt{T}) - \mathcal{S}(\mathtt{B},\mathtt{C})\omega(\mathtt{A})\omega(\mathtt{T})] = 0. \quad (6.64)$$

Contracting 6.64 with respect to $\mathtt{A}$, and $\mathtt{T}$, we have

$$\mathcal{S}(\mathtt{B},\mathtt{C}) - \omega(\mathtt{B})\omega(\mathtt{C}) - g(\mathtt{B},\mathtt{C}) - \omega(\mathtt{B})\omega(\mathtt{C}) + \frac{1}{n-1}[\mathcal{S}(\mathtt{B},\mathtt{C}) + (n-1)\omega(\mathtt{C})\omega(\mathtt{B}) + \mathcal{S}(\mathtt{C},\zeta)\omega(\mathtt{B}) - n\mathcal{S}(\mathtt{B},\mathtt{C}) + \mathcal{S}(\mathtt{B},\mathtt{C})] = 0. \quad (6.65)$$

On simplification of 6.65, it concludes

$$\mathcal{S}(\mathtt{B},\mathtt{C}) = (n-1)g(\mathtt{B},\mathtt{C}). \quad (6.66)$$

This completes the proof. □

7. Lorentzian para-Kenmotsu manifolds with condition $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$

In this part, we explore the behavior of $(LPK)_n$ manifold admitting $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$. We begin this with the subsequent theorem:

Theorem 7.1. *An $(LPK)_n$ manifold is an ω-Einstein manifold if, it satisfies the relation $\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0$.*

Proof. Let us consider that the $(LPK)_n$ manifold admits the condition

$$\mathcal{W}_6(\mathtt{E},\mathtt{F}).\mathcal{R} = 0. \tag{7.67}$$

From the relation 7.67, we have

$$\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathcal{R}(\mathtt{A},\mathtt{B})\mathtt{C} - \mathcal{R}(\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{A},\mathtt{B})\mathtt{C} - \mathcal{R}(\mathtt{A},\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{B})\mathtt{C} - \mathcal{R}(\mathtt{A},\mathtt{B})\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{C} = 0. \tag{7.68}$$

Putting $\mathtt{B} = \zeta$ into the relation 7.68, we have

$$\mathcal{W}_6(\mathtt{E},\mathtt{F})(\mathcal{R}(\mathtt{A},\zeta)\mathtt{C}) - \mathcal{R}(\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{A},\zeta)\mathtt{C} - \mathcal{R}(\mathtt{A},\mathcal{W}_6(\mathtt{E},\mathtt{F})\zeta)\mathtt{C} - \mathcal{R}(\mathtt{A},\zeta)\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{C} = 0. \tag{7.69}$$

Evaluation of the terms of the relation 7.69 in the subsequent manner:
Using 2.7, 2.9, 1.1, 2.13 into first term of 7.69, we get

$$\begin{aligned}\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathcal{R}(\mathtt{A},\zeta)\mathtt{C} = {} & \omega(\mathtt{C})\mathcal{R}(\mathtt{E},\mathtt{F})\mathtt{A} \\ & + \frac{1}{n-1}g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})Q\mathtt{A} - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})\omega(\mathtt{C})\mathtt{E} - g(\mathtt{A},\mathtt{C})\omega(\mathtt{F})\mathtt{E} \\ & + g(\mathtt{A},\mathtt{C})\omega(\mathtt{E})\mathtt{F} - g(\mathtt{A},\mathtt{C})g(\mathtt{E},\mathtt{F})\zeta + g(\mathtt{A},\mathtt{C})\omega(\mathtt{F})\mathtt{E}.\end{aligned} \tag{7.70}$$

Using 2.7, 2.9, and 1.1, into second term of 7.69, we get

$$\begin{aligned}\mathcal{R}(\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{A},\zeta)\mathtt{C} = {} & \omega(\mathtt{C})\mathcal{R}(\mathtt{E},\mathtt{F})\mathtt{A} \\ & + \frac{1}{n-1}g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})Q\mathtt{A} - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})\omega(\mathtt{C})\mathtt{E} - \mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{A},\mathtt{C})\zeta \\ & - \frac{1}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{A},\mathtt{C})\zeta + \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})g(\mathtt{E},\mathtt{C})\zeta.\end{aligned} \tag{7.71}$$

Applying relations 2.7, and 2.13 into third term of 7.69, we get

$$\mathcal{R}(\mathtt{A},\mathcal{W}_6(\mathtt{E},\mathtt{F})\zeta)\mathtt{C} = g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})\mathtt{A} - g(\mathtt{E},\mathtt{F})g(\mathtt{A},\mathtt{C})\zeta - \omega(\mathtt{E})\mathcal{R}(\mathtt{A},\mathtt{F})\mathtt{C}. \tag{7.72}$$

Using 2.7, 2.9, 1.1 and 2.13 into fourth term of 7.69, we get

$$\begin{aligned}\mathcal{R}(\mathtt{A},\zeta)\mathcal{W}_6(\mathtt{E},\mathtt{F})\mathtt{C} &= g(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} - g(\mathtt{E},\mathtt{C})\omega(\mathtt{F})\mathtt{A} \\ &+ g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})\mathtt{A} - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} - \mathcal{K}(\mathtt{E},\mathtt{F},\mathtt{C},\mathtt{A})\zeta \\ &- \frac{1}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{A},\mathtt{C})\zeta + \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})g(\mathtt{E},\mathtt{A})\zeta. \end{aligned} \tag{7.73}$$

Putting the values from 7.70, 7.71, 7.72, and 7.73 into 7.69, it gives

$$\begin{aligned} g(\mathtt{A},\mathtt{C})\omega(\mathtt{E})\mathtt{F} &+ \frac{2}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\mathtt{A},\mathtt{C})\zeta - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{A})g(\mathtt{E},\mathtt{C})\zeta \\ &- 2g(\mathtt{E},\mathtt{F})\omega(\mathtt{C})\mathtt{A} + \omega(\mathtt{E})R(\mathtt{A},\mathtt{F})\mathtt{C} - g(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} + g(\mathtt{E},\mathtt{C})\omega(\mathtt{F})\mathtt{A} \\ &+ \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E})\mathtt{A} - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})g(\mathtt{E},\mathtt{A})\zeta = 0. \end{aligned} \tag{7.74}$$

Contracting 7.74 along $\mathtt{A}$, we have

$$\begin{aligned} g(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) &+ \frac{2}{n-1}g(\mathtt{E},\mathtt{F})\mathcal{S}(\zeta,\mathtt{C}) - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\zeta)g(\mathtt{E},\mathtt{C}) - 2ng(\mathtt{E},\mathtt{F})\omega(\mathtt{C}) + \omega(\mathtt{E})\mathcal{S}(\mathtt{F},\mathtt{C}) \\ &- ng(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) + ng(\mathtt{E},\mathtt{C})\omega(\mathtt{F}) + \frac{n}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) - \frac{1}{n-1}\mathcal{S}(\mathtt{F},\mathtt{C})\omega(\mathtt{E}) = 0. \end{aligned} \tag{7.75}$$

Putting $\mathtt{E} = \zeta$ and making use of 2.9 into 7.75, it provides

$$\mathcal{S}(\mathtt{F},\mathtt{C}) = \frac{(n-1)}{2}g(\mathtt{F},\mathtt{C}) - \frac{(n-1)}{2}\omega(\mathtt{E})\omega(\mathtt{F}). \tag{7.76}$$

This completes the proof. □

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References

[1] Ahmad, M. (2019). On semi-invariant $\zeta^{\perp}$ submanifolds of Lorentzian para-Sasakian Manifolds.International journal of maps in mathematics, 2(1), 89-98.

[2] Atceken, M. (2022). Some results on invariant submanifolds of Lorentzian para-Kenmotsu manifolds. Korean J. Math., 30(1), 175-185.

[3] Haseeb, A., & Prasad, R. (2021). Certain results on Lorentzian para-Kenmotsu manifolds. Bull. Parana.s Math. Soc., 39(3), 201-220.

[4] O'Neill, B. (1983). Semi-Riemannian Geometry with Applications to Relativity. Pure and Applied Mathematics, Vol. 103 (Academic Press, New York).

[5] Pandey, S., Singh, A., & Bahadur, O. (2022). Certain results of Ricci solitons on Three dimensional Lorentzian para-α-Sasakian Manifolds. International journal of maps in mathematics, 5(2), 139-153.

[6] Pokhariyal, G.P.(1982). Study of new curvature tensor in Sasakian manifold. Tensor, N.S., 36 , 222-226.

[7] Pokhariyal, G.P. (1982). Relativistic Significance of curvature tensors. International Journal of Mathematics and Mathematical Sciences, 5(1), 133-139.

[8] Prasad, R., Haseeb, A., Verma, A., & Yadav, V. S. (2024). A study of φ-Ricci symmetric LP-Kenmotsu manifolds. International Journal of Maps in Mathematics, Volume 7, Issue 1, Pages: 33-44.

[9] Prasad, R., Verma A., & Yadav, V. S. (2023). Characterization of the perfect fluid Lorentzian α-para Kenmotsu spacetimes. GANITA, Vol. 73(2), 89-104.

[10] Prasad, R., Verma A., & Yadav, V. S. (2023). Characterization of Φ-symmetric Lorentzian para-Kenmotsu manifolds. FACTA UNIVERSITATIS (NIS) SER. MATH. INFORM. Vol. 38, No 3 635-647 https://doi.org/10.22190/FUMI230314040P

[11] Sari, R., & Vanli, A. T. (2019). Slant submanifolds of a Lorentz Kenmotsu manifold. Mediterr. J. Math., 16, 1-17.

[12] Sharma, R. (2008). Certain results on k-contact and (κ, μ)-contact manifolds. J. Geom., 89, 138-147.

[13] Singh, A., Ahmad, M., Yadav, S.K., & Patel, S. (2024). Some Results on β-Kenmotsu manifolds with a Non-symmetric Non metric connection. International journal of maps in mathematics, 7(1), 20-32.

[14] Sinha, B. B., & Sai Prasad, K. l. (1995). A class of almost para contact metric manifold. Bull. Calcutta Math. Soc., 87, 307-312.

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ON RULED SURFACES BY SMARANDACHE GEOMETRY IN E^3

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Abstract. The paper introduces a series of new ruled surfaces by following the idea of Smarandache geometry according to Frenet frame by taking into account all the possible linear combinations of the frame vectors. The metric properties of each defined ruled surface is examined by computing the 1^{st} and 2^{nd} fundamental forms as well as the curvatures of Gaussian and the mean expressed by the harmonic curvature function. Therefore, the conditions for each surface to be minimal or developable are provided. Moreover, the constraints for the characteristics of the base curve are discussed whether it is geodesic, asymptotic or a curvature line on the generated ruled surface. Finally, the graphical illustrations are presented for each ruled surface with a given appropriate example.

Keywords: Smarandache geometry, Ruled surfaces, Fundamental forms, Principal curvatures, Developable and minimal surfaces, Geodesic, Asymptotic and curvature lines.

2020 Mathematics Subject Classification: 53A04, 53A05.

1. Introduction

Ruled surfaces are widely recognized as the most fundamental and extensively employed objects in the geometric modeling. Researchers utilize this type of surface in various grounds such as computer graphics, architecture, arts, sculpture, manufacturing, etc. The basic definition of a ruled surface is the image of lines' motion on and along a given curve. Therefore,

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a ruled surface is also called the surface of infinitely many lines. Interested readers can refer the main sources [1, 4, 19] to gain a deep insight into about ruled surfaces. Following the importance of this surface kind, researchers defined new ruled surfaces and examined their characterizations. For example, the ruled surfaces were re-visited by means of geodesic curvature and the 2^{nd} fundamental form by [14]. [12] conducted a study on the characteristics of ruled surfaces along the striction curves of a non-cylindrical ruled surface according to Frenet frame. The ruled surfaces according to Bishop frame and their characteristics was examined by [20] and [8]. Further, [13] studied the characteristics for the ruled surfaces with respect to the alternative frame. Moreover, the ruled surfaces generated by rotation minimizing frame (RMF) are investigated in [6] while those by Sannia frame are defined in [5].

Recently, [11] put forth a new way of generating ruled surfaces by taking the advantage of the idea of Smarandache curves which was defined by [21] for Minkowski space, and by [2] for Euclidean space E^3. The method relies on assigning one of the Smarandache curves as a base curve of the surface and utilizes other vector elements of the Frenet frame as the generator line. Thus, she named these newly constructed ruled surfaces as Smarandache ruled surfaces. However, prior to her study, [24] had also discussed the idea of constructing such ruled surfaces. They specifically worked on geodesic conditions of the tangent and normal surfaces with TN-Smarandache curve as a base curve [23]. In addition, the authors examined the geodesics of the binormal surface in [25]. By considering different frames such as alternative, Darboux, Flc (by [3]) and successor frame, Smarandache ruled surfaces were re-defined in [9], [10], [15] and [22], respectively. Moreover, this way of generating such ruled surfaces were benefited in [16, 17, 18] from different point of view most likely by incorporating the Darboux vector. Therefore, in this paper, with the motivation of the given studies, we extend our investigations for the studies in which Smarandache ruled surfaces according to Frenet frame were used. That is we consider all possible combinations of Frenet vectors to construct new ruled surfaces and examine their main characteristics in a more broader perspective.

2. Preliminaries

This section is to recall the primary concepts which we will be using through out the paper. Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve in three dimensional Euclidean space E^3 and denote $\{T, N, B, \kappa, \tau\}$ as its Frenet elements. Then, the definitions of Frenet vectors and the well-known Frenet formulas are given as

$$\begin{aligned} &T=\frac{\gamma'}{\|\gamma'\|},\quad N=B\times T,\quad B=\frac{\gamma'\times\gamma''}{\|\gamma'\times\gamma''\|},\\ &\kappa=\frac{\|\gamma'\times\gamma''\|}{\|\gamma'\|^3},\qquad \tau=\frac{\langle\gamma'\times\gamma'',\gamma'''\rangle}{\|\gamma'\times\gamma''\|^2},\\ &T'=\kappa\nu N,\quad N'=-\kappa\nu T+\tau\nu B,\quad B'=-\tau\nu N, \end{aligned} \tag{2.1}$$

where $\nu = \|\gamma'\|$, and $\langle\ ,\ \rangle$, $\times$ and $\|\ \|$ denote standard inner product, vector product and norm, respectively. κ is the curvature and τ is the torsion of the curve $\gamma(s)$ [1, 4]. Further, a surface χ is ruled if it is formed with the motion of line $X(s)$ on and along a given curve $\gamma(s)$. Thus, a parametrization for a ruled surface is given as follows

$$\chi:\ \psi(s,v)=\gamma(s)+vX(s). \tag{2.2}$$

Here the curve $\gamma(s)$ is called as the base while $X(s)$ is the ruling. The normal vector field for the surface χ is computed by

$$\vec{n}_\psi=\frac{\psi_s\times\psi_v}{\|\psi_s\times\psi_v\|}. \tag{2.3}$$

The 1^{st} and 2^{nd} fundamental forms and the curvatures of Gaussian and mean for the given ruled surface χ are given by

$$I=Eds^2+2Fdsdv+dv^2,\qquad II=Lds^2+2Mdsdv, \tag{2.4}$$

$$K=\frac{-M^2}{E-F^2},\qquad H=\frac{L-2FM}{2(E-F^2)}, \tag{2.5}$$

where the given coefficients are obtained by the following expressions

$$\begin{aligned} &E=\langle\psi_s,\psi_s\rangle\,,\qquad F=\langle\psi_s,\psi_v\rangle\,,\qquad G=\langle\psi_v,\psi_v\rangle\,,\\ &L=\langle\psi_{ss},\vec{n}_\psi\rangle\,,\quad M=\langle\psi_{sv},\vec{n}_\psi\rangle\,,\quad N=\langle\psi_{vv},\vec{n}_\psi\rangle\,. \end{aligned} \tag{2.6}$$

Note that for a ruled surfaces having the ruling of a unit vector, it is always valid that $G=1$ and $N=0$. Therefore, the fundamental forms and the curvatures are expressed in their simplified forms as in the Equation (2.4) and (2.5). Regarding to the given invariants of the surface χ, there exist following definitions:

Definition 2.1. *[1, 4, 19] The surface χ is developable (resp. minimal) if Gaussian (resp. mean) curvature vanishes that is $K=0$ (resp. $H=0$) .*

Moreover, the normal curvature κ_n, the geodesic curvature κ_g and the geodesic torsion τ_g of the surface χ are defined as follows

$$\kappa_n=\langle\gamma'',n_\psi\rangle,\qquad \kappa_g=\langle n_\psi\times T,T'\rangle,\qquad \tau_g=\langle n_\psi\times n'_\psi,T'\rangle. \tag{2.7}$$

There also exist following definitions for given expressions above

Definition 2.2. *[1, 4, 19]*

- $\gamma(s)$ *is asymptotic on the surface* $\chi \iff \kappa_n = 0,$
- $\gamma(s)$ *is geodesic on the surface* $\chi \iff \kappa_g = 0,$
- $\gamma(s)$ *is principal line on the surface* $\chi \iff \tau_g = 0.$

Additionally, we recall the following theorems considered to the specific cases of which the curve $\gamma(s)$ is general or slant helix.

Theorem 2.1. *(Lancret's theorem) A curve is called a general helix* $\iff h = const.$ *, where* $h = \dfrac{\tau}{\kappa}$ *is the harmonic curvature function [19, 26].*

Theorem 2.2. *A curve is called a slant helix* $\iff \sigma = const.$ *where* $\sigma = \dfrac{h'}{\kappa\,(1+h^2)^{\frac{3}{2}}}$, *[7].*

3. The Smarandache Based Ruled Surfaces according to Frenet Frame in E^3

In this section of the paper, it is of interest for us to extend the study of [11] and examine ruled surfaces formed by other possible combinations of Frenet vectors. In addition, the conditions for the base curve to be asymptotic, geodesic and curvature line on the constructed surface are provided, as well. Some special cases are discussed regarding to that the main curve is a general or a slant helix. Thus, twelve of new ruled surfaces all formed by the vectors of Frenet frame are studied by following the idea of Smarandache geometry.

3.1. **Ruled Surfaces with the Base TN− Smarandache Curve.**

Definition 3.1. *Let* $\gamma : s \in I \subset \Re \to E^3$ *be a regular unit speed curve of* C^2 *class and* $\{T,\ N,\ B\}$ *denotes the set of its Frenet vectors. The original definition of* $TN-$ *Smarandache ruled surface introduced in [11] is given as following*

$$ {}^{TN}_{B}\psi(s,v) = \frac{T(s)+N(s)}{\sqrt{2}} + vB(s). \tag{3.8}$$

However, in this study, other two ruled surfaces having the same $TN-$ *curve as a base curve with other Frenet vector as a ruling are considered, which are parameterized in the following*

$$\begin{aligned} {}^{TN}_{T}\psi(s,v) =& \frac{T(s)+N(s)}{\sqrt{2}} + vT(s), \\ {}^{TN}_{N}\psi(s,v) =& \frac{T(s)+N(s)}{\sqrt{2}} + vN(s). \end{aligned} \tag{3.9}$$

In addition, we also examine the conditions for the base curve to be asymptotic, geodesic and curvature line on each ruled surface.

3.1.1. *The characteristics of the ruled surface* ${}^{TN}_{B}\psi(s,v)$.

[11] put forth the following corollaries for the surface ${}^{TN}_{B}\psi(s,v)$

Corollary 3.1. *[11]*

- ${}^{TN}_{B}\psi(s,v)$ *is developable when* $\gamma(s)$ *is a plane curve,*
- ${}^{TN}_{B}\psi(s,v)$ *is minimal if the following relation holds* $\kappa\tau^2\left(1-2v^2\right)+\left(\kappa'+2\kappa^2\right)\left(\sqrt{2}v\tau-\kappa\right)=0.$

Remark 3.1. *We note that in order for* ${}^{TN}_{B}\psi(s,v)$ *be minimal, the following relation should hold*

$$\kappa\tau^2\left(1-2v^2\right)+\sqrt{2}\tau v\left(2\kappa^2+\kappa'\right)-\kappa\left(\sqrt{2}\tau' v+2\kappa^2\right)=0.$$

Now, let us consider the characteristic of the base curve on the ${}^{TN}_{B}\psi(s,v)$, and examine the conditions for it be asymptotic, geodesic and curvature line by associating to $\gamma(s)$.

Theorem 3.1. *The normal curvature* ${}^{TN}_{B}\kappa_n$*, the geodesic curvature* ${}^{TN}_{B}\kappa_g$ *and the geodesic torsion* ${}^{TN}_{B}\tau_g$ *of the* ${}^{TN}_{B}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}^{TN}_{B}\kappa_n =&\frac{\sqrt{2}hv\left(\kappa^2+\kappa'\right)-\tau^2-2\kappa^2}{\sqrt{2}\sqrt{\left(\sqrt{2}hv-1\right)^2+1}},\\
{}^{TN}_{B}\kappa_g =&\frac{\sqrt{2}h'\left(\sqrt{2}-hv\right)-\tau\left(h^2+2\right)\left(\sqrt{2}hv-1\right)}{\left(h^2+2\right)\sqrt{\left(\sqrt{2}hv-1\right)^2+1}},\\
{}^{TN}_{B}\tau_g =&\frac{2vh'\left(h'\sqrt{2}+2\tau hv\right)+\left(\left(\sqrt{2}hv-1\right)\left(h^2+2\right)\tau\kappa-2h'\kappa\right)\left(\sqrt{2}hv-h^2-2\right)}{\left(\left(\sqrt{2}hv-1\right)^2+1\right)\left(h^2+2\right)^{\frac{3}{2}}},
\end{aligned} \tag{3.10}$$

respectively.

Proof. By referring the Equation (2.1), the tangent T_{TN} of $TN-$ Smarandache curve, its derivative and the second order derivative of $TN-$ Smarandache curve are given as

$$\begin{aligned}
T_{TN} =&\frac{-T+N+hB}{\sqrt{h^2+2}},\\
T'_{TN} =&\frac{\left(h'h-\tau h-2\kappa\right)T-\left(h'h+2\kappa+\left(h^2+3\right)\tau h\right)N+\left(2h'+\tau\left(h^2+2\right)\right)B}{\left(h^2+2\right)^{\frac{3}{2}}},\\
\left(\frac{T+N}{\sqrt{2}}\right)'' =&-\frac{\left(\kappa'+\kappa^2\right)T-\left(\kappa'-\kappa^2-\tau^2\right)N-\left(\tau'+\kappa\tau\right)B}{\sqrt{2}}.
\end{aligned} \tag{3.11}$$

Moreover, the derivative of the normal vector of ${}^{TN}_{B}\phi(s,v)$ defined as ${}^{TN}_{B}n=\dfrac{\left(\kappa-v\tau\sqrt{2}\right)T+\kappa N}{\sqrt{2}\sqrt{\kappa^2-\sqrt{2}\kappa v\tau+v^2\tau^2}}$ in [11], but expressed by the harmonic curvature function as ${}^{TN}_{B}n = -\dfrac{\left(\sqrt{2}hv-1\right)T-N}{\sqrt{\left(\sqrt{2}hv-1\right)^2+1}}$ is given in the following

$$
\begin{aligned}
{}^{TN}_{B}n' = &-\left(\frac{h'v+v\tau\left(\sqrt{2}hv-2\right)+\sqrt{2}\kappa}{2\left(h^2v^2-\sqrt{2}hv+1\right)^{\frac{3}{2}}}\right)T\\
&-\left(\frac{h'v\left(\sqrt{2}hv-1\right)}{2\left(h^2v^2-\sqrt{2}hv+1\right)^{\frac{3}{2}}}+\frac{\kappa\left(2hv-\sqrt{2}\right)}{2\sqrt{h^2v^2-\sqrt{2}hv+1}}\right)N\\
&+\left(\frac{\tau\sqrt{2}}{2\sqrt{h^2v^2-\sqrt{2}hv+1}}\right)B.
\end{aligned}
$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.10 of Theorem 3.1, we have the following corollary:

Corollary 3.2.

- *The $TN-$ Smarandache curve of $\gamma(s)$ cannot be asymptotic on ${}^{TN}_{B}\psi(s,v)$.*
- *If $\gamma(s)$ is plane curve, then its corresponding $TN-$ Smarandache curve lies both as geodesic and curvature line on the ruled surface ${}^{TN}_{B}\psi(s,v)$, while the normal curvature simplifies to ${}^{TN}_{B}\kappa_n = -\kappa^2$.*

3.1.2. *The characteristics of the ruled surface ${}^{TN}_{T}\psi(s,v)$.*

Theorem 3.2. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}^{TN}_{T}\psi(s,v)$ are given as following:*

$$
\begin{aligned}
{}^{TN}_{T}I &= \left(\frac{\kappa^2+\tau^2}{2}+\kappa^2\left(v+\frac{1}{\sqrt{2}}\right)^2\right)ds^2-\kappa\sqrt{2}\,dsdv+dv^2,\\
{}^{TN}_{T}II &= \frac{\left(-2\tau\kappa^2\left(v^2+\sqrt{2}v+1\right)+\left(\sqrt{2}v+1\right)\left(\kappa'\tau-\tau'\kappa\right)-\tau^3\right)ds^2+2\kappa\tau\sqrt{2}dsdv}{\sqrt{2}\sqrt{\left(\kappa\left(\sqrt{2}v+1\right)\right)^2+\tau^2}},\\
{}^{TN}_{T}K &= -2\left(\frac{\kappa\tau}{\left(\kappa\left(\sqrt{2}v+1\right)\right)^2+\tau^2}\right)^2,\\
{}^{TN}_{T}H &= \frac{-2\tau\kappa^2v\left(\sqrt{2}+v\right)+\left(\sqrt{2}v+1\right)\left(\kappa'\tau-\tau'\kappa\right)-\tau^3}{\sqrt{2}\left(\left(\kappa\left(\sqrt{2}v+1\right)\right)^2+\tau^2\right)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}_{T}^{TN}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{T}^{TN}\psi(s,v)_s &= \left(\frac{-\kappa}{\sqrt{2}}\right)T + \kappa\left(v + \frac{1}{\sqrt{2}}\right)N + \left(\frac{\tau}{\sqrt{2}}\right)B,\\
{}_{T}^{TN}\psi(s,v)_{ss} &= -\left(\frac{\kappa'}{\sqrt{2}} + \kappa^2\left(v + \frac{1}{\sqrt{2}}\right)\right)T + \left(-\frac{1}{\sqrt{2}}(\kappa^2 + \tau^2 - \kappa') + v\kappa'\right)N\\
&\quad + \left(\frac{1}{\sqrt{2}}\tau' + \left(v + \frac{1}{\sqrt{2}}\right)\kappa\tau\right)B,\\
{}_{T}^{TN}\psi(s,v)_v &= T, \qquad {}_{T}^{TN}\psi(s,v)_{sv} = \kappa N, \qquad {}_{T}^{TN}\psi(s,v)_{vv} = 0.
\end{aligned}
$$

From Equation 2.3, the normal of ${}_{T}^{TN}\psi(s,v)$ is given as

$$
{}_{T}^{TN}n = \frac{\tau N - \kappa\left(\sqrt{2}v + 1\right)B}{\sqrt{\left(\kappa\left(\tau^2 + \sqrt{2}v + 1\right)\right)^2}}.
$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

By using the Definition 2.1 and the Theorem 3.2, the two corollaries given below are valid without the need for proof.

Corollary 3.3.

- ${}_{T}^{TN}\psi(s,v)$ *is both developable and minimal when* $\gamma(s)$ *is a plane curve.*
- ${}_{T}^{TN}\psi(s,v)$ *is minimal if the following relation holds*

$$
-2\tau\kappa^2 v\left(\sqrt{2} + v\right) + \left(\sqrt{2}v + 1\right)\left(\kappa'\tau - \tau'\kappa\right) - \tau^3 = 0.
$$

Theorem 3.3. *The normal curvature* ${}_{T}^{TN}\kappa_n$*, the geodesic curvature* ${}_{T}^{TN}\kappa_g$ *and the geodesic torsion* ${}_{T}^{TN}\tau_g$ *of the* ${}_{T}^{TN}\psi(s,v)$ *surface are given as follows*

$$
\begin{aligned}
{}_{T}^{TN}\kappa_n &= -\frac{\kappa\tau\left(h^2 + \sqrt{2}v + 2\right) + h'\kappa\left(\sqrt{2}v + 1\right) + \sqrt{2}hv\kappa'}{\sqrt{2}\sqrt{h^2 + \left(\sqrt{2}v + 1\right)^2}},\\
{}_{T}^{TN}\kappa_g &= \frac{hh' - (\tau h + 2\kappa)\left(\sqrt{2}v + 1\right)}{(h^2 + 2)\sqrt{h^2 + \left(\sqrt{2}v + 1\right)^2}},\\
{}_{T}^{TN}\tau_g &= \frac{\left(h(h')^2 - 2h'\kappa - \left(h^2 + \sqrt{2}v + 2\right)\left(h^2 + 2\right)\tau\kappa\right)\left(\sqrt{2}v + 1\right) + \left(h^2 + 2v^2 + 1\right)h'\tau h}{\left(h^2 + \left(\sqrt{2}v + 1\right)^2\right)\left(h^2 + 2\right)^{\frac{3}{2}}},
\end{aligned}
\tag{3.12}
$$

respectively.

Proof. Recall the Equation 3.11, since the base curve is still the same $TN-$ Smarandache curve. Moreover, the derivative of the normal of ${}_{T}^{TN}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}_{T}^{TN}n = \dfrac{hN - \left(\sqrt{2}v+1\right)B}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}$ is given in the following

$$
\begin{aligned}
{}_{T}^{TN}n' = & \left(-\frac{\tau}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}\right)T \\
& + \left(\frac{h'\left(2v^2+2\sqrt{2}v+1\right)}{\left(h^2+\left(\sqrt{2}v+1\right)^2\right)^{\frac{3}{2}}} + \frac{\tau\left(\sqrt{2}v+1\right)}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}\right)N \\
& + \left(\frac{hh'\left(\sqrt{2}v+1\right)}{\left(h^2+\left(\sqrt{2}v+1\right)^2\right)^{\frac{3}{2}}} + \frac{h\tau}{\sqrt{h^2+\left(\sqrt{2}v+1\right)^2}}\right)B.
\end{aligned}
$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.12 of Theorem 3.3, we have the following corollary:

Corollary 3.4.

- *The $TN-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}_{T}^{TN}\psi(s,v)$.*
- *If $\gamma(s)$ is a plane curve, then its corresponding $TN-$ Smarandache curve lies both as asymptotic and curvature line on ${}_{T}^{TN}\psi(s,v)$, while the geodesic curvature is expressed by ${}_{T}^{TN}\kappa_g = -\kappa$.*

3.1.3. *The characteristics of the ruled surface ${}_{N}^{TN}\psi(s,v)$.*

Theorem 3.4. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}_{N}^{TN}\psi(s,v)$ are given as following:*

$$
\begin{aligned}
{}_{N}^{TN}I &= \left(\left(\kappa^2+\tau^2\right)\left(v^2+v\sqrt{2}+1\right) - \frac{\tau^2}{2}\right)ds^2 + \kappa\sqrt{2}dsdv + dv^2, \\
{}_{N}^{TN}II &= \left(\frac{\left(\kappa'\tau-\kappa\tau'\right)\left|\sqrt{2}v+1\right|}{\sqrt{2}\sqrt{\kappa^2+\tau^2}}\right)ds^2, \\
{}_{N}^{TN}K &= 0, \\
{}_{N}^{TN}H &= \frac{\left(\kappa'\tau-\kappa\tau'\right)}{\left|2v+\sqrt{2}\right|\left(\kappa^2+\tau^2\right)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}^{TN}_{N}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}^{TN}_{N}\psi(s,v)_s =& -\left(v+\frac{1}{\sqrt{2}}\right)\kappa T + \frac{1}{\sqrt{2}}\kappa N + \left(v+\frac{1}{\sqrt{2}}\right)\tau B,\\
{}^{TN}_{N}\psi(s,v)_{ss} =& -\left(\frac{1}{\sqrt{2}}\kappa^2 + \left(v+\frac{1}{\sqrt{2}}\right)\kappa'\right)T\\
&-\left(\left(v+\frac{1}{\sqrt{2}}\right)\tau^2 - \frac{1}{\sqrt{2}}\kappa' + \left(v+\frac{1}{\sqrt{2}}\right)\kappa^2\right)N\\
&+\left(\left(v+\frac{1}{\sqrt{2}}\right)\tau' + \frac{1}{\sqrt{2}}\kappa\tau\right)B,\\
{}^{TN}_{N}\psi(s,v)_v =& N, \qquad {}^{TN}_{N}\psi(s,v)_{sv} = -\kappa T + \tau B, \qquad {}^{TN}_{N}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal of ${}^{TN}_{N}\psi(s,v)$ is given as

$${}^{TN}_{N}n = -\epsilon_1 \frac{\tau T + \kappa B}{\sqrt{\kappa^2+\tau^2}},$$

where $\epsilon_1 = sign(1+\sqrt{2}v)$. By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

By using the Definition 2.1 and the Theorem 3.4, the following corollaries are clear without the need for proof.

Corollary 3.5.

- *The surface ${}^{TN}_{N}\psi(s,v)$ is always developable.*
- *${}^{TN}_{N}\psi(s,v)$ is minimal when $\gamma(s)$ is a general helix.*

Theorem 3.5. *The normal curvature ${}^{TN}_{N}\kappa_n$, the geodesic curvature ${}^{TN}_{N}\kappa_g$ and the geodesic torsion ${}^{TN}_{N}\tau_g$ of the ${}^{TN}_{N}\psi(s,v)$ surface are given as follows*

$$\begin{aligned}
{}^{TN}_{N}\kappa_n =& -\epsilon_1 \frac{h'\kappa}{\sqrt{2}\sqrt{h^2+1}},\\
{}^{TN}_{N}\kappa_g =& -\epsilon_1 \frac{\tau h\left(h^2+3\right) + h'h + 2\kappa}{\left(h^2+2\right)\sqrt{h^2+1}},\\
{}^{TN}_{N}\tau_g =& -\frac{h'\left(\tau h\left(h^2+3\right) + h'h + 2\kappa\right)}{\left(h^2+2\right)^{\frac{3}{2}}\left(h^2+1\right)},
\end{aligned} \tag{3.13}$$

respectively.

Proof. Recall again the Equation 3.11. Moreover, the derivative of the normal of ${}^{TN}_{N}\psi(s,v)$ expressed by the harmonic curvature function as ${}^{TN}_{N}n = -\epsilon_1 \dfrac{hT+B}{\sqrt{h^2+1}}$ is given in the following

$$ {}^{TN}_{N}n' = -\epsilon_1 \sigma \kappa \left(T - hB\right). $$

By substituting the relations given above into Equation 2.7, the proof is completed. □

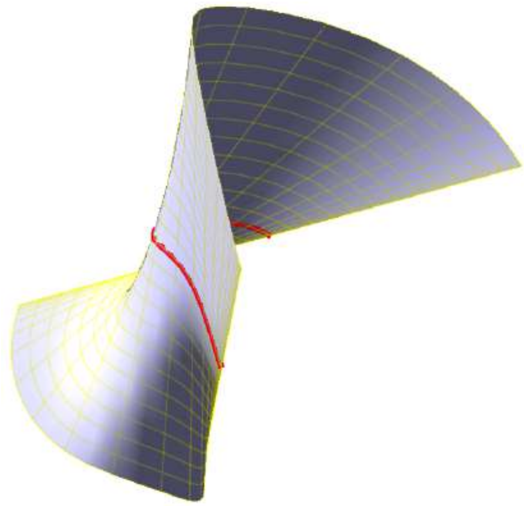
Without the need for proof, from Definition 2.2 and the Equation 3.13 of Theorem 3.5, we have the following remark:

Remark 3.2. *The given corollaries for* ${}^{TN}_{T}\psi(s,v)$ *are the same as of* ${}^{TN}_{N}\psi(s,v)$.

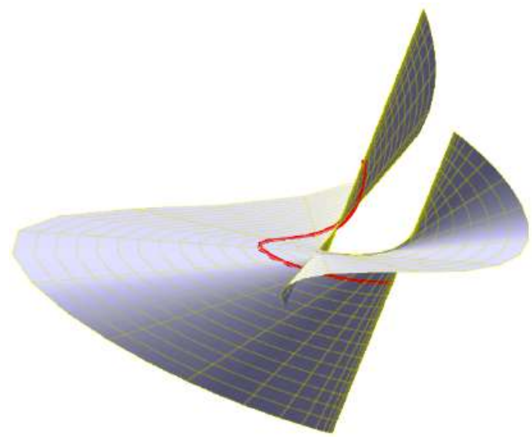
Example 3.1. *Let* $\gamma : [-2\pi, 2\pi] \to E^3$ *be a regular unit speed curve defined by the following parameterization* $\gamma(s) = (cosh(s), sinh(s), 2s)$. *Then, we compute its Frenet apparatus as follows*

$$
\begin{aligned}
T &= \frac{\left(\sinh(s), \cosh(s), 2\right)}{\sqrt{4+\cosh(2s)}}, \quad N = \frac{\left(5\cosh(s), 3\sinh(s), -2\sinh(2s)\right)}{\sqrt{\left(4+\cosh(2s)\right)\left(1+4\cosh(2s)\right)}}, \\
B &= \frac{\left(-2\sinh(s), 2\cosh(s), -1\right)}{\sqrt{1+4\cosh(2s)}}, \quad \kappa = \frac{\sqrt{1+4\cosh(2s)}}{\left(4+\cosh(2s)\right)^{\frac{3}{2}}}, \quad \tau = \frac{2}{1+4\cosh(2s)}.
\end{aligned}
\tag{3.14}
$$

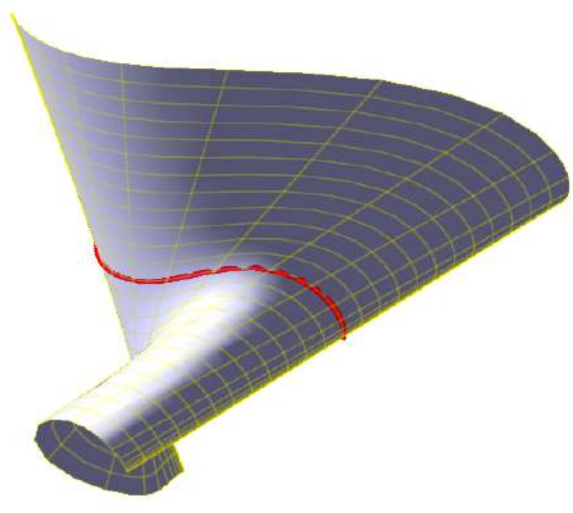
Hence, from the Equations 3.8, 3.9 and 3.14, the ruled surfaces ${}^{TN}_{T}\psi(s,v)$, ${}^{TN}_{N}\psi(s,v)$ *and* ${}^{TN}_{B}\psi(s,v)$ *can be easily obtained (see Fig. 1).*



(a) The ruled surface ${}^{TN}_{T}\psi(s, v)$



(b) The ruled surface ${}^{TN}_{N}\psi(s, v)$



(c) The ruled surface ${}^{TN}_{B}\psi(s, v)$

FIGURE 1. Ruled surfaces with base curve of TN-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

3.2. **Ruled Surfaces with the Base TB− Smarandache Curve.**

Definition 3.2. *Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve of C^2 class and $\{T,\ N,\ B\}$ denotes the set of its Frenet vectors. The original definition of $TB-$ Smarandache* *ruled surface introduced in [11] is as following:*

$$ {}^{TB}_{N}\psi(s,v) = \frac{T+B}{\sqrt{2}} + vN. \tag{3.15}$$

As noted before, the other two ruled surfaces with the base of $TB-$ Smarandache curve and with the ruling of other two Frenet vectors are discussed, which are parameterized in the following

$$\begin{aligned} {}^{TB}_{T}\psi(s,v) &= \frac{T+B}{\sqrt{2}} + vT, \\ {}^{TB}_{B}\psi(s,v) &= \frac{T+B}{\sqrt{2}} + vB. \end{aligned} \tag{3.16}$$

The conditions for the base curve to be asymptotic, geodesic and curvature line on each ruled surface are examined, as well.

3.2.1. *The characteristics of the ruled surface ${}^{TB}_{N}\psi(s,v)$.*

Ouarab, [11] obtained the following corollaries for the ruled surface ${}^{TB}_{N}\psi(s,v)$ as

Corollary 3.6. *[11]*

- ${}^{TB}_{N}\psi(s,v)$ *is always developable.*
- ${}^{TB}_{N}\psi(s,v)$ *is minimal when $\gamma(s)$ is a general helix.*

The base curve characteristics of the ${}^{TB}_{N}\psi(s,v)$ surface associated to $\gamma(s)$ is given with the following theorem.

Theorem 3.6. *The normal curvature ${}^{TB}_{N}\kappa_n$, the geodesic curvature ${}^{TB}_{N}\kappa_g$ and the geodesic torsion ${}^{TB}_{N}\tau_g$ of the ${}^{TB}_{N}\psi(s,v)$ surface are given as follows*

$$ {}^{TB}_{N}\kappa_n = 0, \quad {}^{TB}_{N}\kappa_g = -\epsilon_2\kappa\sqrt{h^2+1}, \quad {}^{TB}_{N}\tau_g = 0, \tag{3.17}$$

respectively, , where $\epsilon_2 = sign(v)$.

Proof. By utilizing the Equation (2.1), the tangent and its derivative, and the second order derivative of the $TB-$ Smarandache curve are given as

$$\begin{aligned} T_{TB} =& \eta N, \qquad T'_{TB} = -\eta\kappa(T-hB), \\ \left(\frac{T+B}{\sqrt{2}}\right)'' =& \frac{\kappa^2(h-1)T - (\tau'-\kappa')N - h\kappa^2(h-1)B}{\sqrt{2}}, \end{aligned} \tag{3.18}$$

where $\eta = sign(\kappa-\tau)$. Moreover, the derivative of the normal of ${}^{TB}_{N}\psi(s,v)$ ruled surface defined as ${}^{TB}_{N}n = \pm\frac{\tau T+\kappa B}{\sqrt{\kappa^2+\tau^2}}$ in [11], but expressed by the harmonic curvature function as ${}^{TB}_{N}n = -\epsilon_2\frac{hT+B}{\sqrt{h^2+1}}$ is given in the following

$$ {}^{TB}_{N}n' = -\epsilon_2\sigma\kappa\,(T-hB).$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.17 of Theorem 3.6, we have the following corollary:

Corollary 3.7.

- *The $TB-$ Smarandache curve of $\gamma(s)$ is always asymptotic and curvature line on ${}^{TB}_{N}\psi(s,v)$, while the geodesic curvature simplifies to ${}^{TB}_{N}\kappa_g = -\epsilon_2\kappa$.*
- *The $TB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}^{TB}_{N}\psi(s,v)$.*

3.2.2. *The characteristics of the ruled surface ${}^{TB}_{T}\psi(s,v)$.*

Theorem 3.7. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}^{TB}_{T}\psi(s,v)$ are given as following:*

$$\begin{aligned}
{}^{TB}_{T}I &= \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa - \frac{1}{\sqrt{2}}\tau\right)^2 ds^2 + dv^2,\\
{}^{TB}_{T}II &= -\,\epsilon_3\left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa - \frac{1}{\sqrt{2}}\tau\right)\tau ds^2,\\
{}^{TB}_{T}K &= 0, \qquad {}^{TB}_{T}H = -\epsilon_3\frac{\tau}{\sqrt{2}\left(\kappa(\sqrt{2}v+1)-\tau\right)}\ ,
\end{aligned}$$

respectively, where $\epsilon_3 = sign(\kappa\left(1+\sqrt{2}v\right)-\tau)$.

Proof. The partial derivatives of ${}^{TB}_{T}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}^{TB}_{T}\psi(s,v)_s &= \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa - \frac{1}{\sqrt{2}}\tau\right)N,\\
{}^{TB}_{T}\psi(s,v)_{ss} &= \left(\frac{1}{\sqrt{2}}\kappa\tau - \left(v+\frac{1}{\sqrt{2}}\right)\kappa^2\right)T + \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa' - \frac{1}{\sqrt{2}}\tau'\right)N\\
&\quad + \left(\left(v+\frac{1}{\sqrt{2}}\right)\kappa\tau - \frac{1}{\sqrt{2}}\tau^2\right)B,\\
{}^{TB}_{T}\psi(s,v)_v &= T, \qquad {}^{TB}_{T}\psi(s,v)_{sv} = \kappa N, \qquad {}^{TB}_{T}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal vector of ${}^{TB}_{T}\psi(s,v)$ is computed as

$${}^{TB}_{T}n = -\epsilon_3 B.$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.8.

- *${}^{TB}_{T}\psi(s,v)$ is always developable.*

- ${}^{TB}_{T}\psi(s,v)$ *is minimal when* $\gamma(s)$ *is a plane curve.*

Theorem 3.8. *The normal curvature* ${}^{TB}_{T}\kappa_n$*, the geodesic curvature* ${}^{TB}_{T}\kappa_g$ *and the geodesic torsion* ${}^{TB}_{T}\tau_g$ *of the* ${}^{TB}_{T}\psi(s,v)$ *surface are given as follows*

$$ {}^{TB}_{T}\kappa_n = \epsilon_3 \frac{\kappa\tau\,(h-1)}{\sqrt{2}}, \qquad {}^{TB}_{T}\kappa_g = -\epsilon_3\kappa, \qquad {}^{TB}_{T}\tau_g = -\eta\kappa\tau, \tag{3.19} $$

respectively.

Proof. Recall the Equation 3.18, since the base is the same $TB-$ Smarandache curve. The derivative of the normal vector of ${}^{TB}_{T}\psi(s,v)$ computed before as ${}^{TB}_{T}n = -\epsilon_3 B$ is given in the following

$$ {}^{TB}_{T}n' = \epsilon_3\tau N. $$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.19 of Theorem 3.8, we have the following corollary:

Corollary 3.9.

- *The* $TB-$ *Smarandache curve of* $\gamma(s)$ *cannot be geodesic on* ${}^{TB}_{T}\psi(s,v)$*.*
- *If* $\gamma(s)$ *is a plane curve , then its corresponding* $TB-$ *Smarandache curve lies both as asymptotic and curvature line on* ${}^{TB}_{T}\psi(s,v)$*.*

3.2.3. *The characteristics of the ruled surface* ${}^{TB}_{B}\psi(s,v)$*.*

Theorem 3.9. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}^{TB}_{B}\psi(s,v)$ *are given as following:*

$$ \begin{aligned} {}^{TB}_{B}I =& \left(\frac{1}{\sqrt{2}}\kappa - \left(v + \frac{1}{\sqrt{2}}\right)\tau\right)^2 ds^2 + dv^2, \\ {}^{TB}_{B}II =& -\epsilon_4\left(\frac{1}{\sqrt{2}}\kappa^2 - \left(v + \frac{1}{\sqrt{2}}\right)\kappa\tau\right) ds^2, \\ {}^{TB}_{B}K =& 0, \\ {}^{TB}_{B}H =& -\epsilon_4 \frac{\kappa}{\sqrt{2}\left(\kappa - \tau\left(\sqrt{2}v+1\right)\right)}, \end{aligned} $$

respectively, where $\epsilon_4 = sign\left(\kappa - \tau\left(\sqrt{2}v+1\right)\right)$*.*

Proof. The partial derivatives of ${}^{TB}_{B}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}^{TB}_{B}\psi(s,v)_s &= \left(\frac{1}{\sqrt{2}}\kappa - \left(v+\frac{1}{\sqrt{2}}\right)\tau\right)N,\\
{}^{TB}_{B}\psi(s,v)_{ss} &= \left(-\frac{1}{\sqrt{2}}\kappa^2 + \left(v+\frac{1}{\sqrt{2}}\right)\kappa\tau\right)T + \left(\frac{1}{\sqrt{2}}\kappa' - \left(v+\frac{1}{\sqrt{2}}\right)\tau'\right)N\\
&\quad + \left(\frac{1}{\sqrt{2}}\kappa\tau - \left(v+\frac{1}{\sqrt{2}}\right)\tau^2\right)B,\\
{}^{TB}_{B}\psi(s,v)_v &= B, \qquad {}^{TN}_{N}\psi(s,v)_{sv} = -\tau N, \qquad {}^{TN}_{N}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal vector of ${}^{TB}_{B}\psi(s,v)$ is computed as

$${}^{TB}_{B}n = \epsilon_4 T.$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.10.

- ${}^{TB}_{B}\psi(s,v)$ *is always developable.*
- ${}^{TB}_{B}\psi(s,v)$ *can not be minimal.*
- *If $\gamma(s)$ is a plane curve, then ${}^{TB}_{B}\psi(s,v)$ is a constant-mean-curvature (CMC) surface.*

Theorem 3.10. *The normal curvature ${}^{TB}_{B}\kappa_n$, the geodesic curvature ${}^{TB}_{B}\kappa_g$ and the geodesic torsion ${}^{TB}_{B}\tau_g$ of the ${}^{TB}_{B}\psi(s,v)$ surface are given as follows*

$${}^{TB}_{B}\kappa_n = \epsilon_4 \frac{\kappa^2(h-1)}{\sqrt{2}}, \qquad {}^{TB}_{B}\kappa_g = \epsilon_4\tau, \qquad {}^{TB}_{B}\tau_g = \eta\kappa\tau, \tag{3.20}$$

respectively.

Proof. By recalling again the Equation 3.18, and taking the derivative of the normal of ${}^{TB}_{B}\psi(s,v)$ computed before as ${}^{TB}_{B}n = -T$, we have

$${}^{TB}_{B}n' = -\kappa N.$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.20 of Theorem 3.10, we have the following corollary:

Corollary 3.11.

- *The $TB-$ Smarandache curve of $\gamma(s)$ cannot be asymptotic on ${}^{TB}_{B}\psi(s,v)$.*

- *If $\gamma(s)$ is a plane curve, then its corresponding $TB-$ Smarandache curve lies both as geodesic and curvature line on ${}_B^{TB}\psi(s,v)$.*

Example 3.2. *By reconsidering the curve given in Example 3.1, and using the Equations 3.14, 3.15 and 3.16 the ruled surfaces ${}_T^{TB}\psi(s,v)$, ${}_N^{TB}\psi(s,v)$ and ${}_B^{TB}\psi(s,v)$ can be easily obtained and illustrated in Fig. 2.*

3.3. **Ruled Surfaces with the Base NB− Smarandache Curve.**

Definition 3.3. *Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve of C^2 class and $\{T,\ N,\ B\}$ denotes the set of its Frenet vectors. The original definition of $NB-$ Smarandache ruled surface introduced in [11] is as following:*

$$ {}_T^{NB}\psi(s,v) = \frac{N+B}{\sqrt{2}} + vT. \tag{3.21}$$

As before, the other two ruled surfaces with the base of $NB-$ Smarandache curve and with the ruling of other two Frenet vectors are discussed, which are parameterized in the following

$$\begin{aligned} {}_N^{NB}\psi(s,v) =& \frac{N+B}{\sqrt{2}} + vN, \\ {}_B^{NB}\psi(s,v) =& \frac{N+B}{\sqrt{2}} + vB. \end{aligned} \tag{3.22}$$

The conditions for the base curve to be asymptotic, geodesic and curvature line on each ruled surface are examined, as well.

3.3.1. *The characteristics of the ruled surface ${}_T^{NB}\psi(s,v)$.*
[11] claimed the following corollaries for the ruled surface ${}_T^{NB}\psi(s,v)$

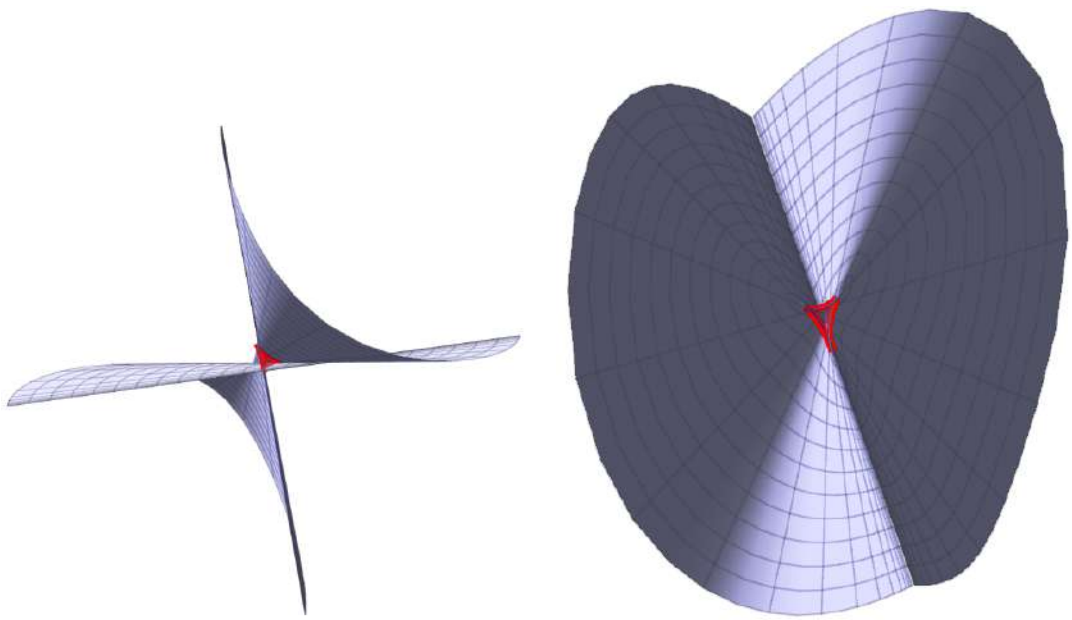
Corollary 3.12. *[11]*

- *If $\gamma(s)$ is a plane curve, then ${}_T^{NB}\psi(s,v)$ is developable, and if it is developable, then it is also minimal.*
- *${}_T^{NB}\psi(s,v)$ is minimal if the following relation holds*

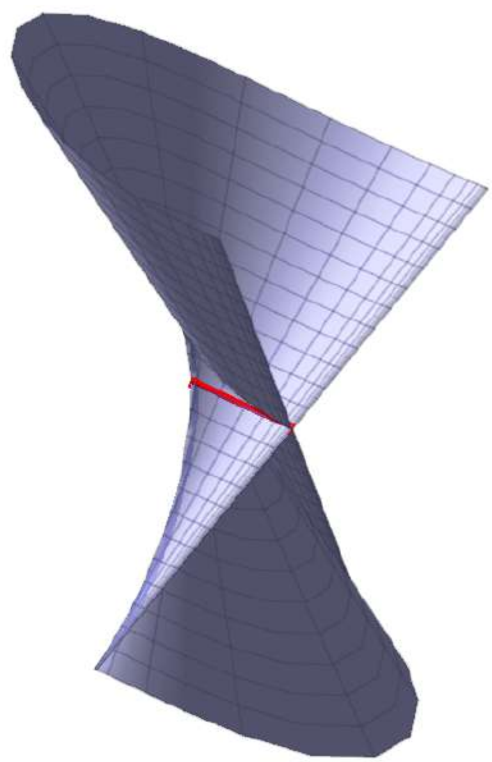
$$\frac{\tau\left(2\kappa^2\tau - 2\tau^2 - \kappa^2\right)}{\sqrt{2}} + v\left(\kappa'\tau + 2\kappa\tau^2 - \kappa^2\tau'\right) - \sqrt{2}v^2\kappa^2\tau = 0.$$

Remark 3.3. *We note that in order for the ruled surface ${}_T^{NB}\psi(s,v)$ be minimal, the following relation should hold*

$$v\sqrt{2}\left(2\kappa\tau^2 + \kappa'\tau - \kappa\tau'\right) - \tau\left(2\tau^2 - \kappa^2\left(1 - 2v^2\right)\right) = 0.$$



(a) The ruled surface ${}^{TB}_{T}\psi(s,v)$ (b) The ruled surface ${}^{TB}_{N}\psi(s,v)$



(c) The ruled surface ${}^{TB}_{B}\psi(s,v)$

FIGURE 2. Ruled surfaces with base curve of TB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

The base curve characteristics of the ${}_{T}^{NB}\psi(s,v)$ surface associated to the curve $\gamma(s)$ is given with the following theorem.

Theorem 3.11. *The normal curvature , the geodesic curvature and the geodesic torsion of the ${}_{T}^{NB}\psi(s,v)$ surface are given as follows*

$$
\begin{aligned}
{}_{T}^{NB}\kappa_n =& \frac{\left(\sqrt{2}v-2h\right)\tau^2-(\tau+h'\kappa)\sqrt{2}v-\tau\kappa}{2\sqrt{h^2-\sqrt{2}hv+v^2}},\\
{}_{T}^{NB}\kappa_g =& \frac{\left(2h^2+1-2\sqrt{2}hv\right)\sqrt{2}\tau+2\left(\left(\sqrt{2}h-v\right)h'-v\kappa\right)}{2\left(2h^2+1\right)\sqrt{h^2-\sqrt{2}hv+v^2}},\\
{}_{T}^{NB}\tau_g =& \frac{\sqrt{2}v(h')^2+\tau\left(2h^2-\sqrt{2}hv+2v^2+1\right)h'}{\left(h^2-\sqrt{2}hv+v^2\right)\left(2h^2+1\right)^{\frac{3}{2}}}+\frac{\left(h-\sqrt{2}v\right)\left(2h^2-\sqrt{2}hv+1\right)\kappa^2}{2\left(h^2-\sqrt{2}hv+v^2\right)\sqrt{2h^2+1}},
\end{aligned}
\tag{3.23}
$$

respectively.

Proof. By referring the Equation (2.1), the tangent and its derivative, and the second order derivative of $NB-$ Smarandache curve are given as

$$
\begin{aligned}
T_{NB} =& -\frac{T+h\left(N-B\right)}{\sqrt{2h^2+1}},\\
T'_{NB} =& \frac{h\left(2h\tau+2h'+\kappa\right)T-\left(2h^3\tau+3h\tau+h'+\kappa\right)N-\left(2h^3\tau+h\tau-h'\right)B}{\left(2h^2+1\right)^{\frac{3}{2}}},\\
\left(\frac{N+B}{\sqrt{2}}\right)'' =& \frac{\left(-\kappa'+\kappa\tau\right)T-\left(\tau^2+h\kappa'+h'\kappa+\kappa^2\right)N-\left(\tau^2-h\kappa'-h'\kappa\right)B}{\sqrt{2}}.
\end{aligned}
\tag{3.24}
$$

Moreover, the derivative of the normal of $NB-$ surface defined as ${}_{T}^{NB}n=\dfrac{\tau N+\left(\tau-\sqrt{2}\kappa v\right)B}{\sqrt{\tau^2+\left(\tau-\sqrt{2}\kappa v\right)^2}}$ in [11], but expressed by the harmonic curvature function as ${}_{T}^{NB}n=\dfrac{hN+\left(h-\sqrt{2}v\right)B}{\sqrt{h^2+\left(h-\sqrt{2}v\right)^2}}$ is given in the following

$$
\begin{aligned}
{}_{T}^{NB}n' =& -\left(\frac{\tau\sqrt{2}}{2\sqrt{h^2-\sqrt{2}hv+v^2}}\right)T+\left(\frac{\tau\left(2v-h\sqrt{2}\right)\left(h^2-\sqrt{2}hv+v^2\right)+h'v\left(\sqrt{2}v-h\right)}{2\left(h^2-\sqrt{2}hv+v^2\right)^{\frac{3}{2}}}\right)N\\
&+\left(\frac{h\left(\left(h^2+v^2\right)\tau\sqrt{2}-v\left(2h\tau-h'\right)\right)}{2\left(h^2-\sqrt{2}hv+v^2\right)^{\frac{3}{2}}}\right)B.
\end{aligned}
$$

By substituting the relations given above into Equation 2.7, the proof is completed. ☐

Without the need for proof, from Definition 2.2 and the Equation 3.23 of Theorem 3.11, we have the following corollary:

Corollary 3.13.

- *The $NB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on ${}_{T}^{NB}\psi(s,v)$.*
- *If $\gamma(s)$ is plane curve, then its corresponding $NB-$ Smarandache curve lies both as asymptotic and curvature line on ${}_{T}^{NB}\psi(s,v)$, while the geodesic curvature simplifies to ${}_{T}^{NB}\kappa_n = -\kappa$.*

3.3.2. *The characteristics of the ruled surface ${}_{N}^{NB}\psi(s,v)$.*

Theorem 3.12. *The 1^{st} and 2^{nd} fundamental forms, and the Gaussian and mean curvature of the ruled surface, ${}_{N}^{NB}\psi(s,v)$ are given as following:*

$$\begin{aligned}
{}_{N}^{NB}I &= \left(\left(v^2 + v\sqrt{2} + 1\right)(\kappa^2 + \tau^2) - \frac{\kappa^2}{2}\right) ds^2 - \tau\sqrt{2}dsdv + dv^2,\\
{}_{N}^{NB}II &= \frac{(\tau\kappa' - \kappa\tau')\left|\sqrt{2}v + 1\right|}{\sqrt{2}\sqrt{\kappa^2 + \tau^2}} ds^2,\\
{}_{N}^{NB}K &= 0,\\
{}_{N}^{NB}H &= \frac{(\kappa'\tau - \kappa\tau')}{\left|2v + \sqrt{2}\right|(\kappa^2 + \tau^2)^{\frac{3}{2}}},
\end{aligned}$$

respectively.

Proof. The partial derivatives of ${}_{N}^{NB}\psi(s,v)$ are given as follows

$$\begin{aligned}
{}_{N}^{NB}\psi(s,v)_s &= -\left(v + \frac{1}{\sqrt{2}}\right)\kappa T - \frac{1}{\sqrt{2}}\tau N + \left(v + \frac{1}{\sqrt{2}}\right)\tau B,\\
{}_{N}^{NB}\psi(s,v)_{ss} &= \left(\frac{1}{\sqrt{2}}\kappa\tau - \left(v + \frac{1}{\sqrt{2}}\right)\kappa'\right) T - \left(\left(\frac{1}{\sqrt{2}} + v\right)(\kappa^2 + \tau^2) + \frac{1}{\sqrt{2}}\tau'\right) N\\
&\quad - \left(\frac{1}{\sqrt{2}}\tau^2 - \left(v + \frac{1}{\sqrt{2}}\right)\tau'\right) B,\\
{}_{N}^{NB}\psi(s,v)_v &= N, \qquad {}_{N}^{NB}\psi(s,v)_{sv} = -\kappa T + \tau B, \qquad {}_{N}^{NB}\psi(s,v)_{vv} = 0.
\end{aligned}$$

From Equation 2.3, the normal vector of ${}_{N}^{NB}\psi(s,v)$ is computed as

$${}_{N}^{NB}n = -\epsilon_1 \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}.$$

Let us remind that $\epsilon_1 = sign(\sqrt{2}v + 1)$ was already defined in the proof of the Theorem (3.4). Then, by recalling Equation 2.6, the components for the fundamental forms can be obtained, and by substituting those in the Equations 2.4 and 2.5, the proof is completed. □

Corollary 3.14.

- ${}^{NB}_{N}\psi(s,v)$ *is always developable.*
- ${}^{NB}_{N}\psi(s,v)$ *is minimal when* $\gamma(s)$ *is a general helix.*

Theorem 3.13. *The normal curvature* ${}^{NB}_{N}\kappa_n$*, the geodesic curvature* ${}^{NB}_{N}\kappa_g$ *and the geodesic torsion* ${}^{NB}_{N}\tau_g$ *of the* ${}^{NB}_{N}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned} {}^{NB}_{N}\kappa_n &= -\epsilon_1 \frac{\kappa h'}{\sqrt{2\left(h^2+1\right)}}, \\ {}^{NB}_{N}\kappa_g &= -\epsilon_1 \frac{\left(\tau h\left(2h^2+3\right)+h'+\kappa\right)}{\left(2h^2+1\right)\sqrt{h^2+1}}, \\ {}^{NB}_{N}\tau_g &= -\frac{h'\left(\tau h\left(2h^2+3\right)+h'+\kappa\right)}{\left(h^2+1\right)\left(2h^2+1\right)^{\frac{3}{2}}}, \end{aligned} \tag{3.25}$$

respectively.

Proof. Recall the Equation 3.24, since the base is the same $NB-$ Smarandache curve. The derivative of the normal vector of ${}^{NB}_{N}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}^{NB}_{N}n = -\epsilon_1 \dfrac{(hT+B)}{\sqrt{h^2+1}}$ is given in the following

$${}^{NB}_{N}n' = -\epsilon_1 \sigma\kappa\left(T - hB\right).$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.25 of Theorem 3.13, we have

Corollary 3.15.

- *The* $NB-$ *Smarandache curve of* $\gamma(s)$ *cannot be geodesic on* ${}^{NB}_{N}\psi(s,v)$*.*
- *If* $\gamma(s)$ *is a plane curve , then its* $NB-$ *Smarandache curve lies both as asymptotic and curvature line on* ${}^{NB}_{N}\psi(s,v)$*, while the geodesic curvature is* ${}^{NB}_{N}\kappa_g = -\kappa$*.*

Remark 3.4. *The two corollaries expressed for the ruled surfaces* ${}^{TN}_{T}\psi(s,v)$ *and* ${}^{TN}_{N}\psi(s,v)$ *are exactly the same for the ruled surface* ${}^{NB}_{N}\psi(s,v)$*.*

3.3.3. *The characteristics of the ruled surface* ${}_{B}^{NB}\psi(s,v)$.

Theorem 3.14. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}_{B}^{NB}\psi(s,v)$ *are given as following:*

$$
\begin{aligned}
{}_{B}^{NB}I &= \left(\frac{\kappa^2}{2} + \left(v^2 + v\sqrt{2} + 1\right)\tau^2\right) ds^2 + \tau\sqrt{2} ds dv + dv^2, \\
{}_{B}^{NB}II &= -\kappa\left(\frac{\left(\tau^2\left(\sqrt{2}v+1\right) - \left(\tau\kappa' + \kappa\tau'\right)\right)\left(\sqrt{2}v+1\right) - \left(\tau^2+\kappa^2\right)}{\sqrt{2}\sqrt{\kappa^2 + \left(\tau\left(\sqrt{2}v+1\right)\right)^2}}\right) ds^2 \\
&\qquad - \frac{2\kappa\tau}{\sqrt{\kappa^2 + \left(\tau\left(\sqrt{2}v+1\right)\right)^2}} ds dv, \\
{}_{B}^{NB}K &= -2\left(\frac{\kappa\tau}{\kappa^2 + \left(\tau\left(\sqrt{2}v+1\right)\right)^2}\right)^2, \\
{}_{B}^{NB}H &= -\frac{\left(\sqrt{2}v+1\right)\left(\kappa\tau' + \tau\kappa'\right) + \left(2\tau^2 v\left(\sqrt{2}+v\right) + \kappa^2\right)\kappa}{\sqrt{2}\left(\kappa^2 + \left(\tau\left(\sqrt{2}v+1\right)\right)^2\right)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}_{B}^{NB}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{B}^{NB}\psi(s,v)_s &= -\frac{\kappa}{\sqrt{2}}T - \tau\left(v + \frac{1}{\sqrt{2}}\right)N + \frac{\tau}{\sqrt{2}}B, \\
{}_{B}^{NB}\psi(s,v)_{ss} &= \left(\left(v + \frac{1}{\sqrt{2}}\right)\kappa\tau - \frac{\kappa'}{\sqrt{2}}\right)T - \left(\frac{\kappa^2+\tau^2}{\sqrt{2}} + \tau'\left(v + \frac{1}{\sqrt{2}}\right)\right)N \\
&\qquad - \left(\tau^2\left(v + \frac{1}{\sqrt{2}}\right) - \frac{\tau'}{\sqrt{2}}\right)B,
\end{aligned}
$$

$$
{}_{B}^{NB}\psi(s,v)_v = B, \qquad {}_{B}^{NB}\psi(s,v)_{sv} = -\tau N, \qquad {}_{B}^{NB}\psi(s,v)_{vv} = 0.
$$

From 2.3, the normal vector of ${}_{B}^{NB}\psi(s,v)$ is computed as

$$
{}_{B}^{NB}n = -\frac{\tau\left(\sqrt{2}v+1\right)T - \kappa N}{\sqrt{\kappa^2 + \left(\tau\left(\sqrt{2}v+1\right)\right)^2}}.
$$

By using the Equation (2.6), the components for the fundamental forms can be obtained. Then, by substituting those in the Equation (2.4) and (2.5), the proof is completed. □

Corollary 3.16.

- ${}_{B}^{NB}\psi(s,v)$ *is developable when* $\gamma(s)$ *is a plane curve.*
- ${}_{B}^{NB}\psi(s,v)$ *is minimal if the following relation holds:*

$$\left(\sqrt{2}v+1\right)\left(\kappa\tau'+\tau\kappa'\right)+\kappa\left(2\tau^2v\left(\sqrt{2}+v\right)+\kappa^2\right)=0.$$

Theorem 3.15. *The normal curvature* ${}_{B}^{NB}\kappa_n$*, the geodesic curvature* ${}_{B}^{NB}\kappa_g$ *and the geodesic torsion* ${}_{B}^{NB}\tau_g$ *of the* ${}_{B}^{NB}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{B}^{NB}\kappa_n =& \frac{\sqrt{2}v\kappa' h-\tau^2\left(\sqrt{2}v+2\right)-h'\kappa-\kappa^2}{\sqrt{2}\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}},\\
{}_{B}^{NB}\kappa_g =& \frac{h'-h\tau\left(2h^2+1\right)\left(\sqrt{2}v+1\right)}{\left(2h^2+1\right)\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}},\\
{}_{B}^{NB}\tau_g =& \frac{\left(\left(2v^2+3\right)\sqrt{2}h^2v+\left(6v^2+1\right)h^2+\sqrt{2}v+1\right)(h')^2}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)\left(2h^2+1\right)^{\frac{3}{2}}}\\
&-\frac{\kappa\left(2h^4\left(\sqrt{2}v+1\right)-h^2\left(2v^2+1\right)-1\right)h'}{\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}\left(2h^2+1\right)^{\frac{3}{2}}}\\
&-\frac{\left(2h^2+\sqrt{2}h^2v+1\right)\left(\sqrt{2}v+1\right)\tau^2}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)\sqrt{2h^2+1}},
\end{aligned} \tag{3.26}$$

respectively.

Proof. By recalling again the Equation 3.24, and taking the derivative of the normal of ${}_{B}^{NB}\psi(s,v)$ expressed by the harmonic curvature function as ${}_{B}^{NB}n=-\dfrac{h\left(\sqrt{2}v+1\right)T-N}{\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}}$, we have

$$
\begin{aligned}
{}^{NB}_{B}n' = &-\left(\frac{\left(\sqrt{2}v+1\right)h'+\kappa\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)^{\frac{3}{2}}}\right)T \\
&-\left(\frac{hh'\left(\sqrt{2}v+1\right)^2+\tau\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)\left(\sqrt{2}v+1\right)}{\left(h^2\left(\sqrt{2}v+1\right)^2+1\right)^{\frac{3}{2}}}\right)N \\
&+\left(\frac{\tau}{\sqrt{h^2\left(\sqrt{2}v+1\right)^2+1}}\right)B.
\end{aligned}
$$

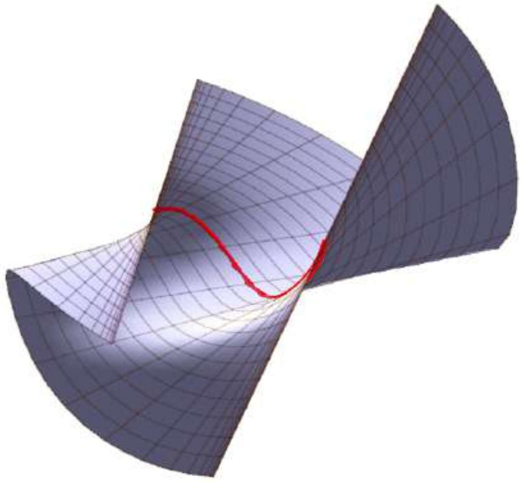
By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.26 of Theorem 3.15, we have the following corollary:

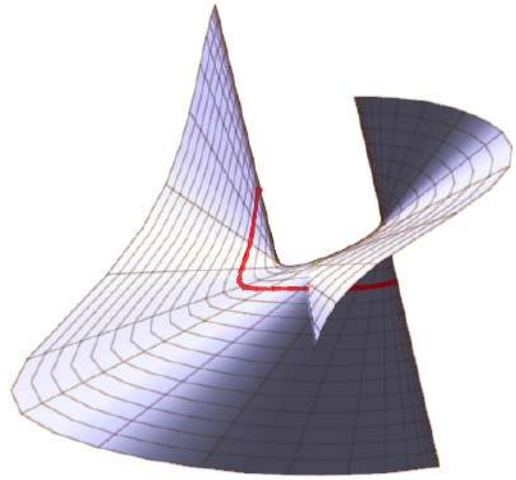
Corollary 3.17.

- *If the curve $\gamma(s)$ is plane curve, then its corresponding $NB-$ Smarandache curve lies both as geodesic and curvature line on the ruled surface ${}^{NB}_{B}\psi(s,v)$, while the normal curvature simplifies to the relation ${}^{NB}_{B}\kappa_n = -\frac{\kappa^2}{\sqrt{2}}$.*
- *The $NB-$ Smarandache curve of $\gamma(s)$ cannot be asymptotic on the ruled surface ${}^{NB}_{B}\psi(s,v)$.*

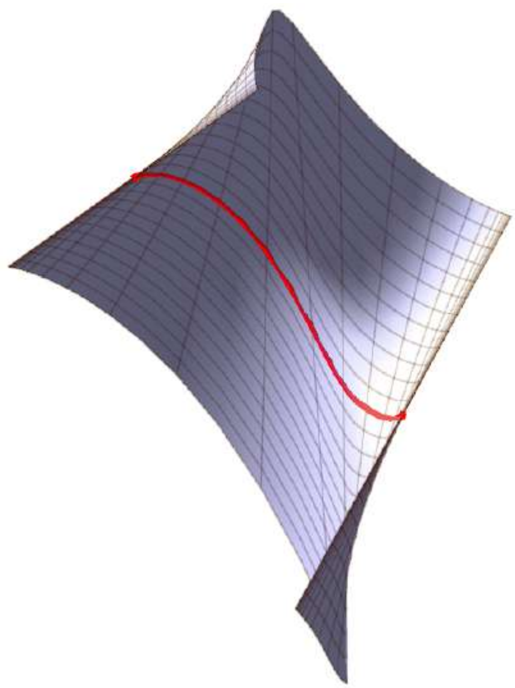
Example 3.3. *By utilizing the same curve as of previous examples, and by applying the Equations 3.14, 3.21 and 3.22 the ruled surfaces ${}^{NB}_{T}\psi(s,v)$, ${}^{NB}_{N}\psi(s,v)$ and ${}^{NB}_{B}\psi(s,v)$ can be easily obtained and illustrated in Fig. 3*



(a) The ruled surface ${}^{NB}_{T}\psi(s,v)$



(b) The ruled surface ${}^{NB}_{N}\psi(s,v)$



(c) The ruled surface ${}^{NB}_{B}\psi(s,v)$

FIGURE 3. Ruled surfaces with base curve of NB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

3.4. **Ruled Surfaces with the Base TNB− Smarandache Curve.**

Definition 3.4. *Let $\gamma : s \in I \subset \Re \to E^3$ be a regular unit speed curve of C^2 class and $\{T,\ N,\ B\}$ denotes the set of its Frenet vectors. Then the ruled surfaces with the base of $TNB-$ Smarandache curve and the generator lines as each one of them are defined as following:*

$$\begin{aligned} {}_{T}^{TNB}\psi(s,v) &= \left(\frac{T+N+B}{\sqrt{3}}\right)+vT,\\ {}_{N}^{TNB}\psi(s,v) &= \left(\frac{T+N+B}{\sqrt{3}}\right)+vN,\\ {}_{B}^{TNB}\psi(s,v) &= \left(\frac{T+N+B}{\sqrt{3}}\right)+vB. \end{aligned} \tag{3.27}$$

3.4.1. *The characteristics of the ruled surface ${}_{T}^{TNB}\psi(s,v)$.*

Theorem 3.16. *The 1^{st} and 2^{nd} fundamental forms, and the curvatures of Gaussian and mean for the ruled surface, ${}_{T}^{TNB}\psi(s,v)$ are given as following:*

$$\begin{aligned} {}_{T}^{TNB}I &= \left(\frac{\kappa^2+\tau^2}{3}+\left(\frac{\kappa-\tau}{\sqrt{3}}+v\kappa\right)^2\right)ds^2-\frac{2\kappa}{\sqrt{3}}dsdv+dv^2,\\ {}_{T}^{TNB}II &= \frac{\left(\left(\sqrt{3}v+1\right)\left(\tau\kappa'-\kappa\tau'\right)-\tau\left(\tau^2+\kappa^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2\right)\right)ds^2+2\sqrt{3}\kappa\tau dsdv}{\sqrt{3}\sqrt{\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2}}, \end{aligned}$$

$$\begin{aligned} {}_{T}^{TNB}K &= -3\left(\frac{\kappa\tau}{\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2}\right)^2,\\ {}_{T}^{TNB}H &= -\frac{\sqrt{3}}{2}\frac{\tau\left(\tau^2+\left(\sqrt{3}v\kappa-\tau\right)\left(2\kappa+\sqrt{3}v\kappa-\tau\right)\right)+\left(\kappa\tau'-\kappa'\tau\right)\left(\sqrt{3}v+1\right)}{\left(\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2\right)^{\frac{3}{2}}}, \end{aligned}$$

respectively.

Proof. The partial derivatives of ${}_{T}^{TNB}\psi(s,v)$ are given as follows

$$\begin{aligned} {}_{T}^{TNB}\psi(s,v)_s &= -\frac{\kappa}{\sqrt{3}}T+\left(\left(\frac{1}{\sqrt{3}}+v\right)\kappa-\frac{\tau}{\sqrt{3}}\right)N+\frac{\tau}{\sqrt{3}}B,\\ {}_{T}^{TNB}\psi(s,v)_{ss} &= \left(-\left(\frac{1}{\sqrt{3}}+v\right)\kappa^2+\frac{\kappa\tau-\kappa'}{\sqrt{3}}\right)T-\left(\frac{\kappa^2+\tau^2+\tau'}{\sqrt{3}}-\left(\frac{1}{\sqrt{3}}+v\right)\kappa'\right)N\\ &\quad+\left(\left(\frac{1}{\sqrt{3}}+v\right)\tau\kappa+\frac{\tau'-\tau^2}{\sqrt{3}}\right)B,\\ {}_{T}^{TNB}\psi(s,v)_v &= T,\qquad {}_{T}^{TNB}\psi(s,v)_{sv}=\kappa N,\qquad {}_{T}^{TNB}\psi(s,v)_{vv}=0. \end{aligned}$$

From Equation 2.3, the normal vector of ${}_{T}^{TNB}\psi(s,v)$ is computed as

$$ {}_{T}^{TNB}n = \frac{\tau N - \left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)B}{\sqrt{\tau^2+\left(\kappa\left(\sqrt{3}v+1\right)-\tau\right)^2}}. $$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.18.

- *If the curve $\gamma(s)$ is a plane curve, then the ruled surface ${}_{T}^{TNB}\psi(s,v)$ is both developable and minimal.*
- *If the curve $\gamma(s)$ is a regular unit speed space curve, then ${}_{T}^{TNB}\psi(s,v)$ is minimal if the following relation holds*

$$ \tau\left(\tau^2+\left(\sqrt{3}v\kappa-\tau\right)\left(2\kappa+\sqrt{3}v\kappa-\tau\right)\right)-\left(\kappa\tau'-\kappa'\tau\right)\left(\sqrt{3}v+1\right)=0. $$

The base curve characteristics of the ${}_{T}^{TNB}\psi(s,v)$ surface associated to the curve $\gamma(s)$ is given with the following theorem.

Theorem 3.17. *The normal curvature ${}_{T}^{TNB}\kappa_n$, the geodesic curvature ${}_{T}^{TNB}\kappa_g$ and the geodesic torsion ${}_{T}^{TNB}\tau_g$ of the ${}_{T}^{TNB}\psi(s,v)$ surface are given as follows*

$$ \begin{aligned} {}_{T}^{TNB}\kappa_n &= -\frac{\left(\sqrt{3}v+1\right)\left(h'\kappa+h\kappa'\right)-h\left(\kappa'+\kappa^2\left(\left(\sqrt{3}v-2h\right)(h-1)-2\right)\right)}{\sqrt{3}\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}, \\ {}_{T}^{TNB}\kappa_g &= -\frac{\left(\sqrt{3}v-2h+1\right)h'+2\kappa\left(\sqrt{3}v-h+1\right)\left(h^2-h+1\right)}{2\left(h^2-h+1\right)\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}, \\ {}_{T}^{TNB}\tau_g &= \frac{\begin{array}{l}(h')^2\left(\sqrt{3}v+1\right)(2h-1) \\ -\kappa h'\left(3hv^2(1-2h)+v\sqrt{3}\left(2h^2-3h+2\right)(h+1)-2\left(h^2-h+1\right)\left(2h^2-1\right)\right)\end{array}}{2\sqrt{2}\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)\left(h^2-h+1\right)^{\frac{3}{2}}} \\ &\quad -\epsilon_5\frac{\kappa\tau\left(\sqrt{3}v-h+1\right)\left((h-1)\left(\sqrt{3}v-2h\right)-2\right)}{\sqrt{2}\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)\sqrt{h^2-h+1}}, \end{aligned} \tag{3.28} $$

respectively, where $\epsilon_5 = sign(h^2-h+1)$.

Proof. By referring the Equation (2.1), the tangent and its derivative, and the second order derivative of $TNB-$ Smarandache curve are given as

$$T_{TNB} = \frac{-T-(h-1)N+hB}{\sqrt{2}\sqrt{h^2-h+1}},$$

$$T'_{TNB} = \frac{\begin{array}{l}\left(h'(2h-1)+2\kappa(h-1)\left(h^2-h+1\right)\right)T\\ -\left(h'(h+1)+2\kappa\left(h^2+1\right)\left(h^2-h+1\right)\right)N\\ -\left(h'(h-2)+2\tau(h-1)\left(h^2-h+1\right)\right)B\end{array}}{2\sqrt{2}(h^2-h+1)^{\frac{3}{2}}}, \tag{3.29}$$

$$\left(\frac{T+N+B}{\sqrt{3}}\right)'' = \frac{\begin{array}{l}-\left(\kappa'-\kappa^2(h-1)\right)T-\left(\kappa'(h-1)+\kappa\left(h'+\kappa\left(h^2+1\right)\right)\right)N\\ +(h\kappa'+\kappa(h'-\tau(h-1)))B\end{array}}{\sqrt{3}}.$$

Moreover, the derivative of the normal of ${}_{T}^{TNB}\psi(s,v)$ ruled surface which is expressed by the harmonic curvature function as ${}_{T}^{TNB}n = \dfrac{hN-\left(\sqrt{3}v-h+1\right)B}{\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}$ is given in the following

$$\begin{aligned}{}_{T}^{TNB}n' = &-\frac{\tau}{\sqrt{\left(\sqrt{3}v-h+1\right)^2+h^2}}T\\ &-\frac{\left(\tau\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)+h'\left(\sqrt{3}v+1\right)\right)\left(\sqrt{3}v-h+1\right)}{\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)^{\frac{3}{2}}}N\\ &-\frac{h\left(\tau\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)+h'\left(\sqrt{3}v+1\right)\right)}{\left(\left(\sqrt{3}v-h+1\right)^2+h^2\right)^{\frac{3}{2}}}B.\end{aligned}$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.28 of Theorem 3.17, we have the following corollary:

Corollary 3.19.

- *The $TNB-$ Smarandache curve of $\gamma(s)$ cannot be geodesic on the ruled surface ${}_{T}^{TNB}\psi(s,v)$.*
- *If the curve $\gamma(s)$ is plane curve, then its corresponding $TNB-$ Smarandache curve lies both as asymptotic and curvature line on the ruled surface ${}_{T}^{TNB}\psi(s,v)$, while the geodesic curvature simplifies to ${}_{T}^{TNB}\kappa_n = -\kappa$.*

3.4.2. *The characteristics of the ruled surface* ${}_{N}^{TNB}\psi(s,v)$.

Theorem 3.18. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}_{N}^{TNB}\psi(s,v)$ *are given as following:*

$$
\begin{aligned}
{}_{N}^{TNB}I &= \left(\left(\frac{1}{\sqrt{3}}+v\right)^2\left(\kappa^2+\tau^2\right)+\frac{(\kappa-\tau)^2}{3}\right)ds^2+\frac{2}{\sqrt{3}}(\kappa-\tau)dsdv+dv^2,\\
{}_{N}^{TNB}II &= \left(\frac{\left|\sqrt{3}v+1\right|(\kappa'\tau-\kappa\tau')}{\sqrt{3}\sqrt{(\tau^2+\kappa^2)}}\right)ds^2,\\
{}_{N}^{TNB}K &= 0,\\
{}_{N}^{TNB}H &= \frac{\sqrt{3}}{2}\frac{\tau\kappa'-\kappa\tau'}{\left|\sqrt{3}v+1\right|(\kappa^2+\tau^2)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}_{N}^{TNB}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{N}^{TNB}\psi(s,v)_s &= -\left(\frac{1}{\sqrt{3}}+v\right)\kappa T+\frac{(\kappa-\tau)}{\sqrt{3}}N+\left(\frac{1}{\sqrt{3}}+v\right)\tau B,\\
{}_{N}^{TNB}\psi(s,v)_{ss} &= -\left(\frac{(\kappa-\tau)\kappa}{\sqrt{3}}+\kappa'\left(\frac{1}{\sqrt{3}}+v\right)\right)T-\left(\left(\frac{1}{\sqrt{3}}+v\right)\left(\tau^2+\kappa^2\right)-\frac{(\kappa'-\tau')}{\sqrt{3}}\right)N\\
&\quad+\left(\left(\frac{1}{\sqrt{3}}+v\right)\tau'+\frac{(\kappa-\tau)\tau}{\sqrt{3}}\right)B,\\
{}_{N}^{TNB}\psi(s,v)_v &= N, \qquad {}_{N}^{TNB}\psi(s,v)_{sv}=-\kappa T+\tau B, \qquad {}_{N}^{TNB}\psi(s,v)_{vv}=0.
\end{aligned}
$$

From Equation 2.3, the normal vector of ${}_{N}^{TNB}\psi(s,v)$ is computed as

$$
{}_{N}^{TNB}n=-\epsilon_6\frac{(\tau T+\kappa B)}{\sqrt{(\tau^2+\kappa^2)}},
$$

where $\epsilon_6=sign(\sqrt{3}v+1)$. By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.20.

- ${}_{N}^{TNB}\psi(s,v)$ *is always developable.*
- ${}_{N}^{TNB}\psi(s,v)$ *is minimal when* $\gamma(s)$ *is a general helix.*

Theorem 3.19. *The normal curvature* ${}_{N}^{TNB}\kappa_n$*, the geodesic curvature* ${}_{N}^{TNB}\kappa_g$ *and the geodesic torsion* ${}_{N}^{TNB}\tau_g$ *of the* ${}_{N}^{TNB}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{N}^{TNB}\kappa_n &= -\epsilon_6\frac{h'\kappa}{\sqrt{3}\sqrt{h^2+1}},\\
{}_{N}^{TNB}\kappa_g &= -\epsilon_6\frac{(h+1)h' + 2\kappa\left(h^2+1\right)\left(h^2-h+1\right)}{2\sqrt{h^2+1}\,(h^2-h+1)},\\
{}_{N}^{TNB}\tau_g &= -\frac{(h+1)(h')^2 + 2\kappa h'\left(h^2+1\right)\left(h^2-h+1\right)}{2\sqrt{2}\,(h^2+1)\,(h^2-h+1)^{\frac{3}{2}}},
\end{aligned} \tag{3.30}$$

respectively.

Proof. By using the Equation 3.29, and taking the derivative of the normal of ${}_{N}^{TNB}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}_{N}^{TNB}n = -\epsilon_6\dfrac{hT+B}{\sqrt{h^2+1}}$ is given in the following

$${}_{N}^{TNB}n' = -\epsilon_6\sigma\kappa\,(T-hB)\,.$$

By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.30 of Theorem 3.19, we have the following corollary:

Corollary 3.21.

- *The* $TNB-$ *Smarandache curve of* $\gamma(s)$ *cannot be geodesic on the ruled surface* ${}_{N}^{TNB}\psi(s,v)$.
- *If* $\gamma(s)$ *is a general helix such that* h *is constant, then its corresponding* $TNB-$ *Smarandache curve lies both as asymptotic and curvature line on* ${}_{N}^{TNB}\psi(s,v)$*, while the geodesic curvature simplifies to* ${}_{N}^{TNB}\kappa_g = -\kappa\sqrt{h^2+1}$.

3.4.3. *The characteristics of the ruled surface* ${}_{B}^{TNB}\psi(s,v)$.

Theorem 3.20. *The* 1^{st} *and* 2^{nd} *fundamental forms, and the curvatures of Gaussian and mean for the ruled surface,* ${}_{B}^{TNB}\psi(s,v)$ *are given as following:*

$$
\begin{aligned}
{}_{B}^{TNB}I &= \left(\frac{\kappa^2+\tau^2}{3} + \left(\frac{1}{\sqrt{3}}(\kappa-\tau) - v\tau\right)^2\right) ds^2 + \frac{2\tau}{\sqrt{3}} ds dv + dv^2,\\
{}_{B}^{TNB}II &= -\frac{\left(\kappa\left(\kappa^2+\tau^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2\right) + \left(\sqrt{3}v+1\right)\left(\kappa\tau'-\tau\kappa'\right)\right) ds^2 - 2\sqrt{3}\kappa\tau ds dv}{\sqrt{3}\sqrt{\kappa^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2}},\\
{}_{B}^{TNB}K &= -3\left(\frac{\kappa\tau}{\kappa^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2}\right)^2,\\
{}_{B}^{TNB}H &= -\frac{\sqrt{3}}{2}\frac{\left(\kappa\tau'-\kappa'\tau\right)\left(\sqrt{3}v+1\right) - \kappa\left(\tau^2-\kappa^2-\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2\right)}{\left(\left(\kappa-\tau\left(v\sqrt{3}+1\right)\right)^2+\kappa^2\right)^{\frac{3}{2}}},
\end{aligned}
$$

respectively.

Proof. The partial derivatives of ${}_{B}^{TNB}\psi(s,v)$ are given as follows

$$
\begin{aligned}
{}_{B}^{TNB}\psi(s,v)_s &= -\frac{1}{\sqrt{3}}\left(\kappa T - \tau B\right) + \left(\frac{1}{\sqrt{3}}(\kappa-\tau) - v\tau\right) N,\\
{}_{B}^{TNB}\psi(s,v)_{ss} &= -\left(\frac{1}{\sqrt{3}}\kappa' + \left(\frac{(\kappa-\tau)}{\sqrt{3}} - v\tau\right)\kappa\right) T\\
&\quad - \left(\frac{1}{\sqrt{3}}\left(\kappa^2+\tau^2-\kappa'\right) + \left(\frac{1}{\sqrt{3}} + v\right)\tau'\right) N\\
&\quad + \left(\frac{1}{\sqrt{3}}\tau' + \left(\frac{1}{\sqrt{3}}(\kappa-\tau) - v\tau\right)\tau\right) B,\\
{}_{B}^{TNB}\psi(s,v)_v &= B, \qquad {}_{B}^{TNB}\psi(s,v)_{sv} = -\tau N, \qquad {}_{B}^{TNB}\psi(s,v)_{vv} = 0.
\end{aligned}
$$

From Equation 2.3, the normal vector of ${}_{B}^{TNB}\psi(s,v)$ is computed as

$$
{}_{B}^{TNB}n = \frac{\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right) T + \kappa N}{\sqrt{\kappa^2+\left(\kappa-\tau\left(\sqrt{3}v+1\right)\right)^2}}.
$$

By recalling Equation 2.6, the components for the fundamental forms can be obtained. Then, by substituting those in Equation 2.4 and 2.5, the proof is completed. □

Corollary 3.22.

- ${}_{B}^{TNB}\psi(s,v)$ *is developable, when* $\gamma(s)$ *is a plane curve.*
- ${}_{B}^{TNB}\psi(s,v)$ *can not be minimal.*
- *If* $\gamma(s)$ *is a plane curve, then* ${}_{B}^{TNB}\psi(s,v)$ *is a constant-mean-curvature (CMC) surface.*

Theorem 3.21. *The normal curvature* ${}_{B}^{TNB}\kappa_n$*, the geodesic curvature* ${}_{B}^{TNB}\kappa_g$ *and the geodesic torsion* ${}_{B}^{TNB}\tau_g$ *of the* ${}_{B}^{TNB}\psi(s,v)$ *surface are given as follows*

$$\begin{aligned}
{}_{B}^{TNB}\kappa_n &= -\frac{h'\kappa-\sqrt{3}h\kappa' v+\kappa^2\left(h\left(h-1\right)\left(\sqrt{3}v+2\right)+2\right)}{\sqrt{3}\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}},\\
{}_{B}^{TNB}\kappa_g &= \frac{h'-\left(h'+2\tau\left(h^2-h+1\right)\right)\left(h\left(\sqrt{3}v+1\right)-1\right)}{2\left(h^2-h+1\right)\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}},\\
{}_{B}^{TNB}\tau_g &= \frac{\begin{array}{c}(2-h)\left((h'')^2\left(\sqrt{3}v+1\right)+\kappa h' 3h^2v^2\right)\\ -\kappa h'\left(\sqrt{3}hv\left(h+1\right)\left(2h^2-3h+2\right)+2\left(h^2-h+1\right)\left(h^2-2\right)\right)\end{array}}{2\sqrt{2}(h^2-h+1)^{\frac{3}{2}}\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)}\\
&\quad-\epsilon_5\frac{\kappa\tau\left(2h^2+\left(\sqrt{3}hv-2\right)\left(h-1\right)\right)\left(h\left(\sqrt{3}v+1\right)-1\right)}{\sqrt{2}\sqrt{h^2-h+1}\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)},
\end{aligned} \tag{3.31}$$

respectively, where $\epsilon_5 = sign(h^2-h+1)$ *as already defined in the Theorem 3.17.*

Proof. By using the Equation 3.29, and taking the derivative of the normal of ${}_{B}^{TNB}\psi(s,v)$ ruled surface expressed by the harmonic curvature function as ${}_{B}^{TNB}n = -\frac{\left(h\left(\sqrt{3}v+1\right)-1\right)T-B}{\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}}$ is given in the following

$$\begin{aligned}
{}_{B}^{TNB}n' = &-\frac{2\kappa+\left(h'+\tau\left(h\left(\sqrt{3}v+1\right)-2\right)\right)\left(\sqrt{3}v+1\right)}{\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)^{\frac{3}{2}}}T\\
&-\frac{\sqrt{3}\left(2\kappa+\left(h'+\tau\left(h\left(\sqrt{3}v+1\right)-2\right)\right)\left(\sqrt{3}v+1\right)\right)\left(h\left(\sqrt{3}v+1\right)-1\right)}{\left(\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1\right)^{\frac{3}{2}}}N\\
&+\frac{\tau}{\sqrt{\left(h\left(\sqrt{3}v+1\right)-1\right)^2+1}}B.
\end{aligned}$$

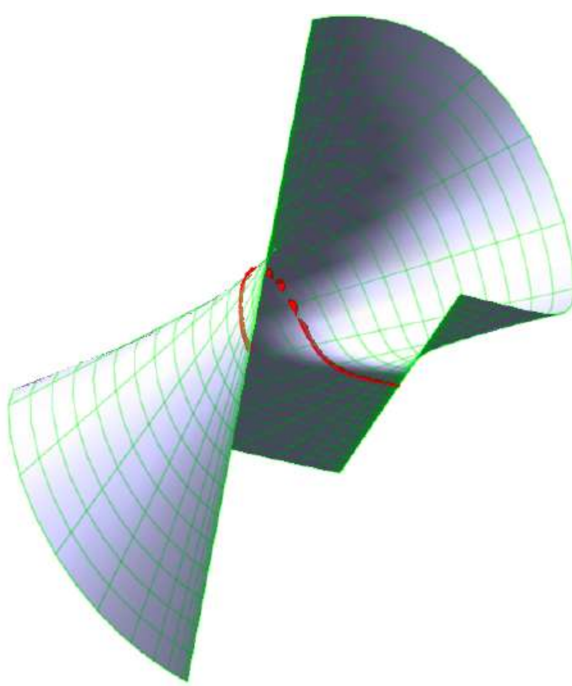
By substituting the relations given above into Equation 2.7, the proof is completed. □

Without the need for proof, from Definition 2.2 and the Equation 3.31 of Theorem 3.21, we have the following corollary:

Corollary 3.23.

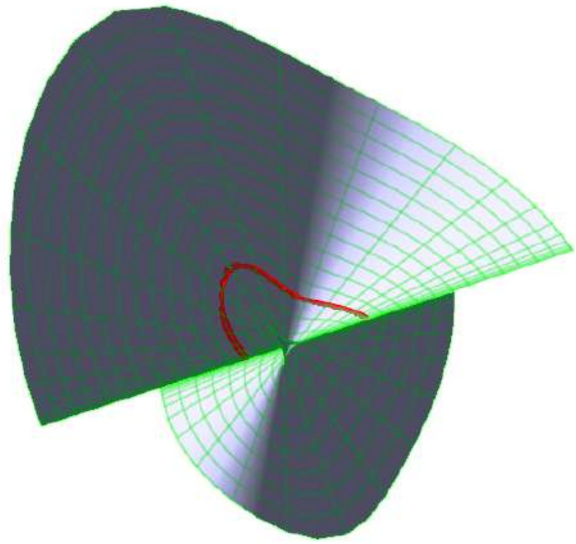
- *The* $TNB-$ *Smarandache curve of* $\gamma(s)$ *cannot be asymptotic on* ${}^{TNB}_{B}\psi(s,v)$.
- *If* $\gamma(s)$ *is plane curve, then its corresponding* $TNB-$ *Smarandache curve lies both as geodesic and curvature line on the ruled surface* ${}^{TNB}_{B}\psi(s,v)$*, while the asymptotic curvature simplifies to* ${}^{TNB}_{B}\kappa_n = -\frac{\kappa}{\sqrt{3}}$.

Example 3.4. *By utilizing the same curve as of previous examples, and by applying the Equations 3.14, and 3.27 the ruled surfaces* ${}^{TNB}_{T}\psi(s,v)$, ${}^{TNB}_{N}\psi(s,v)$ *and* ${}^{TNB}_{B}\psi(s,v)$ *can be easily obtained and illustrated in Fig. 4.*

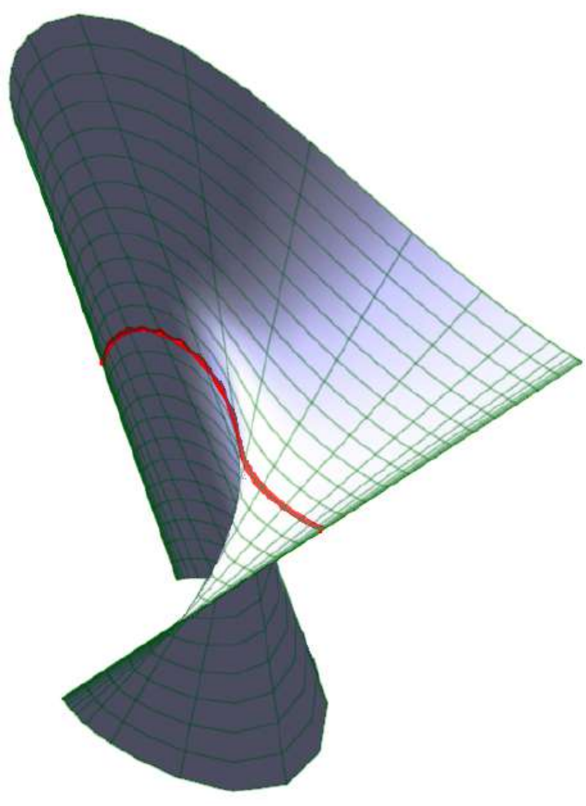


(a) The ruled surface ${}^{TNB}_{T}\psi(s,v)$

FIGURE 4. Ruled surfaces with base curve of TNB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2,2]$



(b) The ruled surface ${}_{N}^{TNB}\psi(s,v)$



(c) The ruled surface ${}_{B}^{TNB}\psi(s,v)$

FIGURE 4. Ruled surfaces with base curve of TNB-Smarandache curve (red) where $s \in [-2\pi, 2\pi]$ and $v \in [-2, 2]$

4. Conclusion

The theory of ruled surfaces plays an important role in the field of geometric modeling, since they are the most preferred ones for computational designs. This study introduces a series of new ruled surfaces and provides some of their metric properties. Such properties as developability and minimality are discussed in terms of the fundamental forms and principal curvatures. Hence, the required conditions are provided for each ruled surface to meet these characteristics. Moreover, asymptotic, geodesic and curvature line characteristics of the each Smarandache curve as a base curve are discussed. This way of generating and characterizing new ruled surfaces as in this study can be extended by referring other orthonormal frames and by using different space forms. Finally, researchers can be interested to examine the dual expressions for these surfaces.

Conflict of interest. The authors declare no conflict of interest.

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References

[1] Abbena, E., Salamon, S., and Gray, A. (2017). *Modern differential geometry of curves and surfaces with Mathematica*. Chapman and Hall/CRC.

[2] Ali, A. T. (2010). Special smarandache curves in the euclidean space. *International Journal of Mathematical Combinatorics*, 2:30–36.

[3] Dede, M. (2018). A new representation of tubular surfaces. *Houston Journal of Mathematics*, 45(3):707–720.

[4] Do Carmo, M. P. (2016). *Differential geometry of curves and surfaces: revised and updated second edition*. Courier Dover Publications.

[5] Eren, K. and Şenyurt, S. (2022). On ruled surfaces with a sannia frame in euclidean 3-space. *Kyungpook Mathematical Journal*, 62(3):509–531.

[6] Güler, F. and Kasap, E. (2016). Ruled surfaces according to rotation minimizing frame. *Math. Sci. Lett.*, 5:33–38.

[7] Izumiya, S. and Takeuchi, N. (2004). New special curves and developable surfaces. *Turkish Journal of Mathematics*, 28(2):153–164.

[8] Masal, M. and Azak, A. Z. (2019). Ruled surfaces according to bishop frame in the euclidean 3-space. *Proceedings of the National Academy of Sciences, India Section A: Physical*

Sciences, 89:415–424.

[9] Ouarab, S. (2021a). Nc-smarandache ruled surface and nw-smarandache ruled surface according to alternative moving frame in e^3. *Journal of Mathematics*, 2021:1–6.

[10] Ouarab, S. (2021b). Smarandache ruled surfaces according to darboux frame in e^3. *Journal of Mathematics*, 2021:1–10.

[11] Ouarab, S. (2022). Corrigendum to "smarandache ruled surfaces according to frenet-serret frame of a regular curve in e^3. In *Abstract and Applied Analysis*, volume 2022. Hindawi.

[12] Ouarab, S. and Chahdi, A. O. (2020). Some characteristic properties of ruled surface with frenet frame of an arbitrary non-cylindrical ruled surface in euclidean 3-space. *International Journal of Applied Physics and Mathematics*, 10(1):16–24.

[13] Ouarab, S., Chahdi, A. O., and Izid, M. (2018). Ruled surfaces with alternative moving frame in euclidean 3-space. *International Journal of Mathematical Sciences and Engineering Applications*, 12(2):43–58.

[14] Sarıoğlugil, A. and Tutar, A. (2007). On ruled surfaces in euclidean space. *Int. J. Contemp. Math. Sci.*, 2(1):1–11.

[15] Şenyurt, S., Ayvacı, K. H., and Canlı, D. (2023). Special smarandache ruled surfaces according to flc frame in e^3. *Applications & Applied Mathematics*, 18(1).

[16] Şenyurt, S., Canlı, D., and Çan, E. (2022a). Smarandache-based ruled surfaces with the darboux vector according to frenet frame in e^3. *Journal of New Theory*, 39:8–18.

[17] Şenyurt, S., Canlı, D., and Çan, E. (2022b). Some special smarandache ruled surfaces by frenet frame in e^3 -i. *Turkish Journal of Science*, 7:31–42.

[18] Şenyurt, S., Canlı, D., Çan, E., and Mazlum, S. G. (2022c). Some special smarandache ruled surfaces by frenet frame in e^3 -ii. *Honam Mathematical Journal*, 44:594–617.

[19] Struik, D. J. (1961). *Lectures on classical differential geometry.* Courier Corporation.

[20] Tunçer, Y. (2015). Ruled surfaces with the bishop frame in euclidean 3–space. *Gen. Math. Notes*, 26:74–83.

[21] Turgut, M. and Yilmaz, S. (2008). Smarandache curves in minkowski spacetime. *International Journal of Mathematical Combinatorics*, 3:51–55.

[22] Uzun, G., Şenyurt, S., and Akdağ, K. Ruled surfaces with $\{\overline{u_1}, \overline{u_3}\}$-smarandache base curve obtained from the successor frame. *Konuralp Journal of Mathematics*, 12(1):28–45.

[23] Yılmaz, A. and Şahin, B. (2018a). On geodesics of the tangent and normal surfaces defined by tn-smarandache curve according to frenet frame. In *16th International Geometry*

Symposium, Manisa, Turkey, pages 1–10.

[24] Yılmaz, A. and Şahin, B. (2018b). On ruled surfaces defined by smarandache curve. In *2nd International Students Science Congress, İzmir, Türkiye*.

[25] Yılmaz, A. and Şahin, B. (2019). On geodesics of the binormal surface defined by smarandache curve. In *3rd International Students Science Congress, İzmir, Türkiye*.

[26] Yilmaz, B., Metin, S., Gok, I., and Yayli, Y. (2019). Harmonic curvature functions of some special curves in galilean 3-space. *Honam Mathematical Journal*, 41(2):301–319.

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SOME CLASSES OF LACUNARY WEAK CONVERGENCE OF SEQUENCES DEFINED BY ORLICZ FUNCTION

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Abstract. In this article, we introduce the notion of difference lacunary weak convergence in sequences defined by an Orlicz function. We examine several algebraic and topological properties and establish some inclusion relationships between these spaces.

Keywords: Weak convergence, Orlicz function, Lacunary convergence.

2020 Mathematics Subject Classification: 40A05, 40F05.

1. Introduction

The idea of weak convergence, first proposed by Banach [1], is a foundational concept in functional analysis, offering a framework for understanding the convergence behavior of sequences in infinite-dimensional spaces. Despite its significance, weak convergence has several limitations, particularly when dealing with more complex sequence structures or when finer convergence criteria are required.

In recent years, researchers like Mahanta and Tripathy [15] have advanced the study of vector-valued sequence spaces by exploring new types of convergence and their implications. Their work has contributed to a deeper understanding of the algebraic and topological properties of these spaces and has led to the development of innovative tools and techniques for analyzing convergence in more generalized contexts. This expanding research underscores the

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continuous evolution and refinement of sequence space theory, addressing the shortcomings of traditional weak convergence and meeting the demands of increasingly complex mathematical analysis.

Freedman et al. [7] conducted pioneering research on lacunary sequences, investigating strongly Cesaro summable and strongly lacunary convergent sequences in the context of a general lacunary sequence θ. Their work uncovered significant connections between these two classes of sequences. Following their initial findings, researchers such as Ercan et al. [5], Gumus [8], Dowari, and Tripathy [2, 3] have further explored various aspects of lacunary sequences, broadening our understanding of their properties and applications. More recently, Tamuli and Tripathy [19, 20] have advanced this field by examining generalized difference lacunary weak convergence of sequences. Their study sheds light on new convergence behaviors and enhances the theoretical framework for analyzing lacunary sequences, highlighting the ongoing development and deepening of this area of research.

Motivation: In recent years, the study of weak convergence in Banach [1] spaces has gained significant attention due to its essential role in various areas of functional analysis, including the theory of distribution, optimization, and approximation methods. The concept of weak convergence was introduced by Banach in the early 20th century, specifically in the 1920s. Banach developed the theory of weak convergence while working in the context of Banach spaces, which are complete normed vector spaces. His work laid the foundation for the study of weak convergence in functional analysis. Fatih Nuray [13] investigated lacunary weak statistical convergence. Motivated by this work, we have investigated some classes of lacunary weak convergent of sequences defined by Orlicz function.

Potential Applications: The work done in this article are on weak convergence. The concept of strong convergence implies weak convergence, but not necessarily conversely. Therefore the work done in this article can be applied for other areas of research, and since, it covers a larger class of sequences.

2. Definition and Preliminaries

The concept of the difference sequence space $Z(\Delta)$ was first introduced by Kizmaz [9], defined as follows:

$$Z(\Delta) = \{x = (x_k) : (\Delta x_k) \in X\},$$

where $\Delta x = (\Delta x_k) = (x_k - x_{k+1})$, for all $k \in \mathbb{N}$.

Later, Et and Colak [6] extended this idea by defining generalized difference sequence spaces, expressed as:

$$Z(\Delta^p) = \{x = (x_k) : (\Delta^p x_k) \in X\},$$

for $Z = \ell_\infty$, c, and c_0, where $\Delta^p x_k = \Delta^{p-1}x_k - \Delta^{p-1}x_{k+1}$ and $\Delta^0 x_k = x_k \; \forall \; k \in \mathbb{N}$.The binomial expansion for this generalized difference operator is provided below:

$$\Delta^p x_k = \sum_{v=0}^{p}(-1)^v \begin{pmatrix} p \\ v \end{pmatrix} x_{k+v}, \text{ for all } k \in \mathbb{N}. \tag{2.1}$$

These generalized difference sequence spaces have been further studied by researchers such as Tripathy [16], Tripathy , Et and Altin [17], among others.

Consider a sequence $\theta = (k_s)$ of positive integers, which is termed lacunary if $k_0 = 0, 0 < k_s < k_{s+1}$, and $h_s = k_s - k_{s-1} \to \infty$ as $s \to \infty$. The intervals determined by θ are denoted by $I_s = (k_{s-1}, k_s)$, and $q_s = k_s/k_{s-1} \; \forall \; s \in N$.

According to Freedman et al., the space of lacunary strongly convergent sequence N_θ is defined as follows: [7]

$$N_\theta = \left\{ x : \lim_{s\to\infty} \frac{1}{h_s} \sum_{i\in I_s} |x_i - L| = 0, \text{ for some } L \right\}.$$

An Orlicz function $\mathcal{H} : [0,\infty) \to [0,\infty)$ is defined such that $\mathcal{H}(0) = 0, \mathcal{H}(x) > 0$ for $x > 0,$ and $\mathcal{H}(x) \to \infty,$ as $x \to \infty.$ This function is continuous, non-decreasing, and convex.

Lindenstrauss and Tzafriri [12] introduced the concept of the Orlicz function to define the sequence space

$$\ell_{\mathcal{H}} = \left\{ (x_i) \in \omega : \sum_{i=1}^{\infty} \mathcal{H}\left(\frac{|x_i|}{\rho}\right) < \infty, \text{ for some } \rho > 0 \right\},$$

where ω denotes the class of all sequences. The norm of the sequence space $\ell_{\mathcal{H}}$ is given by

$$||x|| = \inf \left\{ \rho > 0 : \sum_{i=1}^{\infty} \mathcal{H}\left(\frac{|x_i|}{\rho}\right) \leq 1 \right\},$$

which transforms it into a Banach space, commonly referred to as an Orlicz sequence space. Various researchers, including Tripathy and Esi [18], Parashar and Choudhury [14], Tripathy and Mahanta [15], have explored different forms of Orlicz sequence spaces.

Definition 2.1. *A sequence (x_i) in a normed linear space X is called weakly convergent to an element $L \in X$ if*

$$\lim_{i\to\infty} f(x_i - L) = 0, \textit{ for all } f \in X',$$

where X' represents the continuous dual of X .

Definition 2.2. *A sequence (x_i) in a normed linear space X is said to be lacunary weakly convergent to $L \in X$ if*

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} f(x_i - L) = 0,$$

for all $f \in X'$, where X' is the continuous dual of X. In this context, the notation $\mathcal{D}_\theta^w$ used to denote lacunary weak convergent.

Definition 2.3. *The sequence space $\mathcal{J}$ is termed solid if, for any sequence of scalar (α_i) with $|\alpha_i| \leq 1$ for all $i \in \mathbb{N}$, the condition $(x_i) \in \mathcal{J}$ implies $(\alpha_i x_i) \in \mathcal{J}$.*

Definition 2.4. *A sequence space $\mathcal{J} \subset \omega$ referred to as monotone if it includes all pre-images of its step spaces.*

Definition 2.5. *A sequence space $\mathcal{J} \subset \omega$ is known as symmetric if, whenever $(x_i) \in \mathcal{J}$, the permuted sequence $(x_{\pi(i)})$ also belongs to $\mathcal{J}$, where π is a permutation of $\mathbb{N}$.*

Definition 2.6. *A sequence space $\mathcal{J}$ is said to be convergence free, if x is in $\mathcal{J}$ and if $y_k = 0$ whenever $x_k = 0$, then y is in $\mathcal{J}$*

Lemma 2.1. *A sequence space $\mathcal{J}$ being solid does not necessary imply that $\mathcal{J}$ is monotone.*

Definition 2.7. *An Orlicz function $\mathcal{H}$ satisfies the Δ_2- condition if there exists a constant $T > 0$ such that, for each $z \geq 0$*

$$\mathcal{H}(2z) \leq T\mathcal{H}(z).$$

3. Main Result

In this section we introduce the following classes of sequences and establish result invloving them.

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0 = \left\{ x = (x_k) : \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) = 0, \quad \text{for some} \quad g > 0 \right\};$$

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1 = \left\{ x = (x_k) : \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k - L)|}{g} \right) = 0, \quad \text{for some} \quad L \textit{ and } g > 0 \right\};$$

$$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty = \left\{ x = (x_k) : \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g} \right) < \infty, \quad \text{for some} \quad g > 0 \right\}.$$

We state, without proof, the following result.

Theorem 3.1. *The classes of sequences* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0, [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$ *and* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ *are linear spaces.*

Theorem 3.2. *For any Orlicz function* $\mathcal{H}$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ *is a normed linear space for the given norm*

$$\xi_{\Delta^p}(x) = \sum_{i=1}^{p} |f(x_i)| + \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g}\right) \le 1, s = 1, 2, 3, ... \right\};$$

where the infimum is taken over all $g > 0$.

Proof. Clearly, $\xi_{\Delta^p}(x) = \xi_{\Delta^p}(-x), x = \theta$ implies $\Delta^p x_k = 0$ and as such we have $\mathcal{H}(\theta) = 0$. Therefore $\xi_{\Delta^p}(\theta) = 0$. Conversely support that $\xi_{\Delta^p}(x) = 0$, then

$$\sum_{i=1}^{p} |f(x_i)| + \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g}\right) \le 1, s = 1, 2, 3, ... \right\} = 0.$$

$$\Rightarrow \sum_{i=1}^{p} |f(x_i)| = 0 \text{ and } \inf\left\{ g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g}\right) \le 1, s = 1, 2, 3, ... \right\} = 0.$$

From the first part we have

$$x_i = \bar{\theta}, for\ i = 1, 2, 3, ..., m. \tag{3.2}$$

where, $\bar{\theta}$ is the zero element. In accordance with this second section, there exists some g_ε $(0 < g_\varepsilon < \varepsilon)$ for a given $\varepsilon > 0$. such that

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_\varepsilon}\right) \le 1$$

$$\Rightarrow \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_\varepsilon}\right) \le 1.$$

Thus,

$$\sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{\varepsilon}\right) \le \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_\varepsilon}\right) \le 1.$$

Suppose $\Delta^p x_{c_i} \ne \bar{\theta}$, for each i. Taking $\varepsilon \to 0$, we have $\frac{|f(\Delta^p x_{c_i})|}{\varepsilon} \to \infty$.
It follows that

$$\frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{\varepsilon}\right) \to \infty,$$

as $\varepsilon \to 0$, for $c_i \in I_s$. Hence we arrive at a contradiction. Therefore, $\Delta^p x_{c_i} = \bar{\theta}$, for each $i \in N$. Thus $\Delta^p x_k = \bar{\theta}, \forall k \in N$.

Therefore, it follows from (2.1) and (3.2) that $x_k = \bar{\theta}, \forall k \in N$. Hence $x = \theta$.

Next let $g_1, g_2 > 0$ such that

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_1}\right) \leq 1.$$

and

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_2}\right) \leq 1.$$

Let $g = g_1 + g_2$, then we have

$$\sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p(x_k + y_k)|}{g}\right) \leq 1.$$

Given that the $g's$ are not negative, we have

$$\xi_{\Delta^p}(x+y) = \sum_{i=1}^{p} |f(x_i+y_i)| + \inf\left\{g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p(x_k + y_k))|}{g}\right) \leq 1, s = 1, 2, 3, ...\right\}$$

$\Rightarrow \xi_{\Delta^p}(x + y) \leq \xi_{\Delta^p}(x) + \xi_{\Delta^p}(y)$.

Let $\varphi \neq 0$, and $\varphi \in C$, then

$$\xi_{\Delta^p}(\varphi x) = \sum_{i=1}^{p} |f(\varphi x_i)| + \inf\left\{g > 0 : \sup_s \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p(\varphi x_k))|}{g}\right) \leq 1, s = 1, 2, 3, ...\right\}$$

$$\leq |\varphi| \xi_{\Delta^p}(x).$$

This completes the theorem's proof. □

Theorem 3.3. *The sequence space* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ *is convex.*

Proof. Consider $(x_k), (y_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$. Then from the definition of the space we can write

$$\lim_{s \to \infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_x}\right) < \infty, \quad \text{for some} \quad g_x > 0,$$

$$\lim_{s \to \infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p y_k)|}{g_y}\right) < \infty, \quad \text{for some} \quad g_y > 0.$$

Now, for $z = \lambda x + (1 - \lambda)y$ we have to show that

$$\lim_{s \to \infty} \frac{1}{h_s} \sum_{k \in I_s} \mathcal{H}\left(\frac{|f(\Delta^p(\lambda x_k + (1 - \lambda)y_k)|}{g_z}\right) < \infty, \quad \text{for some} \quad g_z > 0$$

Since $\mathcal{H}$ is convex function, we have

$$\mathcal{H}\left(\frac{|f(\Delta^p(\lambda x_k + (1 - \lambda)y_k)|}{g_z}\right) \leq \lambda \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_x}\right) + (1 - \lambda)\mathcal{H}\left(\frac{|f(\Delta^p y_k)|}{g_y}\right),$$

where $g_z = \lambda g_x + (1-\lambda)g_y$

Now, taking the limit $s \to \infty$:

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p z_k)|}{g_z}\right) \le \lambda \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g_x}\right) + (1-\lambda) \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\Delta^p y_k)|}{g_y}\right)$$

Therefore, $z = \lambda x + (1-\lambda)y \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$.

Hence $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ is convex. □

Theorem 3.4. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Orlicz functions satisfying Δ_2- condition. Then*

(i) $[\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta^p]_\mathcal{G} \subseteq [\mathcal{D}_\theta^w, \mathcal{H}_2.\mathcal{H}_1, \Delta^p]_\mathcal{G}$.

(ii)$[\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta^p]_\mathcal{G} \cap [\mathcal{D}_\theta^w, \mathcal{H}_2, \Delta^p]_\mathcal{G} \subseteq [\mathcal{D}_\theta^w, \mathcal{H}_1 + \mathcal{H}_2, \Delta^p]_\mathcal{G}$, where $\mathcal{G} = 0, 1,$ and ∞.

Proof. We prove it in the case of $\mathcal{G} = 0$, we will apply same methods to the remaining cases.

(i) Let $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}_1, \Delta^p]_0$. Then $\exists\ g > 0$ such that

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}_1\left(\frac{|f(\Delta^p x_k)|}{g}\right) = 0.$$

Let $0 < \varepsilon < 1$ and $0 < \delta < 1$ such that $\mathcal{H}_2(t) < \varepsilon$, for $0 \le t < \delta$.

Let $y_k = \mathcal{H}_1\left(\frac{|f(\Delta^p x_k)|}{g}\right)$ and consider

$$\sum_{k\in I_s} \mathcal{H}_2(y_k) = \sum_1 \mathcal{H}_2(y_k) + \sum_2 \mathcal{H}_2(y_k),$$

where the summations are over $y_k > \delta$ in the second summation and over $y_k \le \delta$ in the first.

Since,

$$\frac{1}{h_s} \sum_1 \mathcal{H}_2(y_k) < \mathcal{H}_2(2) \frac{1}{h_s} \sum_1 (y_k), \tag{3.3}$$

for $y_k > \delta$, we have

$$y_k < 1 + \frac{y_k}{\delta}.$$

Given that $\mathcal{H}_2$ is convex and non-decreasing, it follows thatSince, $\mathcal{H}_2$ is non-decreasing and convex, it follows that

$$\mathcal{H}_2(y_k) < \frac{1}{2}\mathcal{H}_2(2) + \frac{1}{2}\mathcal{H}_2\left(\frac{2y_k}{\delta}\right).$$

Since, $\mathcal{H}_2$ satisfies Δ_2- conditions, we have

$$\mathcal{H}_2(y_k) = K\frac{y_k}{\delta}\mathcal{H}_2(2).$$

Hence,

$$\frac{1}{h_s}\sum_2 \mathcal{H}_2(y_k) \leq max\left(1, K\delta^{-1}\mathcal{H}_2(2)\right)\frac{1}{h_s}\sum_2 y_k. \tag{3.4}$$

Taking limit as $s \to \infty$, from (3.3) and (3.4) we have

$$(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}_2.\mathcal{H}_1, \Delta^p]_0.$$

Similar proof can be shown for the other cases.

(ii) The proof is obvious and omitted. □

Taking $\mathcal{H}_1(x) = x$ and $\mathcal{H}_2 = \mathcal{H}(x)$ in Theorem 3.4(i) we have the following particular case.

Corollary 3.1. $[\mathcal{D}_\theta^w, \Delta^p]_0 \subseteq [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$

Theorem 3.5. *If $p \geq 1$, then $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^{p-1}]_\mathcal{G} \subset [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$ for $\mathcal{G} = 0, 1, \infty$. In general $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^i]_\mathcal{G} \subset [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$ for $i = 0, 1, 2, ..., p-1$.*

Proof Let $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^{p-1}]_0$.

Then we have,

$$\lim_{s\to\infty}\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_k)|}{g}\right) = 0, \text{ for some } g > 0. \tag{3.5}$$

Given that $\mathcal{H}$ is convex and non-decreasing, it follows that

$$\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{2g}\right) = \frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_k - \Delta^{p-1}x_{k+1})|}{2g}\right)$$
$$\leq \left(\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_k)|}{g}\right) - \frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^{p-1}x_{k+1})|}{g}\right)\right)$$

as $s \to \infty$, we have

$$\frac{1}{h_s}\sum_{k\in I_s}\mathcal{H}\left(\frac{|f(\Delta^p x_k)|}{g}\right) = 0, \text{ by (3.5)}$$

which implies $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$.

The remaining cases will proceed in a similar manner.

Proceeding inductively we have, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^i]_\mathcal{G} \subset [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\mathcal{G}$ and $i = 0, 1, ..., p-1$.

The next example strictly follows the inclusion above.

Example 3.1. *Let $\theta = (2^s)$ be a lacunary sequence and $\mathcal{H}(x) = x$. Consider a sequence $(x_k) = (k^{p-1})$. Then $\Delta^p(x_k) = 0, \Delta^{p-1}x_k = (-1)^{m-1}$(m-1)! for all $k \in N$. Therefore $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ but $(x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^{p-1}]_0$*

Theorem 3.6. *The space $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_{\mathcal{G}}$, where, in general, $\mathcal{G} = 0, 1, \infty$ are not solid. The space $[\mathcal{D}_\theta^w, \mathcal{H}]_0$ and $[\mathcal{D}_\theta^w, \mathcal{H}]_\infty$ are solid.*

Proof Let $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}]_0$.

Then there exists $g > 0$ such that

$$\lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(x_k)|}{g}\right) = 0.$$

Let (γ_k) be a sequence of scalars such that $|\gamma_k| \le 1$. Then for each s we can write,

$$\frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\gamma_k x_k)|}{g}\right) \le \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(x_k)|}{g}\right) \tag{3.6}$$

$$\Rightarrow \lim_{s\to\infty} \frac{1}{h_s} \sum_{k\in I_s} \mathcal{H}\left(\frac{|f(\gamma_k x_k)|}{g}\right) = 0.$$

$$\Rightarrow (\gamma_k \alpha_k) \in [\mathcal{D}_\theta^w, \mathcal{H}]_0.$$

From the above inequality (3.6) it follows that $[\mathcal{D}_\theta^w, \mathcal{H}]_\infty$ is solid.

To show that the spaces $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$, $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$ are not solid, in general, we illustrate the following examples.

Example 3.2. *Consider the function $f(x) = x$, $\forall\ x \in R$, and let $X = R$, with $p = 1$,. Let us consider the sequence (x_k), defined by $x_k = k$, $\forall\ k \in N$. Let $\mathcal{H}(x) = x^r$, $r \ge 1$ and the lacunary sequence $\theta = (2^s)$. Then $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$ and $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$. Let $(\gamma_k) = ((-1)^k)$, then $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_1$ and $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_\infty$.*

We consider the following example to show that $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ is not solid in general.

Example 3.3. *Let $X = R$ and consider the function $f(x) = x, \forall\ x \in R$. let $p = 1$, Let us now consider the sequence (x_k), which is defined as $x_k = 1$, $\forall\ k \in N$. Assume that $\mathcal{H}(x) = x^r$, $r = 2$, and that the lacunary sequence is $\theta = (2^s)$. Let $(\gamma_k) = ((-1)^k), \forall\ k \in N$. Then, $(\gamma_k x_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$.*
Thus, the set $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ is not solid.

The following result is a consequence of Lemma 1 and Theorem 6.

Corollary 3.2. *The spaces $[\mathcal{D}_\theta^w, \mathcal{H}]_0$ and $[\mathcal{D}_\theta^w, \mathcal{H}]_\infty$ are monotone.*

Result 1. *The space $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$ is not convergence free.*

Proof The following example makes it obvious.

Example 3.4. *Let* $p = 1, \mathcal{H} = x$ *and* $f(x) = x$. *Consider a lacunary sequence* $\theta = (2^s)$. *Consider a sequence* (x_k) *which is define as*

$$x_k = \frac{1}{2}\left(1 - (-1)^k\right)$$

Then, $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$. *Consider the sequence* (y_k) *defined as*

$$x_k = \begin{cases} k, & \text{if } k \text{ is odd} \\ 0, & \text{if } k \text{ is even.} \end{cases}$$

Then, $(y_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$.

Result 2. *The spaces* $[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_{\mathcal{G}}$, *where* $\mathcal{G} = 0, 1, \infty$ *are not symmetric in general.*

The following example is given to support the previous result.

Example 3.5. *Let* $p = 1$, *let* $X = \mathbb{R}$, *and the function* $f(x) = x$, $\forall x \in \mathbb{R}$, *be considered. Let* $\mathcal{H}(x) = x^2$, *and a lacunary sequence* $\theta = (2^s)$. *Considering the sequence* (x_k) *where* $(x_k) \in [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_0$, *define it as:*

$$x_k = \begin{cases} 1 & \text{if } k = 2^m \text{for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

After rearranging the sequence (x_k) *as follows, let* (y_k) *be considered:*

$$y_k = (x_1, x_2, x_4, x_3, x_8, \dots)$$

Then, $(y_k) \notin [\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_{\mathcal{G}}$, *where* $\mathcal{G} = 0, 1, \infty$.
$[\mathcal{D}_\theta^w, \mathcal{H}, \Delta^p]_{\mathcal{G}}$, *where* $\mathcal{G} = 0, 1, \infty$ *are not symmetric in general.*

4. Conclusion

In this paper, we have introduced and studied the concept of difference lacunary weak convergence in sequences defined by an Orlicz function. Through our exploration, we have thoroughly examined the algebraic and topological properties of these sequences, providing a foundational understanding of their structure and behavior. Additionally, we have established several key inclusion relationships between these newly defined spaces and other known sequence spaces, further enriching the framework within which these sequences operate. Our findings contribute to the broader field of functional analysis, particularly in the study of sequence spaces and Orlicz functions, offering new insights and potential avenues for future research

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References

[1] Banach, S. (1932). Theorie des Operations Lintaires.

[2] Dowari, P. J., & Tripathy, B. C. (2020). Lacunary difference sequences of complex uncertain variables. Methods of Functional Analysis and Topology, 26(4), 327–340.

[3] Dowari, P. J., & Tripathy, B. C. (2021). Lacunary convergence of sequences of complex uncertain variables. Malaysian J. Math. Sci, 15(1), 91-108.

[4] Demirci, I. A., Kişi, Ö., & Gurdal, M. (2024). Lacunary A-statistical convergence of fuzzy variable sequences via Orlicz function. Maejo International Journal of Science and Technology, 18(1), 88-100.

[5] Ercan, S., Altin, A., & Bektas, C. A. (2020). On lacunary weak statistical convergence of order α. Commun. Stat.-Theory. Meth, 49(7), 1653-1664.

[6] Et, M., & Colack, R. (1995). On some generalized difference sequence spaces. Soochow J. Math, 21(4), 377-386.

[7] Freedman, A. R., Sember, J. J. & Raphael, M. (1978). Some Cesàro-type summability spaces. Proceedings of the London Mathematical Society, 3(3), 508-520.

[8] Gumus, H. (2015). Lacunary Weak I-Statistical convergence. Gen. Math. Notes, 28(10), 50-58.

[9] Kizmaz, H. (1981). On certain sequence spaces. Canadian Math. Bull, 24, 169-176.

[10] Kişi, Ö., & Gurdal, M.(2022). Orlicz-lacunary convergent triple sequences and ideal convergence. Communication Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 71(2), 681-600.

[11] Kişi, Ö., & Ünal, H.K. (2021). Rough statistical convergence of difference double sequences in normed linear spaces. Honam Mathematical Journal, 43(1), 47-58.

[12] Lindenstrauss, J., & Tzafriri, L. (1971). On Orlicz sequence spac. Israel J. Math, 101, 379-390.

[13] Nuray, F.(2011). Lacunary weak statistical convergence. Mathematica Bohemica, 136(3), 259-268.

[14] Parashar, S. D., & Choudhary, B. (1994). Sequence spaces defined by Orlicz functions. Israel J. Math, 25, 419-428.

[15] Tripathy, B. C., & Mahanta, S. (1994). On a class of sequences related to the l^p space defined by Orlicz functions. Soochow journal of Mathematics, 25(4), 419-428.

[16] Tripathy, B. C. (2004). Generalized difference paranormed statistically convergent sequence space. Indian J. Pure Appl. Math, 35(5), 655-663.

[17] Tripathy, B. C. (2003). Generalized difference sequence spaces defined by Orlic function in a locally convex space. J. Analysis and Appl, 1(3), 175-192.

[18] Tripathy, B. C., & Esi, A. (2005). Generalized lacunary difference sequence spaces defined by Orlicz functions. Journal Math. Society of the Philippines, 28, 50-57.

[19] Tamuli, B., & Tripathy, B. C. (2024). Generalized difference lacunary weak convergence of sequences. Sahand Communication in Math Analysis, 21(2), 195-206 .

[20] Tamuli, B., & Tripathy, B. C. (2024). Lacunary weak convergence of sequences defined by Orlicz function. J. Appl. Anal, https://doi.org/10.1515/jaa-2024-0027

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$\mathcal{N}(\kappa)$-QUASI-EINSTEIN MANIFOLDS ADMITTING SCHOUTEN TENSOR

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ABSTRACT. In this note, we study $\mathcal{N}(\kappa)$-quasi-Einstein (in short, $\mathcal{N}(\kappa)$-QE) manifolds admitting the Schouten tensor satisfying certain curvature conditions. At last, the existence of an $\mathcal{N}(\kappa)$-QE manifold is verified by an example.

Keywords: k-nullity distribution, $\mathcal{N}(\kappa)$-(QE) manifolds, Schouten tensor.

2020 Mathematics Subject Classification: 53C18, 53C21, 53C25.

1. Introduction

A Riemannian manifold $(\mathcal{M}^n, g)$ is named an Einstein manifold [1] if its $(0,2)$ type Ricci tensor $Ric(\neq 0)$ satisfies: $Ric = \frac{scal}{n} g$, here $scal$ denotes the scalar curvature of $\mathcal{M}^n$. Einstein manifolds play a key role in mathematical physics, Riemannian geometry as well as in general theory of relativity . Due to its significant physical applications in broad perspectives, these manifolds have been explored by many geometers.

An $(\mathcal{M}^n, g)$ is said to be a quasi-Einstein (QE) [3] if its $Ric(\neq 0)$ fulfills

$$Ric(\Upsilon_1, \Upsilon_2) = \vartheta g(\Upsilon_1, \Upsilon_2) + \Phi \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2), \tag{1.1}$$

for some smooth functions $\vartheta, \Phi(\neq 0)$, and 1-form $\mathcal{A}(\neq 0)$ such that

$$g(\Upsilon_1, \ell) = \mathcal{A}(\Upsilon_1), \quad g(\ell, \ell) = \mathcal{A}(\ell) = 1, \tag{1.2}$$

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for any vector field Υ_1 and a unit vector field ℓ named the generator of QE manifold. $\mathcal{A}$ is also called the associated 1-form. It is clear from (1.1) that for $\Phi = 0$, a QE manifold reduce to an Einstein manifold.

From (1.1) and (1.2), we have

$$scal = n\vartheta + \Phi, \quad R(\Upsilon_1) = \vartheta\Upsilon_1 + \Phi\mathcal{A}(\Upsilon_1)\ell, \tag{1.3}$$

where R is the Ricci operator defined by

$$g(R(\Upsilon_1), \Upsilon_2) = Ric(\Upsilon_1, \Upsilon_2), \tag{1.4}$$

for $\Upsilon_1, \Upsilon_2 \in \Gamma(TM)$.

The concept of QE manifolds came into existence during considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces as well as during the study of exact solutions to Einstein's field equations. For example, the Robertson-Walker spacetimes are QE manifolds. Also, QE spacetime can be used as a model of the perfect fluid spacetime in general relativity [13].

The QE manifolds have also been studied by many authors such as Bilal et. al. [2], Chaki [4], De and Ghosh [8], Vasiulla et al. [20] and many others.

Let $\mathcal{K}$ denotes the Riemannian curvature tensor and κ-nullity distribution $\mathcal{N}(\kappa)$ of an $(\mathcal{M}^n, g)$ [19] is defined by

$$\mathcal{N}(\kappa) : p \to \mathcal{N}_p(\kappa) = \{\Upsilon_3 \in T_pM : \mathcal{K}(\Upsilon_1, \Upsilon_2)\Upsilon_3 = \kappa[g(\Upsilon_2, \Upsilon_3)\Upsilon_1 - g(\Upsilon_1, \Upsilon_3)\Upsilon_2]\}, \tag{1.5}$$

for all vector fields Υ_1, Υ_2, $\Upsilon_3 \in \Gamma(\mathcal{TM})$ ($\Gamma(\mathcal{TM})$: the set of all smooth vector fields on $\mathcal{M}^n$) and κ being some smooth function. Also see [12, 14].

From (1.2) and (1.5), we have

$$Ric(\Upsilon_1, \ell) = \kappa(n-1). \tag{1.6}$$

Similarly, κ-nullity distribution $\mathcal{N}(\kappa)$ of a Lorentzian manifold can also be defined. In a QE manifold, if ℓ belongs to some κ-nullity distribution $\mathcal{N}(\kappa)$, then $\mathcal{M}$ is named $\mathcal{N}(\kappa)$-QE manifold [15]. For more detailed study of $\mathcal{N}(\kappa)$-QE manifolds, we refer the papers [5, 6, 18, 22]. Indeed, κ is not arbitrary as the following lemma:

Lemma 1.1. [15] *In an n-dimensional $\mathcal{N}(\kappa)$-QE manifold it follows that*

$$\kappa = \frac{\vartheta + \Phi}{n-1}. \tag{1.7}$$

It is to be noted that in an n-dimensional $\mathcal{N}(\kappa)$-QE manifold [15]

$$\mathcal{K}(\Upsilon_1,\Upsilon_2,\ell)=\frac{\vartheta+\Phi}{n-1}[\mathcal{A}(\Upsilon_2)\Upsilon_1-\mathcal{A}(\Upsilon_1)\Upsilon_2], \tag{1.8}$$

which is equivalent to

$$\mathcal{K}(\ell,\Upsilon_1,\Upsilon_2)=\frac{\vartheta+\Phi}{n-1}[g(\Upsilon_1,\Upsilon_2)\ell-\mathcal{A}(\Upsilon_2)\Upsilon_1]=-\mathcal{K}(\Upsilon_1,\ell,\Upsilon_2). \tag{1.9}$$

Taking $\Upsilon_2=\ell$ in (1.9), we have

$$\mathcal{K}(\ell,\Upsilon_1,\ell)=\frac{\vartheta+\Phi}{n-1}[\mathcal{A}(\Upsilon_1)\ell-\Upsilon_1]. \tag{1.10}$$

A conformally flat QE manifold of dimension n is an $\mathcal{N}(\frac{\vartheta+\Phi}{n-1})$-QE manifold and hence, a QE manifold of dimension 3 is an $\mathcal{N}(\frac{\vartheta+\Phi}{2})$-QE manifold, as demonstrated in [15]. The conformally flat QE manifolds are certain $\mathcal{N}(\kappa)$-QE manifolds [17]. In 2021, Hazra and Sarkar [11] studied certain curvature conditions on $\mathcal{N}(k)$-QE manifolds. The derivation conditions $\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{K}=0$ and $\mathcal{K}(\ell,\Upsilon_1)\cdot Ric=0$ have also been studied in [17]. In [15], the authors studied the derivation conditions $N(\ell,\Upsilon_1)\cdot N=0$ and $N(\ell,\Upsilon_1)\cdot\mathcal{K}=0$ on $\mathcal{N}(\kappa)$-QE manifolds, where N denotes the concircular curvature tensor.

The Weyl conformal curvature tensor $\mathcal{C}$ [7, 9] of an $(\mathcal{M}^n,g)$ is defined by

$$\begin{aligned}\mathcal{C}(\Upsilon_1,\Upsilon_2,\Upsilon_3)=&\ \mathcal{K}(\Upsilon_1,\Upsilon_2,\Upsilon_3)-\frac{1}{n-2}[Ric(\Upsilon_2,\Upsilon_3)\Upsilon_1-Ric(\Upsilon_1,\Upsilon_3)\Upsilon_2\\&+g(\Upsilon_2,\Upsilon_3)R(\Upsilon_1)-g(\Upsilon_1,\Upsilon_3)R(\Upsilon_2)]\\&+\frac{scal}{(n-1)(n-2)}[g(\Upsilon_2,\Upsilon_3)\Upsilon_1-g(\Upsilon_1,\Upsilon_3)\Upsilon_2],\end{aligned} \tag{1.11}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathcal{TM})$. Also, in n-dimensional $\mathcal{N}(\kappa)$-QE manifolds, $\mathcal{C}$ satisfies:

$$\mathcal{C}(\Upsilon_1,\Upsilon_2,\Upsilon_3)=-\frac{\Phi}{n-2}[\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_3)\Upsilon_1-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_3)\Upsilon_2], \tag{1.12}$$

$$\mathcal{C}(\Upsilon_1,\Upsilon_2,\ell)=-\frac{\Phi}{n-2}[\mathcal{A}(\Upsilon_2)\Upsilon_1-\mathcal{A}(\Upsilon_1)\Upsilon_2], \tag{1.13}$$

$$\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3)=0,\quad \mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\ell)=0, \tag{1.14}$$

$$\mathcal{C}(\ell,\Upsilon_2,\Upsilon_3)=-\frac{\Phi}{n-2}[\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_3)\ell-\mathcal{A}(\Upsilon_3)\Upsilon_2], \tag{1.15}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathcal{TM})$.

The projective curvature tensor $\mathcal{P}$ is defined by [10, 21]

$$\mathcal{P}(\Upsilon_1,\Upsilon_2,\Upsilon_3)=\mathcal{K}(\Upsilon_1,\Upsilon_2,\Upsilon_3)-\frac{1}{n-1}[Ric(\Upsilon_2,\Upsilon_3)\Upsilon_1-Ric(\Upsilon_1,\Upsilon_3)\Upsilon_2], \tag{1.16}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_3\in\Gamma(\mathcal{TM})$. Also, in n-dimensional $\mathcal{N}(\kappa)$-QE manifolds, $\mathcal{P}$ satisfies:

$$\mathcal{P}(\Upsilon_1,\Upsilon_2,\ell)=0, \tag{1.17}$$

$$\mathcal{P}(\ell,\Upsilon_1,\Upsilon_2)=\frac{\Phi}{n-1}[g(\Upsilon_1,\Upsilon_2)\ell-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\ell], \tag{1.18}$$

$$\mathcal{A}(\mathcal{P}(\Upsilon_1,\Upsilon_2,\Upsilon_4)=\frac{\Phi}{n-1}[g(\Upsilon_2,\Upsilon_4)\mathcal{A}(\Upsilon_1)-g(\Upsilon_1,\Upsilon_4)\mathcal{A}(\Upsilon_2)], \tag{1.19}$$

for all $\Upsilon_1,\Upsilon_2,\Upsilon_4\in\Gamma(\mathcal{TM})$.

In an $(\mathcal{M}^n,g)$, the Schouten tensor $\mathcal{S}$ is given by [16]

$$S(\Upsilon_1,\Upsilon_2)=\frac{1}{n-2}\Big[Ric(\Upsilon_1,\Upsilon_2)-\frac{scal}{2(n-1)}g(\Upsilon_1,\Upsilon_2)\Big], \tag{1.20}$$

for $\Upsilon_1,\Upsilon_2\in\Gamma(\mathcal{TM})$.

In an $\mathcal{N}(\kappa)$-QE manifold, the Schouten tensor takes the form

$$S(\Upsilon_1,\Upsilon_2)=\frac{1}{n-2}\Big[\Big\{\vartheta-\frac{scal}{2(n-1)}\Big\}g(\Upsilon_1,\Upsilon_2)+\Phi\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\Big]. \tag{1.21}$$

By contracting (1.21) over Υ_1 and Υ_2, we find

$$scal=\frac{1}{n-2}\Big[\Big\{\vartheta-\frac{scal}{2(n-1)}\Big\}n+\Phi\Big]. \tag{1.22}$$

Taking $\Upsilon_1=\ell$ in (1.21), we have

$$S(\ell,\Upsilon_2)=\frac{\mathcal{A}(\Upsilon_2)}{n-2}\Big[\vartheta+\Phi-\frac{scal}{2(n-1)}\Big]. \tag{1.23}$$

2. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{S}=0$

Let us consider an $\mathcal{N}(\kappa)$-QE manifold that satisfies $\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{S}=0$. Then

$$(\mathcal{K}(\ell,\Upsilon_1)\cdot\mathcal{S})(\Upsilon_2,\ell)=-\mathcal{S}(\mathcal{K}(\ell,\Upsilon_1)\Upsilon_2,\ell)-\mathcal{S}(\Upsilon_2,\mathcal{K}(\ell,\Upsilon_1)\ell)=0. \tag{2.24}$$

From (1.9) and (1.21), we find

$$\mathcal{S}(\mathcal{K}(\ell,\Upsilon_1)\Upsilon_2,\ell)=\frac{1}{n-2}\Big[\Big(\vartheta+\Phi-\frac{scal}{2(n-1)}\Big)\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)\Big], \tag{2.25}$$

where $g(\mathcal{K}(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)=\overline{\mathcal{K}}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)$, $\overline{\mathcal{K}}$ is the $(0,4)$ type curvature tensor.

Also, from (1.10) and (1.21), we find

$$\mathcal{S}(\Upsilon_2,\mathcal{K}(\ell,\Upsilon_1)\ell)=-\frac{1}{n-2}\Big[\Big(\vartheta-\frac{scal}{2(n-1)}\Big)\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)\Big], \tag{2.26}$$

where $g(\mathcal{K}(\ell,\Upsilon_1)\ell,\Upsilon_2)=-\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)$ and $g(\mathcal{K}(\ell,\Upsilon_1)\ell,\ell)=0$ being used.

By virtue of (2.25) and (2.26), the relation (2.24) yields,

$$\Big(\frac{\Phi}{n-2}\Big)\overline{\mathcal{K}}(\ell,\Upsilon_1,\Upsilon_2,\ell)=0. \tag{2.27}$$

From (1.9) and (2.27), we have

$$\Big(\frac{\Phi k}{n-2}\Big)(g(\Upsilon_1,\Upsilon_2)-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2))=0, \tag{2.28}$$

Since $\Phi \neq 0$ and $g(\Upsilon_1, \Upsilon_2) \neq \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)$, then we have $\kappa = 0$, i.e., $\vartheta + \Phi = 0$. Conversely, if $\kappa = 0$, then in view of (1.9) and (1.10) $\mathcal{M}$ satisfies $\mathcal{K}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$. Thus, we can state:

Theorem 2.1. *An $\mathcal{N}(\kappa)$-QE manifold satisfies $\mathcal{K}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$ if and only if $\vartheta + \Phi = 0$.*

3. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{P}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$

Let an $\mathcal{N}(\kappa)$-QE manifold satisfies $\mathcal{P}(\ell, \Upsilon_1) \cdot \mathcal{S} = 0$. Then

$$(\mathcal{P}(\ell, \Upsilon_1) \cdot \mathcal{S})(\Upsilon_2, \Upsilon_3) = -\mathcal{S}(P(\ell, \Upsilon_1)\Upsilon_2, \Upsilon_3) - \mathcal{S}(\Upsilon_2, \mathcal{P}(\ell, \Upsilon_1)\Upsilon_3) = 0, \tag{3.29}$$

which in view of (1.18) takes the form

$$\frac{\Phi}{n-1}\Big[g(\Upsilon_1, \Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) \tag{3.30}$$
$$+g(\Upsilon_1, \Upsilon_3)\mathcal{S}(\Upsilon_2, \ell) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_3)\mathcal{S}(\Upsilon_2, \ell)\Big] = 0.$$

Since $\Phi(\neq 0)$, therefore, we have

$$g(\Upsilon_1, \Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\mathcal{S}(\ell, \Upsilon_3) \tag{3.31}$$
$$+g(\Upsilon_1, \Upsilon_3)\mathcal{S}(\Upsilon_2, \ell) - \mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_3)\mathcal{S}(\Upsilon_2, \ell) = 0.$$

In view of (1.23), (3.31) gives

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big)\Big(g(\Upsilon_1, \Upsilon_2)\mathcal{A}(\Upsilon_3) + g(\Upsilon_1, \Upsilon_3)\mathcal{A}(\Upsilon_2) - 2\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_3)\Big) = 0,$$

which by contracting over Υ_1 and Υ_2 gives

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big)\mathcal{A}(\Upsilon_3) = 0. \tag{3.32}$$

This gives $\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) = 0$, as $\mathcal{A}(\Upsilon_3) \neq 0$. Thus, we can state:

Theorem 3.1. *An $\mathcal{N}(\kappa)$-QE manifold $\mathcal{M}$ $(n \geq 3)$ satisfies $\mathcal{P}(\ell, \Upsilon_1).\mathcal{S} = 0$ if and only if $\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) = 0$.*

4. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{S}(\Upsilon_1, \ell) \cdot \mathcal{K} = 0$

Let an $\mathcal{N}(\kappa)$-QE manifold satisfies the condition $(\mathcal{S}(\Upsilon_1, \ell) \cdot \mathcal{K})(\Upsilon_2, \Upsilon_3)\Upsilon_4 = 0$. We know that

$$(\mathcal{S}(\Upsilon_1, \ell) \cdot \mathcal{K})(\Upsilon_2, \Upsilon_3)\Upsilon_4 = ((\Upsilon_1 \wedge_{\mathcal{S}} \ell) \cdot \mathcal{K})(\Upsilon_2, \Upsilon_3)\Upsilon_4, \tag{4.33}$$

where the endomorphism $(\Upsilon_1 \wedge_{\mathcal{S}} \Upsilon_2)\Upsilon_3$ is given by

$$(\Upsilon_1 \wedge_{\mathcal{S}} \Upsilon_2)\Upsilon_3 = \mathcal{S}(\Upsilon_2, \Upsilon_3)\Upsilon_1 - \mathcal{S}(\Upsilon_1, \Upsilon_3)\Upsilon_2. \tag{4.34}$$

Rewriting (4.33) as

$$\begin{aligned}(\mathcal{S}(\Upsilon_1,\ell)\cdot\mathcal{K})(\Upsilon_2,\Upsilon_3)\Upsilon_4 &= ((\Upsilon_1\wedge_{\mathcal{S}}\ell)\mathcal{K})(\Upsilon_2,\Upsilon_3)\Upsilon_4-\mathcal{K}((\Upsilon_1\wedge_{\mathcal{S}}\ell)\Upsilon_2,\Upsilon_3)\Upsilon_4\\ &\quad -\mathcal{K}(\Upsilon_2,(\Upsilon_1\wedge_{\mathcal{S}}\ell)\Upsilon_3)\Upsilon_4-\mathcal{K}(\Upsilon_2,\Upsilon_3)(\Upsilon_1\wedge_{\mathcal{S}}\ell)\Upsilon_4,\end{aligned}$$

which by using $\mathcal{S}(\Upsilon_1,\ell)\cdot\mathcal{K}=0$ and (4.34) turns to

$$\begin{aligned}&\mathcal{S}(\ell,\mathcal{K}(\Upsilon_2,\Upsilon_3)\Upsilon_4)\Upsilon_1-\mathcal{S}(\Upsilon_1,\mathcal{K}(\Upsilon_2,\Upsilon_3)\Upsilon_4)\ell-\mathcal{S}(\ell,\Upsilon_2)\mathcal{K}(\Upsilon_1,\Upsilon_3)\Upsilon_4\\ &+\mathcal{S}(\Upsilon_1,\Upsilon_2)\mathcal{K}(\ell,\Upsilon_3),\Upsilon_4-\mathcal{S}(\ell,\Upsilon_3)\mathcal{K}(\Upsilon_2,\Upsilon_1)\Upsilon_4+\mathcal{S}(\Upsilon_1,\Upsilon_3)\mathcal{K}(\Upsilon_2,\ell)\Upsilon_4 \qquad (4.35)\\ &-\mathcal{S}(\ell,W)\mathcal{K}(\Upsilon_2,\Upsilon_3)\Upsilon_1+\mathcal{S}(\Upsilon_1,\Upsilon_4)\mathcal{K}(\Upsilon_2,\Upsilon_3)\ell=0.\end{aligned}$$

By using (1.8), (1.9) and (1.21) in (4.35), and taking the inner product with ℓ, we have

$$\frac{\Phi\kappa}{n-2}[g(\Upsilon_1,\Upsilon_2)\mathcal{A}(\Upsilon_3)\mathcal{A}(\Upsilon_4)-g(\Upsilon_1,\Upsilon_3)\mathcal{A}(\Upsilon_2)\mathcal{A}(\Upsilon_4)]=0. \qquad (4.36)$$

Putting $\Upsilon_3=\ell$ in (4.36), we have

$$\Phi\kappa\mathcal{A}(\Upsilon_4)[g(\Upsilon_1,\Upsilon_2)-\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)]=0. \qquad (4.37)$$

Since $\Phi(\neq 0)$, $\mathcal{A}(\neq 0)$ and $g(\Upsilon_1,\Upsilon_2)\neq\mathcal{A}(\Upsilon_1)\mathcal{A}(\Upsilon_2)$, then $\kappa=0$. If $\kappa=0$, then the converse is trivial. Thus, we have:

Theorem 4.1. *An $\mathcal{N}(\kappa)$-QE manifold satisfies $\mathcal{S}(\Upsilon_1,\ell)\cdot\mathcal{K}=0$ if and only if $\vartheta+\Phi=0$.*

5. $\mathcal{N}(\kappa)$-QE manifolds satisfying $\mathcal{C}\cdot\mathcal{S}=0$

Let an $\mathcal{N}(\kappa)$-(QE) manifold holds $\mathcal{C}\cdot\mathcal{S}=0$. We know that

$$(\mathcal{C}(\Upsilon_1,\Upsilon_2)\cdot\mathcal{S})(\Upsilon_3,\Upsilon_4)=-\mathcal{S}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)-\mathcal{S}(\Upsilon_3,\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4). \qquad (5.38)$$

Making use of (1.21) in (5.38), we have

$$\begin{aligned}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\cdot\mathcal{S})(\Upsilon_3,\Upsilon_4) &= -\frac{1}{n-2}\Big[\Big\{\vartheta-\frac{scal}{2(n-1)}\Big\}\Big(g(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3,\Upsilon_4)\\ &\quad +g(\Upsilon_3,\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4)\Big)+\Phi\Big(\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3)\mathcal{A}(\Upsilon_4) \qquad (5.39)\\ &\quad +\mathcal{A}(\Upsilon_3)\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4)\Big)\Big],\end{aligned}$$

which by using the symmetric property of the metric tensor, and the skew-symmetric property of $\overline{\mathcal{K}}(\Upsilon_1,\Upsilon_2,\Upsilon_3,\Upsilon_4)$ reduces to

$$(\mathcal{C}(\Upsilon_1,\Upsilon_2)\cdot\mathcal{S})(\Upsilon_3,\Upsilon_4)=-\frac{\Phi}{n-2}\Big(\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_3)\mathcal{A}(\Upsilon_4)+\mathcal{A}(\Upsilon_3)\mathcal{A}(\mathcal{C}(\Upsilon_1,\Upsilon_2)\Upsilon_4)\Big).$$

In view of (1.14), the foregoing equation turns to

$$\mathcal{C}((\Upsilon_1, \Upsilon_2) \cdot \mathcal{S})(\Upsilon_3, \Upsilon_4) = 0. \tag{5.40}$$

Thus, we have:

Theorem 5.1. *In an $\mathcal{N}(\kappa)$-QE manifold, the relation $\mathcal{C} \cdot \mathcal{S} = 0$ holds for all $\Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4$.*

6. Schouten-recurrent $\mathcal{N}(\kappa)$-QE manifolds

In 1952, Patterson [16] proposed the idea of Ricci recurrent manifolds . According to him, an $(\mathcal{M}^n, g)$ is said to be Ricci recurrent if

$$(D_{\Upsilon_1} Ric)(\Upsilon_2, \Upsilon_3) = \mathcal{A}(\Upsilon_1) Ric(\Upsilon_2, \Upsilon_3), \tag{6.41}$$

for some 1-form $\mathcal{A}(\neq 0)$.

An $(\mathcal{M}^n, g)$ is named a Schouten recurrent manifold if its Schouten tensor satisfies

$$(D_{\Upsilon_1} \mathcal{S})(\Upsilon_2, \Upsilon_3) = \mathcal{A}(\Upsilon_1) \mathcal{S}(\Upsilon_2, \Upsilon_3). \tag{6.42}$$

We write

$$(D_{\Upsilon_1} \mathcal{S})(\Upsilon_2, \Upsilon_3) = \Upsilon_1 \mathcal{S}(\Upsilon_2, \Upsilon_3) - \mathcal{S}(D_{\Upsilon_1} \Upsilon_2, \Upsilon_3) - \mathcal{S}(\Upsilon_2, D_{\Upsilon_1} \Upsilon_3). \tag{6.43}$$

From (6.42) and (6.43), we have

$$\Upsilon_1 \mathcal{S}(\Upsilon_2, \Upsilon_3) - \mathcal{S}(D_{\Upsilon_1} \Upsilon_2, \Upsilon_3) - \mathcal{S}(\Upsilon_2, D_{\Upsilon_1} \Upsilon_3) = \mathcal{A}(\Upsilon_1) \mathcal{S}(\Upsilon_2, \Upsilon_3). \tag{6.44}$$

Putting $\Upsilon_2 = \Upsilon_3 = \ell$ in (6.44) and using (1.21), we find

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) \mathcal{A}(\Upsilon_1) = \Upsilon_1 \Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big). \tag{6.45}$$

Thus, we have the following result:

Theorem 6.1. *If $(\mathcal{M}^n, g)$ is a Schouten recurrent $\mathcal{N}(\kappa)$-QE manifold, then*

$$\Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big) \mathcal{A}(\Upsilon_1) = \Upsilon_1 \Big(\vartheta + \Phi - \frac{scal}{2(n-1)}\Big),$$

for all $\Upsilon_1 \in \Gamma(\mathcal{TM})$.

A Schouten recurrent manifold is Schouten symmetric iff $\mathcal{A} = 0$. Thus, we have:

Corollary 6.1. *If $(\mathcal{M}^n, g)$ is a Schouten symmetric $\mathcal{N}(\kappa)$-QE manifold, then $\vartheta + \Phi - \frac{scal}{2(n-1)}$ is constant.*

Corollary 6.2. *If $(\mathcal{M}^n, g)$ is a Schouten symmetric $\mathcal{N}(\kappa)$-QE manifold and if $\vartheta+\Phi-\frac{scal}{2(n-1)}$ is constant, then either $\vartheta+\Phi-\frac{scal}{2(n-1)}=0$ or $(\mathcal{M}^n, g)$ reduces to a Schouten symmetric $\mathcal{N}(\kappa)$-QE manifold.*

7. Example

Define a Riemannian metric g in 4-dimensional space $\mathbb{R}^4$ by

$$ds^2 = g_{ij} d\varkappa^i d\varkappa^j = (1+2p)[(d\varkappa^4)^2 + (d\varkappa^3)^2 + (d\varkappa^2)^2 + (d\varkappa^1)^2], \tag{7.46}$$

where $\varkappa^1, \varkappa^2, \varkappa^3, \varkappa^4$ are non-zero finite and $p = e^{\varkappa^1} k^{-2}$. Then the covariant and contravariant components of the metric tensor are

$$g_{ij} = (1+2p), \ for\ i=j=1,2,3,4, \quad g_{ij}=0, \ otherwise \tag{7.47}$$

and

$$g^{ij} = \frac{1}{1+2p}, \ for\ i=j=1,2,3,4, \quad g^{ij}=0, \ otherwise, \tag{7.48}$$

respectively. The only non-vanishing components of the Christoffel symbols are

$$\begin{aligned} \begin{Bmatrix} 1 \\ 11 \end{Bmatrix} &= \begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 14 \end{Bmatrix} = \frac{p}{1+2p}, \\ \begin{Bmatrix} 1 \\ 22 \end{Bmatrix} &= \begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{-p}{1+2p}. \end{aligned} \tag{7.49}$$

The non-zero derivatives of (7.49) are

$$\begin{aligned} \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 4 \\ 14 \end{Bmatrix} &= \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 11 \end{Bmatrix} = \frac{p}{(1+2p)^2}, \\ \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} &= \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{-p}{(1+2p)^2}. \end{aligned} \tag{7.50}$$

For the Riemannian curvature tensor,

$$\mathcal{K}^l_{ijk} = \underbrace{\begin{vmatrix} \frac{\partial}{\partial \varkappa^j} & \frac{\partial}{\partial \varkappa^k} \\ \begin{Bmatrix} l \\ ij \end{Bmatrix} & \begin{Bmatrix} l \\ ik \end{Bmatrix} \end{vmatrix}}_{=I} + \underbrace{\begin{vmatrix} \begin{Bmatrix} m \\ ik \end{Bmatrix} & \begin{Bmatrix} m \\ ij \end{Bmatrix} \\ \begin{Bmatrix} l \\ mk \end{Bmatrix} & \begin{Bmatrix} l \\ mj \end{Bmatrix} \end{vmatrix}}_{=II}.$$

The non-zero components of (I) are:

$$\mathcal{K}^1_{221} = -\frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 22 \end{Bmatrix} = \frac{p}{(1+2p)^2},$$

$$\mathcal{K}^1_{331} = -\frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 33 \end{Bmatrix} = \frac{p}{(1+2p)^2},$$

$$\mathcal{K}^1_{441} = -\frac{\partial}{\partial \varkappa^1}\begin{Bmatrix} 1 \\ 44 \end{Bmatrix} = \frac{p}{(1+2p)^2}$$

and the non-zero components of (II) are:

$$\mathcal{K}^2_{332} = \begin{Bmatrix} m \\ 32 \end{Bmatrix}\begin{Bmatrix} 2 \\ m3 \end{Bmatrix} - \begin{Bmatrix} m \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 33 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$

$$\mathcal{K}^2_{442} = \begin{Bmatrix} m \\ 42 \end{Bmatrix}\begin{Bmatrix} 2 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ m2 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 2 \\ 12 \end{Bmatrix} = \frac{p^2}{(1+2p)^2},$$

$$\mathcal{K}^3_{443} = \begin{Bmatrix} m \\ 43 \end{Bmatrix}\begin{Bmatrix} 3 \\ m4 \end{Bmatrix} - \begin{Bmatrix} m \\ 44 \end{Bmatrix}\begin{Bmatrix} 3 \\ m3 \end{Bmatrix} = -\begin{Bmatrix} 1 \\ 44 \end{Bmatrix}\begin{Bmatrix} 3 \\ 13 \end{Bmatrix} = \frac{p^2}{(1+2p)^2}.$$

Adding the corresponding components of (I) and (II), we have

$$\mathcal{K}^1_{221} = \mathcal{K}^1_{331} = \mathcal{K}^1_{441} = \frac{p}{(1+2p)^2},$$

$$\mathcal{K}^2_{332} = \mathcal{K}^2_{442} = \mathcal{K}^3_{443} = \frac{p^2}{(1+2p)^2}.$$

Thus, the non-zero components of the curvature tensor, up to symmetry are

$$\overline{K}_{1221} = \overline{K}_{1331} = \overline{K}_{1441} = \frac{p}{1+2p},$$

$$\overline{K}_{2332} = \overline{K}_{2442} = \overline{K}_{3443} = \frac{p^2}{1+2p}$$

and the non-zero components of the Ricci tensor are

$$Ric_{11} = g^{jh}\overline{K}_{1j1h} = g^{22}\overline{K}_{1212} + g^{33}\overline{K}_{1313} + g^{44}\overline{K}_{1414} = \frac{3p}{(1+2p)^2},$$

$$Ric_{22} = g^{jh}\overline{K}_{2j2h} = g^{11}\overline{K}_{2121} + g^{33}\overline{K}_{2323} + g^{44}\overline{K}_{2424} = \frac{p}{(1+2p)},$$

$$Ric_{33} = g^{jh}\overline{K}_{3j3h} = g^{11}\overline{K}_{3131} + g^{22}\overline{K}_{3232} + g^{44}\overline{K}_{3434} = \frac{p}{(1+2p)},$$

$$Ric_{44} = g^{jh}\overline{K}_{4j4h} = g^{11}\overline{K}_{4141} + g^{22}\overline{K}_{4242} + g^{33}\overline{K}_{4343} = \frac{p}{(1+2p)}.$$

The scalar curvature *scal* is

$$scal = \frac{6p(1+p)}{(1+2p)^2}.$$

Let us consider the associated scalars ϑ, Φ are defined by

$$\vartheta = \frac{p}{(1+2p)^2}, \quad \Phi = \frac{2p(1-p)}{(1+2p)^3}$$

and the 1-form

$$\mathcal{A}_i(x) = \begin{cases} \sqrt{1+2p}, & \text{if } i = 1, \\ 0, & \text{otherwise,} \end{cases}$$

where generators are unit vector fields, then from (1.11), we have

$$Ric_{11} = \vartheta g_{11} + \Phi \mathcal{A}_1 \mathcal{A}_1, \tag{7.51}$$

$$Ric_{22} = \vartheta g_{22} + \Phi \mathcal{A}_2 \mathcal{A}_2, \tag{7.52}$$

$$Ric_{33} = \vartheta g_{33} + \Phi \mathcal{A}_3 \mathcal{A}_3, \tag{7.53}$$

$$Ric_{44} = \vartheta g_{44} + \Phi \mathcal{A}_4 \mathcal{A}_4. \tag{7.54}$$

$$\begin{aligned} R.H.S. \quad of \quad (7.51) &= \vartheta g_{11} + \Phi \mathcal{A}_1 \mathcal{A}_1 \\ &= \frac{3p}{(1+2p)^2} \\ &= L.H.S. \quad of \quad (7.51) \end{aligned}$$

By similar way it can be shown that (7.52) to (7.54) are also true. Hence $(\mathbb{R}^4, g)$ is an $\mathcal{N}\left(\frac{p}{(1+2p)^3}\right)$-QE manifold.

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References

[1] Besse, A. L. (1987). Einstein manifolds. Ergeb. Math. Grenzgeb., 3. Folge, Bd. 10. Springer-Verlag, Berlin.

[2] Bilal, M., Vasiulla, M., Haseeb, A., Ahmadini, A. A. H., & Ali, M. (2023). A study of mixed generalized quasi-Einstein spacetimes with applications in general relativity. AIMS Mathematics, 8(10), 24726-24739.

[3] Chaki, M. C., & Maity, R. K. (2000). On quasi-Einstein manifolds. Publ. Math. Debrecen, 57, 297-306.

[4] Chaki, M. C. (2001). On generalized quasi-Einstein manifolds. Publ. Math. Debrecen, 58, 683-691.

[5] Crasmareanu, M. (2012). Parallel tensors and Ricci solitons in $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Indian Journal of Pure and Applied Mathematics, 43(4), 359-369.

[6] Chaubey, S. K. (2017). Existence of $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Facta Universitatis (NIS), Ser. Math. Inform., 32(3), 369-385.

[7] De, U. C., & Shaikh, A. A. (2007). Differential Geometry of Manifolds. Narosa Publishing House (Pvt. Ltd.), New Delhi.

[8] De, U. C. & Ghosh, G. C. (2004). On quasi-Einstein manifolds. Period. Math. Hungar., 48, 223-231.

[9] Haseeb, A. (2015). Some new results on para-Sasakian manifold with a quarter-symmetric metric connection. Facta Universitatis (NIS), Ser. Math. Inform., 30(5), 765-776.

[10] Haseeb, A. (2017). Some results on projective curvature tensor in an ϵ-Kenmotsu manifold. Palestine Journal of Mathematics, 6(special issue: II), 196-203.

[11] Hazra, D., & Sarkar, A. (2021). Quasi-conformal curvature tensor on $\mathcal{N}(k)$-quasi Einstein manifolds. Korean J. Math., 29(4), 801-810.

[12] Mandal P., Shahid M. H., & Yadav S. K. (2024). Conformal Ricci soliton on paracontact metric (κ, μ)-manifolds with Schouten van Kampen connection. Commun. Korean Math. Soc, 39(1), 161-173.

[13] Mantica, C. A., Suh, Y. J., & De, U. C. (2016). A note on generalized Robertson-Walker space-times. Int. J. Geom. Meth. Mod. Phys., 13, 1650079, (9 pages.).

[14] Montano B .C., Erken I. K., & Murathan C. (2012). Nullity conditions in paracontact geometry. Differential Geom. Appl., 30(6), 665-693.

[15] Özgür, C., & Tripathi, M. M. (2007). On the concircular curvature tensor of an $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Math. Pann., 18(1), 95-100.

[16] Patterson, E. M. (1952). Some theorem on Ricci-recurrent space. J. London Math. Soc., 27, 287-295.

[17] Tripathi, M. M., & Kim, J. S. (2007). On $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Commun. Korean Math. Soc., 22(3), 411-417.

[18] Taleshian, A., & Hosseinzadeh, A. A. (2011). Investigation of some conditions on $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Bull. Malays. Math. Sci. Soc., 34(3), 455-464.

[19] Tanno, S. (1988). Ricci curvatures of contact Riemannian manifolds. Tohoku Math. J., 40, 441-448.

[20] Vasiulla, M., Haseeb A., Mofarreh, F. & Ali, M. (2022). Application of mixed generalized quasi-Einstein spacetimes in general relativity. Mathematics, 10, 3749.

[21] Yano, K., & Kon, M. (1984). Structures on Manifolds. Vol. 3, World Scientific Publishing Co., Singapore.

[22] Yang, Y., & Xu, S. (2012), Some conditions of $\mathcal{N}(\kappa)$-quasi Einstein manifolds. Int. J. Dig. Cont.Tech. Appl., 6(8), 144-150.

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POINTWISE BI-SLANT LIGHTLIKE SUBMANIFOLDS OF INDEFINITE NEARLY KÄHLER MANIFOLDS

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Abstract. In this paper, we introduce the notion of pointwise bi-slant lightlike submanifolds of an indefinite nearly Kähler manifold and provide a characterization theorem for the existence of these submanifolds. Following this, we provide a non-trivial example of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds and then derive some conditions for the distributions associated with this class of submanifolds to be involutive. Further, we provide a characterization for a pointwise bi-slant lightlike submanifold of an indefinite nearly Kähler manifold to be a bi-slant lightlike submanifold and investigate the geometry of totally umbilical pointwise bi-slant lightlike submanifold of an indefinite nearly Kähler manifold. Finally, we obtain necessary and sufficient conditions for foliations determined by distributions on pointwise bi-slant lightlike submanifolds of an indefinite nearly Kähler manifold to be totally geodesic.

Keywords: r-lightlike submanifold, Metric connection, Slant distribution, Bi-slant lightlike submanifold.

2020 Mathematics Subject Classification: 53B30, 53B25, 53B35.

1. Introduction

Chen [4, 5] introduced the notion of slant submanifolds of Kähler manifolds as a generalization of holomorphic and totally real submanifolds. Following this, Lotta [15, 16] introduced and studied concept of slant submanifolds in contact manifolds. Further, Carbrerizo et al.

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[1] studied slant submanifolds in Sasakian manifolds. Afterwards, several generalizations of slant submanifolds were introduced and studied by Carriazo [2, 3], Sahin [19] and Papaghuic [18]. Etayo [9] generalized the notion of slant submanifolds to quasi- slant submanifolds of Kähler manifolds. On a similar note, Chen and Garay [6] generalized the notion of slant submanifolds to pointwise slant submanifolds of a Kähler manifold.

Due to interesting applications in study of asymptotically flat spacetimes, even horizon of Kerr and Kruskal black holes, electromagnetic fields, focus of geometers shifted towards the study of geometry of manifolds and submanifolds endowed with indefinite metric. The theory of lightlike submanifolds of semi-Riemannian manifolds was introduced by Bejancu and Duggal [8] which differs from it's non-degenerate counterpart due to non-trivial intersection of tangent and normal bundle. Further, Sahin [20, 22] introduced notion of slant and screen slant lightlike submanifolds of Kähler manifolds. Following this, several generalizations of slant and screen slant submanifolds of indefinite Kähler manifolds were introduced and studied by Shukla et al. [23, 24]. Moreover, slant and screen slant lightlike submanifolds in framework of Contact and indefinite nearly Kähler manifolds were studied as in [21, 12, 14]. Gupta et al. [11] studied pointwise slant lightlike submanifolds of indefinite Kähler manifolds. Further, Kumar et al. [13, 17] studied the theory of screen bi-slant and pointwise bi-slant lightlike submanifolds of indefinite Kähler manifolds. However, the concept of pointwise bi-slant lightlike submanifolds is yet to be explored in indefinite nearly Kähler manifolds.

Therefore, in this paper, we introduce the notion of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds. Then, we give a characterization theorem for the existence of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds and provide a non-trivial example of this class of lightlike submanifolds. We further derive integrability conditions for the distributions associated with these submanifolds and give some conditions for a pointwise bi-slant lightlike submanifold of an indefinite nearly Kähler manifold to be a bi-slant lightlike submanifold. Finally, we investigate the geometry of totally umbilical pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds and obtain necessary and sufficient conditions for foliations determined by distributions on pointwise bi-slant lightlike submanifolds of an indefinite nearly Kähler manifold to be totally geodesic.

2. Preliminaries

Definition 2.1. *Let (N, g) be $m-$dimensional submanifold of semi-Riemannian manifold $(\bar{N}, \bar{g})$ of dimension $(m+n)$ equipped with metric $\bar{g}$ of index $q(\neq 0)$, where, $m, n \geq 1$ and $m+n-1 \geq q \geq 1$. We assume that metric $\bar{g}$ on TN is degenerate, then, metric $\bar{g}$ is degenerate on $TN^{\perp}$ which gives rise to a distribution $Rad(TN) : p \in N \to Rad(T_pN)$ given by $Rad(T_pN) = T_pN \ \cap \ T_pN^{\perp}$. We call N as $r-$lightlike submanifold if $Rad(TN)$ is a smooth distribution of rank $r > 0$ $(1 \leq r \leq n\ )$ on N.*

Let $S(TN)$ and $S(TN^{\perp})$ be non-degenerate subbundles of $Rad(TN)$ in TN and $TN^{\perp}$ respectively such that $TN = Rad(TN) \perp S(TN)$ and $TN^{\perp} = Rad(TN) \perp S(TN^{\perp})$. Moreover, for local coordinate neighbourhood U of N and local frame field $\{\xi_i\}$, $\{i \in 1, 2, ..., r\}$ of $\Gamma(Rad(TN))$, there exists a local null frame $\{N_i\}$ of sections with values in the orthogonal complement of $S(TN^{\perp})$ i.e, $S(TN^{\perp})^{\perp}$ such that

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \text{ for } i, j \in \{1, 2, ..., r\}. \tag{2.1}$$

In view of Theorem (1.3), Chapter 5 (see, [8]), there exists a lightlike transversal vector bundle $ltr(TN)$ complementary to $Rad(TN)$ in $S(TN^{\perp})^{\perp}$ locally spanned by $\{N_i\}$. Next, consider the vector bundle $tr(TN)$ in $T\bar{N}|_N$ defined by

$$tr(TN) = ltr(TN) \perp S(TN^{\perp}),$$

and therefore

$$T\bar{N}|_N = TN \oplus tr(TN) = S(TN) \perp (Rad(TN) \oplus ltr(TN)) \perp S(TN^{\perp}). \tag{2.2}$$

Let $\bar{\nabla}$ be Levi-Civita connection of $\bar{N}$. Then, for $Z_1, Z_2 \in \Gamma(TN)$ and $V \in \Gamma(tr(TN))$, Gauss and Weingarten formulae are given by

$$\overline{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + h(Z_1, Z_2),\ \overline{\nabla}_{Z_1} V = -A_V Z_1 + \nabla^t_{Z_1} V, \tag{2.3}$$

where $\{h(Z_1, Z_2), \nabla^t_{Z_1} V\} \in \Gamma(tr(TN))$, $\{\nabla_{Z_1} Z_2, A_V Z_1\} \in \Gamma(TN)$ and h, A_V represent second fundamental form on $\Gamma(TN)$ and linear shape operator on N respectively. In view of Eq. (2.2), we give the the Gauss and Weingarten formulae as

$$\overline{\nabla}_{Z_1} Z_2 = \nabla_{Z_1} Z_2 + h^l(Z_1, Z_2) + h^s(Z_1, Z_2), \tag{2.4}$$

$$\overline{\nabla}_{Z_1} V = -A_V Z_1 + \nabla^l_{Z_1} LV + \nabla^s_{Z_1} SV + D^l(Z_1, SV) + D^s(Z_1, LV) \tag{2.5}$$

where, $Z_1, Z_2 \in \Gamma(TN)$, $V \in \Gamma(tr(TN))$, h^l and h^s are $\Gamma(ltr(TN))$ and $\Gamma(S(TN^{\perp}))$ valued lightlike second fundamental form and screen second fundamental form of N, ∇^l and ∇^s are lightlike and screen transversal linear connections on N respectively and $D^l : \Gamma(TN) \times \Gamma(S(TN^{\perp})) \to \Gamma(ltr(TN))$, $D^s : \Gamma(TN) \times \Gamma(ltr(TN)) \to \Gamma(S(TN^{\perp}))$ respectively are bilinear mappings, where L and S are projection morphisms onto $ltr(TN)$ and $S(TM^{\perp})$. In particular, if $N \in \Gamma(ltr(TN))$ and $W \in \Gamma(S(TN^{\perp}))$, then, from Eq.(2.5), we have

$$\overline{\nabla}_{Z_1} N = -A_N Z_1 + \nabla^l_{Z_1} N + D^s(Z_1, N) \tag{2.6}$$

and

$$\overline{\nabla}_{Z_1} W = -A_W Z_1 + D^l(Z_1, W) + \nabla^s_{Z_1} W \tag{2.7}$$

From Eqs. (2.4), (2.6) and (2.7), we obtain

$$g(A_W Z_1, Z_2) = \bar{g}(h^s(Z_1, Z_2), W) + \bar{g}(Z_2, D^l(Z_1, W)), \tag{2.8}$$

$$g(A_W Z_1, N) = \bar{g}(D^s(Z_1, N), W). \tag{2.9}$$

Let Q be the projection of TN onto screen distribution $S(TN)$, then using $TN = Rad(TN) \perp S(TN)$, we get

$$\nabla_{Z_1} QZ_2 = \nabla^*_{Z_1} QZ_2 + h^*(Z_1, QZ_2),\ \nabla_{Z_1}\xi = -A^*_{\xi} Z_1 + \nabla^{*t}_{Z_1}\xi, \tag{2.10}$$

where $\xi \in \Gamma(Rad(TN))$, $\{h^*(Z_1, QZ_2), \nabla^{*t}_{Z_1}\xi\} \in \Gamma(Rad(TN))$ and $\{\nabla^*_{Z_1} QZ_2, A^*_{\xi} Z_1\} \in \Gamma(S(TN))$. Also, $h^* : \Gamma(TN) \times \Gamma(S(TN) \to \Gamma(Rad(TN))$ and $A^* : \Gamma(TN) \times \Gamma(Rad(TN) \to \Gamma(S(TN))$ are bilinear forms called second fundamental form and shape operator of distributions $S(TN)$ and $Rad(TN)$ respectively. Moreover, ∇^* and ∇^{*t} denote the induced Levi-Civita connection on $S(TN)$ and $Rad(TN)$ respectively. Then, from Eqs. (2.5), (2.6) and (2.10), we get

$$\bar{g}(h^l(Z_1, QZ_2), \xi) = g(A^*_{\xi} Z_1, QZ_2). \tag{2.11}$$

As $\bar{\nabla}$ is a metric connection on N, therefore for $Z_1, Z_2, Z_3 \in \Gamma(TN)$, one has

$$(\nabla_{Z_1} g)(Z_2, Z_3) = \bar{g}(h^l(Z_1, Z_2), Z_3) + \bar{g}(h^l(Z_1, Z_3), Z_2). \tag{2.12}$$

which shows that the induced connection ∇ on N is not a metric connection.

Definition 2.2. [10] *An indefinite almost Hermitian manifold $(\bar{N}, \bar{J}, \bar{g}, \bar{\nabla})$ is said to be an indefinite nearly Kähler manifold if*

$$\bar{J}^2 = -I,\ \bar{g}(\bar{J}Z_1, \bar{J}Z_2) = \bar{g}(Z_1, Z_2),\ (\overline{\nabla}_{Z_1}\bar{J})Z_2 + (\overline{\nabla}_{Z_2}\bar{J})Z_1 = 0, \tag{2.13}$$

$\forall Z_1, Z_2 \in \Gamma(T\bar{N})$.

3. Pointwise bi-slant Lightlike Submanifolds

In view of Lemmas (3.1) and (3.2) stated by Sahin [20], we introduce the concept of pointwise bi-slant lightlike submanifolds of indefinite nearly Kähler manifolds as follows:

Definition 3.1. *A q-lightlike submanifold N of an indefinite nearly Kähler manifold $\bar{N}$ with index $2q$ is said to be a pointwise bi-slant lightlike submanifold if the following conditions hold:*

(i) $\bar{J}(Rad(TN))$ *is a distribution on* N *such that* $\bar{J}(Rad(TN)) \cap Rad(TN)=\{0\}$.

(ii) *There exists non-degenerate orthogonal distributions* D_1 *and* D_2 *on* N *such that* $S(TN) = (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2$.

(iii) $\bar{J}D_1 \perp D_2$ *and* $\bar{J}D_2 \perp D_1$.

(iv) *For* $p \in U \subset N$ *and each non-zero tangent vector field* $Z \in \Gamma(D_j)_p$ *(for* $j = 1, 2$*), the angle* $(\theta_j)_p$ *between* $\bar{J}Z$ *and the vector space* $(D_j)_p$ *is independent of choice of* $Z \in \Gamma(D_j)_p$.

Note that the angle θ_j is called the slant function on N and the pair $\{\theta_1, \theta_2\}$ is called bi-slant function on N. At each point $p \in U \subset N$, $(\theta_j)_p$ (for $j = 1, 2$) is called the slant angle of the distribution $(D_j)_p$. Moreover, if for $j = 1, 2$, $(D_j)_p \neq \{0\}$ and $(\theta_j)_p \neq 0, \pi/2$, then, the pointwise bi-slant lightlike submanifold is said to be proper.

In view of above definition, the tangent bundle TN of N can be decompsed as:

$$TN = Rad(TN) \perp (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2. \tag{3.14}$$

For $Z \in \Gamma(TN)$, we have,

$$\bar{J}Z = tZ + nZ \tag{3.15}$$

and for $V \in \Gamma(tr(TN))$

$$\bar{J}V = BV + CV, \tag{3.16}$$

where $tZ, BV \in \Gamma(TN)$ and $nZ, CV \in \Gamma(tr(TN))$.

Note: In upcoming sections, we will use **pw.bi-s.l.s.** to denote a pointwise bi-slant lightlike submanifold and an indefinite nearly Kähler manifolds will be denoted by $\bar{N}$, unless otherwise stated.

Consider ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 be the projections of TN on $Rad(TN)$, $\bar{J}(Rad(TN))$, $\bar{J}(ltr(TN))$, D_1 and D_2, respectively. Then, for $Z \in \Gamma(TN)$, we have

$$Z = \phi_1 Z + \phi_2 Z + \phi_3 Z + \phi_4 Z + \phi_5 Z. \tag{3.17}$$

Applying $\bar{J}$ on both sides and using Eq. (3.15), we get

$$\bar{J}Z = \bar{J}\phi_1 Z + \bar{J}\phi_2 Z + \bar{J}\phi_3 Z + t\phi_4 Z + n\phi_4 Z + t\phi_5 Z + n\phi_5 Z, \tag{3.18}$$

where $\bar{J}\phi_1 Z \in \Gamma(\bar{J}(Rad(TN)))$, $\bar{J}\phi_2 Z \in \Gamma(Rad(TN))$ and $\bar{J}\phi_3 Z \in \Gamma(ltr(TN))$.

Lemma 3.1. *For a* ***pw.bi-s.l.s.*** *N of $\bar{N}$, $\{n\phi_4 Z, n\phi_5 Z\} \in \Gamma(S(TN^{\perp}))$, $t\phi_4 Z \in \Gamma(D_1)$ and $t\phi_5 Z \in \Gamma(D_2)$, for $Z \in \Gamma(TN)$.*

Proof. For $\xi \in \Gamma(Rad(TN))$, we have

$$\bar{g}(n\phi_i Z, \xi) = -\bar{g}(\phi_i Z, \bar{J}\xi) = 0 \tag{3.19}$$

for $i = 4, 5$. Therefore, $n\phi_i Z$ has no component in $ltr(TN)$, which implies $\{n\phi_4 Z, n\phi_5 Z\} \in (S(TN^{\perp}))$. On the other hand, Let $N \in \Gamma(ltr(TN))$, then using Eqs.(3.19), (2.13), (3.14) and using the condition $\bar{J}D_2 \perp D_1$, we have
$\bar{g}(t\phi_4 Z, N) = 0 = \bar{g}(t\phi_4 Z, \bar{J}\phi_1 Z) = \bar{g}(t\phi_4 Z, \bar{J}\phi_2 Z) = \bar{g}(t\phi_4 Z, \bar{J}\phi_3 Z) = \bar{g}(t\phi_4 Z, \phi_5 Z)$, which shows that $t\phi_4 Z \in \Gamma(D_1)$. Similarly, using Eqs.(3.19), (2.13), (3.14) and using the condition $\bar{J}D_1 \perp D_2$, it follows that $t\phi_5 Z \in \Gamma(D_2)$.

□

We now provide a classification theorem for the existence of **pw.bi-s.l.s.** N of $\bar{N}$.

Theorem 3.1. *(Existence Theorem) A q-lightlike submanifold N of $\bar{N}$ with index $2q$ is a* ***pw.bi-s.l.s.****, if and only if,*

(i) *$\bar{J}(ltrTN)$ is a distribution on N such that $\bar{J}(ltrTN) \cap Rad(TN) = \{0\}$.*

(ii) *There exists non-degenerate orthogonal distributions D_1 and D_2 on N such that $S(TN) = (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2$.*

(iii) *$\bar{J}D_1 \perp D_2$ and $\bar{J}D_2 \perp D_1$.*

(iv) *for $\{i = 4, 5\}$, there exist functions $\alpha_i \in [0, 1)$ such that $t^2(\phi_i Z) = -\alpha_i(\phi_i Z)$ for all $Z \in \Gamma(S(TN))$, where $\cos^2(\theta_j)_p = \alpha_i$ such that $(\theta_j)_p$ are the respective slant functions of $(D_j)_p$ for $j = i - 3$ and $p \in N$.*

Proof. Let N be a **pw.bi-s.l.s.** of $\bar{N}$. Then, the conditions (ii) and (iii) hold trivially. Let $\bar{J}N' \in \Gamma(Rad(TN))$ for $N' \in \Gamma ltr(TN)$, one has $\bar{J}\bar{J}N' = -N' \in \Gamma(S(TN))$, which is a contradiction. Therefore, we get $\bar{J}N' \notin \Gamma(Rad(TN))$. Again, let $\bar{J}N' \in \Gamma(ltr(TN))$ for $N' \in \Gamma(ltr(TN))$. Choose $\xi \in \Gamma(Rad(TN))$ such that $\bar{g}(N', \xi) = 1$. Then from Eq. (2.13), we derive $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is again a contradiction to our hypothesis. Therefore, $\bar{J}N' \notin \Gamma(ltr(TN))$. Consider $\bar{J}N' \in \Gamma(S(TN^{\perp}))$ for $N' \in \Gamma(ltr(TN))$. Choose $\xi \in \Gamma(Rad(TN))$ such that $\bar{g}(N', \xi) = 1$, then, using Eq. (2.13), we have $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is a contradiction to hypothesis. Therefore, $\bar{J}N' \notin \Gamma(S(TN^{\perp}))$. Thus, we conclude that $\bar{J}(ltr(TN)) \subset S(TN)$ and $\bar{J}(ltr(TN)) \cap Rad(TN)=\{0\}$, which proves condition (i).

As N is **pw.bi-s.l.s.** of $\bar{N}$, the angle between $\bar{J}\phi_i Z$ and D_j at p is constant, therefore for $Z \in \Gamma(S(TN))$ and $p \in N$, we have

$$\cos(\theta_j)_p = \frac{\bar{g}(\bar{J}\phi_i Z, t\phi_i Z)}{|\bar{J}\phi_i Z|\;|t\phi_i Z|} = -\frac{\bar{g}(\phi_i Z, \bar{J}t\phi_i Z)}{|\phi_i Z|\;|t\phi_i Z|} = -\frac{\bar{g}(\phi_i Z, t^2(\phi_i Z))}{|\phi_i Z|\;|t\phi_i Z|}.$$

Since, $\cos(\theta_j)_p = \frac{|t\phi_i Z|}{|\bar{J}(\phi_i Z)|}$, therefore we have

$$\cos^2(\theta_j)_p = -\frac{\bar{g}(\phi_i Z, t^2(\phi_i Z))}{|\phi_i Z|^2}. \tag{3.20}$$

We know that $(\theta_j)_p$ is constant on $(D_j)_p$. Hence, we get

$$\bar{g}(\phi_i Z, t^2\phi_i Z) = -\alpha_i \bar{g}(\phi_i Z, \phi_i Z),$$

which gives $t^2\phi_i Z = -\alpha_i\phi_i Z$ as $g = \bar{g}|_{(D_j)_p \times (D_j)_p}$ is non-degenerate. Hence, (iv) holds.

Conversely, suppose that N be a q-lightlike submanifold of $\bar{N}$ such that the conditions $(i)-(iv)$ are satisfied. Let $\bar{J}\xi \in \Gamma(ltr(TN))$ for $\xi \in \Gamma(Rad(TN))$, one has $\bar{J}\bar{J}\xi = -\xi \in \Gamma(S(TN))$ by condition (i), which is a contradiction. Therefore, we get $\bar{J}\xi \notin \Gamma(ltr(TN))$. Again, let $\bar{J}\xi \in \Gamma(S(TN^{\perp}))$ for $\xi \in \Gamma(Rad(TN))$. Choose $N' \in \Gamma(ltr(TN))$ such that $\bar{g}(N', \xi) = 1$. Then from conditions (i), (ii) and Eq. (2.13), we derive $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is again a contradiction to our hypothesis. Therefore, $\bar{J}\xi \notin \Gamma(S(TN^{\perp}))$. Now, Consider $\bar{J}\xi \in \Gamma(Rad(TN))$ for $\xi \in \Gamma(Rad(TN))$. Choose $N' \in \Gamma(ltr(TN))$ such that $\bar{g}(N', \xi) = 1$, then, using condition (i) and Eq. (2.13), we have $0 = \bar{g}(\bar{J}N', \bar{J}\xi) = \bar{g}(N', \xi) = 1$, which is a contradiction to hypothesis. Therefore, $\bar{J}\xi \notin \Gamma(Rad(TN))$. Thus, we conclude that $\bar{J}(Rad(TN)) \subset S(TN)$ and $\bar{J}(Rad(TN)) \cap Rad(TN)=\{0\}$. Also, by condition (iv), there exists a function α_i such that $t^2\phi_i Z = -\alpha_i\phi_i Z$ for $Z \in \Gamma(S(TN))$. Then using the Eqs.

(2.13) and (3.20), we obtain

$$\cos^2(\theta_j)_p = \frac{g(t(\phi_i Z), t(\phi_i Z))}{g(\phi_i Z, \phi_i Z)}$$
$$= \alpha_i,$$

which shows that the Wirtinger angle is independent of $\phi_i Z \in (D_j)p$. Hence, the theorem is proved. □

Corollary 3.1. *Assume that N be a **pw.bi-s.l.s.** of $\bar{N}$. Then, for $i = 4, 5$ and $j = i - 3$,*

(i) $g(t\phi_i Z_1, t\phi_i Z_2) = \cos^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$,

(ii) $\overline{g}(n\phi_i Z_1, n\phi_i Z_2) = \sin^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$,

where $Z_1, Z_2 \in \Gamma(TN)$.

Proof. Let $Z_1, Z_2 \in \Gamma(TN)$, then we have

$$g(t\phi_i Z_1, t\phi_i Z_2) = \overline{g}(t\phi_i Z_1, t\phi_i Z_2) = -\overline{g}(\phi_i Z_1, t^2\phi_i Z_2) = -\overline{g}(\phi_i Z_1, -\alpha_i\phi_i Z_2) = \alpha_i g(\phi_i Z_1, \phi_i Z_2),$$

which leads to $g(t\phi_i Z_1, t\phi_i Z_2) = \cos^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$. Similarly, consider

$$g(\phi_i Z_1, \phi_i Z_2) = \overline{g}(\bar{J}\phi_i Z_1, \bar{J}\phi_i Z_2) = g(t\phi_i Z_1, t\phi_i Z_2) + \overline{g}(n\phi_i Z_1, n\phi_i Z_2),$$

which gives $\overline{g}(n\phi_i Z_1, n\phi_i Z_2) = \sin^2(\theta_j)_p g(\phi_i Z_1, \phi_i Z_2)$, thus the proof is complete. □

Next, we present a non-trivial example of **pw.bi-s.l.s.** N of an indefinite nearly Kähler manifold $\bar{N}$.

Example 3.1. *Consider N be a 6-dimensional submanifold of $(R^{16}_3, \bar{g})$ with signature $(-,-,-,+,+,+,+,+,+,+,+,+,+,+,+,+)$ given by*

$$x^1 = u^1 = x^4, \quad x^2 = u^2 = -x^3, \quad x^5 = u^3, \quad x^6 = u^3, \quad x^7 = u^4, \quad x^8 = u^4,$$
$$x^9 = u^3u^4, \quad x^{10} = \frac{(u^3)^2}{2} + \frac{(u^4)^2}{2}, \quad x^{11} = u^5, \quad x^{12} = u^5, \quad x^{13} = u^6,$$
$$x^{14} = u^6, \quad x^{15} = u^5u^6, \quad x^{16} = \frac{(u^5)^2}{2} + \frac{(u^6)^2}{2}, \quad u^3 \neq \pm u^4, \quad u^5 \neq \pm u^6.$$

Then TN is spanned by $Z_1, Z_2, Z_3, Z_4, Z_5, Z_6$, where

$$Z_1 = \partial x_1 + \partial x_4, \quad Z_2 = \partial x_2 - \partial x_3,$$

$$Z_3 = \partial x_5 + \partial x_6 + u^4\partial x_9 + u^3\partial x_{10}, \quad Z_4 = \partial x_7 + \partial x_8 + u^3\partial x_9 + u^4\partial x_{10},$$

$$Z_5 = \partial x_{11} + \partial x_{12} + u^6\partial x_{15} + u^5\partial x_{16}, \quad Z_6 = \partial x_{13} + \partial x_{14} + u^5\partial x_{15} + u^6\partial x_{16}.$$

As $Rad(TN) = Span\{Z_1\}$ and $\bar{J}Rad(TN) = Span\{Z_2\}$, where $\bar{J}Z_1 = Z_2$, thus N is a 1-lightlike submanifold with $ltr(TN)$ spanned by $N_1 = \frac{1}{2}\{-\partial x_1 + \partial x_4\}$ and $\bar{J}ltr(TN)$ spanned by $\bar{J}N_1 = \frac{1}{2}\{-\partial x_2 - \partial x_3\}$. On the other hand, by direct calculations, we find that $S(TN^{\perp})$ is spanned by

$$W_1 = -u^3\partial x_5 - u^4\partial x_8 + \partial x_{10}, \quad W_2 = -u^5\partial x_{11} - u^6\partial x_{14} + \partial x_{16}.$$

*Hence, $D_1 = Span\{Z_3, Z_4\}$ and $D_2 = Span\{Z_5, Z_6\}$ are slant distributions with the slant angles $\theta_1 = \frac{(u^4)^2-(u^3)^2}{2+(u^3)^2+(u^4)^2}$ and $\theta_2 = \frac{(u^6)^2-(u^5)^2}{2+(u^5)^2+(u^6)^2}$. Thus, N is a proper **pw.bi-s.l.s.** of R^{16}_3.*

Lemma 3.2. *For a **pw.bi-s.l.s.** N of $\bar{N}$, nD_1 and nD_2 are orthogonal.*

Proof. Since N is **pw.bi-s.l.s.** of $\bar{N}$, therefore using Eq.(3.14) and Eq.(3.16) along with theorem (3.1) for $Z \in \Gamma(TN)$, we have

$$\begin{aligned}
\overline{g}(n\phi_4 Z, n\phi_5 Z) &= \overline{g}(\bar{J}\phi_4 Z - t\phi_4 Z, \bar{J}\phi_5 Z - t\phi_5 Z) \\
&= -\overline{g}(\bar{J}\phi_4 Z, t\phi_5 Z) - \overline{g}(\bar{J}\phi_5 Z, t\phi_4 Z) \\
&= \overline{g}(\phi_4 Z, \bar{J}t\phi_5 Z) + \overline{g}(\phi_5 Z, \bar{J}t\phi_4 Z) \\
&= \overline{g}(\phi_4 Z, t^2\phi_5 Z) + \overline{g}(\phi_5 Z, t^2\phi_4 Z) \\
&= -\cos^2(\theta_2)_p g(\phi_4 Z, \phi_5 Z) - \cos^2(\theta_1)_p g(\phi_5 Z, \phi_4 Z) \\
&= 0,
\end{aligned}$$

which completes the proof. □

In view of Lemma (3.2), there exists a holomorphic subspace $\mu_p \subset S(T_pN^{\perp})$, such that at each $p \in N$, we have

$$S(TN^{\perp}) = nD_1 \perp nD_2 \perp \mu \tag{3.21}$$

and

$$T\bar{N} = S(TN) \perp \{Rad(TN) \oplus ltr(TN)\} \perp nD_1 \perp nD_2 \perp \mu. \tag{3.22}$$

Also, for $V \in \Gamma(tr(TN))$, we have $V = PV + QV$, where $PV \in \Gamma(ltr(TN))$ and $QV \in \Gamma(S(TN^{\perp}))$. Note that as per Eq.(3.21) for $V \in \Gamma(S(TN^{\perp}))$, we have

$$QV = Q_1V + Q_2V + Q_3V,$$

where Q_1, Q_2 and Q_3 denote the projections of $S(TN^{\perp})$ onto nD_1, nD_2 and μ, respectively. Now, applying $\bar{J}$ on both sides, we have

$$\begin{aligned}\bar{J}V &= \bar{J}PV + \bar{J}QV \\ &= \bar{J}PV + BQ_1V + CQ_1V + BQ_2V + CQ_2V + \bar{J}Q_3V,\end{aligned}$$

where, $\bar{J}PV \in \Gamma\bar{J}ltr(TN))$ and using Lemma (3.1), we have $BQ_1V \in \Gamma(D_1)$, $CQ_1V \in \Gamma(S(TN^{\perp}))$, $BQ_2V \in \Gamma(D_2)$, $CQ_2V \in \Gamma(S(TN^{\perp}))$ and $\bar{J}Q_3V \in \Gamma(\mu)$. Then, using Eqs. (2.4), (2.7), (2.13) with (3.15) and (3.16) and equating the components of $Rad(TN)$, $\bar{J}Rad(TN)$, $\bar{J}(ltr(TN))$, D_1, D_2, $ltr(TN)$ and $S(TN)^{\perp}$, we get

$$\begin{aligned}&\phi_1(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_1(\nabla_{Z_1}\bar{J}\phi_2 Z_2) + \phi_1(\nabla_{Z_1}t\phi_4 Z_2) + \phi_1(\nabla_{Z_1}t\phi_5 Z_2) \\ &+\phi_1(\nabla_{Z_2}\bar{J}\phi_1 Z_1) + \phi_1(\nabla_{Z_2}\bar{J}\phi_2 Z_1) + \phi_1(\nabla_{Z_2}t\phi_4 Z_1) + \phi_1(\nabla_{Z_2}t\phi_5 Z_1) \\ &\quad= \phi_1(A_{n\phi_3 Z_2}Z_1) + \phi_1(A_{n\phi_4 Z_2}Z_1) + \phi_1(A_{n\phi_5 Z_2}Z_1) + \phi_1(A_{n\phi_3 Z_1}Z_2) \\ &\qquad+ \phi_1(A_{n\phi_4 Z_1}Z_2) + \phi_1(A_{n\phi_5 Z_1}Z_2) + \bar{J}\phi_2\nabla_{Z_1}Z_2 + \bar{J}\phi_2\nabla_{Z_2}Z_1,\end{aligned} \tag{3.23}$$

$$\begin{aligned}&\phi_2(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_2(\nabla_{Z_1}\bar{J}\phi_2 Z_2) + \phi_2(\nabla_{Z_1}t\phi_4 Z_2) + \phi_2(\nabla_{Z_1}t\phi_5 Z_2) \\ &+\phi_2(\nabla_{Z_2}\bar{J}\phi_1 Z_1) + \phi_2(\nabla_{Z_2}\bar{J}\phi_2 Z_1) + \phi_2(\nabla_{Z_2}t\phi_4 Z_1) + \phi_2(\nabla_{Z_2}t\phi_5 Z_1) \\ &\quad= \phi_2(A_{n\phi_3 Z_2}Z_1) + \phi_2(A_{n\phi_4 Z_2}Z_1) + \phi_2(A_{n\phi_5 Z_2}Z_1) + \phi_2(A_{n\phi_3 Z_1}Z_2) \\ &\qquad+ \phi_2(A_{n\phi_4 Z_1}Z_2) + \phi_2(A_{n\phi_5 Z_1}Z_2) + \bar{J}\phi_1\nabla_{Z_1}Z_2 + \bar{J}\phi_1\nabla_{Z_2}Z_1,\end{aligned} \tag{3.24}$$

$$\begin{aligned}&\phi_3(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_3(\nabla_{Z_1}\bar{J}\phi_2 Z_2) + \phi_3(\nabla_{Z_1}t\phi_4 Z_2) + \phi_3(\nabla_{Z_1}t\phi_5 Z_2) \\ &+\phi_3(\nabla_{Z_2}\bar{J}\phi_1 Z_1) + \phi_3(\nabla_{Z_2}\bar{J}\phi_2 Z_1) + \phi_3(\nabla_{Z_2}t\phi_4 Z_1) + \phi_3(\nabla_{Z_2}t\phi_5 Z_1) \\ &\quad= \phi_3(A_{n\phi_3 Z_2}Z_1) + \phi_3(A_{n\phi_4 Z_2}Z_1) + \phi_3(A_{n\phi_5 Z_2}Z_1) + \phi_3(A_{n\phi_3 Z_1}Z_2) \\ &\qquad+ \phi_3(A_{n\phi_4 Z_1}Z_2) + \phi_3(A_{n\phi_5 Z_1}Z_2) + 2Bh^l(Z_1, Z_2),\end{aligned} \tag{3.25}$$

$$\begin{aligned}&\phi_4(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_4(\nabla_{Z_1}\bar{J}\phi_2 Z_2) + \phi_4(\nabla_{Z_1}t\phi_4 Z_2) + \phi_4(\nabla_{Z_1}t\phi_5 Z_2) \\ &+\phi_4(\nabla_{Z_2}\bar{J}\phi_1 Z_1) + \phi_4(\nabla_{Z_2}\bar{J}\phi_2 Z_1) + \phi_4(\nabla_{Z_2}t\phi_4 Z_1) + \phi_4(\nabla_{Z_2}t\phi_5 Z_1) \\ &\quad= \phi_4(A_{n\phi_3 Z_2}Z_1) + \phi_4(A_{n\phi_4 Z_2}Z_1) + \phi_4(A_{n\phi_5 Z_2}Z_1) + \phi_4(A_{n\phi_3 Z_1}Z_2) \\ &\qquad+ \phi_4(A_{n\phi_4 Z_1}Z_2) + \phi_4(A_{n\phi_5 Z_1}Z_2) + 2BQ_1h^s(Z_1, Z_2) \\ &\qquad+ t\phi_4\nabla_{Z_1}Z_2 + t\phi_4\nabla_{Z_2}Z_1,\end{aligned} \tag{3.26}$$

$$\begin{aligned}
&\phi_5(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_5(\nabla_{Z_1}\bar{J}\phi_2 Z_2) + \phi_5(\nabla_{Z_1}t\phi_4 Z_2) + \phi_5(\nabla_{Z_1}t\phi_5 Z_2) \\
&+\phi_5(\nabla_{Z_2}\bar{J}\phi_1 Z_1) + \phi_5(\nabla_{Z_2}\bar{J}\phi_2 Z_1) + \phi_5(\nabla_{Z_2}t\phi_4 Z_1) + \phi_5(\nabla_{Z_2}t\phi_5 Z_1) \\
&= \phi_5(A_{n\phi_3 Z_2}Z_1) + \phi_5(A_{n\phi_4 Z_2}Z_1) + \phi_5(A_{n\phi_5 Z_2}Z_1) + \phi_5(A_{n\phi_3 Z_1}Z_2) \\
&+ \phi_5(A_{n\phi_4 Z_1}Z_2) + \phi_5(A_{n\phi_5 Z_1}Z_2) + 2BQ_2 h^s(Z_1, Z_2) \\
&+ t\phi_5\nabla_{Z_1}Z_2 + t\phi_5\nabla_{Z_2}Z_1,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
&h^l(Z_2, \bar{J}\phi_1 Z_1) + h^l(Z_2, \bar{J}\phi_2 Z_1) + h^l(Z_2, t\phi_4 Z_1) + h^l(Z_2, t\phi_5 Z_1) \\
&+h^l(Z_1, \bar{J}\phi_1 Z_2) + h^l(Z_1, \bar{J}\phi_2 Z_2) + h^l(Z_1, t\phi_4 Z_2) + h^l(Z_1, t\phi_5 Z_2) \\
&= n\phi_3\nabla_{Z_1}Z_2 + n\phi_3\nabla_{Z_2}Z_1 - \nabla^l_{Z_1}n\phi_3 Z_2 - D^l(Z_1, n\phi_4 Z_2) \\
&- D^l(Z_1, n\phi_5 Z_2) - \nabla^l_{Z_2}n\phi_3 Z_1 - D^l(n\phi_4 Z_1, Z_2) - D^l(Z_2, n\phi_5 Z_1),
\end{aligned} \tag{3.28}$$

and

$$\begin{aligned}
&h^s(Z_2, \bar{J}\phi_1 Z_1) + h^s(Z_2, \bar{J}\phi_2 Z_1) + h^s(Z_2, t\phi_4 Z_1) + h^s(Z_2, t\phi_5 Z_1) \\
&+h^s(Z_1, \bar{J}\phi_1 Z_2) + h^s(Z_1, \bar{J}\phi_2 Z_2) + h^s(Z_1, t\phi_4 Z_2) + h^s(Z_1, t\phi_5 Z_2) \\
&= n\phi_4\nabla_{Z_1}Z_2 + n\phi_5\nabla_{Z_1}Z_2 + n\phi_4\nabla_{Z_2}Z_1 + n\phi_5\nabla_{Z_2}Z_1 \\
&- D^s(Z_1, n\phi_3 Z_2) - \nabla^s_{Z_1}n\phi_4 Z_2 - \nabla^s_{Z_1}n\phi_5 Z_2 \\
&- D^s(Z_2, n\phi_3 Z_1) - \nabla^s_{Z_2}n\phi_4 Z_1 - \nabla^s_{Z_2}n\phi_5 Z_1 \\
&+ 2CQ_1 h^s(Z_1, Z_2) + 2CQ_2 h^s(Z_1, Z_2) + 2CQ_3 h^s(Z_1, Z_2).
\end{aligned} \tag{3.29}$$

Next, we investigate conditions for the distributions associated with **pw.bi-s.l.s.** N of $\bar{N}$ to be involutive.

Theorem 3.2. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution $Rad(TN)$ is involutive, if and only if*

(i) $\phi_1\left(\nabla_{Z_1}\bar{J}\phi_1 Z_2\right) + \phi_1\left(\nabla_{Z_2}\bar{J}\phi_1 Z_1\right) = 2\bar{J}\phi_2\nabla_{Z_2}Z_1,$

(ii) $\phi_4\left(\nabla_{Z_1}\bar{J}\phi_1 Z_2\right) + \phi_4\left(\nabla_{Z_2}\bar{J}\phi_1 Z_1\right) = 2BQ_1 h^s\left(Z_1, Z_2\right) + 2t\phi_4\nabla_{Z_2}Z_1,$

(iii) $\phi_5\left(\nabla_{Z_1}\bar{J}\phi_1 Z_2\right) + \phi_5\left(\nabla_{Z_2}\bar{J}\phi_1 Z_1\right) = 2BQ_2 h^s\left(Z_1, Z_2\right) + 2t\phi_5\nabla_{Z_2}Z_1,$

(iv) $h^l\left(Z_2, \bar{J}\phi_1 Z_1\right) + h^l\left(Z_1, \bar{J}\phi_1 Z_2\right) = 2n\phi_3\nabla_{Z_2}Z_1,$

(v) $h^s\left(Z_2, \bar{J}\phi_1 Z_1\right) + h^s\left(Z_1, \bar{J}\phi_1 Z_2\right) = 2n\phi_4\nabla_{Z_2}Z_1 + 2n\phi_5\nabla_{Z_2}Z_1 + 2CQ_1 h^s(Z_1, Z_2) + 2CQ_2 h^s(Z_1, Z_2) + 2CQ_3 h^s(Z_1, Z_2),$

for any $Z_1, Z_2 \in \Gamma(Rad(TN))$.

Proof. Consider $Z_1, Z_2 \in \Gamma(Rad(TN))$, then using Eqs.(3.23), (3.26), (3.27), (3.28) and (3.29), we have

$$\phi_1(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_1(\nabla_{Z_2}\bar{J}\phi_1 Z_1) = \bar{J}\phi_2[Z_1, Z_2] + 2\bar{J}\phi_2\nabla_{Z_2}Z_1, \tag{3.30}$$

$$\phi_4(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_4(\nabla_{Z_2}\bar{J}\phi_1 Z_1) = 2BQ_1h^s(Z_1, Z_2) + t\phi_4[Z_1, Z_2] + 2t\phi_4\nabla_{Z_2}Z_1, \tag{3.31}$$

$$\phi_5(\nabla_{Z_1}\bar{J}\phi_1 Z_2) + \phi_5(\nabla_{Z_2}\bar{J}\phi_1 Z_1) = 2BQ_2h^s(Z_1, Z_2) + t\phi_5[Z_1, Z_2] + 2t\phi_5\nabla_{Z_2}Z_1, \tag{3.32}$$

$$h^l(Z_2, \bar{J}\phi_1 Z_1) + h^l(Z_1, \bar{J}\phi_1 Z_2) = n\phi_3[Z_1, Z_2] + 2n\phi_3\nabla_{Z_2}Z_1 \tag{3.33}$$

and

$$\begin{aligned} &2CQ_1h^s(Z_1, Z_2) + 2CQ_2h^s(Z_1, Z_2) + 2CQ_3h^s(Z_1, Z_2) + n\phi_4[Z_1, Z_2] + 2n\phi_4\nabla_{Z_2}Z_1 \\ &+ n\phi_5[Z_1, Z_2] + 2n\phi_5\nabla_{Z_2}Z_1 = h^s(Z_2, \bar{J}\phi_1 Z_1) + h^s(Z_1, \bar{J}\phi_1 Z_2). \end{aligned} \tag{3.34}$$

Then the result follows from Eqs. (3.30), (3.31), (3.32), (3.33) and (3.34). □

Theorem 3.3. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution $\bar{J}Rad(TN)$ is involutive, if and only if*

(i) $\phi_2\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_2\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2\bar{J}\phi_1\nabla_{Z_2}Z_1,$

(ii) $\phi_4\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_4\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_1h^s\left(Z_1, Z_2\right) + 2t\phi_4\nabla_{Z_2}Z_1,$

(iii) $\phi_5\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_5\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_2h^s\left(Z_1, Z_2\right) + 2t\phi_5\nabla_{Z_2}Z_1,$

(iv) $h^l\left(Z_2, \bar{J}\phi_2 Z_1\right) + h^l\left(Z_1, \bar{J}\phi_2 Z_2\right) = 2n\phi_3\nabla_{Z_2}Z_1,$

(v) $h^s\left(Z_2, \bar{J}\phi_2 Z_1\right) + h^s\left(Z_1, \bar{J}\phi_2 Z_2\right) = 2n\phi_4\nabla_{Z_2}Z_1 + 2n\phi_5\nabla_{Z_2}Z_1 + 2CQ_1h^s(Z_1, Z_2) + 2CQ_2h^s(Z_1, Z_2) + 2CQ_3h^s(Z_1, Z_2),$

where $Z_1, Z_2 \in \Gamma(\bar{J}Rad(TN))$.

Proof. Let Z_1, $Z_2 \in \Gamma(\bar{J}(Rad(TN))$, then using Eqs.(3.24), (3.26), (3.27), (3.28) and (3.29), we have

$$\phi_2\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_2\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = \bar{J}\phi_1[Z_1, Z_2] + 2\bar{J}\phi_1\nabla_{Z_2}Z_1, \tag{3.35}$$

$$\phi_4\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_4\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_1h^s\left(Z_1, Z_2\right) + t\phi_4[Z_1, Z_2] + 2t\phi_4\nabla_{Z_2}Z_1, \tag{3.36}$$

$$\phi_5\left(\nabla_{Z_1}\bar{J}\phi_2 Z_2\right) + \phi_5\left(\nabla_{Z_2}\bar{J}\phi_2 Z_1\right) = 2BQ_2h^s\left(Z_1, Z_2\right) + t\phi_5[Z_1, Z_2] + 2t\phi_5\nabla_{Z_2}Z_1, \tag{3.37}$$

$$h^l\left(Z_2, \bar{J}\phi_2 Z_1\right) + h^l\left(Z_1, \bar{J}\phi_2 Z_2\right) = n\phi_3[Z_1, Z_2] + 2n\phi_3\nabla_{Z_2}Z_1 \tag{3.38}$$

and

$$n\phi_4[Z_1,Z_2]+2n\phi_4\nabla_{Z_2}Z_1+n\phi_5[Z_1,Z_2]+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)$$
$$+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2)=h^s\left(Z_2,\bar{J}\phi_2Z_1\right)+h^s\left(Z_1,\bar{J}\phi_2Z_2\right). \quad (3.39)$$

The result follows from Eqs. (3.35), (3.36), (3.37), (3.38) and (3.39). □

Theorem 3.4. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution $\bar{J}(ltr(TN))$ is involutive, if and only if*

(i) $\phi_1(A_{n\phi_3Z_2}Z_1)+\phi_1(A_{n\phi_3Z_1}Z_2)+2\bar{J}\phi_2\nabla_{Z_2}Z_1=0,$

(ii) $\phi_2(A_{n\phi_3Z_2}Z_1)+\phi_2(A_{n\phi_3Z_1}Z_2)+2\bar{J}\phi_1\nabla_{Z_2}Z_1=0,$

(iii) $\phi_4(A_{n\phi_3Z_2}Z_1)+\phi_4(A_{n\phi_3Z_1}Z_2)+2BQ_1h^s\left(Z_1,Z_2\right)+2t\phi_4\nabla_{Z_2}Z_1=0,$

(iv) $\phi_5(A_{n\phi_3Z_2}Z_1)+\phi_5(A_{n\phi_3Z_1}Z_2)+2BQ_2h^s\left(Z_1,Z_2\right)+2t\phi_5\nabla_{Z_2}Z_1=0,$

(v) $D^s(Z_1,n\phi_3Z_2)+D^s(Z_2,n\phi_3Z_1)=2n\phi_4\nabla_{Z_2}Z_1+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)$
$+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2),$

for any $Z_1,Z_2\in\Gamma(\bar{J}(ltr(TN))$.

Proof. Consider $Z_1,Z_2\in\Gamma(\bar{J}(ltr(TN))$, then from Eqs.(3.23), (3.24), (3.26), (3.27) and (3.29), we have

$$\phi_1(A_{n\phi_3Z_2}Z_1)+\phi_1(A_{n\phi_3Z_1}Z_2)+\bar{J}\phi_2[Z_1,Z_2]+2\bar{J}\phi_2\nabla_{Z_2}Z_1=0, \quad (3.40)$$

$$\phi_2(A_{n\phi_3Z_2}Z_1)+\phi_2(A_{n\phi_3Z_1}Z_2)+\bar{J}\phi_1[Z_1,Z_2]+2\bar{J}\phi_1\nabla_{Z_2}Z_1=0, \quad (3.41)$$

$$\phi_4(A_{n\phi_3Z_2}Z_1)+\phi_4(A_{n\phi_3Z_1}Z_2)+2BQ_1h^s\left(Z_1,Z_2\right)+t\phi_4[Z_1,Z_2]+2t\phi_4\nabla_{Z_2}Z_1=0, \quad (3.42)$$

$$\phi_5(A_{n\phi_3Z_2}Z_1)+\phi_5(A_{n\phi_3Z_1}Z_2)+2BQ_2h^s\left(Z_1,Z_2\right)+t\phi_5[Z_1,Z_2]+2t\phi_5\nabla_{Z_2}Z_1=0 \quad (3.43)$$

and

$$n\phi_4[Z_1,Z_2]+2n\phi_4\nabla_{Z_2}Z_1+n\phi_5[Z_1,Z_2]+2n\phi_5\nabla_{Z_2}Z_1+2CQ_1h^s(Z_1,Z_2)$$
$$+2CQ_2h^s(Z_1,Z_2)+2CQ_3h^s(Z_1,Z_2)=D^s(Z_1,n\phi_3Z_2)+D^s(Z_2,n\phi_3Z_1). \quad (3.44)$$

Using Eqs. (3.40), (3.41), (3.42), (3.43) and (3.44), the assertion follows directly. □

Theorem 3.5. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution D_1 is involutive, if and only if*

(i) $\phi_1\left(\nabla_{Z_1}t\phi_4Z_2\right)+\phi_1\left(\nabla_{Z_2}t\phi_4Z_1\right)=\phi_1(A_{n\phi_4Z_2}Z_1)+\phi_1(A_{n\phi_4Z_1}Z_2)+2\bar{J}\phi_2\nabla_{Z_2}Z_1,$

(ii) $\phi_2\left(\nabla_{Z_1}t\phi_4Z_2\right)+\phi_2\left(\nabla_{Z_2}t\phi_4Z_1\right)=\phi_2(A_{n\phi_4Z_2}Z_1)+\phi_2(A_{n\phi_4Z_1}Z_2)+2\bar{J}\phi_1\nabla_{Z_2}Z_1,$

(iii) $\phi_5\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_5\left(\nabla_{Z_2} t\phi_4 Z_1\right)=\phi_5(A_{n\phi_4 Z_2} Z_1)+\phi_5(A_{n\phi_4 Z_1} Z_2)+2BQ_2 h^s\left(Z_1, Z_2\right)+2t\phi_5 \nabla_{Z_2} Z_1$,

(iv) $h^l\left(Z_2, t\phi_4 Z_1\right)+h^l\left(Z_1, t\phi_4 Z_2\right)=2n\phi_3 \nabla_{Z_2} Z_1-D^l(Z_1, n\phi_4 Z_2)-D^l(n\phi_4 Z_1, Z_2)$,

(v) $n\phi_4 \nabla_{Z_2} Z_1+n\phi_4 \nabla_{Z_1} Z_2+2n\phi_5 \nabla_{Z_2} Z_1-\nabla^s_{Z_1} n\phi_4 Z_2-\nabla^s_{Z_2} n\phi_4 Z_1+2CQ_1 h^s(Z_1, Z_2)+2CQ_2 h^s(Z_1, Z_2)+2CQ_3 h^s(Z_1, Z_2)=h^s\left(Z_2, t\phi_4 Z_1\right)+h^s\left(Z_1, t\phi_4 Z_2\right)$,

where $Z_1, Z_2 \in \Gamma(D_1)$.

Proof. For $Z_1, Z_2 \in \Gamma(D_1)$, from Eqs.(3.23), (3.24), (3.27), (3.28) and (3.29), we have

$$\phi_1(A_{n\phi_4 Z_2} Z_1)+\phi_1(A_{n\phi_4 Z_1} Z_2)+2\bar{J}\phi_2 \nabla_{Z_2} Z_1+\bar{J}\phi_2[Z_1, Z_2]=\phi_1\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_1\left(\nabla_{Z_2} t\phi_4 Z_1\right), \tag{3.45}$$

$$\phi_2(A_{n\phi_4 Z_2} Z_1)+\phi_2(A_{n\phi_4 Z_1} Z_2)+2\bar{J}\phi_1 \nabla_{Z_2} Z_1+\bar{J}\phi_1[Z_1, Z_2]=\phi_2\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_2\left(\nabla_{Z_1} t\phi_4 Z_2\right), \tag{3.46}$$

$$\phi_5\left(\nabla_{Z_1} t\phi_4 Z_2\right)+\phi_5\left(\nabla_{Z_2} t\phi_4 Z_1\right)=\phi_5(A_{n\phi_4 Z_2} Z_1)+\phi_5(A_{n\phi_4 Z_1} Z_2)+t\phi_5[Z_1, Z_2]+2BQ_2 h^s\left(Z_1, Z_2\right)+2t\phi_5 \nabla_{Z_2} Z_1, \tag{3.47}$$

$$n\phi_3[Z_1, Z_2]+2n\phi_3 \nabla_{Z_2} Z_1-D^l(Z_1, n\phi_4 Z_2)-D^l(n\phi_4 Z_1, Z_2)=h^l\left(Z_2, t\phi_4 Z_1\right)+h^l\left(Z_1, t\phi_4 Z_2\right) \tag{3.48}$$

and

$$n\phi_4 \nabla_{Z_2} Z_1+n\phi_4 \nabla_{Z_1} Z_2+2n\phi_5 \nabla_{Z_2} Z_1-\nabla^s_{Z_1} n\phi_4 Z_2-\nabla^s_{Z_2} n\phi_4 Z_1+2CQ_1 h^s(Z_1, Z_2)+n\phi_5[Z_1, Z_2]+2CQ_2 h^s(Z_1, Z_2)+2CQ_3 h^s(Z_1, Z_2)=h^s\left(Z_2, t\phi_4 Z_1\right)+h^s\left(Z_1, t\phi_4 Z_2\right). \tag{3.49}$$

Then the assertion follows from Eqs. (3.45), (3.46), (3.47), (3.48) and (3.49). □

Theorem 3.6. *For a **pw.bi-s.l.s.** N of $\bar{N}$, the distribution D_2 is involutive, if and only if*

(i) $\phi_1\left(\nabla_{Z_1} t\phi_5 Z_2\right)+\phi_1\left(\nabla_{Z_2} t\phi_5 Z_1\right)=\phi_1(A_{n\phi_5 Z_2} Z_1)+\phi_1(A_{n\phi_5 Z_1} Z_2)+2\bar{J}\phi_2 \nabla_{Z_2} Z_1$,

(ii) $\phi_2\left(\nabla_{Z_1} t\phi_5 Z_2\right)+\phi_2\left(\nabla_{Z_2} t\phi_5 Z_1\right)=\phi_2(A_{n\phi_5 Z_2} Z_1)+\phi_2(A_{n\phi_5 Z_1} Z_2)+2\bar{J}\phi_1 \nabla_{Z_2} Z_1$,

(iii) $\phi_4\left(\nabla_{Z_1} t\phi_5 Z_2\right)+\phi_4\left(\nabla_{Z_2} t\phi_5 Z_1\right)=\phi_4(A_{n\phi_5 Z_2} Z_1)+\phi_4(A_{n\phi_5 Z_1} Z_2)+2BQ_1 h^s\left(Z_1, Z_2\right)+2t\phi_4 \nabla_{Z_2} Z_1$,

(iv) $h^l\left(Z_2, t\phi_5 Z_1\right)+h^l\left(Z_1, t\phi_5 Z_2\right)=2n\phi_3 \nabla_{Z_2} Z_1-D^l(Z_1, n\phi_5 Z_2)-D^l(n\phi_5 Z_1, Z_2)$,

(v) $n\phi_5 \nabla_{Z_2} Z_1+n\phi_5 \nabla_{Z_1} Z_2+2n\phi_4 \nabla_{Z_2} Z_1-\nabla^s_{Z_1} n\phi_5 Z_2-\nabla^s_{Z_2} n\phi_5 Z_1+2CQ_1 h^s(Z_1, Z_2)+2CQ_2 h^s(Z_1, Z_2)+2CQ_3 h^s(Z_1, Z_2)=h^s\left(Z_2, t\phi_4 Z_1\right)+h^s\left(Z_1, t\phi_4 Z_2\right)$, *for any* $Z_1, Z_2 \in \Gamma(D_2)$.

Proof. Let $Z_1, Z_2 \in \Gamma(D_2)$. Using Eqs.(3.23), (3.24), (3.27), (3.28) and (3.29), we have

$$\phi_1(A_{n\phi_5 Z_2}Z_1) + \phi_1(A_{n\phi_5 Z_1}Z_2) + 2\bar{J}\phi_2\nabla_{Z_2}Z_1 + \bar{J}\phi_2[Z_1, Z_2] = \phi_1(\nabla_{Z_1}t\phi_5 Z_2) + \phi_1(\nabla_{Z_2}t\phi_5 Z_1), \quad (3.50)$$

$$\phi_2(A_{n\phi_5 Z_2}Z_1) + \phi_2(A_{n\phi_5 Z_1}Z_2) + 2\bar{J}\phi_1\nabla_{Z_2}Z_1 + \bar{J}\phi_1[Z_1, Z_2] = \phi_2(\nabla_{Z_1}t\phi_5 Z_2) + \phi_2(\nabla_{Z_2}t\phi_5 Z_1), \quad (3.51)$$

$$\phi_4(\nabla_{Z_1}t\phi_5 Z_2) + \phi_4(\nabla_{Z_2}t\phi_5 Z_1) = \phi_4(A_{n\phi_5 Z_2}Z_1) + \phi_4(A_{n\phi_5 Z_1}Z_2) + t\phi_4[Z_1, Z_2] + 2BQ_1h^s(Z_1, Z_2) + 2t\phi_4\nabla_{Z_2}Z_1, \quad (3.52)$$

$$n\phi_3[Z_1, Z_2] + 2n\phi_3\nabla_{Z_2}Z_1 - D^l(Z_1, n\phi_5 Z_2) - D^l(n\phi_5 Z_1, Z_2) = h^l(Z_2, t\phi_5 Z_1) + h^l(Z_1, t\phi_5 Z_2) \quad (3.53)$$

and

$$n\phi_5\nabla_{Z_2}Z_1 + n\phi_5\nabla_{Z_1}Z_2 + 2n\phi_4\nabla_{Z_2}Z_1 - \nabla^s_{Z_1}n\phi_5 Z_2 - \nabla^s_{Z_2}n\phi_5 Z_1 + 2CQ_1h^s(Z_1, Z_2) + n\phi_4[Z_1, Z_2] + 2CQ_2h^s(Z_1, Z_2) + 2CQ_3h^s(Z_1, Z_2) = h^s(Z_2, t\phi_5 Z_1) + h^s(Z_1, t\phi_5 Z_2). \quad (3.54)$$

The result follows from Eqs. (3.50), (3.51), (3.52), (3.53) and (3.54). □

We now give a necessary and sufficient condition for the induced connection on a **pw.bi-s.l.s.** N to be a metric connection.

Theorem 3.7. *Assume that N is a* ***pw.bi-s.l.s.*** *of $\bar{N}$, then ∇ is a metric connection on N, if and only if for each $Z \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$, we have*

(i) $\nabla_Z\bar{J}\xi + \nabla_{\bar{J}\xi}Z \in \Gamma\bar{J}(Rad(TN))$,
(ii) $\nabla_{\bar{J}\xi}tZ - A_{nZ}\bar{J}\xi \in \Gamma(Rad(TN))$,
(iii) $Bh(Z, \bar{J}\xi) = 0$.

Proof. Consider $Z \in \Gamma(TN)$ and $\xi \in \Gamma(Rad(TN))$. Now, by Eqs. (2.13), we have

$$\overline{\nabla}_Z\xi = -\overline{\nabla}_Z\bar{J}^2\xi = -\bar{J}\overline{\nabla}_Z\bar{J}\xi + \overline{\nabla}_{\bar{J}\xi}\bar{J}Z - \bar{J}\overline{\nabla}_{\bar{J}\xi}Z.$$

Then using Eqs. (2.3), (3.15) and (3.16), we get

$$-\bar{J}(\nabla_Z \bar{J}\xi + \nabla_{\bar{J}\xi} Z) + \nabla_{\bar{J}\xi} tZ + h(\bar{J}\xi, tZ) - A_{nZ}\bar{J}\xi + \nabla^t_{\bar{J}\xi} nZ - 2Bh(Z, \bar{J}\xi) - 2Ch(Y, \bar{J}\xi)$$
$$= \nabla_Z \xi + h(Z, \xi),$$

Comparing tangential components on both sides of above equation, we derive

$$\nabla_Z \xi = -t(\nabla_Z \bar{J}\xi + \nabla_{\bar{J}\xi} Z) + \nabla_{\bar{J}\xi} tZ - A_{nZ}\bar{J}\xi - 2Bh(Z, \bar{J}\xi).$$

Hence, $\nabla_Z \xi \in \Gamma(Rad(TN))$, if and only if the conditions (i), (ii) and (iii) are satisfied. □

Lemma 3.3. *Let N be a **pw.bi-s.l.s.** of $\bar{N}$ and $(\theta_j)_p$, $(j = 1, 2)$ be the slant angle. Then, for a unit vector $Z \in \Gamma(D_j)_p$, we have*

$$tZ = \cos(\theta_j)_p(Z)\bar{Z} \tag{3.55}$$

where $\bar{Z}$ represents a unit vector in $(D_j)_p$ such that $g(\bar{Z}, Z) = 0$.

Proof. Let $Z \in \Gamma(D_j)_p$, $(j = 1, 2)$ such that $g(Z, Z) = 1$, then we have

$$\cos(\theta_j)_p(Z) = \frac{|tZ|}{|\bar{J}Z|} = \frac{|tZ|}{|Z|} = |tZ|. \tag{3.56}$$

Now, define $\bar{Z} = \dfrac{tZ}{|tZ|}$, then clearly $|\bar{Z}| = 1$ and $tZ = |tZ|\bar{Z}$. Next from Eq. (3.56), we have

$$tZ = \cos(\theta_j)_p(Z)\bar{Z}.$$

We know that for an indefinite nearly Kähler manifold, $g(\bar{J}Z, Z) = 0$. Using Lemma(3.1) and Eq.(3.15), we get $g(tZ, Z) = 0$. Further, we have $g(\bar{Z}, Z) = g\left(\dfrac{tZ}{|tZ|}, Z\right) = \dfrac{1}{|tZ|} g(tZ, Z) = 0$, which proves the lemma.

□

Definition 3.2. *A q-lightlike submanifold N of an indefinite nearly Kähler manifold $\bar{N}$ with index $2q$ is said to be a bi-slant lightlike submanifold if*

(i) *$\bar{J}(Rad(TN))$ is a distribution on N such that $\bar{J}(Rad(TN)) \cap Rad(TN)$=$\{0\}$.*

(ii) *There exists non-degenerate orthogonal distributions D_1 and D_2 on N such that $S(TN) = (\bar{J}ltr(TN) \oplus \bar{J}Rad(TN)) \perp D_1 \perp D_2$.*

(iii) *$\bar{J}D_1 \perp D_2$ and $\bar{J}D_2 \perp D_1$.*

(iv) *The distribution D_j is slant with slant angle θ_j (for $j = 1, 2$) i.e, for each $p \in N$ and non-zero tangent vector field $Z \in (D_j)_p$, the angle $(\theta_j)_p$ between $\bar{J}Z$ and the vector space $(D_j)_p$ is independent of choice of $Z \in \Gamma(D_j)_p$ and $p \in N$.*

If $D_j \neq \{0\}$ and $\theta_j \neq 0, \pi/2$, then, the bi-slant lightlike submanifold is said to be proper.

Next, we provide conditions for proper **pw.bi-s.l.s.** N of $\bar{N}$ to be a bi-slant lightlike submanifold.

Theorem 3.8. *A proper* ***pw.bi-s.l.s.*** *N of $\bar{N}$ is a bi-slant lightlike submanifold of $\bar{N}$, if and only if,*

$$g(A_{nZ_j}Y, \bar{Z}_j) + g(A_{nY}Z_j, \bar{Z}_j) = 2g(A_{n\bar{Z}_j}Y, Z_j) + g((\nabla_{Z_j}t)Y, \bar{Z}_j),$$

where for $p \in U \subset N$, $Z_j \in \Gamma(D_j)_p$ is a unit vector field, $\bar{Z}_j \in \Gamma(D_j)_p$ is a unit vector field such that $g(\bar{Z}_j, Z_j) = 0$ for $j = 1, 2$ and $Y \in \Gamma(TN)$.

Proof. Assume N be a proper **pw.bi-s.l.s.** of $\bar{N}$ and $Z_j \in \Gamma(D_j)_p$ for $p \in U \subset N$, be unit vector field. For $Y \in \Gamma(TN)$, using Eqs. (3.15), (2.4), (2.5), (2.7) and (3.55), we have

$$\begin{aligned}\overline{\nabla}_Y \bar{J}Z_j = &- sin(\theta_j)_p(Z)Y((\theta_j)_p(Z))\bar{Z}_j + cos(\theta_j)_p(Z)(\nabla_Y \bar{Z}_j + h^l(Y, \bar{Z}_j) + h^s(Y, \bar{Z}_j)) \\ &- A_{nZ_j}Y + \nabla^s_Y nZ_j + D^l(Y, nZ_j) \qquad (3.57)\end{aligned}$$

aand

$$\begin{aligned}\overline{\nabla}_{Z_j}\bar{J}Y = &\nabla_{Z_j}tY + h^l(Z_j, tY) + h^s(Z_j, tY) - A_{nY}Z_j + \nabla^l_{Z_j}L(nY) + \nabla^s_{Z_j}S(nY) \\ &+ D^l(Z_j, S(nY)) + D^s(Z_j, L(nY)). \qquad (3.58)\end{aligned}$$

Again, using Eqs. (2.4), (3.15) ans (3.16), we get

$$\begin{aligned}\bar{J}(\overline{\nabla}_{Z_j}Y) + \bar{J}(\overline{\nabla}_Y Z_j) = &t\nabla_{Z_j}Y + t\nabla_Y Z_j + n\nabla_{Z_j}Y + n\nabla_Y Z_j + 2Bh^l(Z_j, Y) \\ &+ 2Bh^s(Z_j, Y) + 2Ch^s(Z_j, Y). \qquad (3.59)\end{aligned}$$

As $\bar{N}$ is an indefinite nearly kähler manifold, therefore, using Eq. (2.13), we get

$$\overline{\nabla}_Y \bar{J}Z_j + \overline{\nabla}_{Z_j}\bar{J}Y = \bar{J}(\overline{\nabla}_{Z_j}Y) + \bar{J}(\overline{\nabla}_Y Z_j).$$

Using Eqs. (3.57), (3.58) and (3.59) , comparing tangential parts of resulting equation and taking inner product with respect to $\bar{Z}_j \in \Gamma(D_j)_p$, we have

$$\begin{aligned}&- sin(\theta_j)_p(Z)Y((\theta_j)_p(Z)) + cos(\theta_j)_p(Z)g(\nabla_Y \bar{Z}_j, \bar{Z}_j) - g(A_{nZ_j}Y, \bar{Z}_j) - g(A_{nY}Z_j, \bar{Z}_j) \\ &+ g(\nabla_{Z_j}tY, \bar{Z}_j) = g(t\nabla_{Z_j}Y, \bar{Z}_j) + g(t\nabla_Y Z_j, \bar{Z}_j) + 2g(Bh^l(Z_j, Y), \bar{Z}_j) + 2g(Bh^s(Z_j, Y), \bar{Z}_j). \\ &\qquad (3.60)\end{aligned}$$

Next from Eq. (2.12), we have $(\nabla_Y g)(\bar{Z}_j, \bar{Z}_j) = 0$, which gives $g(\nabla_Y \bar{Z}_j, \bar{Z}_j) = 0$. Then consider,

$$\begin{aligned} g(t\nabla_Y Z_j, \bar{Z}_j) &= g\left(t\nabla_Y Z_j, \frac{tZ_j}{|tZ_j|}\right) \\ &= \frac{1}{|tZ_j|} g(t\nabla_Y Z_j, tZ_j) \\ &= \frac{1}{|tZ_j|} \cos^2(\theta_j)_p(Z) g(\nabla_Y Z_j, Z_j) \\ &= 0. \end{aligned} \tag{3.61}$$

Also, from Eq. (2.8), we have

$$\begin{aligned} 2g(Bh^s(Y, Z_j), \bar{Z}_j) = -2\bar{g}(h^s(Y, Z_j), \bar{J}\bar{Z}_j) &= -2\bar{g}(h^s(Y, Z_j), n\bar{Z}_j) \\ &= -2g(A_{n\bar{Z}_j} Y, Z_j). \end{aligned} \tag{3.62}$$

Now, using (3.61) and (3.62) along with the fact that $g(\nabla_Y \bar{W}, \bar{W}) = 0$ in (3.60), we have

$$\begin{aligned} -sin(\theta_j)_p(Z) Y((\theta_j)_p(Z)) =& g(A_{nZ_j} Y, \bar{Z}_j) + g(A_{nY} Z_j, \bar{Z}_j) - g((\nabla_{Z_j} t)Y, \bar{Z}_j) \\ & - 2g(A_{n\bar{Z}_j} Y, Z_j), \end{aligned} \tag{3.63}$$

As N is proper **pw.bi.s.l.s.** of $\bar{N}$, N is a bi-slant lightlike submanifold iff $Y((\theta_j)_p(Z)) = 0$ i.e, θ_j is independent of choice of $p \in N$ which proves the theorem. □

Theorem 3.9. *Assume N be a proper* ***pw.bi-s.l.s.*** *of $\bar{N}$. If*

(i) *there exists $tr(TN)$ which is parallel along TN with respect to metric connection $\overline{\nabla}$.*
(ii) *t is parallel with respect to induced connection ∇ on N.*

Then, N becomes a bi-slant lightlike submanifold of $\bar{N}$.

Proof. Assume that $Y \in \Gamma(TN)$, $Z_j \in \Gamma(D_j)_p$ for $j = 1, 2$, where $p \in U \subset N$. Then, using Lemma (3.1), $\{nZ_j, n\bar{Z}_j\} \in \Gamma(S(TN^\perp)) \subset \Gamma(tr(TN))$. Since $tr(TN)$ is parallel along TN with respect to metric connection $\overline{\nabla}$, we have, $\{\overline{\nabla}_Y nZ_j, \overline{\nabla}_Y n\bar{Z}_j\} \in \Gamma(tr(TN))$ which implies $A_{nZ_j} Y = A_{n\bar{Z}_j} Y = 0$. Similarly, using the fact that $tr(TN)$ is parallel along TN with respect to metric connection $\overline{\nabla}$ and Eq.(2.5), we get $A_{nY} Z_j = 0$. Also, by condition (ii), $(\nabla_{Z_j} t)Y = 0$. As N is proper **pw.bi.s.l.s.** of $\bar{N}$, from Eq. (3.63), $Y((\theta_j)_p(Z)) = 0$ i.e, θ_j is independent of choice of $p \in N$ which proves the theorem. □

Definition 3.3. [7] *A lightlike submanifold (N, g) of a semi-Riemannian $(\bar{N}, \bar{g})$ is called totally umbilical, if there exist a transversal curvature vector field $H \in \Gamma(tr(TN))$ on N such* *that*

$$h(Z_1, Z_2) = H\bar{g}(Z_1, Z_2), \tag{3.64}$$

for $Z_1, Z_2 \in \Gamma(TN)$*. Using Eqs. (2.4) and (2.7), clearly* N *is totally umbilical, if and only if there exist smooth vector fields* $H^l \in \Gamma(ltr(TN))$ *and* $H^s \in \Gamma(S(TN^\perp))$ *such that*

$$h^l(Z_1, Z_2) = H^l g(Z_1, Z_2),\ h^s(Z_1, Z_2) = H^s g(Z_1, Z_2),\ D^l(Z_1, V) = 0, \tag{3.65}$$

for $Z_1, Z_2 \in \Gamma(TN)$ *and* $V \in \Gamma(S(TN^\perp))$*.*

Theorem 3.10. *Assume that* N *be a totally umbilical proper* ***pw.bi-s.l.s.*** *of* $\bar{N}$*. Then* N *becomes a bi-slant lightlike submanifold of* $\bar{N}$*, if* $H^s \in \Gamma(\mu_p)$*.*

Proof. Let $Z \in \Gamma(D_j)_p$ for some $j = 1, 2$ and $p \in U \subset N$. Then, using Eq. (3.64) and Corollary (3.1), we have

$$\overline{\nabla}_{tZ} tZ = \nabla_{tZ} tZ + cos^2(\theta_j)_p g(Z, Z)H.$$

Now, applying $\bar{J}$ on both sides of above equation and using Eqs.(2.7), (3.15) and Theorem (3.1), we get,

$$\begin{aligned} sin2(\theta_j)_p tZ(\theta_j)_p Z - cos^2(\theta_j)_p \nabla_{tZ} Z - A_{ntZ} tZ + \nabla^s_{tZ} ntZ + D^l(tZ, ntZ) = \\ cos^2(\theta_j)_p g(Z, Z)(\bar{J}H^l + \bar{J}H^s) + t\nabla_{tZ} tZ + n\nabla_{tZ} tZ. \end{aligned} \tag{3.66}$$

Comparing transversal components of above equation and taking the inner product of resulting expression with ntZ, we have

$$cos^2(\theta_j)_p g(Z, Z)\bar{g}(CH^s, ntZ) + \bar{g}(n\nabla_{tZ} tZ, ntZ) = \bar{g}(\nabla^s_{tZ} ntZ, ntZ). \tag{3.67}$$

Using the fact that $\bar{\nabla}$ is a metric connection on $\bar{N}$ with respect to $\bar{g}$ along with Eq.(2.7) and Corollary (3.1), we have

$$\bar{g}(\nabla^s_{tZ} ntZ, ntZ) = \frac{1}{2}(sin2(\theta_j)_p g(tZ, tZ) tZ(\theta_j)_p + sin^2(\theta_j)_p \overline{\nabla}_{tZ} \bar{g}(tZ, tZ)). \tag{3.68}$$

Further using Eq.(3.68) in Eq. (3.67) along with Corollary (3.1) and hypothesis that $H^s \in \Gamma(\mu_p)$, we acquire

$$sin^2(\theta_j)_p \bar{g}(\nabla_{tZ} tZ, tZ) = \frac{1}{2}(sin2(\theta_j)_p g(tZ, tZ) tZ(\theta_j)_p + sin^2(\theta_j)_p \overline{\nabla}_{tZ} \bar{g}(tZ, tZ)). \tag{3.69}$$

As $\overline{\nabla}$ is metric connection on $\bar{N}$ with respect to $\bar{g}$, thus we have

$$\overline{\nabla}_{tZ} \bar{g}(tZ, tZ) = 2\bar{g}(\overline{\nabla}_{tZ} tZ, tZ). \tag{3.70}$$

Next using Eq.(3.70) in Eq.(3.69), we get,

$$sin2(\theta_j)_p g(tZ, tZ) tZ(\theta_j)_p = 0.$$

Since g is non-degenerate on $\Gamma(D_j)_p$ and N is proper **pw.bi-s.l.s.**, thus we conclude that $tZ(\theta_j)_p$=0; this shows that θ_j is independent of choice of $p \in N$ which proves the result. □

Definition 3.4. *A lightlike submanifold (N, g, ∇) of $(\bar{N}, \bar{g}, \overline{\nabla})$ is called totally geodesic if any geodesic of N is a geodesic of $\bar{N}$. Using Eq.(2.4), (N, g, ∇) is totally geodesic in $(\bar{N}, \bar{g}, \overline{\nabla})$ if and only if the second fundamental form vanishes on N i.e, $h^l(Z_1, Z_2) = h^s(Z_1, Z_2) = 0$ $\forall Z_1, Z_2 \in \Gamma(TN)$.*

Theorem 3.11. *Assume N is a totally umbilical **pw.bi-s.l.s.** of $\bar{N}$ with $H^s \in \Gamma(\mu)$ and $\overline{\nabla}^s_Z V \in \Gamma(\mu)$ for $V \in \Gamma(S(TN^\perp))$ and $Z \in \Gamma(D_j)_p$ for j=1,2. Then, N is totally geodesic in $\bar{N}$.*

Proof. Let $Z \in \Gamma(D_j)_p$ for some $j = 1, 2$ and $p \in N$. Then from Eq. (2.13), we have $\bar{\nabla}_Z \bar{J} Z = \bar{J} \bar{\nabla}_Z Z$. Further using Eqs. (2.4), (2.7), (3.15) and (3.16), we obtain

$$\begin{aligned} \nabla_Z tZ + h^l(Z, tZ) + h^s(Z, tZ) - A_{nZ} Z + D^l(Z, nZ) + \nabla^s_Z nZ = \\ t\nabla_Z Z + n\nabla_Z Z + Bh^l(Z, Z) + Bh^s(Z, Z) + Ch^s(Z, Z). \end{aligned} \tag{3.71}$$

On comparing the tangential components on both sides of above equation and using Eq.(3.65), we get

$$\nabla_Z tZ - A_{nZ} Z = t\nabla_Z Z + g(Z, Z) BH^l + g(Z, Z) BH^s.$$

taking inner product with $\bar{J}\xi \in \Gamma(Rad(TN))$, where $\xi \in \Gamma(Rad(TN))$, we get

$$\begin{aligned} \bar{g}(\nabla_Z tZ, \bar{J}\xi) - \bar{g}(A_{nZ} Z, \bar{J}\xi) = & \bar{g}(t\nabla_Z Z, \bar{J}\xi) + g(Z, Z)\bar{g}(BH^l, \bar{J}\xi) \\ & + g(Z, Z)\bar{g}(BH^s, \bar{J}\xi). \end{aligned} \tag{3.72}$$

Now, using Eqs.(2.13), (3.15), (2.12) and (3.16), we have

$$\bar{g}(t\nabla_Z Z, \bar{J}\xi) = 0 = \bar{g}(BH^s, \bar{J}\xi) = \bar{g}(A_{nZ} Z, \bar{J}\xi). \tag{3.73}$$

Also,

$$\begin{aligned} \bar{g}(\nabla_Z tZ, \bar{J}\xi) = \bar{g}(\overline{\nabla}_Z tZ, \bar{J}\xi) =& - \bar{g}(\overline{\nabla}_Z \bar{J}tZ, \bar{J}\xi) \\ =& -\bar{g}(\overline{\nabla}_Z t^2 Z, \xi) - \bar{g}(\overline{\nabla}_Z ntZ, \xi) \\ =& \, \bar{g}(h^l(Z, t^2 Z), \xi) \\ =& \, cos^2\theta_p(Z) g(Z, Z)\bar{g}(H^l, \xi). \end{aligned} \tag{3.74}$$

Using Eqs.(3.16) and (2.13), we get

$$\bar{g}(BH^l, \bar{J}\xi) = \bar{g}(H^l, \xi). \tag{3.75}$$

Now, using Eqs.(3.73), (3.74) and (3.75) in (3.72), we have

$$g(Z, Z)\bar{g}(H^l, \xi)(1 + cos^2(\theta_j)_p(Z)) = 0.$$

As g is non-degenerate on $\Gamma(D_j)_p$, therefore one has $\bar{g}(H^l, \xi) = 0$ which further implies that

$$H^l = 0. \tag{3.76}$$

Secondly, On comparing the transversal components of Eq.(3.71) and then considering the inner product of resulting part with $\bar{J}H^s$, we get

$$\bar{g}(\nabla^s_Z nZ, \bar{J}H^s) = g(Z, Z)\bar{g}(H^s, H^s). \tag{3.77}$$

As $\overline{\nabla}$ is a metric connection, we have $(\overline{\nabla}_Z \bar{g})(nZ, \bar{J}H^s) = 0)$, which on using Eq. (2.7) together with hypothesis $H^s \in \Gamma(\mu)$ and $\overline{\nabla}^s_Z V \in \Gamma(\mu)$, for $V \in \Gamma(S(TN^\perp))$ yields that

$$\bar{g}(\nabla^s_Z nZ, \bar{J}H^s) = 0. \tag{3.78}$$

Then using Eq.(3.78) in Eq.(3.77), we get

$$g(Z, Z)\bar{g}(H^s, H^s) = 0.$$

As the slant distribution is non-degenerate, therefore,

$$H^s = 0. \tag{3.79}$$

Thus, the proof follows. □

4. Totally Geodesic Foliations Determined by Distributions

In this section, we investigate the conditions for foliations determined by distributions $Rad(TN), D_1$ and D_2 to be totally geodesic.

Theorem 4.1. *Assume N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, $Rad(TN)$ defines a totally geodesic foliation if and only if*

$$g(A_{n\phi_4 W}Z_1 + A_{n\phi_5 W}Z_1, tZ_2) = g(\nabla_{Z_1}tW + \nabla_{Z_1}tW, tZ_2),$$

for $Z_1, Z_2 \in \Gamma(Rad(TN))$ and $W \in \Gamma(S(TN))$.

Proof. In order to show that $Rad(TN)$ defines a totally geodesic foliation, it is sufficient to show that $\nabla_{Z_1}Z_2 \in \Gamma(Rad(TN))$ for $Z_1, Z_2 \in \Gamma(Rad(TN))$. Using the fact that $\overline{\nabla}$ is a metric connection along with Eq.(2.13) and (2.4), for $Z_1, Z_2 \in \Gamma(Rad(TN))$ and $W \in \Gamma(S(TN))$, we get

$$g(\nabla_{Z_1}Z_2, W) = -\bar{g}(\overline{\nabla}_{Z_1}\bar{J}W, \bar{J}Z_2) - \bar{g}(\overline{\nabla}_W \bar{J}Z_1, \bar{J}Z_2) + \bar{g}(\bar{J}\overline{\nabla}_W Z_1, \bar{J}Z_2). \tag{4.80}$$

Moreover, using Eqs.(3.18) and (3.22) in Eq.(4.80), we get

$$g(\nabla_{Z_1}Z_2, W) = \bar{g}(\nabla_{Z_1}tW + \nabla_W tZ_1 - A_{n\phi_4 W}Z_1 - A_{n\phi_5 W}Z_1, tZ_2),$$

which proves the theorem. □

Theorem 4.2. *Assume N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, D_1 defines a totally geodesic foliation if and only if*

(i) *$\overline{\nabla}_{Z_1}\bar{J}W + \overline{\nabla}_W \bar{J}Z_1$ has no components along D_1, $S(TN^{\perp})$ and $\nabla_W Z_1$ has no component along D_1.*
(ii) *$A_N Z_1$ has no component along D_1.*
(iii) *$\nabla_{Z_1}W'$ has no component along D_1.*
(iv) *$\nabla_{Z_1}V$ has no component along D_1.*

for $Z_1 \in \Gamma(D_1)$, $N \in \Gamma(ltr(TN))$, $W' \in \Gamma\bar{J}(ltr(TN))$, $V \in \Gamma\bar{J}(Rad(TN))$ and $W \in \Gamma(D_2)$.

Proof. Assume $Z_1, Z_2 \in \Gamma(D_1)$. To show that $\nabla_{Z_1}Z_2 \in \Gamma(D_1)$, it is sufficient to show that $\nabla_{Z_1}Z_2$ has no components along $Rad(TN)$, $\bar{J}Rad(TN)$, $\bar{J}ltr(TN)$ and D_2. For $W \in \Gamma(D_2)$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4) and (2.13), we have

$$\bar{g}(\nabla_{Z_1}Z_2, W) = -\bar{g}(\overline{\nabla}_{Z_1}\bar{J}W + \overline{\nabla}_W \bar{J}Z_1, \bar{J}Z_2) + \bar{g}(\nabla_W Z_1, Z_2). \tag{4.81}$$

For $N \in \Gamma(ltr(TN))$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4) and (2.6), we have

$$\bar{g}(\nabla_{Z_1} Z_2, N) = \bar{g}(A_N Z_1, Z_2). \tag{4.82}$$

Also, for $W' \in \Gamma(\bar{J} ltr(TN))$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4), we have

$$\bar{g}(\nabla_{Z_1} Z_2, W') = -\bar{g}(\nabla_{Z_1} W', Z_2). \tag{4.83}$$

Now, consider $V \in \Gamma \bar{J} Rad(TN)$, using the fact that $\overline{\nabla}$ is a metric connection along with Eqs.(2.4), we have

$$\bar{g}(\nabla_{Z_1} Z_2, V) = -\bar{g}(\nabla_{Z_1} V, Z_2). \tag{4.84}$$

hence, the result follows from Eqs.(4.81), (4.82), (4.83) and (4.84). □

Following the same procedure as above, it can easily be shown that

Theorem 4.3. *Assume N be a* ***pw.bi-s.l.s.*** *of $\bar{N}$. Then, D_2 defines a totally geodesic foliation if and only if*

(i) $\overline{\nabla}_{Z_1} \bar{J} W + \overline{\nabla}_W \bar{J} Z_1$ *has no components along D_2, $S(TN^{\perp})$ and $\nabla_W Z_1$ has no component along D_2.*

(ii) $A_N Z_1$ *has no component along D_2.*

(iii) $\nabla_{Z_1} W'$ *has no component along D_2.*

(iv) $\nabla_{Z_1} V$ *has no component along D_2.*

for $Z_1 \in \Gamma(D_2)$, $N \in \Gamma(ltr(TN))$, $W' \in \Gamma \bar{J}(ltr(TN))$, $V \in \Gamma \bar{J}(Rad(TN))$ and $W \in \Gamma(D_1)$.

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References

[1] Cabrerizo, J. L., Carriazo, A., Fernandez, L. M., & Fernandez, M. (2000). Slant submanifolds in Sasakian manifolds. Glasg. Math. J., 42, 125–138.

[2] Carriazo, A. (2002). New developments in slant submanifolds theory. Narosa Publishing House.

[3] Carriazo, A. (2000). Bi-slant immersion. In Proc. ICRAMS (pp. 88–97). Kharagpur, India.

[4] Chen, B. Y. (1990). Slant immersions. Bull. Austral. Math. Soc., 41, 135–147.

[5] Chen, B. Y. (1990). Geometry of slant submanifolds. Katholieke University, Leuven.

[6] Chen, B. Y., & Garay, O. (2012). Pointwise slant submanifolds in almost Hermitian manifolds. Turk. J. Math., 36, 630–640.

[7] Duggal, K. L., & Jin, D. H. (2003). Totally umbilical lightlike submanifolds. Kodai Math. J., 26, 49–68.

[8] Duggal, K. L., & Bejancu, A. (1996). Lightlike submanifolds of semi-Riemannian manifolds and applications (Mathematics and its Applications, Vol. 364). Kluwer Academic Publishers.

[9] Etayo, F. (1998). On quasi-slant submanifolds of an almost Hermitian manifold. Publ. Math. Debrecen, 53, 217–223.

[10] Gray, A. (1970). Nearly Kaehler manifolds. J. Diff. Geom., 4, 283–309.

[11] Gupta, G., Sachdeva, R., Kumar, R., & Nagaich, R. (2018). Pointwise slant lightlike submanifolds of indefinite Kaehler manifolds. Medterr. J. Math., 15(3).

[12] Karmakar, P. (2024). Totally contact umbilical screen-slant and screen-transversal lightlike submanifolds of indefinite Kenmotsu manifold. Mathematica Bohemica, 149(4), 603–613.

[13] Kumar, T., & Kumar, S. (2016). Screen bi-slant lightlike submanifolds of indefinite Kähler manifolds. Balkan J. Geom. Appl., 145, 30–36.

[14] Kumar, T., Pruthi, M., Kumar, S., & Kumar, P. (2021). Geometric characteristics of screen slant lightlike submanifolds of indefinite nearly Kähler manifolds. Balkan J. Geom. Appl., 26, 44–54.

[15] Lotta, A. (1996). Slant submanifolds in contact geometry. Bull. Math. Soc. Roumanie, 39, 183–198.

[16] Lotta, A. (1998). Three-dimensional slant submanifolds of K-contact manifolds. Balkan J. Geom. Appl., 3, 37–51.

[17] Pruthi, M., & Kumar, S. (2023). Pointwise bi-slant lightlike submanifolds and their warped products. J. Geom. Phys., 191(2).

[18] Papaghiuc, N. (1994). Semi-slant submanifolds of Kählerian manifold. Annal. St. Univ. Iasi., 9, 55–61.

[19] Sahin, B. (2009). Warped product submanifolds of a Kähler manifold with a slant factor. Ann. Pol. Math., 95, 207–226.

[20] Sahin, B. (2008). Slant lightlike submanifolds of indefinite Hermitian manifolds. Balkan J. Geom. Appl., 13, 107–119.

[21] Sahin, B. (2012). Slant lightlike submanifolds of indefinite Sasakian manifolds. Filomat, 26, 277–287.

[22] Sahin, B. (2009). Screen slant lightlike submanifolds. Int. Electron. J. Geom., 2, 41–54.

[23] Shukla, S. S., & Yadav, A. (2015). Semi-slant lightlike submanifolds of indefinite Kähler manifolds. Revista De La, 56, 21–37.

[24] Shukla, S. S., & Yadav, A. (2016). Screen pseudo-slant lightlike submanifolds of indefinite Kähler manifolds. Novi Sad J. Math., 46, 147–158.

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A SOLITONIC STUDY ON PARA-SASAKIAN MANIFOLDS ADMITTING SEMI-SYMMETRIC NONMETRIC CONNECTION

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Abstract. In this paper we have introduced a new semi-symmetric nonmetric connection (briefly, SSNM-connection) and established its existence on para-Sasakian manifold. We obtain Riemannian curvature tensor, Ricci tensor, scalar curvature etc. with respect to the SSNM-connection and studied the properties of para-Sasakian manifold with the help of this connection. We also study η-Einstein soliton on para-Sasakian manifolds with respect to this connection and prove that a para-Sasakian manifold admitting η-Einstein soliton with respect to the SSNM-connection is a generalized η-Einstein manifold. Further, we investigate η-Einstein soliton on para-Sasakian manifolds satisfying $\overline{R}.\overline{S} = 0, \overline{S}.\overline{R} = 0$ and $\overline{R}.\overline{R} = 0$, where $\overline{R}$ and $\overline{S}$ are Riemannian curvature tensor and Ricci tensor with respect to the SSNM-connection, respectively. At last, some conclusions are made after observing all the results and an example of 3-dimensional para-Sasakian manifold admitting the SSNM-connection is given in which all the results can be verified easily.

Keywords: Para-Sasakian manifold, Semi-symmetric nonmetric connection, Einstein soliton, η-Einstein soliton.

2020 Mathematics Subject Classification: 53C15, 53C25.

1. Introduction

In 1979, the notion of para-Sasakian (briefly, P-Sasakian) and special para-Sasakian (briefly, SP-Sasakian) manifolds were introduced by Sato and Matsumoto [26]. Later, Adati and Matsumoto investigate some interesting results on P-Sasakian manifolds and SP-Sasakian manifolds in [1]. The properties of para-Sasakian manifold have been studied by many authors. For instance, we see [2, 16, 17, 19, 21, 25, 28] and their references.

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In 1924, Friedmann and Schouten gave the notion of semi-symmetric connection on a differentiable manifold. A linear connection on a differentiable manifold M is said to be semi-symmetric if its torsion tensor T satisfies

$$T(\Lambda_1, \Lambda_2) = \pi(\Lambda_2)\Lambda_1 - \pi(\Lambda_1)\Lambda_2, \tag{1.1}$$

for all Λ_1, $\Lambda_2 \in \chi(M)$, where $\chi(M)$ is the set of all vector fields on M and π is a 1-form associated with the vector field P given by

$$\pi(\Lambda_1) = \delta(\Lambda_1, P),$$

where δ is a metric on M. In 1932, Hayden [14] introduced the semi-symmetric metric connection on a Riemannian manifold and later it was named as Hayden connection. A linear connection ∇ is said to be metric connection if

$$(\nabla_{\Lambda_1}\delta)(\Lambda_2, \Lambda_3) = 0, \tag{1.2}$$

otherwise it is nonmetric. A systematic study of semi-symmetric metric connection was initiated by Yano [31] in 1970. He proved that a Riemannian manifold with respect to the semi-symmetric metric connection has vanishing curvature tensor if and only if it is conformally flat. The study of semi-symmetric metric connection was further developed by Amur and Puzara [4], Binh [5], De [11], Ozgur and Sular [20], Singh and Pandey [27] and many others.

On the other hand, semi-symmetric nonmetric connetion whose torsion is given by (1.1) was introduced by Agashe and Chafle [3] in 1992. They showed that a Riemannian manifold is projectively flat if it's curvature tensor with respect to the SSNM-connection vanishes. This linear connection was further developed by many researchers such as Chaubey and Ojha [9], De and Kamilya [12], De, Han and Zhao [13], Prasad and Singh [22], Prasad and Verma [23] and many others. Recently, in [10], Chaubey and Yieldiz defined a new type of SSNM-connection on Remannian manifolds. They investigated various curvature properties of Riemannian manifold with respect to the SSNM-connection and studied Ricci soliton on Riemannian manifold with respect to this connection. Motivated by their studies, here the SSNM-connection has been introduced on para-Sasakian manifold to study some properties and explore η-Einstein soliton on this manifold.

R. S. Hamilton was the first who introduced the notion of Ricci flow in the early 1980s. His [15] observation on Ricci flow was that it is a tool by which the formation of a manifold can be simplified. It is the process which deforms the metric of a differentiable manifold by smoothing out the irregularities. The equation of Ricci flow is given by

$$\frac{\partial \delta}{\partial t} = -2S, \tag{1.3}$$

where δ is a Riemannian metric, S is Ricci curvature tensor and t being the time. The solitons for the Ricci flow is the self similar solutions of the above partial differential equation, where the metrices at various times differ by a diffeomorphism of the manifold. A triple (δ, V, λ) is used to represent a Ricci soliton regard to Ricci flow, where V is a smooth vector field and λ is a scalar, which satisfies the equation

$$L_V \delta + 2S + 2\lambda\delta = 0, \tag{1.4}$$

where $L_V \delta$ denotes the Lie derivative of δ along the vector field V. A Ricci soliton is said to be shrinking if $\lambda < 0$, steady if $\lambda = 0$ and expanding if $\lambda > 0$. The vector field V is called potential vector field and if it is a gradient of a differentiable function, then the Ricci soliton (δ, V, λ) is said to be a gradient Ricci soliton and the associated differentiable function is named as potential function. Ricci soliton was further studied by many researchers. For instance, we see [8, 18, 24, 29, 30] and their references.

Catino and Mazzieri [7] in 2016 first introduced the notion of Einstein soliton as a generalization of Ricci soliton. An almost contact manifold M with structure $(\phi, \varsigma, \eta, \delta)$ is said to have an Einstein soliton (δ, V, λ) if

$$L_V \delta + 2S + (2\lambda - r)\delta = 0, \tag{1.5}$$

holds, where r being the scalar curvature. The Einstein soliton (δ, V, λ) is said to be shrinking, steady, expanding according as $\lambda < 0, \lambda = 0, \lambda > 0$, respectively. Einstein soliton creates some self-similar solutions of the Einstein flow equation given by

$$\frac{\partial \delta}{\partial t} = -2S + r\delta. \tag{1.6}$$

Again as a generalization of Einstein soliton, the η-Einstein soliton on a Riemannian manifold $M\,(\phi, \varsigma, \eta, \delta)$ was introduced by Blaga [6] and it is given by

$$L_V \delta + 2S + (2\lambda - r)\delta + 2\beta\eta \otimes \eta = 0, \tag{1.7}$$

where, β is some constant. When $\beta = 0$ the notion of η-Einstein soliton simply reduces to the notion of Einstein soliton. And when $\beta \neq 0$, the data $(\delta, V, \lambda, \beta)$ is called proper

η-Einstein soliton on M. The η-Einstein soliton is called shrinking if $\lambda < 0$, steady if $\lambda = 0$, and expanding if $\lambda > 0$.

Definition 1.1. *A para-Sasakian manifold M is called an η-Einstein manifold if its Ricci tensor is of the form*

$$S(\Lambda_2, \Lambda_3) = l_1 \delta(\Lambda_2, \Lambda_3) + l_2 \eta(\Lambda_2)\eta(\Lambda_3),$$

for all $\Lambda_2,\ \Lambda_3 \in \chi(M)$, where l_1, l_2 are scalars.

Definition 1.2. *A para-Sasakian manifold M is called a generalized η-Einstein manifold if its Ricci tensor is of the form*

$$S(\Lambda_2, \Lambda_3) = k_1 \delta(\Lambda_2, \Lambda_3) + k_2 \eta(\Lambda_2)\eta(\Lambda_3) + k_3 \delta(\Lambda_2, \phi\Lambda_3),$$

for all $\Lambda_2,\ \Lambda_3 \in \chi(M)$, where k_1, k_2 and k_3 are scalars.

This paper is structured as follows:

First two sections of the paper has been kept for introduction and preliminaries. In **Section-3**, we introduce semi-symmetric nonmetric connection ($\overline{\nabla}$) on para-Sasakian manifolds. In **Section-4**, we study η-Einstein soliton on para-Sasakian manifold with respect to $\overline{\nabla}$. **Section-5** deals with η-Einstein soliton on para-Sasakian manifold satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$. **Section-6** concerns with η-Einstein soliton on para-Sasakian manifold satisfying $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$. **Section-7** contains η-Einstein soliton on para-Sasakian manifold satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{R} = 0$. **Section-8** contains a non trivial example of three dimensional para-Sasakian manifold admitting semi-symmetric non metric connection.

2. Preliminaries

Let M be an n-dimensional differentiable manifold with structure (ϕ, ς, η), where η is a 1-form, ς is the structure vector field, ϕ is a $(1,1)$-tensor field satisfying [26]

$$\phi^2(\Lambda_1) = \Lambda_1 - \eta(\Lambda_1)\varsigma, \eta(\varsigma) = 1, \tag{2.8}$$

$$\phi(\varsigma) = 0, \eta \circ \phi = 0, \tag{2.9}$$

for all vector field Λ_1 on M is called almost paracontact manifold. If an almost paracontact manifold M with structure (ϕ, ς, η) admits a pseudo-Riemannian metric δ such that [32]

$$\delta(\phi\Lambda_1, \phi\Lambda_2) = -\delta(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2), \tag{2.10}$$

then we say that M is an almost paracontact metric manifold with an almost paracontact metric structure $(\phi, \varsigma, \eta, \delta)$. From (2.10) one can deduce that

$$\begin{aligned} \delta\left(\Lambda_1, \phi\Lambda_2\right) &= -\delta\left(\phi\Lambda_1, \Lambda_2\right), \qquad (2.11) \\ \delta(\Lambda_1, \varsigma) &= \eta(\varsigma). \qquad (2.12) \end{aligned}$$

An almost paracontact metric structure of M becomes a paracontact metric structure [32] if

$$\delta\left(\Lambda_1, \phi\Lambda_2\right) = d\eta(\Lambda_1, \Lambda_2),$$

for all vector fields Λ_1, Λ_2 on M, where

$$d\eta(\Lambda_1, \Lambda_2) = \frac{1}{2}\left\{\Lambda_1\eta(\Lambda_2) - \Lambda_2\eta(\Lambda_1) - \eta([\Lambda_1, \Lambda_2])\right\}.$$

The manifold M is called a para-Sasakian manifold if

$$\left(\nabla_{\Lambda_1}\phi\right)\Lambda_2 = -\delta\left(\Lambda_1, \Lambda_2\right)\varsigma + \eta\left(\Lambda_2\right)\Lambda_1, \qquad (2.13)$$

for any smooth vector fields Λ_1, Λ_2 on M.

In a para-Sasakian manifold the following relations also hold [32]

$$\begin{aligned} \left(\nabla_{\Lambda_1}\eta\right)\Lambda_2 &= \delta\left(\Lambda_1, \phi\Lambda_2\right), \nabla_{\Lambda_1}\varsigma = -\phi\Lambda_1, \qquad (2.14) \\ \eta\left(R\left(\Lambda_1, \Lambda_2\right)\Lambda_3\right) &= \delta(\Lambda_1, \Lambda_3)\eta\left(\Lambda_2\right) - \delta\left(\Lambda_2, \Lambda_3\right)\eta\left(\Lambda_1\right), \qquad (2.15) \\ R\left(\Lambda_1, \Lambda_2\right)\varsigma &= \eta\left(\Lambda_1\right)\Lambda_2 - \eta\left(\Lambda_2\right)\Lambda_1, \qquad (2.16) \\ R(\varsigma, \Lambda_1)\Lambda_2 &= -\delta(\Lambda_1, \Lambda_2)\varsigma + \eta\left(\Lambda_2\right)\Lambda_1, \qquad (2.17) \\ R(\Lambda_1, \varsigma)\Lambda_2 &= \delta(\Lambda_1, \Lambda_2)\varsigma - \eta\left(\Lambda_2\right)\Lambda_1, \qquad (2.18) \\ R(\varsigma, \Lambda_1)\varsigma &= \Lambda_1 - \eta\left(\Lambda_1\right)\varsigma, \qquad (2.19) \\ S\left(\Lambda_1, \varsigma\right) &= -\left(n-1\right)\eta\left(\Lambda_1\right), \qquad (2.20) \\ S\left(\varsigma, \varsigma\right) &= -\left(n-1\right), Q\varsigma = -\left(n-1\right)\varsigma, \qquad (2.21) \\ S\left(\phi\Lambda_1, \phi\Lambda_2\right) &= S\left(\Lambda_1, \Lambda_2\right) + (n-1)\eta\left(\Lambda_1\right)\eta\left(\Lambda_2\right), \qquad (2.22) \end{aligned}$$

for any smooth vector fields Λ_1, Λ_2 and Λ_3 on M.

3. Semi-symmetric nonmetric connection on para-Sasakian manifolds

In this section we get the relation between SSNM-connection and Levi-Civita connection on para-Sasakian manifold M. Then we obtain Riemannian curvature tensor, Ricci curvature tensor, Ricci operator and scalar curvature of M with respect to the SSNM-connection. We also establish here the first Bianchi identity with respect to SSNM-connection on M.

Let $M\ (\phi, \varsigma, \eta, \delta)$ be an n-dimensional para-Sasakian manifold equipped with Levi-Civita connection ∇ corresponding to the Riemannian metric δ. Let a linear connection $\overline{\nabla}$ on M be defined by

$$\overline{\nabla}_{\Lambda_1}\Lambda_2 = \nabla_{\Lambda_1}\Lambda_2 + \frac{1}{2}\left[\eta\left(\Lambda_2\right)\Lambda_1 - \eta\left(\Lambda_1\right)\Lambda_2\right], \tag{3.23}$$

for all $\Lambda_1,\ \Lambda_2 \in \chi\left(M\right).$

Using the fact that ∇ is a metric connection, we have from (3.23) that

$$\begin{aligned}\left(\overline{\nabla}_{\Lambda_1}\delta\right)\left(\Lambda_2, \Lambda_3\right) &= \frac{1}{2}\left[\delta\left(\Lambda_1, \Lambda_2\right)\eta\left(\Lambda_3\right) + \delta\left(\Lambda_1, \Lambda_3\right)\eta\left(\Lambda_2\right)\right] \\ &\quad -\delta\left(\Lambda_2, \Lambda_3\right)\eta\left(\Lambda_1\right),\end{aligned} \tag{3.24}$$

for all $\Lambda_1,\ \Lambda_2,\ \ \Lambda_3 \in \chi\left(M\right).$ Therefore $\overline{\nabla}$ is a nonmetric connection on M. The torsion tensor of $\overline{\nabla}$ is given by

$$\overline{T}\left(\Lambda_1, \Lambda_2\right) = \eta\left(\Lambda_2\right)\Lambda_1 - \eta\left(\Lambda_1\right)\Lambda_2. \tag{3.25}$$

Suppose that the connection $\overline{\nabla}$ defined on M is connected with the Levi-Civita connection ∇ by the relation

$$\overline{\nabla}_{\Lambda_1}\Lambda_2 = \nabla_{\Lambda_1}\Lambda_2 + \mathcal{H}\left(\Lambda_1, \Lambda_2\right), \tag{3.26}$$

where $\mathcal{H}\left(\Lambda_1, \Lambda_2\right)$ is a tensor field of type $(1,1)$. By definition of torsion tensor, we have

$$\overline{T}\left(\Lambda_1, \Lambda_2\right) = \mathcal{H}\left(\Lambda_1, \Lambda_2\right) - \mathcal{H}\left(\Lambda_2, \Lambda_1\right). \tag{3.27}$$

In view of (3.25) and (3.26) we have

$$\begin{aligned}\delta\left(\mathcal{H}\left(\Lambda_1, \Lambda_2\right), \Lambda_3\right) + \delta\left(\mathcal{H}\left(\Lambda_1, \Lambda_3\right), \Lambda_2\right) &= \frac{1}{2}\delta\left(\Lambda_1, \Lambda_2\right)\eta\left(\Lambda_3\right) + \frac{1}{2}\delta\left(\Lambda_1, \Lambda_3\right)\eta\left(\Lambda_2\right) \\ &\quad -\delta\left(\Lambda_2, \Lambda_3\right)\eta\left(\Lambda_1\right),\end{aligned} \tag{3.28}$$

$$\begin{aligned}\delta\left(\mathcal{H}\left(\Lambda_2, \Lambda_1\right), \Lambda_3\right) + \delta\left(\mathcal{H}\left(\Lambda_2, \Lambda_3\right), \Lambda_1\right) &= \frac{1}{2}\delta\left(\Lambda_2, \Lambda_1\right)\eta\left(\Lambda_3\right) + \frac{1}{2}\delta\left(\Lambda_2, \Lambda_3\right)\eta\left(\Lambda_1\right) \\ &\quad -\delta\left(\Lambda_1, \Lambda_3\right)\eta\left(\Lambda_2\right),\end{aligned} \tag{3.29}$$

$$\begin{aligned}\delta\left(\mathcal{H}\left(\Lambda_3, \Lambda_1\right), \Lambda_2\right) + \delta\left(\mathcal{H}\left(\Lambda_3, \Lambda_2\right), \Lambda_1\right) &= \frac{1}{2}\delta\left(\Lambda_3, \Lambda_1\right)\eta\left(\Lambda_2\right). + \frac{1}{2}\delta\left(\Lambda_2, \Lambda_3\right)\eta\left(\Lambda_1\right) \\ &\quad -\delta\left(\Lambda_1, \Lambda_2\right)\eta\left(\Lambda_3\right).\end{aligned} \tag{3.30}$$

In view of (3.27), (3.28), (3.29) and (3.30), we have

$$\begin{aligned}
& \delta\left(\overline{T}(\Lambda_1,\Lambda_2),\Lambda_3\right)+\delta\left(\overline{T}(\Lambda_3,\Lambda_1),\Lambda_2\right)+\delta\left(\overline{T}(\Lambda_3,\Lambda_2),\Lambda_1\right) \\
= & \ \delta(\mathcal{H}(\Lambda_1,\Lambda_2),\Lambda_3)-\delta(\mathcal{H}(\Lambda_2,\Lambda_1),\Lambda_3)+\delta(\mathcal{H}(\Lambda_3,\Lambda_1),\Lambda_2) \\
& -\delta(\mathcal{H}(\Lambda_1,\Lambda_3),\Lambda_2)+\delta(\mathcal{H}(\Lambda_3,\Lambda_2),\Lambda_1)-\delta(\mathcal{H}(\Lambda_2,\Lambda_3),\Lambda_1) \\
= & \ 2\delta(\mathcal{H}(\Lambda_1,\Lambda_2),\Lambda_3)-2\delta(\Lambda_1,\Lambda_2)\eta(\Lambda_3) \\
& +\delta(\Lambda_2,\Lambda_3)\eta(\Lambda_1)-\delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2). \qquad (3.31)
\end{aligned}$$

Setting

$$\delta\left(\overline{T}(\Lambda_3,\Lambda_1),\Lambda_2\right) = \delta(T^*(\Lambda_1,\Lambda_2),\Lambda_3), \qquad (3.32)$$

$$\delta\left(\overline{T}(\Lambda_3,\Lambda_2),\Lambda_1\right) = \delta(T^*(\Lambda_2,\Lambda_1),\Lambda_3), \qquad (3.33)$$

in (3.31), we get

$$\begin{aligned}
& \delta\left(\overline{T}(\Lambda_1,\Lambda_2),\Lambda_3\right)+\delta(T^*(\Lambda_1,\Lambda_2),\Lambda_3)+\delta(T^*(\Lambda_2,\Lambda_1),\Lambda_3) \\
= & \ 2\delta(\mathcal{H}(\Lambda_1,\Lambda_2),\Lambda_3)-2\delta(\Lambda_1,\Lambda_2)\eta(\Lambda_3) \\
& +\delta(\Lambda_2,\Lambda_3)\eta(\Lambda_1)-\delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2), \qquad (3.34)
\end{aligned}$$

which implies that

$$\begin{aligned}
2\mathcal{H}(\Lambda_1,\Lambda_2) = & \ \frac{1}{2}\left[\overline{T}(\Lambda_1,\Lambda_2)+T^*(\Lambda_1,\Lambda_2)+T^*(\Lambda_2,\Lambda_1)\right] \\
& +\delta(\Lambda_1,\Lambda_2)\varsigma+\frac{1}{2}[\eta(\Lambda_2)\Lambda_1-\eta(\Lambda_1)\Lambda_2]. \qquad (3.35)
\end{aligned}$$

From (3.25), (3.32) and (3.33), it follows that

$$T^*(\Lambda_1,\Lambda_2) = -\delta(\Lambda_1,\Lambda_2)\varsigma+\eta(\Lambda_1)\eta(\Lambda_2), \qquad (3.36)$$

$$T^*(\Lambda_2,\Lambda_1) = -\delta(\Lambda_1,\Lambda_2)\varsigma+\eta(\Lambda_1)\eta(\Lambda_2). \qquad (3.37)$$

Substituting (3.25), (3.36) and (3.37) in (3.35), we obtain

$$\mathcal{H}(\Lambda_1,\Lambda_2)=\frac{1}{2}[\eta(\Lambda_2)\Lambda_1-\eta(\Lambda_1)\Lambda_2]. \qquad (3.38)$$

In reference to (3.26) and (3.38), we can easily bring out the equation (3.23).

Theorem 3.1. *There exists a unique semi-symmetric nonmetric connection* $\overline{\nabla}$ *on a para-Sasakian manifold* M *given by (3.23).*

On para-Sasakian manifold the connection $\overline{\nabla}$ has the following properties

$$\left(\overline{\nabla}_{\Lambda_1}\eta\right)\Lambda_2 = -\frac{1}{2}\delta\left(\phi\Lambda_1,\phi\Lambda_2\right), \tag{3.39}$$

$$\overline{\nabla}_{\Lambda_1}\varsigma = -\phi\Lambda_1+\frac{1}{2}\left[\Lambda_1-\eta\left(\Lambda_1\right)\varsigma\right], \tag{3.40}$$

for all $\Lambda_1,\ \Lambda_2\in\chi\left(M\right).$

Let $\overline{R}$ be the Riemannian curvature tensor with respect to SSNM-connection on a para-Sasakian manifold defined as

$$\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3=\overline{\nabla}_{\Lambda_1}\overline{\nabla}_{\Lambda_2}\Lambda_3-\overline{\nabla}_{\Lambda_2}\overline{\nabla}_{\Lambda_1}\Lambda_3-\overline{\nabla}_{[\Lambda_1,\Lambda_2]}\Lambda_3. \tag{3.41}$$

In reference of (2.13), (2.14) and (3.23) we have

$$\begin{aligned}\overline{\nabla}_{\Lambda_1}\overline{\nabla}_{\Lambda_2}\Lambda_3 &= \nabla_{\Lambda_1}\nabla_{\Lambda_2}\Lambda_3+\frac{1}{2}\left[\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2+\eta\left(\nabla_{\Lambda_1}\Lambda_3\right)\Lambda_2+\eta\left(\Lambda_3\right)\nabla_{\Lambda_1}\Lambda_2\right]\\ &\quad -\frac{1}{2}\left[\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3+\eta\left(\nabla_{\Lambda_1}\Lambda_2\right)\Lambda_3+\eta\left(\Lambda_2\right)\nabla_{\Lambda_1}\Lambda_3\right]\\ &\quad +\frac{1}{2}\left[\eta\left(\nabla_{\Lambda_2}\Lambda_3\right)\Lambda_1-\eta\left(\Lambda_1\right)\nabla_{\Lambda_2}\Lambda_3\right]\\ &\quad +\frac{1}{4}\left[\eta\left(\Lambda_1\right)\eta\left(\Lambda_2\right)\Lambda_3-\eta\left(\Lambda_1\right)\eta\left(\Lambda_3\right)\Lambda_2\right], \end{aligned} \tag{3.42}$$

$$\begin{aligned}\overline{\nabla}_{[\Lambda_1,\Lambda_2]}\Lambda_3 &= \nabla_{[\Lambda_1,\Lambda_2]}\Lambda_3+\frac{1}{2}\left[\eta\left(\Lambda_3\right)\nabla_{\Lambda_1}\Lambda_2-\eta\left(\Lambda_3\right)\nabla_{\Lambda_2}\Lambda_1\right]\\ &\quad +\frac{1}{2}\left[\eta\left(\nabla_{\Lambda_2}\Lambda_1\right)\Lambda_3-\eta\left(\nabla_{\Lambda_1}\Lambda_2\right)\Lambda_3\right]. \end{aligned} \tag{3.43}$$

Interchanging Λ_1 and Λ_2 in (3.42) and using it along with (3.42) and (3.43) in (3.41) we get

$$\begin{aligned}\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3 &= R(\Lambda_1,\Lambda_2)\Lambda_3+\frac{1}{2}\left[\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2-\delta\left(\Lambda_2,\phi\Lambda_3\right)\Lambda_1-2\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3\right]\\ &\quad +\frac{1}{4}\left[\eta\left(\Lambda_2\right)\Lambda_1-\eta\left(\Lambda_1\right)\Lambda_2\right]\eta\left(\Lambda_3\right), \end{aligned} \tag{3.44}$$

for all $\Lambda_1,\ \Lambda_2,\ \Lambda_3\in\chi\left(M\right).$

Writing the equation (3.44) by cyclic permutations of Λ_1, Λ_2 and Λ_3 and using first Bianchi identity with respect to Levi-Civita connection we get

$$\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3+\overline{R}(\Lambda_2,\Lambda_3)\Lambda_1+\overline{R}(\Lambda_3,\Lambda_1)\Lambda_2=2\left[\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2-\delta\left(\Lambda_2,\phi\Lambda_3\right)\Lambda_1-\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3\right].$$

Proposition 3.1. *The SSNM-connection satisfies first Bianchi identity if and only if*

$$\delta\left(\Lambda_1,\phi\Lambda_3\right)\Lambda_2=\delta\left(\Lambda_2,\phi\Lambda_3\right)\Lambda_1+\delta\left(\Lambda_1,\phi\Lambda_2\right)\Lambda_3,$$

holds for all $\Lambda_1,\ \Lambda_2$ *and* $\Lambda_3\in\chi\left(M\right).$

Taking inner product of (3.44) with a vector field Λ and contracting over Λ_1 and Λ we get

$$\begin{aligned}\overline{S}(\Lambda_2,\Lambda_3) &= S(\Lambda_2,\Lambda_3)-\frac{1}{2}(n-3)\delta(\Lambda_2,\phi\Lambda_3)\\ &\quad+\frac{1}{4}(n-1)\eta(\Lambda_2)\eta(\Lambda_3), \qquad (3.45)\end{aligned}$$

where $\overline{S}$ denotes Ricci tensor with respect to $\overline{\nabla}$.

Lemma 3.1. *Let M be an n-dimensional para-Sasakian manifold admitting SSNM-connection, then*

$$\begin{aligned}\eta\left(\overline{R}(\Lambda_1,\Lambda_2)\Lambda_3\right) &= \delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2)-\delta(\Lambda_2,\Lambda_3)\eta(\Lambda_1)-\delta(\Lambda_1,\phi\Lambda_2)\eta(\Lambda_3)\\ &\quad \frac{1}{2}\left[\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_2)-\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_1)\right], \qquad (3.46)\\ \overline{R}(\Lambda_1,\Lambda_2)\varsigma &= \frac{3}{4}\left[\eta(\Lambda_1)\Lambda_2-\eta(\Lambda_2)\Lambda_1\right]-\delta(\Lambda_1,\phi\Lambda_2)\varsigma, \qquad (3.47)\end{aligned}$$

$$\begin{aligned}\overline{R}(\varsigma,\Lambda_2)\Lambda_3 &= -\delta(\Lambda_2,\Lambda_3)\varsigma-\frac{1}{2}\delta(\Lambda_2,\phi\Lambda_3)\varsigma\\ &\quad+\frac{3}{4}\eta(\Lambda_3)\Lambda_2+\frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_3)\varsigma, \qquad (3.48)\\ \overline{R}(\Lambda_1,\varsigma)\Lambda_3 &= \delta(\Lambda_1,\Lambda_3)\varsigma+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\varsigma\\ &\quad-\frac{3}{4}\eta(\Lambda_3)\Lambda_1-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\varsigma, \qquad (3.49)\\ \overline{Q}\Lambda_1 &= Q\Lambda_1-\frac{1}{2}(n-3)\phi\Lambda_1+\frac{1}{4}(n-1)\eta(\Lambda_1)\varsigma, \qquad (3.50)\\ \overline{S}(\Lambda_1,\varsigma) &= -\frac{3}{4}(n-1)\eta(\Lambda_1), \qquad (3.51)\\ \overline{Q}\varsigma &= -\frac{3}{4}(n-1)\varsigma, \qquad (3.52)\\ \overline{r} &= r+\frac{1}{4}(n-1)-\frac{1}{2}(n-3)\psi, \qquad (3.53)\end{aligned}$$

for all Λ_1, Λ_2 and $\Lambda_3 \in \chi(M)$, where $\psi = trace(\phi)$ and $\overline{R}$, $\overline{Q}$, $\overline{r}$ denote Riemannian curvature tensor, Ricci operator, scalar curvature with respect to $\overline{\nabla}$, respectively.

Eigen value of Ricci operator with respect to SSNM-connection corresponding to the eigen vector is $-\frac{3}{4}(n-1)$.

4. η-Einstein soliton on para-Sasakian manifold with respect to SSNM-connection

In this section we find the condition of η-Einstein soliton on a para-Sasakian manifold M to be invariant under SSNM-connection. Further, we study η-Einstein soliton on M with respect to SSNM-connection in which the potential vector field being pointwise collinear with the structure vector field of M.

The equation (1.7) with respect to SSNM-connection takes the form

$$0=\left(\overline{L}_{V}\delta\right)(\Lambda_1,\Lambda_2)+2\overline{S}(\Lambda_1,\Lambda_2)+(2\lambda-\overline{r})\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2), \tag{4.54}$$

for all Λ_1, Λ_2, Λ_3, $V\in\chi(M)$. Expanding $\overline{L}_{V}$ and using (3.45), (3.53) in (4.54) we get

$$\begin{aligned}
0 &= \delta(\overline{\nabla}_{\Lambda_1}V,\Lambda_2)+\delta(\Lambda_1,\overline{\nabla}_{\Lambda_2}V)+2\overline{S}(\Lambda_1,\Lambda_2)\\
&\quad+(2\lambda-\overline{r})\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2)\\
&= (L_{V}\delta)(\Lambda_1,\Lambda_2)+2S(\Lambda_1,\Lambda_2)+(2\lambda-r)\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2)\\
&\quad+\left[\eta(V)-\frac{1}{4}(n-1)+\frac{1}{2}(n-3)\psi\right]\delta(\Lambda_1,\Lambda_2)-\frac{1}{2}\delta(V,\Lambda_2)\eta(\Lambda_1)\\
&\quad-\frac{1}{2}\delta(V,\Lambda_1)\eta(\Lambda_2)-(n-3)\delta(\Lambda_1,\phi\Lambda_2)+\frac{1}{2}(n-1)\eta(\Lambda_1)\eta(\Lambda_2).
\end{aligned} \tag{4.55}$$

Theorem 4.1. *An η-Einstein soliton (δ,V,λ,β) on a para-Sasakian manifold M to be invariant under SSNM-connection if and only if*

$$\begin{aligned}
0 &= \left[\eta(V)-\frac{1}{4}(n-1)+\frac{1}{2}(n-3)\psi\right]\delta(\Lambda_1,\Lambda_2)-\frac{1}{2}\delta(V,\Lambda_2)\eta(\Lambda_1)\\
&\quad-\frac{1}{2}\delta(V,\Lambda_1)\eta(\Lambda_2)-(n-3)\delta(\Lambda_1,\phi\Lambda_2)+\frac{1}{2}(n-1)\eta(\Lambda_1)\eta(\Lambda_2),
\end{aligned}$$

holds for Λ_1, Λ_2, Λ_3, $V\in\chi(M)$.

Consider the distribution D on M as $D=\ker\eta$. If $V\in D$, then

$$\eta(V)=0.$$

Taking covariant derivative with respect to ς and using $(\nabla_{\varsigma}\eta)V=0$, we get

$$\eta(\nabla_{\varsigma}V)=0. \tag{4.56}$$

In view of (3.23) and (4.56) we have

$$\eta\left(\nabla^{*}_{\varsigma}V\right)=0. \tag{4.57}$$

After expanding the Lie derivative in (4.54) we get

$$\begin{aligned}
0 &= \delta(\overline{\nabla}_{\Lambda_1}V,\Lambda_2)+\delta(\Lambda_1,\overline{\nabla}_{\Lambda_2}V)+2\overline{S}(\Lambda_1,\Lambda_2)\\
&\quad+(2\lambda-\overline{r})\delta(\Lambda_1,\Lambda_2)+2\beta\eta(\Lambda_1)\eta(\Lambda_2).
\end{aligned} \tag{4.58}$$

Setting $\Lambda_1 = \Lambda_2 = \varsigma$ and using (3.45), (3.53) and (4.57) in (4.58) we obtain

$$r = 2(\lambda + \beta) - \frac{7}{4}(n-1) + \frac{1}{2}(n-3)\psi, \tag{4.59}$$

where $trace(\phi) = \psi$.

Theorem 4.2. *Let M be a para-Sasakian manifold admitting η-Einstein soliton $(\delta, V, \lambda, \beta)$ with respect to SSNM-connection such that $V \in D$, then scalar curvature of M is given by (4.59).*

Setting $V = \varsigma$ in (4.54) we get

$$\begin{aligned} 0 &= \delta(\overline{\nabla}_{\Lambda_1}\varsigma, \Lambda_2) + \delta(\Lambda_1, \overline{\nabla}_{\Lambda_2}\varsigma) + 2\overline{S}(\Lambda_1, \Lambda_2) \\ &\quad + (2\lambda - \overline{r})\delta(\Lambda_1, \Lambda_2) + 2\beta\eta(\Lambda_1)\eta(\Lambda_2). \end{aligned} \tag{4.60}$$

Using (3.40) and (4.60) we obtain

$$\overline{S}(\Lambda_1, \Lambda_2) = -\frac{1}{2}(2\lambda - \overline{r} + 1)\delta(\Lambda_1, \Lambda_2) - \frac{1}{2}(2\beta - 1)\eta(\Lambda_1)\eta(\Lambda_2). \tag{4.61}$$

Using (3.45) and (3.53) in (4.61) we get

$$S(\Lambda_1, \Lambda_2) = k\delta(\Lambda_1, \Lambda_2) + l\eta(\Lambda_1)\eta(\Lambda_2) + m\delta(\Lambda_1, \phi\Lambda_2), \tag{4.62}$$

where

$$\begin{aligned} k &= -\frac{1}{2}\left[2\lambda - r - \frac{1}{4}(n-5) + \frac{1}{2}(n-3)\psi\right], \\ l &= -\frac{1}{4}\left[4\beta + n - 3)\right], \\ m &= -\frac{1}{2}(n-3). \end{aligned}$$

Corollary 4.1. *If a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection, then M is generalized η-Einstein.*

Corollary 4.2. *If a para-Sasakian manifold M contains an η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection such that the structure vector field ς be parallel i.e., $\nabla_{\Lambda_1}\varsigma = 0$, then M is generalized η-Einstein manifold.*

Setting $\Lambda_2 = \varsigma$ and using (3.51) and (3.53) in (4.61) we have

$$r = 2(\lambda + \beta) - \frac{7}{4}(n-1) + \frac{1}{2}(n-3)\psi, \tag{4.63}$$

where $trace(\phi) = \psi$.

Corollary 4.3. *If a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection, then the scalar curvature of M is given by (4.63).*

Putting $\beta = 0$ and $\psi = 0$ in (4.63) we get

$$\lambda = \frac{1}{2}r + \frac{7}{8}(n-1).$$

Corollary 4.4. *Let a para-Sasakian manifold M contain an Einstein soliton $(\delta, \varsigma, \lambda)$ with respect to SSNM-connection, then the soliton is shrinking, steady or expanding if*

$$r < -\frac{7}{4}(n-1), r = -\frac{7}{4}(n-1), r > -\frac{7}{4}(n-1),$$

respectively, provided $trace(\phi) = 0$.

5. η-Einstein soliton on para-Sasakian satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$

The condition that must be satisfied by $\overline{S}$ is

$$\overline{S}(\overline{R}(\varsigma, \Lambda_1)\Lambda_2, \Lambda_3) + \overline{S}(\Lambda_2, \overline{R}(\varsigma, \Lambda_1)\Lambda_3) = 0, \tag{5.64}$$

for all Λ_1, Λ_2, $\Lambda_3 \in \chi(M)$.

Using (3.48) and replacing the expression of $\overline{S}$ from (4.61) in (5.64) we get

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3) + \delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\right] \\ & + \frac{1}{4}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \phi\Lambda_2)\eta(\Lambda_3) + \delta(\Lambda_1, \phi\Lambda_3)\eta(\Lambda_2)\right] \\ & - \frac{3}{8}\left[2\lambda - \overline{r} + 1\right]\left[\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3) + \delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\right] \\ & - \frac{1}{4}\left[2\lambda + 8\beta - \overline{r} - 3\right]\eta(\Lambda_1)\eta(\Lambda_2)\eta(\Lambda_3). \end{aligned} \tag{5.65}$$

Setting $\Lambda_3 = \varsigma$ in (5.65) we get

$$\begin{aligned} 0 \;=\; & \frac{1}{2}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2)\right] \\ & + \frac{1}{4}\left[2(\lambda+\beta) - \overline{r}\right]\left[\delta(\Lambda_1, \phi\Lambda_2)\right] \\ & - \frac{3}{8}\left[2\lambda - \overline{r} + 1\right]\left[\delta(\Lambda_1, \Lambda_2) + \eta(\Lambda_1)\eta(\Lambda_2)\right] \\ & - \frac{1}{4}\left[2\lambda + 8\beta - \overline{r} - 3\right]\eta(\Lambda_1)\eta(\Lambda_2). \end{aligned} \tag{5.66}$$

Contracting (5.66) over Λ_1 and Λ_2 we get

$$\begin{aligned} 0 &= \frac{1}{4}(n-1+2\psi)\lambda + \frac{1}{2}\left[2(n-1)+\psi\right]\beta \\ &\quad -\frac{1}{8}\left[n-1+2\psi\right]\left[r+\frac{1}{4}(n-1)-\frac{1}{2}(n-3)\psi\right] \\ &\quad -\frac{3}{8}(n-1), \end{aligned} \tag{5.67}$$

where $trace(\phi) = \psi$.

Theorem 5.1. *Let a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$, then the soliton constants are given by the equation (5.67).*

Setting $\beta = \psi = 0$ in (5.67) we obtain

$$2\lambda = r + \frac{1}{4}(n+11).$$

Corollary 5.1. *Let a para-Sasakian manifold M contain an Einstein soliton $(\delta, \varsigma, \lambda)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma, \Lambda_1).\overline{S} = 0$, then the soliton is shrinking, steady or expanding if*

$$r < -\frac{1}{4}(n+11), r = -\frac{1}{4}(n+11), r > -\frac{1}{4}(n+11),$$

respectively, provided $trace(\phi) = 0$.

6. η-Einstein soliton on para-Sasakian satisfying $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$

The condition that must be satisfied by $\overline{S}$ is

$$\begin{aligned} 0 &= \overline{S}(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\varsigma - \overline{S}(\varsigma, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\Lambda_1 \\ &\quad + S(\Lambda_1, \Lambda_2)\overline{R}(\varsigma, \Lambda_3)\Lambda_4 - \overline{S}(\varsigma, \Lambda_2)\overline{R}(\Lambda_1, \Lambda_3)\Lambda_4 \\ &\quad + \overline{S}(\Lambda_1, \Lambda_3)\overline{R}(\Lambda_2, \varsigma)\Lambda_4 - \overline{S}(\varsigma, \Lambda_3)\overline{R}(\Lambda_2, \Lambda_1)\Lambda_4 \\ &\quad + \overline{S}(\Lambda_1, \Lambda_4)\overline{R}(\Lambda_2, \Lambda_3)\varsigma - \overline{S}(\varsigma, \Lambda_4)\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1, \end{aligned} \tag{6.68}$$

for all Λ_1, Λ_2, Λ_3, $\Lambda_4 \in \chi(M)$. Taking inner product with ς the relation (6.68) becomes

$$\begin{aligned} 0 &= \overline{S}(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4) - \overline{S}(\varsigma, \overline{R}(\Lambda_2, \Lambda_3)\Lambda_4)\eta(\Lambda_1) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_2)\eta(\overline{R}(\varsigma, \Lambda_3)\Lambda_4) - \overline{S}(\varsigma, \Lambda_2)\eta(\overline{R}(\Lambda_1, \Lambda_3)\Lambda_4) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_3)\eta(\overline{R}(\Lambda_2, \varsigma)\Lambda_4) - \overline{S}(\varsigma, \Lambda_3)\eta(\overline{R}(\Lambda_2, \Lambda_1)\Lambda_4) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_4)\eta(\overline{R}(\Lambda_2, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \Lambda_4)\eta(\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1). \end{aligned} \tag{6.69}$$

Setting $\Lambda_4 = \varsigma$ in (6.69) we obtain

$$\begin{aligned} 0 &= \overline{S}(\Lambda_1, \overline{R}(\Lambda_2, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \overline{R}(\Lambda_2, \Lambda_3)\varsigma)\eta(\Lambda_1) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_2)\eta(\overline{R}(\varsigma, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \Lambda_2)\eta(\overline{R}(\Lambda_1, \Lambda_3)\varsigma) \\ &\quad + \overline{S}(\Lambda_1, \Lambda_3)\eta(\overline{R}(\Lambda_2, \varsigma)\varsigma) - \overline{S}(\varsigma, \Lambda_3)\eta(\overline{R}(\Lambda_2, \Lambda_1)\varsigma) \\ &\quad + \overline{S}(\Lambda_1, \varsigma)\eta(\overline{R}(\Lambda_2, \Lambda_3)\varsigma) - \overline{S}(\varsigma, \varsigma)\eta(\overline{R}(\Lambda_2, \Lambda_3)\Lambda_1). \end{aligned} \tag{6.70}$$

Using (3.46), (3.47), (3.49), (4.61) in (6.70) we get

$$\begin{aligned} 0 &= \frac{3}{8}\left[(2\lambda - \overline{r} + 1)\delta(\Lambda_1, \Lambda_2) + (2\beta - 1)\eta(\Lambda_1)\eta(\Lambda_2)\right]\eta(\Lambda_3) \\ &\quad - \frac{3}{8}\left[(2\lambda - \overline{r} + 1)\delta(\Lambda_1, \Lambda_3) - (2\beta - 1)\eta(\Lambda_1)\eta(\Lambda_3)\right]\eta(\Lambda_2) \\ &\quad + \left(\lambda + \beta - \frac{\overline{r}}{2}\right)\left[\delta(\Lambda_1, \Lambda_2)\eta(\Lambda_3) - \delta(\Lambda_1, \Lambda_3)\eta(\Lambda_2)\right]. \end{aligned} \tag{6.71}$$

Setting $\Lambda_1 = \varsigma$ in (6.71) we get

$$\beta = \frac{1}{2}. \tag{6.72}$$

In view of (4.63) and (6.72) we get

$$\lambda = r + \frac{1}{8}(7n - 11) - \frac{1}{4}(n - 3)\psi, \tag{6.73}$$

where $trace\,(\phi) = \psi$.

Theorem 6.1. *Let a para-Sasakian manifold M admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to SSNM-connection. If M satisfies the equation $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$, then the soliton constants are given by equations (6.72) and (6.73).*

Corollary 6.1. *There exists no Einstein soliton with respect to SSNM-connection on M satisfying $\overline{S}(\varsigma, \Lambda_1).\overline{R} = 0$.*

7. η-Einstein soliton on para-Sasakian satisfying $\overline{R}(\varsigma, \Lambda_1).\overline{R} = 0$.

The condition must be satisfied by R is

$$\begin{aligned} 0 &= \overline{R}(\varsigma, \Lambda_1)\overline{R}(\Lambda_2, \Lambda_3)\Lambda_4 - \overline{R}(\overline{R}(\varsigma, \Lambda_1)\Lambda_2, \Lambda_3)\Lambda_4 \\ &\quad - \overline{R}(\Lambda_2, \overline{R}(\varsigma, \Lambda_1)\Lambda_3)\Lambda_4 - \overline{R}(\Lambda_2, \Lambda_3)\overline{R}(\varsigma, \Lambda_1)\Lambda_4. \end{aligned} \tag{7.74}$$

Using (3.44), (3.46), (3.47) and (3.48) in (7.74) we get

$$
\begin{aligned}
0 \;=\; & -\delta(\Lambda_1, \overline{R}(\Lambda_2,\Lambda_3)\Lambda_4)\varsigma - \frac{1}{2}\delta(\Lambda_1, \phi\overline{R}(\Lambda_2,\Lambda_3)\Lambda_4)\varsigma + \frac{3}{4}\eta(\overline{R}(\Lambda_2,\Lambda_3)\Lambda_4)\Lambda_1 \\
& + \frac{1}{4}\eta(\overline{R}(\Lambda_2,\Lambda_3)\Lambda_4)\eta(\Lambda_1)\varsigma - \frac{3}{4}\eta(\Lambda_1)\overline{R}(\Lambda_1,\Lambda_3)\Lambda_4 - \frac{3}{4}\eta(\Lambda_4)\overline{R}(\Lambda_2,\Lambda_3)\Lambda_1 \\
& - \delta(\Lambda_1,\Lambda_2)\left[\delta(\Lambda_3,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_3,\phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_3 - \frac{1}{4}\eta(\Lambda_3)\eta(\Lambda_4)\varsigma\right] \\
& - \frac{1}{2}\delta(\Lambda_1,\phi\Lambda_2)\left[\delta(\Lambda_3,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_3,\phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_3 - \frac{1}{4}\eta(\Lambda_3)\eta(\Lambda_4)\varsigma\right] \\
& + \frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_2)\left[\delta(\Lambda_3,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_3,\phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_3 - \frac{1}{4}\eta(\Lambda_3)\eta(\Lambda_4)\varsigma\right] \\
& - \frac{3}{4}\left[\eta(\Lambda_3)\Lambda_2 - \eta(\Lambda_2)\Lambda_3\right]\left[\delta(\Lambda_1,\Lambda_4) + \frac{1}{2}\delta(\Lambda_1,\phi\Lambda_4) - \frac{1}{4}\eta(\Lambda_4)\eta(\Lambda_1)\right]
\end{aligned}
$$

$$
\begin{aligned}
& -\delta(\Lambda_2,\phi\Lambda_3)\left[\delta(\Lambda_1,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_1,\phi\Lambda_4)\varsigma - \frac{1}{4}\eta(\Lambda_4)\eta(\Lambda_1)\varsigma\right] - \frac{3}{4}\eta(\Lambda_3)\overline{R}(\Lambda_2,\Lambda_1)\Lambda_4 \\
& + \delta(\Lambda_1,\Lambda_3)\left[\delta(\Lambda_2,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_2,\phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_2 - \frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_4)\varsigma\right] \\
& + \frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\left[\delta(\Lambda_2,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_2,\phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_2 - \frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_4)\varsigma\right] \\
& - \frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\left[\delta(\Lambda_2,\Lambda_4)\varsigma + \frac{1}{2}\delta(\Lambda_2,\phi\Lambda_4)\varsigma - \frac{3}{4}\eta(\Lambda_4)\Lambda_2 - \frac{1}{4}\eta(\Lambda_2)\eta(\Lambda_4)\varsigma\right]. \qquad (7.75)
\end{aligned}
$$

Setting $V = \varsigma$ in (7.75) we get

$$
\begin{aligned}
0 \;=\; & -\frac{3}{4}\delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2)\varsigma + \frac{3}{4}\delta(\Lambda_1,\Lambda_2)\eta(\Lambda_3)\varsigma + \frac{3}{4}\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_1)\varsigma \\
& - \frac{3}{8}\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_2)\varsigma + \frac{3}{8}\delta(\Lambda_1,\phi\Lambda_2)\eta(\Lambda_3)\varsigma - \frac{3}{4}\delta(\Lambda_2,\phi\Lambda_3)\Lambda_1 \\
& + \frac{3}{4}\left[\delta(\Lambda_1,\Lambda_2) + \frac{1}{2}\delta(\Lambda_1,\phi\Lambda_2) - \frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_2)\right]\left[-\eta(\Lambda_3)\varsigma + \Lambda_3\right] \\
& - \frac{3}{4}\left[\frac{3}{4}\left\{\eta(\Lambda_1)\Lambda_3 - \eta(\Lambda_3)\Lambda_1\right\} - \delta(\Lambda_1,\phi\Lambda_3)\varsigma\right]\eta(\Lambda_2) \\
& + \frac{3}{4}\left[\frac{3}{4}\left\{\eta(\Lambda_2)\Lambda_3 - \eta(\Lambda_3)\Lambda_2\right\} - \delta(\Lambda_2,\phi\Lambda_3)\varsigma\right]\eta(\Lambda_1) - \frac{3}{4}\overline{R}(\Lambda_2,\Lambda_3)\Lambda_1 \\
& + \frac{3}{4}\left[\delta(\Lambda_1,\Lambda_3) + \frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3) - \frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\right]\left[-\eta(\Lambda_2)\varsigma + \Lambda_2\right] \\
& - \frac{3}{4}\left[\frac{3}{4}\left\{\eta(\Lambda_2)\Lambda_1 - \eta(\Lambda_1)\Lambda_2\right\} - \delta(\Lambda_2,\phi\Lambda_1)\varsigma\right]\eta(\Lambda_3). \qquad (7.76)
\end{aligned}
$$

Taking inner product of (7.76) with a vector field Λ_5 we get

$$\begin{aligned} 0 \;=\; & -\frac{3}{4}\delta(\Lambda_1,\Lambda_3)\eta(\Lambda_2)\eta(\Lambda_5)+\frac{3}{4}\delta(\Lambda_1,\Lambda_2)\eta(\Lambda_3)\eta(\Lambda_5)+\frac{3}{4}\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_1)\eta(\Lambda_5)\\ & -\frac{3}{8}\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_2)\eta(\Lambda_5)+\frac{3}{8}\delta(\Lambda_1,\phi\Lambda_2)\eta(\Lambda_3)\eta(\Lambda_5)-\frac{3}{4}\delta(\Lambda_2,\phi\Lambda_3)\delta(\Lambda_1,\Lambda_5)\\ & +\frac{3}{4}\left[\delta(\Lambda_1,\Lambda_2)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_2)-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_2)\right][-\eta(\Lambda_3)\eta(\Lambda_5)+\delta(\Lambda_3,\Lambda_5)]\\ & -\frac{3}{4}\left[\frac{3}{4}\{\eta(\Lambda_1)\delta(\Lambda_3,\Lambda_5)-\eta(\Lambda_3)\delta(\Lambda_1,\Lambda_5)\}-\delta(\Lambda_1,\phi\Lambda_3)\eta(\Lambda_5)\right]\eta(\Lambda_2)\\ & +\frac{3}{4}\left[\frac{3}{4}\{\eta(\Lambda_2)\delta(\Lambda_3,\Lambda_5)-\eta(\Lambda_3)\delta(\Lambda_2,\Lambda_5)\}-\delta(\Lambda_2,\phi\Lambda_3)\eta(\Lambda_5)\right]\eta(\Lambda_1)\\ & +\frac{3}{4}\left[\delta(\Lambda_1,\Lambda_3)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)-\frac{1}{4}\eta(\Lambda_1)\eta(\Lambda_3)\right][-\eta(\Lambda_2)\eta(\Lambda_5)+\delta(\Lambda_2,\Lambda_5)]\\ & -\frac{3}{4}\left[\frac{3}{4}\{\eta(\Lambda_2)\delta(\Lambda_1,\Lambda_5)-\eta(\Lambda_1)\delta(\Lambda_2,\Lambda_5)\}-\delta(\Lambda_2,\phi\Lambda_1)\eta(\Lambda_5)\right]\eta(\Lambda_3)\\ & -\frac{3}{4}\delta(\overline{R}(\Lambda_2,\Lambda_3)\Lambda_1,\Lambda_5). \end{aligned} \tag{7.77}$$

Contracting (7.77) over Λ_2 and Λ_5 we obtain

$$\overline{S}(\Lambda_1,\Lambda_3)=-(n-1)\left[\delta(\Lambda_1,\Lambda_3)+\frac{1}{2}\eta(\Lambda_1)\eta(\Lambda_3)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\right]. \tag{7.78}$$

Using (4.61) in (7.78) we get

$$\begin{aligned} 0 \;=\; & \frac{1}{2}(2\lambda-\overline{r}+1)\delta(\Lambda_1,\Lambda_2)+\frac{1}{2}\left(2\beta-1\right)\eta(\Lambda_1)\eta(\Lambda_2)\\ & -(n-1)\left[\delta(\Lambda_1,\Lambda_3)+\frac{1}{2}\eta(\Lambda_1)\eta(\Lambda_3)+\frac{1}{2}\delta(\Lambda_1,\phi\Lambda_3)\right]. \end{aligned} \tag{7.79}$$

Setting $\Lambda_2=\varsigma$ in (7.79) we have

$$2(\lambda+\beta)=r+\frac{13}{4}(n-1)-\frac{1}{2}(n-3)\psi, \tag{7.80}$$

where $trace\,(\phi)=\psi$.

Theorem 7.1. *Let a para-Sasakian manifold M admits η-Einstein soliton $(\delta,\varsigma,\lambda,\beta)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma,\Lambda_1).\overline{R}=0$, then the relation between the soliton constants are given by equation (7.80).*

Setting $\beta=0$ in (7.80) we get

$$\lambda=\frac{1}{2}r+\frac{13}{8}(n-1)-\frac{1}{4}(n-3)\psi.$$

Corollary 7.1. *Let a para-Sasakian manifold M contain an Einstein soliton $(\delta, \varsigma, \lambda)$ with respect to SSNM-connection. If M satisfies the equation $\overline{R}(\varsigma, \Lambda_1).\overline{R} = 0$, then the soliton is shrinking, steady and expanding if*

$$r < -\frac{13}{4}(n-1), r = -\frac{13}{4}(n-1), r > -\frac{13}{4}(n-1),$$

respectively, provided $trace(\phi) = 0$.

8. Example of para-Sasakian manifold admitting SSNM-connection

Let us consider 3-dimensional manifold

$$M^3 = \left\{(x, y, z) \in R^3\right\},$$

where (x, y, z) are the standard co-ordinates in R^3. We choose the linearly independent vector fields

$$E_1 = e^x \frac{\partial}{\partial y}, E_2 = e^x \left(\frac{\partial}{\partial y} - \frac{\partial}{\partial z}\right), E_3 = -\frac{\partial}{\partial x}.$$

Let g be the pseudo Riemannian metric defined by $g(E_i, E_j) = 0$, $if\ i \neq j\ for\ i, j = 1, 2, 3,$ and $g(E_1, E_1) = -1, g(E_2, E_2) = -1, g(E_3, E_3) = 1$

Let η be the 1-form defined by $\eta(X) = g(X, E_3)$ for any $X \in \chi\left(M^3\right)$. Let ϕ be the $(1,1)$ tensor field defined by

$$\phi E_1 = E_1, \phi E_2 = E_2, \phi E_3 = 0. \tag{8.81}$$

$$trace(\phi) = \sum_{i=1}^{3} g(E_i, \phi E_i) = -2 \tag{8.82}$$

Let X, Y, $Z \in \chi\left(M^3\right)$ be given by

$$\begin{aligned} X &= x_1E_1 + x_2E_2 + x_3E_3, \\ Y &= y_1E_1 + y_2E_2 + y_3E_3, \\ Z &= z_1E_1 + z_2E_2 + z_3E_3. \end{aligned}$$

Then, we have

$$\begin{aligned} g(X, Y) &= x_1y_1 + x_2y_2 + x_3y_3, \\ \eta(X) &= x_3, \\ g(\phi X, \phi Y) &= x_1y_1 + x_2y_2. \end{aligned}$$

Using the linearity of g and ϕ, $\eta(E_3) = 1, \phi^2 X = X - \eta(X) E_3$ and $g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y)$ for all X, $Y \in \chi(M)$. We have

$$\begin{aligned}[E_1, E_2] &= 0, [E_1, E_3] = -E_1, [E_2, E_3] = E_2, \\ [E_2, E_1] &= 0, [E_3, E_1] = E_1, [E_3, E_2] = -E_2.\end{aligned}$$

Let the Levi-Civita connection with respect to g be ∇, then using Koszul formula we get the following

$$\begin{pmatrix} \nabla_{E_1} E_1 & \nabla_{E_1} E_2 & \nabla_{E_1} E_3 \\ \nabla_{E_2} E_1 & \nabla_{E_2} E_2 & \nabla_{E_2} E_3 \\ \nabla_{E_3} E_1 & \nabla_{E_3} E_2 & \nabla_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

From the above results we see that the structure (ϕ, ξ, η, g) satisfies

$$(\nabla_X \phi) Y = -g(X,Y)\xi + \eta(Y) X,$$

for all X, $Y \in \chi(M^3)$, where $\eta(\xi) = \eta(E_3) = 1$. Hence $M^3(\phi, \xi, \eta, g)$ is a para-Sasakian manifold.

The components of Riemannian curvature tensor of M^3 are given by

$$\begin{pmatrix} R(E_1,E_2)E_2 & R(E_1,E_3)E_3 & R(E_1,E_2)E_3 \\ R(E_2,E_1)E_1 & R(E_2,E_3)E_3 & R(E_2,E_3)E_1 \\ R(E_3,E_1)E_1 & R(E_3,E_2)E_2 & R(E_3,E_1)E_2 \end{pmatrix} = \begin{pmatrix} -E_1 & -E_1 & 0 \\ E_2 & E_2 & 0 \\ E_3 & E_3 & 0 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 are given by

$$S(E_1,E_1) = S(E_3,E_3) = 0, S(E_2,E_2) = 2. \tag{8.83}$$

Therefore the scalar curvature of M^3 is

$$r = \sum_{i=1}^{3} S(E_i, E_i) = 2. \tag{8.84}$$

Using (3.23) we have the following values of $\overline{\nabla}$:

$$\begin{pmatrix} \overline{\nabla}_{E_1} E_1 & \overline{\nabla}_{E_1} E_2 & \overline{\nabla}_{E_1} E_3 \\ \overline{\nabla}_{E_2} E_1 & \overline{\nabla}_{E_2} E_2 & \overline{\nabla}_{E_2} E_3 \\ \overline{\nabla}_{E_3} E_1 & \overline{\nabla}_{E_3} E_2 & \overline{\nabla}_{E_3} E_3 \end{pmatrix} = \begin{pmatrix} -E_3 & 0 & -\frac{1}{2}E_1 \\ 0 & E_3 & -\frac{1}{2}E_2 \\ \frac{1}{2}E_1 & \frac{1}{2}E_2 & 0 \end{pmatrix}.$$

By the help of (3.41) and above matrix we get the components of Riemannian curvature tensor of M^3 with respect to SSNM-connection as follows

$$\begin{pmatrix} \overline{R}(E_1,E_2)E_1 & \overline{R}(E_1,E_3)E_1 & \overline{R}(E_2,E_3)E_1 \\ \overline{R}(E_1,E_2)E_2 & \overline{R}(E_1,E_3)E_2 & \overline{R}(E_2,E_3)E_2 \\ \overline{R}(E_1,E_2)E_3 & \overline{R}(E_1,E_3)E_3 & \overline{R}(E_2,E_3)E_3 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}E_2 & -\frac{3}{2}E_3 & 0 \\ -\frac{1}{2}E_1 & 0 & -\frac{1}{2}E_3 \\ 0 & -\frac{1}{4}E_1 & \frac{1}{4}E_2 \end{pmatrix}.$$

The components of Ricci curvature tensor of M^3 with respect to SSNM-connection are given by

$$\overline{S}(E_1, E_1) = \overline{S}(E_2, E_2) = 1, \overline{S}(E_3, E_3) = \frac{1}{2}. \tag{8.85}$$

Therefore the scalar curvature of M^3 with respect to SSNM-connection is

$$\overline{r} = \sum_{i=1}^{3} S(E_i, E_i) = \frac{5}{2}. \tag{8.86}$$

In view of (8.82), (8.84) and (8.86) we have

$$\begin{aligned} \overline{r} &= \frac{5}{2} \\ &= 2 + \frac{1}{4}(3-1) - \frac{1}{2}(3-3).(-2) \\ &= r + \frac{1}{4}(n-1) - \frac{1}{2}(n-3)\psi, \end{aligned}$$

which verifies the relation (3.53). Similarly, we can verify all the results obtained.

9. Conclusion

From the results obtained in this paper we can conclude that if a para-Sasakian manifold $M(\phi, \varsigma, \eta, \delta)$ admits η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to semi-symmetric nonmetric connection, then M is generalized η-Einstein manifold. We also conclude that if a para-Sasakian manifold M admitting η-Einstein soliton $(\delta, \varsigma, \lambda, \beta)$ with respect to semi-symmetric nonmetric connection satisfies $\overline{R}.\overline{S} = 0, \overline{S}.\overline{R} = 0$ and $\overline{R}.\overline{R} = 0$, then the soliton constants depend on scalar curvature of M and trace of the function ϕ on M

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References

[1] Adati, A., & Matsumoto, K. (1997). On conformally recurrent and conformally symmetric P-Sasakian manifolds. TRU Math., 13, 25–32.

[2] Adati, A., & Miyazawa, T. (1979). On P-Sasakian manifolds satisfying certain conditions. Tensor, N.S., 33, 173–178.

[3] Agashe, N. S., & Chafle, M. R. (1992). A semi-symmetric non-metric connection on a Riemannian manifold. Indian J. Pure Appl. Math., 23, 399–409.

[4] Amur, K., & Pujar, S. S. (1978). On submanifolds of a Riemannian manifold admitting a metric semi-symmetric connection. Tensor, N.S., 32, 35–38.

[5] Binh, T. Q. (1990). On semi-symmetric connections. Periodic Math. Hungarica, 21, 101–107.

[6] Blaga, A. M. (2018). On gradient η-Einstein solitons. Kragujev. J. Math., 42(2), 229–237.

[7] Catino, G., & Mazzieri, L. (2016). Gradient Einstein solitons. Nonlinear Anal., 132, 66–94.

[8] Cho, J. T., & Kimura, M. (2009). Ricci solitons and real hypersurfaces in a complex space form. Tohoku Math. J., 61(2), 205–212.

[9] Chaubey, S. K., & Ojha, R. H. (2012). On semi-symmetric non-metric connection on a Riemannian manifold. Filomat, 26, 63–69.

[10] Chaybey, S. K., & Yieldiz, A. (2019). Riemannian manifold admitting a new type of semi-symmetric non-metric connection. Turk. J. Math., 43, 1887–1904.

[11] De, U. C. (1990). On a type of semi-symmetric metric connection on a Riemannian manifold. Indian J. Pure Appl. Math., 21, 334–338.

[12] De, U. C., & Kamilya, D. (1995). Hypersurfaces of a Riemannian manifold with semi-symmetric non-metric connection. J. Indian Inst. Sci., 75, 707–710.

[13] De, U. C., Han, Y. L., & Zhao, P. B. (2006). A special type of semi-symmetric non-metric connection on a Riemannian manifold. Facta Univ. (NIS), 31(2), 529–541.

[14] Hayden, H. A. (1932). Subspaces of space with torsion. Proc. London Math. Soc., 34, 27–50.

[15] Hamilton, R. S. (1988). The Ricci flow on surfaces. In Math. and General Relativity (American Math. Soc. Contemp. Math., 7(1), 232–262).

[16] Mandal, K., & De, U. C. (2015). Quarter symmetric metric connection in a P-Sasakian manifold. Annals of West University of Timisoara-Mathematics and Comp. Sci., 53(1), 137–150.

[17] Matsumoto, K., Ianus, S., & Mihai, I. (1986). On P-Sasakian manifolds which admit certain tensor-fields. Publicationes Mathematicae-Debrecen, 33, 199–204.

[18] Nagaraja, H. G., & Premalatha, C. R. (2012). Ricci solitons in Kenmotsu manifolds. J. of Mathematical Analysis, 3(2), 18–24.

[19] Ozgur, C. (2005). On a class of para-Sasakian manifolds. Turkish Journal of Mathematics, 29(3), 249–258.

[20] Ozgur, C., & Sular, S. (2011). Wrapped product manifolds with semi-symmetric metric connections. Taiwan J. Math., 15, 1701–1719.

[21] Ozgur, C., & Tripathi, M. M. (2007). On P-Sasakian manifolds satisfying certain conditions on concircular curvature tensor. Turkish Journal of Mathematics, 31(3), 171–179.

[22] Prasad, B., & Singh, S. C. (2006). Some properties of semi-symmetric non-metric connection in a Riemannian manifold. Jour. Pure Math., 23, 121–134.

[23] Prasad, B., & Verma, R. K. (2004). On a type of semi-symmetric non-metric connection on a Riemannian manifold. Bull. Call. Math. Soc., 96(6), 483–488.

[24] Reddy, V. V., Sharma, R., & Sivaramkrishan, S. (2007). Space times through Hawking-Ellis construction with a background Riemannian metric. Class. Quant. Grav., 24, 3339–3345.

[25] Sasaki, S., & Hatakeyama, Y. (1961). On differentiable manifolds with certain structures which are closely related to almost contact structures II. Tohoku Math. J., 13, 281–294.

[26] Sato, I., & Matsumoto, K. (1979). On P-Sasakian manifolds satisfying certain conditions. Tensor N. S., 33, 173–178.

[27] Singh, R. N., & Pandey, M. K. (2008). On a type of semi-symmetric non-metric connection on a Riemannian manifold. Bull. Call. Math. Soc., 96(6), 179–184.

[28] Shukla, S. S., & Shukla, M. K. (2010). On ϕ-symmetric para-Sasakian manifolds. Int. J. Math. Analysis, 16(4), 761–769.

[29] Sharma, R. (2008). Certain results on K-contact and (k,μ)-contact manifolds. J. of Geometry, 89, 138–147.

[30] Tripathi, M. M. (2008). Ricci solitons in contact metric manifold. ArXiv: 0801.4222 [math.DG].

[31] Yano, K. (1970). On a semi-symmetric metric connection. Rev. Roum. Math. Pures Appl., 15, 1570–1586.

[32] Zamkovoy, S. (2009). Canonical connection on paracontact manifolds. Ann. Global Anal. Geom., 36, 37–60.

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ON HYPERCYCLICITY OF WEIGHTED COMPOSITION OPERATORS ON STEIN MANIFOLDS

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Abstract. In this manuscript, we study the hypercyclicity of weighted composition operators defined on the set of holomorphic complex functions on a connected Stein n-manifold $\mathbf{M}$. We show that a weighted composition operator $\mathbf{C}_{\psi,\omega}$ (associated to a holomorphic self-map ψ and a holomorphic function ω on $\mathbf{M}$) is hypercyclic with respect to an increasing sequence $(n_l)_l$ of natural numbers if and only if at every $p \in \mathbf{M}$ we have $\omega(p) \neq 0$ and the self-map ψ is injective without any fixed points in $\mathbf{M}$, $\psi(\mathbf{M})$ is a Runge domain and for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$ there is a positive integer number k such that the sets C and $\psi^{[n_k]}(C)$ are separable in $\mathbf{M}$.

Keywords: Holomorphic, Composition operators, Hypercyclic, Convex.

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1. Introduction

Let U be a domain in the complex plane $\mathbb{C}$, and $\mathbb{H}(\mathrm{U})$ be the space of holomorphic complex functions in U. The space $\mathbb{H}(\mathrm{U})$ is endowed with the topology of locally uniform convergence, under which it becomes a complete separable metric space. We are interested in proving the existence of dense orbits for composition operators on $\mathbb{H}(\mathrm{U})$. If ψ is a holomorphic self-map on U, then the composition operator associated to ψ is defined as $\mathbf{C}_{\psi}(f) = f \circ \psi$ for every $f \in \mathbb{H}(\mathrm{U})$.

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The first step has been taken in 1929 by Birkhoff ([7]) when he proved that there exists an entire function $\lambda : \mathbb{C} \mapsto \mathbb{C}$ such that $\{\lambda \circ t_n\}_{n\in\mathbf{N}}$ forms a dense set in $\mathbb{H}(\mathbb{C})$, where $\{t_n\}_{n=1}^{\infty}$ is the sequence of $\mathbb{C}$-automorphisms defined by $t_n : z \mapsto z + n$. The function λ is called universal.

Gethner and Shapiro have studied universal vectors for operators on spaces of holomorphic functions in 1987 ([13]). In 90s, the subject of cyclic composition operators has been discussed by many researchers ([8, 9, 10, 14]). In the same decade, some generalizations to hypercyclic operators have also been studied ([15, 20, 21]).

In 2001, Shapiro studied the dynamics of linear operators ([22]) which followed by Grose-Erdman in 2003 ([16]). As a concrete example, Bernal-Gonzales has studied the universal entire functions for affine endomorphisms on $\mathbb{C}^n$ in 2005.

A class of linear fractional maps of the ball and its composition operators has been considered by Bayart in 2007 ([5]). One can find the continuation of research progress on the hypercyclicity of operators in the references [6, 11, 18, 24]. Between them, the manuscript [24] has a special importance because it discuss on the hypercyclicity of composition operators associated to some holomorphic self-maps defined on an important class of complex manifolds namely Stein manifolds. The important properties of Stein manifolds can be found in [24].

The weighted composition operators associated to some holomorphic self-maps have been interested in some recent researches (see for instance [1, 2, 3, 4, 23]). Also, in [19], the authors have studied the dynamics of weighted composition operators on Stein manifolds, where the maps and functions are defined on a Stein manifold.

In this paper, we consider a holomorphic self-map $\psi \in \mathcal{O}(\mathbf{M})$ defined on a connected Stein n-manifold $\mathbf{M}$ and a holomorphic function $\omega \in \mathbb{H}(\mathbf{M})$. We study the hypercyclicity of weighted composition operator $\mathbf{C}_{\psi,\omega} : \mathbb{H}(\mathbf{M}) \to \mathbb{H}(\mathbf{M})$ defined by rule $\mathbf{C}_{\psi,\omega}(f) := \omega \cdot (f \circ \psi)$ with respect to an increasing sequence of natural numbers.

We prove that $\mathbf{C}_{\psi,\omega}$ is hypercyclic if and only if for every $p \in \mathbf{M}$, $\omega(p) \neq 0$ and ψ is univalent without fixed points in $\mathbf{M}$, $\psi(\mathbf{M})$ is a Runge domain and for every compact holomorphically convex set $C \subset \mathbf{M}$ there is an integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and their sum is $\mathbf{M}$-convex.

In the study of hypercyclicity of $\mathbf{C}_{\psi,\omega}$, which is connected with some approximation theorems, one can use two well-known theorems namely the Runge Theorem and Oka-Weil Theorem.

2. Preliminaries

In this section, we present the preliminary concepts and notations from [1, 2, 5, 6, 17, 24]. We denote the family of all open subsets of a given topological space X by $\mathbf{Op}(X)$ and the family of all compact subsets of X by $\mathbf{Cp}(X)$. As usual, $\mathcal{C}(X,Y)$ denotes the set of all continuous maps between two topological spaces X and Y.

Definition 2.1. For every $C \in \mathbf{Cp}(X)$ and $U \in \mathbf{Op}(Y)$, the set of functions $f \in \mathcal{C}(X,Y)$ satisfying condition $f(C) \subset U$ is denoted by $\mathcal{V}(C,U)$. The topology generated by subbase

$$\triangle := \{\mathcal{V}(C,U) | C \in \mathbf{Cp}(X), U \in \mathbf{Op}(Y)\}$$

is called the *compact-open topology* on $\mathcal{C}(X,Y)$.

We note that $\triangle$ does not always form a base for a topology on $\mathcal{C}(X,Y)$. The compact-open topology (which is applied in homotopy theory and functional analysis) was introduced by Ralph Fox in 1945 [12].

A continuous map $f \in \mathcal{C}(X,Y)$ is said to be *proper* if each connected component of $f^{-1}(K)$ is compact for every $K \in \mathbf{Cp}(Y)$.

Definition 2.2. Let X be a topological vector space and $\{\alpha_r : X \to X\}_{r=1}^{\infty}$ be a sequence of continuous self-maps on X.

(1) $\{\alpha_r\}_{r=1}^{\infty}$ is called *topologically transitive* if for every non-empty $U, V \in \mathbf{Op}(X)$ there exists r_0 such that $\alpha_{r_0}(U) \cap V \neq \emptyset$.
(2) A point $p \in X$ is said to be an *universal element* for $\{\alpha_r\}_{r=1}^{\infty}$ if the sequence $\{\alpha_r(p)\}_{r=1}^{\infty}$ of points is dense in X.
(3) A point $p \in X$ is said to be an *weakly universal element* for $\{\alpha_r\}_{r=1}^{\infty}$ if the sequence $\{\alpha_r(p)\}_{r=1}^{\infty}$ of points is dense in X with respect to the weak topology of X.
(4) The sequence $\{\alpha_r\}_{r=1}^{\infty}$ is said to be *universal* if it admits a universal element.
(5) The sequence $\{\alpha_r\}_{r=1}^{\infty}$ is said to be *weakly universal* if it admits a weakly universal element.

Definition 2.3. Let X be a topological vector space and $\alpha : X \to X$ be a continuous self-map on X.

(1) The iterations of α is defined by $\alpha^{[1]} = \alpha$, $\alpha^{[2]} = \alpha \circ \alpha$ and $\alpha^{[r+1]} = \alpha \circ \alpha^{[r]}$ for integer number $r \geq 2$.

(2) We say that α is *hypercyclic with respect to an increasing sequence* $\{r_k\}_{k=1}^{\infty} \subset \mathbb{N}$ if the sequence $\{\alpha^{[r_k]}\}_{k=1}^{\infty}$ is universal.

(3) We say that α is *weakly hypercyclic with respect to an increasing sequence* $\{r_k\}_{k=1}^{\infty} \subset \mathbb{N}$ if the sequence $\{\alpha^{[r_k]}\}_{k=1}^{\infty}$ is weakly universal.

(4) α is called *hypercyclic* if it is hypercyclic with respect to the full sequence $\{r\}_{r=1}^{\infty}$.

(5) α is called *weakly hypercyclic* if it is hypercyclic with respect to the full sequence $\{r\}_{r=1}^{\infty}$.

Here, we recall an essential theorem from [15] which gives a necessary and sufficient condition for topological transitivity of a sequence of continuous linear maps on a separable Fréchet space using the set of its universal elements. Remember that, a Fréchet space is a complete locally convex metrizable topological vector space.

Theorem 2.1. Let $\mathbf{F}$ be separable Fréchet space and $\{\alpha_r\}_{r=1}^{\infty}$ be a sequence of continuous self-maps on $\mathbf{F}$. This sequence is topologically transitive if and only if the set of its universal elements is dense in $\mathbf{F}$. Moreover, in this case the set of universal elements for $\{\alpha_r\}_{r=1}^{\infty}$ is a dense G_δ-subset of $\mathbf{F}$.

Also, we recall another useful theorem from [15] in this context.

Theorem 2.2. Let $\mathbf{F}$ be separable Fréchet space and $\{\alpha_r\}_{r=1}^{\infty}$ be a sequence of continuous self-maps on $\mathbf{F}$. If α_r has dense range in $\mathbf{F}$ for each $r \in \mathbb{N}$ and the sequence $\{\alpha_r\}_{r=1}^{\infty}$ is commuting (i.e. for every $r, s \in \mathbb{N}$, we have $\alpha_r \circ \alpha_s = \alpha_s \circ \alpha_r$), then the set of universal elements of $\{\alpha_r\}_{r=1}^{\infty}$ is empty or dense in $\mathbf{F}$.

The hypercyclicity of a bounded linear map α on a Fréchet space $\mathbf{F}$ means that for a vector $\mathbf{v} \in \mathbf{F}$, its orbit (i.e. $Orb(\alpha, \mathbf{v}) = \{\alpha^{[r]}(\mathbf{v})\}_{r=1}^{\infty}$) is dense in $\mathbf{F}$. By these theorems we get a corollary that allows us to investigate topological transitivity instead of hypercyclicity. Also, Theorem 3 in [15] has a similar argument.

Corollary 2.1. Let X be a separable Fréchet space, let $\alpha : X \to X$ be a continuous map, and let $\{r_k\}_{k=1}^{\infty} \subset \mathbb{N}$ be an increasing sequence. Then, α is hypercyclic w.r.t. $\{r_k\}_{k=1}^{\infty}$ if and only if the sequence $\{\alpha^{[r_k]}\}_{k=1}^{\infty}$ is topologically transitive.

Now, we introduce the Stein manifold which plays main role in this paper.

Definition 2.4. A complex manifold $\mathbf{M}$ of (finite) dimension n is called a *Stein manifold*, if it satisfies the following four conditions:

(1) $\mathbf{M}$ admits a *compact exhaustion*, which means that, there is a sequence $(C_r)_{r=1}^{\infty}$ of compact subsets of $\mathbf{M}$ such that $\mathbf{M} = \bigcup_{r=l}^{\infty} C_r$ and for each r, $C_r \subset (C_{r+1})^0$.

(2) $\widehat{C}_{\mathbf{M}} \in \mathbf{Cp}(\mathbf{M})$ for every $C \in \mathbf{Cp}(\mathbf{M})$, where

$$\widehat{C}_{\mathbf{M}} := \{p \in \mathbf{M} : |f(p)| \leq \sup_{C} |f|, \forall f \in \mathcal{O}(\mathbf{M})\}$$

is the holomorphic hull of C.

(3) $\mathbb{H}(\mathbf{M})$ separates points in $\mathbf{M}$, i.e. for each two distinct points $p, q \in \mathbf{M}$, there exists $f \in \mathbb{H}(\mathbf{M})$ with $f(p) \neq f(q)$,

(4) For each $p \in \mathbf{M}$ there exists a map $F \in \mathcal{O}(\mathbf{M}, \mathbb{C}^n)$ such that the derivative of F at p is an isomorphism.

Definition 2.5. Let $\mathbf{M}$ be a Stein n-manifold.

(1) A $C \in \mathbf{Cp}(\mathbf{M})$ is said to be $\mathbf{M}$-convex (equivalently, holomorphically convex) if $\widehat{C}_{\mathbf{M}} = C$.

(2) In special case $\mathbf{M} = \mathbb{C}^n$, $\widehat{C}_{\mathbf{M}}$ is denoted with shorter symbol $\widehat{C}$ and is called the polynomial hull of C.

(3) A $C \in \mathbf{Cp}(\mathbb{C}^n)$ is called polynomially convex if $C = \widehat{C}$.

For two finite-dimensional complex manifolds $\mathbf{M}$, $\mathbf{N}$, the notation $\mathcal{O}(\mathbf{M}, \mathbf{N})$ denotes the set of all holomorphic maps $\phi : \mathbf{M} \to \mathbf{N}$. In special cases, we use simple notations $\mathcal{O}(\mathbf{M}) := \mathcal{O}(\mathbf{M}, \mathbf{M})$ and $\mathbb{H}(\mathbf{M}) := \mathcal{O}(\mathbf{M}, \mathbb{C})$. A holomorphic function on an open subset of the complex plane is called univalent if it is injective.

Definition 2.6.

(1) We say that a sequence of holomorphic maps $\{\phi_k \in \mathcal{O}(\mathbf{M}, \mathbf{N})\}_{k=1}^{\infty}$ is *compactly divergent* (in $\mathcal{O}(\mathbf{M}, \mathbf{N})$) if for each $C \in \mathbf{Cp}(\mathbf{M})$ and $K \in \mathbf{Cp}(\mathbf{N})$ there is k_0 such that $\phi_k(C) \cap K = \emptyset$ for all $k \geqslant k_0$.

(2) The sequence $\{\phi_k \in \mathcal{O}(\mathbf{M}, \mathbf{N})\}_{k=1}^{\infty}$ is said to be *run-away* (in $\mathcal{O}(\mathbf{M}, \mathbf{N})$) if for each $C \in \mathbf{Cp}(\mathbf{M})$ and $K \in \mathbf{Cp}(\mathbf{N})$, there is k_0 such that $\phi_{k_0}(C) \cap K = \emptyset$. In the case $\mathbf{M} = \mathbf{N}$, it is always enough to consider the situation when $C = K$.

When $\mathbf{M}$ and $\mathbf{N}$ admit compact exhaustions, the sequence $\{\phi_k\}_{k=1}^{\infty}$ is run-away if and only if it has a compactly divergent subsequence.

A holomorphic map $f \in \mathcal{O}(\mathbf{M}, \mathbf{N})$ between to complex manifold is called *regular* if its derivative is a monomorphism at each point of $\mathbf{M}$.

A *Runge domain* in a Stein Manifold $\mathbf{M}$ is a domain $U \subset \mathbf{M}$ such that every function $f \in \mathbb{H}(U)$ can be approximated uniformly on U by a sequence of members of $\mathbb{H}(\mathbf{M})$. By the well-known Oka-Weil theorem, on every compact $\mathbf{M}$-convex subset $C \subset \mathbf{M}$, every holomorphic function (i.e. holomorphic on a neighborhood of C) can be approximated uniformly by functions from $\mathbb{H}(\mathbf{M})$.

Remark 2.1. By condition (1) of Definition 2.4, a Stein manifold $\mathbf{M}$ has a compact exhaustion $\{C_k\}_{k=1}^{\infty}$ such that $\bigcup_{k=l}^{\infty} C_k = \mathbf{M}$ and for each k, $C_k \subset (C_{k+1})^0$. So, we can take a sequence of semi-norms $\{p_k : \mathbb{H}(\mathbf{M}) \to \mathbb{R}\}_{k=1}^{\infty}$ defined by $p_k(f) := \sup\{|f(p)| p \in C_k\}$, which gives the topology of $\mathbb{H}(\mathbf{M})$. So, $\mathbb{H}(\mathbf{M})$ with this topology is a separable Fréchet space (see [23, 24]). This observation allows us to use Corollary 2.1 for the space $X = \mathbb{H}(\mathbf{M})$, with $\mathbf{M}$ being a connected Stein manifold.

Remark 2.2. By theorem from [24], a domain U in a connected Stein manifold $\mathbf{M}$ is a Runge domain if and only if every compact subset $C \subset U$ satisfies $\widehat{C}_{\mathbf{M}} = \widehat{C}_U$. Also, that condition is equivalent to equality $\widehat{C}_{\mathbf{M}} \cap U = \widehat{C}_U$ for every compact subset $C \subset U$.

For every locally compact topological space X, the usual compactification with one point $\infty_X \notin X$ is denoted by $X_c = X \cup \{\infty_X\}$.

It is clear that, if a continuous self-map α defined on a topological vector space X is hypercyclic, then any universal element of $\{\alpha^{[r]}\}_{r=1}^{\infty}$ is a hypercyclic vector. Finally, we have a useful lemma which guarantees that the adjoint operator of a weakly hypercyclic operator on a topological vector space dose not have any eigenvector.

Lemma 2.1. The adjoint operator of a weakly hypercyclic operator on a topological vector space does not have any eigenvector.

Proof. Let α be a weakly hypercyclic linear self-map on a topological vector space X. Clearly, α is 1-weakly. Hence, α^* does not have any eigenvectors by Proposition 3.2 in [11]. □

The following well-known theorems ([24]) characterizes the Runge domains in a Stein manifold $\mathbf{M}$ in the language of holomorphic hulls.

Theorem 2.3. Let U be a Stein manifold which is a domain of a connected Stein manifold $\mathbf{M}$. Then, the following conditions are equivalent:

(1) The domain U is a Runge domain in $\mathbf{M}$.
(2) $\widehat{C}_{\mathbf{M}} = \widehat{C}_U$ for every compact subset $C \subset U$.

(3) $\widehat{C}_{\mathbf{M}} \cap U = \widehat{C}_U$ for every compact subset $C \subset U$.

Theorem 2.4. Let C and D be two compact subsets of a connected Stein manifold $\mathbf{M}$. Then the following conditions are equivalent:

(1) C and D are separable in $\mathbf{M}$.

(2) There exist open and disjoint subsets $U, V \subset \mathbf{M}$ such that $\widehat{C}_{\mathbf{M}} \subset U$, $\widehat{D}_{\mathbf{M}} \subset V$ and $(\widehat{C \cup D})_{\mathbf{M}} \subset U \cup V$.

(3) $\widehat{C}_{\mathbf{M}} \cap \widehat{D}_{\mathbf{M}} = \emptyset$ and $(\widehat{C \cup D})_{\mathbf{M}} = \widehat{C}_{\mathbf{M}} \cup \widehat{D}_{\mathbf{M}}$.

In particular, if C and D are disjoint and $\mathbf{M}$-convex, then $C \cup D$ is $\mathbf{M}$-convex if and only if C and D are separable in $\mathbf{M}$.

Corollary 2.2. Let C and D be two disjoint compact subsets of a connected Stein manifold $\mathbf{M}$ such that $C \cup D$ is $\mathbf{M}$-convex. Then C and D are both $\mathbf{M}$-convex.

3. Main results

In this section, we choose a $\psi \in \mathcal{O}(\mathbf{M})$ and a weight function $\omega \in \mathbb{H}(\mathbf{M})$ on a connected Stein n-manifold $\mathbf{M}$. Some necessary conditions for hypercyclicity of the weighted composition operator $\mathbf{C}_{\psi,\omega}$ with respect to an increasing sequence of natural numbers $\{n_k\}_{k=1}^{\infty}$ are presented.

Proposition 3.1. Let $\{n_k\}_{k=1}^{\infty}$ be an increasing sequence of natural numbers, $\mathbf{M}$ be a connected Stein n-manifold, $\omega \in \mathbb{H}(\mathbf{M})$ and $\psi \in \mathcal{O}(\mathbf{M})$. If the weighted composition operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic with respect to $\{n_k\}_{k=1}^{\infty}$, then the following conditions hold:

(1) $\omega \neq 0$ on $\mathbf{M}$ and ψ has no fixed point in $\mathbf{M}$.

(2) ψ is injective.

(3) $\psi(\mathbf{M})$ is a Runge domain w.r.t. $\mathbf{M}$.

(4) The sequence $\{\psi^{[n_k]}\}_{k=1}^{\infty}$ is run-away.

Proof.

(1) Remember that $\mathbb{H}(\mathbf{M})$ is a separable Fréchet space and the point evaluation linear functional $\mathcal{E}_p : \mathbb{H}(\mathbf{M}) \to \mathbb{C}$ (at each point $p \in \mathbf{M}$) defined by $\mathcal{E}_p(h) := h(p)$ is continuous. The adjoint of $\mathbf{C}_{\psi,\omega}$ satisfies the following equality

$$\mathbf{C}^*_{\psi,\omega}(\mathcal{E}_p)(h) = \mathcal{E}_p \circ \mathbf{C}_{\psi,\omega}(h) = \mathcal{E}_p(\omega \cdot (h \circ \psi)) = \omega(p) \cdot (h \circ \psi)(p).$$

So, $\mathbf{C}^*_{\psi,\omega}$ has an eigenvalue if $\omega(p) = 0$ or $\psi(p) = p$ and then, in these two cases $\mathbf{C}_{\psi,\omega}$ can not be hypercyclic.

(2) Since $\mathbf{C}_{\psi,\omega}$ is hypercyclic with respect to $\{n_k\}_{k=1}^{\infty}$, it admits a hypercyclic vector $g \in \mathbb{H}(\mathbf{M})$. So, for each $h \in Orb(\mathbf{C}_{\psi,\omega}, g)$ there exists a positive integer k such that

$$h = (\mathbf{C}^{[n_k]}_{\psi,\omega}(g)) = \prod_{j=0}^{n_k-1} \mathbf{C}^{[j]}_{\psi}(\omega) \cdot \mathbf{C}^{[n_k]}_{\psi} g = \omega \cdot (\prod_{j=1}^{n_k-1} \omega \circ \psi^j) \cdot (g \circ \psi^{n_k}).$$

Assuming $\psi(p) = \psi(q)$ for two distinct points $p, q \in \mathbf{M}$, we get $\frac{1}{\omega(p)} h(p) = \frac{1}{\omega(q)} h(q)$ and then

$$\frac{1}{\omega(p)} \mathcal{E}_p(h) = \frac{1}{\omega(q)} \mathcal{E}_q(h) \tag{3.1}$$

for every $h \in Orb(\mathbf{C}_{\psi,\omega}, g)$. So, by continuity of $\frac{1}{\omega(p)}\mathcal{E}_p$ and $\frac{1}{\omega(q)}\mathcal{E}_q$, it follows that the equality (3.1) holds for every $h \in \overline{Orb(\mathbf{C}_{\psi,\omega}, g)} = \mathbb{H}(\mathbf{M})$. Therefore, $\frac{1}{\omega(p)}\mathcal{E}_p = \frac{1}{\omega(q)}\mathcal{E}_q$ on $\mathbb{H}(\mathbf{M})$.

Now, putting $g = 1$, we get $\frac{1}{\omega(p)}\mathcal{E}_p(1) = \frac{1}{\omega(q)}\mathcal{E}_q(1)$ which gives $\omega(p) = \omega(q)$. Therefore, the equality $h(p) = h(q)$ holds for every $g \in \mathbb{H}(\mathbf{M})$, which by condition (3) in Definition 2.4, implies that $p = q$. So, ψ is injective.

(3) It is enough to prove that the subset of restrictions $\{h|_{\psi(\mathbf{M})} : h \in \mathcal{O}(\mathbf{M})\}$ is dense in $\mathcal{O}(\psi(\mathbf{M}))$.

If $h \in \mathcal{O}(\psi(\mathbf{M}))$, then $h \circ \psi$ is holomorphic on $\mathbf{M}$, so there is a subsequence $\{n_{l_k}\}_{k=1}^{\infty}$ of $\{n_k\}_{k=1}^{\infty}$ such that $g \circ \psi^{[n_{l_k}]} \to g \circ \psi$ on $\mathbf{M}$ (where, $g \in \mathbb{H}(\mathbf{M})$ is a hypercyclic vector for $C_{\psi,\omega}$ with respect to $\{n_k\}_{k=1}^{\infty}$). Hence $f \circ \psi^{[n_{l_k}-1]} \to h$ on $\psi(\mathbf{M})$, as the mapping ψ is a biholomorphism on its image.

(4) Let $K \subset \mathbf{M}$ be compact. For each positive integer k, there exists a positive integer n_{l_k} such that $|f \circ \psi^{[n_{l_k}]} - k| \le \frac{1}{k}$ on K. So, for a big enough k, we have

$$\inf\{|f(z)| : z \in \psi^{[n_{l_k}]}(K)\} = \inf\{|(f \circ \psi^{[n_{l_k}]})(z)| : z \in K\} \ge k - \frac{1}{k} > \sup\{|f(z)| : z \in K\}.$$

Hence, $\psi^{[n_{l_k}]}(K) \cap K = \emptyset$.

□

Remark 3.1. It follows from the equivalence of conditions in Remark 2.2 and theorems 2.3 and 2.4 that ψ maps every $\mathbf{M}$-convex compact $C \subset \mathbf{M}$ onto an $\mathbf{M}$-convex compact set. Also, it implies that for any natural number n the set $\psi^{[n]}(C)$ is $\mathbf{M}$-convex.

It is natural to ask whether the necessary conditions given by Proposition 3.1 are sufficient. In [18], it is shown that if $\mathbf{M}$ is a simply connected or an infinitely connected planar domain or a special type of higher-dimensional Stein manifolds, then the mentioned property holds. But in general the above necessary conditions are not sufficient, as we can see using a simple example $\mathbf{M} = \mathbb{D}_*$ and $\psi(z) = \frac{1}{2}z$ then by Theorems 4.6 the operator C_ψ is not hypercyclic, although it satisfies the conditions (1), (2), (3).

Here, we prefer to re-describe the topology of $\mathcal{O}(\mathbf{M})$ and the concept of topologically transitivity of weighted composition operators.

For every $K \in \mathbf{Cp}(\mathbf{M})$ and $f_0 \in \mathbb{H}(\mathbf{M})$ and positive real number ϵ, the ϵ-neighborhood of f_0 is defined by

$$N_\epsilon^K(f_0) := \{f \in \mathbb{H}(\mathbf{M}) : \forall y \in K, |f(y) - f_0(y)| < \epsilon\}.$$

The family of all such a neighborhoods forms a basis of the topology of $\mathbb{H}(\mathbf{M})$.

With the aim of using Corollary 2.1, so let us first clear the topological transitivity of the sequence $(\mathbf{C}_{\psi,\omega}^{[n_l]})_l$.

Let $\psi \in \mathcal{O}(\mathbf{M})$ be an injective holomorphic self-map and $0 \neq \omega \in \mathbb{H}(\mathbf{M})$. The sequence $(\mathbf{C}_{\psi,\omega}^{[n_l]})_{l=1}^{\infty}$ is topologically transitive if and only if for every $\epsilon > 0$, $g, h \in \mathbb{H}(\mathbf{M})$ and $K \in \mathbf{Cp}(\mathbf{M})$ there are natural number k and function $f \in \mathbb{H}(\mathbf{M})$ such that $|f - g| < \epsilon$ and $|\mathbf{C}_{\psi,\omega}^{[n_k]}(f) - h| < \epsilon$ on K.

As the mapping ψ is injective and ω in non-zero, the above condition has another form:

$$|f - g| < \epsilon \texttt{ on } K \textit{ and } |f - [\prod_{j=0}^{k-1} \mathbf{C}_\psi^j(\omega)]^{-1} \cdot h \circ \psi^{[-n_k]}| < \epsilon \texttt{ on } \psi^{[n_k]}(K). \tag{3.2}$$

Since $\mathbf{M}$ is a Stein manifold, we can restrict to considering only $\mathbf{M}$-convex sets.

Theorem 3.1. Let $\mathbf{M}$ be a connected Stein manifold, $\psi \in \mathcal{O}(\mathbf{M})$, $\omega \in \mathbb{H}(\mathbf{M})$ and the weighted composition operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic on $\mathcal{O}(\mathbf{M})$. Then for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$, there exists positive integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and the set $C \cup \psi^{[n]}(C)$ is $\mathbf{M}$-convex.

Proof. Suppose that $\mathbf{C}_{\psi,\omega}$ is hypercyclic. In view of Corollary 2.1, the condition 3.2 holds. Fix an $\mathbf{M}$-convex compact set $C \subset \mathbf{M}$. By Remark 3.1 we get that the set $\psi^{[n]}(C)$ is $\mathbf{M}$-convex. Using the condition 3.2 for $g = 0$, $h = 1$ and $\epsilon = \frac{1}{2}$, we get that there are $f \in \mathcal{O}(\mathbf{M})$ and $k \in \mathbf{N}$ such that $f(C) \subset \frac{1}{2}\mathbb{D}$ and $\frac{\lambda}{2}(\psi^{[k]}(C)) \subset (1 + \frac{1}{2}\mathbb{D})$ where $\lambda = sup_C[\prod_{j=0}^{k-1} \mathbf{C}_\psi^{[j]}(\omega)]$. This implies that C and $\frac{\lambda}{2}\psi^{[k]}(C)$ are separable in $\mathbf{M}$, so by Lemma 2.9 in [24], the sum $C \cup \frac{\lambda}{2}\psi^{[k]}(C)$ is $\mathbf{M}$-convex. $\square$

Theorem 3.2. Let $\mathbf{M}$ be a connected Stein manifold, $\psi \in \mathcal{O}(\mathbf{M})$, $\omega \in \mathbb{H}(\mathbf{M})$ and the following conditions hold:

(1) for every $p \in \mathbf{M}$, $\omega(p) \neq 0$ and ψ is an injective self-map without fixed point in $\mathbf{M}$.
(2) for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$, there exists positive integer n such that $C \cap \psi^{[n]}(C) = \emptyset$ and the set $C \cup \psi^{[n]}(C)$ is $\mathbf{M}$-convex.

Then, the weighted composition operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic on $\mathbb{H}(\mathbf{M})$.

Proof. Assume that $\{C_n\}_{n=1}^{\infty}$ be an exhaustion of $\mathbf{M}$. Without lose of generality, we can assume that every C_n is $\mathbf{M}$-convex. Since the compact-open topology on $\mathbb{H}(\mathbf{M})$ is independent of the chosen exhaustion, we can endow $\mathbb{H}(\mathbf{M})$ with the topology induced by the semi-norms on $\mathbb{H}(\mathbf{M})$ defined by $p_n(f) := \sup\{|f(p)| : p \in C_n\}$. Let $U, V \subset \mathbb{H}(\mathbf{M})$ be non-empty open sets and fix $f \in U$ and $g \in V$. By definition of compact-open topology of $\mathbb{H}(\mathbf{M})$, there is a closed ball $B \subset \mathbf{M}$ (with respect to the Carathéodory pseudo-distance as can be seen in [24]) and a positive real number ϵ such that, every $h_1 \in U$ satisfies $\sup_{p\in B} |f(p) - h_1(p)| < \epsilon$ and similarly every $h_2 \in V$ satisfies $\sup_{p\in B} |g(p) - h_2(p)| < \epsilon$.

Now, assume that D be another closed ball such that $B \subset D^{\circ}$. Since ψ is an injective self-map without fixed point on $\mathbf{M}$, then the function f is holomorphic on some neighborhood of D, and the function $\frac{g\circ(\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0-1}(\omega\circ(\psi^{[k]})^{-1})}$ is holomorphic on some neighborhood of $\psi^{[n_0]}(D)$.

By assumption (2), there exists n_0 such that $D \cap \psi^{[n_0]}(D) = \emptyset$ and the compact set $K := D \cup \psi^{[n_0]}(D)$ is $\mathbf{M}$-convex (by Oka-Weil theorem), there exists a holomorphic function $h \in \mathbb{H}(\mathbf{M})$ such that $\sup_{z\in D} |f(z) - h(z)| < \epsilon$ and

$$\sup_{y\in\psi^{[n_0]}(D)} |\frac{g \circ (\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})}(y) - h(y)| < \frac{\epsilon}{M}.$$

where $M := \max_{y\in\psi^{[n_0]}(D)} |\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})(y)|$.

Hence $\sup_{z\in K} |f(z) - h(z)| < \epsilon$ and

$$\begin{aligned}
&\sup_{z\in K} |g(z) - (|K_{\psi,\omega})^{[n_0]}h(z)| \\
&= \sup_{z\in K} |\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})(y)(\frac{g \circ (\psi^{[n_0]})^{-1}}{\prod_{k=1}^{n_0}(\omega \circ (\psi^{[k]})^{-1})}(y) - h(y))| < \epsilon,
\end{aligned}$$

where $y := \psi^{[n_0]}(z)$. This shows that $h \in U$ and $(\mathbf{C}_{\psi,\omega})^{[n_0]}h \in V$, so that $\mathbf{C}_{\psi,\omega}$ is topologically transitive. Since $\mathbb{H}(\mathbf{M})$ is a separable Fréchet space, $\mathbf{C}_{\psi,\omega}$ is hypercyclic. □

Theorem 3.3. Let $\mathbf{M}$ be a connected Stein manifold, $\psi \in \mathcal{O}(\mathbf{M})$ and $\omega \in \mathbb{H}(\mathbf{M})$ and $\{n_l\}_{l=1}^{\infty}$ be an increasing sequence of positive integer numbers. Then the operator $\mathbf{C}_{\psi,\omega}$ is hypercyclic w.r.t. $(n_l)_l$ if and only if for every $p \in \mathbf{M}$, $\omega(p) \neq 0$ and ψ is injective without fixed points in $\mathbf{M}$, $\psi(\mathbf{M})$ is a Runge domain w.r.t. $\mathbf{M}$ and for every $\mathbf{M}$-convex compact subset $C \subset \mathbf{M}$ there is a positive integer number k such that the sets C and $\psi^{[n_k]}(C)$ are separable in $\mathbf{M}$.

Proof. Sufficiency in both parts follows from Theorem 3.1 and Theorem 3.2. If the sets C and $\psi^{[n_l]}(C)$ are separable in $\mathbf{M}$, since $\psi(\mathbf{M})$ is a Runge domain in $\mathbf{M}$ and C is $\mathbf{M}$-convex, then $\psi^{[n_l]}(C)$ is $\mathbf{M}$-convex and by a Lemma from [24] their sum is $\mathbf{M}$-convex . Necessity in both parts follows directly from Theorem 3.1. □

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References

[1] Akbarbaglu, I., & Azimi, M. R. (2022). Universal family of translations on weighted Orlicz spaces. Positivity, 26(1), 1–21.

[2] Akbarbaglu, I., Azimi, M. R., & Kumar, V. (2021). Topologically transitive sequence of cosine operators on Orlicz spaces. Ann. Funct. Anal., 12(1), 3–14.

[3] Azimi, M. R., & Farmani, M. (2022). Subspace-hypercyclicity of conditional weighted translations on locally compact groups. Positivity, 26(3), 1–20.

[4] Azimi, M. R., & Jabbarzadeh, M. R. (2022). Hypercyclicity of weighted composition operators on L^p-spaces. Mediterr. J. Math., 19, Paper No.164, 1–16.

[5] Bayart, F. (2007). A class of linear fractional maps of the ball and its composition operators. Adv. Math., 209, 649–665.

[6] Bayart, F., & Matheron, É. (2009). Dynamics of linear operators. Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge.

[7] Birkhoff, G. D. (1929). Démonstration d'un théorème élémentaire sur les fonctions entières. C.R. Acad. Sci. Paris, 189, 473–475.

[8] Bourdon, P. S., & Shapiro, J. H. (1990). Cyclic composition operators on H_2. Proc. Symp. Pure Math., 51(2), 43–53.

[9] Bourdon, P. S., & Shapiro, J. H. (1997). Cyclic phenomena for composition operators. Mem. Amer. Math. Soc., 596.

[10] Chan, K. C., & Shapiro, J. H. (1991). The cyclic behavior of translation operators on Hilbert spaces of entire functions. Indiana Univ. Math. J., 40, 1421–1449.

[11] Feldman, N. S. (2012). n-Weakly hypercyclic and n-weakly supercyclic operators. J. Funct. Anal., 263, 2255–2299.

[12] Fox, R. H. (1945). On topologies for function spaces. Bull. Amer. Math. Soc., 51(6), 429–433. https://doi.org/10.1090/S0002-9904-1945-08370-0

[13] Gethner, R. M., & Shapiro, J. H. (1987). Universal vectors for operators on spaces of holomorphic functions. Proc. Amer. Math. Soc., 100, 281–288.

[14] Godefroy, G., & Shapiro, J. H. (1991). Operators with dense invariant cyclic vector manifolds. J. Funct. Anal., 98, 229–269.

[15] Grosse-Erdmann, K. G. (1999). Universal families and hypercyclic operators. Bull. Amer. Math. Soc. (N. S.), 36, 345–381.

[16] Grosse-Erdmann, K. G. (2003). Recent developments in hypercyclicity. Rev. R. Acad. Cien. Serie A. Mat., 97(2), 273–286.

[17] Grosse-Erdmann, K. G., & Manguillot, A. P. (2011). Linear chaos. Universitext. Springer, London.

[18] Grosse-Erdmann, K. G., & Mortini, R. (2009). Universal functions for composition operators with nonautomorphic symbol. J. Anal. Math., 107, 355–376.

[19] Pashaie, F., Azimi, M. R., & Shahidi, S. M. Dynamics of weighted composition operators on Stein manifolds. (Submitted).

[20] Salas, H. N. (1995). Hypercyclic weighted shifts. Trans. Amer. Math. Soc., 347, 993–1004.

[21] Shapiro, J. H. (1993). Composition operators and classical function theory. Springer-Verlag, New York.

[22] Shapiro, J. H. (2001). Notes on dynamics of linear operators. http://www.math.msu.edu/shapiro

[23] Yousefi, B., & Rezaei, H. (2007). Hypercyclic property of weighted composition operators. Proc. Amer. Math. Soc., 135, 3263–3271.

[24] Zajac, S. (2016). Hypercyclicity of composition operators in Stein manifolds. Proc. Amer. Math. Soc., 144(9), 3991–4000.

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ON GENERALIZED CLOSED $QTAG$-MODULES

AYAZUL HASAN *, MOHD NOMAN ALI , AND VINIT KUMAR SHARMA

Abstract. This paper explores the concept of closed modules by utilizing the notion of h-topology within the context of $QTAG$-modules. In addition, we delve into the intricate relationships between different types of submodules and Ulm invariants, shedding light on their interconnected roles within the closures. This investigation aims to provide a deeper understanding of these algebraic structures and their dynamic interactions.

Keywords: QTAG-modules, Closures, Isotype submodules, Ulm invariants.

2020 Mathematics Subject Classification: Primary: 16K20, Secondary: 13C12, 13C13.

1. Introduction

One of the rapidly developing areas of research in module theory is the study of TAG-modules. The idea was first introduced by Singh [15] in 1976. Moreover, module theory has also witnessed a surge of interest in recent research, with the TAG-module being an intriguing area of investigation, which is one of the variations of torsion Abelian groups in modules. Over time, many researchers have extensively studied torsion Abelian groups and its numerous variants, as evidenced by a range of notable studies found in [3, 11, 17].

Consider the following two conditions on a module M over an arbitrary (associative, unitary) ring R.

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"(i) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(ii) Given any two uniserial submodules U_1 and U_2 of a homomorphic image of M, for any submodule N of U_1, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided that the composition length $d(U_1/N) \leq d(U_2/\phi(N))$ holds."

When M over a ring R is a module and satisfies conditions (i) and (ii), it is called a TAG-module, and when M over a ring R has condition (i) only, it is called a $QTAG$-module. Following up on his investigation in [15], Singh [16] published a paper in 1987 titled "Abelian groups like modules," which naturally led to the introduction of the concept of $QTAG$-module, which has since generated interest in the field of module theory. The study was then followed by numerous developments on the topic. In recent years, this exploration for $QTAG$-modules has regained the interests of some authors, and a lot of interesting results on $QTAG$-modules of many torsion Abelian groups have been obtained during the course of this quest (see, for example, [1, 2, 9, 10] and the references cited therein). Many such advances in the theory of torsion Abelian groups exhibit characteristics of the earlier developments, which is not surprising. The current work contributes to the understanding of the structure of $QTAG$-modules and is a logical extension of the studies carried out in [18]. Another useful source on the explored subject is [4] (see [19], too) as well. For some other interesting generalizations of the topic mentioned here, the reader can see in [5, 6].

2. Preliminaries

Throughout the present paper, unless specified something else, let us assume that all rings R into consideration are associative with unity and modules M are unital $QTAG$-modules, written additively, as is the custom when studying them. All other not explicitly explained herein notions and notations are well-known and mainly follow those from [7] and [8]. A module M is called uniform if the intersection of any two of its nonzero submodules is nonzero. An element a in M is called uniform if aR is a nonzero uniform module. Standardly, the decomposition length of any module M with a unique decomposition series is denoted by $d(M)$. In addition, the exponent of a uniform element a of M, denoted by the symbol $e(a)$, is equal to $d(aR)$. As usual, for such a module M, we state the height of a in M as $H_M(a) = \sup\{d(bR/aR) : b \in M,\ a \in bR \text{ and } b \text{ uniform}\}$. Likewise, for $k \geq 0$, $H_k(M) = \{a \in M|\ H_M(a) \geq k\}$ represents the submodule of M that is generated by the elements that have at least k heights. The module M is h-divisible if $M = M^1 = \cap_{k=0}^{\infty} H_k(M)$, where M^1 is the submodule of M generated by uniform elements of M of infinite height. The module M is h-reduced if it does not contain any h-divisible submodule. The topology of M, which admits as a base of neighborhoods of zero, is known as the h-topology. This topology has the submodules $H_k(M)$ for some k. In this fashion, a submodule S of M is named the closure in M if $\overline{S} = \cap_{k=0}^{\infty}(S + H_k(M))$. With this in hand, we say that a submodule S of M is closed with respect to the h-topology provided that $\overline{S} = S$ and h-dense in M if $\overline{S} = M$. By closed module M, we mean those modules which do not have any element of infinite height and has a limit in M for every Cauchy sequence. Moreover, the sum of all the simple submodules of M is called the socle of M, denoted by $Soc(M)$.

Furthermore, we assemble some basic concepts which are crucial in the following development. For pertinent results related to these concepts, we refer the reader to [13, 14] (see [12] too). By analogy, for every ordinal σ, one can define the infinite height $H_\sigma(M)$ as follows: $H_\sigma(M) = H_1(H_{\sigma-1}(M))$ if σ is non-limit, or $H_\sigma(M) = \cap_{\gamma<\sigma} H_\gamma(M)$ otherwise. Usually, $H_\sigma(M)$ denotes the submodule consisting of all elements of M with height $\geq \sigma$. This submodule is also called σ^{th}-Ulm submodule of M. In particular, $H_\omega(M)$ will be the first Ulm submodule of M, i.e., the set of elements of infinite height. A submodule S of M is said to be σ-pure if, for all ordinal γ, there exists an ordinal σ (depending on S) such that $H_\gamma(M) \cap S = H_\gamma(S)$. Besides, a submodule S of M is termed isotype, if it is σ-pure for every ordinal σ. The cardinality of the minimal generating set of M is denoted by the symbol $g(M)$. For all ordinals σ, $f_M(\sigma) = g\big(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M))\big)$ is called the σ^{th}-Ulm invariant of M.

Finally, the project is organized as follows. In the previous section, we have explored the subject's background. The current section, i.e. here, looks at the topics's related notions. The study of generalized closed modules is discussed in the next section, and important results and distinctive properties of closures as well as Ulm invariants are presented. In the final section, we list some interesting left-open questions.

3. Main results

It is well-known that the direct sum of countably generated modules and the closed modules are determined up to isomorphism by their Ulm invariants. The latter type of modules can be characterized as the closed submodule of the closure of a direct sum of uniserial modules. This closure is considered with respect to an h-topology (cf. [1]) which is defined for modules without elements of infinite height. One of the main goals of this article is to extend the concept of h-topology, and thereby to include modules of arbitrary countable length for investigating the generalized closed modules.

The following notions are our major tools.

Let γ be an ordinal and M an h-reduced $QTAG$-module, we define a descending chain of submodules $H_\gamma(M)$ by

$$H_\gamma(M) = \begin{cases} H_1(H_\alpha(M)), & \text{if } \gamma = \alpha + 1 \\ \cap_{\alpha<\gamma} H_\alpha(M), & \text{if } \gamma \text{ is a limit ordinal.} \end{cases}$$

Since all modules are assumed to be h-reduced, there is an ordinal β such that $H_\gamma(M) = 0$ for $\gamma \geq \beta$. The smallest such β is usually referred to as the length of M. When length$(M) \leq \omega$, M is said to contain no elements of infinite height.

Let η be the first limit ordinal greater than or equal to the length of a $QTAG$-module M. Then a h-topology can be constructed using the submodules $H_\gamma(M)$, for $\gamma < \eta$, as a base for the neighborhoods of the identity. This extension of the h-topology is known as the natural topology.

We start here with a new useful criterion for a submodule to be isotype.

Proposition 3.1. *Suppose that γ is an ordinal. Then a submodule S is an isotype submodule of a $QTAG$-module M if $H_\gamma(M) \cap S \subset H_\gamma(S)$ implies $H_{\gamma+1}(M) \cap S \subset H_{\gamma+1}(M)$.*

Proof. The proof is by induction on γ in conjunction with $H_\gamma(M) \cap S \supset H_\gamma(S)$. Clearly, if γ is a limit ordinal and $H_\alpha(M) \cap S = H_\alpha(S)$ for all $\alpha < \gamma$, then

$$\begin{aligned} H_\gamma(M) \cap S &= (\cap_{\alpha<\gamma} H_\alpha(M)) \cap S, \\ &= \cap_{\alpha<\gamma} H_\alpha(S), \\ &= H_\gamma(S), \end{aligned}$$

which allows us to infer that S is isotype in M for each ordinal γ. □

In light of the previous construction, we obtain the following.

Proposition 3.2. *Let N be a submodule of a $QTAG$-module M of countable length β. Then there exists an isotype submodule S of M such that $N \subset S \subset M$ and $g(N) = g(S)$.*

Proof. Foremost, we construct inductively a chain of submodules S^k such that $S = \cup S^k$ for some positive integer k. Now, we set $S^o = N$, then there exist equations $x' = y$ with

$d(xR/x'R) = 1$ for some $x \in M$ and $y \in H_{\gamma+1}(M) \cap S^{k-1}$. However, we observe that these equations do not have a solution for $x \in S$. Among all such equations, let $T^{\gamma,k}$ be one solution of $H_\gamma(M)$, for each ordinal $\gamma < \beta$. In fact, denote by T^k the module generated by the elements of $\cup_{\gamma<\beta} T^{\gamma,k}$ and define $S^k = S^{k-1} + T^k$.

Next, assume that $H_\gamma(M) \cap S = H_\gamma(S)$ and choose $y \in H_{\gamma+1}(M) \cap S$ such that $y \in S^k$. Then by the definition of $T^{\gamma,k+1}$, there exists an element z such that $z \in T^{\gamma,k+1}$ and $z' = y$ where $d(zR/z'R) = 1$. By hypothesis on $H_\gamma(M)$, we have $T^{\gamma,k+1} \subset (H_\gamma(M) \cap S)$ and $H_\gamma(M) \cap S = H_\gamma(S)$. Therefore, we obviously observe that $y \in H_{\gamma+1}(S)$ and $H_{\gamma+1}(M) \cap S \subset H_{\gamma+1}(S)$. By appealing to the same reasoning as in Proposition 3.1, one may infer that the assertion follows. □

The following technicality is pivotal.

Proposition 3.3. *Let M be a QTAG-module. If $\overline{M}$ is the closure of M, then M is isotype in $\overline{M}$.*

Proof. Suppose that $H_\gamma(\overline{M}) \cap M = H_\gamma(M)$ and choose $x \in H_{\gamma+1}(\overline{M}) \cap M$. Then there exists a uniform element $y \in H_\gamma(\overline{M})$ such that $x = y'$ where $d(yR/y'R) = 1$. If $\{y_k\}$ is a sequence in M, then its limit y is also an element of M and $y_k - y$ is an element of $H_\gamma(\overline{M})$ for every k. This, in turn, implies that $y_k \in H_\gamma(\overline{M}) \cap M = H_\gamma(M)$. Therefore, $y'_k - x$ and y'_k are in $H_{\gamma+1}(M)$ such that $d(y_kR/y'_kR) = 1$. Hence, it consequently follows that $x \in H_{\gamma+1}(M)$, and the result follows from Proposition 3.1. □

The next statement is pretty simple but useful.

Proposition 3.4. *Let S be an isotype submodule of a QTAG-module M which is h-dense in M. Then S and M have equal lengths.*

Proof. Let β_1 and β_2 be the lengths of S and M, respectively. Clearly $\beta_1 \leq \beta_2$. Now, if x is a nonzero uniform element of $H_\gamma(M)$ for $\gamma < \beta_1$ and $\{y_k\}$ is a sequence in S converging to x, then $y_k - x \in H_\gamma(M)$ for every k. This gives that $y_k \in H_\gamma(M) \cap S = H_\gamma(S) = 0$ for every k and means that $H_\gamma(M) = 0$ for all $\gamma \geq \beta_1$. Consequently, as early checked, $\beta_1 = \beta_2$. The proof is over. □

We now will explore the closureness for the submodule classes.

Theorem 3.1. *Suppose M is a QTAG-module and γ is an ordianl. If $\overline{M}$ is the closure of M, then $\overline{H_\gamma(M)} = H_\gamma(\overline{M})$.*

Proof. Let x be a uniform element of $H_\gamma(\overline{M})$ with a sequence $\{x_k\}$ in M converging to x. Then there exists an integer t such that $x_k - x \in H_\gamma(\overline{M})$ for $k \geq t$. Setting $y_k = x_{t+k}$. Indeed, this gives a sequence in $H_\gamma(M)$ converging to x. Thereby, because of the closureness of M, it follows at one that $H_\gamma(\overline{M}) \subset \overline{H_\gamma(M)}$.

Turning to the opposite part-half, we shall prove at first by induction on γ. First, if $\gamma = 0$, it is obvious to assume the result holds for $\alpha < \gamma$. Now, we have two cases to consider. First, if γ is a limit ordinal, then

$$\begin{aligned}\overline{H_\gamma(M)} &= \overline{\cap_{\alpha<\gamma} H_\alpha(M)},\\ &\subset \cap_{\alpha<\gamma}\overline{H_\alpha(M)},\\ &= \cap_{\alpha<\gamma} H_\alpha(\overline{M}),\\ &= H_\gamma(\overline{M}),\end{aligned}$$

so that $\overline{H_\gamma(M)} \subset H_\gamma(\overline{M})$, and we are done. For the remaining case, if γ is not a limit ordinal, then we write $\gamma = \alpha + 1$ and choose a sequence of ordinals γ_k with length of M as a supremum such that $\gamma_{k+1} > \gamma_k > \gamma$. Let $x \in \overline{H_\gamma(M)}$, we observe that a subsequence of sequence in $H_\gamma(M)$ converging to x, and we obtain a sequence $\{x_k\}$ in $H_\gamma(M)$ such that

$$\lim_{k\to\infty} x_k = x \text{ and } x_{k+1} - x_k \in H_{\gamma_{k+1}+1}(M)$$

for each k. Suppose $a \in H_\alpha(M)$ such that $a' = u$ where $d(aR/a'R) = 1$ and choose $z \in H_\beta(M)$ such that $z' = v - a'$ where $d(zR/z'R) = d(aR/a'R) = 1$. Setting $b = z + a$. This gives that $b' = v$, $b - a = z \in H_\beta(M)$, and $b \in H_\alpha(M)$ where $d(bR/b'R) = 1$. On continuing same process in this manner, one may see that there exists a sequence $\{z_k\}$ in $H_\alpha(M)$ such that $z'_k = x_n$ where $d(z_kR/z'_kR) = 1$. Let c be the limit of $\{z_k\}$ in $\overline{H_\alpha(M)}$. Then $c \in H_\alpha(\overline{M})$ and $c' = x \in H_\gamma(\overline{M})$ where $d(cR/c'R) = 1$. Thus, $\overline{H_\gamma(M)} \subset H_\gamma(\overline{M})$, and the result follows. □

The next two statements are worthy of noticing.

Corollary 3.1. *Suppose M is a QTAG-module and γ is an ordinal such that length of M is greater than γ. Then $\overline{M} = M + \overline{H_\gamma(M)}$.*

Proof. First, we take $x \in \overline{M}$, and let $\{x_n\}$ be a sequence in M which converges to x. Then there exists an integer t such that $x_t - x_k \in H_\gamma(M)$ for $k > t$ and length $(M) > \gamma$. By setting $y_k = x_t - x_{t+k}$, one may see that a sequence in $H_\gamma(M)$. Let y be the limit of $\{y_k\}$ in $\overline{H_\gamma(M)}$. Then $x = x_t - y$, and we are done. □

Corollary 3.2. *Suppose M is a $QTAG$-module and γ is an ordinal. Then M is closed if and only if $H_\gamma(M)$ is closed, provided length of $M > \gamma$.*

Proof. Assume that M is closed; i.e., $M = \overline{M}$. In accordance with Theorem 3.1, we subsequently deduce that

$$H_\gamma(M) = H_\gamma(\overline{M}) = \overline{H_\gamma(M)}$$

for each ordinal γ. This allows us to infer that $H_\gamma(M)$ is closed, thus completing the first half.

Conversely, we presume now that $H_\gamma(M)$ is closed for some $\gamma <$ length(M). Hence, $\overline{H_\gamma(M)} = H_\gamma(M)$ and so in conjunction with Corollary 3.1, we get

$$\overline{M} = M + \overline{H_\gamma(M)} = M + H_\gamma(M).$$

Consequently, it is plainly seen that M is closed, and we are finished. □

Our next example show that in the above corollaries, the requirement that length of $M > \gamma$ cannot be removed.

Example 3.1. Let

$$f : Soc(M/H_\gamma(M)) \to H_\gamma(M)/H_{\gamma+1}(M)$$

be a homomorphism such that $\ker(f) = \overline{M}/\overline{H_\gamma(M)}$. Let S be a submodule of $Soc(M/H_\gamma(M))$, then $f : Soc(M/H_\gamma(M)) \to H_\gamma(M)/H_{\gamma+1}(M)$ is an isomorphism. Obviously, $\overline{H_\gamma(M)} \in \overline{M}$, and $S = \Sigma_{\gamma \leq k < t}(x_t - y_k)$, for some integers t and k. Now, for every ordinal γ, let K_γ be a submodule of $\overline{H_\gamma(M)}$ and L_γ the image of K_γ in $\overline{M}/\overline{H_\gamma(M)}$. Then, $U = \Sigma_{\gamma+\omega \leq k < t}(x_t - y_k)$ is a direct sum of uniserial modules. Since $S \cap U = 0$, we have $S + U \leq \Sigma_{\gamma+t \leq \gamma+\omega}\ x_t$ and $\overline{S} + \overline{U} \leq \Sigma_{\gamma+k \leq \gamma+\omega}\ y_k$. Putting these inequalities together, we obtain the desired claim.

We come now to our main theorem on closed modules.

Theorem 3.2. *Let M_i $(i \in I)$ be a system of $QTAG$-modules. Then $M = \oplus_{i \in I} M_i$ is closed if and only if there exists an ordinal $\gamma <$ length(M) such that the family $\mathcal{F} = \{i \in I :$ length$(M_i) > \gamma\}$ is of nonzero finite cardinality and for each $i \in \mathcal{F}$, M_i is closed.*

Proof. To prove necessity, let M be a closed module. If there is no γ such that $\mathcal{F}$ has nonzero finite cardinality, then there exists an increasing sequence $\beta_{i_k} <$ length(M_{i_k}) such that $\lim_{k\to\infty} \beta_{i_k} =$ length(M_{i_k}). Choose $0 \neq x_{i_k} \in Soc(H_{\beta_{i_k}}(M_{i_k}))$ and let $y_t = \oplus_{k=1}^{t} x_{i_k}$.

However, if $\{y_t\}$ is a sequence in M, then one sees that y is an element of M and $y = \oplus_{i\in J} x_i$, where J is a finite subset of I. Let r be an integer such that $i_r \in \mathcal{F}$. For $t > r$, we have

$$H_M(y - \oplus_{k=1}^{t} x_{i_k}) = \min\{H_{M_i}(\phi_i(y - \oplus_{k=1}^{t} x_{i_k}))\} \leq H_{M_{i_k}}(x_{i_k}),$$

where $\phi_i : M \to M_i$ is the projection map. Thus, $H_M(y - y_t) \leq H_{M_i}(x_{i_k})$ for some t, which is an absurd, so we pursued the contradiction. Therefore, there exists an ordinal γ such that $\mathcal{F}$ has the proper cardinality.

Let $\{x_{i_0}^k\}$ be a sequence in M_{i_0} for $i_0 \in \mathcal{F}$, and since M is closed, one verifies that the sequence has a limit in M, say $y = x_{i_0} + \oplus_{i\in I-i_0} x_i$. But $H_M(x - x_{i_0}^k) \leq H_M(x_i)$ implies $x_i = 0$ for $i \in I - i_0$, and besides, that $\lim_{k\to\infty} H_M(x - x_{i_0}^k) = H_M(0)$, which is greater than the height of any nonzero element of M. So, it follows that $x \in M_{i_0}$. This surely means that M_i is closed for $i \in \mathcal{F}$, as wanted.

To show now the truthfulness of sufficiency, let us assume that $\mathcal{F}$ has cardinality a positive integer with each M_i, for $i \in \mathcal{F}$, closed. In order to show that M is closed, it suffices to show that every bounded sequence in M has a limit in M. In order to do this, suppose $\{y_k\}$ is such a sequence and let $y_k = \oplus_{i\in I} x_i^k$, then $\{x_i^k\}$ is a sequence in the h-topology induced in M_i by the natural topology of M. In case that $i \in I - \mathcal{F}$, we can observe that the induced h-topology is discrete. So, for some k, we find $\{x_i^k\}$ is constant and has a limit x_i. But, in this case, $i \in \mathcal{F}$, we then can identify that the induced h-topology is either discrete or the natural topology of M_i. And since M_i is closed, there exists a limit x_i in M_i, and hence $y = \oplus_{i\in I} x_i$ is the limit of $\{y_k\}$. The proof is completed. □

The following gives a great deal of information about the *Ulm* invariants.

Theorem 3.3. *Let S be an isotype submodule of a QTAG-module M which is h-dense in M. Then M and S have the same Ulm invariants.*

Proof. Let the injection $S \to M$ induces a map

$$\phi : (H_\gamma(S) \cap Soc(S))/(H_{\gamma+1}(S) \cap Soc(S)) \to (H_\gamma(M) \cap Soc(M))/(H_{\gamma+1}(M) \cap Soc(M))$$

for every ordinal γ. Then

$$\begin{aligned}(H_\gamma(M) \cap Soc(S)) \cap H_{\gamma+1}(M) \cap Soc(M) &= (H_\gamma(S) \cap H_{\gamma+1}(M)) \cap Soc(S),\\ &= (S \cap H_{\gamma+1}(M)) \cap Soc(S),\\ &= H_{\gamma+1}(S) \cap Soc(S),\end{aligned}$$

and so ϕ is a monomorphism. This shows that $f_S(\gamma) \leq f_M(\gamma)$.

On the other hand, if x is any uniform element of $H_\gamma(M) \cap Soc(M)$, there exists a sequence $\{x_k\}$ in S such that it has x as a limit. By adding terms and constructing subsequences, let us assume that all the elements of $\{x_k\}$ are in S such that $e(x_k) = 1$. Then, for some k, we have $x_k - x \in (H_{\gamma+1}(M) \cap Soc(M))$ which yields

$$\phi(x_k + (H_{\gamma+1}(S) \cap Soc(S))) = x + (H_{\gamma+1}(M) + Soc(M)),$$

and ϕ is an epimorphism. Thus, $f_S(\gamma) = f_M(\gamma)$ for each ordinal γ. □

Mimicking the method demonstrated above, we record the following consequence.

Corollary 3.3. *If S is an isotype, h-dense submodule of a $QTAG$-module M, then*

$$M/H_\gamma(M) \cong S/H_\gamma(S)$$

for all $\gamma < length(M)$.

We are now ready to give our desired example.

Example 3.2. Let U and V be the $QTAG$-modules having same Ulm invariants and length $\omega + 1$. In fact, as U and V are direct sum of uniserial modules, we have that $(Soc(U) + H_\omega(U))/H_\omega(U)$ is countably generated and $(Soc(V) + H_\omega(V))/H_\omega(V)$ is not countably generated. Indeed, there exists a countably generated module P of length ω^2 with $f_\gamma(P) = 1$ for all ordinals $\gamma < \omega^2$. Applying Corollary 3.3 appointed above, and the countability of P, we see that $\overline{P}/H_\omega(\overline{P})$ is countably generated, where $\overline{P}$ is the closure of P in U and V, respectively. Let us decompose $M = \overline{P} \oplus U$ and $S = \overline{P} \oplus V$. Then M and S are closed modules with same Ulm invariants. Therefore, $(Soc(M) + H_\omega(M))/H_\omega(M)$ is countably generated and $(Soc(S)+H_\omega(S))/H_\omega(S)$ is not countably generated. Consequently, $M \ncong S$, as claimed.

Remark 3.1. *Since an isomorphism between two closed modules M and S carried $Soc(H_\gamma(M))$ isomorphically to $Soc(H_\gamma(S))$ for each ordinal γ. Therefore, the natural topological structure is preserved by isomorphisms, and such maps are actually homomorpshims.*

We continue with the significant characterization of a closed module.

Theorem 3.4. *A closed $QTAG$-module containing a direct sum of countably generated modules which form an isotype, h-dense submodule is determined up to isomorphism by its Ulm invariants.*

Proof. Let M_1 and M_2 be two closed $QTAG$-modules having the same Ulm invariants and containing submodules S_1 and S_2 possessing the desired properties. By hypothesis and consulting with Theorem 3.3, we inspect that $f_{S_1}(\gamma) = f_{M_1}(\gamma) = f_{M_2}(\gamma) = f_{S_2}(\gamma)$ for each ordinal γ. However, it is easily verified that M_1 is isomorphic to M_2. In fact, since M_1 and M_2 are the direct sum of countably generated modules, we detect that S_1 is isomorphic to S_2, and we are done. □

The above theorem leads to the analysis of determining which closed modules contain a h-dense, isotype submodule, which is a direct sum of countably generated modules.

Analysis. In accordance with [1], we construct a closed module without elements of infinite height. In fact, for a closed module of length less than or equal to the first countable ordinal ω, any of its basic submodules is the h-dense, isotype submodule. However, this is not valid for closed modules of greater length. Letting M be a closed module of countable length β containing the h-dense, isotype submodule. According to Corollary 3.3, it is plainly seen that $M/H_\gamma(M)$ must be a direct sum of countably generated modules for all $\gamma < \beta$. If $M/H_\gamma(M)$ is countably generated for all $\gamma < \beta$, the situation is the following.

Theorem 3.5. *Let M be a $QTAG$-module such that $M/H_\gamma(M)$ is countably genrated for some ordinal γ. Then there exists a countably generated, h-dense, and isotype submodule S of M, provided length of $M > \gamma$.*

Proof. Let us assume that length$(M) = \beta$. If $\gamma \geq \beta$, we are done. For the remaining case $\gamma < \beta$, we choose a set of representatives $\{x_{\gamma,k}\}_{k \in Z^+}$ of the countably generated module $M/H_\gamma(M)$ and let $\mathcal{F} = \cup_{\gamma<\beta}\{x_{\gamma,k}\}_{k\in Z^+}$. Since β is countable, we obtain that $\mathcal{F}$ is countable, and then $\mathcal{F}$ generates a countably generated module in M. Having in mind Proposition 3.2, one infers that a countably generated isotype submodule S of M containing $\mathcal{F}$.

Now we choose $x \in M$ and constructing a sequence in $\mathcal{F}$ converging to x. After this, let us find a sequence $\{\gamma_t\}$ of ordinals whose limit is β. Then for each t choose the representative x_t from $\{x_{\gamma_t,k}\}_{k\in Z^+}$ such that x is in the same coset as x_t modulo M_{γ_t}. Since the ordinals γ_t converge to β, we have $\lim_{t\to\infty} x_t = x$ and the proof is completed. □

So, the leitmotif of this article is the utilization of the above material to explore the countability of quotient modules as follows: If the $QTAG$-module M has a countable length β and $M/H_\gamma(M)$ is countably generated, for some ordinal $\gamma < \beta$. This state is known as the countability property. Therefore, we have the following direct consequences of Theorems 3.4 and 3.5, respectively.

Corollary 3.4. *Closed $QTAG$-modules with the countability property are determined up to isomorphism by their Ulm invariants.*

Corollary 3.5. *If M is a closed $QTAG$-module with the countability property, then M is determined up to isomorphism by its Ulm invariants.*

We continue with an observation on the above two corollaries.

Example 3.3. Let M be a $QTAG$-module such that $M = \oplus_\gamma Soc(H_\gamma(M))$ is the decomposition of a closed module, then M is determined by its Ulm invariants if and only if $Soc(H_\gamma(M))$ is determined by its Ulm invariants. It is readily checked that for every submodule S of $Soc(H_\gamma(M))$, we get that $\text{Hom}(S, \oplus_\gamma Soc(H_\gamma(M))) = 0$, which is an essential submodule of M. This means that countability property is not sufficient in order to find an isomorphism. In accordance with Theorem 3.5, one may see that there exists a countably generated, h-dense, and isotype submodule L of M such that it is a direct sum of uniserial modules, which is a closed $QTAG$-module, as required.

We will now argue the following theorem.

Theorem 3.6. *Suppose that M_1 is a $QTAG$-module with the countability property and that M_2 is a countably generated with $f_\gamma(M_1) = f_\gamma(M_2)$ for some ordinal γ. Then M_1 can be embedded as an isotype submodule S of $\overline{M_2}$ such that $M_1 \supset M_2$.*

Proof. The existence of a countably generated, h-dense, isotype submodule S of M_1 is guaranteed by Theorem 3.5, so hypothesis M_2 does exist. S and M_1 can considered submodules of $\overline{S}$ by means of the standard topological map that embeds a space in its closure. The h-denseness of M_1 in $\overline{S}$ follows from the fact that $S \subset M_1 \subset \overline{S}$. By applying Proposition 3.1, the isotype property of M_1 can be demonstrated. Due to the equality of Ulm invariants, S and M_2 are isomorphic, and this map can be extended to $\overline{S}$ and $\overline{M_2}$ to give the desired embedding map. □

The following lemma determines the cardinality of a closed module that meets the countability property.

Lemma 3.1. *Suppose M is a countably generated $QTAG$-module. If $\overline{M}$ is the closure of M, then $g(\overline{M}) = 2^{\aleph_0}$.*

Proof. Since $g(M) = \aleph_0$, we obtain the number of Cauchy sequences in $M \leq 2^{\aleph_0}$ and thus $g(\overline{M}) \leq 2^{\aleph_0}$. So, what remains to show is the inequality $g(\overline{M}) \geq 2^{\aleph_0}$. For this purpose, choose a sequence of ordinals γ_k with a length of M as a supremum such that $Soc(H_{\gamma_k}(M)) \subset Soc(H_{\gamma_k+1}(M))$ for some $k \geq 0$. Then, there exists a sequence $\{x_k\}$ of elements in M such that $x_k \in Soc(H_{\gamma_k}(M)) - Soc(H_{\gamma_k+1}(M))$.

Let $A = (a_1, a_2, \dots)$ be the set of all $\aleph_0$ tuples, where a_k is 0 and 1. Now, define a map $f : A \to \overline{M}$ such that $f(A) = \lim y_k$ where $y_k = \oplus_{n=1}^{k} a_n x_n$. Let a and a' be two distinct elements of A with r the first n such that $a_n \neq a'_n$. Then, for $k > r$, we have

$$y_k - y'_k = x_r + \oplus_{n=r+1}^{k}(a_n x_k - a' k x_k) \notin H_{\gamma+1}(M).$$

Therefore, $f(a) \neq f(a')$. This gives that f is one-one and means that $2^{\aleph_0} = g(A) \leq g(\overline{M})$, as promised. □

We finish off with a statement which explores when a direct sum of countably generated modules has a length equal to ω_1, the first uncountable ordinal.

Theorem 3.7. *Let M_i $(i \in I)$ be a system of QTAG-modules, and let $M = \oplus_{i \in I} M_i$ be the direct sum of countably generated modules. Then M is a closed module under natural topology, provided length$(M) = \omega_1$.*

Proof. Let J be the set of countable ordinals, and let $\{x_\alpha\}_{\alpha \in J}$ be a Cauchy sequence. Then for each $i \in I$, one sees that $\{\phi_i(x_\alpha)\}_{\alpha \in J}$ is a Cauchy sequence, where $\phi_i : M \to M_i$ is the projection map. Therefore, for every ordinal γ_i, there will exist $\alpha_i \in J$ such that

$$\phi_i(x_\alpha) - \phi_i(x_\beta) \in M_{\gamma_i} \cap M_{i_{\gamma_i}} = 0$$

and $\gamma_i = \text{length}(M_i)$, for $\alpha, \beta > \alpha_i$.

Let us assume in a way of contradiction that the set $\mathcal{F} = \{i \in I : x_i \neq 0\}$ is not finite. Then there exists a sequence $\{i_k\}_{k \in Z^+}$ in $\mathcal{F}$ such that $\eta = \lim_{k \to \infty}\{\alpha_{i_k}\}$ where η is any countable ordinal. If, however, a countable ordinal $\alpha > \eta$, then $\phi_{i_k}(x_\alpha) = x_{i_k} \neq 0$ for some k, thus contradicting to our choice. So, $\mathcal{F}$ is a finite set. Letting σ be the countable ordinal with a countable ordinal α_σ such that $x_\alpha - x_\beta \in M_\sigma$ for $\alpha, \beta > \alpha_\sigma$. Thus, $\phi_i(x_\alpha) - \phi_i(x_\beta) \in M_\sigma$ for each $i \in I$. In case that $i \in \mathcal{F}$ and $\beta > \alpha_i$, we can observe that $\phi_i(x_\beta) = x_i$, and that $\phi_i(x_\alpha) - x_i \in M_\sigma$ for all $\alpha > \alpha_\sigma$. But in this case, $i \notin \mathcal{F}$ and $\beta > \alpha_i$, we may deduce that $\phi_i(x_\beta) = 0$, and that $\phi_i(x_{\alpha_i}) \in M_\sigma$ for all $\alpha > \alpha_\sigma$. Finally, in the remaining case, it can be inferred that $x_\alpha = \oplus_{i \in \mathcal{F}} x_i \in M_\sigma$ for all $\alpha > \alpha_\sigma$. This surely means that $\lim x_\alpha = \oplus_{i \in \mathcal{F}} x_i \in M$. Consequently, every Cauchy sequence in M converges in M, as formulated. □

4. Conclusion and Open problems

In this project, we examined different types of submodules and Ulm invariants via the notions of h-topology and closed modules. The intriguing properties of these notions and their interrelationships are explored, and some connections are investigated between the σ^{th}-submodules and the closures existing in the literature (see, for example, Theorem 3.1, etc.). We further revealed that a necessary and sufficient condition for a direct sum of $QTAG$-modules to be closed modules can be developed in terms of a direct summands, as detailed in Theorem 3.2. Moreover, we demonstrated that if a direct sum of countably generated modules has a length equal to ω_1, the first uncountable ordinal, then the module is a closed module under the natural topology, which occurred in Theorem 3.7.

In future work, we will study certain invariants by utilizing closed modules and h-topology via $QTAG$-modules. Also, we will generate a new countability property from $QTAG$-modules and other types of submodules in the literature. We close the work with certain challenging problems which are worthwhile for a further study.

Problem 4.1. Find the necessary (and sufficient) conditions under which a direct sum of a closed module is again a closed module?

Problem 4.2. Can closed modules be characterized by certain Ulm invariants?

Problem 4.3. Is it true that every $QTAG$-module of countable length γ with the countability property is isotype?

Problem 4.4. For a $QTAG$-module M of countable type, does it follow that $M \oplus M$ is a closed module?

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References

[1] Ali, M.N., Sharma, V.K., & Hasan, A. (2024). QTAG-modules whose h-pure-S-high submodules have closure. J. Math. Res. Appl., 44(1), 18-24.

[2] Ali, M.N., Sharma, V.K., & Hasan, A. (2024). Closures of high submodules of QTAG-modules. Creat. Math. Inform., 33(2), 129-136.

[3] Benabdallah, K., & Singh, S. (1983). On torsion Abelian groups like modules. Lect. Notes Math., 1006, 639-653.

[4] Breaz, S., & Calugareanu, G. (2006). Self-c-injective Abelian groups. Rend. Sem. Mat. Univ. Padova, 116, 193-203.

[5] Danchev, P.V. (2002). Isomorphism of commutative group algebras of closed p-groups and p-local algebraically compact Abelian groups. Proc. Amer. Math. Soc., 130(7), 1937-1941.

[6] Danchev, P.V. (2003). Quasi-closed primary components in Abelian group rings. Tamkang J. Math., 34(1), 87-92.

[7] Fuchs, L. (1970). Infinite Abelian Groups. Volume I, Pure Appl. Math. 36, Academic Press, New York.

[8] Fuchs, L. (1973). Infinite Abelian Groups. Volume II, Pure Appl. Math. 36, Academic Press, New York.

[9] Hasan, A., & Rafiquddin. (2022). On completeness in QTAG-modules. Palest. J. Math., 11(2), 335-341.

[10] Hasan, A., & Mba, J.C. (2022). On QTAG-modules having all N-high submodules h-pure. Mathematics, 10(19), 3523.

[11] Khan, M.Z. (1979). Modules behaving like torsion Abelian groups. Canad. Math. Bull., 22(4), 449-457.

[12] Mehdi, A., Abbasi, M.Y., & Mehdi, F. (2005). On some structure theorems of QTAG-modules of countable Ulm type. South East Asian J. Math. Math. Sci, 3(3), 103–110.

[13] Mehran, H.A., & Singh, S. (1985). Ulm Kaplansky invariant for TAG-modules. Comm. Algebra, 13(2), 355-373.

[14] Mehran, H.A., & Singh, S. (1986). On σ-pure submodules of QTAG-modules. Arch. Math., 46(6), 501-510.

[15] Singh, S. (1976). Some decomposition theorems in Abelian groups and their generalizations. Ring Theory: Proceedings of Ohio University Conference, Marcel Dekker, New York, 25, 183-189.

[16] Singh, S. (1987). Abelian groups like modules. Act. Math. Hung, 50, 85-95.

[17] Singh, S., & Khan, M.Z. (1998). TAG-modules with complement submodules h-pure. Internat. J. Math. Math. Sci., 21(4), 801-814.

[18] Waller, J.D. (1968). Generalized torsion complete groups. In: Études sur les Groupes abéliens/Studies on Abelian Groups, Springer, Berlin, Heidelberg, 345-456.

[19] Warfield, R.B. (1975). A classification theorem for Abelian p-groups. Trans. Amer. Math. Soc., 210, 149-168.

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STATISTICAL COMPACTNESS OF TOPOLOGICAL SPACES CONFINED BY WEIGHT FUNCTIONS

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ABSTRACT. A mapping of the form $\varrho : \mathbb{N} \longrightarrow [0, \infty)$ satisfying $\lim_{n\to\infty} \varrho(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{\varrho(n)} \neq 0$ is called a weight function. By incorporating weight functions into the statistical framework, we come up with a new notion called weighted statistical compactness that extends the traditional notion of compactness. The paper involves studying the compactness properties via sequences and relationship between compactness variations. We also look into the nature of weighted statistical compactness with in sub-space and under open continuous onto maps. Weighted statistical compactness has also been given a finite intersection-like characterization.

Keywords: Countable compactness, s-compactness, Asymptotic density, Weighted density, Finite intersection property.

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1. Introduction

A group of points in a topological space are called dense when they are widely distributed throughout the space. The distances between the points are often used to calculate this density in a metric space. To determine the natural density (also known as asymptotic density) of a subset $A \subseteq \mathbb{N}$, one can measure how closely spaced out the points in A are in $\mathbb{N}$, $\mathbb{N}$ being all natural numbers set. It is described as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in A \subseteq \mathbb{N}\}|.$$

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H. Fast [11], Schoenberg [13] expanded the sense of conventional convergence to statistical convergence by utilizing the concept of asymptotic density. In a space X, a sequence $\{z_n\}$ converges statistically to a point z if the natural density of the collection $\{n \in \mathbb{N} : z_n \in U\}$ (i.e., the part of the sequence's elements that fall within U) converges to 1 as n tends to infinity for every open set U containing z. i.e., $\delta(\{n \in \mathbb{N} : z_n \in U\}) = 1$ or equivalently, $\delta(\{n \in \mathbb{N} : z_n \notin U\}) = 0$ [12]. In 2012, Bhunia et al. [9] strengthened the idea that real sequences s-converge by using asymptotic density of order α, where $0 < \alpha < 1$. s^α-convergence restricts the notion of statistical convergence in topology. It introduces a parameter α that is important in characterizing the specific convergence behavior of sequences. Here, α represents a parameter influencing the convergence rate of sub-sequences, providing a more nuanced understanding of convergence than is achievable with traditional reasoning. A sequence $\{y_n\}$, s^α-converges to a point y in a space X if each open set U that contains y, produces

$$\delta^\alpha(\{n \in \mathbb{N} : x_n \notin U\}) = \lim_{n\to\infty} \frac{|\{n \in \mathbb{N} : x_n \notin U\}|}{n^\alpha} = 0.$$

Compactness and other covering features have been a very interesting topic for many mathematicians [4, 5, 6, 7] for a long period of time. Compactness in a topological space is a fundamental property that encapsulates what it means to be 'finite' in a general sense. Compact topological spaces are those that have a finite sub-cover for every open cover. Stated differently, regardless matter how we choose to cover the space, there is always a finite number of open sets that cover the entire space. Compactness has many implications and applications in the mathematical domains of analysis, geometry, and topology, to name a few. The notions of boundedness and finiteness are naturally extended from metric spaces to more general topological spaces. Other types of compactness, such as sequentially compact space, pseudo-compact space, and St-compact space, s^α-compact space [2, 3, 8] have been studied by many authors.

A mapping ϱ defined in the form $\varrho : \mathbb{N} \longrightarrow [0, \infty)$ such that $\lim_{n\to\infty} \varrho(n) = \infty$ and $\lim_{n\to\infty} \frac{n}{\varrho(n)} \neq 0$ is called a weight function [1]. For example, $h : \mathbb{N} \longrightarrow [0, \infty)$ such that $h(n) = n^\alpha$, where $0 < \alpha \leq 1$, $\varrho : \mathbb{N} \longrightarrow [0, \infty)$ such that $\varrho(n) = log(1 + n)$ are weight functions. Adem et al. [1] studied the concept of weighted convergence for real sequences. A sequence $\{y_n\}$ is said to s_ϱ-converge to the point y in a space X if each neighborhood U of y produces

$$\delta_\varrho(\{n \in \mathbb{N} : x_n \notin U\}) = \lim_{n\to\infty} \frac{|\{n \in \mathbb{N} : x_n \notin U\}|}{\varrho(n)} = 0.$$

In this paper, we continue our study of the weighted density in order to identify a topological property related to compactness.

2. Preliminaries

This part provides a quick overview of the basic tools and mathematical ideas needed to understand the main conclusions. Unless otherwise indicated, this paper does not presuppose any separation axioms; a space will refer to a topological space in this paper. We refer to [10] for further concepts and symbols.

Definition 2.1. [10] *If each open covering of a space X posses a finite sub-cover, then that space is said to be countably compact.*

Definition 2.2. [10] *A countable family $\mathcal{F} = \{F_s\}_{s\in\mathbb{N}}$ whose elements are subsets of a set X is stated to have finite intersection property (FIP), if $\bigcap_{i=1}^{n} F_{s_i} \neq \emptyset$ and $\mathcal{F} \neq \emptyset$ for every finite set $\{s_1, s_2, s_3, \ldots, s_n\} \subseteq \mathbb{N}$.*

Theorem 2.1. [10] *Every collection of closed subsets of a space X having FIP produces non-empty intersection if and only if X is compact.*

Definition 2.3. [8] *A statistical compact (or s-compact) space is a topological space X in which every countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ posses a sub-cover $\mathcal{V} = \{U_{m_k} : k \in \mathbb{N}\}$ for which $\delta\left(\{m_k : U_{m_k} \in \mathcal{V}\}\right) = 0$.*

Definition 2.4. [8] *A countable family $\mathcal{F} = \{F_s\}_{s\in\mathbb{N}}$ whose elements are subsets of a set X is stated to have δ_r-intersection property if $\bigcap_{n\in S} F_n \neq \emptyset$ for every subset $S \subseteq \mathbb{N}$ with $\delta(S) = r$ and $\mathcal{F} \neq \emptyset$.*

Theorem 2.2. [8] *Every collection of closed subsets of a space X having δ_0-intersection property produces non-empty intersection if and only if X is s-compact.*

3. s_ϱ-Compact space

Using the concept of weighted density, we want to find a covering criteria that lies somewhere between countable compactness and statistical compactness.

Definition 3.1. *Let X be a space with the weight function ϱ. If every countable open covering $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ of X posses a sub cover $\mathcal{Q} = \{P_{n_k} : k \in \mathbb{N}\}$ for which $\delta_\varrho(\{n_k \in \mathbb{N} : P_{n_k} \in \mathcal{Q}\}) = 0$, then X is stated as a weighted statistical compact space (or shortly s_ϱ-compact space).*

Theorem 3.1. *Every countably compact space is an s_ϱ-compact space.*

Proof. Let X be a countably compact space. Then, every countable open cover $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ has a finite sub cover $\mathcal{Q} = \{P_{n_1}, P_{n_2}, P_{n_3}, \ldots, P_{n_k}\}$. As the sub cover is a finite, the set $\{n_1, n_2, \cdots, n_k\} \subseteq \mathbb{N}$ is finite having weighted density zero. Hence, the space X is an s_ϱ-compact space. □

Example 3.1. *There exists a non-compact, s_ϱ-compact space.*

Let $X = \{(a, b) : a^2 + b^2 < 1\}$ and $\tau = \{A_n = \{(a, b) : a^2 + b^2 < 1 - \frac{1}{n}\} : n \in \mathbb{N}\} \cup \{\emptyset, X\}$. Clearly, X is a topological space. Consider the weight function $\varrho : \mathbb{N} \to [0, \infty)$ such that $\varrho(n) = log(1 + n)$. For an arbitrary countable open cover $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$, we can choose a sub sequence $\mathcal{V} = \{U_{n_k} : k \in \mathbb{N}\}$ such that $U_{n_1} = U_1$ and $U_{n_{k+1}} \supseteq U_{n_k}$ for all $k \in \mathbb{N}$. So $\mathcal{V}$ is an increasing sub sequence of $\mathcal{U}$ that covers X. Now, we choose a sub sequence $\mathcal{W}$ of $\mathcal{V}$ as $\mathcal{W} = \{U_{n_{k^k}} : k \in \mathbb{N}\}$. It is clear that $\bigcup \mathcal{W} = X$ and

$$\delta_\varrho(\{n_k \in \mathbb{N} : U_{n_k} \in \mathcal{W}\}) = \lim_{k\to\infty} \frac{|\{k^k : k \in \mathbb{N}\}|}{log(1 + k^k)}$$

$$= \lim_{k\to\infty} \frac{k}{log(1 + k^k)} = \lim_{k\to\infty} \frac{1 + k^k}{k^k(1 + logk)} = 0.$$

But $\mathcal{W}$ is a sub sequence of $\mathcal{V}$ and $\mathcal{V}$ is a sub sequence of $\mathcal{U}$. So, $\mathcal{W}$ is a sub cover of $\mathcal{U}$ such that $\delta_\varrho(\{n_k \in \mathbb{N} : U_{n_k} \in \mathcal{W}\}) = 0$. Thus, X is an s_ϱ compact space.

Now, consider the open cover $\mathcal{A} = \{A_n = \{(a, b) : a^2 + b^2 < 1 - \frac{1}{n}\} : n \in \mathbb{N}\}$ and if possible suppose that it has a finite sub cover $\mathcal{A}' = \{A_{n_1}, A_{n_2}, A_{n_3}, \ldots, A_{n_k}\}$. We take $n_{max} = max\{n_1, n_2, n_3, \ldots, n_k\}$. Therefore, $\bigcup \mathcal{A}' = A_{n_{max}} = \{(a, b) : a^2 + b^2 < (1 - \frac{1}{n_{max}})\}$. The portion $\{(a, b) : (1 - \frac{1}{n_{max}}) \le a^2 + b^2 < 1\}$ remains uncovered, which is a contradiction. So, $\mathcal{A}$ can not have a finite sub cover. Thus, (X, τ) is not compact.

Theorem 3.2. *Every s_ϱ-compact space is an s-compact space.*

Proof. Let X be a space having s_ϱ-compactness. Then, every countable open covering $\mathcal{P} = \{P_n : n \in \mathbb{N}\}$ of X posses a sub cover $\mathcal{Q} = \{P_{n_k} : k \in \mathbb{N}\}$ such that $\delta_\varrho(\{n_k \in \mathbb{N} : P_{n_k} \in \mathcal{Q}\}) = 0$. But $\delta(\{n_k \in \mathbb{N} : P_{n_k} \in \mathcal{Q}\}) \le \delta_\varrho(\{n_k \in \mathbb{N} : P_{n_k} \in \mathcal{Q}\}) = 0$. So, X is a statistical compact space. Hence, every s_ϱ-compact space is an s-compact space. □

Open Problem 3.3. *Does there exists a topological space which is statistical compact but not s_ϱ-compact?*

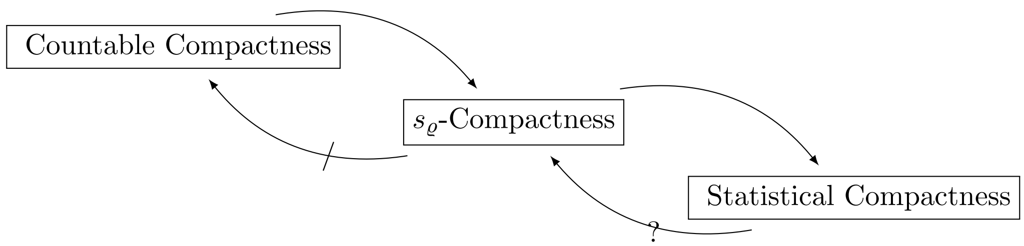

Figure 1. The relationship between compactness variations.

Theorem 3.4. *s_ϱ-compactness is a closed hereditary property.*

Proof. Let (X, τ) be a s_ϱ-compact topological space and (B, τ_B) be a closed sub-space of X and let $\mathcal{U} = \{U_n \in \tau_B : n \in \mathbb{N}\}$ be a covering of (B, τ_B).

$$\text{Therefore, } B = \bigcup_{n\in\mathbb{N}} U_n = \bigcup \mathcal{U}.$$

Now, for every $n \in \mathbb{N}$, we can find a τ-open set V_n for which $U_n = B \cap V_n$.

$$\text{Therefore, } B = \bigcup_{n\in\mathbb{N}} U_n \subseteq \bigcup_{n\in\mathbb{N}} V_n.$$

Consider $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$, where

$$W_n = \begin{cases} X \setminus B & \text{if } n = 1, \\ V_{n-1} & \text{otherwise,} \end{cases}$$

So, $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ is a countably infinite cover of the s_ϱ-compact space X. So, we can find a sub cover $\mathcal{P} = \{W_{n_k} : k \in \mathbb{N}\}$ having $\delta_\varrho(\{n_k : W_{n_k} \in \mathcal{P}\}) = 0$. Let $\mathcal{P}_B = \{B \cap W_{n_k} : k \in \mathbb{N}\}$, then $\mathcal{U}$ have a sub cover $\mathcal{P}_B$ covering B. Now, if $W_1 \notin \mathcal{P}$, then $\{n_k : W_{n_k} \in \mathcal{P}\} = \{n_k : B \cap W_{n_k} \in \mathcal{P}_B\}$ and $\delta_\varrho(\{n_k : W_{n_k} \in \mathcal{P}\}) = \delta_\varrho(\{n_k : B \cap W_{n_k} \in \mathcal{P}_B\}) = 0$. If $W_1 \in \mathcal{P}$, then $|\{n_k : W_{n_k} \in \mathcal{P}\}| = |\{n_k : B \cap W_{n_k} \in \mathcal{P}_B\}| - 1$.

$$\text{So, } \delta_\varrho(\{n_k : W_{n_k} \in \mathcal{P}\}) = \delta_\varrho(\{n_k : B \cap W_{n_k} \in \mathcal{P}_B\}) = 0.$$

Hence, (B, τ_B) is an s_ϱ-compact space.

□

Theorem 3.5. *If $B \subseteq X$ and (B, τ_B) is an s_ϱ-compact closed sub-space of a topological space X, then for every family of open sets $\{W_n : n \in \mathbb{N}\}$ of X such that $B \subseteq \bigcup W_n$ for all $n \in \mathbb{N}$ there exists a subset $P \subseteq \mathbb{N}$ with $\delta_\varrho(P) = 0$ such that $B \subseteq \bigcup_{n\in P} W_n$.*

Proof. Let $\{W_n : n \in \mathbb{N}\}$ be a family of open subsets of X ensuring $\bigcup_{n\in\mathbb{N}} W_n \supseteq B$. Then, $B = \bigcup_{n\in\mathbb{N}}(B \cap W_n)$, which implies that B is covered by $\{B \cap W_n : n \in \mathbb{N}\}$, a collection of τ_B-open sets. Also, (B, τ_B) is an s_ϱ-compact space. Therefore, there exists a set $P \subseteq \mathbb{N}$ with $\delta_\varrho(P) = 0$ such that $B = \bigcup_{n\in P}(B \cap W_n)$. Thus, $B \subseteq \bigcup_{n\in P} W_n$. Hence, for every family of open sets $\{W_n : n \in \mathbb{N}\}$ of X such that $B \subseteq \bigcup W_n$ for all $n \in \mathbb{N}$ there exists a subset $P \subseteq \mathbb{N}$ with $\delta_\varrho(P) = 0$ such that $B \subseteq \bigcup_{n\in P} W_n$. □

Theorem 3.6. *Let W be an open subset of a topological space X and consider the weight function ϱ. If a family $\{G_m : m \in \mathbb{N}\}$ of closed subsets of X consists at least one s_ϱ-compact set (say G_{m_0}) such that $\bigcap_{m\in\mathbb{N}} G_m \subseteq W$, then there exists $P \subseteq \mathbb{N}$ with $\delta_\varrho(P) = 0$ and $\bigcap_{m\in P} G_m \subseteq W \cup G^c_{m_0}$.*

Proof. Let G_{m_0} be an s_ϱ-compact set in the family $\{G_m : m \in \mathbb{N}\}$. As $W \in \tau$ is open, so W^c is closed. Thus, $W^c \cap G_{m_0} = G_{m_0} \setminus W$. Since $G_{m_0} \setminus W \subseteq G_{m_0}$ and G_{m_0} is an s_ϱ-compact set so by Theorem 3.4, $G_{m_0} \setminus W$ is an s_ϱ-compact set. Let $B = G_{m_0} \setminus W$. $\{W_m = X \setminus G_m : m \in \mathbb{N}\}$ is family of open sets.

$$\text{Now, } \bigcup_{m\in\mathbb{N}} W_m = \bigcup_{m\in\mathbb{N}} (X \setminus G_m) = X \setminus \bigcap_{m\in\mathbb{N}} G_m.$$
$$\text{and } X \setminus \bigcap_{m\in\mathbb{N}} G_m \supseteq X \setminus W \supseteq G_{m_0} \setminus W = B.$$

So, $B \subseteq \bigcup_{m\in\mathbb{N}} W_m$. But, $B = G_{m_0} \setminus W$ is an s_ϱ-compact space. Therefore, by Theorem 3.5, there exists $P \subseteq \mathbb{N}$ with $\delta_\varrho(P) = 0$ such that $B \subseteq \bigcup_{m\in P} W_m$. So, $G_{m_0} \setminus W = (X \setminus W) \cap G_{m_0} \subseteq \bigcup_{m\in P} W_m$.

$$\text{Thus, } (X \setminus W) \cap (X \setminus G^c_{m_0}) \subseteq \bigcup_{m\in P} X \setminus G_m = X \setminus \bigcap_{m\in P} G_m.$$
$$\text{Therefore, } X \setminus (W \cup G^c_{m_0}) \subseteq X \setminus \bigcap_{m\in P} G_m.$$

Hence, $\bigcap_{m\in P} G_m \subseteq W \cup G^c_{m_0}$. □

Theorem 3.7. *Let $\{(X_m, \tau_m) : m = 1, 2, \cdots, s\}$ be a finite collection of s_ϱ-compact subspaces of X such that $X = \bigcup_{m=1}^{s} X_m$. Then, X is an s_ϱ-compact space.*

Proof. Let (X_m, τ_m) be an s_ϱ-compact sub-space of X for $m = 1, 2, 3, \ldots, s$ such that $X = \bigcup_{m=1}^{s} X_m$ and let $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ be a countable open cover of X. Then, $\mathcal{W}_m = \{X_m \cap W_n : n \in \mathbb{N}\}$ are countable open covers of (X_m, τ_m) where $m = 1, 2, \cdots, s$. Therefore, there exist $\mathcal{V}_m = \{X_m \cap W_{n_k} : k \in \mathbb{N}\}$ for every $\mathcal{W}_m$ of (X_m, τ_m) such that $\delta_\varrho(\{n_k \in \mathbb{N} : X_m \cap W_{n_k} \in \mathcal{V}_m\}) = 0$.

Now,

$$\delta_\varrho(\bigcup_{m=1}^{s} \{n_k \in \mathbb{N} : X_m \cap W_{n_k} \in \mathcal{V}_m\}) \leq \sum_{m=1}^{s} \delta_\varrho(\{n_k \in \mathbb{N} : X_m \cap W_{n_k} \in \mathcal{V}_m) = 0.$$

Also, $\bigcup_{m=1}^{s} \mathcal{V}_m$ are covers of X. So, $X \subseteq \bigcup_{m=1}^{s} \mathcal{V}_m \subseteq \bigcup_{m=1}^{s}\{W_{n_k} : k \in \mathbb{N}$ and $X_m \cap W_{n_k} \in \mathcal{V}_m\} = \mathcal{P}$. Thus, $\mathcal{P}$ is a sub cover of $\mathcal{W}$ such that $\delta_\varrho(\{n_k : W_{n_k} \in \mathcal{P}\}) = 0$. Hence, X is an s_ϱ-compact space. □

Theorem 3.8. *s_ϱ-compactness is preserved under surjective open continuous mapping.*

Proof. Consider a surjective map $f : (X, \tau) \longrightarrow (Y, \sigma)$ which is both open and continuous, where X is an s_ϱ-compact space, ϱ being a weight function.

Suppose that $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$ is a random covering of Y, $\mathcal{A}$ being countable and elements of $\mathcal{A}$ being open. Then, $Y = \bigcup_{n \in \mathbb{N}} A_n$. So, $f^{-1}(Y) = f^{-1}(\bigcup_{n \in \mathbb{N}} A_n)$. Thus, $X = \bigcup_{n \in \mathbb{N}} f^{-1}(A_n)$. So, X is covered by $\{f^{-1}(A_n) : n \in \mathbb{N}\}$, which is an open covering (f being continuous and $\mathcal{A} = \{A_n\}_{n \in \mathbb{N}}$ being open). Also, X is an s_ϱ-compact space. Therefore, we will get a countable sub covering $\{f^{-1}(A_{n_k})\}_{k \in \mathbb{N}}$ of X having $\delta_\varrho(\{n_k : k \in \mathbb{N}\}) = 0$. Therefore, $\bigcup_{n_k \in \mathbb{N}}\{f^{-1}(A_{n_k})\} = X$ that implies $f(\bigcup_{n_k \in \mathbb{N}}\{f^{-1}(A_{n_k})\}) = f(X) = Y$. i.e., $Y = \bigcup_{n_k \in \mathbb{N}}\{A_{n_k}\}$. Thus, Y is covered by $\{A_{n_k}\}_{k \in \mathbb{N}}$ which is a subset of $\{A_n\}_{n \in \mathbb{N}}$ with $\delta_\varrho(\{n_k : k \in \mathbb{N}\}) = 0$.

Hence, Y also have the s_ϱ-compactness property. □

Under the effect of weighted statistical density, now we search for a finite intersection like attributes for s_ϱ compactness.

Definition 3.2. *A countable family $\mathcal{D} = \{D_n : n \in \mathbb{N}\} \subseteq \mathcal{P}(X)$ is stated to posses ${}_\varrho\Delta_r$-intersection property (${}_\varrho\Delta_r$-IP) if $\mathcal{D} \neq \emptyset$ and for each $P \subseteq \mathbb{N}$ having $\delta_\varrho(P) = r$, gives $\bigcap_{n \in P} D_n \neq \emptyset$.*

Theorem 3.9. *Every countable collection of closed subsets of X having ${}_\varrho\Delta_0$-IP produces non-empty intersection if the space X is s_ϱ-compact and vice versa.*

Proof. Let X be a space having s_ϱ-compactness and $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ be an arbitrary family of closed subsets of X with ${}_\varrho\Delta_0$-intersection property .

If possible, let us consider $\bigcap_{n \in \mathbb{N}} D_n = \emptyset$ and $F_n = X \setminus D_n$. Then, $X = X \setminus \bigcap_{n \in \mathbb{N}} D_n =$

$\bigcup_{n\in\mathbb{N}} X \setminus D_n = \bigcup_{n\in\mathbb{N}} F_n$. Therefore, X is covered by $\{F_n\}_{n\in\mathbb{N}}$, elements of the collection $\{F_n\}_{n\in\mathbb{N}}$ being open sets. But X is an s_ϱ-compact space. Thus, we can obtain a $P \subseteq \mathbb{N}$ having $\delta_\varrho(P) = 0$ and $\bigcup_{n\in P} F_n = X$. Now,

$$X = \bigcup_{n\in P} F_n = \bigcup_{n\in P} X \setminus D_n = X \setminus \bigcap_{n\in P} D_n.$$

Therefore, $\bigcap_{n\in P} D_n = \emptyset$, that leads to a contradiction. So, $\mathcal{D} = \{D_n : n \in \mathbb{N}\}$ has ${}_g\Delta_0$-intersection property.

Conversely, let $\mathcal{W} = \{W_n : n \in \mathbb{N}\}$ be a countable open cover of X. Now, $\mathcal{D} = \{D_n = X \setminus W_n : n \in \mathbb{N}\}$ is a countable family of closed sets. Now,

$$\bigcap_{n\in\mathbb{N}} D_n = \bigcap_{n\in\mathbb{N}} (X \setminus W_n) = X \setminus \bigcup_{n\in\mathbb{N}} W_n = \emptyset.$$

By contra positive process of our assumption it does not have ${}_\varrho\Delta_0$-intersection property. Therefore, there exists $P \subseteq \mathbb{N}$ with $\delta_\varrho(P) = 0$ and $\bigcap_{n\in P} D_n = \emptyset$. So, $\bigcap_{n\in P}(X \setminus W_n) = \emptyset$ i.e., $X \setminus \bigcup_{n\in P} W_n = \emptyset$. Thus, $X = \bigcup_{n\in P} W_n$.

Hence, X is an s_ϱ-compact space. □

4. Conclusion remarks

s_ϱ compact space serves as an intermediate between countable compactness and statistical compactness. This compactness property is preserved under closed sub-space and open continuous surjection. s_ϱ compactness can be characterized in terms of families of closed sets by means of ${}_\varrho\Delta_0$ intersection property.

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References

[1] Adem, A. A., & Altinok, M. (2020). Weighted statistical convergence of real valued sequences. Facta Universitatis Ser. Math. Info., 35(3), 887-898.

[2] Bal, P., & Kočinac, L. D. R. (2020). On selectively star-ccc spaces. Topology Appl., 281, Article No. 107184.

[3] Bal, P., & De, R. (2023). On strongly star semi-compactness of topological spaces. Khayyam J. Math., 9, 54-60.

[4] Bal, P., & Bhowmik, S. (2017). On R-Star-Lindelöf Spaces. Palest. J. Math., 6(2), 480-486.

[5] Bal, P., Bhowmik, S., & Gauld, D. (2018). On Selectively Star-Lindelöf Properties. J. Indian Math. Soc., 85(3-4), 291-304.

[6] Bal, P., & Rakshit, D. (2023). A Variation of the Class of Statistical γ-Covers. Topol. Algebra Appl., 11, Article No. 20230101.

[7] Bal, P, & Sarkar, S. (2023). On strongly star g-compactness of topological spaces. Tatra Mt. Math. Publ, 85, 89-100.

[8] Bal, P., Sarkar, S., & Rakshit, D. (2024). On Statistical Compactness. Iranian J. Math. Sc. Info., (to appear).

[9] Bhunia, S., Das, P., & Pal, S. K. (2012). Restricting statistical convergence. Acta Math. Hungar., 134 (1-2), 153-161.

[10] Engelking, R. (1989). General Topology. Sigma Series in Pure Mathematics, Revised and complete ed., Berlin: Heldermann.

[11] Fast, H. (1951). Sur la convergence statistique. Colloq. Math., 2, 241-244.

[12] Maio, G. D., & Kočinac, L. D. R. (2008). Statistical convergence in topology. Topology Appl., 156, 28-45.

[13] Schoenberg, I. J. (1959). Integrability of certain functions and related summability methods. Amer. Math. Monthly, 66, 361-375.

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KENMOTSU MANIFOLDS COUPLED WITH η-ρ-EINSTEIN SOLITONS ADMITTING AN EXTENDED $\mathcal{M}$-PROJECTIVE CURVATURE TENSOR

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Abstract. The object of the present paper is to study some curvature conditions on Kenmotsu manifolds. Initially, we analyze the condition ξ-$\mathcal{M}^e$ projective flat and φ-$\mathcal{M}^e$ semi-symmetric on Kenmotsu manifolds coupled with an η-ρ-Einstein soliton. Subsequently, we elaborate the conditions $\mathcal{M}^e \cdot \mathcal{R}=0$, $\mathcal{M}^e \cdot \mathcal{M}^e=0$ and $\mathcal{M}^e \cdot \mathcal{Q}=0$ on Kenmotsu manifolds in view of an η-ρ-Einstein soliton, where $\mathcal{M}^e$ is the extended $\mathcal{M}$-projective curvature tensor. In addition, we verify the results with a concrete example.

Keywords: Kenmotsu manifolds, Extended $\mathcal{M}$-projective curvature tensor $\mathcal{M}^e$, ξ-$\mathcal{M}^e$ projectively flat, φ-$\mathcal{M}^e$ semi symmetric, η-ρ-Einstein soliton and η-Einstein manifolds.
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1. Introduction

The product of an almost contact manifold $\mathbb{M}$ and the real line $\mathbb{R}$ carries a natural almost complex structure. However if one takes $\mathbb{M}$ to be an almost contact metric manifold and supposes that the product metric $\mathbb{G}$ on $\mathbb{M} \times \mathbb{R}$ is Kaehlerian, then the structure on $\mathbb{M}$ is cosymplectic [15] and not Sasakian. On the other hand Oubina [18] pointed out that if the conformally related metric $e^{2t}\mathbb{G}, t$ being the coordinate on $\mathbb{R}$, is Kaehlerian, then $\mathbb{M}$ is Sasakian and conversely. In [22], S. Tanno classified connected almost contact metric

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manifolds whose automorphism groups possess the maximum dimension. For such a manifold $\mathbb{M}$, the sectional curvature of plane sections containing ξ is a constant, say c. If $c > 0$, $\mathbb{M}$ is a homogeneous Sasakian manifold of constant sectional curvature. If $c = 0$, $\mathbb{M}$ is the product of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature. If $c < 0$, $\mathbb{M}$ is a warped product space $\mathbb{R} \times_f \mathcal{C}^n$. In 1972, Kenmotsu studied a class of contact Riemannian manifolds that satisfy specific conditions [17]. We call it Kenmotsu manifold. If a Kenmotsu manifold satisfies the condition $\mathcal{R}(X,Y)\cdot\mathcal{R}$=0, it must have a constant curvature of -1, where $\mathcal{R}$ denotes the Riemannian curvature tensor and $\mathcal{R}(X,Y)$ refers to the tensor algebra derivation at each point in the tangent vectors X,Y. Kenmotsu manifolds have been studied by many authors such as (see, [3], [4], [20], [12], [19], [8], [9], [21], [13],[11],[10],[24],[31]) and many others. The metric g on $(\mathbb{M}, g)$ is called a ρ-Einstein soliton if there is a smooth vector field $\mathbb{V}$ such that [2]:

$$\mathcal{S} + \frac{1}{2}\mathcal{L}_{\mathbb{V}}g = (\gamma_1 + \rho r)g, \tag{1.1}$$

where $\mathcal{L}_{\mathbb{V}}$ and r denote the Lie derivative and Ricci scalar respectively, where $\rho \neq 0$, $\gamma_1 \in \mathbb{R}$. As usual ρ-Einstein soliton is steady for γ_1=0, shrinking for $\gamma_1 > 0$ and expanding for $\gamma_1 < 0$. A new type of soliton called η-ρ -Einstein soliton which is a generalization of ρ-Einstein soliton given by

$$\mathcal{S} + \frac{1}{2}\mathcal{L}_{\mathbb{V}}g = (\gamma_1 + \rho r)g + \gamma_2\eta \otimes \eta, \tag{1.2}$$

where γ_1, $\gamma_2 \in \mathbb{R}$. Analogous to equation (1.2), we recall η-ρ-Einstein soliton and so equation (1.2) takes the form

$$\mathcal{S} + Hess(\psi) = (\gamma_1 + \rho r)g + \gamma_2\eta \otimes \eta. \tag{1.3}$$

As η-ρ-Einstein soliton (or gradient η-ρ-Einstein soliton) can be classified as (i) ρ-Einstein soliton (or gradient ρ-Einstein soliton) [2] if γ_2=0, (ii) η-Einstein soliton (or gradient η-Einstein soliton) [14] if ρ=$\frac{1}{2}$, (iii) η-traceless Ricci soliton (or gradient η-traceless Ricci soliton) if ρ=$\frac{1}{2n+1}$, (iv) η-Schouten soliton (or gradient η-Schouten soliton) [23] if ρ=$\frac{1}{4n}$. In this sequel many authors have been studied Kenmostu manifold with reference to different type of solitons (see,[5], [6], [7], [27], [26], [29],[28],[32],[30]) and many others.

More specific, the Lie derivative $(\mathcal{L}_\xi g)(H_1, H_2)$ given by

$$(\mathcal{L}_\xi g)(H_1, H_2) = g(\nabla_{H_1}\xi, H_2) + g(H_1, \nabla_{H_2}\xi). \tag{1.4}$$

The work of the paper is organized as follows: After the introduction, in section 2, we carried out the basic exposition on Kenmotsu manifold. In section 3, we analyze ξ-$\mathcal{M}^e$

projectively flat Kenmotsu manifold and deduce the interesting result coupled with an η-ρ-Einstein soliton. In section 4 we take up φ-$\mathcal{M}^e$ semi-symmetric in Kenmotsu manifold admitting an η-ρ-Einstein soliton. Again in section 5, 6 and 7 we discuss the some curvature conditions namely, $\mathcal{M}^e \cdot \mathcal{R}$=0, $\mathcal{M}^e \cdot \mathcal{M}^e$=0 and $\mathcal{M}^e \cdot \mathcal{Q}$=0 on such manifold and we verifies the results by suitable example. The conclusion of the work is given in the last section 8.

2. Preliminaries

Let $(\mathbb{M}^{2n+1}, \varphi, \xi, \eta, g)$ be an $(2n+1)$-dimensional almost contact metric manifold, where φ is a $(1,1)$-tensor field, ξ is the structure vector field, η is a 1-form and g is the Riemannian metric. It is well known that the (φ, ξ, η, g) structure satisfies the conditions [1]:

$$\varphi^2 H_1 = -H_1 + \eta(H_1)\xi,\ \eta(\xi) = 1,\ \varphi\xi = 0, \tag{2.5}$$

$$g(H_1, \xi) = \eta(H_1),\ \eta(\varphi H_1) = 0, \tag{2.6}$$

$$g(\varphi H_1, \varphi H_2) = g(H_1, H_2) - \eta(H_1)\eta(H_2), \tag{2.7}$$

$$g(\varphi H_1, H_2) = -g(H_1, \varphi H_2), \tag{2.8}$$

for any $H_1, H_2 \in \chi(\mathbb{M})$. If moreover

$$(\nabla_{H_1}\varphi)H_2 = g(\varphi H_1, H_2)\xi - \eta(H_2)\varphi H_1, \tag{2.9}$$

$$\nabla_{H_1}\xi = H_1 - \eta(H_1)\xi, \tag{2.10}$$

where ∇ denotes the Levi-Civita connection on $(\mathbb{M}^{2n+1}, g)$, then $(\mathbb{M}^{2n+1}, \varphi, \xi, \eta, g)$ is called a Kenmotsu manifold. In this case, it is well known that [17]:

$$\mathcal{R}(H_1, H_2)H_3 = g(H_1, H_3)H_2 - g(H_2, H_3)H_1, \tag{2.11}$$

$$\mathcal{R}(H_1, H_2)\xi = \eta(H_1)H_2 - \eta(H_2)H_1, \tag{2.12}$$

$$\mathcal{R}(H_1, \xi)H_3 = g(H_1, H_3)\xi - \eta(H_3)H_1, \tag{2.13}$$

$$\mathcal{R}(\xi, H_2)H_3 = \eta(H_3)H_2 - g(H_2, H_3)\xi, \tag{2.14}$$

$$\mathcal{S}(\varphi X, \varphi Y) = \mathcal{S}(X, Y) + 2n\eta(X)\eta(Y), \tag{2.15}$$

$$\mathcal{S}(H_1, \xi) = -2n\eta(H_1), \tag{2.16}$$

$$\mathcal{S}(\xi, \xi) = -2n, \tag{2.17}$$

$$\mathcal{Q}\xi = -2n\xi, \tag{2.18}$$

$\forall\ H_1, H_1 \in \chi(\mathbb{M})$. According to [25], the $\mathcal{M}$-projective curvature tensor and the extended $\mathcal{M}$-projective curvature tensor $\mathcal{M}^e$ on $(\mathbb{M}^{2n+1}, g)$ are defined by

$$\begin{aligned} \mathcal{M}(H_1, H_2)H_3 &= \mathcal{R}(H_1, H_2)H_3 - \frac{1}{4n}[\mathcal{S}(H_2, H_3)H_1 - \mathcal{S}(H_1, H_3)H_2 \\ &+ g(H_2, H_3)\mathcal{Q}H_1 - g(H_1, H_3)\mathcal{Q}H_2], \end{aligned} \tag{2.19}$$

$$\begin{aligned} \mathcal{M}^e(H_1, H_2)H_3 &= \mathcal{M}(H_1, H_2)H_3 - \eta(H_1)\mathcal{M}(\xi, H_2)H_3 \\ &- \eta(H_2)\mathcal{M}(H_1, \xi)H_3 - \eta(H_3)\mathcal{M}(H_1, H_2)\xi, \end{aligned} \tag{2.20}$$

for any $H_1,\ H_2,\ H_3 \in \chi(\mathbb{M})$. Now using the Eq. (2.12), (2.13),(2.14),(2.16), (2.17) and (2.18) we get from (2.19) that

$$\begin{aligned} \mathcal{M}(H_1, H_2)\xi &= \eta(H_1)H_2 - \eta(H_2)H_1 \\ &- \frac{1}{4n}[2n\eta(H_1)H_2 - 2n\eta(H_2)H_1 + \eta(H_2)\mathcal{Q}H_1 \\ &- \eta(H_1)\mathcal{Q}H_2], \end{aligned} \tag{2.21}$$

$$\begin{aligned} \mathcal{M}(\xi, H_2)H_3 &= \eta(H_3)H_2 - g(H_2, H_3)\xi \\ &- \frac{1}{4n}[\mathcal{S}(H_2, H_3)\xi + 2n\eta(H_3)H_2 - 2ng(H_2, H_3)\xi \\ &- \eta(H_3)\mathcal{Q}H_2], \end{aligned} \tag{2.22}$$

$$\begin{aligned} \mathcal{M}(H_1, \xi)H_3 &= g(H_1, H_3)\xi - \eta(H_3)H_1 \\ &- \frac{1}{4n}[-2n\eta(H_3)H_1 - \mathcal{S}(H_1, H_3)\xi + \eta(H_3)\mathcal{Q}H_1 \\ &+ 2ng(H_1, H_3)\xi]. \end{aligned} \tag{2.23}$$

Also, taking H_3=ξ in (2.20) we yield

$$\mathcal{M}^e(H_1, H_2)\xi = -\eta(H_1)\mathcal{M}(\xi, H_2)\xi - \eta(H_2)\mathcal{M}(H_1, \xi)\xi. \tag{2.24}$$

For fix H_1=ξ in (2.21) along with (2.5) and (2.18), we get

$$\mathcal{M}(\xi, H_2)\xi = \frac{1}{2}H_2 + \frac{1}{4n}\mathcal{Q}H_2. \tag{2.25}$$

Again by substituting H_2=ξ in (2.21) and using (2.5) and (2.18), we have

$$\mathcal{M}(H_1, \xi)\xi = -\frac{1}{2}H_1 - \frac{1}{4n}\mathcal{Q}H_1. \tag{2.26}$$

Using (2.25) and (2.26) in (2.24), we obtain

$$\mathcal{M}^e(H_1, H_2)\xi = -\frac{1}{2}\eta(H_1)H_2 - \frac{1}{4n}\eta(H_1)\mathcal{Q}H_2 + \frac{1}{2}\eta(H_2)H_1 + \frac{1}{4n}\eta(H_2)\mathcal{Q}H_1. \tag{2.27}$$

Taking H_1=ξ in (2.27) and using (2.5) and (2.18), we get

$$\mathcal{M}^e(\xi, H_2)\xi = -\frac{1}{2}H_2 - \frac{1}{4n}\mathcal{Q}H_2. \tag{2.28}$$

Again taking H_1=ξ in (2.20) and using (2.5), we obtain

$$\mathcal{M}^e(\xi, H_2)H_3 = -\eta(H_2)\mathcal{M}(\xi, \xi)H_3 - \eta(H_3)\mathcal{M}(\xi, H_2)\xi. \tag{2.29}$$

For fix, H_2=ξ in (2.22) and using (2.6), (2.16) and (2.18), we yield

$$\mathcal{M}(\xi, \xi)H_3 = 0. \tag{2.30}$$

With the help of (2.25) and (2.30), Eq.(2.29) reduces to

$$\mathcal{M}^e(\xi, H_2)H_3 = -\frac{1}{2}\eta(H_3)H_2 - \frac{1}{4n}\eta(H_3)\mathcal{Q}H_2. \tag{2.31}$$

Similarly, one can get

$$\mathcal{M}^e(H_1, \xi)H_3 = \frac{1}{2}\eta(H_3)H_1 + \frac{1}{4n}\eta(H_3)\mathcal{Q}H_1. \tag{2.32}$$

Definition 2.1. *An almost contact manifold* $(\mathbb{M}^{2n+1}, g)$ *is said to be an* η*-Einstein if its Ricci tensor* $\mathcal{S}$ *has the form*

$$\mathcal{S} = \mathcal{A}g + \mathcal{B}\eta \otimes \eta, \tag{2.33}$$

where $\mathcal{A}$ *and* $\mathcal{B}$ *are constants. If* $\mathcal{B}$=0, *then it is identified as Einstein and if* $\mathcal{A}$=0, *it is know as special type of* η*-Einstein.*

3. ξ-$\mathcal{M}^e$-projectively flat Kenmotsu Manifolds

Definition 3.1. *An* $(2n + 1)$*-dimensional manifold is said to be* ξ-$\mathcal{M}^e$ *projectively flat if it fulfills the condition*

$$\mathcal{M}^e(H_1, H_2)\xi = 0, \tag{3.34}$$

for all $H_1, H_2 \in \chi(\mathbb{M})$.

Theorem 3.1. *A* ξ-$\mathcal{M}^e$ *projectively flat Kenmotsu manifold* $(\mathbb{M}^{2n+1}, g)$ *is an Einstein manifold.*

Proof. Let $(\mathbb{M}^{2n+1}, g)$ be ξ-$\mathcal{M}^e$ projectively flat. Then from (2.20), we have

$$\eta(H_1)\mathcal{M}(\xi, H_2)\xi + \eta(H_2)\mathcal{M}(H_1, \xi)\xi = 0. \tag{3.35}$$

Using (2.23) and H_3=ξ in (2.22), we obtain from (3.35) that

$$\begin{aligned} \eta(H_1)[\eta(\xi)H_2 \quad &- \quad g(H_2,\xi)\xi - \frac{1}{4n}\{\mathcal{S}(H_2,\xi)\xi + 2n\eta(\xi)H_2 - 2ng(H_2,\xi)\xi - \eta(\xi)\mathcal{Q}H_2\}] \\ &+ \quad \eta(H_2)[g(H_1,\xi)\xi - \eta(\xi)H_1 - \frac{1}{4n}\{-\mathcal{S}(H_1,\xi)\xi \\ &- \quad 2nH_1 + \eta(\xi)\mathcal{Q}H_1 + 2ng(H_1,\xi)\xi\}] = 0. \end{aligned} \tag{3.36}$$

With the help of (2.5), (2.6) and (2.16), Eq. (3.36), reduces to

$$\frac{1}{2}\{\eta(H_1)H_2 - \eta(H_2)H_1\} - \frac{1}{4n}\{\eta(H_2)\mathcal{Q}H_1 - \eta(H_1)\mathcal{Q}H_2\} = 0. \tag{3.37}$$

Taking H_2=ξ in (3.37) we yield

$$\mathcal{Q}H_1 = -2nH_1, \tag{3.38}$$

which implies

$$\mathcal{S}(H_1, H_4) = -2ng(H_1, H_4). \tag{3.39}$$

□

Thus the Theorem 3.1 is completed.

Theorem 3.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-ρ-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady, or shrinking as $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, or $r > \frac{-2n}{\rho}$.*

Proof. Also from (1.2), we have

$$\mathcal{S}(H_1, H_2) + \frac{1}{2}(\mathcal{L}_{\mathbb{V}}g)(H_1, H_2) = (\gamma_1 + \rho r)g(H_1, H_2) + \gamma_2\eta(H_1)\eta(H_2). \tag{3.40}$$

Taking trace after putting H_1=H_2=e_i, $1 \leq i \leq 2n+1$ in (3.40), we get

$$\mathcal{S}(e_i, e_i) + \frac{1}{2}(\mathcal{L}_{\mathbb{V}}g)(e_i, e_i) = (\gamma_1 + \rho r)g(e_i, e_i) + \gamma_2\eta(e_i)\eta(e_i). \tag{3.41}$$

Using (3.39) in (3.41), we obtain

$$\text{div}\mathbb{V} = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2. \tag{3.42}$$

If $\mathbb{V}$ is solenoidal, i.e., div$\mathbb{V}$=0, then (3.42) implies that

$$\gamma_1 = -[2n + \frac{\gamma_2}{(2n+1)} + \rho r]. \tag{3.43}$$

□

So the proof of Theorem 3.2 is finished. Utilizing the Theorem 3.2, we state the following Corollary.

Corollary 3.1. *If a ξ-$\mathcal{M}^e$ protectively flat Kenmotsu manifold admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 3.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady, or shrinking as $r < -4n$, $r = -4n$, or $r > -4n$.*

Corollary 3.3. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady, or reducing as $r < -2n(2n+1)$, $r = -2n(2n+1)$, or $r > -2n(2n+1)$.*

Corollary 3.4. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1}, g)$. Then $\mathbb{V}$ is solenoidal if and only if the soliton is growing, steady, or shrinking as $r < -8n^2$, $r = -8n^2$, or $r > -8n^2$.*

Again, if $\mathbb{V}$=grad(f), where f is a smooth function on $(\mathbb{M}^{2n+1}, g)$. Then from equation (3.42) we yield the following result.

Theorem 3.3. *If the metric g of a $(\mathbb{M}^{2n+1}, g)$ satisfies an η-ρ-Einstein soliton $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2.$$

4. φ-$\mathcal{M}^e$ semi-symmetric on Kenmotsu manifold

Definition 4.1. *An $(2n+1)$-dimensional manifold is said to be φ-$\mathcal{M}^e$ semi-symmetric if it fulfills the criterion*

$$\mathcal{M}^e(H_1, H_2) \cdot \varphi = 0, \tag{4.44}$$

for all $H_1, H_2 \in \chi(\mathbb{M})$.

Theorem 4.1. *A φ-$\mathcal{M}^e$ semi-symmetric Kenmotsu manifold $(\mathbb{M}^{2n+1}, g)$ is an Einstein manifold.*

Proof. The condition $\mathcal{M}^e(H_1,H_2)\cdot\varphi$=0 on $(\mathbb{M}^{2n+1},g)$ from (2.20) implies that

$$(\mathcal{M}^e(H_1,H_2)\cdot\varphi)H_3=\mathcal{M}^e(H_1,H_2)\varphi H_3-\varphi\mathcal{M}^e(H_1,H_2)H_3=0. \tag{4.45}$$

for any vector fields $H_1,H_2,H_3\in\chi(\mathbb{M})$.

Since form (2.20) we have

$$\begin{aligned}\mathcal{M}^e(H_1,H_2)\varphi H_3 &= \mathcal{M}(H_1,H_2)\varphi H_3-\eta(H_1)M(\xi,H_2)\varphi H_3\\ &- \eta(H_2)\mathcal{M}(H_1,\xi)\varphi H_3-\eta(\varphi H_3)\mathcal{M}(H_1,H_2)\xi.\end{aligned} \tag{4.46}$$

Using (2.6), (2.11), (2.19), (2.22), and (2.23) in (4.46), we get

$$\begin{aligned}\mathcal{M}^e(H_1,H_2)\varphi H_3 &= g(H_1,\varphi H_3)H_2-g(H_2,\varphi H_3)H_1\\ &- \frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)H_1-\mathcal{S}(H_1,\varphi H_3)H_2+\eta(H_2)\mathcal{Q}H_1-\eta(H_1)\mathcal{Q}H_2\}\\ &- \eta(H_1)[-g(H_2,\varphi H_3)\xi-\frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)\xi-2ng(H_2,\varphi H_3)\xi\}]\\ &- \eta(H_2)[g(H_1,\varphi H_3)\xi-\frac{1}{4n}\{-\mathcal{S}(H_1,\varphi H_3)\xi+2ng(H_1,\varphi H_3)\xi\}].\end{aligned} \tag{4.47}$$

Again,

$$\begin{aligned}\varphi\mathcal{M}^e(H_1,H_2)H_3 &= \varphi\mathcal{M}(H_1,H_2)H_3-\eta(H_1)\varphi\mathcal{M}(\xi,H_2)H_3\\ &- \eta(H_2)\varphi\mathcal{M}(H_1,\xi)H_3-\eta(H_3)\varphi\mathcal{M}(H_1,H_2)\xi.\end{aligned} \tag{4.48}$$

Using (2.5), (2.11), (2.12), (2.19), (2.21), (2.22), and (2.23) in (4.48), we have

$$\begin{aligned}\varphi\mathcal{M}^e(H_1,H_2)H_3 = {}& g(H_1,H_3)\varphi H_2-g(H_2,H_3)\varphi H_1\\ &-\frac{1}{4n}\{\mathcal{S}(H_2,H_3)\varphi H_1-\mathcal{S}(H_1,H_3)\varphi H_2+g(H_2,H_3)\mathcal{Q}\varphi H_1-g(H_1,H_3)\mathcal{Q}\varphi H_2\}\\ &-\eta(H_1)[\eta(H_3)\varphi H_2-\frac{1}{4n}\{2n\eta(H_3)\varphi H_2-\eta(H_3)\mathcal{S}\varphi H_2\}]\\ &-\eta(H_2)[-\eta(H_3)\varphi H_1-\frac{1}{4n}\{-2n\eta(H_3)\varphi H_1+\eta(H_3)\mathcal{Q}\varphi H_1\}]\\ &-\eta(H_3)[\eta(H_1)\varphi H_2-\eta(H_2)\varphi H_1]\\ &+\frac{\eta(H_3)}{4n}\{2n\eta(H_1)\varphi H_2-2n\eta(H_2)\varphi H_1+\eta(H_2)\mathcal{Q}\varphi H_1-\eta(H_1)\mathcal{Q}\varphi H_2\}.\end{aligned} \tag{4.49}$$

Using (4.47) and (4.48) in (4.45), we get

$$\begin{aligned}
&g(H_1,\varphi H_3)H_2 - g(H_2,\varphi H_3)H_1 \\
&\quad - \frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)H_1 - \mathcal{S}(H_1,\varphi H_3)H_2 + \eta(H_2)\mathcal{Q}H_1 - \eta(H_1)\mathcal{Q}H_2\} \\
&\quad - \eta(H_1)[-g(H_2,\varphi H_3)\xi - \frac{1}{4n}\{\mathcal{S}(H_2,\varphi H_3)\xi - 2ng(H_2,\varphi H_3)\xi\}] \\
&\quad - \eta(H_2)[g(H_1,\varphi H_3)\xi - \frac{1}{4n}\{-\mathcal{S}(H_1,\varphi H_3)\xi + 2ng(H_1,\varphi H_3)\xi\}] \\
&\quad - [g(H_1,H_3)\varphi H_2 - g(H_2,H_3)\varphi H_1] \\
&\quad + \frac{1}{4n}[\mathcal{S}(H_2,H_3)\varphi H_1 - \mathcal{S}(H_1,H_3)\varphi H_2] \\
&\quad + \frac{1}{4n}[g(H_2,H_3)\mathcal{Q}\varphi H_1 - g(H_1,H_3)\mathcal{Q}\varphi H_2] \\
&\quad + \eta(H_1)[\eta(H_3)\varphi H_2 - \frac{1}{4n}\{2n\eta(H_3)\varphi H_2 - \eta(H_3)\mathcal{Q}\varphi H_2\}] \\
&\quad + \eta(H_2)[-\eta(H_3)\varphi H_1 - \frac{1}{4n}\{-2n\eta(H_3)\varphi H_1 + \eta(H_3)\mathcal{Q}\varphi H_1\}] \\
&\quad + \eta(H_3)[\eta(H_1)\varphi H_2 - \eta(H_2)\varphi H_1] \\
&\quad - \frac{\eta(H_3)}{4n}[2n\eta(H_1)\varphi H_2 - 2n\eta(H_2)\varphi H_1 + \eta(H_2)\mathcal{Q}\varphi H_1 - \eta(H_1)\mathcal{Q}\varphi H_2] = 0.
\end{aligned} \tag{4.50}$$

Taking H_2=ξ in (4.50) and using (2.5), (2.6), (2.16), (2.18), we have

$$\begin{aligned}
\frac{1}{4n}\{2\mathcal{S}(H_1,\varphi H_3)\xi \quad &- \quad 2ng(H_1,\varphi H_3)\xi - 2n\eta(H_3)\varphi H_1 + \eta(H_3)\mathcal{Q}\varphi H_1\} \\
&+ \quad \eta(H_3)\varphi H_1 = 0.
\end{aligned} \tag{4.51}$$

For fix, $H_3 = \xi$ in (4.51) and using (2.5), we obtain

$$\mathcal{Q}\varphi H_1 = -2n\varphi H_1. \tag{4.52}$$

Replacing H_1 by φH_1 in (4.52) and using (2.5), (2.18), one can get

$$\mathcal{Q}H_1 = -2nH_1, \tag{4.53}$$

which implies that

$$\mathcal{S}(H_1,H_4) = -2ng(H_1,H_4). \tag{4.54}$$

□

Therefore, the Theorem 4.1 is completed.

Like wise section 3, we reflect the following result:

Theorem 4.2. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η-ρ-*Einstein soliton on* $(\mathbb{M}^{2n+1}, g)$. *Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, *or* $r > \frac{-2n}{\rho}$.

Corollary 4.1. *If a* φ-$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold admits an* η-ρ-*Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 4.2. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η-*Einstein soliton on* φ-$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < -4n$, $r = -4n$, *or* $r > -4n$.

Corollary 4.3. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η-*traceless soliton on* φ-$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold. Then* $\mathbb{V}$ *is solenoidal if and only if it is expanding, steady or shrinking as* $r < -2n(2n+1)$, $r = -2n(2n+1)$, *or* $r > -2n(2n+1)$.

Corollary 4.4. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η-*Schouten soliton on* φ-$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < -8n^2$, $r = -8n^2$, *or* $r > -8n^2$.

Theorem 4.3. *If the metric* g *of a* $(2n+1)$-*dimensional* φ-$\mathcal{M}^e$ *semi-symmetric Kenmotsu manifold admits* η-ρ-*Einstein soliton* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, *where* $\mathbb{V}$ *is gradient of smooth function* f, *then the Laplace equation satisfied by* f *is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2.$$

5. Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{R}$=0

Theorem 5.1. *If a* $(2n+1)$-*dimensional Kenmotsu manifold satisfying the condition* $\mathcal{M}^e \cdot \mathcal{R}$=0, *then* $(\mathbb{M}^{2n+1}, g)$ *is an Einstein manifold.*

Proof. Let $(\mathbb{M}^{2n+1}, g)$ satisfies the condition $\mathcal{M}^e \cdot \mathcal{R}$=0. Then from [11], we have

$$\begin{aligned}\mathcal{M}^e(\xi, U)\mathcal{R}(H_1, H_2)H_3 \quad &- \quad \mathcal{R}(\mathcal{M}^e(\xi, U)H_1, H_2)H_3 \\ &- \quad \mathcal{R}(H_1, \mathcal{M}^e(\xi, U)H_2)H_3 \\ &- \quad \mathcal{R}(H_1, H_2)\mathcal{M}^e(\xi, U)H_3 = 0. \qquad (5.55)\end{aligned}$$

Taking H_3=ξ in (5.55) and using (2.12), we get

$$\eta(\mathcal{M}^e(\xi, U)H_1)H_2 - \eta(\mathcal{M}^e(\xi, U)H_2)H_1 + \mathcal{R}(H_1, H_2)\mathcal{M}^e(\xi, U)\xi = 0. \qquad (5.56)$$

Using (2.28), (2.31) in (5.56) and then using (2.6), (2.11), (2.16), we obtain

$$\frac{1}{2}\{g(H_1,U)H_2 - g(H_2,U)H_1\} + \frac{1}{4n}\{\mathcal{S}(H_1,U)H_2 - \mathcal{S}(H_2,U)H_1\} = 0. \tag{5.57}$$

Replacing H_2=ξ in (5.57), using (2.6) and (2.16), we yield

$$\mathcal{S}(H_1,U)\xi + 2ng(H_1,U)\xi = 0. \tag{5.58}$$

Taking the inner product of (5.58) with ξ and using (2.5), we obtain

$$\mathcal{S}(H_1,U) = -2ng(H_1,U). \tag{5.59}$$

□

So, the proof of the Theorem 5.1 is completed.

Therefore, as section 4, we state that

Theorem 5.2. *If $(g,\mathbb{V},\rho,\gamma_1,\gamma_2)$ be an η-ρ-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e\cdot\mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r<\frac{-2n}{\rho}$, $r=\frac{-2n}{\rho}$, or $r>\frac{-2n}{\rho}$.*

Corollary 5.1. *If a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e\cdot\mathcal{R}$=0 admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n+\rho r).$$

Corollary 5.2. *Let $(g,\mathbb{V},\rho,\gamma_1,\gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1},g)$ satisfying the condition $\mathcal{M}^e\cdot\mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r<-4n$, $r=-4n$, or $r>-4n$.*

Corollary 5.3. *Let $(g,\mathbb{V},\rho,\gamma_1,\gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1},g)$ satisfying the condition $\mathcal{M}^e\cdot\mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady or shrinking as $r<-2n(2n+1)$, $r=-2n(2n+1)$, or $r>-2n(2n+1)$.*

Corollary 5.4. *Let $(g,\mathbb{V},\rho,\gamma_1,\gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1},g)$ satisfying the condition $\mathcal{M}^e\cdot\mathcal{R}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r<-8n^2$, $r=-8n^2$, or $r>-8n^2$.*

Theorem 5.3. *If the metric g of a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e\cdot\mathcal{R}$=0 admits η-ρ-Einstein soliton $(g,\mathbb{V},\rho,\gamma_1,\gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows:*

$$\nabla(f) = (2n+\gamma_1+\rho r)(2n+1)+\gamma_2.$$

6. Kenmotsu Manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0

Theorem 6.1. *If a $(2n+1)$-dimensional Kenmotsu manifold satisfies the condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0, then $(\mathbb{M}^{2n+1}, g)$ is an Einstein manifold.*

Proof. The condition $\mathcal{M}^e \cdot \mathcal{M}^e$=0 on $(\mathbb{M}^{2n+1}, g)$ implies that

$$\begin{aligned}\mathcal{M}^e(\xi,U)\mathcal{M}^e(H_1,H_2)H_3 &- \mathcal{M}^e(\mathcal{M}^e(\xi,U)H_1,H_2)H_3\\ &- \mathcal{M}^e(H_1,\mathcal{M}^e(\xi,U)H_2)H_3\\ &- \mathcal{M}^e(H_1,H_2)\mathcal{M}^e(\xi,U)H_3 = 0. \qquad (6.60)\end{aligned}$$

Taking H_3=ξ in (6.60), we get

$$\begin{aligned}\mathcal{M}^e(\xi,U)\mathcal{M}^e(H_1,H_2)\xi &- \mathcal{M}^e(\mathcal{M}^e(\xi,U)H_1,H_2)\xi\\ &- \mathcal{M}^e(H_1,\mathcal{M}^e(\xi,U)H_2)\xi\\ &- \mathcal{M}^e(H_1,H_2)\mathcal{M}^e(\xi,U)\xi = 0. \qquad (6.61)\end{aligned}$$

Using (2.27), (2.28) and (2.31) in (6.61), we have

$$\begin{aligned}-\frac{1}{2}\eta(H_1)\mathcal{M}^e(\xi,U)H_2 &- \frac{1}{4n}\eta(H_1)\mathcal{M}^e(\xi,U)\mathcal{Q}H_2 + \frac{1}{2}\eta(H_2)\mathcal{M}^e(\xi,U)H_1\\ &+ \frac{1}{4n}\eta(H_2)\mathcal{M}^e(\xi,U)\mathcal{Q}H_1 + \frac{1}{2}\eta(H_1)\mathcal{M}^e(U,H_2)\xi\\ &+ \frac{1}{4n}\eta(H_1)\mathcal{M}^e(\mathcal{Q}U,H_2)\xi + \frac{1}{2}\eta(H_2)\mathcal{M}^e(H_1,U)\xi\\ &+ \frac{1}{4n}\eta(H_2)\mathcal{M}^e(H_1,\mathcal{Q}U)\xi + \frac{1}{2}\mathcal{M}^e(H_1,H_2)U\\ &+ \frac{1}{4n}\mathcal{M}^e(H_1,H_2)\mathcal{Q}U = 0. \qquad (6.62)\end{aligned}$$

Taking H_2=ξ in (6.62) and using (2.5), (2.18), (2.31) and (2.32), we get

$$\begin{aligned}-\frac{1}{8n}\eta(\mathcal{Q}H_1)U &- \frac{1}{16n^2}\eta(\mathcal{Q}H_1)\mathcal{Q}U + \frac{1}{4}\eta(H_1)U + \frac{1}{8n}\eta(H_1)\mathcal{Q}U\\ &+ \frac{1}{2}\eta(U)H_1 + \frac{1}{4n}\eta(U)\mathcal{Q}H_1 + \frac{1}{4n}\eta(\mathcal{Q}U)H_1\\ &+ \frac{1}{8n^2}\eta(\mathcal{Q}U)\mathcal{Q}H_1 = 0, \qquad (6.63)\end{aligned}$$

which implies that $\eta(H_1) \neq 0$, therefore equation (6.63) turns into

$$\mathcal{S}(U,H_4) = -2ng(U,H_4). \qquad (6.64)$$

□

Thus the proof of the Theorem 6.1 is completed.

As per section 5, we reflect the outcome

Theorem 6.2. *If* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-*ρ*-Einstein soliton on a* $(2n+1)$*-dimensional Kenmotsu manifold satisfying the condition* $\mathcal{M}^e \cdot \mathcal{M}^e$=0*. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$*, or* $r > \frac{-2n}{\rho}$*.*

Corollary 6.1. *If an* $(2n+1)$*-dimensional Kenmotsu manifold satisfying the condition* $\mathcal{M}^e \cdot \mathcal{M}^e$=0 *admits an* η*-*ρ*-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 6.2. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-Einstein soliton on* $(\mathbb{M}^{2n+1}, g)$ *satisfying the condition* $\mathcal{M}^e \cdot \mathcal{M}^e$=0*. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < -4n$*,* $r = -4n$*, or* $r > -4n$*.*

Corollary 6.3. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-traceless soliton on* $(\mathbb{M}^{2n+1}, g)$ *satisfying the condition* $\mathcal{M}^e \cdot \mathcal{M}^e$=0*. Then* $\mathbb{V}$ *is solenoidal if and only if it is expanding, steady or shrinking as* $r < -2n(2n+1)$*,* $r = -2n(2n+1)$*, or* $r > -2n(2n+1)$*.*

Corollary 6.4. *Let* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ *be an* η*-Schouten soliton on* $(\mathbb{M}^{2n+1}, g)$ *satisfying the condition* $\mathcal{M}^e \cdot \mathcal{M}^e$=0*. Then* $\mathbb{V}$ *is solenoidal if and only if the soliton is expanding, steady or shrinking as* $r < -8n^2$*,* $r = -8n^2$*, or* $r > -8n^2$*.*

Theorem 6.3. *If the metric* g *of a* $(2n+1)$*-dimensional Kenmotsu manifold satisfying the condition* $\mathcal{M}^e \cdot \mathcal{M}^e$=0 *admits* η*-*ρ*-Einstein soliton* $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$*, where* $\mathbb{V}$ *is gradient of smooth function* f*, then the Laplace equation satisfied by* f *is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n+1) + \gamma_2.$$

7. Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0

Theorem 7.1. *If a* $(2n+1)$ *dimensional Kenmotsu manifold satisfies the condition* $\mathcal{M}^e \cdot \mathcal{Q}$=0*, then* $(\mathbb{M}^{2n+1}, g)$ *is an Einstein manifold.*

Proof. The condition $\mathcal{M}^e \cdot \mathcal{Q}$=0 on $(\mathbb{M}^{2n+1}, g)$ implies that

$$\mathcal{M}^e(H_1, H_2)\mathcal{Q}H_3 - \mathcal{Q}(\mathcal{M}^e(H_1, H_2)H_3) = 0. \tag{7.65}$$

Taking H_2=ξ in (7.65), we get

$$\mathcal{M}^e(H_1, \xi)\mathcal{Q}H_3 - \mathcal{Q}(\mathcal{M}^e(H_1, \xi)H_3) = 0. \tag{7.66}$$

Using (2.32) in (7.66), we have

$$\frac{1}{2}\eta(\mathcal{Q}H_3)H_1 + \frac{1}{4n}\eta(\mathcal{Q}H_3)\mathcal{Q}H_1 - \mathcal{Q}[\frac{1}{2}\eta(\mathcal{Q}H_3)H_1 + \frac{1}{4n}\eta(\mathcal{Q}H_3)\mathcal{Q}H_1] = 0. \tag{7.67}$$

By virtue of (2.16), we get from (7.67) that

$$n\eta(H_3)H_1 + \eta(H_3)\mathcal{Q}H_1 + \mathcal{Q}(\frac{1}{4n}\eta(H_3)\mathcal{Q}H_1) = 0, \tag{7.68}$$

which implies that

$$n\eta(H_3)H_1 + \frac{1}{2}\eta(H_3)\mathcal{Q}H_1 = 0. \tag{7.69}$$

Now, taking the inner product of (7.69) with H_4, we obtain

$$n\eta(H_3)g(H_1, H_4) + \frac{1}{2}\eta(H_3)\mathcal{S}(H_1, H_4) = 0, \tag{7.70}$$

which implies that $\eta(H_3) \neq 0$, thus from (7.70) we yield

$$\mathcal{S}(H_1, H_4) = -2ng(H_1, H_4). \tag{7.71}$$

□

Thus the Theorem 7.1 is finished.

Following Section 6, we derive:

Theorem 7.2. *If $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-ρ-Einstein soliton on a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < \frac{-2n}{\rho}$, $r = \frac{-2n}{\rho}$, or $r > \frac{-2n}{\rho}$.*

Corollary 7.1. *If an $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0 admits an η-ρ-Einstein soliton then*

$$\gamma_1 = \frac{1}{2n+1}[div\mathbb{V} - \gamma_2] - (2n + \rho r).$$

Corollary 7.2. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Einstein soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -4n$, $r = -4n$, or $r > -4n$.*

Corollary 7.3. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-traceless soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if it is expanding, steady or shrinking as $r < -2n(2n+1)$, $r = -2n(2n+1)$, or $r > -2n(2n+1)$.*

Corollary 7.4. *Let $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$ be an η-Schouten soliton on $(\mathbb{M}^{2n+1}, g)$ satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0. Then $\mathbb{V}$ is solenoidal if and only if the soliton is expanding, steady or shrinking as $r < -8n^2$, $r = -8n^2$, or $r > -8n^2$.*

Theorem 7.3. *If the metric g of a $(2n+1)$-dimensional Kenmotsu manifold satisfying the condition $\mathcal{M}^e \cdot \mathcal{Q}$=0 admits η-ρ-Einstein soliton $(g, \mathbb{V}, \rho, \gamma_1, \gamma_2)$, where $\mathbb{V}$ is gradient of smooth function f, then the Laplace equation satisfied by f is as follows:*

$$\nabla(f) = (2n + \gamma_1 + \rho r)(2n + 1) + \gamma_2.$$

8. An Example

The notion of Ricci η-parallelity for Sasakian manifolds was introduced by M. Kon [16]. In [8] the authors proved that a three-dimensional Kenmotsu manifold has η-parallel Ricci tensor if and only if it is of constant scalar curvature. So, we verify the theorem obtained in [8] by a concrete example.

Let a 3-dimensional manifold $\mathbb{M} = \{(h_1, h_2, h_3) \in \mathbb{R}^3 : h_3 \neq 0\}$, where (h_1, h_2, h_3) are the standard coordinates and the linearly independent vector fields in $\mathbb{R}^3$ as follows

$$p_1 = e^{h_3}\frac{\partial}{\partial h_1}, \quad p_2 = e^{h_3}\frac{\partial}{\partial h_2}, \quad p_3 = -\frac{\partial}{\partial h_3}.$$

We defined the Riemannian metric g by

$$g(p_i, p_j) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let φ be a $(1,1)$ tensor field defined by

$$\varphi(p_1) = -p_2, \quad \varphi(p_2) = p_1, \quad \varphi(p_3) = 0.$$

If η denote the 1-form defined by $\eta(H_1) = g(H_1, p_3)$ for any $H_1 \in \mathcal{X}(M)$. Then we have

$$\varphi^2 H_1 = -H_1 + \eta(H_1)p_3, \ \eta(p_3) = 1,$$

$$g(\varphi H_1, \varphi H_2) = g(H_1, H_2) - \eta(H_1)\eta(H_2),$$

for any $H_2 \in \chi(\mathbb{M})$. Then for p_3=ξ, the structure (φ, ξ, η, g) establish an almost contact metric structure on $\mathbb{M}^3$.

Let ∇ be the Levi-Civita connection with respect to g. We have

$$[p_1, p_2] = 0, \quad [p_2, p_3] = p_2, \quad [p_3, p_1] = -p_1.$$

Using Koszul's formula, we can obtain

$$\nabla_{p_1}p_1 = -p_1, \quad \nabla_{p_2}p_1 = 0, \quad \nabla_{p_3}p_1 = 0,$$

$$\nabla_{p_1}p_2 = 0, \quad \nabla_{p_2}p_2 = -p_3, \quad \nabla_{p_3}p_2 = 0,$$

$$\nabla_{p_1}p_3 = p_1, \quad \nabla_{p_2}p_3 = p_2, \quad \nabla_{p_3}p_3 = 0.$$

As per above consequence for p_3=ξ, the manifold satisfies $\nabla_{H_1}\xi$=$H_1 - \eta(H_1)\xi$. Therefore, it can be classified as a Kenmotsu manifold.

Now, the components of curvature tensor $\mathcal{R}$ are as follows

$$\mathcal{R}(p_1, p_2)p_3 = 0, \quad \mathcal{R}(p_2, p_3)p_3 = -p_2, \quad \mathcal{R}(p_1, p_3)p_3 = -p_1,$$

$$\mathcal{R}(p_1, p_2)p_2 = -p_1, \quad \mathcal{R}(p_2, p_3)p_2 = -p_3, \quad \mathcal{R}(p_1, p_3)p_2 = 0,$$

$$\mathcal{R}(p_1, p_2)p_1 = 0, \quad \mathcal{R}(p_2, p_3)p_1 = 0, \quad \mathcal{R}(p_1, p_3)p_1 = p_1.$$

Also the Ricci tensor $\mathcal{S}$, one can get

$$\mathcal{S}(p_1, p_1) = \mathcal{S}(p_2, p_2) = \mathcal{S}(p_3, p_3) = -2.$$

Again, we can easily verify the following

$$\nabla_{H_1}\mathcal{S}(\varphi p_1, \varphi p_2) = 0, \ \nabla_{H_1}\mathcal{S}(\varphi p_2, \varphi p_3) = 0, \ \nabla_{H_1}\mathcal{S}(\varphi p_1, \varphi p_1) = 0,$$

$$\nabla_{H_1}\mathcal{S}(\varphi p_1, \varphi p_3) = 0, \ \nabla_{H_1}\mathcal{S}(\varphi p_3, \varphi p_1) = 0, \ \nabla_{H_1}\mathcal{S}(\varphi p_2, \varphi p_2) = 0,$$

$$\nabla_{H_1}\mathcal{S}(\varphi p_2, \varphi p_1) = 0, \ \nabla_{H_1}\mathcal{S}(\varphi p_3, \varphi p_2) = 0, \ \nabla_{H_1}\mathcal{S}(\varphi p_3, \varphi p_3) = 0.$$

Therefore, we conclude that $\nabla_{H_1}\mathcal{S}(\varphi H_2, \varphi H_3) = 0$, for all $H_1, H_2, H_3 \in \chi(\mathbb{M})$.

So, the Ricci tensor is η-parallel. Also, the scalar curvature of the manifold is -6, then the Theorems $3.1, 4.1, 5.1, 6.1$ and 7.1 are effectively satisfied by this example.

9. Conclusion

As a generalization of ρ-Einstein soliton [2], we study a new type soliton is called an η-ρ-Einstein soliton and gradient η-ρ-Einstein soliton on a $(2n + 1)$-dimensional Kenmotsu manifold admitting extended $\mathcal{M}$-Projective curvature tensor.The study of such new types of solitons is of significant interest from different fields due to its wide applications in general relativity, cosmology, quantum field theory, string theory, thermodynamics, mathematical physics, etc. That is why, we depict some geometrical properties of an η-ρ-Einstein soliton and gradient η-ρ-Einstein soliton on such manifold.

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References

[1] Bair, D. E. (1976). Contact manifolds in Riemannian Geometry. Lecture Notes in Math., Vol. 509. Berlin: Springer.

[2] Catino, G., & Mazzieri, L. (2016). Gradient Einstein solitons. Nonlinear Anal., 13(2), 66–94.

[3] Chattopadhyay, K., Bhattacharyya, A., & Debnath, D. (2018). A study of spacetimes with vanishing $\mathcal{M}$-projective curvature tensor. Journal of The Tensor Society of India, 12, 23–31.

[4] Chaubey, S. K., & Ojha, R. H. (2010). On the m-projective curvature tensor of a Kenmotsu manifold. Differential Geometry-Dynamical Systems, 12, 52–60.

[5] Chaubey, S. K., & Yadav, S. K. (2018). Study of Kenmotsu manifolds with semi-symmetric metric connection. Universal Journal of Mathematics and Application, 1(2), 89–97.

[6] Chaubey, S. K., Lee, J. W., & Yadav, S. K. (2019). Riemannian manifolds with a semi-symmetric metric P-connection. Journal of Korea Mathematical Society, 56(4), 1113–1129.

[7] Chaubey, S. K., Yadav, S. K., & Garvandha, M. (2022). Kenmotsu manifolds admitting a non-symmetric non-metric connection. Int. J. of IT, Res. & App., 1(3), 11–14.

[8] De, U. C., & Pathak, G. (2004). On 3-dimensional Kenmotsu manifolds. Indian Journal of Pure and Applied Mathematics, 35(2), 159–165.

[9] De, K., & De, U. C. (2013). Conharmonic curvature tensor on Kenmotsu manifolds. Bulletin of the Transilvania University of Brasov. Series III: Mathematics and Computer Science, 9–22.

[10] De, A. (2010). On Kenmotsu manifold. Bulletin of Mathematical Analysis and Applications, 2(3), 1–6.

[11] Gurupadavva, I. A., & Bagewadi, C. S. (2020). A study on W_8-curvature tensor in Kenmotsu manifolds. Int. J. Math. and Appl., 8(10), 27–34.

[12] Haseeb, H. (2017). Some results on projective curvature tensor in an η-Kenmotsu manifold. Palestine J. Math., 6, 196–203.

[13] Halil İ., Yoldaş, & Erol, Y. (2021). Some notes on Kenmotsu manifold. Facta Universitatis, Series: Mathematics and Informatics, 949–961.

[14] Hamilton, R. S. (1988). The Ricci flow on surfaces. Mathematics and General Relativity, Contemp. Math., 71, 237–261.

[15] Ianus, S., & Smaranda, D. (1997). Some remarkable structures on the product of an almost contact metric manifold with the real line. Papers from the National Coll. on Geometry and Topology, Univ. Timisoara, 107–110.

[16] Kon, M. (1976). Invariant submanifolds in Sasakian manifolds. Mathematische Annalen, 219, 277–290.

[17] Kenmotsu, K. (1972). A class of almost contact Riemannian manifolds. Tohoku Mathematical Journal, Second Series, 24(1), 93–103.

[18] Oubina, A. (1985). New classes of contact metric structures. Publ. Math. Debrecen, 32(4), 187–193.

[19] Özgür, C., & De, U. C. (2006). On the quasi-conformal curvature tensor of a Kenmotsu manifold. Mathematica Pannonica, 17(2), 221–228.

[20] Pakize, U., Suleyman, D., & Mehmet, A. (2022). Some curvature characterizations on Kenmotsu metric spaces. Gulf Journal of Mathematics, 13(2), 78–86.

[21] Singh, R. N., Pandey, S. K., & Pandey, G. (2013). On W_2-curvature tensor in a Kenmotsu manifold. Tamsui Oxf. J. Inf. Math. Sci., 29(2), 129–141.

[22] Tanno, S. (1969). The automorphism groups of almost contact Riemannian manifolds. Tohoku Math. J., 21, 21–38.

[23] Valter, B. (2022). On complete gradient Schouten solitons. Nonlinear Analysis, 221, 112883.

[24] Yıldız, A., De, U. C., & Acet, B. E. (2009). On Kenmotsu manifolds satisfying certain curvature conditions. SUT Journal of Mathematics, 45(2), 89–101.

[25] Yano, K., & Kon, M. (1984). Structures on manifolds. World Scientific, Singapore, Vol. 10.

[26] Yadav, S. K., Chaubey, S. K., & Prasad, R. (2020). On Kenmotsu manifolds with a semi-symmetric metric connection. Facta Universitatis (NIS) Ser. Math. Inform., 35(1), 101–119.

[27] Yadav, S. K., & Suthar, D. L. (2023). Kenmotsu manifolds with quarter symmetric non-metric connections. Montes Taurus J. Pure Appl. Math., 5(1), 78–89.

[28] Yadav, S. K., Haseeb, A., & Yildiz, A. (2024). Conformal η-Ricci-Yamabe solitons on submanifolds of an $(LCS)_n$-manifold admitting a quarter-symmetric metric connection. Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat., 73(3), 1–19.

[29] Yadav, S. K., & Yildiz, A. (2022). On $\mathcal{Q}$-curvature tensor in 3-dimensional f-Kenmotsu manifolds. Universal Journal of Mathematics and Applications, 5(3), 96–106.

[30] Yadav, S. K., & De, U. C. (2025). Kaehleian Norden spacetime admitting η-ρ-Einstein solitons. Honam Mathematical J., 47(1), 44–69.

[31] Chaubey, S. K., Prasad, R., Yadav, S. K., & Pankaj. (2024). Kenmotsu manifolds admit a semi-symmetric metric connection. Palestine Journal of Mathematics, 13(4), 623–636.

[32] Yıldırım, Ü., Atçeken, M., & Dirik, S. (2019). A normal paracontact metric manifold satisfying some conditions on the $\mathcal{M}$-projective curvature tensor. Konuralp Journal of Mathematics, 7(1), 217–221.

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DOMINANCE INDEX- A NEW PERSPECTIVE FOR DECISION-MAKING USING DUAL HESITANT FUZZY SOFT SETS

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Abstract. The dual hesitant fuzzy soft set (DHFSS), a hybrid structure of a dual hesitant fuzzy set and a soft set, is highly effective in handling membership and non-membership values using a set of possible values. This article explores an entirely different application of DHFSS for representing preliminary data involved in decision-making problems. Moreover, an innovative measure for comparing DHFSSs, namely the Dominance Index, which determines the dominance of one DHFSS over another, is presented. Furthermore, a linear algebraic approach, integrated with the Dominance Index of a dual hesitant fuzzy element, is proposed for solving decision-making problems. Finally, a real-life decision-making problem involving the evaluation of mobile tower work sites based on the performance of their workers is presented and solved using the proposed method to demonstrate its applicability.

Keywords: Decision-Making, Dual Hesitant Fuzzy Soft Set, Dominance Index, Ranking, MCDM.

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1. Introduction

The main objectives of research on hesitant fuzzy sets and their related hybrid structures are the construction of methods for solving Multi-Criteria Decision-Making(MCDM) problems. Since Torra [1] proposed the hesitant fuzzy set in 2010, many MCDM problems

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have been solved using structures such as hesitant fuzzy sets [2, 3, 4, 5], dual hesitant fuzzy sets [6, 7, 8], hesitant fuzzy soft sets [9, 10] and dual hesitant fuzzy soft sets [11, 12, 13]. MCDM problems have an inevitable place in most real-life situations. It is understood that an MCDM problem deals with the evaluation of a set of alternatives based on a set of decision criteria. This paper provides an innovative method for presenting data of an MCDM problem using a dual hesitant fuzzy soft set and also develops a method for processing that data, and thereby arriving at a reliable decision. MCDM problems can be categorized into three types based on the nature of its criteria, as (i) all categories are crisp, (ii) all are fuzzy, and (iii) mixed type. Among these, this paper focuses is of the second type because the criteria of the problem presented here exhibit some hesitancy.

Among those structures handling fuzziness and uncertainty, the dual hesitant fuzzy soft set seems to be a promising tool in MCDM due to its ability to simultaneously handle fuzziness, parametrization, hesitancy, and non-membership. As a starting point in developing the concept of the dual hesitant fuzzy soft set, Y. He [11] developed a method to encompass and solve decision-making problems using a dual hesitant fuzzy soft set. Further, he also proposed a technique for ranking the alternatives in the problem. Following this, several studies [14, 16] have been conducted in the area of including the introduction of process such as the proposal of concepts like distance [17], similarity [17], aggregation operators [18], and correlation coefficients [12] for comparing two dual hesitant fuzzy soft sets. Recently, studies on weighted hesitant fuzzy soft sets [31] have been developed, where a weight vector is assigned to all possible membership degrees of each element.

Decision-making problems are often associated with uncertainty and imprecision that cannot be effectively solved using classical fuzzy set models alone. Dual hesitant fuzzy sets extend hesitant fuzzy sets by considering multiple membership and non-membership values, thus providing a more comprehensive representation of uncertainty. However, in many real-world scenarios, decision-making problems involve multiple parameters that must be evaluated simultaneously. Therefore, considering more flexible and adaptive models is essential. Soft-set theory offers a parameterized approach that provides a systematic and effective mechanism for dealing with uncertainty in decision-making. Integrating soft sets into the dual hesitant fuzzy model enables more efficient modeling of multi-criteria decision problems, concurrent treatment of multiple attributes and their associated uncertainties, and improving the flexibility and adaptability of decision models to better reflect the complexity of the real world.

Usually, a decision-making problem involves several alternatives, their parameters, and evaluations corresponding to different parameters of each alternative. The goal is to rank these alternatives based on assessments. However, in real-life situations, each alternative may consist of a structure that includes other types of alternatives and their parameters, making the ranking process more complex. The dominance index is a widely used measure in decision-making frameworks to compare and evaluate alternatives, particularly in fuzzy and hesitant fuzzy environments. It determines the extent to which one alternative dominates another. Early work by Zadeh [32] on fuzzy sets laid the foundation for dominance-based comparisons, which were later extended to hesitant fuzzy sets and dual hesitant fuzzy sets, a more refined representation of uncertainty was achieved, leading to the development of the dominance index for comparing DHFSS elements.

Fuzzy sets [32] and soft set [33] frameworks have made significant progress, leading to the development of various generalized models that extend traditional approaches to solve more complex decision-making problems. (2,1)-fuzzy sets [34] introduce a more refined approach by incorporating weighted aggregate operators, enhancing their applicability in multi-criteria decision-making (MCDM) methods. Similarly, (3,2)-fuzzy sets [35] extend this concept to higher dimensions and find application in topology and optimal choice theory, enabling more sophisticated modeling of uncertainty in decision systems. A further generalization is provided by (m, n)-fuzzy set [36, 37], which establish a generalized framework for orthopair fuzzy sets and provide a robust framework for addressing MCDM problems. Furthermore, (a, b)-fuzzy soft sets [38] represent a new class of fuzzy soft sets that consider multiple attributes, thus improving the decision-making process by incorporating a broader range of evaluations. Finally, K_m^n-Rung picture fuzzy sets extend traditional fuzzy models by including multiple degrees of membership, non-membership, and hesitation. This makes them suitable for capturing complex uncertainties in real-world problems. These contributions pave the way for more flexible and powerful tools in decision-making and significantly enrich the theoretical foundations of fuzzy and soft-set frameworks.

After depicting the data using dual hesitant fuzzy soft sets, which are the building blocks of the problem, the next challenge was to compare the dual hesitant fuzzy elements efficiently. Here, the authors made use of the fact that a dual hesitant fuzzy set is an extension of hesitant fuzzy set. After exploring various approaches for comparing hesitant fuzzy elements like aggregation method [19, 20], entropy method [5, 21] distance and similarity measure method [22, 23] etc., the authors concluded inclusion measure approach is the most suitable one for this purpose. Speaking of the inclusion measure, it has a long history. It has originated from the so-called relation subset-hood. The inclusion measure is a relation that can be seen as the fuzzification of the crisp inclusion relation. It is a very useful tool for comparing objects in a wide range of fields such as fuzzy sets [24], intuitionistic fuzzy sets [25, 26], hesitant fuzzy sets [27], interval neutrosophic sets [28], etc. Using the techniques of inclusion measure, the authors have developed a method to quantify the dominance of one object over another. Since this dominance index fails to satisfy the transitivity condition, only a pairwise comparison is possible. Here, the authors have modified this approach in accordance with their purpose. The endogenous cardinalization [29] provided by this approach enables researchers to quantify each object's achievement in addition to merely ranking them. In this paper, the researchers have also depicted an evaluation problem to illustrate the practicability of their approach.

This paper is organized as follows: The first section discusses some concepts that are needed for the further sequel. The second and third sections introduces the dual hesitant fuzzy Maclaurin symmetric mean and the weighted dual hesitant fuzzy Maclaurin symmetric mean. A partial order and hybrid monotonic inclusion measure for dual hesitant fuzzy elements are presented in the fourth section. After that, we move on to discussing the methodology for ranking the objects in an evaluation problem. In the final section, we present a real-life problem to demonstrate the efficacy of the proposed method.

2. Preliminaries

This section provides essential definitions and background concepts that serve as the foundation for the remainder of this article. Also, throughout this paper, HFS, DHFS, DHFSS and DHFE stands for hesitant fuzzy set, dual hesitant fuzzy set, dual hesitant fuzzy soft set and dual hesitant fuzzy element respectively.

2.1. **HFS, DHFS, and DHFSS:** The following are definitions, associated concepts and supporting examples for HFS, DHFS, and DHFSS.

Definition 2.1. [1] *Let X be a reference set, a hesitant fuzzy set (HFS) E on X is defined in terms of a function* h *that when applied to X returns a subset of* $[0,1]$.
To be easily understood, Xu and Xia [15] *expressed an HFS by the following mathematical form:*

$$\mathrm{E} = \{< \mathrm{x}, \mathrm{h}(\mathrm{x}) > / \mathrm{x} \in \mathrm{X}\},$$

where $h(x)$ is a set of some values in $[0,1]$, denoting the possible membership degrees of the element $x \in X$ to the set E. For convenience, Xu and Xia [15] called $h(x)$ a hesitant fuzzy element (HFE).

Definition 2.2. [2] *Let X be a fixed set, then a dual hesitant fuzzy set* ($\mathtt{DHFS}$) $\mathtt{D}$ *on X is described as:*

$$\mathtt{D} = \{< \mathtt{x}, \mathtt{h}(\mathtt{x}), \mathtt{g}(\mathtt{x}) >, \mathtt{x} \in \mathtt{X}\},$$

in which $\mathtt{h}(\mathtt{x})$ *and* $\mathtt{g}(\mathtt{x})$ *are two sets of some values in* $[0,1]$ *denoting the possible membership degrees and nonmembership degrees of the element* $\mathtt{x} \in \mathtt{X}$ *to the set* $\mathtt{D}$, *respectively with the conditions:*

$$0 \leq \gamma, \eta \leq 1, \quad 0 \leq \gamma^+ + \eta^+ \leq 1,$$

where $\gamma \in \mathtt{h}(\mathtt{x}), \eta \in \mathtt{g}(\mathtt{x}), \gamma^+ \in \mathtt{h}^+(\mathtt{x}) = \cup_{\gamma \in \mathtt{h}(\mathtt{x})} \mathtt{max}\{\gamma\}$ *and* $\eta^+ \in g^+(x) = \cup_{\eta \in g(x)} max\{\eta\}$, *for all $x \in X$. For convenience, the pair* $\mathtt{d}(\mathtt{x}) = (\mathtt{h}(\mathtt{x}), \mathtt{g}(\mathtt{x}))$ *is called a dual hesitant fuzzy element(*$\mathtt{DHFE}$*), denoted by* $\mathtt{d} = (\mathtt{h}, \mathtt{g})$.

Denote by $\mathtt{DHFS}(\mathtt{U})$, *the set of all Dual Hesitant fuzzy sets over* $\mathtt{U}$.

Definition 2.3. [3] *Let* $(\mathtt{U}, \mathtt{E})$ *be a soft universe and* $\mathtt{A} \subseteq \mathtt{E}$. *A pair* $Ç = (\tilde{\mathtt{F}}, \mathtt{A})$ *is called a Dual hesitant fuzzy soft set (DHFSS) over* $\mathtt{U}$, *where* $\tilde{\mathtt{F}}$ *is a mapping given by* $\tilde{\mathtt{F}} : \mathtt{A} \to \mathtt{DHFS}(\mathtt{U})$. *In general* $\tilde{\mathtt{F}}(\mathtt{e})$ *can be written as,*

$$\tilde{\mathtt{F}}(\mathtt{e}) = \{< \mathtt{x}, \mathtt{h}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x}), \mathtt{g}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x}) > / \mathtt{x} \in \mathtt{U}\},$$

where $\mathtt{h}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x})$ and $\mathtt{g}_{\tilde{\mathtt{F}}(\mathtt{e})}(\mathtt{x})$ *are two sets of some values in* $[0,1]$, *denoting the possible membership degrees and non membership degrees that object* $\mathtt{x}$ *holds on parameter* $\mathtt{e}$, *respectively.*

To represent dual hesitant fuzzy soft sets concisely, Y.He [3] proposed a tabular representation, which is depicted in the following example in detail.

Example 2.1. [3] *Let* $\mathtt{U}$ *be a set of four participants performing dance program, which is denoted by* $\mathtt{U} = \{\mathtt{x}_1, \mathtt{x}_2, \mathtt{x}_3, \mathtt{x}_4\}$. *Let* $\mathtt{E}$ *be a parameter set, where*

$$\mathtt{E} = \{\mathtt{e}_1, \mathtt{e}_2, \mathtt{e}_3\} = \{\mathtt{confident}, \mathtt{creative}, \mathtt{graceful}\}.$$

Suppose that there are three judges who are invited to evaluate the membership degrees and non-membership degrees of a candidate $\mathtt{x}_\mathtt{j}$ *to a parameter* $\mathtt{e}_\mathtt{i}$ *with several possible values in* $[0,1]$. *Then the tabular representation of dual hesitant fuzzy soft set* $Ç = (\tilde{\mathtt{F}}, \mathtt{A})$ *defined as below by Table 2.1 gives the evaluation of the performance of candidates by three judges.*

TABLE 2.1. Tabular Representation of dual hesitant fuzzy soft set $\c{G} = (\tilde{\mathtt{F}}, \mathtt{A})$

U	e_1	e_2	e_3
x_1	{.6,.7,.8}{.3,.2,.1}	{.5,.6,.4}{.4,.3,.2}	{.4,.4,.3}{.7,.6,.6}
x_2	{.4,.5,.6}{.3,.2,.1}	{.5,.4,.3}{.5,.3,.3}	{.5,.7,.7}{.3,.2,.2}
x_3	{.8,.7,.7}{.2,.1,.1}	{.7,.8,.8}{.2,.2,.1}	{.5,.6,.7}{.3,.2,.1}
x_4	{.3,.4,.4}{.6,.5,.4}	{.5,.6,.6}{.4,.3,.2}	{.7,.6,.8}{.2,.1,.1}

To compare the DHFEs, Zhu et al.[2] introduced the following comparison laws:

Definition 2.4. [2] *The score and accuracy function of a* DHFE $\mathtt{d} = (\mathtt{h}, \mathtt{g})$ *are*

$$\mathtt{s_d} = (1/\#\mathtt{h}) \sum_{\gamma \in \mathtt{h}} \gamma - (1/\#\mathtt{g}) \sum_{\eta \in \mathtt{g}} \eta$$

and

$$\mathtt{p_d} = (1/\#\mathtt{h}) \sum_{\gamma \in \mathtt{h}} \gamma + (1/\#\mathtt{g}) \sum_{\eta \in \mathtt{g}} \eta$$

respectively, where $\#\mathtt{h}$ *and* $\#\mathtt{g}$ *are the number of elements in* h *and* g *respectively, then*

i. *if* $\mathtt{s}_{\mathtt{d}_1} > \mathtt{s}_{\mathtt{d}_2}$, *then* $\mathtt{d}_1$ *is superior to* $\mathtt{d}_2$

ii. *if* $\mathtt{s}_{\mathtt{d}_1} = \mathtt{s}_{\mathtt{d}_2}$, *then*

  1. *if* $p_{\mathtt{d}_1} = p_{\mathtt{d}_2}$, *then* $\mathtt{d}_1$ *is equivalent to* d_2, *denoted by* $\mathtt{d}_1 \sim \mathtt{d}_2$
  2. *if* $p_{\mathtt{d}_1} > p_{\mathtt{d}_2}$, *then* $\mathtt{d}_1$ *is superior than* $\mathtt{d}_2$, *denoted by* $\mathtt{d}_1 \succ \mathtt{d}_2$

In [2], Zhu et al. proposed the following operational laws for DHFEs :

Definition 2.5. [2] *Let* $d = (h, g)$, $d_1 = (h_1, g_1)$ *and* $d_2 = (h_2, g_2)$ *be three* DHFE*s, then*

(1) $d_1 \oplus d_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{\{\gamma_1 + \gamma_2 - \gamma_1\gamma_2\}, \{\eta_1\eta_2\}\}$

(2) $d_1 \otimes d_2 = \bigcup_{\gamma_1 \in h_1, \gamma_2 \in h_2, \eta_1 \in g_1, \eta_2 \in g_2} \{\{\gamma_1\gamma_2\}, \{\eta_1 + \eta_2 - \eta_1\eta_2\}\}$

(3) $nd = \bigcup_{\gamma \in h, \eta \in g} \{\{1 - (1-\gamma)^n\}, \{\eta^n\}\}$

(4) $d^n = \bigcup_{\gamma \in h, \eta \in g} \{\{\gamma^n\}, \{1 - (1-\eta)^n\}\}$

The following assumptions are made in the rest of the paper:

* Elements of h and g are arranged in increasing order.
* $\mathbb{H}$ denote the set of all finite subsets of $[0, 1]$ whose elements are arranged in increasing order.
* $\mathtt{d} = (\mathtt{h}, \mathtt{g}; \mathtt{l}, \mathtt{l}')$ represents a dual hesitant fuzzy element $(\mathtt{h}, \mathtt{g}) \in \mathbb{H} \times \mathbb{H}$, with $\mathtt{l}(\mathtt{h}) = \mathtt{l}$ and $\mathtt{l}(\mathtt{g}) = \mathtt{l}'$.

2.2. **The Maclaurin Symmetric Mean.** Due to its ability to capture the inter-relationship among the multi-input arguments, the Maclaurin symmetric mean (MSM), introduced by Maclaurin [4], has a prominent place in the list of aggregation operators. The MSM is defined as follows:

Definition 2.6. [4] *Let* $a_i, \{i = 1, 2, ..., n\}$ *be a collection of non-negative real numbers, and* $k \in \{1, 2, \ldots, n\}$. *If*

$$\mathtt{MSM}^{(k)}(a_1, a_2, ..., a_n) = \left(\frac{\sum\limits_{1 \le i_1 < i_2 < ... < i_k \le n} \left(\prod\limits_{j=1}^{k} a_{i_j} \right)}{C_n^k} \right)^{1/k},$$

then $\mathtt{MSM}^{(k)}$ *is called the Maclaurin symmetric mean (*MSM*), where* $(i_1, i_2, ..., i_k)$ *traverse through all the k-tuples combinations of* $(1, 2, \ldots, n)$, *and* C_n^k *is the binomial coefficient.*

In 2015, Quin et al.[5] extend the notion of MSM to hesitant fuzzy environment and defined hesitant fuzzy Maclaurin symmetric mean (HFMSM) as follows:

Definition 2.7. [4] *Let* $\mathtt{h_i}, (\mathtt{i} = 1, 2, ..., \mathtt{n})$ *be a collection of* HFEs *and* $k = 1, 2, ..., n$. *If*

$$\mathtt{HFMSM}^{(k)}(h_1, h_2, ..., h_n) = \left(\frac{\bigoplus\limits_{1 \le i_1 < i_2 < ... < i_k \le n} \left(\bigotimes\limits_{j=1}^{k} h_{i_j} \right)}{C_n^k} \right)^{1/k},$$

then $\mathtt{HFMSM}^{(k)}$ *is called the hesitant fuzzy Maclaurin symmetric mean (*HFMSM*) operator.*

3. The Dual Hesitant Fuzzy Maclaurin Symmetric Mean

The evaluation problem we discussed in this paper has dual hesitant fuzzy soft framework. Meanwhile, we need an aggregation operator that reflect the inter-relationship among the arguments. As Maclaurin symmetric mean is a right candidate for this purpose, we define dual hesitant fuzzy Maclaurin symmetric mean in this section as follows.

Definition 3.1. *Let* $\mathtt{d_j} = (\mathtt{h_j}, \mathtt{g_j}), (\mathtt{j} = 1, 2, ..., \mathtt{n})$ *be a group of* DHFEs *and* $\mathtt{k} \in \{1, 2, ..., \mathtt{n}\}$. *If*

$$\mathtt{DHFMSM}^{(k)}(\mathtt{d_1}, \mathtt{d_2}, ..., \mathtt{d_n}) = \left(\frac{\bigoplus\limits_{1 \le i_1 < i_2 < ... < i_k \le n, i_j \in \mathbb{Z},\ \forall j=1 \text{ to } k} \left(\bigotimes\limits_{j=1}^{k} \mathtt{d_{i_j}} \right)}{\mathtt{C_n^k}} \right)^{1/k},$$

where C_n^k *denote the number of combinations of n things taken k at a time. Then* $\mathtt{DHFMSM}^{(k)}$ *is called the dual hesitant fuzzy Maclaurin symmetric mean (*DHFMSM*) operator. Here* $\overset{k}{\underset{j=1}{\otimes}} d_{i_j}$ *reflects the interrelationship among* $d_{i_1}, d_{i_2}, ..., d_{i_k}$.

The following theorem exhibits a nice representation for the DHFMSM operator.

Theorem 3.1. Let $d_j = (h_j, g_j), (j = 1, 2, ..., n)$ be a collection of DHFEs and $k \in \{1, 2, ..., n\}$, then the aggregated value of $d_j, j = 1, 2, ..., n$ using the proposed DHFMSM operator is again a DHFE, given by

$$\mathtt{DHFMSM}^{(k)}(d_1, d_2, ..., d_n) = (\bar{h}, \bar{g}),$$

where,

$$\bar{h} = \bigcup_{\gamma_1 \in h_1, ..., \gamma_n \in h_n} \left\{ \left(1 - \left[\prod_{(i_1, i_2, ..., i_k) \in S} \left(1 - \prod_{j=1}^{k} \gamma_{i_j} \right) \right]^{\frac{1}{C_n^k}} \right)^{1/k} \right\}$$

and

$$\bar{g} = \bigcup_{\eta_1 \in g_1, ..., \eta_n \in g_n} \left\{ 1 - \left(1 - \left[\prod_{(i_1, i_2, ..., i_k) \in S} \left[1 - \prod_{j=1}^{k} (1 - \eta_{i_j}) \right] \right]^{\frac{1}{C_n^k}} \right)^{1/k} \right\}$$

where $S = \{(i_1, i_2, ..., i_k) \in \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z} / 1 \le i_1 < i_2 < ... < i_k \le n\}$ and C_n^k denote the number of combinations of n things taken k at a time.

Proof. Using the operations of DHFEs given by definition 2.6(1-4), we have

$$\overset{k}{\underset{j=1}{\otimes}} d_{i_j} = \bigcup_{\substack{\gamma_{i_j} \in h_{i_j} \\ \eta_{i_j} \in g_{i_j}}} \left\{ \left\{ \prod_{j=1}^{k} \gamma_{i_j} \right\}, \left\{ 1 - \prod_{j=1}^{k} (1 - \eta_{i_j}) \right\} \right\}$$

Eventually, we have

$$\underset{1 \le i_1 < i_2 < ... < i_k \le n}{\oplus} \overset{k}{\underset{j=1}{\otimes}} d_{i_j} = \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ 1 - \prod_{(i_1, i_2, ..., i_k) \in S} \left(1 - \prod_{j=1}^{k} \gamma_{i_j} \right) \right\}, \left\{ \prod_{(i_1, i_2, ..., i_k) \in S} \left(1 - \prod_{j=1}^{k} (1 - \eta_{i_j}) \right) \right\} \right\}$$

and

$$\frac{\underset{1 \le i_1 < i_2 < ... < i_k \le n}{\oplus} \overset{k}{\underset{j=1}{\otimes}} d_{i_j}}{C_n^k} = \bigcup_{\gamma_i \in h_i, \eta_i \in g_i} \left\{ \left\{ 1 - \left[\prod_{(i_1, i_2, ..., i_k) \in S} \left(1 - \prod_{j=1}^{k} \gamma_{i_j} \right) \right]^{1/C_n^k} \right\}, \right.$$

$$\left\{\prod_{(i_1,i_2,\ldots,i_k)\in S}\left[1-\prod_{j=1}^{k}(1-\eta_{i_j})\right]\right\}^{1/C_n^k}\Bigg\}$$

Therefore,

$$\left(\frac{\underset{1\le i_1<\ldots<i_k\le n}{\oplus}\overset{k}{\underset{j=1}{\otimes}} d_{i_j}}{C_n^k}\right)^{1/k} = \bigcup_{\gamma_i\in h_i,\eta_i\in g_i}\left\{\left\{\left(1-\left[\prod_{(i_1,\ldots,i_k)\in S}\left(1-\prod_{j=1}^{k}\gamma_{i_j}\right)\right]^{\frac{1}{C_n^k}}\right)^{1/k}\right\},\right.$$

$$\left.\left\{1-\left(1-\left[\prod_{(i_1,\ldots,i_k)\in S}\left[1-\prod_{j=1}^{k}(1-\eta_{i_j})\right]\right]^{\frac{1}{C_n^k}}\right)^{1/k}\right\}\right\}.$$

This completes the proof. □

The weighted dual hesitant fuzzy Maclaurin symmetric mean

In DHFMSM operator, every DHFE receives the same importance. But real-life decision-making situations demand different priorities for parameters and categories. So we have to incorporate the concept of weights in DHFMSM operator. Therefore, in this section, we shall propose the weighted dual hesitant fuzzy Maclaurin symmetric mean operator, which is defined as follows:

Definition 3.2. *Let* $d_j = (h_j, g_j), (j = 1, 2, ..., n)$ *be a collection of* DHFEs *and* $k \in \{1, 2, ..., n\}$. *Let* $w = (w_1, w_2, ..., w_n)^T$ *is the weight vetor, where* w_j *indicates the degree of importance of* d_j, *satisfying* $w_j \in [0, 1],\ j = 1, 2, ..., n$ *and* $\sum_{j=1}^{n} w_j = 1$. *If*

$$WDHFMSM_w^{(k)}(d_1, d_2, ..., d_n) = \left(\frac{\bigoplus_{1\le i_1<i_2<\ldots<i_k\le n}\left(\bigotimes_{j=1}^{k}(d_{i_j})^{w_{i_j}}\right)}{C_n^k}\right)^{1/k},$$

then WDHFMSM $^{(k)}$ *is called the weighted dual hesitant fuzzy Maclaurin symmetric mean (*WDHFMSM*) operator.*

According to the operations of DHFEs exhibited in section 2, we can derive the following theorem.

Theorem 3.2. Let $d_j = (h_j, g_j), (j = 1, 2, ..., n)$ be a collection of DHFEs and $k \in \{1, 2, ..., n\}$, then the aggregated value of $d_j, j = 1, 2, ..., n$ using the WDHFMSM operator is also a DHFE, given by

$$\mathtt{WDHFMSM}_{\mathtt{w}}^{(\mathtt{k})}(\mathtt{d}_1, \mathtt{d}_2, ..., \mathtt{d}_\mathtt{n}) = (\bar{\mathtt{h}}, \bar{\mathtt{g}}),$$

where,

$$\bar{\mathtt{h}} = \bigcup_{\gamma_1 \in \mathtt{h}_1, ..., \gamma_\mathtt{n} \in \mathtt{h}_\mathtt{n}} \left\{ \left(1 - \prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left[1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} \gamma_{\mathtt{i}_\mathtt{j}}^{\mathtt{w}_{\mathtt{i}_\mathtt{j}}} \right]^{\frac{1}{C_\mathtt{n}^\mathtt{k}}} \right)^{1/\mathtt{k}} \right\}$$

and

$$\bar{\mathtt{g}} = \bigcup_{\eta_1 \in \mathtt{g}_1, ..., \eta_\mathtt{n} \in \mathtt{g}_\mathtt{n}} \left\{ 1 - \left(1 - \prod_{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathtt{S}} \left[1 - \prod_{\mathtt{j}=1}^{\mathtt{k}} (1 - \eta_{\mathtt{i}_\mathtt{j}})^{\mathtt{w}_{\mathtt{i}_\mathtt{j}}} \right]^{\frac{1}{C_\mathtt{n}^\mathtt{k}}} \right)^{1/\mathtt{k}} \right\}$$

where $\mathtt{S} = \{(\mathtt{i}_1, \mathtt{i}_2, ..., \mathtt{i}_\mathtt{k}) \in \mathbb{Z} \times \mathbb{Z} \times ... \times \mathbb{Z} / 1 \le \mathtt{i}_1 < \mathtt{i}_2 < ... < \mathtt{i}_\mathtt{k} \le \mathtt{n}\}$.
Here $\bigotimes_{\mathtt{j}=1}^{\mathtt{k}} (\mathtt{d}_{\mathtt{i}_\mathtt{j}})^{\mathtt{w}_{\mathtt{i}_\mathtt{j}}}$ reflects the inter-relationship among $\mathtt{d}_{\mathtt{i}_1}, \mathtt{d}_{\mathtt{i}_2}, ..., \mathtt{d}_{\mathtt{i}_\mathtt{k}}$.

4. A novel partial order and hybrid monotonic inclusion measures for DHFEs

As previously mentioned, the framework of our evaluation problem is based on dual hesitant fuzzy soft sets. To effectively compare such sets, it is necessary to define an order relation on the set of DHFEs. In [12], Zhang et al. proposed a partial order for hesitant fuzzy elements (HFEs) using disjunctive semantic interpretation. In this work, we extend their approach to the dual hesitant fuzzy context.

Definition 4.1. *Let* $\mathtt{d}_1 = (\mathtt{h}_1, \mathtt{g}_1; \mathtt{l}_1, \mathtt{l}_1^{'}), \mathtt{d}_2 = (\mathtt{h}_2, \mathtt{g}_2; \mathtt{l}_2, \mathtt{l}_2^{'})$ *be two* DHFEs. *We define an order relation* $\le^{\S}$ *between* $\mathtt{d}_1$ *and* $\mathtt{d}_2$ *as follows:*

$$\mathtt{d}_1 \le^{\S} \mathtt{d}_2 \quad \textit{iff} \quad \begin{cases} \mathtt{h}_1^\mathtt{i} \le \mathtt{h}_2^\mathtt{i}, \ \forall\ \mathtt{i} = 1 \cdots \mathtt{l}_1 \quad \textit{if} \quad \mathtt{l}_1 \le \mathtt{l}_2 \\ \\ \mathtt{h}_1^{\mathtt{l}_1 - \mathtt{l}_2 + \mathtt{i}} \le \mathtt{h}_2^\mathtt{i}, \ \forall\ \mathtt{i} = 1 \cdots \mathtt{l}_2 \quad \textit{Otherwise} \\ \\ \qquad \textit{and} \\ \\ \mathtt{g}_1^\mathtt{j} \ge \mathtt{g}_2^\mathtt{j}, \ \forall\ \mathtt{j} = 1 \cdots \mathtt{l}_2{'} \quad \textit{if} \quad \mathtt{l}_1^{'} \ge \mathtt{l}_2^{'} \\ \\ \mathtt{g}_1^\mathtt{j} \ge \mathtt{g}_2^{\mathtt{l}_2^{'} - \mathtt{l}_1^{'} + \mathtt{j}}, \ \forall\ \mathtt{j} = 1 \cdots \mathtt{l}_1^{'} \quad \textit{Otherwise} \end{cases}$$

For any two DHFS A and B on X, A$\subseteq^{\S}$ B iff $\mathtt{d}_\mathtt{A}(\mathtt{x}) \le^{\S} \mathtt{d}_\mathtt{B}(\mathtt{x}), \forall \mathtt{x} \in \mathtt{X}$. The ordered set is denoted by (DHFS(X),$\subseteq^{\S}$). We can easily prove the following theorem.

Theorem 4.1. $(\mathbb{H}\times\mathbb{H}, \leq^{\S})$ is a partially ordered set. Moreover, $\subseteq^{\S}$ is a partial order on $\mathtt{DHFS(X)}$.

Proof.

(1) Reflexive: Clearly the reflexive property hold for $\leq^{\S}$.

(2) Antisymmetric: Let $d_1 \leq^{\S} d_2$ and $d_2 \leq^{\S} d_1$, where $d_1 = (h_1, g_1; l_1, l_1')$ and $d_2 = (h_2, g_2; l_2, l_2')$. Now, the anti-symmetry of $\leq^{\S}$ can be easily proved using the monotonicity property of h as well as g and using the definition of $\leq^{\S}$. To prove $d_1 = d_2$, we have to consider the following four cases:

(i): $l_1 \leq l_2$ and $l_1' \geq l_2'$.

(ii): $l_1 \leq l_2$ and $l_1' \leq l_2'$.

(iii): $l_1 \geq l_2$ and $l_1' \leq l_2'$.

(iv): $l_1 \geq l_2$ and $l_1' \geq l_2'$.

Case(i): From $d_1 \leq^{\S} d_2, l_1 \leq l_2\ l_1' \geq l_2'$, we get, $h_1^i \leq h_2^i, \forall i = 1 \cdots$. l_1 and $g_1^j \geq g_2^j, \forall j = 1 \cdots l_2'$. Also, from $d_2 \leq^{\S} d_1$, we get $h_2^{l_2-l_1+i} \leq h_1^i,\ \forall\ i = 1 \cdots l_1$, and $g_2^j \geq g_1^{l_1'-l_2'+j},\ \forall\ j = 1 \cdots l_2'$. By increasing property of h_2, it follows that $h_1^i \leq h_2^i \leq h_2^{l_2-l_1+i} \leq h_1^i,\ \forall\ i = 1 \cdots l_1$.
From this, it is evident that $l_1 = l_2$ and $h_1^i = h_2^i,\ \forall\ i = 1 \cdots l_1$.
Again by increasing property of g_1, we get $g_1^j \geq g_2^j \geq g_1^{l_1'-l_2'+j} \geq g_1^j,\ \forall\ j = 1 \cdots l_2'$. From the above inequalities, it is clear that $l_1' = l_2'$ and $g_1^j = g_2^j,\ \forall\ \ j = 1 \cdots l_1'$. Thus we prove $d_1 = d_2$. Here we depict only one of the four above cases; others can be proved similarly.

(3) Transitive: Let $d_1 \leq^{\S} d_2$ and $d_2 \leq^{\S} d_3$, where $d_1 = (h_1, g_1; l_1, l_1')$, $d_2 = (h_2, g_2; l_2, l_2')$, $d_3 = (h_3, g_3; l_3, l_3')$. We claim that $d_1 \leq^{\S} d_3$. We can easily prove our claim using the transitivity property of $\leq$, monotonicity property of h as well as g and the definition of $\leq^{\S}$. Here we have to consider several cases, but it is only a routine calculations. So we demonstrate only one case and others are left to the reader. Suppose $l_3 \leq l_1 \leq l_2$ and $l_3' \leq l_1' \leq l_2'$. From $d_1 \leq^{\S} d_2, l_1 \leq l_2$ and $l_1' \leq l_2'$, we get, $h_1^i \leq h_2^i,\ \forall\ i = 1 \cdots l_1$ and $g_1^j \geq g_2^{l_2'-l_1'+j},\ \forall\ j = 1 \cdots l_1'$. Also, from $d_2 \leq^{\S} d_3, l_3 \leq l_2$ $l_3' \leq l_2'$, we get, $h_2^{l_2-l_3+i} \leq h_3^i,\ \ \forall\ i = 1 \cdots l_3$ and $g_2^j \geq g_3^j,\ \forall\ j = 1 \cdots l_3'$. Applying the increasing property of h_2 and g_2 together with the inequality $l_2 - l_3 \geq l_1 - l_3$, we get
$h_1^{l_1-l_3+i} \leq h_2^{l_1-l_3+i} \leq h_2^{l_2-l_3+i} \leq h_3^i,\ \forall\ i = 1 \cdots l_3$ and

$g_1^j \geq g_2^{l_2'-l_1'+j} \geq g_2^j \geq g_3^j$, $\forall$ $j = 1 \cdots l_3'$. From the observations $l_1 \geq l_3$ and $l_1' \geq l_3'$, transitivity is obvious. Hence the proof.

□

It is well known that a partially ordered set generally contains elements that are not mutually comparable. The presence of such elements naturally leads to the need for an inclusion measure. Therefore, we define an inclusion measure on the partially ordered set $(\mathbb{H}\times\mathbb{H}, \leqslant^{\S})$. In the following, we first provide an axiomatic definition of the inclusion measure. According to H. Y. Zhang [41], hybrid monotonicity is essential for a rational generalization of inclusion measures. Accordingly, we define a hybrid monotonic inclusion measure on $(\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$.

Definition 4.2. *Let* $d_1, d_2 \in (\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$. *A real number* $\mathfrak{Inc}(d_1, d_2) \in [0, 1]$ *is called an HM inclusion measure between* d_1 *and* d_2, *if* $\mathfrak{Inc}(d_1, d_2)$ *satisfies the following properties.*

(ID1): $\mathfrak{Inc}(d_1, d_2) = 1$ *if and only if* $d_1 \leq^{\S} d_2$

(ID2): *If* $d = \overline{1}$, *then* $\mathfrak{Inc}(d, d^c) = 0$, *where* $\overline{1} = (\{1\}, \{0\})$

(ID3): *If* $d_1 \leq^{\S} d_2$, *then for any* $d_3 \in (\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$, $\mathfrak{Inc}(d_3, d_1) \leq \mathfrak{Inc}(d_3, d_2)$, $\mathfrak{Inc}(d_2, d_3) \leq \mathfrak{Inc}(d_1, d_3)$

To study the structure of an inclusion measure, axiomatic approach is the best choice. But, our aim is to use inclusion measure in a decision making problem. So we are more interested in constructive approach. In the following section, we present a concrete example for inclusion measure which satisfies our proposed axioms.

Proposition 4.1. *For* $d_1 = (h_1, g_1; l_1, l_1')$ *and* $d_2 = (h_2, g_2; l_2, l_2') \in (\mathbb{H} \times \mathbb{H}, \leqslant^{\S})$, *let*

$$\mathfrak{Inc}(d_1, d_2) = s_{(\, d_{\mathcal{L}}(d_1, d_2)\,)}$$

where s_d *is the score function of the* DHFE d *and* $d_{\mathcal{L}}(d_1, d_2) = (h, g)$ *is a* DHFE, *called* $\mathcal{L}$-*subsethood index of* d_1 *and* d_2, *where,*

$$h = \begin{cases} \bigcup\limits_{i=1}^{l_1} \mathfrak{I}_L(h_1^i, h_2^i), & \text{if} \quad l_1 \leq l_2 \\ \bigcup\limits_{i=1}^{l_2} \mathfrak{I}_L(h_1^{l_1-l_2+i}, h_2^i), & \text{Otherwise} \end{cases}$$

and

$$\mathtt{g} = \begin{cases} \bigcup_{\mathtt{j}=1}^{\mathtt{l}_1^{'}} \Im_{\mathtt{L}}(\mathtt{g}_2^{\mathtt{l}_2^{'}-\mathtt{l}_1^{'}+\mathtt{j}}, \mathtt{g}_1^{\mathtt{j}}), & \textit{if} \quad \mathtt{l}_1^{'} \leq \mathtt{l}_2^{'} \\ \bigcup_{\mathtt{j}=1}^{\mathtt{l}_2^{'}} \Im_{\mathtt{L}}(\mathtt{g}_2^{\mathtt{j}}, \mathtt{g}_1^{\mathtt{j}}), & \textit{Otherwise} \end{cases}$$

Here, $\Im_{\mathtt{L}}(\mathtt{x}, \mathtt{y}) = \mathtt{min}(1,\ 1 - \mathtt{x} + \mathtt{y})$ *is the well-known R-implicator based on Lukasiewicz t-norm, viz., Lukasiewicz implicator, proposed by* [40]*. Then* $\mathfrak{Inc}(d_1, d_2)$ *is an HM-inclusion measure for* DHFE.

Proof. We can easily verify the axiomatic requirements (ID1), (ID2) and (ID3) of HM-inclusion measure for DHFE. Hence Q.E.D. □

The concept of HM-inclusion measure was defined in this section intending to use it in our evaluation problem, but there we want a DHFE. We know HM-inclusion measure is not a DHFE. So we have decided to use the $\mathcal{L}$-subset hood index instead of HM-inclusion measure in the evaluation problem. The necessity of DHFE into the evaluation problem had led us to take this decision.

In the following section, we discuss our evaluation problem and develop a methodology for ranking the objects in the problem by make use of the proposed definitions in this paper.

5. A novel methodology for ranking objects in an evaluation problem

The decision-making problem is being described as follows. $\mathtt{q}$ objects $\mathtt{F}_1, \mathtt{F}_2, ..., \mathtt{F}_\mathtt{q}$ shall be compared in our problem. Here each $\mathtt{F}_\mathtt{i}$ may be a branch of a company or a project under a vendor. Further to this, each of the q objects will be characterized as the parameterized collection of subsets of the universal set U, where U consists of categories of workers belonging to the object $\mathtt{F}_\mathtt{i}$. Let $\mathtt{U} = \Big\{x_1, x_2, ..., x_n.\Big\}$ be the universal set and $\mathtt{E} = \Big\{e_1, e_2, ..., e_m\Big\}$ be the parameter set. Here, E consists of parameters which are defined by experts in the relevant field. Moreover, the character of parameters of this problem is fuzzy. Also, the universal set U and the parameter set E are one and the same for all the q objects in this problem. Nevertheless, the number of workers in each category $\mathtt{x}_\mathtt{s}$ in different $\mathtt{F}_\mathtt{i}$ may vary. Also, note that the number of workers in distinct categories $\mathtt{x}_\mathtt{s},\ \mathtt{s} \in \{1, 2, ..., \mathtt{n}\}$ in the same object $\mathtt{F}_\mathtt{i}$ may be different. From these observations, we arrived at the conclusion that the evaluation of categories $\mathtt{x}_\mathtt{s},\ \mathtt{s} \in \{1, 2, ..., \mathtt{n}\}$ for the parameters $\mathtt{e}_\mathtt{r},\ \mathtt{r} \in \{1, 2, ..., \mathtt{m}\}$ in the object $\mathtt{F}_\mathtt{i},\ \mathtt{i} \in \{1, 2, ..., \mathtt{q}\}$ can be better presented by using an HFE or a DHFE. Since the provision for assigning negative mark is an added benefit for an assessment procedure, dual hesitant fuzzy element seems to be a better representative than hesitant fuzzy element. Thus, we constructed a dual hesitant fuzzy soft set $(F_i, A),\ i \in \{1, 2, ..., q\}$ for describing the object $F_i,\ i \in \{1, 2, ..., q\}$. Further, $(F_i, A),\ i \in \{1, 2, ..., q\}$ can be implicitly described as

$$(F_i, A) = \left\{ d_{F_i(e_r)}(x_s) = \left(h_{F_i(e_r)}(x_s), g_{F_i(e_r)}(x_s)\right) / \ r = 1, ..., m \text{ and } s = 1, ..., n. \right\}$$

and we denote $(F_i, A),\ i \in \{1, 2, ..., q\}$ by simply $F_i,\ i \in \{1, 2, ..., q\}$. Here $h_{F_i(e_r)}(x_s)$ and $g_{F_i(e_r)}(x_s)$ represent the sets of memberships and non-memberships of the workers in the category x_s to the set describing the parameter e_r, respectively. We develop the following method for ranking these $F_i,\ i = 1, 2, ..., q$ by being motivated from the work of Herrero[29].

Step 1:: Consider two objects F_i and F_j. Form the collection of $\mathcal{L}$-subsethood indexes, viz.,

$\{d_{\mathcal{L}}\left(d_{F_j(e_r)}(x_s), d_{F_i(e_r)}(x_s)\right) :\ r = 1, ..., m\ ;\ s = 1, ..., n\}$.

Step 2:: In this step, we fix s. i.e., we consider the category x_s. Here a weight vector for the parameters of this category must be defined by the decision makers, viz., $w^{(s)}$ such that $w^{(s)} = \left(w_1^{(s)}, w_2^{(s)}, ..., w_m^{(s)}\right)$, $w_r^{(s)} \in [0, 1]$ and $\sum_{r=1}^{m} w_r^{(s)} = 1$, where $w_r^{(s)}$ indicates the importance of the parameter e_r to the alternative x_s. Then using the WDHFMSM operator and the weight vector $w^{(s)}$, the $\mathcal{L}$-subsethood indexes are aggregated as follows:

$$\left(\frac{\bigoplus_{1 \le r_1 < r_2 < ... < r_k \le m} \left(\bigotimes_{t=1}^{k} \left[d_{\mathcal{L}}\left(d_{F_j(e_{r_t})}(x_s), d_{F_i(e_{r_t})}(x_s)\right)\right]^{w_{r_t}} \right)}{C_m^k} \right)^{1/k},$$

which is a DHFE denoted by $\delta^{(s)}(F_j, F_i)$.

Step 3:: Repeat step 2 for each $x_s,\ s \in \{1, 2, ..., n\}$ and the collection $\{\delta^{(s)}(F_j, F_i) :\ s = 1, ..., n\}$ are formed.

Step 4:: Before proceeding further, the weight vector deciding the importance of categories should be determined by the decision makers.

Let it be $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$, $\lambda_s \in [0, 1]$ and $\sum_{s=1}^{n} \lambda_s = 1$, where λ_s indicates the importance of the alternative x_s. Note that this weight vector λ is same for all the objects $F_i,\ i \in \{1, 2, ..., q\}$.

The Weighted geometric aggregation mean operator, proposed by Xia[15], could be used here for final aggregation, viz., $\bigotimes_{s=1}^{n} \left(\delta^{(s)}(F_j, F_i)\right)^{\lambda_s}$, which is again a DHFE denoted by $\delta(i, j)$. In our problem, we want to pay more attention to arguments having too high or too low performance. It justifies our decision of choosing WGM operator.

Step 5:: Now we find out the status of the DHFE $\delta(\mathtt{i},\mathtt{j})$, i.e., $\mathtt{S}_{(\delta(\mathtt{i},\mathtt{j}))}$, and it is denoted by $\mathcal{D}(\mathtt{F_i},\mathtt{F_j})$. Since $\mathcal{D}(\mathtt{F_i},\mathtt{F_j})$ is ultimately derived from $\mathcal{L}$-subsethood index of $\mathtt{F_j}$ over $\mathtt{F_i}$, $\mathcal{D}(\mathtt{F_i},\mathtt{F_j})$ gives out the degree of dominance of $\mathtt{F_i}$ relative to $\mathtt{F_j}$. Hence it will be called dominance index of $(\mathtt{F_i},\mathtt{A})$ over $(\mathtt{F_j},\mathtt{A})$.

Step 6:: Repeat steps 1-5 for $\mathtt{i},\mathtt{j} = 1,2,3,...,\mathtt{q}$, $\mathtt{i} \neq \mathtt{j}$. Thus we find out all combinations of dominance index and let us denote this collection by P, viz., $P = \{\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) : i,j = 1,2,...,q \ ; \ i \neq j\}$. Then P can be viewed as a comprehensive form of our evaluation problem.

Here we discuss the following remarks about the dominance index.

Remark 5.1. (i) $0 \leq \mathcal{D}(\mathtt{F_i},\mathtt{F_j}) \leq 1$.

(ii) $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) = 1 \Rightarrow \mathtt{F_i}$ *is completely dominant with respect to* $\mathtt{F_j}$ *in all aspects.*

(iii) $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) = 0 \Rightarrow$ *Stunning performance by the first object* $\mathtt{F_i}$ *while no performance at all by the second object* $\mathtt{F_j}$*. In a real working site, this will never happen. So in this paper, without loss of generality, we assume* $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) > 0$.

(iv) *For a fixed j, if* $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) = 1, \forall \mathtt{i},\ \mathtt{i} \neq \mathtt{j}$, *then* $\mathtt{F_j}$ *is inferior to every other objects. In that case* $\mathtt{F_j}$ *can be eliminated from further evaluation process and can be given the last rank. This remark shall be used later in this paper.*

Besides the above-said properties, this dominance index could be used for the pairwise comparison of objects. i.e., $\mathcal{D}(\mathtt{F_i},\mathtt{F_j}) \leq \mathcal{D}(\mathtt{F_j},\mathtt{F_i}) \Rightarrow \mathtt{F_i} \leq \mathtt{F_j}$ or literally, $\mathtt{F_j}$ dominates $\mathtt{F_i}$.

We know inclusion measure doesn't satisfy the transitive relation, and also the dominance index is derived from inclusion measure. So that, this newly introduced measure 'dominance index' is not suitable for the comparison of more than two objects. If this measure is being utilized in our problem, we need to extend this into more general settings which involve more than two objects. Towards this aim, some definitions are proposed as follows.

Definition 5.1. *Relative dominance of* $\mathtt{F_i}$ *with respect to* $\mathtt{F_j}$ *is given by*

$$\mathtt{R}(\mathtt{i},\mathtt{j}) = \frac{\mathcal{D}(\mathtt{F_i},\mathtt{F_j})}{\sum_{k=1,k\neq i}^{q} \mathcal{D}(\mathtt{F_k},\mathtt{F_i})} \quad , \qquad \mathtt{i},\mathtt{j} = 1,2,3,...,\mathtt{q} \ , \ \mathtt{i} \neq \mathtt{j}.$$

Now, the net dominance of $\mathtt{F_i}$ can be defined as the weighted average of its relative dominance.

Definition 5.2. *Net dominance of* $\mathtt{F_i}$ *is given by*

$$\mathtt{N}(\mathtt{i}) = \sum_{\mathtt{j}=1,\mathtt{j}\neq\mathtt{i}}^{\mathtt{q}} \mathtt{w_j}\mathtt{R}(\mathtt{i},\mathtt{j}), \quad \mathtt{i} = 1,2,3,...,\mathtt{q}.$$

where $\mathtt{w_j}$ *is a measure of the importance of the object* $\mathtt{F_j}, \mathtt{j} = 1, 2, 3, ..., \mathtt{q}, \mathtt{i} \neq \mathtt{j}$.

Here we wish to mention one thing. Both $\mathtt{N(i)}$ and $\mathtt{w_i}$ give out the rank of the object $\mathtt{F_i}$. Initially, both of them are unknown, and our aim is to find the rank of the object $\mathtt{F_i}$ which may be $\mathtt{N(i)}$ or $\mathtt{w_i}$. If there exist an invariant system of weights $(\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ satisfying $(\mathtt{N(1)}, \mathtt{N(2)}, ..., \mathtt{N(q)}) = (\mathtt{w_1}, \mathtt{w_2}, ..., \mathtt{w_q}) = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$, and $\mathtt{N(i)} = \sum_{\mathtt{j=1,j\neq i}}^{\mathtt{q}} \mathtt{w_j R(i,j)}, \mathtt{i} = 1, ..., \mathtt{q}$, then we are succeeded in this journey. This will be achieved by applying a little bit theories of linear algebra here. For that, a matrix $\mathtt{P}^* = [\mathtt{p_{ij}}]$ will be constructed as follows:

$$\mathtt{p_{ij}} = \begin{cases} \mathcal{D}(\mathtt{F_i}, \mathtt{F_j}), & i \neq j \\ (q-1) - \sum_{k=1,k\neq i}^{q} \mathcal{D}(\mathtt{F_k}, \mathtt{F_i}), & i = j \end{cases}$$

For the matrix $\mathtt{P}^*$, the following observations have been made.

- $\mathtt{P}^*$ is a positive matrix. This observation results from the properties of dominance index discussed earlier.
- Each column sum of $\mathtt{P}^*$ is q-1.
- $\mathtt{P}^*$ is an irreducible matrix.

Using the matrix $\mathtt{P}^*$, we can construct an eigenvalue problem $\mathtt{P^*X} = \lambda\mathtt{X}$, $\mathtt{X} = \begin{bmatrix} w_1 & w_2 & ... & w_q \end{bmatrix}^{\mathtt{T}}$, which is equivalent to the comprehensive form of the problem $\mathtt{P}$. From now on, we consider this eigenvalue problem instead of $\mathtt{P}$. Such a transformation gives the benefit of solving the evaluation problem consistently and uniquely. From the characteristics of $\mathtt{P}^*$, it is clear that $\mathtt{q} - 1$ is the unique dominant eigenvalue of $\mathtt{P}^*$. According to Perron-Frobenius theorem, the matrix $\mathtt{P}^*$ has a strictly positive eigenvector corresponding to the eigenvalue $\mathtt{q} - 1$, viz., $\mathtt{V} = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ with $\mathtt{P^*V} = (\mathtt{q} - 1)\mathtt{V}$, where $\mathtt{v_i} = \dfrac{\sum_{\mathtt{j=1,j\neq i}}^{\mathtt{q}}(\mathtt{v_j} * \mathcal{D}(\mathtt{F_i}, \mathtt{F_j}))}{\sum_{\mathtt{k=1,k\neq i}}^{\mathtt{q}} \mathcal{D}(\mathtt{F_k}, \mathtt{F_i})}, \mathtt{i} = 1, 2, 3, ..., \mathtt{q}$. Also, we know that this eigen vector is unique up to scalar multiplication. So we can make this eigen vector unique by imposing the condition $\sum_{\mathtt{i=1}}^{\mathtt{n}} \mathtt{v_i} = \mathtt{q}$. Thus, a unique and consistent system of weights $(\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ satisfying $(\mathtt{N(1)}, \mathtt{N(2)}, ..., \mathtt{N(q)}) = (\mathtt{w_1}, \mathtt{w_2}, ..., \mathtt{w_q}) = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ and $\mathtt{N(i)} = \sum_{\mathtt{j=1,j\neq i}}^{\mathtt{q}} \mathtt{w_j R(i,j)}$, $\mathtt{i} = 1, 2, 3, ..., \mathtt{q}$ have been obtained.

This vector $\mathtt{V} = (\mathtt{v_1}, \mathtt{v_2}, ..., \mathtt{v_q})$ is called the worth vector [29] associated with our evaluation problem $\mathtt{P}$. Usually, in a ranking method, the decision-maker consider which one is better than the other; nevertheless, they need not calculate how much better it is. But, here we need this feature. We think that the differences between preferences are also important. Here, Herrero's worth vector provide this feature. Each component of this vector gives the worth associated with a respective object. In other words, it cardinalizes the objects. According to Herrero [29], the worth vector provides not only a complete ranking of the objects under consideration but also an endogenous cardinalization that allows a quantitative estimate of their differences. We can now use the following observations of Herrero [29].

- $v_i > v_j \Rightarrow$ the object F_i is dominant with respect to the object F_j.
- The condition $\sum_{i=1}^{q} v_i = q$ allows us to identify the objects which are above or below the average.
- There exist a consistent evaluation function f which associates an evaluation problem P to its worth vector.i.e., $f(P) = V$, where $V = (v_1, v_2, ..., v_q)$ satisfying $P^*V = (q-1)V$ and $\sum_{i=1}^{q} v_i = q$. This function f enable us to handle distinct evaluation problems consistently and uniquely.

For ranking, we adopt the following steps.

Step 1:: First sort out v_i's in descending order.

Step 2:: Let the sorted vectors be $u_1, u_2, ..., u_q$.

Step 3:: If $u_i = v_j$, then the rank of the object F_j is i. Also, its worth is v_j. Repeat this for every $i = 1, 2, 3, ..., q$. Thus our evaluation have been completed .

To assess the performance of the proposed method, in the following section, we depict a problem of continuous evaluation of workers in a mobile tower construction site.

Practical Example. SU Square Projects and Infrastructures (P) Ltd - an ISO 9001: 2008 certified company - is primarily engaged in the construction and maintenance of mobile communication towers for various passive infrastructure providers in the telecom sector in Kerala. The promoters of the company aim to provide optimal coverage to rural and mountainous areas throughout Kerala and southern India. To achieve this, they have developed several strategies to implement targeted and efficient actions.

As part of its business expansion, the company's promoters have decided to evaluate workers based on a predetermined performance package. The primary objective of this initiative is to minimize the time required to complete tower installations without compromising quality.

Through this evaluation, they aim to tap into the full potential of each employee. To foster healthy competition, they have decided to rank the various sites according to worker performance. In addition, gifts have been included in the package as incentives, directly linked to site rankings. This initiative has understandably generated interest and enthusiasm among the workers.

If this ranking can be cardinalized, that is, expressed in numerical terms, the distribution of perks can be carried out in a more consistent and objective manner. It is worth recalling here that our proposed ranking method allows for such cardinalization, thereby enhancing its practical applicability. As the next step, we proceed to analyze the compatibility of the problem's structure with the proposed framework.

Suppose $\mathtt{F}_1, \mathtt{F}_2, ..., \mathtt{F}_\mathtt{q}$ are $\mathtt{q}$ sites considered for evaluation. The selection of the appropriate assessment criteria is an inevitable part of the evaluation process. This selection should be made by experts in the respective fields. In the background of years of experience, the promoters select the appropriate parameters for the evaluation. The list of parameters and their descriptions are given in Table 5.2. Let the set of parameters be denoted by A, that is, $\mathtt{A} = \{\mathtt{TAT}, \mathtt{Quality}, \mathtt{Safety}, \mathtt{Costing}\}$.

TABLE 5.2. Parameters List

Parameters	**Descriptions**	**Notation**
Turnaround Time (TAT)	The time taken to complete a particular project	TAT
Quality of Work	Maintaining required quality	Quality
Complying with Safety Norms	Every workforce must comply, e.g., wearing safety helmets, safety shoes, using barricades, signboards, etc.	Safety
Project Costing	Costs incurred for a particular project, including material and labor costs	Costing

Next, our discussion turns to the workforce. Each tower-working-site would have needed different categories of workers. Those categories of workers provided by the promoters are as shown in Table 5.3.

Table 5.3. List of categories of workers together with their description

Categories of workers	Details about no. of workers	Notation	Weights
Civil Engineer	One engineer per site	CE	.2
Electrical Engineer	One engineer per site	EE	.25
Mason	Each site will have two, three or four masons	MN	.15
Helper (Mason)	The number of helpers depends upon the number of masons. A site requires six masons including helpers	HM	.05
Electrician	Each site will have three electricians.	EN	.1
Trainees(Electrician)	For helping electricians,there are two trainees.	ET	.05
Riggers	Normally 6 riggers per site, but for Roof Top Towers(RTT) it is 7	RS	.08
Head load workers for each site is 10	The number of head load workers	HL	.07
Concrete Labors	A site requires 20 concrete labors but RTT needs only 10	CL	.05

Here we would like to indicate some important points. From the description of categories of workers, it is clear that each category may have more than one members. So to get a better picture, we have to evaluate them individually. Also, note that different categories may contain a distinct number of deputies. Further, the number of employees belongs to the same category in two distinct sites may not be the same. These observations have led us to choose the hesitant fuzzy elements as an appropriate structure for representing the evaluation of a category based on a parameter. The provision for assigning negative marking is an added benefit for an assessment. So that, the dual hesitant fuzzy element seems to be the better representative rather than the hesitant fuzzy element. Thus we arrive at the conclusion that the dual hesitant fuzzy soft set is used for exhibiting the evaluation details of a mobile tower site. This demonstration shall be described as follows. Here, $\mathtt{U} = \{\mathtt{CE}, \mathtt{EE}, \mathtt{MN}, \mathtt{HM}, \mathtt{EN}, \mathtt{ET}, \mathtt{RS}, \mathtt{HL}, \mathtt{CL}\}$, the set of category names of workers, are taken as the universal set. The dual hesitant fuzzy soft set $(\mathtt{F_1}, \mathtt{A})$ represents the evaluation measurements of $\mathtt{F_1}$. This DHFSS can be briefly described as follows. $\mathtt{F_1}$ is a mapping given by $\mathtt{F_1} : \mathtt{A} \rightarrow \mathtt{DHFS(U)}$. Here, $(\mathtt{F_1}, \mathtt{A}) = \{\mathtt{F_1(TAT)}, \mathtt{F_1(Quality)}, \mathtt{F_1(Safety)}, \mathtt{F_1(Costing)}\}$, where each $\mathtt{F_1}(.)$ is a dual hesitant fuzzy set. To get better clarification, we discuss the case of a particular $\mathtt{F_1}(.)$, viz., $\mathtt{F_1(TAT)}$. Here $\mathtt{F_1(TAT)}$ is a dual hesitant fuzzy set which assigns to each member of U a dual hesitant fuzzy element. For example, Riggers, $\mathtt{RS} \in \mathtt{U}$, the corresponding dual hesitant fuzzy element is $\mathtt{d}_{\mathtt{F_1(TAT)}}(\mathtt{RS}) = (\mathtt{h}_{\mathtt{F_1(TAT)}}(\mathtt{RS}), \mathtt{g}_{\mathtt{F_1(TAT)}}(\mathtt{RS}))$ where $\mathtt{h}_{\mathtt{F_1(TAT)}}(\mathtt{RS})$ is a finite subset of [0,1] consisting of either 6 or 7 entries which represents the evaluation given to Riggers working at $\mathtt{F_1}$ for TAT. In other words, $\mathtt{h}_{\mathtt{F_1(TAT)}}(\mathtt{RS})$ gives the membership of Riggers to the set which describes TAT. Similarly, $\mathtt{g}_{\mathtt{F_1(TAT)}}(\mathtt{RS})$ provides the non-membership of Riggers to the set which describes TAT. We know elements of DHFEs are arranged in increasing order. Here also, the marks obtained by different Riggers working at $\mathtt{F_1}$ could be arranged in increasing order. Our evaluation is about sites and not about employees. So that, there is no ambiguity in arranging the marks in this manner.

In a similar manner, we construct $\mathtt{F_1(Quality)}, \mathtt{F_1(Safety)}, \mathtt{F_1(Costing)}$ and thus formed $(\mathtt{F_1}, \mathtt{A})$, denoted by $\mathtt{F_1}$ Likewise we build $(\mathtt{F_2}, \mathtt{A})$ for site $\mathtt{F_2}$, $(\mathtt{F_3}, \mathtt{A})$ for site $\mathtt{F_3}$, and $(\mathtt{F_4}, \mathtt{A})$ for site $\mathtt{F_4}$, which are denoted by $\mathtt{F_2}, \mathtt{F_3}, \mathtt{F_4}$ respectively. In this way, we have accommodated successfully all the information provided by the experts. Now, by all means, we have been convinced that the proposed method is the suitable method for this problem. Thus, we are moving onto solving the problem using the proposed method.

TABLE 5.4. Tabular representation of dual hesitant fuzzy soft set $F_1 = \left(\tilde{F_1}, A\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.9} {.1}	{.98} {.101}	{.96} {.001}	{.97} {.12}
EE	{.85} {.1}	{.88} {.01}	{.89} {.11}	{.84} {.005}
MN	{.91,.92} {.2}	{.93,.98} {.1}	{..94,.95} {.22}	{.99,.992} {.101}
HM	{.85,.86,.868,.869} {.2}	{.886,.89,.895,.93} {.13}	{.92,.94,.949,.952} {.1}	{..981,.983,.985,.989} {.09}
EN	{.9,.92,.923} {.103}	{..941,.942,.948} {.141}	{.891,.892,.894} {.12}	{..86,.864,.865} {.2}
ET	{.92,.95} {.201}	{.93,.96} {.138}	{.91,.93} {.17}	{..98,.99} {.142}
RS	{.91,.92,925,.927, .93,.934} {..005}	{.96,.964,.967,.972, .974,.98} {.02}	{.81,.83,.836,.84, .847,.85} {.001,.003}	{.91,.913,.924,.926, .929,.93} {.1}
HL	{.71,.714,.719,.723, .725,.727,.738,.739, .74,.743} {.07}	{.732,.736,.74,.745, .749,.76,.762,.765, .769,.78} {.156}	{..813,.824,.83,.845, .86,.864,.869,.87, .881,.883} {.1}	{..91,.913,.915,.95, .98,.981,.982,.984, .985,.989} {.09}
CL	{..73,.734,.735,.738, .739,.741,.742,.746, .749,.75,.752,.753, .755,.757,.758,.76, .762,.763,.765,.768} {.102}	{.81,.814,.815,.82, .834,.836,.838,.841, .843,.845,.846,.849, .852,.856,.859,.862, .864,.868,.87,.89} {.18}	{.91,.913,.915,.921, .924,.926,.928,.929, .932,.934,.935,.937, .938,.94,.942,.943, .944,.945,.946,.95} {.21}	{.87,.876,.877,.88, .884,.886,.887,.89, .892,.893,.895,.9, .92,.93,.95,.97, .972,.975,.977,.979} {.08}

TABLE 5.5. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_2} = \left(\tilde{\mathtt{F_2}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.8} {.1}	{..72} {.2}	{.7} {.07}	{.8} {.11}
EE	{.6} {.18}	{.65} {.121}	{..61} {.109}	{.63} {.123}
MN	{.7,.72,.74,.746} {.28}	{..77,.792,.798,.81} {.105}	{.75,.756,.7567,.761} {.127}	{.791,.794,.81,.83} {.174}
HM	{..52,.54} {.108}	{.59,.61} {.113}	{..61,.63} {.101}	{.67,.69} {.161}
EN	{..73,.74,.75} {.28}	{..716,.723,.74} {.26}	{.74,.743,.745} {.23}	{..732,.735,.761} {.21}
ET	{.62,.65} {.102}	{.68,.685} {.195}	{..645,..672} {.138}	{.656,.692} {.124}
RS	{.634,.639,.642,.645, .649,.651,.657} {.101}	{.636,.654,.672,.675, .681,.692,.71} {.131}	{..621,.628,.634,.637, .639,.64,.642} {.159}	{..624,.628,.637,.645, .676,.684,.692} {.128}
HL	{.71,.718,.72,.723, .727,.734,.74,.749, .752,.76} {.197}	{..76,.762,.763,.771, .772,.78,.794,.799, .88,.89} {.111}	{.81,.83,.85,.872, .876,.88,.882,.884, .887,.89} {.17}	{..83,.832,.834,.847, .849,.852,.853,.855, .857,.86} {.09}
CL	{..61,.672,.689,.692, .694,.696,.698,.71, .72,.726} {.001}	{.52,.525,.529,.531, .535,.538,.542,.545, .549,.559} {.007}	{.61,.68,.694,.712, .724,.75,.758,.778, .79,.81} {.0012}	{.634,.691,.695,.724, .728,.729,.73,.739, .743,.745} {.0089}

TABLE 5.6. Tabular representation of dual hesitant fuzzy soft set $F_3 = \left(\tilde{F_3}, A\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.8} {.12}	{.82} {.23}	{.9} {.116}	{.85} {.017}
EE	{.9} {.101}	{.92} {.119}	{.85} {.181}	{.93} {.192}
MN	{.73,.81,.84,.91} {.21}	{.82,.85,.87,.89} {.101}	{.61,.82,.84,.89} {.10001}	{.91,.935,.94,.942} {.002}
HM	{.812,.823} {.11}	{.71,.75} {.15}	{.52,.61} {.05}	{.92,.95} {.07}
EN	{.93,.941,.95} {.1002}	{.941,.945,.95} {.001}	{.95,.953,.96} {.023}	{.936,.939,.95} {.008}
ET	{.82,.85} {.071}	{.836,.851} {.082}	{.72,.75} {.14}	{.91,.95} {.21}
RS	{.71,.73,.79,.85, .88,.92,.95} {.106}	{.91,.912,.918,.923, .934,.94,.945} {.11}	{.81,.815,.819,.821, .828,.834,.836} {.105}	{.852,.854,.86,.865, .872,.874,.88} {.009}
HL	{.71,.712,.734,.745, .82,.832,.838,.84, .85,.9} {.006}	{.81,.812,.815,.832, .86,.865,.881,.92, .925,.928} {.001}	{.91,.915,.918,.923, .927,.932,.938,.941, .943,.95} {.006}	{.8,.82,.824,.83, .835,.839,.85,.853, .86,.868} {.08}
CL	{.9,.913,.917,.924, .931,.936,.939,.94, .948,.95} {.172}	{.82,.825,.828,.831, .833,.84,.852,.853, .864,.87} {.025}	{.71,.72,.724,.73, .738,.74,.742,.744, .75,.752} {.173}	{.78,.81,.85,.91, .92,.925,.93,.938, .942,.945} {.087}

TABLE 5.7. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_4} = \left(\tilde{\mathtt{F_4}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.6} {.102}	{.7} {.189}	{.6} {.076}	{.9} {.045}
EE	{.95} {.108}	{.98} {.009}	{.8} {.0004}	{.92} {.153}
MN	{.8,.85,.87} {.11}	{.76,.81,.9} {.137}	{.9,.91,.93} {.023}	{.84,.89,.9} {.001}
HM	{.6,.65,.67} {.1}	{.8,.89,.92} {.02}	{.92,.94,.96} {.03}	{.87,.88,.89} {.132}
EN	{.78,.79,.85} {.076}	{.82,.84,.87} {.23}	{.74,.78,.79} {.2}	{.91,.92,.95} {.13}
ET	{.65,.68} {.122}	{.85,.87} {.114}	{.91,.94} {.176}	{.92,.96} {.13}
RS	{.84,.87,.89,.92, .94,.956,.97} {.106}	{.91,.912,.919,.923, .928,.934,.95} {.12}	{.71,.73,.74,.752, .759,.76,.769,.928, .934,.95} {.104}	{.91,.918,.92,.924, .928,.93,.934} {.113}
HL	{.65,.676,.685,.694, .725,.738,.824,.839, .841,.852} {.21}	{.91,.934,.943,.952, .956,.959,.964,.968, .969,.97} {.008}	{.7,.75,.78,.791, .792,.798,.82,.83, .845,.852} {.2}	{.9,.923,.941,.949, .95,.954,.958,.961, .962,.97} {..089}
CL	{.918,.925,93,.934, .938,.942,.946,.948, .951,.953} {.06}	{.94,.941,.943,.947, .952,.954,.957,.961, .962,.964} {.087}	{.71,.78,.79,.794, .799,.81,.845,.848, .849,.852} {.1}	{.92,.94,.945,.947, .952,.953,.957,.959, .962,.963} {.01}

TABLE 5.8. Tabular representation of dual hesitant fuzzy soft set $F_5 = \left(\tilde{F_5}, A\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.4} {.1}	{.32} {.07}	{.38} {.13}	{.42} {.12}
EE	{.2} {.12}	{.3} {.04}	{.12} {.021}	{.45} {.118}
MN	{.31,.342,.36,.41} {.1}	{.36,.378,.394} {.17}	{.24,.245,.253,.26} {.03}	{.41,.423,.445,.45} {.071}
HM	{.2,.45} {.01}	{.35,.42} {.12}	{.13,.32} {.13}	{.34,.39} {.232}
EN	{.42,.435,.44} {.16}	{.38,.382,.39} {.13}	{.24,.28,.3} {.11}	{.45,.49,.53} {.18}
ET	{.51,.56} {.22}	{.43,.47} {.14}	{.27,.292} {.16}	{.53,.58} {.103}
RS	{.23,.274,.282,.287, .29,.3,.31} {.006}	{.31,.312,.313,.317, .32,.325,.327,.329} {.102}	{.134,.178,.18,.193, .195,.198,.21} {.17}	{.42,.43,.436,.442, .445,.456,.458} {.023}
HL	{.21,.223,.23,.242, .245,.251,.267,.35, .4,.42} {.211}	{.31,.33,.37,.48, .51,.53,.57,.59,.62} {.108}	{.12,.124,.127,.132, .135,.182,.24,.29, .3,.34} {.12}	{.41,.414,.418,.423, .425,.43,.478,.48, .482,.485} {..019}
CL	{.34,.345,.348,.352, .354,.357,.389,.392, .395,.41} {.16}	{.43,.434,.437,.439, .448,.449,.452,.46, .47,.475} {.17}	{.21,.213,.215,.237, .297,.299,.32,.33, .375,.39} {.012}	{.13,.17,.19,.21, .214,.218,.24,.248, .25,.27} {.11}

TABLE 5.9. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_6} = \left(\tilde{\mathtt{F_6}}, \mathtt{A}\right)$

U/E	TAT	Quality	Safety	Costing
CE	{.1} {.001}	{.01} {.0007}	{.2} {.103}	{.12} {.012}
EE	{.05} {.012}	{.03} {.004}	{.13} {.0021}	{.15} {.107}
MN	{.04,.07,.09,.11} {.009}	{.02,.03,.07,.08} {.107}	{.04,.05,.09,.11} {.13}	{.1,.11,.13,.14} {.061}
HM	{.13,.18} {.009}	{.15,.19} {.106}	{.13,.17} {.036}	{.14,.17} {.152}
EN	{.132,.135,.137} {.034}	{.048,.05,.08} {.15}	{.14,.148,.152} {.196}	{.136,.145,.178} {.007}
ET	{.046,.078} {.002}	{.021,.042} {.001}	{.091,.098} {.109}	{.34,.41} {.121}
RS	{.12,.124,.183,.24, .29,.34,.42} {.016}	{.042,.09,.098,.13, .139,.14,.172} {.202}	{.012,.017,.019,.034, .039,.052} {.107}	{.02,.026,.031,.038, .042,.043,.047} {.003}
HL	{.013,.016,.018,.021, .023,.026,.031,.034, .035,.039} {.101}	{.024,.026,.028,.029, .032,.034,.036,.038, .039,.043} {.1}	{.049,.051,.053,.054, .061,.08,.123,.137, .139,.152} {.0012}	{.52,.525,.585,.592, .61,.618,.624,.63, .631,.637} {.19}
CL	{.14,.145,.1452,.151, .159,.162,.168,.169, .171,.172} {.1006}	{.18,.183,.184,.192, .199,.23,.234,.24, .26,.263} {.1007}	{.04,.045,.053,.078, .092,.098,.13,.132, .153,.157} {.0124}	{.2,.22,.234,.247, .249,.25,.253,.257, .259,.261} {.011}

TABLE 5.10. Tabular representation of dual hesitant fuzzy soft set $\mathtt{F_7} = (\tilde{\mathtt{F_7}}, \mathtt{A})$

U/E	TAT	Quality	Safety	Costing
CE	{.95} {.01}	{.99} {.00002}	{.97} {.0001}	{.99} {.0101}
EE	{.97} {.001}	{.99} {.004}	{.91} {.0501}	{.94} {.008}
MN	{.94,.95,.96} {.001}	{.97,.99,.999} {.107}	{.97,.98,.99} {.0103}	{.997,.998,.999} {.0401}
HM	{.91,.913,.914} {.0071}	{.912,.913,.915} {.0102}	{.96,.967,.98} {.103}	{.991,.993,.995} {.0232}
EN	{.961,.964,.978} {.016}	{.97,.973,.975} {.019}	{.965,.969,.972} {.011}	{.961,.964,.967} {.018}
ET	{.952,.96} {.022}	{.961,.972} {.014}	{.948,.952} {.016}	{.993,.997} {.0103}
RS	{.967,.968,.969,.97, .972,.974,.976} {.003}	{.981,.982,.983,.985, .987,.989,.99} {.105}	{.86,.87,.89,.9, .92,.93,.94} {.007}	{.964,.966,.967,.97, .971,.973,.98} {.0203}
HL	{.88,.882,.885,.887, .89,.892,.895,.897, .91,.93} {.011}	{.961,.963,.967,.969, .971,.973,.975,.977, .981,.983} {.008}	{.951,.953,.955,.957, .958,.96,.963,.965, .967,.969} {.0102}	{.986,.988,.989,.991, .992,.994,.995,.997, .998,.999} {.0019}
CL	{.951,.953,954,.955, .956,.958,.959,.961, .962,.964} {.00409}	{.962,.963,.964,.965, .967,.968,.969,.97, .971,.972} {.0024}	{.951,.952,.953,.954, .955,.956,.957,.959, .962,.967} {.0014}	{.981,.9823,.983,.9834, .9835,.984,.9842,.9844, .9846,.985} {.0001}

TABLE 5.11. Weight vectors which decide the importance of parameters in each category.

U/E	TAT	Quality	Safety	Costing
Civil Engineer	.35	.2	.1	.35
Electrical Engineer	.35	.2	.1	.35
Mason	.3	.25	.15	.3
Helper(Mason)	.35	.1	.2	.35
Electrician	.3	.2	.2	.3
Trainees(Electrician)	.3	.2	.2	.3
Riggers	.3	.25	.25	.2
Headload workers	.3	.1	.2	.4
Concrete labors	.3	.15	.15	.4

The tabular representation of `F1`, `F2`, `F3`, ..., `F7` respectively, formed from the information provided by the promoters, are shown in tables 4, 5, 6, ..., 10. Recall that the weight vector for categories is shown in the last column of Table 5.3.
i.e., $\lambda = (0.2, 0.25, 0.15, 0.05, 0.1, 0.05, 0.08, 0.07, 0.05)$.

Table 5.11 provides the weight assigned by experts for the parameters in each category. In this table, each row represents the weight vector for the respective category in that row. For example, the first row corresponds to the weight vector $w^{(1)} = (0.35, 0.2, 0.1, 0.35)$ for `civilengineer`. That is, for the `civilengineer`, 0.35 weight is given for `TAT`, 0.2 for `Quality`, 0.1 for `Safety`, and 0.35 for `Costing`. Similarly, the fifth row gives the weight vector $w^{(5)} = (0.3, 0.2, 0.2, 0.3)$ for `Electrician`, the seventh row provides the weight vector $w^{(7)} = (0.3, 0.25, 0.25, 0.2)$ for the `Riggers`, and so on.

Thus the building blocks of the evaluation, namely, weights and evaluations, were obtained. Now the authors proceeded to construct P, the comprehensive form of the problem. For that passes steps 1 through 6 and calculated $D(F_i, F_j)$, $i, j = 1, 2, 3, 4$ $i \neq j$. Further, the matrix P^* can be constructed by using the comprehensive problem P, which is as shown below.

$$\mathrm{P}^* = \begin{bmatrix} .7263177 & .981574 & .977197 & .976024 & 1 & .981677 & .968049 \\ .949087 & .4481468 & .958576 & .960466 & 1 & 1 & .93128 \\ .986 & 1 & .6628048 & .998439 & 1 & 1 & .977028 \\ .9175843 & .9348083 & .9285823 & .5822471 & .9372493 & .9372493 & .9093323 \\ .7509 & .8298589 & .7658639 & .7717499 & .1227487 & .9283619 & .7206219 \\ .670111 & .805612 & .706976 & .711074 & .940002 & .1527118 & .614004 \\ 1 & 1 & 1 & 1 & 1 & 1 & .8796848 \end{bmatrix}$$

P^* instead of P is demonstrated because of limited space. The authors created an eigen-value problem $P^*X = \lambda X$, using this P^*. From the previous discussion, it is obvious that 6 (that is., $q-1$) is the dominant eigenvalue. The objective is to determine an eigen-vector of this dominant eigenvalue. Here the authors are looking for the unique eigen vector $(v_1, v_2, ..., v_q)$ satisfying the condition $\sum_{i=1}^{q} v_i = q$. There are numerous methods and corresponding softwares available in the literature for finding out the eigen vector of an eigen-value problem. Since the authors needed eigenvector corresponding to the dominant eigenvalue, they adopted the power method and developed a C++ program for generating the required unique eigenvector associated with the dominant eigenvalue 6. The normalised eigen-vector, namely, the worth vector, calculated by this program is given as $(1.0968442, 1.0348668, 1.09787431.0221596, 0.8260924, 0.7783165, 1.143846)$. Then, went through the ranking procedure and obtained the ranking as

$$F7 > F3 > F1 > F2 > F4 > F5 > F6.$$

The ranking of sites together with their worth is exhibited in Table 5.12:

Table 5.12. Ranking of Sites

Site:	F_7	F_3	F_1	F_2	F_4	F_5	F_6
WORTH:	1.1438	1.0979	1.0968	1.0349	1.0222	0.8261	0.7783
RANK:	1	2	3	4	5	6	7

This information will be equipped promoters to distribute perks based on the pre-announced package (that is, distribute perk based on their worth) and which will improve the work quality of employees positively in subsequent.

TABLE 6.13. Ranking of Sites after removing Site F_7

Site:	F_3	F_1	F_2	F_4	F_5	F_6
WORTH:	1.104812	1.097866	1.080376	1.023899	0.878530	0.805518
RANK:	1	2	3	4	5	6

6. Discussion

Let us examine the significance of non-membership values in this problem. To do so, we exclude all non-membership values and recalculate the worth vector, which now becomes $(1.089, 1.011452, 1.07204, 1.06576, 0.8835705, 0.7626358, 1.1159)$. Previously, the worth of F_5 was 0.8261, but after the omission, it is increased to 0.8835705 indicating that non-membership values contribute to a decrease in worth. These observations clearly highlight the impact of non-membership values on the overall ranking. Since perks are awarded in proportion to worth, it is the collective responsibility of all employees at the site to ensure that no one engages in actions that contribute to non-membership values. Such vigilance helps minimize the risk of violating strictly prohibited rules.

Next, the authors discuss Remark 5.1. From P^*, we get $D(F_7, F_i) = 1,\ \forall i \neq 7$, which implies that F_7 is completely dominant with respect to $F_i,\ \forall i \neq 7$ in all aspects. By our earlier calculations, the rank of F_7 is one. This result coincides with remark 5.1(iii). To verify the second statement of remark 5.1 (iv), the authors eliminate F_7 and calculate the worth vector. The new ranking is as shown in Table 6.13. If the rank of each of the above six sites is incremented by one position and site F_7 is assigned the first rank, then it can be seen that this will coincide with the previous ranking; this verifies the remark 5.1(iv). However, if one needs the worth of F_7 in addition to just ranking, this site must be included in the ranking procedure. Another noteworthy thing is that the omission of F_7 increases the worth of other sites.

7. Conclusion

In this paper, the authors have developed an innovative method based on Linear Algebra, for solving a real-life decision-making problem. By choosing the dual hesitant fuzzy soft set as the framework, the problem becomes quite handy. By implementing the eigen-value concepts, the solution becomes more reliable and precise. This method is suitable for the evaluation of unrelated data. The authors have also presented a practical application for their proposed method which necessarily depicts the effectiveness of the method.

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References

[1] Torra, V. (2010). Hesitant fuzzy sets. International Journal of Intelligent Systems, 25(6), 529–539.

[2] Zhu, B., Xu, Z., & Xia, M. (2012). Dual hesitant fuzzy sets. Journal of Applied Mathematics, 2012(1), 879629.

[3] He, Y. P. (2016). An approach to dual hesitant fuzzy soft set based on decision making. In Fuzzy Systems & Operations Research and Management (pp. 339–349). Springer International Publishing.

[4] Maclaurin, C. (1729). A second letter to Martin Folkes, Esq.; concerning the roots of equations, with demonstration of other rules of algebra. Philosophical Transactions of the Royal Society of London, Series A, 1729(36), 59–96.

[5] Qin, J., Liu, X., & Pedrycz, W. (2015). Hesitant fuzzy Maclaurin symmetric mean operators and its application to multiple-attribute decision making. International Journal of Fuzzy Systems, 17, 509–520.

[6] Yu, D., Li, D. F., & Merigó, J. M. (2016). Dual hesitant fuzzy group decision making method and its application to supplier selection. International Journal of Machine Learning and Cybernetics, 7, 819–831.

[7] Ye, J. (2014). Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making. Applied Mathematical Modelling, 38(2), 659–666.

[8] Xu, Y., Rui, D., & Wang, H. (2015). Dual hesitant fuzzy interaction operators and their application to group decision making. Journal of Industrial and Production Engineering, 32(4), 273–290.

[9] Wang, F., Li, X., & Chen, X. (2014). Hesitant fuzzy soft set and its applications in multi-criteria decision making. Journal of Applied Mathematics, 2014(1), 643785.

[10] Babitha, K. V., & John, S. J. (2013). Hesitant fuzzy soft sets. Journal of New Results in Science, 2(3).

[11] He, Y. P. (2016). An approach to dual hesitant fuzzy soft set based on decision making. In Fuzzy Systems & Operations Research and Management (pp. 339–349). Springer International Publishing.

[12] Arora, R., & Garg, H. (2018). A robust correlation coefficient measure of dual hesitant fuzzy soft sets and their application in decision making. Engineering Applications of Artificial Intelligence, 72, 80–92.

[13] Garg, H., & Arora, R. (2018). Dual hesitant fuzzy soft aggregation operators and their application in decision-making. Cognitive Computation, 10(5), 769–789.

[14] Zhang, H., & He, Y. (2018). A rough set-based method for dual hesitant fuzzy soft sets based on decision making. Journal of Intelligent & Fuzzy Systems, 35(3), 3437–3450.

[15] Xu, Z., & Xia, M. (2011). Distance and similarity measures for hesitant fuzzy sets. Information Sciences, 181(11), 2128–2138.

[16] Zhang, H., & Yang, S. (2016). Inclusion measure for typical hesitant fuzzy sets, the relative similarity measure and fuzzy entropy. Soft Computing, 20, 1277–1287.

[17] Garg, H., & Arora, R. (2017). Distance and similarity measures for dual hesitant fuzzy soft sets and their applications in multicriteria decision making problem. International Journal for Uncertainty Quantification, 7(3).

[18] Garg, H., & Arora, R. (2020). Maclaurin symmetric mean aggregation operators based on t-norm operations for the dual hesitant fuzzy soft set. Journal of Ambient Intelligence and Humanized Computing, 11, 375–410.

[19] Yu, D., Zhang, W., & Xu, Y. (2013). Group decision making under hesitant fuzzy environment with application to personnel evaluation. Knowledge-Based Systems, 52, 1–10.

[20] Qin, J., Liu, X., & Pedrycz, W. (2016). Frank aggregation operators and their application to hesitant fuzzy multiple attribute decision making. Applied Soft Computing, 41, 428–452.

[21] Hu, J., Yang, Y., Zhang, X., & Chen, X. (2018). Similarity and entropy measures for hesitant fuzzy sets. International Transactions in Operational Research, 25(3), 857–886.

[22] Li, D., Zeng, W., & Li, J. (2015). New distance and similarity measures on hesitant fuzzy sets and their applications in multiple criteria decision making. Engineering Applications of Artificial Intelligence, 40, 11–16.

[23] Zeng, W., Li, D., & Yin, Q. (2016). Distance and similarity measures between hesitant fuzzy sets and their application in pattern recognition. Pattern Recognition Letters, 84, 267–271.

[24] Zhang, H. Y., & Zhang, W. X. (2009). Hybrid monotonic inclusion measure and its use in measuring similarity and distance between fuzzy sets. Fuzzy Sets and Systems, 160(1), 107–118.

[25] Grzegorzewski, P. (2011). On possible and necessary inclusion of intuitionistic fuzzy sets. Information Sciences, 181(2), 342–350.

[26] Zhang, H. Y., Yang, S. Y., & Yue, Z. W. (2016). On inclusion measures of intuitionistic and interval-valued intuitionistic fuzzy values and their applications to group decision making. International Journal of Machine Learning and Cybernetics, 7, 833–843.

[27] Zhang, H., & Yang, S. (2016). Inclusion measure for typical hesitant fuzzy sets, the relative similarity measure and fuzzy entropy. Soft Computing, 20, 1277–1287.

[28] Sahin, R., & Karabacak, M. (2015). A multi-attribute decision making method based on inclusion measure for interval neutrosophic sets. International Journal of Engineering and Applied Sciences, 2(2), 13–15.

[29] Herrero, C., & Villar, A. (2013). On the comparison of group performance with categorical data. PLoS One, 8(12), e84784.

[30] Alcantud, J. C. R., de Andrés Calle, R., & Torrecillas, M. J. M. (2016). Hesitant fuzzy worth: An innovative ranking methodology for hesitant fuzzy subsets. Applied Soft Computing, 38, 232–243.

[31] Wen, X. (2023). Weighted hesitant fuzzy soft set and its application in group decision making. Granular Computing, 8(6), 1583–1605.

[32] Zadeh, L. A. (1965). Fuzzy sets. Information and Control, 8(3), 338–353.

[33] Molodtsov, D. (1999). Soft set theory—first results. Computers & Mathematics with Applications, 37(4–5), 19–31.

[34] Al-shami, T. M. (2023). (2,1)-Fuzzy sets: Properties, weighted aggregated operators and their applications to multi-criteria decision-making methods. Complex & Intelligent Systems, 9(2), 1687–1705.

[35] Ibrahim, H. Z., Al-Shami, T. M., & Elbarbary, O. G. (2021). (3,2)-Fuzzy sets and their applications to topology and optimal choices. Computational Intelligence and Neuroscience, 2021, 1272266.

[36] Al-shami, T. M., & Mhemdi, A. (2023). Generalized frame for orthopair fuzzy sets: (m,n)-fuzzy sets and their applications to multi-criteria decision-making methods. Information, 14(1), 56.

[37] Sivadas, A., John, S. J., & Athira, T. M. (2024). (p,q)-fuzzy aggregation operators and their applications to decision-making. The Journal of Analysis, 2024, 1–30.

[38] Al-shami, T. M., Alcantud, J. C. R., & Mhemdi, A. (2023). New generalization of fuzzy soft sets: (a,b)-Fuzzy soft sets. AIMS Mathematics, 8(2), 2995–3025.

[39] Ibrahim, H. Z., Al-shami, T. M., Arar, M., & Hosny, M. (2024). k_m^n-Rung picture fuzzy information in a modern approach to multi-attribute group decision-making. Complex & Intelligent Systems, 10(2), 2605–2625.

[40] Smets, P., & Magrez, P. (1987). Implication in fuzzy logic. International Journal of Approximate Reasoning, 1(4), 327–347.

[41] Yang, S. Y., Zhang, H. Y., & Yue, Z. W. (2013). Inclusion measure and its use in measuring similarity and distance measure between hesitant fuzzy sets. IEEE International Conference on Granular Computing (GrC). IEEE.

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ANALYTICAL EXPLORATION OF WEYL-CONFORMAL CURVATURE TENSOR IN LORENTZIAN β-KENMOTSU MANIFOLDS ENDOWED WITH GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. This paper investigates the conformal curvature properties of *Lorentzian β-Kenmotsu (LβK) manifolds* admitting a *generalized Tanaka-Webster (g-TW) connection*. We begin by establishing the fundamental preliminaries of $L\beta K$ manifolds and exploring their curvature properties under the influence of g-TW connection. The study then focuses on specific curvature conditions, namely $\widetilde{R}\cdot\widetilde{S}=0$, $\widetilde{S}\cdot\widetilde{R}=0$, conformal flatness, ζ-conformal flatness, and pseudo-conformal flatness, to examine their geometric and structural implications. Additionally, we construct an explicit example of a *3-dimensional $L\beta K$ manifold* that admits a g-TW connection, providing concrete validation of our theoretical results. The findings contribute to the broader understanding of curvature behaviors in almost contact pseudo-Riemannian geometry and extend the study of non-Riemannian connections in Lorentzian manifolds.

Keywords: Lorentzian β-Kenmotsu manifolds, Generalized Tanaka-Webster connection, Weyl-conformal curvature tensor, Generalized η-Einstein manifolds.

2020 Mathematics Subject Classification: Primary: 53C05, 53C15, 53C25, Secondary: 53D10.

1. Introduction

The *Tanaka–Webster connection* was introduced by *Tanno* [16] as a generalization of the well-known connection formulated in the late 1970s by *Tanaka* [15] and independently by

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Webster [19]. This connection coincides with the classical *Tanaka–Webster connection* when the associated *CR-structure* is integrable. It is defined as the canonical affine connection on a *non-degenerate, pseudo-Hermitian CR-manifold*.

For a *real hypersurface* in a *Kähler manifold* endowed with an *almost contact structure* (ϕ, ζ, η, g), *Cho* [3, 4] adapted *Tanno's g-Tanaka–Webster connection* for a nonzero real constant k. Utilizing this connection, several researchers have explored various geometric properties of *real hypersurfaces in complex space forms* [17].

A *Riemannian manifold* is termed *semisymmetric* if its curvature tensor satisfies

$$R(\mathcal{H}_1, \mathcal{H}_2) \cdot R = 0, \tag{1.1}$$

where $R(\mathcal{H}_1, \mathcal{H}_2)$ is regarded as a field of linear operators acting on R. It is well established that the class of *semisymmetric manifolds* properly contains *locally symmetric manifolds* (where $\nabla R = 0$). The concept of *semisymmetry* in Riemannian geometry was first investigated by *E. Cartan, A. Lichnerowicz, R. S. Couty, and N. S. Sinjukov*.

A Riemannian manifold is called *Ricci semisymmetric* if its curvature tensor satisfies

$$R(\mathcal{H}_1, \mathcal{H}_2) \cdot S = 0, \tag{1.2}$$

where S denotes the *Ricci tensor* of type $(0, 2)$. The class of *Ricci semisymmetric manifolds* contains *Ricci symmetric manifolds* (where $\nabla S = 0$) as a proper subset. Several researchers have studied these manifolds extensively. It is known that every *semisymmetric manifold* is *Ricci semisymmetric*, but the converse does not always hold. However, under certain additional conditions, the equations

$$R(\mathcal{H}_1, \mathcal{H}_2) \cdot R = 0 \quad \text{and} \quad R(\mathcal{H}_1, \mathcal{H}_2) \cdot S = 0$$

become equivalent. *Szabó* classified *semisymmetric manifolds* locally in [14], while fundamental studies in this area were carried out by *Szabó* [14], *Boeckx et al.* [2], and *Kowalski* [6].

One notable example of a curvature condition related to *semisymmetry* is

$$Q \cdot R = 0, \tag{1.3}$$

where Q is the *Ricci operator* defined by

$$S(\mathcal{H}_1, \mathcal{H}_2) = g(Q\mathcal{H}_1, \mathcal{H}_2).$$

Such curvature conditions naturally extend to *pseudosymmetry-type* conditions. The condition $Q \cdot R = 0$ was extensively studied by *Verstraelen et al.* in [18].

Several properties on $\mathcal{M}_\beta$ and the g-TW connection have also been researched by numerous geometers, such as ([1, 7, 8, 9, 10, 11, 12, 13]). Inspired by these foundational works, the present paper aims to *characterize Lorentzian β-Kenmotsu manifolds admitting the generalized Tanaka–Webster connection.*

The arrangement of this paper is structured as follows: Section 2 presents the fundamental definitions and preliminary results related to Lorentzian β-Kenmotsu (LβK) manifolds. We introduce the structure equations and discuss essential properties that will be used in subsequent sections. In section 3, we explore the curvature properties of a $L\beta K$ manifold admitting the generalized Tanaka-Webster (g-TW) connection. We derive explicit expressions for the curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{S}$ with respect to g-TW connection and establish some interesting geometric properties. Section 4 investigates the condition $\widetilde{R} \cdot \widetilde{S} = 0$ in a $L\beta K$ manifold equipped with g-TW connection. We demonstrate that under this condition, the manifold becomes a generalized η-Einstein manifold with respect to the g-TW connection. In section 5, we analyze the condition $\widetilde{S} \cdot \widetilde{R} = 0$ and establish that the $L\beta K$ manifold satisfying this curvature restriction is also a generalized η-Einstein manifold with respect to g-TW connection. Section 6 is devoted to the study of conformally flat $L\beta K$ manifolds under the influence of g-TW connection. We prove that such manifolds naturally admit a generalized η-Einstein structure with respect to g-TW connection. In section 7, we focus on ζ-conformally flat $L\beta K$ manifolds and derive certain interesting curvature properties arising from this condition. Section 8 examines the notion of pseudo-conformal flatness in the framework of $L\beta K$ manifolds. Finally, in section 9, we construct an explicit example of a 3-dimensional $L\beta K$ manifold admitting g-TW connection and verify that it satisfies the curvature conditions discussed in the previous sections. This structured approach ensures a coherent development of our results, highlighting the interplay between various curvature conditions and the geometry of Lorentzian β-Kenmotsu manifolds.

2. Preliminaries

A $(2n+1)$-dimensional differentiable manifold is termed as $L\beta K$ manifold ($\mathcal{M}_\beta$), if it possesses a $(1,1)$-tensor field ϕ, a contravariant vector field ζ, a covariant vector field η and a Lorentzian metric g satisfying

$$\phi^2\mathcal{H}_1 = \mathcal{H}_1 + \eta(\mathcal{H}_1)\zeta, \quad g(\mathcal{H}_1, \zeta) = \eta(\mathcal{H}_1), \tag{2.4}$$

$$\eta(\zeta) = -1, \quad \phi(\zeta) = 0, \quad \eta(\phi\mathcal{H}_1) = 0, \tag{2.5}$$

$$g(\phi\mathcal{H}_1, \phi\mathcal{H}_2) = g(\mathcal{H}_1, \mathcal{H}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{2.6}$$

$$g(\phi\mathcal{H}_1, \mathcal{H}_2) = g(\mathcal{H}_1, \phi\mathcal{H}_2), \tag{2.7}$$

for all vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}_\beta$. Furthermore, $\mathcal{M}_\beta$ satisfies

$$\nabla_{\mathcal{H}_1}\zeta = \beta[\mathcal{H}_1 - \eta(\mathcal{H}_1)\zeta], \tag{2.8}$$

$$(\nabla_{\mathcal{H}_1}\eta)(\mathcal{H}_2) = \beta[g(\mathcal{H}_1, \mathcal{H}_2) - \eta(\mathcal{H}_1)\eta(\mathcal{H}_2)], \tag{2.9}$$

$$(\nabla_{\mathcal{H}_1}\phi)(\mathcal{H}_2) = \beta[g(\phi\mathcal{H}_1, \mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\phi\mathcal{H}_1], \tag{2.10}$$

where ∇ represents the covariant differentiation operator with respect to the Lorentzian metric g. Moreover, on $\mathcal{M}_\beta$, the following relations hold

$$\eta(R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3) = \beta^2[g(\mathcal{H}_1, \mathcal{H}_3)\eta(\mathcal{H}_2) - g(\mathcal{H}_2, \mathcal{H}_3)\eta(\mathcal{H}_1)], \tag{2.11}$$

$$R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = \beta^2[g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2 - g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1], \tag{2.12}$$

$$R(\zeta, \mathcal{H}_1)\mathcal{H}_2 = \beta^2[\eta(\mathcal{H}_2)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_2)\zeta], \tag{2.13}$$

$$R(\mathcal{H}_1, \mathcal{H}_2)\zeta = \beta^2[\eta(\mathcal{H}_1)\mathcal{H}_2 - \eta(\mathcal{H}_2)\mathcal{H}_1], \tag{2.14}$$

$$S(\mathcal{H}_1, \zeta) = -2n\beta^2\eta(\mathcal{H}_1), \tag{2.15}$$

$$Q\mathcal{H}_1 = -2n\beta^2\mathcal{H}_1, \quad Q\zeta = -2n\beta^2\zeta, \tag{2.16}$$

$$S(\zeta, \zeta) = 2n\beta^2, \tag{2.17}$$

$$g(Q\mathcal{H}_1, \mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2) = -2n\beta^2 g(\mathcal{H}_1, \mathcal{H}_2), \tag{2.18}$$

$$S(\phi\mathcal{H}_1, \phi\mathcal{H}_2) = S(\mathcal{H}_1, \mathcal{H}_2) - 2n\beta^2\eta(\mathcal{H}_1)\eta(\mathcal{H}_2), \tag{2.19}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$, where R, S and Q stand for the curvature tensor, the Ricci tensor and the Ricci operator on $\mathcal{M}_\beta$, respectively.

Let $\{e_1, e_2, e_3, \ldots, e_n = \zeta\}$ be an orthonormal basis for the tangent space at any point on the manifold $\mathcal{M}_\beta$. The Ricci tensor S and the scalar curvature r of the manifold are given by the following expression

$$S(\mathcal{H}_1, \mathcal{H}_2) = \sum_{i=1}^{2n+1} \varepsilon_i g(R(e_i, \mathcal{H}_1)\mathcal{H}_2, e_i), \tag{2.20}$$

where ε_i are the signs corresponding to the metric signature.

On $L\beta K$-manifolds, the scalar curvature r is given by

$$r = \sum_{i=1}^{2n+1} \varepsilon_i S(e_i, e_i), \tag{2.21}$$

where ε_i are the signs corresponding to the metric signature. Additionally, we have

$$g(\mathcal{H}_1, \mathcal{H}_2) = \sum_{i=1}^{2n+1} \varepsilon_i g(\mathcal{H}_1, e_i) g(\mathcal{H}_2, e_i), \tag{2.22}$$

where $\mathcal{H}_1, \mathcal{H}_2 \in \chi(\mathcal{M}_\beta)$ and $\varepsilon_i = g(e_i, e_i) = \pm 1$.

Definition 2.1 *A $L\beta K$-manifold $\mathcal{M}_\beta$ is referred to as a generalized η-Einstein manifold if its Ricci tensor S takes the form*

$$S(\mathcal{H}_1, \mathcal{H}_2) = \nu_1 g(\mathcal{H}_1, \mathcal{H}_2) + \nu_2 \eta(\mathcal{H}_1)\eta(\mathcal{H}_2) + \nu_3 \Phi(\mathcal{H}_1, \mathcal{H}_2), \tag{2.23}$$

where $\Phi(\mathcal{H}_1, \mathcal{H}_2) = g(\phi\mathcal{H}_1, \mathcal{H}_2)$ is the fundamental 2-form of the manifold $\mathcal{M}_\beta$ and ν_1, ν_2, ν_3 are smooth functions on $\mathcal{M}_\beta$.

If $\nu_3 = 0$, then $\mathcal{M}_\beta$ is said to be an η-Einstein manifold.

If $\nu_2 = 0, \nu_3 = 0$, then $\mathcal{M}_\beta$ is said to be an Einstein manifold.

Definition 2.2 *In a $(2n+1)$-dimensional $(n > 1)$ almost contact metric manifold, the Weyl-conformal curvature tensor $\mathcal{C}$ (also known as conformal curvature tensor) with respect to the Levi-Civita connection is defined as follows (see [20]):*

$$\begin{aligned}\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 - \frac{1}{(2n-1)}\Big[S(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - S(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2 + g(\mathcal{H}_2, \mathcal{H}_3)Q\mathcal{H}_1 \\ - g(\mathcal{H}_1, \mathcal{H}_3)Q\mathcal{H}_2\Big] + \frac{r}{2n(2n-1)}\Big[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2\Big]. \end{aligned} \tag{2.24}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$, R and r represent the curvature tensor and the scalar curvature with respect to the Levi-Civita connection, respectively.

Definition 2.3 *The sectional curvature $\kappa(\mathcal{H}_1, \mathcal{H}_2)$ of a manifold is given by*

$$\kappa(\mathcal{H}_1, \mathcal{H}_2) = -\frac{R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2)}{g(\mathcal{H}_1, \mathcal{H}_1)g(\mathcal{H}_2, \mathcal{H}_2) - g(\mathcal{H}_1, \mathcal{H}_2)^2}, \tag{2.25}$$

where $R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2)$ represents the associated curvature tensor.

3. Curvature properties of a $L\beta K$ manifold admitting g-TW connection

The g-TW connection $\widetilde{\nabla}$, associated with the Levi-Civita connection ∇, is defined by [16, 5]

$$\widetilde{\nabla}_{\mathcal{H}_1}\mathcal{H}_2 = \nabla_{\mathcal{H}_1}\mathcal{H}_2 + (\nabla_{\mathcal{H}_1}\eta)(\mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\nabla_{\mathcal{H}_1}\zeta - \eta(\mathcal{H}_1)\phi\mathcal{H}_2, \tag{3.26}$$

for any vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}_\beta$. Using (2.8) and (2.9) in (3.26), we obtain

$$\widetilde{\nabla}_{\mathcal{H}_1}\mathcal{H}_2 = \nabla_{\mathcal{H}_1}\mathcal{H}_2 + \beta g(\mathcal{H}_1, \mathcal{H}_2)\zeta - \beta\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\phi\mathcal{H}_2, \tag{3.27}$$

for all smooth vector fields $\mathcal{H}_1$ and $\mathcal{H}_2$ on $\mathcal{M}_\beta$.
Substituting $\mathcal{H}_2 = \zeta$ in (3.27), we have

$$\widetilde{\nabla}_{\mathcal{H}_1}\zeta = 2\beta\mathcal{H}_1. \tag{3.28}$$

Let $\widetilde{R}$ and R denote the curvature tensors of $\mathcal{M}_\beta$ with respect to the connections $\widetilde{\nabla}$ and ∇, respectively. The curvature tensor of a $(2n+1)$-dimensional $L\beta K$ manifold with respect to the g-TW connection $\widetilde{\nabla}$ is defined by

$$\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = \widetilde{\nabla}_{\mathcal{H}_1}\widetilde{\nabla}_{\mathcal{H}_2}\mathcal{H}_3 - \widetilde{\nabla}_{\mathcal{H}_2}\widetilde{\nabla}_{\mathcal{H}_1}\mathcal{H}_3 - \widetilde{\nabla}_{[\mathcal{H}_1, \mathcal{H}_2]}\mathcal{H}_3. \tag{3.29}$$

By virtue of (3.27) in (3.29), we obtain

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = {} & R(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \rho\eta(\mathcal{H}_1)[g(\mathcal{H}_2, \mathcal{H}_3)\zeta - \eta(\mathcal{H}_3)\mathcal{H}_2] \\ & - \rho\eta(\mathcal{H}_2)[g(\mathcal{H}_1, \mathcal{H}_3)\zeta - \eta(\mathcal{H}_3)\mathcal{H}_1] + 3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2] \\ & - 2\beta[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3)\zeta - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)\zeta],\end{aligned} \tag{3.30}$$

where $\rho = \zeta\beta$ and $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ are any vector fields on $\mathcal{M}_\beta$.
By taking the inner product of (3.30) with the vector field $\mathcal{H}_4$, we have

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = {} & R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) + \rho\eta(\mathcal{H}_1)[\eta(\mathcal{H}_4)g(\mathcal{H}_2, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ & - \rho\eta(\mathcal{H}_2)[\eta(\mathcal{H}_4)g(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4)] \\ & + 3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4) - g(\mathcal{H}_1, \mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ & - 2\beta\eta(\mathcal{H}_4)[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)],\end{aligned} \tag{3.31}$$

where $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = g(\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3, \mathcal{H}_4)$ is the curvature tensor associated with $\widetilde{\nabla}$.

Let $\{e_1, e_2, e_3, \ldots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at any point of the manifold $\mathcal{M}_\beta$. By setting $\mathcal{H}_1 = \mathcal{H}_4 = e_i$ in (3.31) and summing over i for $1 \leq i \leq (2n+1)$, we obtain

$$\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3) = S(\mathcal{H}_2, \mathcal{H}_3) + (6n\beta^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3) + (2n-1)\rho\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) - 2\beta\Phi(\mathcal{H}_2, \mathcal{H}_3), \tag{3.32}$$

for all vector fields $\mathcal{H}_2, \mathcal{H}_3$ on $\mathcal{M}_\beta$, where $\widetilde{S}$ and S denote the Ricci tensor of $\mathcal{M}_\beta$ with respect to the connections $\widetilde{\nabla}$ and ∇ respectively.

Using (3.32) , the Ricci operator $\widetilde{Q}$ with respect to the connection $\widetilde{\nabla}$ is determined by

$$\widetilde{Q}\mathcal{H}_2 = Q\mathcal{H}_2 + (6n\beta^2 - \rho)\mathcal{H}_2 + (2n-1)\rho\eta(\mathcal{H}_2)\zeta - 2\beta\phi\mathcal{H}_2. \tag{3.33}$$

Let $\widetilde{r}$ and r denote the scalar curvature of $\mathcal{M}_\beta$ with respect to the connections $\widetilde{\nabla}$ and ∇, respectively. Let $\{e_1, e_2, e_3, \ldots, e_{2n+1}\}$ be a local orthonormal basis of the tangent space at any point of the manifold $\mathcal{M}_\beta$. By setting $\mathcal{H}_2 = \mathcal{H}_3 = e_i$ in (3.32) and summing over i for $1 \leq i \leq (2n+1)$, we obtain

$$\widetilde{r} = r + 6n(2n+1)\beta^2 - 4n\rho - 2\beta\psi, \tag{3.34}$$

where $\psi = trace(\phi)$.
From above discussion, we state the following:

Theorem 3.1 *In a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$, the following holds:*
(i) The curvature tensor $\widetilde{R}$, Ricci tensor $\widetilde{S}$, Ricci operator $\widetilde{Q}$, and scalar curvature $\widetilde{r}$ with respect to $\widetilde{\nabla}$ are given by (3.30), (3.32), (3.33), and (3.34) respectively,
(ii) $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{H}_2, \mathcal{H}_1)\mathcal{H}_3 = 0$,
(iii) $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 + \widetilde{R}(\mathcal{H}_3, \mathcal{H}_1)\mathcal{H}_2 = 0$,
(iv) The Ricci tensor $\widetilde{S}(\mathcal{H}_1, \mathcal{H}_2)$ is symmetric in nature.

Now, let $\mathcal{M}_\beta$ be a Ricci flat with respect to the g-TW connection $\widetilde{\nabla}$. Then from (3.32), we lead to

$$S(\mathcal{H}_2, \mathcal{H}_3) = -(6n\beta^2 - \rho)g(\mathcal{H}_2, \mathcal{H}_3) - (2n-1)\rho\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) + 2\beta\Phi(\mathcal{H}_2, \mathcal{H}_3), \tag{3.35}$$

where $\rho = \zeta\beta$ and $\Phi(\mathcal{H}_2, \mathcal{H}_3) = g(\mathcal{H}_2, \phi\mathcal{H}_3)$.
This leads to the following result:

Theorem 3.2 *A $L\beta K$ manifold $\mathcal{M}_\beta$ is Ricci flat with respect to the g-TW connection $\widetilde{\widetilde{\nabla}}$ if and only if it is a generalized η-Einstein manifold with respect to the Levi-Civita connection ∇.*

Now, if $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0$, then by virtue of (3.31), we have

$$\begin{aligned} R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = &-\rho\eta(\mathcal{H}_1)[\eta(\mathcal{H}_4)g(\mathcal{H}_2, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ &+ \rho\eta(\mathcal{H}_2)[\eta(\mathcal{H}_4)g(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4)] \\ &- 3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4) - g(\mathcal{H}_1, \mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)] \\ &+ 2\beta\eta(\mathcal{H}_4)[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \phi\mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \phi\mathcal{H}_3)], \end{aligned} \tag{3.36}$$

Let $\zeta^\perp = \{\mathcal{H}_1 : g(\mathcal{H}_1, \zeta) = 0, \forall \mathcal{H}_1 \in \chi(\mathcal{M}_\beta)\}$ denotes a $(2n+1)$-dimensional distribution orthogonal to ζ, then for any $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \in \zeta^\perp$, (3.36) takes the form

$$R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4) = -3\beta^2[g(\mathcal{H}_2, \mathcal{H}_3)g(\mathcal{H}_1, \mathcal{H}_4) - g(\mathcal{H}_1, \mathcal{H}_3)g(\mathcal{H}_2, \mathcal{H}_4)]. \tag{3.37}$$

Thus, we can state the following:

Theorem 3.3 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$. The curvature tensor of $\mathcal{M}_\beta$ determined by $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \in \zeta^\perp$ with respect to $\widetilde{\widetilde{\nabla}}$ vanishes if and only if $\mathcal{M}_\beta$ with respect to the Levi-Civita connection ∇ is isomorphic to the hyperbolic space $H^{2n+1}(-3\beta^2)$.*

Replacing $\mathcal{H}_3$ by $\mathcal{H}_1$ and $\mathcal{H}_4$ by $\mathcal{H}_2$ in (3.37), we have

$$\kappa(\mathcal{H}_1, \mathcal{H}_2) = -\frac{R(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_2)}{g(\mathcal{H}_1, \mathcal{H}_1)g(\mathcal{H}_2, \mathcal{H}_2) - g(\mathcal{H}_1, \mathcal{H}_2)^2} = -3\beta^2. \tag{3.38}$$

Hence, we obtain the following result:

Corollary 3.1 *If $\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\mathcal{H}_3 = 0$ in a $L\beta K$ manifold, then the sectional curvature of the plane section determined by $\mathcal{H}_1, \mathcal{H}_2 \in \zeta^\perp$ is $-3\beta^2$.*

Furthermore, we obtain the following results:

Lemma 3.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional LβK manifold admitting g-TW connection $\widetilde{\nabla}$, then we have the following*

(i) $\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\zeta = (2\beta^2-\rho)[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2]$,

(ii) $\widetilde{R}(\zeta,\mathcal{H}_1)\mathcal{H}_2 = (2\beta^2-\rho)[g(\mathcal{H}_1,\mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\mathcal{H}_1] - 2\beta\Phi(\mathcal{H}_1,\mathcal{H}_2)\zeta$,

(iii) $\widetilde{R}(\mathcal{H}_1,\zeta)\mathcal{H}_2 = -(2\beta^2-\rho)[g(\mathcal{H}_1,\mathcal{H}_2)\zeta - \eta(\mathcal{H}_2)\mathcal{H}_1] + 2\beta\Phi(\mathcal{H}_1,\mathcal{H}_2)\zeta$,

(iv) $\widetilde{R}(\zeta,\mathcal{H}_1)\zeta = (2\beta^2-\rho)\phi^2\mathcal{H}_1$,

(v) $\widetilde{S}(\mathcal{H}_1,\zeta) = 2n(2\beta^2-\rho)\eta(\mathcal{H}_1)$,

(vi) $\widetilde{Q}\zeta = 2n(2\beta^2-\rho)\zeta$,

(vii) $\eta(\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3) = (2\beta^2-\rho)[g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1) - g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)] + 2\beta[\eta(\mathcal{H}_2)\Phi(\mathcal{H}_1,\mathcal{H}_3) - \eta(\mathcal{H}_1)\Phi(\mathcal{H}_2,\mathcal{H}_3)]$,

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$.

Now, we define conformal curvature tensor with respect to g-TW connection $\widetilde{\nabla}$.

Definition 3.1 *The conformal curvature tensor $\widetilde{\mathcal{C}}$ for a (2n+1)-dimensional LβK manifold $\mathcal{M}_\beta$ admitting g-TW connection is defined as*

$$\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 = \widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 - \frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1,\mathcal{H}_3)\mathcal{H}_2 + g(\mathcal{H}_2,\mathcal{H}_3)\widetilde{Q}\mathcal{H}_1 - g(\mathcal{H}_1,\mathcal{H}_3)\widetilde{Q}\mathcal{H}_2\Big] + \frac{\widetilde{r}}{2n(2n-1)}\Big[g(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1 - g(\mathcal{H}_1,\mathcal{H}_3)\mathcal{H}_2\Big]. \tag{3.39}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$. Here $\widetilde{R}$, $\widetilde{S}$ and $\widetilde{r}$ are the Riemannian curvature tensor, Ricci tensor and the scalar curvature with respect to the connection $\widetilde{\nabla}$, respectively on $\mathcal{M}_\beta$.

Also, we can state the following:

Lemma 3.2 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional LβK manifold admitting g-TW connection $\widetilde{\nabla}$. Let $\widetilde{\mathcal{C}}$ be the conformal curvature tensor with respect to $\widetilde{\nabla}$. Then, we have the following*

(i) $\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 + \widetilde{\mathcal{C}}(\mathcal{H}_2,\mathcal{H}_1)\mathcal{H}_3 = 0$,

(ii) $\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3 + \widetilde{\mathcal{C}}(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1 + \widetilde{\mathcal{C}}(\mathcal{H}_3,\mathcal{H}_1)\mathcal{H}_2 = 0$,

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$.

4. Lorentzian β-Kenmotsu manifold admitting g-TW connection satisfying $\widetilde{R}\cdot\widetilde{S}=0$ condition

Let us consider a $L\beta K$ manifold admitting g-TW connection satisfying the condition

$$\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\cdot\widetilde{S}=0, \tag{4.40}$$

for any vector fields $\mathcal{H}_1,\mathcal{H}_2$ on $\mathcal{M}_\beta$.

From (4.40), we infer

$$(\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\cdot\widetilde{S})(\mathcal{F}_1,\mathcal{F}_2)=\widetilde{S}(\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_1,\mathcal{F}_2)+\widetilde{S}(\mathcal{F}_1,\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{F}_2)=0, \tag{4.41}$$

for any vector fields $\mathcal{H}_1,\mathcal{H}_2,\mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{M}_\beta$.

Substituting $\mathcal{H}_1=\zeta$ in (4.41), we have

$$\widetilde{S}(\widetilde{R}(\zeta,\mathcal{H}_2)\mathcal{F}_1,\mathcal{F}_2)+\widetilde{S}(\mathcal{F}_1,\widetilde{R}(\zeta,\mathcal{H}_2)\mathcal{F}_2)=0, \tag{4.42}$$

By virtue of (3.30), we have

$$\widetilde{S}(\widetilde{R}(\zeta,\mathcal{H}_2)\mathcal{F}_1,\mathcal{F}_2)=(2\beta^2-\rho)[g(\mathcal{H}_2,\mathcal{F}_1)\widetilde{S}(\zeta,\mathcal{F}_2)-\eta(\mathcal{F}_1)\widetilde{S}(\mathcal{H}_2,\mathcal{F}_2)]-2\beta\Phi(\mathcal{H}_2,\mathcal{F}_1)\widetilde{S}(\zeta,\mathcal{F}_2), \tag{4.43}$$

and

$$\widetilde{S}(\mathcal{F}_1,\widetilde{R}(\zeta,\mathcal{H}_2)\mathcal{F}_2)=(2\beta^2-\rho)[g(\mathcal{H}_2,\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1,\zeta)-\eta(\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1,\mathcal{H}_2)]-2\beta\Phi(\mathcal{H}_2,\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1,\zeta), \tag{4.44}$$

where $\Phi(\mathcal{H}_2,\mathcal{F}_1)=g(\mathcal{H}_2,\phi\mathcal{F}_1)$ and $\Phi(\mathcal{H}_2,\mathcal{F}_2)=g(\mathcal{H}_2,\phi\mathcal{F}_2)$.

Substituting (4.43) and (4.44) in (4.42), we obtain

$$\begin{aligned}(2\beta^2-\rho)[g(\mathcal{H}_2,\mathcal{F}_1)\widetilde{S}(\zeta,\mathcal{F}_2)-\eta(\mathcal{F}_1)\widetilde{S}(\mathcal{H}_2,\mathcal{F}_2)+g(\mathcal{H}_2,\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1,\zeta)-\eta(\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1,\mathcal{H}_2)]\\ -2\beta[\Phi(\mathcal{H}_2,\mathcal{F}_1)\widetilde{S}(\zeta,\mathcal{F}_2)+\Phi(\mathcal{H}_2,\mathcal{F}_2)\widetilde{S}(\mathcal{F}_1,\zeta)]=0.\end{aligned} \tag{4.45}$$

Setting $\mathcal{F}_1=\zeta$ in (4.45) and on further simplification, we have

$$\widetilde{S}(\mathcal{H}_2,\mathcal{F}_2)=2n(2\beta^2-\rho)g(\mathcal{H}_2,\mathcal{F}_2)-4n\beta\Phi(\mathcal{H}_2,\mathcal{F}_2). \tag{4.46}$$

Contracting above, we have

$$\widetilde{r}=2n(2n+1)(2\beta^2-\rho)-4n\beta\psi, \tag{4.47}$$

where $\psi = trace(\phi)$.

By virtue of (3.32) in (4.46), we obtain

$$S(\mathcal{H}_2, \mathcal{F}_2) = -[2n\beta^2 + (2n-1)\rho]g(\mathcal{H}_2, \mathcal{F}_2) - (2n-1)\rho\eta(\mathcal{H}_2)\eta(\mathcal{F}_2) - 2(2n-1)\beta\Phi(\mathcal{H}_2, \mathcal{F}_2). \tag{4.48}$$

Contracting above, we have

$$r = -2n(2n+1)\beta^2 - 2(2n-1)[n\rho + \beta\psi]. \tag{4.49}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 4.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$ satisfying $\widetilde{R} \cdot \widetilde{S} = 0$ condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to $\widetilde{\nabla}$ whose Ricci tensor is of the form (4.46) and having scalar curvature $\widetilde{r}$ of the form (4.47), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (4.48) and having scalar curvature r of the form (4.49).

5. Lorentzian β-Kenmotsu manifold admitting g-TW connection satisfying $\widetilde{S} \cdot \widetilde{R} = 0$ condition

Let us consider a $L\beta K$ manifold admitting g-TW connection satisfying the condition

$$(\widetilde{S}(\mathcal{H}_1, \mathcal{H}_2) \cdot \widetilde{R})(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3 = 0, \tag{5.50}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{F}_1$ and $\mathcal{F}_2$ on $\mathcal{M}_\beta$.

From(5.50), we infer that

$$(\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3 + \widetilde{R}((\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{F}_1, (\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{F}_2)\mathcal{H}_3 + \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)(\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{H}_3 = 0, \tag{5.51}$$

where the endomorphism $\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2$ is defined by

$$(\mathcal{H}_1 \wedge_{\widetilde{S}} \mathcal{H}_2)\mathcal{H}_3 = \widetilde{S}(\mathcal{H}_2, \mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\mathcal{H}_2. \tag{5.52}$$

Substituting $\mathcal{H}_2 = \zeta$ in (5.51) and on further simplification, we obtain

$$\begin{aligned}\widetilde{S}(\zeta, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3)\zeta + \widetilde{S}(\zeta, \mathcal{F}_1)\widetilde{R}(\mathcal{H}_1, \mathcal{F}_2)\mathcal{H}_3 \\ -\widetilde{S}(\mathcal{H}_1, \mathcal{F}_1)\widetilde{R}(\zeta, \mathcal{F}_2)\mathcal{H}_3 + \widetilde{S}(\zeta, \mathcal{F}_2)\widetilde{R}(\mathcal{F}_1, \mathcal{H}_1)\mathcal{H}_3 - \widetilde{S}(\mathcal{H}_1, \mathcal{F}_2)\widetilde{R}(\mathcal{F}_1, \zeta)\mathcal{H}_3 \\ +\widetilde{S}(\zeta, \mathcal{H}_3)\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\zeta = 0. \qquad (5.53)\end{aligned}$$

Taking inner product of (5.53) with ζ, we have

$$\begin{aligned}\widetilde{S}(\zeta, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3)\eta(\mathcal{H}_1) + \widetilde{S}(\mathcal{H}_1, \widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_3) + \widetilde{S}(\zeta, \mathcal{F}_1)\eta(\widetilde{R}(\mathcal{H}_1, \mathcal{F}_2)\mathcal{H}_3) \\ -\widetilde{S}(\mathcal{H}_1, \mathcal{F}_1)\eta(\widetilde{R}(\zeta, \mathcal{F}_2)\mathcal{H}_3) + \widetilde{S}(\zeta, \mathcal{F}_2)\eta(\widetilde{R}(\mathcal{F}_1, \mathcal{H}_1)\mathcal{H}_3) - \widetilde{S}(\mathcal{H}_1, \mathcal{F}_2)\eta(\widetilde{R}(\mathcal{F}_1, \zeta)\mathcal{H}_3) \\ +\widetilde{S}(\zeta, \mathcal{H}_3)\eta(\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\mathcal{H}_1) - \widetilde{S}(\mathcal{H}_1, \mathcal{H}_3)\eta(\widetilde{R}(\mathcal{F}_1, \mathcal{F}_2)\zeta) = 0. \qquad (5.54)\end{aligned}$$

Setting $\mathcal{F}_1 = \mathcal{H}_3 = \zeta$ in (5.54) and on simplification, we have

$$\begin{aligned}(2\beta^2 - \rho)[\widetilde{S}(\mathcal{H}_1, \mathcal{F}_2) + \eta(\mathcal{F}_2)\widetilde{S}(\mathcal{H}_1, \zeta)] + 2n(2\beta^2 - \rho)^2[g(\mathcal{H}_1, \mathcal{F}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{F}_2)] \\ -4n\beta(2\beta^2 - \rho)\Phi(\mathcal{H}_1, \mathcal{F}_2) = 0. \qquad (5.55)\end{aligned}$$

From (3.32), we have

$$\widetilde{S}(\mathcal{H}_1, \zeta) = 2n(2\beta^2 - \rho)\eta(\mathcal{H}_1). \qquad (5.56)$$

Using (5.56) in (5.55), we obtain

$$\widetilde{S}(\mathcal{H}_1, \mathcal{F}_2) = -2n(2\beta^2 - \rho)g(\mathcal{H}_1, \mathcal{F}_2) - 4n(2\beta^2 - \rho)\eta(\mathcal{H}_1)\eta(\mathcal{F}_2) + 4n\beta\Phi(\mathcal{H}_1, \mathcal{F}_2). \qquad (5.57)$$

Contracting above, we have

$$\widetilde{r} = -2n(2n - 1)(2\beta^2 - \rho) + 4n\beta\psi, \qquad (5.58)$$

where $\psi = trace(\phi)$.

Furthermore, using (3.32) in (5.57), we obtain

$$\begin{aligned}S(\mathcal{H}_1, \mathcal{F}_2) = [(2n + 1)\rho - 10n\beta^2]g(\mathcal{H}_1, \mathcal{F}_2) + [(2n + 1)\rho - 8n\beta^2]\eta(\mathcal{H}_1)\eta(\mathcal{F}_2) \\ + 2(2n + 1)\beta\Phi(\mathcal{H}_1, \mathcal{F}_2). \qquad (5.59)\end{aligned}$$

Contracting above, we have

$$r = 2n(2n + 1)\rho - 2n(10n + 1)\beta^2 + 2(2n + 1)\beta\psi. \qquad (5.60)$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 5.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$ satisfying $\widetilde{S}\cdot\widetilde{R}=0$ condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to $\widetilde{\widetilde{\nabla}}$ whose Ricci tensor is of the form (5.57) and having scalar curvature $\widetilde{r}$ of the form (5.58), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (5.59) and having scalar curvature r of the form (5.60).

6. Conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we examine conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$.

Definition 6.1 *A $L\beta K$ manifold is said to be conformally flat with respect to g-TW connection $\widetilde{\widetilde{\nabla}}$ if it satisfies*

$$\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3=0, \tag{6.61}$$

for any vector fields $\mathcal{H}_1,\mathcal{H}_2$ and $\mathcal{H}_3$ on $\mathcal{M}_\beta$.

By virtue of (6.61) in (3.39), we obtain

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3=&\frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1-\widetilde{S}(\mathcal{H}_1,\mathcal{H}_3)\mathcal{H}_2+g(\mathcal{H}_2,\mathcal{H}_3)\widetilde{Q}\mathcal{H}_1-g(\mathcal{H}_1,\mathcal{H}_3)\widetilde{Q}\mathcal{H}_2\Big]\\&-\frac{\widetilde{r}}{2n(2n-1)}\Big[g(\mathcal{H}_2,\mathcal{H}_3)\mathcal{H}_1-g(\mathcal{H}_1,\mathcal{H}_3)\mathcal{H}_2\Big].\end{aligned} \tag{6.62}$$

Taking inner product of (6.62) with ζ and on further simplification, we have

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\zeta)=&\frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-\widetilde{S}(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)\Big]\\&+\left[\frac{4n^2(2\beta^2-\rho)-\widetilde{r}}{(2n-1)}\right][g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)].\end{aligned} \tag{6.63}$$

Further, on substituting $\mathcal{H}_4=\zeta$ in (3.31) and using (2.12), we obtain

$$\begin{aligned}\widetilde{R}(\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\zeta)=&(2\beta^2-\rho)[g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)]\\&+2\beta[\eta(\mathcal{H}_2)g(\mathcal{H}_1,\phi\mathcal{H}_3)-\eta(\mathcal{H}_1)g(\mathcal{H}_2,\phi\mathcal{H}_3)].\end{aligned} \tag{6.64}$$

Using (6.64) in (6.63), we infer

$$\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-\widetilde{S}(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)=\left[\frac{\widetilde{r}-2n(2\beta^2-\rho)}{2n}\right][g(\mathcal{H}_2,\mathcal{H}_3)\eta(\mathcal{H}_1)-g(\mathcal{H}_1,\mathcal{H}_3)\eta(\mathcal{H}_2)]$$
$$+2(2n-1)\beta\left[\eta(\mathcal{H}_2)g(\mathcal{H}_1,\phi\mathcal{H}_3)-\eta(\mathcal{H}_1)g(\mathcal{H}_2,\phi\mathcal{H}_3)\right]. \tag{6.65}$$

Assuming $\mathcal{H}_1=\zeta$ in (6.65) and on further simplification, we have

$$\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)=\left[\frac{\widetilde{r}-2n(2\beta^2-\rho)}{2n}\right]g(\mathcal{H}_2,\mathcal{H}_3)+\left[\frac{\widetilde{r}-2n(2n+1)(2\beta^2-\rho)}{2n}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)$$
$$-2(2n-1)\beta\Phi(\mathcal{H}_2,\mathcal{H}_3), \tag{6.66}$$

where $\Phi(\mathcal{H}_2,\mathcal{H}_3)=g(\mathcal{H}_2,\phi\mathcal{H}_3)$. Using (3.32) in (6.66), we obtain

$$S(\mathcal{H}_2,\mathcal{H}_3)=\left[\frac{r+2n\beta^2-2\beta\psi}{2n}\right]g(\mathcal{H}_2,\mathcal{H}_3)+\left[\frac{r+2n(2n+1)\beta^2-2\beta\psi}{2n}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3)$$
$$-4(n-1)\beta\Phi(\mathcal{H}_2,\mathcal{H}_3). \tag{6.67}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 6.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$ satisfying conformally flat condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to $\widetilde{\nabla}$ whose Ricci tensor is of the form (6.66), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (6.67).

7. ζ-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we examine ζ-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection $\widetilde{\nabla}$.

Definition 7.1 *A $L\beta K$ manifold is said to be ζ-conformally flat with respect to g-TW connection $\widetilde{\nabla}$ if it satisfies*

$$\widetilde{\mathcal{C}}(\mathcal{H}_1,\mathcal{H}_2)\zeta=0, \tag{7.68}$$

for any vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}_\beta$.

Setting $\mathcal{H}_3 = \zeta$ in (3.39) and using (7.68), we obtain

$$\widetilde{R}(\mathcal{H}_1, \mathcal{H}_2)\zeta = \frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2, \zeta)\mathcal{H}_1 - \widetilde{S}(\mathcal{H}_1, \zeta)\mathcal{H}_2 + \eta(\mathcal{H}_2)\widetilde{Q}\mathcal{H}_1 - \eta(\mathcal{H}_1)\widetilde{Q}\mathcal{H}_2\Big] - \frac{\widetilde{r}}{2n(2n-1)}\Big[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2\Big]. \tag{7.69}$$

On further simplification, we have

$$\eta(\mathcal{H}_2)\widetilde{Q}\mathcal{H}_1 - \eta(\mathcal{H}_1)\widetilde{Q}\mathcal{H}_2 = \left[\frac{\widetilde{r} - 2n(2\beta^2 - \rho)}{2n}\right]\Big[\eta(\mathcal{H}_2)\mathcal{H}_1 - \eta(\mathcal{H}_1)\mathcal{H}_2\Big]. \tag{7.70}$$

Taking inner product of (7.70) with $\mathcal{H}_3$, we have

$$\eta(\mathcal{H}_2)\widetilde{S}(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_1)\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3) = \left[\frac{\widetilde{r} - 2n(2\beta^2 - \rho)}{2n}\right]\Big[\eta(\mathcal{H}_2)g(\mathcal{H}_1, \mathcal{H}_3) - \eta(\mathcal{H}_1)g(\mathcal{H}_2, \mathcal{H}_3)\Big]. \tag{7.71}$$

Substituting $\mathcal{H}_1 = \zeta$ in (7.71), we obtain

$$\widetilde{S}(\mathcal{H}_2, \mathcal{H}_3) = \left[\frac{\widetilde{r} - 2n(2\beta^2 - \rho)}{2n}\right]g(\mathcal{H}_2, \mathcal{H}_3) + \left[\frac{\widetilde{r} - 2n(2n+1)(2\beta^2 - \rho)}{2n}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3). \tag{7.72}$$

Using (3.32) in (7.72), we have

$$S(\mathcal{H}_2, \mathcal{H}_3) = \left[\frac{r + 2n\beta^2 - 2\beta\psi}{2n}\right]g(\mathcal{H}_2, \mathcal{H}_3) + \left[\frac{r + 2n(2n+1)\beta^2 - 2\beta\psi}{2n}\right]\eta(\mathcal{H}_2)\eta(\mathcal{H}_3) + 2\beta\Phi(\mathcal{H}_2, \mathcal{H}_3), \tag{7.73}$$

where $\Phi(\mathcal{H}_2, \mathcal{H}_3) = g(\mathcal{H}_2, \phi\mathcal{H}_3)$.

Thus, based on the discussion above, we can present the following theorem:

Theorem 7.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\nabla}$ satisfying ζ-conformally flat condition. Then we have the following:*
(i) $\mathcal{M}_\beta$ is an η-Einstein manifold with respect to $\widetilde{\nabla}$ whose Ricci tensor is of the form (7.72), and
(ii) $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (7.73).

8. Pseudo-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we examine pseudo-conformally flat Lorentzian β-Kenmotsu manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$.

Definition 8.1 *A $L\beta K$ manifold is said to be pseudo-conformally flat with respect to g-TW connection $\widetilde{\widetilde{\nabla}}$ if it satisfies*

$$g(\widetilde{\mathcal{C}}(\phi\mathcal{H}_1,\mathcal{H}_2)\mathcal{H}_3,\phi\mathcal{H}_4)=0, \tag{8.74}$$

for any vector fields $\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3$ and $\mathcal{H}_4$ on $\mathcal{M}_\beta$.

By virtue of (3.39) and (8.74), we have

$$\begin{aligned}\widetilde{R}(\phi\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\phi\mathcal{H}_4)=&\frac{1}{(2n-1)}\Big[\widetilde{S}(\mathcal{H}_2,\mathcal{H}_3)g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)-\widetilde{S}(\phi\mathcal{H}_1,\mathcal{H}_3)g(\mathcal{H}_2,\phi\mathcal{H}_4)\\&+g(\mathcal{H}_2,\mathcal{H}_3)\widetilde{S}(\phi\mathcal{H}_1,\phi\mathcal{H}_4)-g(\phi\mathcal{H}_1,\mathcal{H}_3)\widetilde{S}(\mathcal{H}_2,\phi\mathcal{H}_4)\Big]\\&-\frac{\widetilde{r}}{2n(2n-1)}\left[g(\mathcal{H}_2,\mathcal{H}_3)g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)-g(\phi\mathcal{H}_1,\mathcal{H}_3)g(\mathcal{H}_2,\phi\mathcal{H}_4)\right].\end{aligned} \tag{8.75}$$

Let $\{e_1,e_2,e_3,\ldots,e_{2n+1}\}$ be a local orthonormal basis of the tangent space at any point of the manifold $\mathcal{M}_\beta$. By setting $\mathcal{H}_2=\mathcal{H}_3=e_i$ in (8.75) and summing over i for $1\leq i\leq(2n+1)$, we obtain

$$(2n+1)\widetilde{r}g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)=0. \tag{8.76}$$

Since $(2n+1)\neq 0$, therefore

$$\widetilde{r}g(\phi\mathcal{H}_1,\phi\mathcal{H}_4)=0. \tag{8.77}$$

By virtue of (2.6), we have

$$\widetilde{r}\left[g(\mathcal{H}_1,\mathcal{H}_4)+\eta(\mathcal{H}_1)\eta(\mathcal{H}_4)\right]=0. \tag{8.78}$$

Replacing $\mathcal{H}_1$ by $\widetilde{Q}\mathcal{H}_1$ in (8.78), we have

$$\widetilde{r}\left[\widetilde{S}(\mathcal{H}_1,\mathcal{H}_4)+2n(2\beta^2-\rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_4)\right]=0. \tag{8.79}$$

From above, we infer following cases:

Case I: If $\widetilde{r}=0$. Then from (3.34), we obtain

$$r=-6n(2n+1)\beta^2+4n\rho+2\beta\psi, \tag{8.80}$$

where $\psi = trace(\phi)$.

Case II: If $\widetilde{r}\neq 0$. Then from (8.79), we have

$$\widetilde{S}(\mathcal{H}_1,\mathcal{H}_4)=-2n(2\beta^2-\rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_4). \tag{8.81}$$

Contracting above, we infer

$$\widetilde{r}=2n(2\beta^2-\rho). \tag{8.82}$$

Using (3.32) in (8.81), we obtain

$$S(\mathcal{H}_1,\mathcal{H}_4)=-(6n\beta^2-\rho)g(\mathcal{H}_1,\mathcal{H}_4)-(4n\beta^2-\rho)\eta(\mathcal{H}_1)\eta(\mathcal{H}_4)+2\beta\Phi(\mathcal{H}_1,\mathcal{H}_4), \tag{8.83}$$

where $\Phi(\mathcal{H}_1,\mathcal{H}_4)=g(\mathcal{H}_1,\phi\mathcal{H}_4)$.
Contracting above, we have

$$r=-2n(6n+1)\beta^2+2n\rho+2\beta\psi. \tag{8.84}$$

Thus, based on the discussion above, we can present the following theorem:

Theorem 8.1 *Let $\mathcal{M}_\beta$ be a (2n+1)-dimensional $L\beta K$ manifold admitting g-TW connection $\widetilde{\widetilde{\nabla}}$ satisfying pseudo-conformally flat condition. Then we have the following:*
(i) The scalar curvature $\widetilde{r}$ with respect to $\widetilde{\widetilde{\nabla}}$ vanishes. Moreover, the scalar curvature r with respect to Levi-Civita connection ∇ is of the form (8.80), or
(ii) $\mathcal{M}_\beta$ is an η-Einstein manifold with respect to $\widetilde{\widetilde{\nabla}}$ whose Ricci tensor is of the form (8.81) and having scalar curvature $\widetilde{r}$ of the form (8.82). Moreover, $\mathcal{M}_\beta$ is a generalized η-Einstein manifold with respect to Levi-Civita connection ∇ whose Ricci tensor is of the form (8.83) and having scalar curvature of the form (8.84).

9. Example of a three-dimensional Lorentzian β-Kenmotsu manifold admitting g-TW connection

In this section, we illustrate an example of a three-dimensional Lorentzian β-Kenmotsu manifold. Consider the three-dimensional manifold

$$\mathcal{M}^3=\{(x,y,z)\in\mathbb{R}^3 : z>0\}$$

where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. We define the vector fields

$$\vartheta_1 = e^{-z}\frac{\partial}{\partial x}, \quad \vartheta_2 = e^{-z}\frac{\partial}{\partial y}, \quad \vartheta_3 = e^{-z}\frac{\partial}{\partial z} = \zeta,$$

which remain linearly independent at each point in M.

The Lorentzian metric g is given by

$$g(\vartheta_1, \vartheta_1) = 1, \quad g(\vartheta_2, \vartheta_2) = 1, \quad g(\vartheta_3, \vartheta_3) = -1,$$

$$g(\vartheta_1, \vartheta_2) = g(\vartheta_2, \vartheta_3) = g(\vartheta_3, \vartheta_1) = 0,$$

which can be expressed as

$$g = e^{2z}(dx \otimes dx + dy \otimes dy - dz \otimes dz).$$

Let the 1-form η satisfy

$$\eta(\mathcal{H}_1) = g(\mathcal{H}_1, \vartheta_3)$$

The $(1, 1)$-tensor field ϕ is defined as

$$\phi(\vartheta_1) = -\vartheta_2, \quad \phi(\vartheta_2) = -\vartheta_1, \quad \phi(\vartheta_3) = 0.$$

For any vector fields $\mathcal{H}_1, \mathcal{H}_2$ on $\mathcal{M}^3$, the following conditions hold:

$$\phi^2(\mathcal{H}_1) = \mathcal{H}_1 + \eta(\mathcal{H}_1)\vartheta_3,$$

$$g(\phi\mathcal{H}_1, \phi\mathcal{H}_2) = g(\mathcal{H}_1, \mathcal{H}_2) + \eta(\mathcal{H}_1)\eta(\mathcal{H}_2).$$

Thus, the structure $\mathcal{M}^3(\phi, \zeta, \eta, g)$ forms an almost contact metric structure on $\mathcal{M}^3$, where we set $\vartheta_3 = \zeta$.

The Lie brackets of the vector fields are computed as follows:

$$[\vartheta_1, \vartheta_3] = e^{-z}\vartheta_1, \quad [\vartheta_1, \vartheta_2] = 0, \quad [\vartheta_2, \vartheta_3] = e^{-z}\vartheta_2.$$

Using Koszul's formula, the Levi-Civita connection ∇ is obtained as

$$\begin{cases} \nabla_{\vartheta_1}\vartheta_1 = e^{-z}\vartheta_3, \quad \nabla_{\vartheta_2}\vartheta_1 = 0, \quad \nabla_{\vartheta_3}\vartheta_1 = 0, \\ \nabla_{\vartheta_1}\vartheta_2 = 0, \quad \nabla_{\vartheta_2}\vartheta_2 = e^{-z}\vartheta_3, \quad \nabla_{\vartheta_3}\vartheta_2 = 0, \\ \nabla_{\vartheta_1}\vartheta_3 = 0, \quad \nabla_{\vartheta_2}\vartheta_3 = 0, \quad \nabla_{\vartheta_3}\vartheta_3 = 0. \end{cases} \tag{9.85}$$

From the above results, setting $\beta = e^{-z}$, we conclude that $\mathcal{M}^3(\phi, \zeta, \eta, g)$ defines a $\mathcal{M}_\beta$ structure in dimension three. From (3.27) and (9.85), we obtain

$$\begin{cases} \widetilde{\nabla}_{\vartheta_1}\vartheta_1 = 2e^{-z}\vartheta_3, \quad \widetilde{\nabla}_{\vartheta_2}\vartheta_1 = 0, \quad \widetilde{\nabla}_{\vartheta_3}\vartheta_1 = -\vartheta_2, \\ \widetilde{\nabla}_{\vartheta_1}\vartheta_2 = 0, \quad \widetilde{\nabla}_{\vartheta_2}\vartheta_2 = 2e^{-z}\vartheta_3, \quad \widetilde{\nabla}_{\vartheta_3}\vartheta_2 = -\vartheta_1, \\ \widetilde{\nabla}_{\vartheta_1}\vartheta_3 = e^{-z}\vartheta_1, \quad \widetilde{\nabla}_{\vartheta_2}\vartheta_3 = e^{-z}\vartheta_2, \quad \widetilde{\nabla}_{\vartheta_3}\vartheta_3 = 0. \end{cases} \tag{9.86}$$

The components of the curvature tensor with respect to the Levi-Civita connection ∇ are given by:

$$\begin{cases} R(\vartheta_1,\vartheta_2)\vartheta_1 = e^{-2z}\vartheta_2, \quad R(\vartheta_2,\vartheta_3)\vartheta_1 = 0, \quad R(\vartheta_1,\vartheta_3)\vartheta_1 = e^{-2z}\vartheta_3, \\ R(\vartheta_1,\vartheta_2)\vartheta_2 = -e^{-2z}\vartheta_1, \quad R(\vartheta_2,\vartheta_3)\vartheta_2 = e^{-2z}\vartheta_3, \quad R(\vartheta_1,\vartheta_3)\vartheta_2 = 0, \\ R(\vartheta_1,\vartheta_2)\vartheta_3 = 0, \quad R(\vartheta_2,\vartheta_3)\vartheta_3 = e^{-2z}\vartheta_2, \quad R(\vartheta_1,\vartheta_3)\vartheta_3 = e^{-2z}\vartheta_1. \end{cases} \tag{9.87}$$

The components of the curvature tensor with respect to the g-TW connection $\widetilde{\nabla}$ are given by:

$$\begin{cases} \widetilde{R}(\vartheta_1,\vartheta_2)\vartheta_1 & = -2e^{-2z}\vartheta_2, \quad \widetilde{R}(\vartheta_2,\vartheta_3)\vartheta_1 = -2e^{-z}\vartheta_3, \quad \widetilde{R}(\vartheta_1,\vartheta_3)\vartheta_1 = -2e^{-2z}\vartheta_3 + \rho\vartheta_3, \\ \widetilde{R}(\vartheta_1,\vartheta_2)\vartheta_2 & = 2e^{-2z}\vartheta_1, \quad \widetilde{R}(\vartheta_2,\vartheta_3)\vartheta_2 = -2e^{-2z}\vartheta_3 + \rho\vartheta_3, \quad \widetilde{R}(\vartheta_1,\vartheta_3)\vartheta_2 = -2e^{-z}\vartheta_3, \\ \widetilde{R}(\vartheta_1,\vartheta_2)\vartheta_3 & = 0, \quad \widetilde{R}(\vartheta_2,\vartheta_3)\vartheta_3 = -2e^{-2z}\vartheta_2 + \rho\vartheta_2, \quad \widetilde{R}(\vartheta_1,\vartheta_3)\vartheta_3 = -2e^{-2z}\vartheta_1 + \rho\vartheta_1. \end{cases} \tag{9.88}$$

From (9.87), the non-vanishing components of Ricci tensor with respect to Levi-Civita connection ∇ is as follows

$$S(\vartheta_1,\vartheta_1) = -2e^{-2z}, \quad S(\vartheta_2,\vartheta_2) = -2e^{-2z}, \quad S(\vartheta_3,\vartheta_3) = 2e^{-2z}, \tag{9.89}$$

which implies that the scalar curvature r with respect to ∇ can be evaluated by

$$r = \sum_{i=1}^{3} \varepsilon_i S(e_i, e_i) = -6e^{-2z}. \tag{9.90}$$

Furthermore, from (9.88), the non-vanishing components of Ricci tensor with respect to the g-TW connection $\widetilde{\nabla}$ are given as

$$\widetilde{S}(\vartheta_1,\vartheta_1) = 4e^{-2z} - \rho, \quad \widetilde{S}(\vartheta_2,\vartheta_2) = 4e^{-2z} - \rho, \quad \widetilde{S}(\vartheta_3,\vartheta_3) = -4e^{-2z} + 2\rho, \tag{9.91}$$

which implies that the scalar curvature $\widetilde{r}$ with respect to $\widetilde{\nabla}$ can be evaluated by

$$\widetilde{r} = \sum_{i=1}^{3} \varepsilon_i \widetilde{S}(e_i, e_i) = 12e^{-2z} - 4\rho. \tag{9.92}$$

which can also be verified from (3.34) where ψ can be evaluated as

$$\psi = trace(\phi) = \sum_{i=1}^{3} \varepsilon_i \Phi(e_i, e_i) = 0. \tag{9.93}$$

10. Conclusion

In this paper, we conducted a comprehensive study of Lorentzian β-Kenmotsu ($L\beta K$) manifolds equipped with the generalized Tanaka-Webster (g-TW) connection. Beginning with fundamental definitions and preliminary results, we established the essential structure equations and derived explicit expressions for the curvature tensor $\widetilde{R}$ and the Ricci tensor $\widetilde{S}$ in this setting. Our analysis revealed several significant geometric properties, including the conditions under which an $L\beta K$ manifold admitting the g-TW connection becomes a generalized η-Einstein manifold.

We demonstrated that a $L\beta K$ manifold satisfies crucial curvature identities, such as the symmetry and skew-symmetry of the curvature tensor, and explored conditions like $\widetilde{R} \cdot \widetilde{S} = 0$ and $\widetilde{S} \cdot \widetilde{R} = 0$, under which the manifold naturally admits a generalized η-Einstein structure. Further, we investigated the geometric implications of conformally flat and ζ-conformally flat conditions, showing that such manifolds inherently exhibit the generalized η-Einstein property with respect to the g-TW connection. Additionally, we examined the notion of pseudo-conformal flatness in $L\beta K$ manifolds, establishing key results regarding scalar curvature and the structure of the Ricci tensor.

To solidify our theoretical findings, we provided an explicit example of a three-dimensional $L\beta K$ manifold equipped with the g-TW connection and verified that it satisfies the curvature conditions discussed throughout the paper. This study offers new insights into the geometric nature of Lorentzian β-Kenmotsu manifolds and their curvature properties under different structural constraints. The results presented here open pathways for further research, including extensions to higher-dimensional cases, the study of additional curvature conditions, and potential applications in mathematical physics and relativity.

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References

[1] Ahmad, M., Haseeb, A., & Jun, J.B. (2019). *Quasi-concircular curvature tensor on a Lorentzian β-Kenmotsu manifold*. Journal of Chungcheong Mathematical Society, 32(3), 281–293. https://doi.org/10.14403/jcms.2019.32.3.281

[2] Boeckx, E., Kowalski, O., & Vanhecke, L. (1996). Riemannian manifolds of conullity two. World Scientific Publishing, Singapore.

[3] Cho, J.T. (1999). CR-structures on real hypersurfaces of a complex space form. Publications Mathematicae, 54, 473–487. https://doi.org/10.5486/pmd.1999.2081

[4] Cho, J.T. (2008). Pseudo-Einstein CR-structures on real hypersurfaces in a complex space form. Hokkaido Mathematical Journal, 37, 1–17. https://doi.org/10.14492/hokmj/1253539581

[5] Ghosh, G., & De, U.C. (2017). Kenmotsu manifolds with generalized Tanaka-Webster connection. Publications de l'Institut Mathématique-Beograd, 102, 221–230. https://doi.org/10.2298/PIM1716221G

[6] Kowalski, O. (1996). An explicit classification of 3-dimensional Riemannian spaces satisfying $R(X,Y)\cdot R = 0$. Czechoslovak Mathematical Journal, 46(121), 427–474. https://doi.org/10.21136/CMJ.1996.127308

[7] Mishra, A.K., Prajapati, P., Rajan, & Singh, G.P. (2024). On M-projective curvature tensor of Lorentzian β-Kenmotsu manifold. Bulletin of the Transilvania University of Brasov, 4(66), 201–214. https://doi.org/10.31926/but.mif.2024.4.66.2.12

[8] Prakasha, D.G., Bagewadi, C.S., & Basavarajappa, N.S. (2008). On Lorentzian β-Kenmotsu manifolds. International Journal of Mathematical Analysis, 2(19), 919–927.

[9] Singh, A., Ahmad, M., Yadav, S. K., & Patel, S. (2024). Some results on β-Kenmotsu manifolds with a non-symmetric non-metric connection. *International Journal of Maps in Mathematics*, 7(1), 20–32.

[10] Singh, A., Das, L. S., Pankaj, P., & Patel, S. (2024). Hyperbolic Kenmotsu manifolds admitting a semi-symmetric non-metric connection. *Facta Universitatis (Niš), Series: Mathematics and Informatics*, 39(1), 123–139.

[11] Singh, A., Kishor, S., & Kumar, L. (2025). Ricci soliton in an (ε)-para-Sasakian manifold admitting conharmonic curvature tensor. *Filomat*, 39(1), 83–96.

[12] Singh, A., Prasad, R., & Kumar, L. (2025). Lorentzian β-Kenmotsu manifold admitting generalized Tanaka–Webster connection. *International Journal of Maps in Mathematics*, 8(1), 227–246.

[13] Singh, G.P., Prajapati, P., Mishra, A.K., & Rajan. (2024). Generalized B-curvature tensor within the framework of Lorentzian β-Kenmotsu manifold. International Journal of Geometric Methods in Modern Physics, 21(2), 2450125 (14 pages). https://doi.org/10.1142/S0219887824501251

[14] Szabó, Z.I. (1982). Structure theorems on Riemannian spaces satisfying $R(X,Y)\cdot R = 0$, the local version. Journal of Differential Geometry, 17, 531–582.

[15] Tanaka, N. (1976). On non-degenerate real hypersurface, graded Lie algebra and Cartan connections. Japanese Journal of Mathematics, New Series 2, 131–190. https://doi.org/10.4099/math1924.2.131

[16] Tanno, S. (1969). The automorphism groups of almost contact Riemannian manifold. Tohoku Mathematical Journal, 21, 21–38. https://doi.org/10.2748/tmj/1178243031

[17] Takagi, R. (1975). Real hypersurfaces in complex projective space with constant principal curvatures. Journal of the Mathematical Society of Japan, 27, 45–53. https://doi.org/10.2969/jmsj/02710043

[18] Verheyen, P., & Verstraelen, L. (1985). A new intrinsic characterization of hypercylinders in Euclidean spaces. Kyungpook Mathematical Journal, 25, 1–4.

[19] Webster, S.M. (1978). Pseudohermitian structures on a real hypersurface. Journal of Differential Geometry, 13, 25–41. https://doi.org/10.4310/jdg/1214434345

[20] Yano, K., & Kon, M. (1984). Structures on manifolds. Series in Mathematics, Vol. 3. World Scientific Publishing, Singapore.

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SEMI-SYMMETRIC STATISTICAL MANIFOLDS

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Abstract. This paper studies semi-symmetric statistical manifolds (3S-manifolds for short) to generalise semi-Weyl manifolds. We prove that this class of manifolds is invariant under the conformal change of metrics. We show that every 3S-structure $(g, \omega, \omega^*, \nabla)$ on a Riemannian manifold (M, g) induces a statistical structure $(g, \widetilde{\nabla})$ on M and we find necessary and sufficient conditions for ∇ and $\widetilde{\nabla}$ to have the same sectional curvature. In addition, the analogue of the statistical Curvature is defined for 3S structures and its properties are investigated. We also give a method to construct 3S structures on a warped product manifold from 3S structures on the fiber and base manifolds.

Keywords: Semi-symmetric connection, Semi-Weyl structure, Dual connection, Semi-symmetric statistical manifold, 3S-manifold.

2020 Mathematics Subject Classification: 053C23, 53C25, 53C44, 53C50.

1. Introduction

Statistical Manifolds were introduced by S.L. Lauritzen [14] and lie at the confluence of some research areas such as Information Geometry, Affine Geometry and Hessian Geometry (see [2, 11, 18] and references therein). Therefore, statistical manifolds have been intensively studied in several contexts where they are also associated with additional structures and lead to other concepts such as statistical holomorphic structures, statistical Sasakian structures, statistical submanifolds etc. (See [10, 9, 8, 16] and references therein.)

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There have been several attempts to generalize statistical structures which have led for instance to quasi-statistical structures, Weyl, semi-Weyl, quasi-semi-weyl structures and semi-symmetric non-metric statistical structures (see [1, 15, 6] and references therein). But, all these generalizations failed to satisfy some nice properties such as invariance under conformal changing of the Riemannian metric.

In this paper, we introduce and study Semi-Symmetric statistical manifolds (3S-manifolds for short), $(M, g, \omega, \omega^*, \nabla)$ which form a subclass of quasi-semi-Weyl manifolds, such that both ∇ and ∇^* are semi-symmetric connections. It appears that unlike the other generalizations of statistical manifolds [1, 15], the 3S-manifolds are invariant under the conformal changing of the metric. Other good results are obtained and compared, when possible, to the existing ones in a statistical setting.

The paper is organized as follows: The next Section is devoted to the preliminaries where we recall basic definitions and properties of statistical structures that we need in the sequel of the paper. Section 3 deals with 3S structures. We show that non-trivial statistical structures can be generated from 3S structures and vice-versa. And when the so-called 3S-mean vector field is torse-forming, the 3S connection is of constant sectional curvature if and only if its associated statistical connection is of the same constant sectional curvature. In the setting of this paper, α-connections associated to 3S-connections are introduced and studied and it is shown that surprisingly, the curvatures relations obtained for α-3S-connections are similar to those of the classical statistical setting. The section ends with the study of the analogue of the statistical curvature for the 3S connections and the condition for the statistical sectional curvature to be constant. In Section 4, we briefly show that a submanifold of a 3S-manifold is also a 3S-manifold. Finally, Section 5 deals with the warped product of 3S-manifolds. We give a way to construct a 3S structure on a warped product, starting with 3S structures on fiber and the base manifolds.

2. Preliminaries

In the present section, we give some basic definitions and fundamental formulae useful in the sequel.

In what follows, M denotes a smooth manifold, g a Riemannian metric on M, ∇^g the Levi-Civita connection of g, and ∇ an affine connection on M. Throughout the paper, we shall denote the tangent bundle of M by TM, its cotangent bundle by T^*M and the set of smooth sections of TM (respectively, of T^*M) by $\mathfrak{X}(M)$ (respectively, by $\Omega^1(M)$). We will also write T^{∇} to denote the torsion of ∇. From now on, for any $\omega \in \Omega^1(M)$, we denote by S_ω the tensor

$$S_\omega = \omega \otimes I - I \otimes \omega,$$

where $I : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the identity map.

The dual connection of ∇ with respect to g is the unique affine connection ∇^* on M such that:

$$Xg(Y,Z) = g(\nabla_X Y, Z) + g(\nabla^*_X Z, Y), \tag{2.1}$$

for all $X, Y, Z \in \mathfrak{X}(M)$. In this case the triplet (g, ∇, ∇^*) is called a dualistic structure on M.

Definition 2.1 ([12]). *A connection ∇ is said to be semi-symmetric affine connection if there exists a $1-$form ω such that $T^{\nabla} = S_w$.*

Definition 2.2 ([10, 6]). *The pair (g, ∇) is called a statistical structure on M (and (M, g, ∇) a statistical manifold) when ∇ and its dual ∇^* are torsion-free affine connections. This is equivalent to saying that ∇ is torsion-free and the cubic form $C = \nabla g$ is totally symmetric, that is*

$$(\nabla_X g)(Y,Z) = (\nabla_Y g)(X,Z) \tag{2.2}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 2.3 ([17]). *Let ∇ be a torsion-free affine connection and let ω a 1-form on M. The triplet (g, ω, ∇) is called a Weyl structure on M (and (M, g, ω, ∇) a Weyl manifold) if*

$$(\nabla_X g)(Y,Z) = -\omega(X)g(Y,Z), \tag{2.3}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Definition 2.4 ([1]). *Let ∇ be a torsion-free affine connection and let ω be a 1-form on M. Then, (g, ω, ∇) is called a semi-Weyl structure on M (and (M, g, ω, ∇) a semi-Weyl manifold) if*

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = -g(S_\omega(X,Y), Z), \tag{2.4}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In (2.3) and (2.4) when $\omega = 0$ one finds $\nabla = \nabla^g$. Hence statistical structure, Weyl structure and semi-Weyl structure may be regarded as generalizations of the Levi-Civita connection. A direct computation show that for any affine connection ∇ one has:

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = g(T^{\nabla^*}(X,Y) - T^{\nabla}(X,Y), Z). \tag{2.5}$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

Remark 2.1. *Observe that when (M, g, ω, ∇) is a semi-Weyl manifold, the dual connection ∇^* is rather semi-symmetric.*

Definition 2.5 ([1])**.** *Let ∇ be an affine connection on M with torsion tensor T^{∇} and let ω be a 1-form. Then, (g, ω, ∇) is called a quasi-semi- Weyl structure on M (and (M, g, ω, ∇) a quasi-semi-Weyl manifold) if*

$$(\nabla_X g)(Y,Z) - (\nabla_Y g)(X,Z) = -g(T^{\nabla}(X,Y) + S_\omega(X,Y), Z),$$

for all $X, Y, Z \in \mathfrak{X}(M)$.

In the next section we are interested in quasi-semi-Weyl structures (g, ω, ∇) such that ∇ is semi-symmetric.

3. Semi-symmetric statistical manifolds

Semi-symmetric metric connections on statistical manifolds have been introduced and studied in [12, 13, 3]. Such a connection is also called statistical semi-symmetric connection in [13]. In this mentioned papers, the dual connection of a statistical semi-symmetric connection ∇ has the same torsion as ∇. We keep the same terminology to define something more general. In what follows, the torsion of the dual of ∇ may be different from the one of ∇.

Definition 3.1. *The pair (g, ∇) is called a semi-symmetric statistical structure on M (and (M, g, ∇) a semi-symmetric statistical manifold) if there are two 1-forms ω and ω^* on M such that*

$$T^{\nabla} = S_\omega \quad \text{and} \quad T^{\nabla^*} = S_{\omega^*}.$$

To be short, such a structure will be called a 3S-structure and the connection ∇ a 3S-connection.

Semi-symmetric statistical structure is a generalization of semi-symmetric metric connection studied in [12, 13]. When $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M, we set

$$\omega^S = \frac{1}{2}(\omega + \omega^*), \tag{3.6}$$

and we call it the 3S-mean 1-form (for short the mean 1-form). The vector field V, g-associated to ω^S is called the mean vector field. The 3S-structure $(g, \omega, \omega^*, \nabla)$ is said to be closed when ω^S is closed.

Example 3.1. *We have the following simple examples of 3S-structures:*

1. *A statistical structure (g, ∇) is a 3S-structure and is called the trivial 3S-structure, where $\omega = \omega^* = 0$.*
2. *A semi-Weyl structure (g, ω, ∇) is a 3S-structure.*
3. *If (g, ∇) is a statistical structure then for all 1-form ω on M, $(g, \omega, -\omega, \overline{\nabla} =: \nabla + \omega \otimes I)$ is a 3S-structure. Just observe that the dual connection $\overline{\nabla}^*$ of $\overline{\nabla}$ with respect to g is given by $\overline{\nabla}^* = \nabla^* + \omega^* \otimes I$ with $\omega^* = -\omega$.*

It is proven in [6] that, if f is a function on M and ω_1, ω_2 are 1-forms g-associated to the vector fields ξ_1, ξ_2 on M respectively, then there exists a unique affine connection ∇ which satisfies the following equations:

$$T^{\nabla}(X, Y) = \omega_1(X)Y - \omega_1(Y)X, \tag{3.7}$$

$$(\nabla_X g)(Y, Z) = f\big(\omega_2(Y)g(X, Z) + \omega_2(Z)g(X, Y)\big). \tag{3.8}$$

Such a connection has been called semi-symmetric non-metric connection and is given by

$$\nabla_X Y = \nabla^g_X Y + \omega_1(Y)X - g(X, Y)\xi_1 - fg(X, Y)\xi_2.$$

Remark 3.1. *All semi-symmetric non-metric affine connection are 3S-connections. But a 3S-connection needs not to be semi-symmetric non-metric affine connection.*

Indeed, let ∇ be an affine connection which satisfies (3.7) and (3.8). Since X, Y, Z are arbitrary in (3.8), so we have

$$(\nabla_Y g)(X, Z) = f\big(\omega_2(X)g(Y, Z) + \omega_2(Z)g(X, Y)\big). \tag{3.9}$$

Using (3.8) and (3.9) we get

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = g(f\omega_2(Y)X - f\omega_2(X)Y, Z). \tag{3.10}$$

Moreover, using (3.7), (3.8) and (3.10) we have

$$T^{\nabla^*}(X,Y) = (f\omega_2 - \omega_1)(Y)X - (f\omega_2 - \omega_1)(X)Y.$$

Hence $T^{\nabla} = S_{\omega_1}$ and $T^{\nabla^*} = S_{\omega_1 - f\omega_2}$.

Conversely, take h a function on M. Then $(g, dh, -dh, \nabla^g + dh \otimes I)$ is 3S-structure on M but

$$(\nabla_X g)(Y,Z) = -2(dh)(X)g(Y,Z),$$

which shows that ∇ is not a semi-symmetric non-metric affine connection.

The following theorem shows that unlike statistical structures, 3S-structures are preserved under conformal changing of g.

Theorem 3.1. *Let $\widetilde{g}$ be a conformal metric of g such that for a function h on M we have, $\widetilde{g} = e^h g$. Then, $(g, \omega, \omega^*, \nabla)$ is 3S-structure on M if and only if $(\widetilde{g}, \omega, \omega^* + dh, \nabla)$ is 3S-structure on M.*

Proof. Let X, Y, Z be three vector fields on M, ∇^d the dual connection of ∇ with respect to $\widetilde{g}$ and ∇^* the dual connection of ∇ with respect to g. From the duality condition of ∇ with respect to $\widetilde{g}$ we have

$$X\widetilde{g}(Y,Z) = \widetilde{g}(\nabla_X Y, Z) + \widetilde{g}(Y, \nabla^d_X Z),$$

that is

$$Xg(Y,Z) + X(h)g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla^d_X Z). \tag{3.11}$$

As X, Y and Z are arbitrary, from (3.11) we have

$$Zg(X,Y) + Z(h)g(X,Y) = g(\nabla_Z Y, X) + g(Y, \nabla^d_Z X). \tag{3.12}$$

Take (3.11) and substract (3.12), then

$$\begin{aligned} &Xg(Y,Z) - Z(h)g(X,Y) + X(h)g(Y,Z) - Zg(X,Y) \\ &= g(\nabla_X Y, Z) - g(\nabla_Z Y, X) + g(T^{\nabla^d}(X,Z) + [X,Z], Y). \end{aligned}$$

That is

$$\begin{aligned} &g(Y, \nabla^*_X Z) - g(Y, \nabla^*_Z X) + g(Y, X(h)Z - Z(h)X) \\ &= g(T^{\nabla^d}(X,Z) + [X,Z], Y). \end{aligned}$$

Therefore, we have

$$T^{\nabla^d}(X,Z) = T^{\nabla^*}(X,Z) + S_{dh}(X,Z).$$

Thus $T^{\nabla^d} = S_{\omega^*+dh}$ if and only if $T^{\nabla^*} = S_{\omega^*}$. This ends the proof. □

One can easily prove the following:

Theorem 3.2. *Let (g, M) be a Riemannian manifold, ∇ a connection on M, η, ω and ω^* some 1-forms on M such that $2\omega^S = \omega + \omega^*$. Set V and ζ vector fields g-associated to ω^S and η respectively, i.e., $g(X, V) = \omega^S(X)$ and $g(X, \zeta) = \eta(X)$. The pair $(g, \widetilde{\nabla})$ where*

$$\widetilde{\nabla} = \nabla - \omega \otimes I + \eta \otimes I + I \otimes \eta + g(.,.)(\zeta - 2V), \tag{3.13}$$

is a statistical structure on M if and only if $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M. Moreover, the dual of $\widetilde{\nabla}$ with respect to g is given by

$$(\widetilde{\nabla})^* = \nabla^* + \omega \otimes I - \eta \otimes I - I \otimes \eta + 2I \otimes \omega^S - g(.,.)\zeta, \tag{3.14}$$

where ∇^* is the dual of ∇ with respect to g.

Taking $\eta = 0$ in (3.13), one obtains

$$\widetilde{\nabla} = \nabla - \omega \otimes I - 2g(.,.)V, \tag{3.15}$$

$$(\widetilde{\nabla})^* = \nabla^* + \omega \otimes I + 2I \otimes \omega^S. \tag{3.16}$$

$\widetilde{\nabla}$ is called the statistical connection with respect to ω and associated to ∇.

For a 3S-structure $(g, \omega, \omega^*, \nabla)$ on M, the mean vector field V is called torse-forming with respect to ω if

$$\nabla_X V = \omega(X)V,$$

for any X in $\mathfrak{X}(M)$.

Example 3.2. *We consider the manifold $M = \{(x, y) \in \mathbb{R}^2, x; y > 0\}$ equipped with its canonical metric $g_0 = dx^2 + dy^2$ and we set the function $h(x, y) = -\frac{1}{2}ln(x + y)$ on M. The gradient ∇h of h with respect to g_0 is given by*

$$\nabla h = -\frac{1}{2(x+y)}(\vec{i} + \vec{j}),$$

where $(\vec{i}, \vec{j}) = (\partial_x, \partial_y)$ is the canonical basis of $\mathfrak{X}(M)$. From Theorem 3.1, we see that $(M, \widetilde{g}, \omega, \omega^*, \nabla^{g_0})$ is a 3S-manifold, that is $(M, \widetilde{g}, \omega^*, \omega, (\nabla^{g_0})^*)$ is a 3S-manifold, where $\widetilde{g} = e^{2h}g_0$, $\omega = 0$, $\omega^* = 2dh$, ∇^{g_0} is a Levi-Civita connection with respect to g_0 and the dual $(\nabla^{g_0})^*$ of ∇^{g_0} with respect to $\widetilde{g}$ is given by

$$(\nabla^{g_0})^* = \nabla^{g_0} + \omega^* \otimes I. \tag{3.17}$$

Moreover, we have $\omega^S = dh$*, that is* $(\omega^S)^{\sharp_{\widetilde{g}}} = V$*. From*

$$\widetilde{g}((\omega^S)^{\sharp_{\widetilde{g}}}, X) = X(h) = g_0((dh)^{\sharp_{g_0}}, X),$$

we get $V = e^{-2h}\nabla h$*. From (3.17), we have*

$$(\nabla^{g_0})^*_X V = e^{-2h}\nabla^{g_0}_X \nabla h, \tag{3.18}$$

for any vector field X *in* M*. Since* $(\nabla^{g_0})_{\vec{i}}\vec{i} = (\nabla^{g_0})_{\vec{i}}\vec{j} = (\nabla^{g_0})_{\vec{j}}\vec{i} = (\nabla^{g_0})_{\vec{j}}\vec{j} = \vec{0}$*, we have*

$$(\nabla^{g_0})_{\vec{i}}\nabla h = -\frac{1}{x+y}\nabla h \ \ \text{and} \ \ (\nabla^{g_0})_{\vec{j}}\nabla h = -\frac{1}{x+y}\nabla h. \tag{3.19}$$

So, using (3.19), for any vector field $X = X^1\vec{i} + X^2\vec{j}$ *in* M*, we have*

$$\begin{aligned}\nabla^{g_0}_X \nabla h &= -\frac{1}{x+y}[X^1 + X^2]\nabla h \\ &= 2g_0(\nabla h, X)\nabla h \\ &= 2dh(X)\nabla h \\ &= \omega^*(X)\nabla h.\end{aligned}$$

From (3.18), we get

$$(\nabla^{g_0})^*_X V = \omega^*(X)V.$$

Thus, V *is a torse-forming vector on the 3S-structure* $(M, \widetilde{g}, \omega^*, \omega, (\nabla^{g_0})^*)$ *with respect to* ω^**.*

Proposition 3.1. *For a 3S-structure* $(g, \omega, \omega^*, \nabla)$ *on* M*, for any 1-form* η*, we set* $\rho = \eta - \omega$ *and* $\Omega = \zeta - 2V$*, where* V *and* ζ *are vector fields associated with* ω^S *and* η *with respect to* g *respectively. The relationship between the curvature* R *of* $(g, \omega, \omega^*, \nabla)$ *and the curvature* $\widetilde{R}$ *of the statistical structure* $(g, \widetilde{\nabla})$ *associated to* $(g, \omega, \omega^*, \nabla)$ *is given by*

$$\begin{aligned}\widetilde{R}(X,Y)Z = R(X,Y)Z &- (d\omega)(X,Y)Z + (d\eta)(X,Y)Z \\ &+ g(Y,Z)\nabla_X\Omega - g(X,Z)\nabla_Y\Omega + \{g(Y,Z)\rho(X) - g(X,Z)\rho(Y)\}\Omega \\ &+ \big\{X\eta(Z) - \eta(X)\eta(Z) + \omega(X)\eta(Z) - \eta(\nabla_X Z) - \eta(\Omega)g(X,Z)\big\}Y \\ &- \big\{Y\eta(Z) - \eta(Y)\eta(Z) + \omega(Y)\eta(Z) - \eta(\nabla_Y Z) - \eta(\Omega)g(Y,Z)\big\}X,\end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(M)$*.*

Proposition 3.2. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. We assume that the sectional curvatures of ∇ and $\widetilde{\nabla}$ defined as (3.15) are well-defined. Then, these sectional curvatures are the same if and only if the mean vector field V is torse-forming with respect to ω.*

Proof. Let X, Y be two linear independent vector fields on M. Let $\widetilde{R}$ and R the curvature of $\widetilde{\nabla}$ and ∇ respectively. From proposition 3.1, taking $\eta = 0$ we get

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - (d\omega)(X,Y)Z \\ &\quad + 2g(Y,Z)\{\omega(X)V - \nabla_X V\} - 2g(X,Z)\{\omega(Y)V - \nabla_Y V\}.\end{aligned}$$

Also we have

$$\widetilde{R}(X,Y,Y,X) = R(X,Y,Y,X) + 2g(X, \omega(X)V - \nabla_X V), \tag{3.20}$$

where $\widetilde{R}(X,Y,Y,X) = g(\widetilde{R}(X,Y)Y,X)$ and $R(X,Y,Y,X) = g(R(X,Y)Y,X)$. Moreover, from (3.20), $\widetilde{R}(X,Y,Y,X) = R(X,Y,Y,X)$ if and only if the vector field V is torse-forming with respect ω. $\square$

We define the tensor S by

$$S = \nabla - \nabla^g. \tag{3.21}$$

Lemma 3.1. *Let ∇ be an affine connection on M. Then, we have*

$$g(X, S(Y,Z)) - g(Y, S(X,Z)) = g(T^{\nabla^*}(X,Y), Z), \tag{3.22}$$

$$S(X,Y) - S(Y,X) = T^{\nabla}(X,Y), \tag{3.23}$$

for all vectors fields X, Y, Z on M.

Proof. Let X, Y, Z be three vectors fields on M. From the duality condition of ∇^g

$$Xg(Y,Z) = g(\nabla^g_X Y, Z) + g(Y, \nabla^g_X Z),$$

and eq. (3.21), we get

$$g(\nabla^*_X Y, Z) = g(\nabla^g_X Y, Z) - g(Y, S(X,Z)). \tag{3.24}$$

Therefore, from (3.24) we obtain

$$g(X, S(Y,Z)) - g(Y, S(X,Z)) = g(T^{\nabla^*}(X,Y), Z).$$

Moreover,

$$S(X,Y)-S(Y,X)=\nabla_X Y-\nabla^g_X Y-\nabla_Y X+\nabla^g_Y X$$
$$=T^{\nabla}(X,Y).$$

□

Proposition 3.3. *Let ∇ be an affine connection on M. Then, $(g,\omega,\omega^*,\nabla)$ is 3S-structure on M if and only if*

$$S(X,Y)-S(Y,X)=S_\omega(X,Y), \tag{3.25}$$

$$g(X,S(Y,Z))-g(Y,S(X,Z))=g(S_{\omega^*}(X,Y),Z). \tag{3.26}$$

Proof. This follows easily from lemma 3.1. □

If $(g,\omega,\omega^*,\nabla)$ is 3S-structure on M, from Proposition 3.3 we give the Levi-Civita connection ∇^g in terms of g, mean 1-form ω^S, mean vector field V, ∇ and ∇^*. For this purpose, we set

$$K^{\omega,\omega^*}(X,Y)=\omega^S(Y)X-g(X,Y)V,$$

Proposition 3.4. *If $(g,\omega,\omega^*,\nabla)$ is a 3S-structure on M, then the Levi-Civita connection ∇^g is given by:*

$$\nabla^g_X Y=\frac{1}{2}\big(\nabla_X Y+\nabla^*_X Y\big)+K^{\omega,\omega^*}(X,Y), \tag{3.27}$$

for all vector fields X,Y on M.

Proof. Let X,Y, Z be vector fields on M, ξ and ξ^* be vector fields g-associated to ω and ω^* respectively. From the duality condition of ∇ and ∇^g with respect to g we obtain

$$g(Z,S(X,Y))=g(Y,\nabla^g_X Z-\nabla^*_X Z). \tag{3.28}$$

From (3.25) and (3.28) we get

$$g(Z,S(Y,X))=g(Y,\nabla^g_X Z-\nabla^*_X Z)+g(Y,-\omega(X)Z+g(X,Z)\xi). \tag{3.29}$$

As (3.26) is equivalent to

$$g(Z,S(Y,X))=g(Y,S(Z,X))+g(Y,\omega^*(Z)X-g(Z,X)\xi^*), \tag{3.30}$$

then from (3.29) and (3.30) we have

$$g(Y, S(Z,X)) + g(Y, \omega^*(Z)X - g(Z,X)\xi^*)$$
$$= g(Y, \nabla^g_X Z - \nabla^*_X Z - \omega(X)Z + g(X,Z)\xi),$$

that is

$$S(Z,X) = \nabla^g_X Z - \nabla^*_X Z - \omega(X)Z - \omega^*(Z)X + g(X,Z)(\xi + \xi^*). \tag{3.31}$$

According to (3.25) and (3.31) we have

$$S(X,Z) + S_\omega(Z,X) = \nabla^g_X Z - \nabla^*_X Z - \omega(X)Z - \omega^*(Z)X + g(X,Z)(\xi + \xi^*),$$

that is

$$S(X,Z) = \nabla^g_X Z - \nabla^*_X Z - (\omega + \omega^*)(Z)X + g(X,Z)(\xi + \xi^*). \tag{3.32}$$

Using (3.21) and (3.32), we get

$$2\nabla^g_X Z = \nabla_X Z + \nabla^*_X Z + (\omega + \omega^*)(Z)X - g(X,Z)(\xi + \xi^*).$$

□

Corollary 3.1. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, the dual connection ∇^* of ∇ is given by*

$$S = \nabla^g - \nabla^* - 2K^{\omega,\omega^*} = \frac{1}{2}(\nabla - \nabla^*) - K^{\omega,\omega^*}. \tag{3.33}$$

For a given totally symmetric (0, 3)-tensor field, we can define mutually dual semi-symmetric affine connections.

Proposition 3.5. *Assume that (g, M) is a Riemannian manifold, C is a totally symmetric (0, 3)-tensor field on M, ω and ω^* two 1-forms on M such that $2\omega^S = \omega + \omega^*$ and V the vector field g-associated to ω^S. We define the mapping ∇ and ∇^* by*

$$g(\nabla_X Y, Z) = g(\nabla^g_X Y, Z) - \frac{1}{2}C(X,Y,Z) + g(Z, \omega(X)Y + 2g(X,Y)V), \tag{3.34}$$
$$g(\nabla^*_X Y, Z) = g(\nabla^g_X Y, Z) + \frac{1}{2}C(X,Y,Z) - g(Z, \omega(X)Y + 2\omega^S(Y)X). \tag{3.35}$$

Then ∇ and ∇^* are mutually dual connections. Moreover $(g, \omega, \omega^*, \nabla)$ is 3S-structure on M.

Proposition 3.6. *Let ∇ and ∇^* be mutually dual connections on M with respect to g. Let C be a (0, 3)-tensor field on M, $2\omega^S = \omega + \omega^*$ where ω, ω^* are two 1-forms on M and V the vector field g-associated with ω^S. We suppose that ∇ and C verify (3.34) and $(g, \omega, \omega^*, \nabla)$ is 3S-structure on M. Then C is totally symmetric.*

Proof. We set $S = \nabla - \nabla^g$ and $\widetilde{S} = \widetilde{\nabla} - \nabla^g$ such that ∇ and $\widetilde{\nabla}$ verify (3.15). Let $X, Y, Z \in \mathfrak{X}(M)$. Since ∇ and C verify (3.34), so we get

$$C(X,Y,Z) = -2g(S(X,Y) - \omega(X)Y - 2g(X,Y), Z). \tag{3.36}$$

From (3.36), we have

$$C(X,Y,Z) = -2g(\widetilde{S}(X,Y), Z).$$

$(g, \widetilde{\nabla})$ is statistical structure implies that $C(X,Y,Z) = (\widetilde{\nabla}_X g)(Y,Z)$, thus, C is totally symmetric. ∎

3.1. **Some results on the alpha-connections of a 3S-structure.** For $\alpha \in \mathbb{R}$, we define a family of connections $\nabla^{(\alpha)}$ by,

$$\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*. \tag{3.37}$$

$\nabla^{(\alpha)}$ is called an α-connection of dualistic structure (∇, ∇^*). The dual of $\nabla^{(\alpha)}$ with respect to g is given by $\nabla^{(-\alpha)}$.

If $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M, we set

$$\omega_\alpha = \frac{1+\alpha}{2}\omega + \frac{1-\alpha}{2}\omega^* \quad \text{and} \quad \omega_\alpha^* = \omega_{-\alpha} = \frac{1-\alpha}{2}\omega + \frac{1+\alpha}{2}\omega^*. \tag{3.38}$$

In particular,

$$\omega_0 = \omega^g, \omega_1 = \omega \ \text{ and } \ \omega_{-1} = \omega^*. \tag{3.39}$$

We define the tensors K by

$$K = \nabla^* - \nabla. \tag{3.40}$$

Proposition 3.7. *If $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M, then $(g, \omega_\alpha, \omega_\alpha^*, \nabla^{(\alpha)})$ is a 3S-structure.*

Proof. Using (3.37) and (3.38) we get

$$T^{\nabla^{(\alpha)}} = S_{\omega_\alpha} \ \text{ and } \ T^{\nabla^{(-\alpha)}} = S_{\omega_\alpha^*}.$$

∎

When $(g, \omega, \omega^*, \nabla)$ is a 3S-structure on M we have the following equality,

$$\nabla^{(\alpha)} = \nabla^g - \frac{\alpha}{2}K - K^{\omega,\omega^*}. \tag{3.41}$$

In particular

$$\nabla = \nabla^g - \frac{1}{2}K - K^{\omega,\omega^*}, \tag{3.42}$$

$$\nabla^* = \nabla^g + \frac{1}{2}K - K^{\omega,\omega^*}. \tag{3.43}$$

From (3.40) we get

$$K(X,Y) - K(Y,X) = S_{\omega^*}(X,Y) - S_\omega(X,Y). \tag{3.44}$$

It was shown in [3] that for a pair of conjugate connections, their curvature tensors satisfy

$$g(R(X,Y)Z,T) + g(Z,R^*(X,Y)T) = 0, \tag{3.45}$$

and more generally

$$g(R^{(\alpha)}(X,Y)Z,T) + g(Z,R^{(-\alpha)}(X,Y)T) = 0 \tag{3.46}$$

where $R^{(-\alpha)} = R^{*(\alpha)}$.

Denote $K(X,Y,Z) = (\nabla_X g)(Y,Z)$, where

$$(\nabla_X g)(Y,Z) = Xg(Y,Z) - g(\nabla_X Y,Z) - g(Y,\nabla_X Z).$$

The 3-tensor $K(.,.,.)$ is called the cubic form and satisfies $K(.,Y,Z) = K(.,Z,Y)$ by its definition.

Recall the difference tensor $K(X,Y)$ introduced in (3.40), which can be verified to be related to $K(X,Y,Z)$ via

$$g(K(X,Y),Z) = K(X,Y,Z). \tag{3.47}$$

Proposition 3.8. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M, the curvature tensor $R^{(\alpha)}$ for the α-connection $\nabla^{(\alpha)}$ satisfies*

$$\begin{aligned} R^{(\alpha)}(X,Y)Z &= \frac{1+\alpha}{2}R(X,Y)Z + \frac{1-\alpha}{2}R^*(X,Y)Z \\ &+ \frac{1-\alpha^2}{4}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big). \end{aligned} \tag{3.48}$$

Proof. We assume that $(g,\omega,\omega^*,\nabla)$ is a 3S-structure on M. By definition of the α-connection and of curvature tensor

$$\begin{aligned} R^{(\alpha)}(X,Y)Z &= \left(\frac{1+\alpha}{2}\right)^2 R(X,Y)Z + \left(\frac{1-\alpha}{2}\right)^2 R^*(X,Y)Z \\ &\quad + \left(\frac{1-\alpha^2}{4}\right)\Big(\nabla_X\nabla^*_Y Z + \nabla^*_X\nabla_Y Z - \nabla_Y\nabla^*_X Z \\ &\quad - \nabla^*_Y\nabla_X Z - \nabla_{[X,Y]}Z - \nabla^*_{[X,Y]}Z\Big). \end{aligned} \tag{3.49}$$

From (3.42) and (3.43) we have

$$\begin{aligned} \nabla_X\nabla^*_Y Z &= \nabla^g_X\nabla^g_Y Z - \frac{1}{2}K(X,\nabla^g_Y Z) - K^{\omega,\omega^*}(X,\nabla^g_Y Z) \\ &\quad + \frac{1}{2}\nabla^g_X K(Y,Z) - \frac{1}{4}K(X,K(Y,Z)) - \frac{1}{2}K^{\omega,\omega^*}(X,K(Y,Z)) \\ &\quad - \nabla^g_X K^{\omega,\omega^*}(Y,Z) + \frac{1}{2}K(X,K^{\omega,\omega^*}(Y,Z)) + K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)), \end{aligned}$$

$$\begin{aligned} -\nabla_Y\nabla^*_X Z &= -\nabla^g_Y\nabla^g_X Z + \frac{1}{2}K(Y,\nabla^g_X Z) + K^{\omega,\omega^*}(Y,\nabla^g_X Z) \\ &\quad - \frac{1}{2}\nabla^g_Y K(X,Z) + \frac{1}{4}K(Y,K(X,Z)) + \frac{1}{2}K^{\omega,\omega^*}(Y,K(X,Z)) \\ &\quad + \nabla^g_Y K^{\omega,\omega^*}(X,Z) - \frac{1}{2}K(Y,K^{\omega,\omega^*}(X,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z)), \end{aligned}$$

$$\begin{aligned} \nabla^*_X\nabla_Y Z &= \nabla^g_X\nabla^g_Y Z + \frac{1}{2}K(X,\nabla^g_Y Z) - K^{\omega,\omega^*}(X,\nabla^g_Y Z) \\ &\quad - \frac{1}{2}\nabla^g_X K(Y,Z) - \frac{1}{4}K(X,K(Y,Z)) + \frac{1}{2}K^{\omega,\omega^*}(X,K(Y,Z)) \\ &\quad - \nabla^g_X K^{\omega,\omega^*}(Y,Z) - \frac{1}{2}K(X,K^{\omega,\omega^*}(Y,Z)) + K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)), \end{aligned}$$

$$\begin{aligned} -\nabla^*_Y\nabla_X Z &= -\nabla^g_Y\nabla^g_X Z - \frac{1}{2}K(Y,\nabla^g_X Z) + K^{\omega,\omega^*}(Y,\nabla^g_X Z) \\ &\quad + \frac{1}{2}\nabla^g_Y K(X,Z) + \frac{1}{4}K(Y,K(X,Z)) - \frac{1}{2}K^{\omega,\omega^*}(Y,K(X,Z)) \\ &\quad + \nabla^g_Y K^{\omega,\omega^*}(X,Z) + \frac{1}{2}K(Y,K^{\omega,\omega^*}(X,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z)) \end{aligned}$$

and

$$-\nabla_{[X,Y]}Z - \nabla^*_{[X,Y]}Z = 2K^{\omega,\omega^*}([X,Y],Z) - 2\nabla^g_{[X,Y]}Z.$$

Therefore, the last parentheses in (3.49) become

$$
\begin{aligned}
&2R^g(X,Y)Z + \frac{1}{2}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big)\\
&+ 2\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big) - 2K^{\omega,\omega^*}(X,\nabla^g_Y Z)\\
&+ 2K^{\omega,\omega^*}(Y,\nabla^g_X Z) - 2\nabla^g_X K^{\omega,\omega^*}(Y,Z) + 2\nabla^g_Y K^{\omega,\omega^*}(X,Z) + 2K^{\omega,\omega^*}([X,Y],Z),
\end{aligned}
$$

where R^g is the Riemann curvature tensor, i.e, the curvature of the Levi–Civita connection ∇^g. This last expression can simplify to

$$
\begin{aligned}
&2R^g(X,Y)Z + \frac{1}{2}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big)\\
&+ 2\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big)\\
&+ 2\Big((\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\Big).
\end{aligned}
$$

Therefore, (3.49) becomes

$$
\begin{aligned}
R^{(\alpha)}(X,Y)Z = &\left(\frac{1+\alpha}{2}\right)^2 R(X,Y)Z + \left(\frac{1-\alpha}{2}\right)^2 R^*(X,Y)Z\\
&+ \frac{1-\alpha^2}{2}\Big(R^g(X,Y)Z + \frac{1}{4}\big(K(Y,K(X,Z)) - K(X,K(Y,Z))\big)\Big)\\
&+ \frac{1-\alpha^2}{2}\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big)\\
&+ \frac{1-\alpha^2}{2}\Big((\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\Big).
\end{aligned}
\tag{3.50}
$$

Taking $\alpha = 0$ and using (3.50) we get

$$
\begin{aligned}
R^{(0)}(X,Y)Z = &\frac{1}{4}R(X,Y)Z + \frac{1}{4}R^*(X,Y)Z + \frac{1}{8}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big)\\
&+ \frac{1}{2}R^g(X,Y)Z + \frac{1}{2}\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\Big)\\
&+ \frac{1}{2}\Big((\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\Big).
\end{aligned}
\tag{3.51}
$$

Taking $\alpha = 0$ and using (3.41) we get

$$
\begin{aligned}
R^{(0)}(X,Y)Z = &R^g(X,Y)Z + K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,Z))\\
&+ K^{\omega,\omega^*}(Y,\nabla^g_X Z) - K^{\omega,\omega^*}(X,\nabla^g_Y Z) + \nabla^g_Y K^{\omega,\omega^*}(X,Z)\\
&- \nabla^g_X K^{\omega,\omega^*}(Y,Z) + K^{\omega,\omega^*}([X,Y],Z),
\end{aligned}
$$

that is

$$R^{(0)}(X,Y)Z = R^g(X,Y)Z + K^{\omega,\omega^*}(X, K^\omega_g(Y,Z)) - K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,Z)) + (\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z). \tag{3.52}$$

Using (3.51) and (3.52) we obtain

$$R^g(X,Y)Z = \frac{1}{2}R(X,Y)Z + \frac{1}{2}R^*(X,Y)Z + \frac{1}{4}\big(K(Y,K(X,Z)) - K(X,K(Y,Z))\big) + K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,Z)) - K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,Z)) + (\nabla^g_X K^{\omega,\omega^*})(Y,Z) - (\nabla^g_Y K^{\omega,\omega^*})(X,Z). \tag{3.53}$$

Using (3.53) in (3.50) leads to :

$$R^{(\alpha)}(X,Y)Z = \frac{1+\alpha}{2}R(X,Y)Z + \frac{1-\alpha}{2}R^*(X,Y)Z + \frac{1-\alpha^2}{4}\Big(K(Y,K(X,Z)) - K(X,K(Y,Z))\Big).$$

□

From (3.48)

$$R^{(\alpha)}(X,Y)Z - R^{(-\alpha)}(X,Y)Z = \alpha(R(X,Y)Z - R^*(X,Y)Z). \tag{3.54}$$

Remark 3.2. *Surprisinely, the transformation $R \longmapsto R^{(\alpha)}$ from (3.48) is the same form for 3S-structure and statistical structure* [21].

Remark 3.3. *In the theory of Semi-Symmetric statistical manifold, from (3.52), we get $R^{(0)} \neq R^g$.*

3.2. **Statistical curvature of semi-Symmetric statistical manifolds.** In this section, we firstly give symmetry properties of curvatures R, R^* and give these properties for the statistical curvature R^S of semi-symmetric statistical manifolds.

Lemma 3.2. *For a 3S-structure* $(g,\omega,\omega^*,\nabla)$ *on* M*, the following formulas hold for* $X,Y,Z,T\in\mathfrak{X}(M)$*:*

$$R(X,Y)Z=-R(Y,X)Z, \tag{3.55}$$

$$g(R(X,Y)Z,T)+g(R(Y,X)Z,T)=0, \tag{3.56}$$

$$R(X,Y)Z+R(Y,Z)X+R(Z,X)Y=(d\omega(X,Y))Z$$
$$+(d\omega(Y,Z))X+(d\omega(Z,X))Y, \tag{3.57}$$

$$g(R(X,Y)Z,T)+g(R(X,Y)T,Z)=2g\Big((\nabla^g_XS)(Y,T)-(\nabla^g_YS)(X,T),Z\Big)$$
$$+2g\Big((\nabla^g_XK^{\omega,\omega^*})(Y,T)-(\nabla^g_YK^{\omega,\omega^*})(X,T),Z\Big)+4g\Big(K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,T))$$
$$-K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T)),Z\Big)+2g\Big(K^{\omega,\omega^*}(Y,S(X,T))-K^{\omega,\omega^*}(X,S(Y,T)),Z\Big)$$
$$+2g\Big(S(Y,K^{\omega,\omega^*}(X,T))-S(X,K^{\omega,\omega^*}(Y,T)),Z\Big), \tag{3.58}$$

and

$$R^*(X,Y)Z=-R^*(Y,X)Z, \tag{3.59}$$

$$g(R^*(X,Y)Z,T)+g(R^*(Y,X)Z,T)=0, \tag{3.60}$$

$$R^*(X,Y)Z+R^*(Y,Z)X+R^*(Z,X)Y=(d\omega^*(X,Y))Z$$
$$+(d\omega^*(Y,Z))X+(d\omega^*(Z,X))Y, \tag{3.61}$$

$$g(R^*(X,Y)Z,T)+g(R^*(X,Y)T,Z)=2g\Big((\nabla^g_YS)(X,T)-(\nabla^g_XS)(Y,T),Z\Big)$$
$$+2g\Big((\nabla^g_YK^{\omega,\omega^*})(X,T)-(\nabla^g_XK^{\omega,\omega^*})(Y,T),Z\Big)+4g\Big(K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T))$$
$$-K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,T)),Z\Big)+2g\Big(K^{\omega,\omega^*}(X,S(Y,T))-K^{\omega,\omega^*}(Y,S(X,T)),Z\Big)$$
$$+2g\Big(S(X,K^{\omega,\omega^*}(Y,T))-S(Y,K^{\omega,\omega^*}(X,T)),Z\Big). \tag{3.62}$$

Proof. For the proof of (3.55), (3.56), (3.57), (3.59), (3.60) and (3.61) refer to [12]. We now, prove (3.58) and (3.62) which depend on the tensor K^{ω,ω^*}.

Using (3.33) we get

$$\begin{aligned}\nabla^*_X\nabla^*_YT=\nabla_X\nabla_YT-2\nabla_XS(Y,T)-2\nabla_XK^{\omega,\omega^*}(Y,T)-2S(X,\nabla_YT)\\+4S(X,S(Y,T))+4S(X,K^{\omega,\omega^*}(Y,T))-2K^{\omega,\omega^*}(X,\nabla_YT)\\+4K^{\omega,\omega^*}(X,S(Y,T))+4K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T)).\end{aligned}$$

Using (3.21), the last equation becomes

$$\begin{aligned}\nabla^*_X\nabla^*_Y T &= \nabla_X\nabla_Y T - 2\nabla^g_X S(Y,T) - 2\nabla^g_X K^{\omega,\omega^*}(Y,T) - 2S(X,\nabla^g_Y T)\\ &\quad - 2K^{\omega,\omega^*}(X,\nabla^g_Y T) + 2S(X,K^{\omega,\omega^*}(Y,T)) + 2K^{\omega,\omega^*}(X,S(Y,T))\\ &\quad + 4K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T)).\end{aligned} \tag{3.63}$$

Moreover,

$$\begin{aligned}-\nabla^*_Y\nabla^*_X T &= -\nabla_Y\nabla_X T + 2\nabla^g_Y S(X,T) + 2\nabla^g_Y K^{\omega,\omega^*}(X,T) + 2S(Y,\nabla^g_X T)\\ &\quad + 2K^{\omega,\omega^*}(Y,\nabla^g_X T) - 2S(Y,K^{\omega,\omega^*}(X,T)) - 2K^{\omega,\omega^*}(Y,S(X,T))\\ &\quad - 4K^{\omega,\omega^*}(Y,K(X,T)).\end{aligned} \tag{3.64}$$

Since $[X,Y] = \nabla^g_X Y - \nabla^g_Y X$, we get

$$\begin{aligned}-\nabla^*_{[X,Y]}T &= -\nabla_{[X,Y]}T + 2S(\nabla^g_X Y,T) - 2S(Y,\nabla^g_X T)\\ &\quad + 2K^{\omega,\omega^*}(\nabla^g_X Y,T) - 2K^{\omega,\omega^*}(Y,\nabla^g_X T).\end{aligned} \tag{3.65}$$

Summing up (3.63), (3.64) and (3.65), we obtain

$$\begin{aligned}R^*(X,Y)T &= R(X,Y)T - 2(\nabla^g_X S)(Y,T) + 2(\nabla^g_Y S)(X,T) - 2(\nabla^g_X K^{\omega,\omega^*})(Y,T)\\ &\quad + (\nabla^g_Y K^{\omega,\omega^*})(X,T) - 4K^{\omega,\omega^*}(Y,K^{\omega,\omega^*}(X,T)) + 4K^{\omega,\omega^*}(X,K^{\omega,\omega^*}(Y,T))\\ &\quad - 2K^{\omega,\omega^*}(Y,S(X,T)) + 2K^{\omega,\omega^*}(X,S(Y,T)) - 2S(Y,K^{\omega,\omega^*}(X,T))\\ &\quad + 2S(X,K^{\omega,\omega^*}(Y,T)).\end{aligned} \tag{3.66}$$

Using (3.45) and (3.66) we get (3.58). Similarly, the proof can be done for Eq. (3.62). □

For a 3S-structure $(g,\omega,\omega^*,\nabla)$ on M, we denote the curvature tensor of ∇ by R^∇ or R for short, and R^{∇^*} by R^* in the similar fashion. We define

$$R^S(X,Y)Z = \frac{1}{2}\{R(X,Y)Z + R^*(X,Y)Z\}, \tag{3.67}$$

for all $X,Y,Z \in \mathfrak{X}(M)$, and call R^S the semi-symmetric statistical curvature tensor field of $(g,\omega,\omega^*,\nabla)$. We define the (0, 4) tensor $\overline{R}$, $\overline{R}^*$ and $\overline{R}^S$ by

$$\begin{aligned}&\overline{R}(X,Y,Z,T) = g(R(X,Y)Z,T), \quad \overline{R}^*(X,Y,Z,T) = g(R^*(X,Y)Z,T),\\ &\overline{R}^S(X,Y,Z,T) = \frac{1}{2}\{\overline{R}(X,Y,Z,T) + \overline{R}^*(X,Y,Z,T)\}.\end{aligned}$$

From Lemma 3.2, we can state the following.

Proposition 3.9. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, we get*

$$\begin{aligned} R^S(X,Y)Z = R^g(X,Y)Z &+ S(X, S(Y,Z)) - S(Y, S(X,Z)) \\ &+ S(X, K^{\omega,\omega^*}(Y,Z)) - S(Y, K^{\omega,\omega^*}(X,Z)) \\ &+ K^{\omega,\omega^*}(X, S(Y,Z)) - K^{\omega,\omega^*}(Y, S(X,Z)) \\ &- (\nabla^g_X K^{\omega,\omega^*})(Y,Z) + (\nabla^g_Y K^{\omega,\omega^*})(X,Z) \\ &+ 2\{K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y,(X,Z))\}. \end{aligned}$$

Proof. Using (3.21) in $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$, we get

$$\begin{aligned} R(X,Y)Z = R^g(X,Y)Z &+ (\nabla^g_X S)(Y,Z) - (\nabla^g_Y S)(X,Z) \\ &+ S(X, S(Y,Z)) - S(Y, S(X,Z)). \end{aligned} \tag{3.68}$$

Similarly, using (3.33) $R^*(X,Y)Z = \nabla^*_X \nabla^*_Y Z - \nabla^*_Y \nabla^*_X Z - \nabla^*_{[X,Y]} Z$, we get

$$\begin{aligned} R^*(X,Y)Z = R^g(X,Y)Z &- (\nabla^g_X S)(Y,Z) + (\nabla^g_Y S)(X,Z) \\ &+ S(X, S(Y,Z)) - S(Y, S(X,Z)) \\ &+ 2\{K^{\omega,\omega^*}(X, S(Y,Z)) - K^{\omega,\omega^*}(Y, S(X,Z))\} \\ &+ 2\{(\nabla^g_Y K^{\omega,\omega^*})(X,Z) - (\nabla^g_X K^{\omega,\omega^*})(Y,Z)\} \\ &+ 4\{K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,Z))\}. \end{aligned} \tag{3.69}$$

And finally, using (3.68) and (3.69) in (3.67), we reach that

$$\begin{aligned} R^S(X,Y)Z = R^g(X,Y)Z &+ S(X, S(Y,Z)) - S(Y, S(X,Z)) \\ &+ S(X, K^{\omega,\omega^*}(Y,Z)) - S(Y, K^{\omega,\omega^*}(X,Z)) \\ &+ K^{\omega,\omega^*}(X, S(Y,Z)) - K^{\omega,\omega^*}(Y, S(X,Z)) \\ &- (\nabla^g_X K^{\omega,\omega^*})(Y,Z) + (\nabla^g_Y K^{\omega,\omega^*})(X,Z) \\ &+ 2\{K^{\omega,\omega^*}(X, K^{\omega,\omega^*}(Y,Z)) - K^{\omega,\omega^*}(Y, K^{\omega,\omega^*}(X,Z))\}. \end{aligned}$$

□

Remark 3.4. *If a 3S-structure $(g, \omega, \omega^*, \nabla)$ is a statistical structure, that is, $\omega = \omega^* = 0$, then, the semi-symmetric statistical curvature tensor field R^S is given by*

$$R^S(X,Y)Z = R^g(X,Y)Z + S(X, S(Y,Z)) - S(Y, S(X,Z)),$$

and coincides with the statistical curvature tensor. Thus, the semi-symmetric statistical curvature tensor field R^S is a generalization of the statistical curvature tensor field.

The following series of lemmas and theorem give the symmetrical properties of R^S for a 3S-structure $(g, \omega, \omega^*, \nabla)$.

Lemma 3.3. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, the semi-symmetric statistical curvature field R^S satisfies the following property.*

$$\begin{aligned} &R^S(X,Y)Z + R^S(Y,Z)X + R^S(Z,X)Y \\ &= (d\omega^S(X,Y))Z + (d\omega^S(Y,Z))X + (d\omega^S(Z,X))Y. \end{aligned} \tag{3.70}$$

Proof. It follows from Lemma 3.2. □

Hence, from Lemma 3.3 we can state the following lemma :

Lemma 3.4. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, we have*

$$\begin{aligned} &\overline{R}^S(Z,X,Y,T) + \overline{R}^S(T,Y,Z,X) + \overline{R}^S(X,Z,T,Y) \\ &+ \overline{R}^S(Y,T,X,Z) = 2g(Z,T)(d\omega^S)(X,Y) + 2g(X,T)(d\omega^S)(Y,Z) \\ &+ 2g(X,Y)(d\omega^S)(Z,T) + 2g(Y,Z)(d\omega^S)(T,X). \end{aligned}$$

Proof. Using the Bianchi identity of R and R^* given by the lemma 3.2 we get

$$\begin{aligned} &g(R(X,Y)Z,T) + g(R(Y,Z)X,T) + g(R(Z,X)Y,T) \\ &= g((d\omega)(X,Y)Z + (d\omega)(Y,Z)X + (d\omega)(Z,X)Y, T), \end{aligned}$$

$$\begin{aligned} &g(R^*(X,Y)T,Z) + g(R^*(Y,T)X,Z) + g(R^*(T,X)Y,Z) \\ &= g((d\omega^*)(X,Y)T + (d\omega^*)(Y,T)X + (d\omega^*)(T,X)Y, Z), \end{aligned}$$

$$\begin{aligned} &g(R(X,Z)T,Y) + g(R(Z,T)X,Y) + g(R(T,X)Z,Y) \\ &= g((d\omega)(X,Z)T + (d\omega)(Z,T)X + (d\omega)(T,X)Z, Y), \end{aligned}$$

and

$$\begin{aligned} &g(R^*(Y,Z)T,X) + g(R^*(Z,T)Y,X) + g(R^*(T,Y)Z,X) \\ &= g((d\omega^*)(Y,Z)T + (d\omega^*)(Z,T)Y + (d\omega^*)(T,Y)Z, X). \end{aligned}$$

Using (3.45) and summing up these equations, we get

$$\begin{aligned}&\overline{R}(Z,X,Y,T)+\overline{R}^*(T,Y,Z,X)+\overline{R}(X,Z,T,Y)\\&+\overline{R}^*(Y,T,X,Z)=2g(Z,T)(d\omega^S)(X,Y)+2g(X,T)(d\omega^S)(Y,Z)\\&+2g(X,Y)(d\omega^S)(Z,T)+2g(Y,Z)(d\omega^S)(T,X).\end{aligned}\tag{3.71}$$

Moreover, using similar reasoning as above, we have the following dual version of (3.71)

$$\begin{aligned}&\overline{R}^*(Z,X,Y,T)+\overline{R}(T,Y,Z,X)+\overline{R}^*(X,Z,T,Y)\\&+\overline{R}(Y,T,X,Z)=2g(Z,T)(d\omega^S)(X,Y)+2g(X,T)(d\omega^S)(Y,Z)\\&+2g(X,Y)(d\omega^S)(Z,T)+2g(Y,Z)(d\omega^S)(T,X).\end{aligned}\tag{3.72}$$

Summing up (3.71) and (3.72) we get

$$\begin{aligned}&2\overline{R}^S(Z,X,Y,T)+2\overline{R}^S(T,Y,Z,X)+2\overline{R}^S(X,Z,T,Y)\\&+2\overline{R}^S(Y,T,X,Z)=4g(Z,T)(d\omega^S)(X,Y)+4g(X,T)(d\omega^S)(Y,Z)\\&+4g(X,Y)(d\omega^S)(Z,T)+4g(Y,Z)(d\omega^S)(T,X).\end{aligned}$$

We deduce the result.

□

Proposition 3.10. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M. Then, we have*

$$\begin{aligned}&\overline{R}^S(X,Z,T,Y)-\overline{R}^S(T,Y,X,Z)=g(Z,T)(d\omega^S)(X,Y)\\&+g(X,T)(d\omega^S)(Y,Z)+g(X,Y)(d\omega^S)(Z,T)+g(Y,Z)(d\omega^S)(T,X).\end{aligned}\tag{3.73}$$

Proof. Using (3.55), (3.59) and (3.77) we have

$$\begin{aligned}\overline{R}^S(Z,X,Y,T)+\overline{R}^S(X,Z,T,Y)&=-\overline{R}^S(X,Z,Y,T)+\overline{R}^S(X,Z,T,Y)\\&=\overline{R}^S(X,Z,T,Y)+\overline{R}^S(X,Z,T,Y)\\&=2\overline{R}^S(X,Z,T,Y).\end{aligned}$$

Therefore,

$$\overline{R}^S(Z,X,Y,T)+\overline{R}^S(X,Z,T,Y)=2\overline{R}^S(X,Z,T,Y),\tag{3.74}$$

$$\overline{R}^S(T,Y,Z,X)+\overline{R}^S(Y,T,X,Z)=-2\overline{R}^S(T,Y,X,Z).\tag{3.75}$$

Using the lemma 3.4, (3.74) and (3.75) we have the result. □

Proposition 3.11. *For a 3S-structure $(g, \omega, \omega^*, \nabla)$ on M, we have following equations*

$$g(R^S(X,Y)Z,T) + g(R^S(Y,X)Z,T) = 0, \tag{3.76}$$

$$g(R^S(X,Y)Z,T) + g(R^S(X,Y)T,Z) = 0. \tag{3.77}$$

Moreover, if ω^S is closed, we get

$$R^S(X,Y)Z + R^S(Y,Z)X + R^S(Z,X)Y = 0, \tag{3.78}$$

$$g(R^S(X,Y)Z,T) - g(R^S(Z,T)X,Y) = 0. \tag{3.79}$$

Proof. Summing up Eqs. (3.58) and (3.62) we get (3.77). □

Remark 3.5. *Using the preceding properties of the statistical curvature R^S, we easily check that the statistical sectional curvature given as (3.83), is well defined.*

Proposition 3.12. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M, V its mean vector field and $\widetilde{\nabla}$ its associated statistical connection with respect to ω. If V is torse-forming with respect to ω, then the statistical sectional curvatures of ∇ and $\widetilde{\nabla}$ are the same if and only if*

$$\omega(X)\omega^S(X) + 4\omega^S(X)\omega^S(X) - 2X\omega^S(X) + 2\omega^S(\nabla^*_X X) = 0,$$

for any unitary vector field X.

Proof. From (3.15) and (3.16) we get

$$\begin{aligned}\widetilde{R}(X,Y)Z &= R(X,Y)Z - (d\omega)(X,Y)Z \\ &\quad + 2g(Y,Z)\{\omega(X)V - \nabla_X V\} - 2g(X,Z)\{\omega(Y)V - \nabla_Y V\},\end{aligned}$$

and

$$\begin{aligned}(\widetilde{R})^*(X,Y)Z &= R^*(X,Y)Z + (d\omega)(X,Y)Z + \{-\omega(X)\omega^S(Z) - 4\omega^S(X)\omega^*(Z) + 2X\omega^S(Z) \\ &\quad - 2\omega^S(\nabla^*_X Z)\}Y + \{\omega(Y)\omega^S(Z) + 4\omega^S(Y)\omega^*(Z) - 2Y\omega^S(Z) + 2\omega^S(\nabla^*_Y Z)\}X.\end{aligned}$$

After summing up these equations we get

$$\begin{aligned}2\widetilde{R}^S(X,Y)Z &= 2R^S(X,Y)Z + 2g(Y,Z)\{\omega(X)V - \nabla_X V\} - 2g(X,Z)\{\omega(Y)V - \nabla_Y V\} \\ &\quad + \Big\{ - \omega(X)\omega^S(Z) - 4\omega^S(X)\omega^S(Z) + 2X\omega^S(Z) - 2\omega^S(\nabla^S_X Z)\Big\}Y \\ &\quad + \Big\{\omega(Y)\omega^S(Z) + 4\omega^S(Y)\omega^S(Z) - 2Y\omega^S(Z) + 2\omega^S(\nabla^*_Y Z)\Big\}X.\end{aligned} \tag{3.80}$$

From (3.80), since the mean vector field V is torse-forming with respect to ω then we have

$$2\widetilde{R}^S(X,Y)Z = 2R^S(X,Y)Z + \Big\{ -\omega(X)\omega^S(Z) - 4\omega^S(X)\omega^S(Z) + 2X\omega^S(Z) - 2\omega^S(\nabla^S_X Z)\Big\}Y$$
$$+ \Big\{\omega(Y)\omega^S(Z) + 4\omega^S(Y)\omega^S(Z) - 2Y\omega^S(Z) + 2\omega^S(\nabla^*_Y Z)\Big\}X. \quad (3.81)$$

Taking $\{X,Y\}$ an orthonormal basis on M. Then (3.81) becomes

$$2g(\widetilde{R}^S(X,Y)Y,X) = 2g(R^S(X,Y)Y,X) + \omega(Y)\omega^S(Y) + 4\omega^S(Y)\omega^S(Y) - 2Y\omega^S(Y) + 2\omega^S(\nabla^*_Y Y).$$

This ends the proof. □

A multilinear function $F : Tp(M)^4 \longrightarrow \mathbb{R}$ is curvature-like provided F has the symmetries stated in Proposition 3.11. Thus, $F(X,Y,Y,X) = 0$ for all $X,Y \in T_p(M)$ spanning a nondegenerate plane implies F vanishes on M [19].

Theorem 3.3. *A 3S-structure $(g,\omega,\omega^*,\nabla)$ on M is of constant semi-symmetric statistical sectional curvature $k \in \mathbb{R}$ if and only if*

$$R^S(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} + \frac{1}{2}\{d\omega^S(Y,Z)X - d\omega^S(X,Z)Y\}$$
$$+ \frac{1}{2}\{g(Y,Z)d\omega^S(X,.)^\sharp - g(X,Z)d\omega^S(Y,.)^\sharp\}, \quad (3.82)$$

for $X,Y,Z \in \Gamma(TM)$.

Proof. Let X,Y,Z and T be vector fields on M. We set

$$\Omega(X,Y)Z = k\{g(Y,Z)X - g(X,Z)Y\} + \frac{1}{2}\{d\omega^S(Y,Z)X - d\omega^S(X,Z)Y\}$$
$$+ \frac{1}{2}\{g(Y,Z)d\omega^S(X,.)^\sharp - g(X,Z)d\omega^S(Y,.)^\sharp\}.$$

It is easy to see that Ω verifies (3.70), (3.73), (3.76) and (3.77). We now set

$$F(X,Y,Z,T) = g(R^S(X,Y)Z,T) - g(\Omega(X,Y)Z,T).$$

Since R^S and Ω verify (3.70), (3.73), (3.76) and (3.77) then F is also curvature-like. That is, F verifies the symmetries properties given by proposition 3.11. Moreover, by hypothesis, we have $F(X,Y,Y,X) = 0$. Thus, F vanishes on M, that is $R^S(X,Y)Z = \Omega(X,Y)Z$.

Conversely, if R^S verify (3.82), it is easy to see that the semi-symmetric statistical sectional curvature tensor field given by

$$K(\pi) = \frac{g(R^S(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \quad (3.83)$$

is equal to k, where π is the 2-dimensional plane spanned by X and Y.

□

The Ricci curvature tensor with respect to ∇, denoted by $R\mathrm{ic}^{\nabla}$ is defined by

$$\mathrm{Ric}^{\nabla}(X,Y) = \mathrm{tr}\{Z \longmapsto R^{\nabla}(X,Z)Y\}.$$

We denote $R\mathrm{ic}^{\nabla}$ by Ric, $R\mathrm{ic}^{\nabla^*}$ by $R\mathrm{ic}^*$ and $R\mathrm{ic}^{\nabla^g}$ by $R\mathrm{ic}^g$ respectively, for short. For a 3S-structure $(g,\omega,\omega^*,\nabla)$, we can also define the Ricci curvature tensor relative to the semi-symmetric statistical curvature tensor field R^S as follows.

Definition 3.2. *Let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M and R^S the semi-symmetric statistical curvature tensor field of ∇.*

1. *The 3S-Ricci curvature tensor, denoted by Ric^S, is defined as*

$$Ric^S(X,Y) = tr_g\{Z \longmapsto R^S(X,Z)Y\}.$$

2. *The 3S-Ricci curvature tensor field Ric^S of $(M,g,\omega,\omega^*,\nabla)$ is said to be 3S-Einstein manifold if there exists $\lambda \in \mathbb{R}$ such that*

$$Ric^S = \lambda g.$$

3. *The 3S-scalar curvature, denoted by ρ^S, is defined as*

$$\rho^S = tr_g Ric^S = \sum_{i=1}^{n} Ric^S(e_i,e_i),$$

where $\{e_1,e_2,...,e_n\}$ is an orthonormal basis of T_xM with respect to g, for $x \in M$.

Remark 3.6. *For a 3S-structure $(g,\omega,\omega^*,\nabla)$ on M, it is easy to see that $\rho^S = \rho = \rho^*$, where*

$$\rho = tr_g Ric, \qquad \rho^* = tr_g Ric^*.$$

From proposition 3.10, we obtain the following corollary.

Corollary 3.2. *Let (M,g) be a Riemannian manifold of dimension n, let $(g,\omega,\omega^*,\nabla)$ be a 3S-structure on M. Then, the Ricci tensor Ric^S of $(g,\omega,\omega^*,\nabla)$ verifies*

$$Ric^S(X,Y) - Ric^S(Y,X) = (2-n)(d\omega^S)(X,Y).$$

Proof. Let $\{e_1, e_2, ..., e_n\}$ be an orthonormal basis of the tangent space at any point p of the 3S-structure manifold. Taking $Z = Y = e_i$ in (3.73), summing over i, $1 \leq i \leq n$, we get

$$\begin{aligned}
Ric^S(X,T) - Ric^S(T,X) &= \sum_{i=1}^{n} \Big(\overline{R}^S(X,e_i,T,e_i) - \overline{R}^S(T,e_i,X,e_i)\Big) \\
&= \sum_{i=1}^{n} \Big(g(e_i,T)(d\omega^S)(X,e_i) + g(X,T)(d\omega^S)(e_i,e_i) \\
&+ g(X,e_i)(d\omega^S)(e_i,T) + g(e_i,e_i)(d\omega^S)(T,X)\Big) \\
&= \sum_{i=1}^{n} \Big((d\omega^S)(X,g(e_i,T)e_i) + (d\omega^S)(g(X,e_i)e_i,T) \\
&+ g(e_i,e_i)(d\omega^S)(T,X)\Big) \\
&= (d\omega^S)(X,T) + (d\omega^S)(g(X,T) + n(d\omega^S)(T,X) \\
&= (2-n)(d\omega^S)(X,T).
\end{aligned}$$

□

From Corollary 3.2, we obtain the following Corollaries :

Corollary 3.3. *Let (M, g) be a Riemannian manifold of dimension 2. The semi-symmetric statistical Ricci tensor of a 3S-structure on M is always symmetric.*

Corollary 3.4. *Let (M, g) be a Riemannian manifold of dimension $n \geq 3$, let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M. Then, the semi-symmetric statistical Ricci tensor Ric^S of $(g, \omega, \omega^*, \nabla)$ is symmetric if and only if ω^S is closed. Thus, for a statistical manifold (M, g, ∇), the statistical Ricci tensor Ric^S is always symmetric.*

Proposition 3.13. *Let (M, g) be a Riemannian manifold of dimension n, let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M such that the 3S-sectional curvature of R^S is constant $k \in \mathbb{R}$. Then, we have*

$$Ric^S(X,Y) = k(1-n)g(X,Y) + (1-\frac{n}{2})d\omega^S(X,Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

Proof. Let $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
\mathrm{Ric}^S(X,Y) &= \sum_{i=1}^{n} g(R^S(X,e_i)Y,e_i) \\
&= \sum_{i=1}^{n} g\Big(k\{g(e_i,Y)X - g(X,Y)e_i\} + \frac{1}{2}\{d\omega^S(e_i,Y)X - d\omega^S(X,Y)e_i\} \\
&+ \frac{1}{2}\{g(e_i,Y)d\omega^S(X,.)^\sharp - g(X,Y)d\omega^S(e_i,.)^\sharp\}, e_i\Big) \\
&= k\sum_{i=1}^{n}\{g(e_i,Y)g(X,e_i) - g(X,Y)g(e_i,e_i)\} + \frac{1}{2}\sum_{i=1}^{n} d\omega^S(e_i,Y)g(X,e_i) \\
&- \frac{1}{2}\sum_{i=1}^{n} d\omega^S(X,Y)g(e_i,e_i) + \frac{1}{2}\sum_{i=1}^{n} g(e_i,Y)d\omega^S(X,e_i).
\end{aligned}$$

That is:

$$\begin{aligned}
\mathrm{Ric}^S(X,Y) &= k\sum_{i=1}^{n}\{g(X,g(e_i,Y)e_i) - g(X,Y)\} + \frac{1}{2}\sum_{i=1}^{n} d\omega^S(g(X,e_i)e_i,Y) \\
&- \frac{n}{2}d\omega^S(X,Y) + \frac{1}{2}\sum_{i=1}^{n} d\omega^S(X,g(e_i,Y)e_i) \\
&= kg(X,Y) - nkg(X,Y) + \frac{1}{2}d\omega^S(X,Y) - \frac{n}{2}d\omega^S(X,Y) + \frac{1}{2}d\omega^S(X,Y) \\
&= k(1-n)g(X,Y) + (1-\frac{n}{2})d\omega^S(X,Y).
\end{aligned}$$

□

Corollary 3.5. *Let $(M, g, \omega, \omega^*, \nabla)$ be a 3S-manifold such that its 3S-sectional curvature is constant. Then, M is 3S-Einstein if ω^S is closed.*

Definition 3.3. *Let $(M, g, \omega, \omega^*, \nabla)$ be a 3S-manifold and Ric^S its 3S-Ricci tensor. The symmetrized $\widetilde{R}ic^S$ of Ric^S is given by*

$$\widetilde{R}ic^S(X,Y) = \frac{1}{2}\{Ric^S(X,Y) + Ric^S(Y,X)\},$$

for all vector fields X, Y on M.

Definition 3.4. *Let $(g, \omega, \omega^*, \nabla)$ be a 3S-structure on M, Ric^S the 3S-Ricci curvature tensor field. $(M, g, \omega, \omega^*, \nabla)$ is said to be symmetrically Einstein manifold if the symmetrized $\widetilde{R}ic^S$ of Ric^S is of the form*

$$\widetilde{R}ic^S = \lambda g,$$

where $\lambda = \frac{\rho^S}{n}$ is a constant.

Corollary 3.6. *If a 3S-manifold $(M, g, \omega, \omega^*, \nabla)$ of dimension n is of constant 3S-sectional curvature, then $(M, g, \omega, \omega^*, \nabla)$ is symmetrically Einstein manifold.*

Proof. Let $X, Y \in \mathfrak{X}(M)$, from the proposition 3.11 we get

$$\widetilde{R}\mathrm{ic}^S(X, Y) = k(1-n)g(X, Y).$$

We also said that $(M, g, \omega, \omega^*, \nabla)$ is symmetrically Einstein manifold. □

4. Semi-symmetric statistical submanifolds

Let $(\overline{M}, \overline{g})$ be a Riemannian manifold, $(\overline{M}, \overline{g}, \overline{\omega}, \overline{\omega}^*, \overline{\nabla})$ be a 3S-manifold, M be a submanifold of $\overline{M}$, g the induced metric $\overline{g}$ and two 1-forms ω, ω^* such that

$$\omega(X) = \overline{\omega}(X), \omega^*(X) = \overline{\omega}^*(X),$$

for all vector field X on M.

Let $TM^{\perp}$ be the normal bundle of M in $\overline{M}$ with respect to $\overline{g}$. We define the second fundamental form of M for $\overline{\nabla}$ and $\overline{\nabla}^*$ by

$$h(X, Y) = (\overline{\nabla}_X Y)^{\perp}, \tag{4.84}$$

$$h^*(X, Y) = (\overline{\nabla}^*_X Y)^{\perp} \tag{4.85}$$

respectively, where $()^{\perp}$ denotes the orthogonal projection on the normal bundle $TM^{\perp}$.
Let consider the connections ∇ and ∇^* given by

$$\nabla_X Y = (\overline{\nabla}_X Y)^{\top}, \tag{4.86}$$

$$\nabla^*_X Y = (\overline{\nabla}^*_X Y)^{\top}. \tag{4.87}$$

It is well known in the literature that ∇ and ∇^* are dual connections with respect to g and h, h^* are bilinear and symmetric.

Proposition 4.1. *When $(\overline{M}, \overline{g}, \overline{\omega}, \overline{\omega}^*, \overline{\nabla})$ is a 3S-manifold, the induced structure $(M, g, \omega, \omega^*, \nabla)$ on a submanifold M is a 3S-manifold.*

Proof. Let $X, Y \in \mathfrak{X}(M)$. From (4.84) and (4.86) we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y). \tag{4.88}$$

Since $\overline{\nabla}$ has semi-symmetric connection, we can write

$$\begin{aligned}\overline{\omega}(X)Y - \overline{\omega}(Y)X &= \overline{\nabla}_X Y - \overline{\nabla}_Y X - [X,Y] \\ &= \nabla_X Y + h(X,Y) - \nabla_Y X + h(Y,X) - [X,Y] \\ &= T^{\nabla}(X,Y) + h(X,Y) - h(Y,X). \end{aligned} \tag{4.89}$$

Grouping the normal and tangential components of (4.89), we have

$$T^{\nabla}(X,Y) = \omega(X)Y - \omega(Y)X. \tag{4.90}$$

This equation show that ∇ has semi-symmetric. The proof for ∇^* follows immediately by substituting ∇^* for ∇ in the preceding argument, and we obtain

$$T^{\nabla^*}(X,Y) = \omega^*(X)Y - \omega^*(Y)X. \tag{4.91}$$

□

From (4.90) and (4.91), $(M, g, \omega, \omega^*, \nabla)$ is a 3S-manifold.

5. Semi-symmetric dualistic structures on warped product spaces

In this section, we give a method to construct 3S-structures on warped product manifolds, starting from 3S-strutures on the fiber and base manifolds.

Let $(M; g)$ and $(N; h)$ be two Riemannian manifolds of dimension m and n respectively and $f \in \mathcal{C}^\infty(M)$ a positive function on M. The warped product of $(M; g)$ and $(N; h)$, with warping function f, is the $(m+n)$-dimensional manifold $M \times N$ endowed with the metric G_f given by:

$$G_f = \pi^* g + (f \circ \pi)^2 \sigma^* h,$$

where π^* and σ^* are the pull-backs of the projections π and σ of $M \times N$ on M and N respectively.

This warped product is sometime denoted by $M \times_f N$, but for simplicity we keep $M \times N$ in the sequel.

The tangent space $T_{(p;q)}(M \times N)$ at a point $(p; q) \in M \times N$ is isomorph to the direct sum $T_pM \oplus T_qN$. Let $\mathcal{L}_H(M)$ (resp. $\mathcal{L}_V(N)$) denote the set of the horizontal lifts (resp. the vertical lifts) to $T(M \times N)$ of all the tangent vectors on M. (resp. on N). Hence from [20], we have the following:

$$\Gamma(T(M \times N)) \simeq \mathcal{L}_H(M) \oplus \mathcal{L}_V(N),$$

and thus a vector field A on $M \times N$ can be written as

$$A = X + U; \text{ with } X \in \mathcal{L}_H(M) \text{ and } U \in \mathcal{L}_V(N).$$

For any vector field $X \in \mathcal{L}_H(M)$ we denote $\pi_*(X)$ by $\overline{X}$, and for any vector field $U \in \mathcal{L}_V(N)$ we denote $\sigma_*(U)$ by $\widetilde{U}$. Furthermore we denote the horizontal lift on $M \times N$ of a vector field $X \in \Gamma(TM)$ by $(X)^H$, and the vertical lift on $M \times N$ of a vector field $U \in \Gamma(TN)$ by $(U)^V$.

Obviously

$$\pi_*(\mathcal{L}_H(M)) = \Gamma(TM) \text{ and } \sigma_*(\mathcal{L}_V(N)) = \Gamma(TN).$$

Let $(G_f; \widetilde{D}; \widetilde{D}^*)$ be a dualistic structure on $M \times N$. For $X; Y, Z \in \mathcal{L}_H(M)$ and $U; V, W \in \mathcal{L}_V(N)$ we define the following four connections [20] :

$${}^M\widetilde{\nabla}_{\overline{X}}\overline{Y} = \pi_*(\widetilde{D}_X Y) \text{ and } {}^M\widetilde{\nabla}'_{\overline{X}}\overline{Y} = \pi_*(\widetilde{D}^*_X Y),$$

and

$${}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{V} = \sigma_*(\widetilde{D}_U V) \text{ and } {}^N\widetilde{\nabla}'_{\widetilde{U}}\widetilde{V} = \sigma_*(\widetilde{D}^*_U V).$$

We also recall that[19] :

$$\overline{X}G_f(\overline{Y}, \overline{Z}) \circ \pi = XG_f(Y, Z) \text{ and } \widetilde{U}G_f(\widetilde{V}, \widetilde{W}) \circ \sigma = UG_f(V, W).$$

Hence we have the following result from [20]:

Proposition 5.1. *The triplet* $(g;{}^M \widetilde{\nabla};{}^M \widetilde{\nabla}')$ *is a dualistic structure on* M *and the triplet* $(h;{}^N \widetilde{\nabla};{}^N \widetilde{\nabla}')$ *is a dualistic structure on* N*; that is*

$${}^M\widetilde{\nabla}' = {}^M \widetilde{\nabla}^* w.r.t.g \text{ and } {}^N\widetilde{\nabla}' = {}^N \widetilde{\nabla}^* w.r.t.h.$$

Conversely, in [20], it has been given a method to construct statistical structure on the warped product, starting from statistical structures on the fiber and the base manifolds as it follows:

Let $(g, {}^M \widetilde{\nabla}, {}^M \widetilde{\nabla}^*)$ and $(h, {}^N \widetilde{\nabla}, {}^N \widetilde{\nabla}^*)$ be dualistic structures on M and N respectively. For all $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$ we set:

i) $\widetilde{D}_X Y = ({}^M\widetilde{\nabla}_{\overline{X}}\overline{Y})^H$

ii) $\widetilde{D}_X U = \widetilde{D}_U X = \frac{X(f)}{f} U$

iii) $\widetilde{D}_U V = -\frac{<U,V>}{f} grad(f) + ({}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{V})^V$

and

a) $\widetilde{D}^*_X Y = ({}^M\widetilde{\nabla}^*_{\overline{X}}\overline{Y})^H$

b) $\widetilde{D}^*_X U = \widetilde{D}^*_U X = \frac{X(f)}{f} U$

c) $\widetilde{D}^*_U W = -\frac{<U,W>}{f} grad(f) + ({}^N\widetilde{\nabla}^*_{\widetilde{U}}\widetilde{W})^V$,

where we simplify the notation by writing f for $f \circ \pi$ and $grad(f)$ for $grad(f \circ \pi)$, and we denote by $<,>$ the inner product w.r.t. G_f. Obviously $\widetilde{D}$ and $\widetilde{D}^*$ define affine connections on $T(M \times N)$ and the following proposition holds:

Proposition 5.2. *The triplet $(G_f, \widetilde{D}, \widetilde{D}^*)$ is a dualistic structure on $M \times N$.*

We call $(G_f, \widetilde{D}, \widetilde{D}^*)$ the dualistic structure on $M \times N$ induced from $(g, {}^M\widetilde{\nabla}, {}^M\widetilde{\nabla}^*)$ on M and $(h, {}^N\widetilde{\nabla}, {}^N\widetilde{\nabla}^*)$ on N.

Now, using [5, 7], and the fact that $[X, U] = 0$, $[\overline{X}, \overline{Y}]^H = [X, Y]$, $[\widetilde{U}, \widetilde{W}]^V = [U, W]$ for all $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$, one obtains the following result.

Corollary 5.1. *If $(g, {}^M\widetilde{\nabla}, {}^M\widetilde{\nabla}^*)$ and $(h, {}^N\widetilde{\nabla}, {}^N\widetilde{\nabla}^*)$ are statistical structures on M and N respectively, then $(G_f, \widetilde{D}, \widetilde{D}^*)$ is statistical structure on $M \times N$.*

Inspired by the reasoning from [5, 7, 20] we introduce here similar but different construction for 3S-structures on the warped product. We first notice that a 3S-structure on $M \times N$ projects to 3S-structures on M and N.

Let $(G_f; \eta, \eta*, D)$ be a 3S-structure on $M \times N$, ${}^M\nabla$ and ${}^N\nabla$ the connections such that

$${}^M\nabla_{\overline{X}}\overline{Y} = \pi_*(D_X Y) \text{ and } {}^N\nabla_{\widetilde{U}}\widetilde{W} = \sigma_*(D^*{}_U W),$$

for all $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$, where D^* is the dual connection of D with respect to G_f. Let ${}^M\nabla^*$ and ${}^N\nabla^*$ be defined by

$${}^M\nabla^*_{\overline{X}}\overline{Y} = \pi_*(D^*_X Y) \text{ and } {}^N\nabla^*_{\widetilde{U}}\widetilde{W} = \sigma_*(D^*_U W),$$

The last two connections are dual connections of ${}^M\nabla$ and ${}^N\nabla$ respectively. We set

$$\omega(\overline{X}) = \eta(X), \omega^*(\overline{X}) = \eta^*(X), \widetilde{\omega}(\widetilde{U}) = \eta(U) \text{ and } \widetilde{\omega}^*(\widetilde{U}) = \eta^*(U),$$

for all $X \in \mathcal{L}_H(M)$ and $U \in \mathcal{L}_V(N)$. Then the following holds:

Proposition 5.3. *$(M, g, \omega, \omega^*, {}^M\nabla)$ and $(N, h, \widetilde{\omega}, \widetilde{\omega}^*, {}^N\nabla)$ are 3S-manifolds.*

Proof. Direct computations give $T^{{}^M\nabla}(\overline{X}, \overline{Y}) = \eta(X)\overline{Y} - \eta(Y)\overline{X}$ and $T^{{}^M\nabla^*}(\overline{X}, \overline{Y}) = \eta^*(X)\overline{Y} - \eta^*(Y)\overline{X}$. Thus, $(M, g, \omega, \omega^*, {}^M\nabla)$ is a 3S-manifold. Similarly for $(N, h, \widetilde{\omega}, \widetilde{\omega}^*, {}^N\nabla)$. □

We now give a converse of the preceding construction. Assume that $(g, \omega, \omega^*, {}^M\nabla)$ and $(h, \widetilde{\omega}, \widetilde{\omega}^*, {}^N\nabla)$ are 3S-structures on M and N respectively and let ${}^M\widetilde{\nabla}$ and ${}^N\widetilde{\nabla}$ be the affine connections on M and N respectively defined by

$$
\begin{aligned}
{}^M\widetilde{\nabla}_{\overline{X}}\overline{Y} &= {}^M\nabla_{\overline{X}}\overline{Y} - \omega(\overline{X})\overline{Y} - 2g(\overline{X}, \overline{Y})\, {}^M V, \\
{}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{W} &= {}^N\nabla_{\widetilde{U}}\widetilde{W} - \widetilde{\omega}(\widetilde{U})\widetilde{W} - 2h(\widetilde{U}, \widetilde{W})\, {}^N V,
\end{aligned}
$$

where ${}^M V$ and ${}^N V$ are the vector field g-associated and h-associated with $\omega^S = \frac{1}{2}(\omega + \omega^*)$ and $\widetilde{\omega}^S = \frac{1}{2}(\widetilde{\omega} + \widetilde{\omega}^*)$ respectively. Let η and η^* be the 1-forms defined by:

$$\eta = \omega \oplus \widetilde{\omega} \ \text{ and } \ \eta^* = \omega^* \oplus \widetilde{\omega}^*$$

namely, $\eta(X+U) = \omega(\overline{X}) + \widetilde{\omega}(\widetilde{U})$ and $\eta^*(X+U) = \omega^*(\overline{X}) + \widetilde{\omega}^*(\widetilde{U})$ for all $X \in \mathcal{L}_H(M)$ and $U \in \mathcal{L}_V(N)$. We set V^S to be the vector field G_f-associated with $\Omega^S = \frac{1}{2}(\eta + \eta^*)$.

It is easy to see from Theorem 3.2 that $(M, g, {}^M\widetilde{\nabla})$ and $(N, h, {}^N\widetilde{\nabla})$ are statistical manifolds. Then, from corollary 5.1, define a connection $\widetilde{D}$ on $M \times N$ by the following formula:

i) $\widetilde{D}_X Y = ({}^M\widetilde{\nabla}_{\overline{X}}\overline{Y})^H$
ii) $\widetilde{D}_X U = \widetilde{D}_U X = \frac{X(f)}{f} U$
iii) $\widetilde{D}_U V = -\frac{<U,V>}{f} grad(f) + ({}^N\widetilde{\nabla}_{\widetilde{U}}\widetilde{V})^V$

and

a) $\widetilde{D}^*_X Y = ({}^M\widetilde{\nabla}^*_{\overline{X}}\overline{Y})^H$
b) $\widetilde{D}^*_X U = \widetilde{D}^*_U X = \frac{X(f)}{f} U$
c) $\widetilde{D}^*_U W = -\frac{<U,W>}{f} grad(f) + ({}^N\widetilde{\nabla}^*_{\widetilde{U}}\widetilde{W})^V$,

Then, $(G_f, \widetilde{D}, \widetilde{D}^*)$ is a statistical structure on $M \times N$.

From this, we deduce a 3S-structure as it follows:

Proposition 5.4. *Let D be the connection defined by*

$$D = \widetilde{D} + \eta \otimes I + 2G_f(.,.)V^S. \tag{5.92}$$

Then, (G_f, η, η^*, D) is 3S-structure on $M \times N$.

Proof. Let T^D and T^{D^*} be the torsion tensors of D and D^* respectively, where D^* is the dual connection of D with respect to G_f. From 3.2 we have

$$D^* = \widetilde{D}^* - \eta \otimes I - 2I \otimes \Omega^S. \tag{5.93}$$

Let $A = X + U$ and $B = Y + W$ such that $X, Y \in \mathcal{L}_H(M)$ and $U, W \in \mathcal{L}_V(N)$. Since $G_f(X, W) = 0 = G_f(U, Y)$ and $[X, W] = 0$, $[U, Y] = 0$, $[\overline{X}, \overline{Y}]^H = [X, Y]$, $[\widetilde{U}, \widetilde{W}]^V = [U, W]$ [4] and $\widetilde{D}$ and $\widetilde{D}^*$ are torsion-free connections on $M \times N$, we get:

$$T^D(A, B) = \eta(A)B - \eta(B)A, \quad T^{D^*}(A, B) = \eta^*(A)B - \eta^*(B)A.$$

Thus, (G_f, η, η^*, D) is a 3S-structure on $M \times N$. □

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References

[1] Adara, M., & Blaga, A. N. (2022). On Statistical and Semi-Weyl Manifolds Admitting Torsion. Mathematics 10 990.

[2] Amari, S., & Nagaoka, H. (2000). Methods of Information Geometry. In Transl. Math. Monogr. Amer. Math. Soc. 191, .

[3] Asali, F., & Kazemi Balgeshir, M.B. (2024). On statistical generalized recurrent manifolds. Journal of Finsler Geometry and its Applications, 5(2), 101–115

[4] Chen, B. Y. (2017). Differential Geometry of Warped Product Manifolds and Submanifolds.

[5] Diallo, A. S (2013). Dualistic Structures on Doubly Warped Product Manifolds. International Electronic Journal of Geometry, 6(1), 41–45

[6] Diallo, A. S., Massamba, F. (2019). Conjugate semi-symmetric non-metric connections. Ser. Math. Inform. 34(5), 823-–836.

[7] Diallo, A. S., & Todjihounde, L. (2015). Dualistic Structures on Twisted Product Manifolds. Global Journal of Advanced Research on Classical and Modern Geometries, 4(1), 35–43.

[8] Furuhata, H., & Hasegawa, L. (2016). Submanifolds Theory in Holomorphic Statistical Manifolds, In Dragomiv, S., Shahid, M.H., Al Solamy, F.R. (eds) Geometry of Cauchy-Riemann Submanifolds, Springer, Singapore , 179–215.

[9] Furuhata, H. (2009). Hypersurfaces in Statistical Manifolds. Differential Geom. Appl. 27 , 420–429.

[10] Furuhata, H., Izumi, H., Yukihiko, O., & Kimitake, Mohammad, H. S. (2017). Sasakian Statistical Manifolds, J. Geom. Phys. 117 , 179–186.

[11] Hirohiko, S. (2007). The Geometry of Hessian Structures. World Scientific Publishing, Co.Pte. Ltd.

[12] Kazemi Balgeshir, M. B., & Salahvarzi, S. (2021). Curvature of semi-symmetric metric connection on statistical manifold. Commun. Korean Math. Soc. 36(1), 149–164.

[13] Kazemi Balgeshir, M. B., Panahi Gharehkoshan, S., & Ilmakchi, M. (2024). Statistical manifolds equipped with semi-symmetric connection and Ricci-soliton equations. Reviews in Mathematical Physics. 2450055 (12 pages)

[14] Lauritzen, S. L. (1987). Statistical manifolds, In Differential Geometry in Statistical Inferences, IMS Lectures Notes. Monogr. Ser. 10, Int. Math. Statist. Hayward California, , 96–163.

[15] Matsuzoe, H. (2001). Geometry of semi-Weyl manifolds and Weyl manifolds. Kyushu J. Math. 55 , 107-–117.

[16] Meli, C. B., Ngakeu, F., & Olea, B. (2023). Statistical Structures Arising in Null Submanifolds. Rev. Real. Acad. Exactas Fis. Nat. A-Mat. 117(1), 48.

[17] Norden, A.P. (1950). Affinely Connected Spaces; Nauka: Moscow, Russian, (In Russian).

[18] Nomizu, K., & Sasaki, T. (1994). Affine Differential Geomtry. Cambridge University Press, Cambridge.

[19] O'Neill, B. (1983). *Semi-Riemannian geometry*. Academic Press, New-York.

[20] Todjihounde, L. (2006). Dualistic Structures on Warped Product Manifolds. Differential Geometry-Dynamical Systems, Balkan Society of Geometers, Geometry Balkan Press 8 , 278–284.

[21] Zhang J. (2007). A note on curvature of α-connections of a statistical manifold. The Institute of Statistical Mathematics, Tokyo.

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η-RICCI-BOURGUIGNON SOLITONS ON K-PARACONTACT AND PARACONTACT METRIC $(\kappa \neq -1,\ \mu)$-MANIFOLDS

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Abstract. In this study, we focus on $(2n+1)$-dimensional K-paracontact manifolds admitting η-Ricci-Bourguignon solitons and gradient η-Ricci-Bourguignon solitons. We then completely present the classification of a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1,\ \mu)$-manifold that admits a gradient η-Ricci-Bourguignon soliton. Finally, we construct examples that provide our results.

Keywords: K-paracontact manifolds, Paracontact (κ, μ)-manifolds, η-Ricci-Bourguignon solitons, Gradient η-Ricci-Bourguignon solitons.

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1. Introduction and Motivations

Geometric flows represent a powerful tool for the topological classification of manifolds, providing profound insights into their structural intricacies through the study of metric evolution over time. In this process, questions concerning the short- and long-term behavior of metrics such as whether they smooth out or develop singularities come to the forefront. Moreover, geometric flows have significant applications in physical theories, including general relativity and quantum gravity, particularly in modeling the dynamics of the universe's geometric structure. In this context, self-similar solutions to the flow, known as solitons (e.g., Ricci solitons), play a critical role in understanding the long-term behavior of the flow and

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contribute to the identification of stable or special geometric structures. This is particularly evident in the case of the Poincaré conjecture, a century-old problem that was resolved in the early 2000's by Perelman through the use of Ricci flows, [18], [19], [20]. Ricci solitons were instrumental in resolving the Poincaré conjecture, a problem that had been debated for more than a hundred years. Thus, given a geometric flow, it is natural to study the solitons associated to that flow. As a result of this, in 1981, Hamilton [11] introduced *Ricci flow* by

$$\frac{\partial}{\partial t} g\left(t\right) = -2Rc\left(t\right),$$

where Rc represents Ricci tensor of type $(0,2)$ and g is the time dependent metric of the space evolving under the flow.

Hamilton [12] also defined *Yamabe flow* as follows.

$$\frac{\partial}{\partial t} g\left(t\right) = -r\left(t\right) g\left(t\right),$$

where $r(t)$ represents the scalar curvature of the metric $g\left(t\right)$.

In 1981, a new geometric flow, named *Ricci-Bourguignon flow*, was introduced and extended the Ricci flow notation by Bourguignon [3] as follows:

$$\frac{\partial}{\partial t} g\left(t\right) = -2\left(Rc\left(t\right) - \rho r\left(t\right) g\left(t\right)\right), \tag{1.1}$$

where $\rho \in \mathbb{R}$.

Einstein flow [6] is given by

$$\frac{\partial}{\partial t} g\left(t\right) = -2(Rc\left(t\right) - \frac{r\left(t\right)}{2} g\left(t\right)).$$

Moreover, Ricci-Bourguignon flow is known as a generalization of Einstein flow. Depending on the choice of ρ, the Ricci-Bourguignon flow may turn to certain geometric flows, namely, for $\rho = \frac{1}{2}$ this flow turn to be Einstein flow, for $\rho = \frac{1}{2}(n-1)$ it will turn to the Schouten flow and for $\rho = 0$ it will turn to the famous Ricci flow.

The solutions of (1.1) are called *Ricci-Bourguignon solitons (RB-solitons) or ρ-Einstein solitons* which are given in [9] by the following

$$\mathcal{L}_{\mathcal{W}} g + 2\left(Rc - \rho r g\right) = 2\lambda g, \tag{1.2}$$

where λ is a constant and $\mathcal{L}$ denotes the Lie derivative. λ and $\mathcal{W}$ are called soliton constant and potential vector field, respectively. If λ is a smooth function, then it is called *almost Ricci-Bourguignon soliton* [9]. The Ricci-Bourguignon soliton, a prominent concept in Riemannian geometry, arises as a solution to the Einstein field equations in the context of general relativity. These solitons, which have garnered considerable attention in recent years, play a crucial role in Riemannian geometry. Interestingly, Ricci Bourguignon solitons are critical points of the Ricci flow, and studying the flow's behavior near a soliton provides important insights into the global geometry of the manifold. Ricci-Bourguignon soliton is called *trivial* if $\mathcal{W}$ is zero or a Killing vector field (i.e. $\mathcal{L}_{\mathcal{W}}g = 0$). If $\rho = 0$ in (1.2), a *Ricci soliton* (a solution of the Ricci flow) is obtained. Theoretical physicists are fascinated by Ricci solitons because of their link to string theory and the fact that the soliton equation represents a particular instance of the Einstein field equations. A Ricci soliton extends the concept of an Einstein metric when there is a smooth, non-zero vector field $\mathcal{W}$ and a constant λ. Recently, numerous researchers have examined Ricci solitons and gradient Ricci solitons on certain types of three-dimensional almost contact metric manifolds. For instance, the study of Ricci solitons and gradient Ricci solitons on three-dimensional normal almost contact metric manifolds is investigated in [8]. Additionally, a comprehensive classification of Ricci solitons on three-dimensional Kenmotsu manifolds is provided in [7] and [10].

The solutions of the Einstein flow are *Einstein solitons* and Einstein solitons are given by

$$\mathcal{L}_{\mathcal{W}}g + 2(Rc - \frac{1}{2}rg) = 2\lambda g.$$

A generalization of Einstein soliton is *RB soliton (or ρ-Einstein soliton).* Also a generalization of Ricci-Bourguignon flow is *η-Ricci-Bourguignon flow* which is given by

$$\frac{\partial}{\partial t} g\left(t\right) = -2\left(Rc\left(t\right) - \rho r\left(t\right) g\left(t\right) - \sigma \eta\left(t\right) \otimes \eta\left(t\right)\right), \tag{1.3}$$

where σ and ρ are real numbers.

An essential aspect of studying any geometric flow is analyzing its associated solitons, which produce self-similar solutions to the flow and frequently serve as models for singularities. Motivated by the concept of Ricci solitons, it is intriguing to explore special solutions of the flow (1.3) which is known as a generalization of Ricci-Bourguignon soliton is *η-Ricci-Bourguignon soliton (η-RB soliton)* and is given by

$$\mathcal{L}_{\mathcal{W}}g + 2\left(Rc - \rho r g - \sigma \eta \otimes \eta\right) = 2\lambda g, \tag{1.4}$$

where σ and ρ are real numbers, if λ and σ are smooth functions, it is called an *almost η-Ricci-Bourguignon soliton* [2]. For $\rho = \frac{1}{2}$, the soliton reduces to *η-Einstein soliton* and for $\rho = 0$, it is *η-Ricci-soliton*.

The soliton is shrinking, steady or expanding according as $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$, respectively.

If the potential vector field $\mathcal{W}$ is the gradient of a smooth function f, denoted by ∇f, then (1.4) can be written

$$Hessf + (Rc - \rho rg - \sigma\eta \otimes \eta) = \lambda g, \tag{1.5}$$

where $Hessf$ is the Hessian of f. (1.5) is called a *gradient η-Ricci-Bourguignon soliton*.

A significant amount of work has been contributed by various researchers to explore the geometric properties of Ricci-Bourguignon solitons. For instance, in [5], Catino et al. investigated the Ricci-Bourguignon solitons, where they discussed important rigidity results. In recent year, in [22] Shaikh et al. demonstrated that a compact gradient Ricci-Bourguignon soliton with constant scalar curvature is isometric to the Euclidean sphere. A similar result was established for a gradient Ricci-Bourguignon soliton with a vector field of bounded norm, subject to additional conditions. [21].

Recently, it is worth to mention that in [15] Mandal et al. studied η-Ricci-Bourguignon solitons on K-contact and contact (κ, μ)-manifolds. Also, in [16], Mandal et al. investigated η-Ricci-Bourguignon solitons on three-dimensional almost coKaehler manifolds. Blaga and Ozgur [1] worked on submanifolds as almost η-Ricci Bourguignon solitons.

As far as our knowledge goes, η-Ricci-Bourguignon solitons and gradient η-Ricci-Bourguignon solitons on K-paracontact manifolds and paracontact $(\kappa \neq -1, \mu)$-manifolds are not studied by the researchers. This manuscript will fill these gaps.

This paper is structured as follows: In Section 2, we review some concepts essential for the discussion. Section 3 focuses on $(2n+1)$-dimensional K-paracontact manifolds which admit η-Ricci-Bourguignon solitons and gradient η-Ricci-Bourguignon solitons. We proved that if a $(2n+1)$-dimensional K-paracontact manifold admits an η-Ricci-Bourguignon soliton whose potential vector field being collinear with ξ, we showed that the manifold is η-Einstein and then the scalar curvature $r = -2n\,(2n+1+\sigma)$ is constant. Also we proved that if a $(2n+1)$-dimensional K-paracontact manifold admits a gradient η-Ricci-Bourguignon soliton, then the scalar curvature is constant and the manifold is η-Einstein. In Section 4, we completely give the classification of a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1, \mu)$-manifold that admits a gradient η-Ricci-Bourguignon soliton.

Finally, we construct examples which verifies our results.

2. Preliminaries

In this section, we review various concepts and results that will be essential for the rest of the paper.

A smooth manifold M^{2n+1}has an *almost paracontact structure* (ϕ, ξ, η) if it possesses a tensor field ϕ of type $(1,1)$, a vector field ξ, and a 1-form η that satisfy the compatibility conditions listed below.

$i)\phi(\xi) = 0, \eta \circ \phi = 0,$

$ii)\eta(\xi) = 1, \phi^2 = id - \eta \otimes \xi,$

$iii)$the tensor field ϕ gives rise to an almost paracomplex structure on each fibre of the horizontal distribution $\mathcal{D} = Ker\eta$ [13]

A differentiable manifold M^{2n+1} equipped with an almost paracontact structure is referred to as an *almost paracontact manifold*.

A direct implication of the definition of an almost paracontact structure is that the endomorphism ϕ has rank $2n$.

If a manifold M^{2n+1} endowed with (ϕ, ξ, η)-structure possesses a pseudo-Riemannian metric g such that

$$g(\phi\zeta_1, \phi\zeta_2) = -g(\zeta_1, \zeta_2) + \eta(\zeta_1)\eta(\zeta_2), \tag{2.6}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$, then we say that M^{2n+1} has an *almost paracontact metric structure* and g is called *compatible metric*. The differentiable manifold M^{2n+1} given by the almost paracontact metric structure is called an *almost paracontact metric manifold*. Any metric g that is compatible with a given almost paracontact structure must have a signature of $(n+1, n)$.

Within the framework of almost paracontact manifolds, the tensor $N^{(1)}$ of type $(1,2)$ can be introduced by

$$N^{(1)}(\zeta_1, \zeta_2) = [\phi, \phi](\zeta_1, \zeta_2) - 2d\eta(\zeta_1, \zeta_2)\xi$$

where

$$[\phi, \phi](\zeta_1, \zeta_2) = \phi^2[\zeta_1, \zeta_2] + [\phi\zeta_1, \phi\zeta_2] - \phi[\phi\zeta_1, \zeta_2] - \phi[\zeta_1, \phi\zeta_2]$$

is the Nijenhuis torsion of ϕ. The almost paracontact manifold is designated as *normal*, when $N^{(1)} = 0$ [23].

Setting $\zeta_2 = \xi$, we have $g(\zeta_1, \xi) = \eta(\zeta_1)$. From here and (2.6) follows

$$g(\phi\zeta_1, \zeta_2) = -g(\zeta_1, \phi\zeta_2).$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$. In an almost paracontact metric manifold, an orthogonal basis always exists. $\{\zeta_{11}, ..., \zeta_{1n}, \zeta_{21}, ..., \zeta_{2n}, \xi\}$, namely ϕ-basis, such that $g(\zeta_{1i}, \zeta_{1j}) = -g(\zeta_{2i}, \zeta_{2j}) = \delta_{ij}$ and $\phi\zeta_{1i} = \zeta_{2i}$, for any $i, j \in \{1, ..., n\}$.

The fundamental 2-form is defined by

$$\Phi(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2),$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

If $d\eta(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2)$ (where $d\eta(\zeta_1, \zeta_2) = \frac{1}{2}(\zeta_1\eta(\zeta_2) - \zeta_2\eta(\zeta_1) - \eta[\zeta_1, \zeta_2])$), then η is a paracontact form and the almost paracontact metric manifold is said to be *paracontact metric manifold*.

Lemma 2.1. [23] *On a paracontact metric manifold M^{2n+1}, $h = \frac{1}{2}\mathcal{L}_\xi\phi$ is a symmetric operator and satisfy the followings:*

$$\begin{aligned} trh &= tr\phi h = 0,\ h\xi = 0,\ h\phi + \phi h = 0, \\ \nabla_{\zeta_1}\xi &= -\phi\zeta_1 + \phi h\zeta_1, \qquad (2.7) \\ Rc(\xi, \xi) &= -2n + trh^2, \end{aligned}$$

for all vector field $\zeta_1 \in \Gamma(M^{2n+1})$, tr is the trace operator.

It is important to note that h is equal to zero if and only if the vector field ξ is Killing. When ξ is Killing, the paracontact metric manifold is referred to as a *K-paracontact manifold*. A normal almost paracontact metric manifold is said to be *para-Sasakian manifold* if $\Phi = d\eta$. Furthermore, a para-Sasakian manifold is also K-paracontact, with the reverse holding true solely in a three-dimensional [23].

An almost paracontact metric manifold is called *η-Einstein* if its Ricci tensor Rc takes the form of

$$Rc = ag + b\eta \otimes \eta$$

where a and b are smooth functions on the manifold.

For a K-paracontact manifold M^{2n+1}, we have the following relations [23]

$$\nabla_{\zeta_1}\xi = -\phi\zeta_1, \qquad (2.8)$$

$$R(\xi, \zeta_1)\zeta_2 = -g(\zeta_1, \zeta_2)\xi + \eta(\zeta_2)\zeta_1, \qquad (2.9)$$

$$Rc(\zeta_1, \xi) = -2n\eta(\zeta_1), \qquad (2.10)$$

$$R(\xi, \zeta_1)\zeta_2 = (\nabla_{\zeta_1}\phi)\zeta_2, \qquad (2.11)$$

$$R(\zeta_1, \xi)\xi = -\zeta_1 + \eta(\zeta_1)\xi, \qquad (2.12)$$

$$(\nabla_{\phi\zeta_1}\phi)\phi\zeta_2 - (\nabla_{\zeta_1}\phi)\zeta_2 = 2g(\zeta_1, \zeta_2)\xi - (\zeta_1 + \eta(\zeta_1)\xi)\eta(\zeta_2), \qquad (2.13)$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$, where Q is the *Ricci operator* defined by $g(Q\zeta_1, \zeta_2) = Rc(\zeta_1, \zeta_2)$.

Also followings hold on a $(2n+1)$-dimensional K-paracontact manifold [17],

$$(\nabla_{\zeta_1} Q)\,\xi = Q\phi\zeta_1 + 2n\phi\zeta_1 \tag{2.14}$$

and

$$(\nabla_{\xi} Q)\,\zeta_1 = Q\phi\zeta_1 - \phi Q\zeta_1 \tag{2.15}$$

for all vector field $\zeta_1 \in \Gamma(M^{2n+1})$.

On a $(2n+1)$-dimensional paracontact metric manifold, the notion of (κ, μ)-*nullity distribution* is given by

$$N(\kappa,\mu) : p \to N_p(\kappa,\mu) = \left[\begin{array}{c} \zeta_3 \in T_pM : R(\zeta_1,\zeta_2)\,\zeta_3 = \kappa\,(g(\zeta_2,\zeta_3)\,\zeta_1 - g(\zeta_1,\zeta_3)\,\zeta_2) \\ +\mu\,(g(\zeta_2,\zeta_3)\,h\zeta_1 - g(\zeta_1,\zeta_3)\,h\zeta_2)\,, \end{array} \right]$$

for every vector fields $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(M^{2n+1})$ and $\kappa,\ \mu \in \mathbb{R}$. If ξ belongs to above distribution, namely,

$$R(\zeta_1,\zeta_2)\,\xi = \kappa\,(\eta(\zeta_2)\,\zeta_1 - \eta(\zeta_1)\,\zeta_2) + \mu\,(\eta(\zeta_2)\,h\zeta_1 - \eta(\zeta_1)\,h\zeta_2)\,, \tag{2.16}$$

then the paracontact metric manifold is called a paracontact metric (κ,μ)-manifold. When $\mu = 0$, a paracontact metric (κ,μ)-manifold reduces to $N(\kappa)$-paracontact metric manifold [4].

Lemma 2.2. [4]*Let M^{2n+1} be a paracontact metric (κ,μ)-manifold, then the following identities hold:*

$$h^2\zeta_1 = (1+\kappa)\,\phi^2\zeta_1, \tag{2.17}$$

$$\begin{aligned} R(\xi,\zeta_1)\,\zeta_2 &= \kappa\,[g(\zeta_1,\zeta_2)\,\xi - \eta(\zeta_2)\,\zeta_1] \\ &\quad +\mu\,[g(h\zeta_1,\zeta_2)\,\xi - \eta(\zeta_2)\,h\zeta_1]\,, \end{aligned} \tag{2.18}$$

$$(\nabla_{\zeta_1}\eta)\,\zeta_2 = g(\zeta_1 - h\zeta_1, \phi\zeta_2)\,, \tag{2.19}$$

for all vector fields $\zeta_1,\ \zeta_2 \in \Gamma(M^{2n+1})$.

Lemma 2.3. [4]*Let M^{2n+1} be a paracontact metric $(\kappa \neq -1, \mu)$-manifold, then the following identities hold:*

$$(\nabla_{\zeta_1}\phi)\,\zeta_2 = -g(\zeta_1 - h\zeta_1, \zeta_2)\,\xi + \eta(\zeta_2)\,(\zeta_1 - h\zeta_1)\ \ , \tag{2.20}$$

$$Rc(\zeta_1,\xi) = 2n\kappa\eta(\zeta_1)\,, \tag{2.21}$$

$$Rc(\zeta_1,\zeta_2) = [2(1-n)+n\mu]\,g(\zeta_1,\zeta_2)+[2(n-1)+\mu]\,g(h\zeta_1,\zeta_2) + [2(n-1)+n(2\kappa-\mu)]\,\eta(\zeta_1)\,\eta(\zeta_2), \tag{2.22}$$

$$(\nabla_{\zeta_1}h)\,\zeta_2 = -[(1+\kappa)\,g(\zeta_1,\phi\zeta_2)+g(\zeta_1,\phi h\zeta_2)]\,\xi + \eta(\zeta_2)\,[(1+\kappa)\,\phi\zeta_1-\phi h\zeta_1]-\mu\eta(\zeta_1)\,\phi h\zeta_2, \tag{2.23}$$

$$Q\xi = 2n\kappa\xi, \tag{2.24}$$

$$r = 2n\,[2(1-n)+\kappa+n\mu], \tag{2.25}$$

for all vector fields $\zeta_1,\ \zeta_2 \in \Gamma(M^{2n+1})$.

Theorem 2.1. [24] *Let* M^{2n+1} *be a paracontact metric manifold and suppose that* $R(\zeta_1,\zeta_2)\,\xi = 0$ *for all vector fields* ζ_1 *and* ζ_2. *Then locally* M^{2n+1} *is the product of a flat* $(n+1)$*-dimensional manifold and* n*-dimensional manifold of negative constant curvature equal to* -4, *for* $n > 1$ *and its locally flat for* $n = 1$.

Lemma 2.4. *On a paracontact metric* (κ,μ)*-manifold* M^{2n+1}, *we have*

$$(\nabla_\xi h)\,\zeta_1 = \mu h\phi\zeta_1, \tag{2.26}$$

$$(\nabla_\xi Q)\,\zeta_1 = \mu\,[2(n-1)+\mu]\,h\phi\zeta_1, \tag{2.27}$$

$$(\nabla_{\zeta_1}Q)\,\xi = Q(\phi\zeta_1-\phi h\zeta_1)-2n\kappa(\phi\zeta_1-\phi h\zeta_1),\ \kappa \neq -1 \tag{2.28}$$

for all vector field $\zeta_1 \in \Gamma(M^{2n+1})$.

Proof. If we write $\zeta_1 = \xi$ in (2.23), we obtain (2.26).

From (2.22), we get

$$Q\zeta_1 = [2(1-n)+n\mu]\,\zeta_1+[2(n-1)+\mu]\,h\zeta_1+[2(n-1)+n(2\kappa-\mu)]\,\eta(\zeta_1)\,\xi. \tag{2.29}$$

If we take the covariant derivative of (2.29) along ξ and use (2.26), we have (2.27). If we take the covariant derivative of (2.24) along ζ_1 and use (2.7), we obtain (2.28). □

3. η-Ricci-Bourguignon and Gradient η-Ricci-Bourguignon Solitons on K-Paracontact Manifolds

In this section, we will investigate η-Ricci-Bourguignon and Gradient η-Ricci-Bourguignon solitons on K-paracontact manifolds.

Theorem 3.1. *Let M^{2n+1} be a K-paracontact manifold. If M^{2n+1} admits an η-Ricci-Bourguignon soliton whose potential vector field being collinear with ξ, the manifold is η-Einstein and the scalar curvature $r = -2n(2n+1+\sigma)$ is constant.*

Proof. Now assume that $\mathcal{W}=f\xi$, where f is a smooth function. Letting $\mathcal{W}$ by $f\xi$ and using (2.8) in (1.4), we get

$$Rc(\zeta_1, \zeta_2) + \frac{1}{2}(\zeta_1(f)\eta(\zeta_2) + \zeta_2(f)\eta(\zeta_1)) = (\lambda + \rho r) g(\zeta_1, \zeta_2) + \sigma\eta(\zeta_1)\eta(\zeta_2). \quad (3.30)$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

Putting ζ_2 by ξ in (3.30), we have

$$Rc(\zeta_1, \xi) + \frac{1}{2}(\zeta_1(f) + \xi(f)\eta(\zeta_1)) = (\lambda + \rho r)\eta(\zeta_1) + \sigma\eta(\zeta_1). \quad (3.31)$$

Using (2.10) in (3.31), we get

$$gradf = (2(\lambda + \rho r) + 2\sigma - \xi(f) + 4n)\xi. \quad (3.32)$$

On the other hand putting $\zeta_1 = \zeta_2 = \xi$ and using again (2.10) in (3.30), we have

$$-2n + \xi(f) = \lambda + \rho r + \sigma. \quad (3.33)$$

If we use (3.33) in (3.32), we obtain

$$gradf = \xi(f)\xi. \quad (3.34)$$

If we take the covariant derivative of (3.34) along ζ_1 and using (2.8), we get

$$g(\nabla_{\zeta_1} gradf, \zeta_2) = \xi(f)g(\nabla_{\zeta_1}\xi, \zeta_2) + \zeta_1(\xi(f))\eta(\zeta_2) \quad (3.35)$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

By using $g(\nabla_{\zeta_1} gradf, \zeta_2) = g(\nabla_{\zeta_2} gradf, \zeta_1)$ we have

$$\zeta_1(\xi(f))\eta(\zeta_2) - \zeta_2(\xi(f))\eta(\zeta_1) = -2\xi(f)d\eta(\zeta_1, \zeta_2) \quad (3.36)$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$. Putting ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (3.36), we obtain $\xi(f) = 0$, because of $d\eta \neq 0$. So from (3.34), we have $gradf = 0$, namely f is constant and so the manifold is η-Einstein.

Let $\{w_i\}$ $(1 \leq i \leq 2n+1)$ be an orthonormal basis. Taking the summation over i for $\zeta_1 = \zeta_2 = w_i$ in (3.30), we obtain

$$r = (\lambda + \rho r)(2n+1) + \sigma. \tag{3.37}$$

Using (3.33) in (3.37), we get

$$r = -2n(2n+1+\sigma)$$

which completes the proof. □

Theorem 3.2. *If a $(2n+1)$-dimensional K-paracontact manifold admits a gradient η-Ricci-Bourguignon soliton, then the scalar curvature is constant and the manifold is η-Einstein.*

Proof. By virtue of (1.5), we have

$$\nabla_{\zeta_1} gradf = -Q\zeta_1 + (\lambda + \rho r)\zeta_1 + \sigma\eta(\zeta_1)\xi. \tag{3.38}$$

Taking the covariant derivative of (3.38) with ζ_2 and using (2.8), we get

$$\nabla_{\zeta_2}\nabla_{\zeta_1} gradf = -\nabla_{\zeta_2} Q\zeta_1 + (\lambda + \rho r)\nabla_{\zeta_2}\zeta_1 + \rho\zeta_2(r)\zeta_1 + \sigma(\nabla_{\zeta_2}\eta(\zeta_1)\xi - \eta(\zeta_1)\phi\zeta_2). \tag{3.39}$$

Interchanging ζ_1 and ζ_2 in the last equation, we derive

$$\nabla_{\zeta_1}\nabla_{\zeta_2} gradf = -\nabla_{\zeta_1} Q\zeta_2 + (\lambda + \rho r)\nabla_{\zeta_1}\zeta_2 + \rho\zeta_1(r)\zeta_2 + \sigma(\nabla_{\zeta_1}\eta(\zeta_2)\xi - \eta(\zeta_2)\phi\zeta_1). \tag{3.40}$$

From (3.38), we obtain

$$\nabla_{[\zeta_1,\zeta_2]} gradf = -Q[\zeta_1,\zeta_2] + (\lambda + \rho r)[\zeta_1,\zeta_2] + \sigma\eta([\zeta_1,\zeta_2])\xi. \tag{3.41}$$

In the view of (3.39), (3.40) and (3.41), we can compute

$$\begin{aligned} R(\zeta_1,\zeta_2) gradf &= -(\nabla_{\zeta_1} Q)\zeta_2 + (\nabla_{\zeta_2} Q)\zeta_1 + \rho(\zeta_1(r)\zeta_2 - \zeta_2(r)\zeta_1) \\ &\quad +\sigma(-2g(\phi\zeta_1,\zeta_2)\xi + \eta(\zeta_1)\phi\zeta_2 - \eta(\zeta_2)\phi\zeta_1). \end{aligned} \tag{3.42}$$

Contracting the last equation over ζ_1 and using

$$divQ\zeta_2 = \sum_{i=1}^{2n+1} \varepsilon_i g((\nabla_{w_i} Q)\zeta_2, w_i) = \frac{1}{2}\zeta_2(r).$$

We conclude that

$$Rc(\zeta_2, gradf) = \left(\frac{1}{2} - 2n\rho\right)\zeta_2(r). \tag{3.43}$$

By (2.10), we have

$$Rc(gradf, \xi) = -2n\xi(f). \tag{3.44}$$

Since ξ is Killing, $\xi(r) = 0$. Putting $\zeta_2 = \xi$ in (3.43) and using (3.44), we get $\xi(f) = 0$.

Taking the inner product of (3.42) with ξ and using equation (2.14) , we obtain

$$\begin{aligned} g\left(R\left(gradf,\xi\right)\zeta_1,\zeta_2\right) &= g\left(Q\phi\zeta_2,\zeta_1\right) - g\left(Q\phi\zeta_1,\zeta_2\right) - 2\left(2n+\sigma\right)g\left(\phi\zeta_1,\zeta_2\right) \\ &\quad +\rho\left[\zeta_1\left(r\right)\eta\left(\zeta_2\right) - \zeta_2\left(r\right)\eta\left(\zeta_1\right)\right]. \end{aligned} \tag{3.45}$$

Replacing ζ_1 by ξ in (3.45) and using the fact that $\xi(r) = 0$ and $\xi(f) = 0$, equations (2.9) and (2.10), we have

$$\zeta_2\left(f - \rho r\right) = 0,$$

this leads to the conclusion that $f - \rho r$ is a constant.

Substituting $\zeta_2 = \xi$ in (3.42) and taking the inner product with ζ_2 and using (2.11), (2.14) and (2.15) we get

$$g((\nabla_{\zeta_1}\phi)\zeta_2, gradf) = -\left(2n+\sigma\right)g\left(\phi\zeta_1,\zeta_2\right) - g\left(\phi Q\zeta_1,\zeta_2\right) + \rho\zeta_1\left(r\right)\eta\left(\zeta_2\right). \tag{3.46}$$

First, if we replace ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (3.46) and then subtract (3.46) from the obtained equation, we obtain following equation

$$Q\phi\zeta_1 + \phi Q\zeta_1 = -2\left(2n+\sigma\right)\phi\zeta_1, \tag{3.47}$$

by using (2.13) and $\xi(f) = 0$.

Let $\{w_i\}$ $(1 \leq i \leq 2n+1)$ be an orthonormal basis, after writing $\zeta_1 = w_i$ in (3.47), we have

$$Q\phi w_i + \phi Q w_i = -2\left(2n+\sigma\right)\phi w_i. \tag{3.48}$$

Moreover, we can calculate following

$$g\left(\phi Q w_i, \phi w_i\right) = -g\left(Q w_i, \phi^2 w_i\right) = -g\left(Q w_i, w_i\right). \tag{3.49}$$

By virtue of (3.48) and (3.49), we get

$$\begin{aligned} r &= Rc\left(\xi,\xi\right) + \sum_{i=1}^{n}\left\{Rc\left(w_i,w_i\right) - Rc\left(\phi w_i,\phi w_i\right)\right\} \\ &= -2n + \sum_{i=1}^{n}\left\{-g\left(\phi Q w_i + Q\phi w_i, \phi w_i\right)\right\} \\ &= -2n\left(2n+1\right) - 2n\sigma. \end{aligned}$$

constant, so from $f - \rho r$ is constant, we have f is constant. Hence from $\mathcal{W}=gradf$, $\mathcal{W}=0$. By (1.5), the manifold is η-Einstein. This concludes the proof. □

4. Gradient η-Ricci-Bourguignon Solitons on Paracontact $(\kappa \neq -1, \mu)$-Manifolds

In this section, we will investigate gradient η-Ricci-Bourguignon solitons on paracontact metric $(\kappa \neq -1, \mu)$-manifolds.

Lemma 4.1. *If a $(2n+1)$-dimensional paracontact metric $(\kappa \neq -1, \mu)$-manifold admits a gradient η-Ricci-Bourguignon soliton, then we have*

$$\kappa (2 - \mu) = \mu (n+1) + \sigma. \tag{4.50}$$

Proof. By virtue of (1.5), we have

$$\nabla_{\zeta_1} gradf + Q\zeta_1 = (\lambda + \rho r) \zeta_1 + \sigma \eta (\zeta_1) \xi. \tag{4.51}$$

Taking the covariant derivative of (4.51) with ζ_2 and using (2.7), we get

$$\nabla_{\zeta_2} \nabla_{\zeta_1} gradf + \nabla_{\zeta_2} Q\zeta_1 = (\lambda + \rho r) \nabla_{\zeta_2} \zeta_1 + \sigma (\nabla_{\zeta_2} \eta (\zeta_1) \xi - \eta (\zeta_1) \phi \zeta_2 + \eta (\zeta_1) \phi h \zeta_2). \tag{4.52}$$

Interchanging ζ_1 and ζ_2 in the last equation, we obtain

$$\nabla_{\zeta_1} \nabla_{\zeta_2} gradf + \nabla_{\zeta_1} Q\zeta_2 = (\lambda + \rho r) \nabla_{\zeta_1} \zeta_2 + \sigma (\nabla_{\zeta_1} \eta (\zeta_2) \xi - \eta (\zeta_2) \phi \zeta_1 + \eta (\zeta_2) \phi h \zeta_1). \tag{4.53}$$

From (4.51), we have

$$\nabla_{[\zeta_1, \zeta_2]} gradf + Q [\zeta_1, \zeta_2] = (\lambda + \rho r) [\zeta_1, \zeta_2] + \sigma \eta ([\zeta_1, \zeta_2]) \xi. \tag{4.54}$$

In the view of (4.52), (4.53) and (4.54), we can compute

$$\begin{aligned} R (\zeta_1, \zeta_2) gradf &= - (\nabla_{\zeta_1} Q) \zeta_2 + (\nabla_{\zeta_2} Q) \zeta_1 \\ &\quad + \sigma (2g (\zeta_1, \phi \zeta_2) \xi + \eta (\zeta_1) \phi \zeta_2 - \eta (\zeta_1) \phi h \zeta_2 - \eta (\zeta_2) \phi \zeta_1 + \eta (\zeta_2) \phi h \zeta_1). \end{aligned} \tag{4.55}$$

Using (2.28) in (4.55), we obtain

$$\begin{aligned} g (R (\zeta_1, \zeta_2) gradf, \xi) &= g ((Q\phi + \phi Q) \zeta_2, \zeta_1) - g ((Q\phi h + h\phi Q) \zeta_2, \zeta_1) \\ &\quad - 4n\kappa g (\phi \zeta_2, \zeta_1) + 2\sigma g (\zeta_1, \phi \zeta_2). \end{aligned} \tag{4.56}$$

Putting ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (4.56) and using the fact that $R (\phi\zeta_1, \phi\zeta_2) \xi = 0$ from (2.16), we get

$$0 = \phi (- (Q\phi + \phi Q) \phi \zeta_1 + (Q\phi h + h\phi Q) \phi \zeta_1 + 4n\kappa \zeta_1 - 2\sigma \zeta_1). \tag{4.57}$$

From (2.29), we can compute

$$\phi\left(Q\phi+\phi Q\right)\phi\zeta_1 = 2\left(2\left(1-n\right)+n\mu\right)\phi\zeta_1. \tag{4.58}$$

$$\phi\left(Q\phi h+h\phi Q\right)\phi\zeta_1 = -2\left(\kappa+1\right)\left(2\left(n-1\right)+\mu\right)\phi\zeta_1. \tag{4.59}$$

If we use (4.58) and (4.59) in (4.57), we get (4.50). □

Theorem 4.1. *If a $(2n+1)$-dimensional paracontact metric $(\kappa\neq-1,\mu)$-manifold admits a gradient η-Ricci-Bourguignon soliton, then either*

i) The manifold is η-Einstein, $\kappa=0$, $\mu=2(1-n)$, $r=4n(1-n^2)$, or

ii) The manifold is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4 for $n>1$ and its locally flat for $n=1$, or

iii) The manifold is η-Einstein, $\kappa=\frac{1-n^2}{n}+\frac{\sigma}{2n}$, $\mu=2(1-n)$, $r=2(1-n^2)(1+2n)+\sigma$, or

iv) The manifold is paracontact metric $\left(\kappa>-1,\ \mu=\pm\frac{\kappa}{\sqrt{\kappa+1}}\right)$-manifold.

Proof. Substituting $\zeta_1=\xi$ in (4.55) and then using (2.27) and (2.28), we get

$$R\left(\xi,\zeta_2\right)gradf=-\mu\left[2\left(n-1\right)+\mu\right]h\phi\zeta_2+Q\left(\phi\zeta_2-\phi h\zeta_2\right)-2n\kappa\left(\phi\zeta_2-\phi h\zeta_2\right)+\sigma\left(\phi\zeta_2-\phi h\zeta_2\right). \tag{4.60}$$

Putting $\zeta_1=\zeta_2$, $\zeta_2=gradf$ in (2.18), we obtain

$$R\left(\xi,\zeta_2\right)gradf=\kappa\left[\zeta_2\left(f\right)\xi-\xi\left(f\right)\zeta_2\right]+\mu\left[\left(h\zeta_2\right)\left(f\right)\xi-\xi\left(f\right)h\zeta_2\right]. \tag{4.61}$$

By equating the right-hand sides of equations (4.60) and (4.61) and subsequently taking the inner product of the resulting equation with ξ, we obtain

$$\kappa\left[\zeta_2\left(f\right)-\xi\left(f\right)\eta\left(\zeta_2\right)\right]+\mu\left[\left(h\zeta_2\right)\left(f\right)\right]=0. \tag{4.62}$$

If we substitute ζ_2 by $h\zeta_2$ in (4.62) and use (2.17), we get

$$\kappa\left(h\zeta_2\right)\left(f\right)+\mu\left(\kappa+1\right)\left[\zeta_2\left(f\right)-\eta\left(\zeta_2\right)\xi\left(f\right)\right]=0. \tag{4.63}$$

Combining (4.62) and (4.63), we obtain

$$\left[\zeta_2\left(f\right)-\xi\left(f\right)\eta\left(\zeta_2\right)\right]\left[\kappa^2-\mu^2\left(\kappa+1\right)\right]=0. \tag{4.64}$$

Contracting (4.55) over ζ_1 and using

$$divQ\zeta_2=\sum_{i=1}^{2n+1}\varepsilon_i g((\nabla_{w_i}Q)\zeta_2,w_i)=\frac{1}{2}\zeta_2(r).$$

We conclude that

$$Rc\left(\zeta_2, gradf\right) = 0. \tag{4.65}$$

In the view of (2.22) and (4.65), we get

$$\begin{aligned} 0 &= [2(1-n)+n\mu]\, g(\zeta_1, gradf) + [2(n-1)+\mu]\, g(h\zeta_1, gradf) \\ &\quad + [2(n-1)+n(2\kappa-\mu)]\, \eta(\zeta_1)\, \eta(gradf). \end{aligned} \tag{4.66}$$

Substituting $\zeta_1 = \xi$ in (4.66), we have

$$2n\kappa\xi(f) = 0.$$

This gives either $\kappa = 0$, or $\xi(f) = 0$.

Case 1: Let $\kappa = 0$. From (4.64), we have

$$[gradf - \xi(f)\,\xi]\,\mu^2 = 0. \tag{4.67}$$

By (4.67), we have followings:

Case 1a: Let $\mu \neq 0$. So we obtain

$$gradf = \xi(f)\,\xi. \tag{4.68}$$

If we take the covariant derivative of (4.68) along ζ_1 and using (2.7), we get

$$g\left(\nabla_{\zeta_1} gradf, \zeta_2\right) = \xi(f) g\left(\nabla_{\zeta_1}\xi, \zeta_2\right) + \zeta_1(\xi(f))\eta(\zeta_2) \tag{4.69}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$.

By using $g\left(\nabla_{\zeta_1} gradf, \zeta_2\right) = g\left(\nabla_{\zeta_2} gradf, \zeta_1\right)$ we have

$$\zeta_1(\xi(f))\eta(\zeta_2) - \zeta_2(\xi(f))\eta(\zeta_1) = -2\xi(f)d\eta(\zeta_1, \zeta_2) \tag{4.70}$$

for all vector fields $\zeta_1, \zeta_2 \in \Gamma(M^{2n+1})$. Putting ζ_1 by $\phi\zeta_1$ and ζ_2 by $\phi\zeta_2$ in (4.70), we obtain $\xi(f) = 0$, because of $d\eta \neq 0$. So from (4.68), we have $gradf = 0$, namely f is constant and so the manifold is η-Einstein. So from (2.29), we obtain $\mu = 2(1-n)$. Let $\{w_i\}$ $(1 \leq i \leq 2n+1)$ be an orthonormal basis. Taking the summation over i for $\zeta_1 = \zeta_2 = w_i$ in (2.29), we obtain $r = 4n\left(1-n^2\right)$. Note that in this subcase the scalar curvature can not be positive.

Case 1b: Let $\mu = 0$. So we can use Theorem 2.1.

Case 2: Let $\xi(f) = 0$. By (4.64) we have

$$gradf\left(\kappa^2 - \mu^2(\kappa+1)\right) = 0. \tag{4.71}$$

By (4.71), we have followings:

Case 2a: Let $gradf = 0$. Namely f is constant. So the manifold is η-Einstein. So from (2.29), we obtain $\mu = 2(1-n)$. Using this in (4.50), we get $\kappa = \frac{1-n^2}{n} + \frac{\sigma}{2n}$. Let $\{w_i\}$ $(1 \le i \le 2n+1)$ be an orthonormal basis. Taking the summation over i for $\zeta_1 = \zeta_2 = w_i$ in (2.29), we obtain $r = 2(1-n^2)(1+2n) + \sigma$.

Case 2b: Let $\kappa^2 - \mu^2(\kappa+1) = 0$. We want to remind that $\kappa \neq -1$. It means that $\kappa > -1$ or $\kappa < -1$. Firstly let us suppose that $\kappa < -1$. In this case we say that $\kappa = 0$ and $\mu = 0$. But this case is contradiction with the assumption that $\kappa < -1$. Therefore, κ must be bigger than -1. Now, from $\kappa^2 - \mu^2(\kappa+1) = 0$, we obtain $\mu = \pm\frac{\kappa}{\sqrt{\kappa+1}}$ Namely the manifold is paracontact metric $\left(\kappa > -1,\, \mu = \pm\frac{\kappa}{\sqrt{\kappa+1}}\right)$-manifold. This concludes the proof. □

5. Examples

Example 5.1. *We consider the three-dimensional manifold M. Define the almost paracontact structure (ϕ, ξ, η) on M by*

$$\phi\xi = 0,\ \phi w_1 = w_2,\ \phi w_2 = w_1, \xi = w_3.$$

We have

$$[w_1, w_3] = 0, \quad [w_2, w_3] = 0, \quad [w_1, w_2] = -2\xi.$$

Let g be the semi-Riemannian metric defined by

$$g(w_2, w_2) = -1,\ g(w_1, w_1) = g(\xi, \xi) = 1,\ g(w_i, w_j) = 0,\ i \neq j$$

where $i, j = 1, 2, 3$. Let ∇ be the Levi-Civita connection with respect to g. Then by Koszul formula

$$\begin{aligned}
\nabla_{w_1} w_1 &= 0,\ \nabla_{w_2} w_1 = \xi,\ \nabla_{w_3} w_1 = -w_2, \\
\nabla_{w_1} w_2 &= -\xi,\ \nabla_{w_{w_2}} w_2 = 0,\ \nabla_{w_3} w_2 = -w_1, \\
\nabla_{w_1} w_3 &= -w_2,\ \nabla_{w_2} w_3 = -w_1, \nabla_{w_3} w_3 = 0.
\end{aligned}$$

It is easy to see that M is a K-paracontact manifold. The components of the curvature tensor are

$$\begin{aligned}
R(w_1, w_2)w_2 &= -3w_1, \quad R(w_1, w_2)w_3 = 0,\ R(w_3, w_2)w_2 = \xi, \\
R(w_1, w_3)w_3 &= -w_1,\ R(w_2, w_3)w_3 = -w_2, \quad R(w_1, w_3)w_2 = 0, \\
R(w_2, w_1)w_1 &= 3w_2,\ R(w_3, w_1)w_1 = -\xi, \quad R(w_2, w_3)w_1 = 0.
\end{aligned}$$

Using the components of the curvature tensor, we obtain

$$Rc(w_1, w_1) = 2,\ Rc(w_2, w_2) = -2,\ Rc(\xi, \xi) = -2$$

In view of above relations, we have $r = S(w_1, w_1) - S(w_2, w_2) + S(\xi, \xi) = 2$. *Using (1.4), we have*

$$Rc(w_1, w_1) = \lambda + \rho r = 2,\ Rc(w_2, w_2) = -(\lambda + \rho r) = -2,\ Rc(\xi, \xi) = \lambda + \rho r + \sigma = -2. \quad (5.72)$$

From (5.72), we get $\lambda + 2\rho = 2$ *and* $\sigma = -4$. *Hence we see that* M *admits an* η*-Ricci-Bourguignon soliton with* $\sigma = -4$, *for* $\mathcal{W}{=}f\xi$, f *constant.* M *is also* η*-Einstein manifold and verifies Theorem 3.1. Also the soliton is shrinking, steady or expanding according as* $2(1-\rho) > 0$, $2(1-\rho) = 0$ *and* $2(1-\rho) < 0$, *respectively.*

We used [14] while constructing following examples.

Example 5.2. *Let* M *be a three-dimensional manifold.* $w_1 = w$, $w_2 = \phi w$ *and* $w_3 = \xi$ *are vector fields such that*

$$[w, \xi] = (\tilde{\lambda} - 1)\phi w, \quad [\phi w, \xi] = -(\tilde{\lambda} + 1)w, \quad [w, \phi w] = 2\xi.$$

The semi-Riemannian metric g *is defined by*

$$g(w, w) = -1,\ \ g(\phi w, \phi w) = g(\xi, \xi) = 1,\ g(w_i, w_j) = 0,\ i \neq j$$

where $i, j = 1, 2, 3$. *The* 1*-form* η *is defined by*

$$\eta(\zeta_1) = g(\zeta_1, \xi)$$

for all ζ_1 *on* M. *Let* ϕ *be the* $(1,1)$*-tensor field defined by*

$$\phi\xi = 0,\ \phi w_1 = w_2,\ \phi w_2 = w_1.$$

Then,

$$\begin{aligned} \eta(\xi) &= 1,\ \phi^2(\zeta_1) = \zeta_1 - \eta(\zeta_1)\xi \\ g(\phi\zeta_1, \phi\zeta_2) &= -g(\zeta_1, \zeta_2) + \eta(\zeta_1)\eta(\zeta_2),\ d\eta(\zeta_1, \zeta_2) = g(\zeta_1, \phi\zeta_2), \end{aligned}$$

for any vector fields ζ_1, ζ_2 on M. Hence (ϕ, ξ, η, g) defines a paracontact structure. Let ∇ be the Levi-Civita connection on M, then by Koszul's formula, we obtain

$$\begin{aligned}
\nabla_w w &= 0,\ \nabla_{\phi w} w = -(\tilde{\lambda}+1)\xi,\ \nabla_\xi w = 0,\\
\nabla_w \phi w &= (1-\tilde{\lambda})\xi,\ \nabla_{\phi w}\phi w = 0,\ \nabla_\xi \phi w = 0,\\
\nabla_w \xi &= (\tilde{\lambda}-1)\phi w,\ \ \nabla_{\phi w}\xi = -(\tilde{\lambda}+1)w,\ \nabla_\xi \xi = 0.
\end{aligned}$$

Using the covariant derivatives, we compute the components of the Riemannian curvature tensor:

$$\begin{aligned}
R(w,\phi w)\phi w &= (1-\tilde{\lambda}^2)w,\ \ R(\phi w,\xi)\xi = (\tilde{\lambda}^2-1)\phi w,\ R(w,\phi w)\xi = 0,\\
R(w,\xi)\xi &= (\tilde{\lambda}^2-1)w,\ R(\xi,w)w = (1-\tilde{\lambda}^2)\xi,\quad R(w,\xi)\phi w = 0,\\
R(\phi w,w)w &= (\tilde{\lambda}^2-1)\phi e,\ R(\xi,\phi w)\phi w = (\tilde{\lambda}^2-1)\xi\ ,\ R(\phi w,\xi)w = 0.
\end{aligned}$$

Also, the followings are valid:

$$hw = \tilde{\lambda}w,\ \ h\phi w = -\tilde{\lambda}\phi w,\ \ h\xi = 0.$$

$$\begin{aligned}
Qw &= (1-\tilde{\lambda}^2+\frac{r}{2})w,\\
Q\phi w &= (1-\tilde{\lambda}^2+\frac{r}{2})\phi w,\\
Q\xi &= 2(\tilde{\lambda}^2-1)\xi.
\end{aligned} \tag{5.73}$$

Thus, the manifold is a ($\kappa \neq -1$,0)-paracontact metric manifold with $\kappa = \tilde{\lambda}^2 - 1 > -1$.

From the components of the Riemannian curvature tensor, we derive $Rc(w,w) = 0$, $Rc(\phi w,\phi w) = 0$, $Rc(\xi,\xi) = 2\tilde{\lambda}^2 - 2$. Hence, the scalar curvature $r = 2(\tilde{\lambda}^2-1) = 2\kappa$. Then, using this, (1.5) and (5.73) we get

$$(-1+\tilde{\lambda}^2-\frac{r}{2}+\lambda+\rho r)w = 0,\ (-1+\tilde{\lambda}^2-\frac{r}{2}+\lambda+\rho r)\phi w = 0, (-2\tilde{\lambda}^2+2+\lambda+\rho r+\sigma)\xi = 0. \tag{5.74}$$

By (5.74), we get $\lambda+\rho r = 1-\tilde{\lambda}^2+\frac{r}{2}$ and $r = \sigma$. If we use $r = 2(\tilde{\lambda}^2-1)$ in the last equation we have $\lambda+\rho r = 0$. Hence we see that M admits gradient η-Ricci-Bourguignon soliton with $\sigma = 2(\tilde{\lambda}^2-1) = r$ and constant f. M is also η-Einstein manifold and verifies Theorem 4.1.

Example 5.3. *Let M be a three-dimensional manifold. $w_1 = w$, $w_2 = \phi w$ and $w_3 = \xi$ are vector fields such that*

$$[w,\xi] = 2w - \phi w,\ \ [\phi w,\xi] = -w - 2\phi w,\ \ [w,\phi w] = 2\xi.$$

The semi-Riemannian metric g is defined by

$$g(w,w)=-1,\ \ g(\phi w,\phi w)=g(\xi,\xi)=1,\ g(w_i,w_j)=0,\ i\neq j$$

where $i,j=1,2,3$. The 1-form η is defined by

$$\eta(\zeta_1)=g(\zeta_1,\xi)$$

for all ζ_1 on M. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi\xi=0,\ \phi w_1=w_2,\ \phi w_2=w_1.$$

Then,

$$\begin{aligned}\eta(\xi) &= 1,\ \phi^2(\zeta_1)=\zeta_1-\eta(\zeta_1)\xi\\ g(\phi\zeta_1,\phi\zeta_2) &= -g(\zeta_1,\zeta_2)+\eta(\zeta_1)\eta(\zeta_2),\ d\eta(\zeta_1,\zeta_2)=g(\zeta_1,\phi\zeta_2),\end{aligned}$$

for any vector fields ζ_1, ζ_2 on M. Hence (ϕ,ξ,η,g) defines a paracontact structure. Let ∇ be the Levi-Civita connection on M, then by Koszul's formula, we obtain

$$\begin{aligned}\nabla_w w &= 2\xi,\ \nabla_{\phi w}w=-\xi,\quad \nabla_\xi w=0,\\ \nabla_w\phi w &= \xi,\ \nabla_{\phi w}\phi w=2\xi,\ \nabla_\xi\phi w=0,\\ \nabla_w\xi &= -\phi w+2w,\quad \nabla_{\phi w}\xi=-w-2\phi w,\ \nabla_\xi\xi=0.\end{aligned}$$

Using the covariant derivatives, we compute the components of the Riemannian curvature tensor:

$$\begin{aligned}R(w,\phi w)\phi w &= 5w,\ R(\phi w,\xi)\xi=-5\phi w,\ R(w,\phi w)\xi=0,\\ R(w,\xi)\xi &= -5w,\quad R(\xi,w)w=5\xi,\ R(w,\xi)\phi w=0,\\ R(\phi w,w)w &= -5\phi w,\ R(\xi,\phi w)\phi w=-5\xi,\ R(\phi w,\xi)w=0.\end{aligned}$$

Also, the followings are valid:

$$hw=\tilde{\lambda}\phi w,\ \ h\phi w=-\tilde{\lambda}w,\ \ h\xi=0.$$

$$\begin{aligned}Qw &= (5+\frac{r}{2})w,\\ Q\phi w &= (5+\frac{r}{2})\phi w,\\ Q\xi &= -10\xi.\end{aligned}\tag{5.75}$$

Thus, the manifold is a $(\kappa \neq -1{,}0)$*-paracontact metric manifold with* $\kappa = -5 < -1$.

From the components of the Riemannian curvature tensor, we derive $Rc(w,w) = 0$, $Rc(\phi w, \phi w) = 0$, $Rc(\xi,\xi) = -10$. *Hence, the scalar curvature* $r = -10 = 2\kappa$. *Then, using this, (1.5) and (5.75) we get*

$$(\lambda - 10\rho)w = 0,\ (\lambda - 10\rho)\phi w = 0,\ (10 + \lambda - 10\rho + \sigma)\xi = 0. \tag{5.76}$$

By (5.76), we get $\lambda - 10\rho = 0$ *and* $r = \sigma = -10$. *Hence we see that* M *admits a gradient* η*-Ricci-Bourguignon soliton with* $\sigma = -10$ *and constant* f. M *is also* η*-Einstein manifold and verifies Theorem 4.1.*

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References

[1] Blaga, A., & Ozgur, C. (2022). Remarks on submanifolds as almost η-Ricci Bourguignon solitons. Facta Universitatis Series: Mathematics and Informatics, 37(2), 397–407.

[2] Blaga, A. M., & Tastan, H. M. (2021). Some results on almost η-Ricci-Bourguignon solitons. Journal of Geometry and Physics, 168, Article 104316.

[3] Bourguignon, J. P. (1981). Ricci curvature and Einstein metrics. In Global Differential Geometry and Global Analysis (Lecture Notes in Mathematics, Vol. 838, pp. 42–63).

[4] Cappelletti Montano, B., Küpeli Erken, İ., & Murathan, C. (2012). Nullity conditions in paracontact geometry. Differential Geometry and Its Applications, 30, 665–693.

[5] Catino, G., Cremaschi, L., Djadli, Z., Mantegazza, C., & Mazzieri, L. (2017). The Ricci-Bourguignon flow. Pacific Journal of Mathematics, 287, 337–370.

[6] Catino, G., & Mazzieri, L. (2016). Gradient Einstein solitons. Nonlinear Analysis, 132, 66–94.

[7] Cho, J. T. (2013). Almost contact 3-manifolds and Ricci solitons. International Journal of Geometric Methods in Modern Physics, 10(1), 1220022. https://doi.org/10.1142/S0219887812200228

[8] De, U. C., Turan, M., Yildiz, A., & De, A. (2012). Ricci solitons and gradient Ricci solitons on 3-dimensional normal almost contact metric manifolds. Publications Mathematicae Debrecen, 80(1-2), 127–142.

[9] Dwivedi, S. (2021). Some results on Ricci-Bourguignon solitons and almost solitons. Canadian Mathematical Bulletin, 64, 591–604.

[10] Ghosh, A. (2011). Kenmotsu 3-metric as a Ricci soliton. Chaos, Solitons & Fractals, 44(8), 647–650.

[11] Hamilton, R. S. (1982). Three-manifolds with positive Ricci curvature. Journal of Differential Geometry, 17(2), 255–306.

[12] Hamilton, R. S. (1988). The Ricci flow on surfaces. Contemporary Mathematics, 71, 237–262.

[13] Kaneyuki, S., & Konzai, M. (1985). Paracomplex structures and affine symmetric spaces. Tokyo Journal of Mathematics, 8, 301–318.

[14] Kupeli Erken, İ., & Murathan, C. (2017). A study of three-dimensional paracontact $(\tilde{\kappa}, \tilde{\mu}, \tilde{\nu})$-spaces. International Journal of Geometric Methods in Modern Physics, 14(7), 1750106.

[15] Mandal, T., De, U. C., Khan, M. A., & Khan, M. N. I. (2024). A study on contact metric manifolds admitting a type of solitons. Journal of Mathematics, 8516906.

[16] Mandal, T., De, U. C., & Sarkar, A. (2024). η-Ricci-Bourguignon solitons on three-dimensional (almost) coKähler manifolds. Mathematical Methods in the Applied Sciences, 1–14.

[17] Özkan, M., & Küpeli Erken, İ. (2025). Fischer-Marsden Conjecture on K-paracontact manifolds and Quasi-para-Sasakian manifolds. Communications Faculty of Sciences University of Ankara Series A1: Mathematics and Statistics, 74(1), 68–78.

[18] Perelman, G. (2002). The entropy formula for the Ricci flow and its geometric applications. arXiv preprint math.DG/0211159.

[19] Perelman, G. (2003). Ricci flow with surgery on three-manifolds. arXiv preprint math.DG/0303109.

[20] Perelman, G. (2003). Finite extinction time for the solutions to the Ricci flow on certain three manifolds. arXiv preprint math.DG/0307245.

[21] Shaikh, A. A., Cunha, A. W., & Mandal, P. (2021). Some characterizations of ρ-Einstein solitons. Journal of Geometry and Physics, 166, 104270.

[22] Shaikh, A. A., Mondal, C. K., & Mandal, P. (2021). Compact gradient ρ-Einstein soliton is isometric to the Euclidean sphere. Indian Journal of Pure and Applied Mathematics, 52, 335–339.

[23] Zamkovoy, S. (2009). Canonical connections on paracontact manifolds. Annals of Global Analysis and Geometry, 36(1), 37–60.

[24] Zamkovoy, S., & Tzanov, V. (2009). Non-existence of flat paracontact metric structures in dimension greater than or equal to five. Annuaire Université de Sofia Faculté de Mathématiques et Informatique, 100, 27–34.

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CONFORMAL SOLITONS IN RELATIVISTIC MAGNETO-FLUID SPACETIMES WITH ANTI-TORQUED VECTOR FIELDS

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Abstract. The kinematic and dynamic properties of relativistic spacetime in the context of relativity can be modelled by three distinct classes: shrinking, steady, and expanding. This physical framework bears a resemblance to conformal Ricci flow, where solitons serve as fixed points. Notably, within the solar system, the gravitational effects predicted by Ricci flow align with those of Einstein's gravity, ensuring consistency with all classical tests. In this article, we investigate conformal solitons, which extend the concept of Ricci solitons, within the framework of a magnetized spacetime manifold equipped with an anti-torqued vector field ζ. An anti-torqued vector field is defined as one that resists rotational deformation within the fluid-spacetime structure, effectively encoding a type of constrained rotational symmetry relevant in magneto-fluid dynamics. We demonstrate that whether these conformal solitons are steady, expanding, or shrinking depends on intricate relationships among key physical parameters, including magnetic permeability, magneto-fluid density, isotropic pressure, magnetic flux, and the strength of the magnetic field.

Keywords: Soliton, Spacetime, Energy momentum tensor.

2020 Mathematics Subject Classification: 53B30, 53C50, 53C80.

1. Introduction

In modern physics, space and time are inseparable, at least in the process of representing physical things through ourselves, where these two dimensions play an important role in imagining and conceptualizing the connections of all physical things. In 1915, Einstein

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developed the theory of gravity known as general relativity, which exposes the fundamental role of the physics and geometry of spacetime. It plays an important role in Engineering when applied to day to day life. If we consider general relativity, then the space-time in the four-dimensional pseudo-Riemannian manifold with Lorentzian metric (M^4, g), where g is considered to be perfectly liquid space-time. Perfect fluids are used in cosmology to model the idealized distributions of matter. It is defined by various thermodynamical variables (variables are: particle number density, energy density, pressure, temperature, and entropy per particle). These variables are spacetime scalar fields whose values represent measurements made in the rest frame of the isotropic or star.

On the other hand, the Ricci flow was first introduced by Hamilton [9]. Over the last decades, many differential geometers progressively studied Ricci flow [3, 14]. Fischer [8] proposed a modified version known as conformal Ricci flow, which differs from the classical Ricci flow in its constraints. While the original Ricci flow preserves unit volume, the conformal Ricci flow instead imposes a scalar curvature constraint. Interestingly, the conformal Ricci flow equations exhibit structural similarities to the Navier-Stokes equations in fluid dynamics. In this analogy, the time-dependent scalar field p acts as a conformal pressure—unlike physical pressure, which ensures fluid incompressibility, conformal pressure influences the deformation of the metric under the flow. The fixed points of this system correspond to Einstein metrics with a specific constant $\frac{-1}{n}$. Building on these concepts, Catino and Mazzieri [6] introduced Einstein solitons, which provide self-similar solutions to the Einstein flow. Extending this framework, Roy et al. [18] developed the notion of conformal Einstein solitons. Both conformal Ricci and conformal Einstein solitons generate self-similar solutions, offering a deeper understanding of geometric flows in mathematical physics. The conformal Ricci and conformal Einstein flow respectively are given by:

$$\frac{\partial g}{\partial t} = -2(S + \frac{g}{n}) - \phi g \quad and \quad r = -1 \quad and \quad \frac{\partial g}{\partial t} = -2(S - \frac{r}{2}g). \tag{1.1}$$

A matter is assumed to be fluid, having pressure, density, and kinematic and dynamical quantities like verticity, shear, velocity, acceleration, and expansion [25, 1]. The energy-momentum tensor acts a big role in the matter content of spacetime (universe). The energy-momentum tensor applications are cosmology and stellar structure, and examples are electromagnetism and scalar field theory. The study of the kinematic and dynamic nature of relativistic space-time application in relativity has a physical model of three classes, namely: shrinking, steady, and expanding. Such a physical model are similar to conformal flow. Also, for the solar system, conformal flow gravity effects are not different from Einstein's gravity, and hence it obeys all the classical tests.

Over the last decades, many differential geometers [19, 17] progressively studied the various geometric flows in Relativistic perfect fluid spacetime (briefly RPFS). The study of Ricci solitons and their geometric properties in RPFS was first explored by Ali and Ahsan [2]. Subsequently, Blaga [5] investigated the geometric characteristics of RPFS in the context of Ricci solitons, Einstein solitons, and their extensions—namely, π-Ricci solitons and π-Einstein solitons. Further contributions were made in [26], where the authors examined Ricci soliton structures in RPFS with a torse-forming timelike velocity vector field ζ. D. Siddiqi and A. Siddiqui [23] later analyzed the geometric structure of RPFS using conformal Ricci solitons. Siddiqi and De [24] extended these investigations to relativistic magneto-fluid spacetimes (RMFS). More recently, Praveena et al. [16, 15] studied Ricci, Einstein, and conformal Ricci solitons in almost pseudo-symmetric K$\ddot{a}$hlerian and Kähler-Norden spacetimes, incorporating various curvature tensors. Additionally, Bhattacharyya et al. [18] examined conformal Einstein solitons in para-K$\ddot{a}$hler manifolds.

Inspired by these developments, the present work explores the geometric behavior of conformal Ricci and Einstein flows in RMFS with an anti-torqued vector field.

2. Relativistic magneto fluid spacetime

A relativistic magneto-fluid (RMF) is a continuum medium whose physical state can be fully described by several key parameters: the fluid's rest frame, mass density, isotropic pressure, magnetic flux, and magnetic field strength. In general relativity, such magneto-fluids serve as fundamental models for idealized matter distributions, including stellar interiors and homogeneous cosmological models.

The RMF framework makes several simplifying assumptions - the medium exhibits zero shear stress, negligible viscosity, and no thermal conduction. Mathematically, its behavior is governed by a magnetic energy-momentum tensor T with specific symmetric properties that capture these physical characteristics. This formulation provides a valuable theoretical tool for analyzing relativistic plasma systems where electromagnetic and gravitational interactions play equally important roles. T is in the form [13, 12]:

$$\mathfrak{T} = \rho g + (\sigma + \rho)\mathcal{A} \otimes \mathcal{A} + \nu\{\mathcal{H}\left(\mathcal{A} \otimes \mathcal{A} + \frac{1}{2}g\right) - \mathcal{B} \otimes \mathcal{B}\}, \tag{2.2}$$

where $\nu, \sigma, \rho, \mathcal{B}, \mathcal{H}$ are the magnetic permeability, magneto-fluid density, isotopic pressure, magnetic flux, strength of the magnetic field, respectively and $\mathcal{A}(\cdot) = g(\cdot, \zeta)$, $g(\cdot, \xi) = \mathcal{B}(\cdot)$ are two non-zero 1-forms. Also, ζ and ξ, are unit timelike vector field ζ such that $g(\zeta, \zeta) = -1$ and spacelike magnetic flux vector field ξ such that $g(\xi, \xi) = 1$. Therefore, ζ and ξ are orthogonal vector fields generate the magneto-fluid spacetime.

Einstein's gravitational equation with cosmological constant is given as [12]

$$k\mathfrak{T} = S + \left(\lambda - \frac{r}{2}\right) g, \tag{2.3}$$

for any $E, F \in \chi(M)$, where λ, k are the cosmological constant and gravitational constant, respectively.

In view of (2.2), equation (2.3) takes the form

$$\begin{aligned} S &= \left[-\lambda + \frac{r}{2} + k\left(\frac{\nu\mathcal{H}}{2} + \rho\right)\right] g \\ &\quad + k(\nu\mathcal{H} + \sigma + \rho)\mathcal{A} \otimes \mathcal{A} - k\nu\mathcal{B} \otimes \mathcal{B}. \end{aligned} \tag{2.4}$$

3. Characteristics of relativistic magneto fluid spacetime with anti-torqued vector field

Let (M^4, g) be a relativistic magneto fluid spacetime (briefly $RMFS$) satisfying (2.4). Contracting the equation (2.4) provides

$$r = 4\lambda - k[\nu(\mathcal{H} - 1) + 3\rho - \sigma]. \tag{3.5}$$

Using the above equation in (2.4), we have

$$\begin{aligned} S(E, F) &= \left(\lambda + \frac{k}{2}(\nu + \sigma - \rho)\right) g(E, F) + k(\nu\mathcal{H} + \sigma + \rho)\mathcal{A}(E)\mathcal{A}(F) \\ &\quad - k\nu\mathcal{B}(E)\mathcal{B}(F), \end{aligned} \tag{3.6}$$

which also implies

$$QE = aE + b\mathcal{A}(E)\zeta + c\mathcal{B}(E)\xi, \tag{3.7}$$

where $a = \lambda + \frac{k}{2}(\nu + \sigma - \rho), b = k(\nu\mathcal{H} + \sigma + \rho), c = -k\nu$.

We consider the special case when ζ is an anti-torqued vector field [7] of the form:

$$\nabla_E \zeta = f(E - \mathcal{A}(E)\zeta), \tag{3.8}$$

for a vector field E on M^4, where $\mathcal{A}$ is one form dual to unit anti-torqued vector field and f is a non-zero smooth function.

Theorem 3.1. *On a $RMFS$ with an anti-torqued vector field ζ, the following relations hold:*

$$(\nabla_X \mathcal{A})(E) = f[g(E,F) - \mathcal{A}(E)\mathcal{A}(F)], \tag{3.9}$$

$$\mathcal{A}(\nabla_\zeta \zeta) = 2, \qquad \nabla_\zeta \zeta = 2\zeta, \tag{3.10}$$

$$R(E,F)\zeta = f^2[\mathcal{A}(E)F - \mathcal{A}(F)E] + E(f)[F - \mathcal{A}(F)\zeta] - F(f)[E - \mathcal{A}(E)\zeta], \tag{3.11}$$

$$R(E,\zeta)\zeta = f^2[E + \mathcal{A}(E)\zeta] + 2E(f)\zeta - \zeta(f)[E - \mathcal{A}(E)\zeta], \tag{3.12}$$

$$\begin{aligned}\mathcal{A}(R(E,F)D) &= f^2[\mathcal{A}(F)g(X,D) - \mathcal{A}(E)g(F,D)] - E(f)[g(F,D) - \mathcal{A}(F)\mathcal{A}(D)] \\ &\quad + F(f)[g(E,D) - \mathcal{A}(E)\mathcal{A}(D)],\end{aligned} \tag{3.13}$$

$$(\pounds_\zeta g)(E,F) = 2f[g(E,F) - \mathcal{A}(E)\mathcal{A}(F)]. \tag{3.14}$$

Proof. Compute $(\nabla_E \mathcal{A})(F) = E(\mathcal{A}(F)) - \mathcal{A}(\nabla_E F) = E(g(F,\zeta)) - g(\nabla_E F, \zeta) = g(F, \nabla_E \zeta) = f[g(E,F) - \mathcal{A}(E)\mathcal{A}(F)]$. Specifically, $(\nabla_\zeta \mathcal{A})E = 0$. The relation (3.9) can be obtained by (3.8).

Now, utilizing (3.8) in $R(E,F)\zeta = \nabla_E \nabla_F \zeta - \nabla_F \nabla_E \zeta - \nabla_{[E,F]}\zeta$ and from direct computation we obtain the relation (3.11). Additionally (3.12) and (3.13) follows from (3.11). Now differentiating g along ζ, then by simple calculation we get (3.14). □

4. Conformal Ricci soliton in a $RMFS$

This section is devoted to studying the conformal Ricci soliton in the context of $RMFS$. Conformal Ricci solitons, which are defined as [4]:

$$\pounds_V g + 2S + \left[2\Lambda - \left(\pi + \frac{2}{n}\right)\right] g = 0, \tag{4.15}$$

where S, π, Λ are the Ricci tensor, the conformal pressure, a constant respectively and $\pounds_V$ is the Lie-derivative operator along the vector field V on spacetime. The conformal Ricci soliton becomes shrinking (resp. steady, expanding) for $\Lambda < 0$ (resp. $\Lambda = 0$, $\Lambda > 0$).

Taking ζ instead of V in (4.15) and then using (3.14) yields

$$S(E,F) = -\left[\Lambda - \frac{1}{2}\left(\pi + \frac{1}{2}\right) + f\right] g(E,F) + f\mathcal{A}(E)\mathcal{A}(F).$$

Making use of (2.4) in the above equation, we obtain

$$\begin{aligned}&\left[-\lambda + \frac{r}{2} + k\left(\frac{\nu\mathcal{H}}{2} + \rho\right)\right] g(E,F) + k(\nu\mathcal{H} + \sigma + \rho)\mathcal{A}(E)\mathcal{A}(F) \\ &-k\nu\mathcal{B}(E)\mathcal{B}(F) = -\left[\Lambda - \frac{1}{2}\left(\pi + \frac{1}{2}\right) + f\right] g(E,F) + f\mathcal{A}(E)\mathcal{A}(F).\end{aligned}$$

Setting $E = F = \zeta$ in the foregoing equation and then making use of (3.5) yields

$$\Lambda = -\lambda + k\nu\left(\mathcal{H} - \frac{1}{2}\right) + \frac{3}{2}k\rho + \frac{k\sigma}{2} + \frac{\pi}{2} - 2f + \frac{1}{4}. \tag{4.16}$$

Theorem 4.1. *A RMFS with anti-torqued vector field ζ admitting a conformal Ricci soliton is shrinking, steady, or expanding accordingly cosmological constant $\lambda \lesseqgtr k\nu\left(\mathcal{H} - \frac{1}{2}\right) + \frac{3}{2}k\rho + \frac{k\sigma}{2} + \frac{\pi}{2} - 2f + \frac{1}{4}$ respectively.*

Let us consider a spacetime in the absence of a cosmological constant i.e. $\lambda = 0$. Then it yields $S(\zeta,\zeta) = \frac{k}{2}[\nu(2\mathcal{H} - 1) + \sigma + 3\rho]$. If the characteristic vector field is timelike then in a spacetime $S(\zeta,\zeta) > 0$, which implies $\nu(2\mathcal{H} - 1) + \sigma + 3\rho > 0$, the spacetime obeys the cosmic strong force condition.
In view of the above converse and Eq. (4.16), we can state the following theorem.

Theorem 4.2. *A RMFS with anti-torqued vector field ζ admitting a conformal Ricci soliton which satisfies timelike convergence condition in the absence of a cosmological constant is expanding.*

5. Conformal $\mathcal{A}$-Ricci Soliton in a *RMFS*

Consider the equation

$$\pounds_V g + 2S + \left[2\Lambda - \left(\pi + \frac{2}{n}\right)\right]g + 2\Omega\mathcal{A}\otimes\mathcal{A} = 0, \tag{5.17}$$

where Λ, Ω are real constants and π, S are same as defined in (4.15). The quadruple $(g, \zeta, \Lambda, \Omega)$ which satisfy the equation (5.17) is said to be a conformal $\mathcal{A}$-Ricci soliton in M [21]. In particular if $\Omega = 0$, then it reduces to a conformal Ricci soliton [4] and it becomes shrinking (resp. steady, expanding) for $\Lambda < 0$ (resp. $\Lambda = 0$, $\Lambda > 0$) [9].
Writing the Lie derivative $\pounds_\zeta g$ explicitly, we have $\pounds_\zeta g = g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta)$. Then (5.17) takes the form

$$S(E,F) = -\left[\Lambda - \frac{1}{2}\left(\pi + \frac{1}{2}\right)\right]g(E,F) - \Omega\mathcal{A}(E)\mathcal{A}(F) - \frac{1}{2}[g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta)], \tag{5.18}$$

for any $E, F \in \chi(M^4)$.
From (2.4) and (5.18), we have

$$\left[\lambda + \frac{k}{2}(\nu + \sigma - \rho) + \Lambda - \frac{1}{2}\left(\pi + \frac{1}{2}\right)\right]g(E,F) + [k(\nu\mathcal{H} + \sigma + \rho) + \Omega]\mathcal{A}(E)\mathcal{A}(F)$$
$$-k\nu\mathcal{B}(E)\mathcal{B}(F) + \frac{1}{2}[g(\nabla_E\zeta, F) + g(E, \nabla_F\zeta)] = 0. \tag{5.19}$$

Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field and $\zeta = \sum_{i=1}^{4} \zeta^i e_i$. We have $\sum_{i=1}^{4} \epsilon_{ii}(\zeta^i)^2 = -1$ and multiplying (5.19) by ϵ_{ii} and summing over i for $E = F = e_i$, we obtain

$$4\Lambda - \Omega = -4\lambda + k[\nu(\mathcal{H} - 1) - \sigma + 3\rho) + 2\pi + 1 - div\zeta. \tag{5.20}$$

Plugging $E = F = \zeta$ in (5.19), we obtain

$$\Lambda - \Omega = -\lambda + \frac{k}{2}[\nu(2\mathcal{H} - 1) + \sigma + 3\rho] + \frac{1}{2}\left(\pi + \frac{1}{2}\right). \tag{5.21}$$

On solving (5.20) and (5.21), we have

$$\Lambda = -\lambda - \frac{k}{2}\left(\frac{\nu}{3} + \sigma - \rho\right) + \frac{\pi}{2} + \frac{1}{4} - \frac{div\zeta}{3},$$

$$\Omega = -k\left[\nu\left(\mathcal{H} - \frac{1}{3}\right) + \sigma + \rho\right] - \frac{div\zeta}{3}.$$

Thus, we have the following theorem:

Theorem 5.1. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and let $\mathcal{A}$ be the g-dual 1-form of the gradient vector field $\zeta = grad(\phi)$ with $g(\zeta,\zeta) = -1$. If (5.17) define a conformal $\mathcal{A}$-Ricci soliton in M^4, then the Laplacian equation satisfied by ϕ becomes*

$$\Delta(\phi) = -3\Omega - k\left[\nu\left(\mathcal{H} - \frac{1}{3}\right) + \sigma + \rho\right].$$

Remark 5.1. *If $\Omega = 0$ in (5.17), then we obtain the conformal Ricci soliton with $\Lambda = -\lambda + k\left[\nu\left(\mathcal{H} + \frac{1}{6}\right) + \frac{\sigma+\rho}{2}\right] + \frac{1}{2}\left(\pi + \frac{1}{2}\right)$, which is expanding, steady, or shrinking accordingly*

$$\lambda \underset{>}{\overset{\leq}{=}} k\left[\nu\left(\mathcal{H} + \frac{1}{6}\right) + \frac{\sigma + \rho}{2}\right] + \frac{1}{2}\left(\pi + \frac{1}{2}\right)$$

respectively.

6. Conformal Einstein Soliton in a $RMFS$

Consider the equation

$$\pounds_V g + 2S + \left[2\Lambda - r + \left(\pi + \frac{2}{n}\right)\right] g = 0, \tag{6.22}$$

where $g, \xi, \Lambda, S, r, \mathcal{A}$ are same as defined in (4.15) and π is a scalar non-dynamical field. The triplet (g, ζ, Λ) which satisfy the equation (6.22) is said to be a conformal Einstein soliton in M [18]. It is called shrinking (resp. steady or expanding) for $\Lambda < 0$ (resp. $\Lambda = 0$ or $\Lambda > 0$). Taking ζ instead of V in (6.22) and then making use of (3.14) yields

$$S(E, F) = -\left[\Lambda - \frac{r}{2} + \frac{1}{2}\left(\pi + \frac{1}{2}\right) + f\right] g(E, F) + f\mathcal{A}(E)\mathcal{A}(F).$$

Utilizing (2.4) in the foregoing equation, one can easily obtain

$$\left[-\lambda+\frac{r}{2}+k\left(\frac{\nu\mathcal{H}}{2}+\rho\right)\right]g(E,F)+k(\nu\mathcal{H}+\sigma+\rho)\mathcal{A}(E)\mathcal{A}(F)-k\nu\mathcal{B}(E)\mathcal{B}(F)$$
$$=-\left[\Lambda-\frac{r}{2}+\frac{1}{2}\left(\pi+\frac{1}{2}\right)+f\right]g(E,F)+f\mathcal{A}(E)\mathcal{A}(F).$$

Setting $E=F=\zeta$ in the above equation provides

$$\Lambda=\lambda+k\left(\frac{\nu\mathcal{H}}{2}+\sigma\right)-\frac{\pi}{2}-2f-\frac{1}{4}.$$

Theorem 6.1. *A RMFS with anti-torqued vector field ζ admitting a conformal Einstein soliton is shrinking, steady, or expanding accordingly cosmological constant $\lambda \gtreqless +\frac{\pi}{2}+2f+\frac{1}{4}-k\left(\frac{\nu\mathcal{H}}{2}+\sigma\right)$ respectively.*

7. Conformal $\mathcal{A}$-Einstein Soliton in a $RMFS$

Consider the equation

$$\pounds_V g+2S+\left[2\Lambda-r+\left(\pi+\frac{2}{n}\right)\right]g+2\Omega\mathcal{A}\otimes\mathcal{A}=0, \tag{7.23}$$

where Λ, Ω are real constants and r,π,S are same as defined in (6.22). The quadruple (g,ζ,Λ,Ω) which satisfy the equation (7.23) is said to be a conformal $\mathcal{A}$-Einstein soliton in M. In particular if $\Omega=0$, (g,ζ,Λ) is a conformal Einstein soliton [18] and it becomes shrinking (resp. steady, expanding) for $\Lambda<0$ (resp. $\Lambda=0$, $\Lambda>0$) [9].

Writing the Lie derivative $\pounds_\zeta g$ explicitly, we have $\pounds_\zeta g=g(\nabla_E\zeta,F)+g(E,\nabla_F\zeta)$ and from (7.23) we obtain:

$$S(E,F)=-\left[\Lambda-\frac{r}{2}+\frac{1}{2}\left(\pi+\frac{1}{2}\right)\right]g(E,F)-\Omega\mathcal{A}(E)\mathcal{A}(F)-\frac{1}{2}[g(\nabla_E\zeta,F)+g(E,\nabla_F\zeta)], \tag{7.24}$$

for any $E,F\in\chi(M^4)$.

From (2.4) and (7.24), we have

$$\left[-\lambda+k\left(\frac{\nu\mathcal{H}}{2}+\rho\right)+\Lambda-\frac{1}{2}\left(\pi+\frac{1}{2}\right)\right]g(E,F)+[k(\nu\mathcal{H}+\sigma+\rho)+\Omega]\mathcal{A}(E)\mathcal{A}(F)$$
$$-k\nu\mathcal{B}(E)\mathcal{B}(F)+\frac{1}{2}[g(\nabla_E\zeta,F)+g(E,\nabla_F\zeta)]=0. \tag{7.25}$$

Consider $\{e_i\}_{1\leq i\leq 4}$ an orthonormal frame field and $\zeta=\sum_{i=1}^4\zeta^i e_i$. We have $\sum_{i=1}^4\epsilon_{ii}(\zeta^i)^2=-1$ and multiplying (7.25) by ϵ_{ii} and summing over i for $X=Y=e_i$, we obtain

$$4\Lambda-\Omega=4\lambda+k(\nu\mathcal{H}+3\rho+\nu+\sigma)+2\pi+1-div\zeta. \tag{7.26}$$

Plugging $E = F = \zeta$ in (7.25), we obtain

$$\Lambda - \Omega = \lambda + k(\frac{\nu\mathcal{H}}{2} + \sigma) + \left(\frac{\pi}{2} + \frac{1}{4}\right). \tag{7.27}$$

On solving (7.26) and (7.27), we have

$$\Lambda = \lambda + k\left[\frac{\nu\mathcal{H}}{6} + \rho + \frac{\nu}{3}\right] + \frac{\pi}{2} + \frac{1}{4} - \frac{div\zeta}{3},$$

$$\Omega = -k\left(\frac{\nu\mathcal{H}}{3} - \sigma + \rho - \frac{\nu}{3}\right) - \frac{div\zeta}{3}.$$

Thus, we have the following theorem:

Theorem 7.1. *Let (M^4, g) be a 4-dimensional pseudo-Riemannaian manifold and let $\mathcal{A}$ be the g-dual 1-form of the gradient vector field $\zeta = grad(\phi)$ with $g(\zeta,\zeta) = -1$. If (7.23) define a conformal $\mathcal{A}$-Einstein soliton in M^4, then the Laplacian equation satisfied by ϕ becomes*

$$\Delta(\phi) = -3\left[\Omega + k\left(\frac{\nu\mathcal{H}}{3} - \sigma + \rho - \frac{\nu}{3}\right)\right].$$

Remark 7.1. *If $\Omega = 0$ in (7.23), then we obtain the conformal Ricci soliton with* $\Lambda = \lambda + k\left[\frac{\nu\mathcal{H}}{2} - \sigma + 2\rho\right] + \frac{\pi}{2} + \frac{1}{4}$, *which is expanding, steady or shrinking accordingly*

$$\lambda \begin{matrix} \geq \\ = \\ < \end{matrix} - k\left[\frac{\nu\mathcal{H}}{2} - \sigma + 2\rho\right] - \frac{\pi}{2} - \frac{1}{4}$$

respectively.

8. Conclusion

In the framework of general relativity, the energy-momentum tensor T fundamentally characterizes the matter distribution within spacetime. Conventional cosmological models typically represent the universe's matter content as a perfect fluid within a 4-dimensional Lorentzian manifold. Within this paradigm, Einstein's field equations serve as the foundational tool for constructing viable cosmological models.

Relativistic magneto-fluid spacetime (RMFS) models hold particular significance across multiple disciplines, including astrophysics, nuclear physics, and plasma physics. Recent investigations have revealed that geometric flows provide powerful tools for characterizing the intrinsic structures of RMFS. Of special interest are soliton solutions - those metric configurations evolving through dilations and diffeomorphisms, which emerge naturally in the singularity analysis of these flows. These self-similar solutions find applications not only in physics but also in chemistry, biology, and economics (see [20], [27], [28]).

This work systematically examines various classes of solitons in RMFS endowed with an anti-torqued vector field. We establish precise conditions under which these solitons exhibit expanding, steady, or shrinking behavior. Furthermore, we derive the Laplace equation for such RMFS configurations admitting conformal $\mathcal{A}$-Ricci and $\mathcal{A}$-Einstein solitons.

The investigation of conformal solitons gains additional importance from the remarkable similarity between conformal Ricci flow equations and the Navier-Stokes equations of fluid dynamics. In this correspondence, the time-dependent scalar field p functions as a conformal pressure - distinct from conventional fluid pressure that preserves incompressibility, as it directly influences metric deformation under the flow.

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References

[1] Ahsan, Z. (2017). Tensors: Mathematics of Differential Geometry and Relativity. Delhi: PHI Learning Pvt. Ltd.

[2] Ali, M., & Ahsan, Z. (2014). Ricci solitons and symmetries of space-time manifold of general relativity. Journal of Advanced Research in Classical and Modern Geometry, 1(2), 75–84.

[3] Almia, P., & Upreti, J. (2023). Certain properties of η-Ricci soliton on η-Einstein para-Kenmotsu manifolds. Filomat, 37(28), 9575–9585.

[4] Basu, N., & Bhattacharyya, A. (2015). Conformal Ricci soliton in Kenmotsu manifold. Global Journal of Advanced Research in Classical and Modern Geometry, 4, 15–21.

[5] Blaga, A. M. (2020). Solitons and geometrical structures in a perfect fluid spacetime. Rocky Mountain Journal of Mathematics. https://projecteuclid.org/euclid.rmim

[6] Catino, G., & Mazzieri, L. (2016). Gradient Einstein solitons. Nonlinear Analysis, 132, 66–94.

[7] Chen, B. Y. (2017). Classification of torqued vector fields and its applications to Ricci solitons. Kragujevac Journal of Mathematics, 41, 239–250.

[8] Fischer, A. E. (2004). An introduction to conformal Ricci flow. Classical and Quantum Gravity, 21, S171–S218.

[9] Hamilton, R. S. (1988). The Ricci flow on surfaces. In Mathematics and General Relativity (Santa Cruz, CA, 1986) (Contemporary Mathematics, Vol. 71, pp. 237–262). American Mathematical Society.

[10] Haseeb, A., & Khan, M. A. (2022). Conformal π-Ricci-Yamabe solitons within the framework of ϵ-LP-Sasakian 3-manifolds. Advances in Mathematical Physics, 2, 1–8.

[11] Kaigorodov, V. R. (1983). The curvature structure of spacetime. Problems of Geometry, 14, 177–204.

[12] O'Neill, B. (1983). Semi-Riemannian geometry with applications to relativity (Pure and Applied Mathematics). New York: Academic Press.

[13] Novello, M., & Rebouças, M. J. (1978). The stability of a rotating universe. Astrophysical Journal, 225, 719–724.

[14] Pandey, S., Almia, P., & Upreti, J. (2025). Investigation of $*$-Yamabe conformal soliton on LP-Kenmotsu manifolds. International Journal of Maps in Mathematics, 8(1), 106–124.

[15] Praveena, M. M., Bagewadi, C. S., & Siddesha, M. S. (2022). Solitons of Kählerian Norden space-time manifolds. Communications of the Korean Mathematical Society, 37(3), 813–824.

[16] Praveena, M. M., Bagewadi, C. S., & Krishnamurthy, M. R. (2021). Solitons of Kählerian space-time manifolds. International Journal of Geometric Methods in Modern Physics, 18, Article 2150021. https://doi.org/10.1142/S0219887821500213

[17] Siddesha, M. S., Praveena, M. M., & Madhumohana, R. A. B. (2025). Kählerian Norden space-time manifolds and Ricci-Yamabe solitons. Palestine Journal of Mathematics, 14(1), 968–976.

[18] Roy, S., Dey, S., & Bhattacharyya, A. (2021). Conformal Einstein soliton within the framework of para-Kähler manifolds. Differential Geometry - Dynamical Systems, 23, 235–243.

[19] Roy, S., Dey, S., & Unal, B. (2025). Characterizations of relativistic magneto-fluid spacetimes admitting Einstein soliton. Filomat, 39(4), 1235–1245.

[20] Sandhu, R. S., Geojou, T. T., & Tamenbaun, A. R. (2016). Ricci curvature: An economic indicator for market fragility and system risk. Scientific Advances, 1501495.

[21] Siddiqi, M. D. (2018). Conformal η-Ricci solitons in δ-Lorentzian trans-sasakian manifold. International Journal of Maps in Mathematics, 1, 15–34.

[22] Siddiqi, M. D., & Akyol, M. A. (2020). π-Ricci-Yamabe soliton on Riemannian submersions from Riemannian manifolds. arXiv:2004.14124. https://doi.org/10.48550/arXiv.2004.14124

[23] Siddiqi, M. D., & Siddiqi, S. A. (2020). Conformal Ricci soliton and geometrical structure in a perfect fluid space-time. International Journal of Geometric Methods in Modern Physics, 17, Article 2050083. https://doi.org/10.1142/S0219887820500838

[24] Siddiqi, M. D., & De, U. C. (2021). Relativistic magneto-fluid spacetimes. Journal of Geometry and Physics, 170, 104370.

[25] Stephani, H. (1982). General relativity: An introduction to the theory of gravitational field. Cambridge: Cambridge University Press.

[26] Venkatesha, & Kumara, H. A. (2019). Ricci solitons and geometrical structure in a perfect fluid space-time with Torse-forming vector field. Afrika Matematika, 30, 725–736.

[27] Ivancevic, V. G., & Ivancevic, T. T. (2011). Ricci flow and nonlinear reaction diffusion systems in biology, chemistry, and physics. Nonlinear Dynamics, 65, 35–54.

[28] Graf, W. (2007). Ricci Flow Gravity. PMC Physics A, 1–13.

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ON GEOMETRY OF THE PARALLEL SURFACE OF THE TUBE SURFACE GIVEN BY THE FLC FRAME IN EUCLIDEAN 3-SPACE

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Abstract. In this study, first, the parallel surfaces of the tube surfaces given with the Flc frame are defined. By calculating the Gaussian and mean curvatures of these parallel surfaces, it was found the conditions developable and minimal. Afterwards, the conditions for parameter curves on the parallel surface to be asymptotic, geodesic and curvature lines were investigated. It has been proven that the tube and parallel tube surface preserve the Gaussian transform. Finally, examples of these surfaces are given.

Keywords: Parallel surface, Tubular surface, Flc frame, Gaussian transform.

2020 Mathematics Subject Classification: 53A05, 53A10.

1. Introduction

It is known that two surfaces with a common normal are called parallel surfaces. Parallel surfaces have various uses in the design field and in the modeling of forging casting molds [25]. It has been one of the surfaces that has been the focus of attention of many mathematicians from past to present,[22, 8, 9, 10, 1, 11]. A large number of papers and books have been published in the literature which deal with parallel surfaces in both Minkowski space and Euclidean space. Kılıç showed that if a parallel transformation on E^n is a connection-preserving transformation, the fundamental curvatures of the underlying surface are constant [14]. Taleshian used Euler's theorem to examine the orthogonal curvatures of parallel hypersurfaces and stated that if the parallel transformation preserves the second fundamental form,

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the fundamental hypersurface defines a hyperplane [23]. Fukui and Hasegawa studied the singularities of parallel surfaces [12]. Önder and Kızıltuğ gave the relations between Bertrand and Mannheim partner D-curves on parallel surfaces in 3-dimensional Minkowski space [19]. Dede, Ekici and Çöken first defined parallel surfaces in Galilean space and examined the relationship between them, and then obtained the first, second fundamental forms and Gaussian, mean curvatures of the parallel surface depending on the first, second fundamental forms and Gaussian, mean curvatures of the main surface [5]. Savcı studied the relationship between the Darboux frame, geodesic curvatures, normal curvatures, and geodesic torsions of the curves lying on the parallel surface pair, showed that the parallel surface of a non-developable ruled surface is not a ruled surface, and obtained that the parallel surface of a Weingarten ruled surface is also a ruled Weingarten surface [20].

Craig worked on parallel surfaces of the ellipsoid [2]. Eisenhart wrote a section on parallel surfaces in his work “A treatise on the differential geometry of curves and surfaces” [7]. Nizamoğlu stated that the parallel ruled surface is a curve that depends on a parameter and gave some geometrical properties of such a surface [18]. Hacısalihoğlu and Tarakcı defined surfaces with constant ridge distance and showed that a parallel surface is a special case of a surface with constant ridge distance [24]. Again, Hacısalihoğlu and Yaşar studied the parallel surface of a hypersurface in Lorentz space and obtained new characterizations [27]. Çöken, Çiftçi and Ekici worked on parallel surfaces of timelike ruled surfaces [3]. Dae Won Yoon studied parallel Weingarten surfaces in Euclidean space and showed that for a surface to be a Weingarten surface, it is necessary and sufficient that its parallel surface is also a Weingarten surface [28]. In recent years, Kızıltuğ has taken a curve on a surface and obtained the image of this curve on a parallel surface and examined the characteristic features of this curve on the parallel surface [15, 16, 17]. Ünlütürk and Özüsağlam showed that the image of a curve that is geodesic on M by normal transformation in Minkowski 3-space on the parallel surface M_r is also a geodesic [26].

Given any curve in three-dimensional Euclidean space, an orthonormal vector system called the Frenet frame can be established at every point of this curve. The Frenet frame defines the curvature and torsion functions of the curve that characterize the curve. However, the disadvantage of this frame is that the Frenet frame cannot be established at points where the second derivative of the curve is zero. With the Flc frame defined by Dede in 2019, the singular points occurring in the second derivative of the curve were eliminated and a new frame was established. This shows that the Flc frame can be established along the curve, including the points where the Frenet frame cannot be established. Thus, the deformation on the surfaces created by taking this frame as a reference was also minimized, [4].

In this study, the parallel surfaces of tube surfaces defined using the Flc frame are first introduced. The Gaussian and mean curvatures of these parallel surfaces are calculated to determine the conditions under which they are developable or minimal. Next, the criteria for the parameter curves on the parallel surfaces to be asymptotic, geodesic, or curvature lines are analyzed. It is also demonstrated that both the tube surface and its parallel surfaces preserve the Gaussian transform. Finally, examples of these surfaces are provided.

2. Preliminaries

In this section, we remind some basic concepts that will be used throughout the paper. Let $\lambda = \lambda(t)$ be a regular space curve satisfying non-degenerate condition $\lambda'(t) \wedge \lambda''(t) \neq 0$. Then, the orthonormal vector system called Frenet frame is defined by

$$T(t) = \frac{\lambda'(t)}{\|\lambda'(t)\|}, \;\; B(t) = \frac{\lambda'(t) \wedge \lambda''(t)}{\|\lambda'(t) \wedge \lambda''(t)\|}, \;\; N(t) = B(t) \wedge T(t)$$

where T is tangent, N is principal normal, and B is binormal vector field. The Frenet formulas are given by

$$T' = \kappa\eta N, \;\; N' = -\kappa\eta T + \tau\eta B, \;\; B' = -\tau\eta N \;, \;\; \left\|\lambda'\right\| = \eta$$

where the curvature κ and torsion τ of the curve are, [4]

$$\kappa = \frac{\left\|\lambda'(t) \wedge \lambda''(t)\right\|}{\|\lambda'(t)\|^3}, \qquad \tau = \frac{\left\langle \lambda'(t) \wedge \lambda''(t), \lambda'''(t) \right\rangle}{\|\lambda'(t) \wedge \lambda''(t)\|^2}.$$

The n^{th} degree polynomial with parameter t is defined as

$$P(t) = \lambda_n t^n + \lambda_{n-1} t^{n-1} + ... + \lambda_1 t^1 + \lambda_0, \qquad \lambda_n \neq 0$$

where $n \in \mathbb{N}_0$, $\lambda_i \in \mathbb{R}$, $(0 \leq i \leq n)$, [4]. Now let us define a curve such that, $\lambda : [a, b] \to E^n, \quad \lambda(t) = (\lambda_1(t), \lambda_2(t), ..., \lambda_n(t))$. If each $\lambda_i(t)$ are polynomials for $1 \leq i \leq n$, then $\lambda_t \in \mathbb{R}[s]$ is defined to be an $n-$dimensional polynomial curve [4]. The degree of such a polynomial curve as $\lambda(t)$ is given by

$$\deg \lambda(t) = max \left\{ \deg \left(\lambda_1(t)\right), \deg \left(\lambda_2(t)\right), ..., \deg \left(\lambda_n(t)\right) \right\}.$$

The definition of the Flc frame of a polynomial space curve $\lambda = \lambda(t)$ given by Dede in [4] is as follows

$$T(t) = \frac{\lambda'(t)}{\|\lambda'(t)\|}, \quad D_1(t) = \frac{\lambda'(t) \wedge \lambda^{(n)}(t)}{\left\|\lambda'(t) \wedge \lambda^{(n)}(t)\right\|}, \quad D_2(t) = D_1(t) \wedge T(t)$$

where the prime $'$ indicates the differentiation with respect to s and $^{(n)}$ stands for the n^{th} derivative. The new vectors D_1 and D_2 are called binormal-like vector and normal-like vector, respectively. The curvatures of the Flc-frame d_1, d_2, and d_3 are given by

$$d_1 = \frac{\langle T', D_2\rangle}{\eta}, \quad d_2 = \frac{\langle T', D_1\rangle}{\eta}, \quad d_3 = \frac{\langle D_2', D_1\rangle}{\eta}$$

where $\|\lambda'\| = \eta$. The local rate of change of the Flc-frame called as the Frenet-like formulas can be expressed in the following form

$$\begin{bmatrix} T' \\ D_2' \\ D_1' \end{bmatrix} = \eta \begin{bmatrix} 0 & d_1 & d_2 \\ -d_1 & 0 & d_3 \\ -d_2 & -d_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix}.$$

The relationship between the Frenet and Frenet like frame (Flc) is given by

$$\begin{bmatrix} T \\ D_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

and the relations between the curvatures of two frames are

$$d_1 = \kappa cos\theta, \qquad d_2 = -\kappa sin\theta, \qquad \theta = arctan\left(-\frac{d_2}{d_1}\right), \qquad d_3 = \frac{d\theta}{\eta} + \tau$$

where $\theta = \sphericalangle(N, D_2)$. Let E^3 be a 3-dimensional Euclidean space provided with the metric given by

$$< X, X >= dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E^3. Recall that, the norm of an arbitrary vector $X \in E^3$ is given by $\|X\| = \sqrt{< X, X >}$, [13]. The parametric equation of a parallel surface is given as: Let M_1 and M_2 be two surfaces in 3-dimensional Euclidean space and the unit normal vector field of M_1 be Z. If there is a function f defined as

$$f : M_1 \longrightarrow M_2, f(P) = P + rZ_p$$

where r is a constant number, then the surfaces M_1 and M_2 are called parallel surfaces. Given the surface M,

$$M_r = \{P + rZ_p : P \in M, r \in \mathbf{R} \text{ and r=constant } \}$$

the set M_r given by the equation is a surface parallel to M. The normal vector field of the surface M_r is computed as

$$N_{M_r}(t,\theta) = \frac{M_{r_t} \wedge M_{r_\theta}}{\|M_{r_t} \wedge M_{r_\theta}\|}.$$

In addition, the first and second fundamental forms of the surface M_r are given by

$$I = Edt^2 + 2Fdtd\theta + Gd\theta^2,$$

$$II = Ldt^2 + 2Mdtd\theta + Nd\theta^2$$

while the Gaussian and mean curvatures are

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{EN - 2FM + GL}{2(EG - F^2)}$$

where the coefficients are found by following:

$$E =< M_{r_t}, M_{r_t} >, \quad F =< M_{r_t}, M_{r_\theta} >, \quad G =< M_{r_\theta}, M_{r_\theta} >,$$

$$L =< M_{r_{tt}}, N_{M_r} >, \quad N =< M_{r_{t\theta}}, N_{M_r} >, \quad M =< M_{r_{\theta\theta}}, N_{M_r} > .$$

Concerning the Gaussian and mean curvatures, the following definitions exist

- A surface is said to be developable and has parabolic points if the Gaussian curvature vanishes,
- A surface is said to have hyperbolic (resp. elliptic) points, if it has a negative (resp. positive) Gaussian curvature,
- A surface is said to be minimal if the mean curvature vanishes, [6].

3. On geometry of the parallel surface of the tube surface given by the Flc frame in Euclidean 3-space

Let $M(t)$ be a polynomial space curve of degree n. We can parametrize a tubular surface generated by an Flc-frame as follows

$$K(t,\theta) = M\,(t) + r\,[\cos\theta D_2(t) + \sin\theta D_1(t)] \tag{3.1}$$

where $\theta \in [0, 2\pi)$, $r \in \mathbb{R}$ is the radius of the tubular surface and the curve $M(t)$ is the center curve of the tubular surface, [4]. The derivatives according to parameters t and θ of the tubular surface $K(t,\theta)$ are, respectively,

$$K_t = \nu(1 - r(\cos\theta d_1 + \sin\theta d_2))T - \nu r \sin\theta d_3 D_2 + \nu r \cos\theta d_3 D_1,$$
$$K_\theta = -r\sin\theta D_2 + r\cos\theta D_1.$$

The normal vector field of the tubular surface $K(t,\theta)$ is obtained as

$$N(t,\theta) = cos\theta D_2 + sin\theta D_1. \tag{3.2}$$

If the parallel surface of the tube surface $K(t,\theta)$ is represented by $K_P(t,\theta)$, the equation of this surface is defined as

$$K_P(t,\theta) = K(t,\theta) + \varepsilon N(t,\theta).$$

If the expressions (3.1) and (3.2) are written here, the expression of the parallel surface $K_P(t,\theta)$ with respect to the Flc frame becomes,

$$\begin{aligned} K_p(t,\theta) &= K(t,\theta) + \varepsilon N(t,\theta) \\ &= M(t) + (r+\varepsilon)\left[\cos\theta D_2(t) + \sin\theta D_1(t)\right]. \end{aligned}$$

If the first order partial derivatives of the surface $K_p(t,\theta)$ are taken with respect to the parameters t and θ

$$\begin{aligned} K_{p_t} =& \nu(1 - (r+\varepsilon)(\cos\theta d_1 + \sin\theta d_2))T - \nu(r+\varepsilon)\sin\theta d_3 D_2 \\ &+ \nu(r+\varepsilon)\cos\theta d_3 D_1, \\ K_{p_\theta} =& -(r+\varepsilon)\sin\theta D_2 + (r+\varepsilon)\cos\theta D_1, \end{aligned}$$

is found. Here the unit normal vector of the surface is

$$N_p(t,\theta) = \frac{K_{p_t} \wedge K_{p_\theta}}{\|K_{p_t} \wedge K_{p_\theta}\|} = cos\theta D_2 + sin\theta D_1.$$

The coefficients of the first fundamental form of the surface are as follows

$$\begin{aligned} E_p &=< K_{p_t}, K_{p_t} >= \nu^2\left[1 - (r+\varepsilon)(\cos\theta d_1 + \sin\theta d_2)\right]^2 + \nu^2(r+\varepsilon)^2 {d_3}^2, \\ F_p &< K_{p_t}, K_{p_\theta} >= \nu(r+\varepsilon)^2 d_3, \\ G_p &=< K_{p_\theta}, K_{p_\theta} >= (r+\varepsilon)^2. \end{aligned} \tag{3.3}$$

The second-order partial derivatives of the surface $K_p(t,\theta)$ are as follows:

$$\begin{aligned} K_{p_{tt}} =& [\nu^2(r+\varepsilon)d_3(sin\theta d_1 - cos\theta d_2) - \nu(r+\varepsilon)(cos\theta d_1' + sin\theta d_2') \\ & - \nu'(r+\varepsilon)(cos\theta d_1 + sin\theta d_2) + \nu']T \\ & - [\nu^2(r+\varepsilon)cos\theta(d_1^2+d_3^2) + \nu^2(r+\varepsilon)d_1d_2 sin\theta + (r+\varepsilon)sin\theta(vd_3)' - \nu^2 d_1]D_2 \\ & - [\nu^2(r+\varepsilon)sin\theta(d_2^2+d_3^2) + \nu^2(r+\varepsilon)d_1d_2 cos\theta + (r+\varepsilon)cos\theta(vd_3)' - \nu^2 d_2]D_1, \end{aligned} \tag{3.4}$$

$$\begin{aligned} K_{p_{t\theta}} =& \nu(r+\varepsilon)(sin\theta d_1 - cos\theta d_2)T - \nu(r+\varepsilon)cos\theta d_3 D_2 - \nu(r+\varepsilon)sin\theta d_3 D_1, \\ K_{p_{\theta\theta}} =& -(r+\varepsilon)cos\theta D_2 - (r+\varepsilon)sin\theta D_1. \end{aligned} \tag{3.5}$$

The coefficients of the first fundamental form of the surface are written as follows

$$\begin{aligned} e_p = < K_{p_{tt}}, N_p > =& \nu^2(d_1cos\theta + d_2sin\theta) - \nu^2(r+\varepsilon)(d_1cos\theta + d_2sin\theta)^2 \\ & - \nu^2(r+\varepsilon)d_3^2, \end{aligned} \tag{3.6}$$

$$f_p = < K_{p_{t\theta}}, N_p > = -\nu(r+\varepsilon)d_3, \tag{3.7}$$

$$g_p = < K_{p_{\theta\theta}}, N_p > = -(r+\varepsilon). \tag{3.8}$$

With the help of these expressions, the Gaussian curvature $\mathbb{K}_p$ and the mean curvature $\mathbb{H}_p$ of the parallel surface $K_p(t,\theta)$ are written as follows, respectively:

$$\mathbb{K}_p = \frac{-cos\theta d_1 - sin\theta d_2}{(r+\varepsilon)[1-(r+\varepsilon)(cos\theta d_1 + sin\theta d_2)]},$$

$$\mathbb{H}_p = \frac{1-2(r+\varepsilon)(cos\theta d_1 + sin\theta d_2)}{2(r+\varepsilon)[1-(r+\varepsilon)(cos\theta d_1 + sin\theta d_2)]}.$$

Theorem 3.1. *Singular points of the parallel surface $K_p(t,\theta)$ satisfy the equation*

$$cos\theta_0 d_1 + sin\theta_0 d_2 = \frac{1}{r+\varepsilon}.$$

Proof. For the parallel surface $K_p(t,\theta)$ to have singular points at the point (t_0,θ_0) ,

$$\|K_{p_t} \wedge K_{p_\theta}\|(t_0,\theta_0) = 0.$$

If the necessary operations are carried out from here, the following is obtained:

$$\|K_{p_t} \wedge K_{p_\theta}\|(t_0, \theta_0) = 0 \Rightarrow \nu(r+\varepsilon)\left(\cos\theta d_1(r+\varepsilon) + \sin\theta d_2(r+\varepsilon) - 1\right) = 0$$

$$\Rightarrow (r+\varepsilon)cos\theta_0 d_1 + (r+\varepsilon)sin\theta_0 d_2 = 1$$

$$\Rightarrow cos\theta_0 d_1 + sin\theta_0 d_2 = \frac{1}{r+\varepsilon}.$$

□

Corollary 3.1. *In particular, if $\theta_0 = 0$ is taken, then $d_1 = \frac{1}{r+\varepsilon}$. In this case, the locus of singular points of the surface is a curve of the form*

$$K_p(t, 0) = M(t) + (r+\varepsilon)D_2(t).$$

Corollary 3.2. *If $\theta_0 = \frac{\pi}{2}$ or $\theta_0 = \frac{3\pi}{2}$ is taken, then $d_2 = \frac{1}{r+\varepsilon}$. In this case, the geometric locus of the singular points of the surface is a curve of the form*

$$K_p(t, \frac{\pi}{2}) = M(t) + (r+\varepsilon)D_1(t),$$

$$K_p(t, \frac{3\pi}{2}) = M(t) - (r+\varepsilon)D_1(t).$$

Theorem 3.2. *For $K_p(t,\theta)$ parallel surface:*

(i) *t parametric curves are asymptotic if and only if*

$$(r+\varepsilon)d_3^2 + (r+\varepsilon)(cos\theta d_1 + sin\theta d_2)^2 = cos\theta d_1 + sin\theta d_2.$$

(ii) *The parameter curves θ are not asymptotic curves.*

Proof. (i) For the parameter curves of the parallel surface $K_p(t,\theta)$ to be asymptotic curves, it is necessary and sufficient that $e_p = 0$. From the equation (3.6) we find:

$$e_p = 0 \Rightarrow \nu^2(d_1 cos\theta + d_2 sin\theta) - \nu^2(r+\varepsilon)(d_1 cos\theta + d_2 sin\theta)^2 - \nu^2(r+\varepsilon)d_3^2 = 0$$

$$\Rightarrow (r+\varepsilon)d_3^2 + (r+\varepsilon)(d_1 cos\theta + d_2 sin\theta)^2 = d_1 cos\theta + d_2 sin\theta.$$

(ii) For the θ parameter curves of the parallel surface $K_p(t,\theta)$ to be asymptotic curves, the necessary and sufficient condition is that $g_p = 0$. From the equation (3.8), since $g_p = -r - \varepsilon$ and $r, \varepsilon \neq 0$, the θ parameter curves cannot be asymptotic.

□

Theorem 3.3. *For $K_p(t,\theta)$ parallel surface:*

(i) *t parametric curves are geodesic if and only if*

$$v^2 d_1 d_2 (r+\varepsilon) cos2\theta - v^2 cos\theta sin\theta (r+\varepsilon)(d_1^2 - d_2^2)$$
$$- v^2 (cos\theta d_2 - sin\theta d_1) - (r+\varepsilon)(v d_3)' = 0,$$
$$\big(cos\theta + sin\theta\big)\Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$
$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) = 0.$$

(ii) *The θ parameter curves are always geodesic.*

Proof. (i) The necessary and sufficient condition for the parameter curves t of the parallel surface $K_p(t,\theta)$ to be geodesic curves is that $N_p \wedge K_{p_{tt}} = 0$. From the equations (3.2) and (3.4), the vector $N_p \wedge K_{p_{tt}}$ is given by

$$N_p \wedge K_{p_{tt}} = \Big(v^2 d_1 d_2 (r+\varepsilon) cos2\theta - v^2 cos\theta sin\theta (r+\varepsilon)(d_1^2 - d_2^2)$$
$$- v^2 (cos\theta d_2 - sin\theta d_1) - (r+\varepsilon)(v d_3)'\Big) T(t)$$
$$+ sin\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$
$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) D_2(t)$$
$$- cos\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$
$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) D_1(t).$$

For $N_p \wedge K_{p_{tt}} = 0$ the coefficients must be zero. Therefore,

$$v^2 d_1 d_2 (r+\varepsilon) cos2\theta - v^2 cos\theta sin\theta (r+\varepsilon)(d_1^2 - d_2^2)$$
$$- v^2 (cos\theta d_2 - sin\theta d_1) - (r+\varepsilon)(v d_3)' = 0,$$
$$sin\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$
$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) = 0,$$
$$cos\theta \Big(v^2 (r+\varepsilon) d_3 (cos\theta d_2 - sin\theta d_1) + v(r+\varepsilon)(cos\theta d_1' + sin\theta d_2')$$
$$+ (r+\varepsilon) v' (cos\theta d_1 + sin\theta d_2) - v'\Big) = 0.$$

If these equations are arranged, the desired result is obtained.

(ii) The necessary and sufficient condition for the parameter curves of the parallel surface $K_p(t,\theta)$ to be geodesic curves is that $N_p \wedge K_{p_{\theta\theta}} = 0$. Using the equations (3.2) and (3.5), $N_p \wedge K_{p_{\theta\theta}} = 0$ means that the θ parameter curves are always geodesic.

□

Theorem 3.4. *Let the parallel surface $K_p(t,\theta)$ be given. In order for the parameter curves on the surface to be lines of curvature, the necessary and sufficient condition is that $d_3 = 0$.*

Proof. For the parameter curves of the parallel surface $K_p(t,\theta)$ to be lines of curvature, it is necessary and sufficient that $F_p = f_p = 0$. From the equations (3.3) and (3.7) we write

$$\nu(r+\varepsilon)^2 d_3 = 0 \qquad \text{and} \qquad -\nu(r+\varepsilon)d_3 = 0.$$

Here $d_3 = 0$ since $\nu, r \neq 0$. □

In this case, the following result can be given:

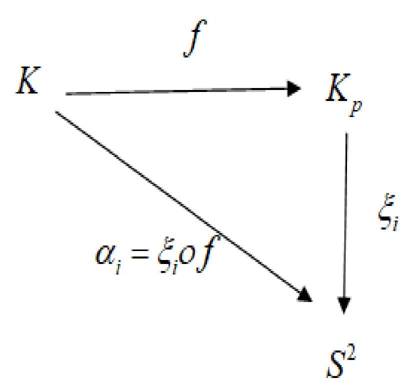
Corollary 3.3. *If the parameter curves t and θ on the parallel surface $K_p(t,\theta)$ are planar, these curves are the curvature lines of the surface.*

Theorem 3.5. *Let (K, K_p) be the pair of parallel surfaces in $\mathbb{E}^3$. There is a relation between the Gaussian transformations,*

$$\eta = \eta_p$$

where the unit normal vectors of the surfaces K and K_p are N and N_p, respectively.

Proof. Let the coordinates of the unit normal vectors K and K_p be $\alpha_i = (\alpha_1, \alpha_2, \alpha_3)$ and $\xi_i = (\xi_1, \xi_2, \xi_3)$, respectively.



$$\eta : K \longrightarrow S^2$$
$$X \longrightarrow \eta(X) = \sum_{i=1}^{3} \alpha_i(X) \frac{\partial}{\partial Y_i|_X}$$

is the Gaussian transform of the surface K. On the other hand, the Gaussian transform of the surface K_p is as follows, where $f : K \longrightarrow K_p$ is the parallel transform:

$$\begin{aligned}
&\eta_p : K_p \longrightarrow S^2 \\
f(X) \longrightarrow \eta_p(f(X)) &= \sum_{i=1}^{n} \xi_i(f(X)) \frac{\partial}{\partial Y_i}|_{f(X)} \\
&= \sum_{i=1}^{n} (\xi_i \circ f(X)) \frac{\partial}{\partial Y_i}|_{f(X)} \\
&= \sum_{i=1}^{n} \alpha_i(X) \frac{\partial}{\partial Y_i}|_{f(X)} \\
&= \eta_p(X)
\end{aligned}$$

Since the Gaussian transformation will be provided for $\forall X \in K$, $\eta = \eta_p$ is obtained. □

Example 3.1. *Let $M : I \longrightarrow \mathbb{R}^3$ be a polynomial curve with center curve $M(t) = (t, t^8, t^9)$. If the derivatives of the curve $M(t)$ are calculated, it is as follows:*

$$\begin{aligned}
M'(t) &= (1, 8t^7, 9t^8), \\
M''(t) &= (0, 56t^6, 72t^7), \\
M^{(9)}(t) &= (0, 0, 362880).
\end{aligned}$$

The Flc frame vectors of the polynomial curve $M(t)$ are found as follows, respectively:

$$\begin{aligned}
T(t) &= \frac{M'(t)}{\|M'(t)\|} = \Big(\frac{1}{\sqrt{81t^{16} + 64t^{14} + 1}}, \frac{8t^7}{\sqrt{81t^{16} + 64t^{14} + 1}}, \frac{9t^8}{\sqrt{81t^{16} + 64t^{14} + 1}} \Big), \\
D_1(t) &= \frac{M'(t) \times M^{(9)}(t)}{\|M'(t) \times M^{(9)}(t)\|} = \Big(\frac{8t^7}{\sqrt{64t^{14} + 1}}, -\frac{1}{\sqrt{64t^{14} + 1}}, 0 \Big), \\
D_2(t) &= T(t) \times D_1(t) \\
&= \Big(-\frac{9t^8}{\sqrt{64t^{14} + 1}\sqrt{81t^{16} + 64t^{14} + 1}}, -\frac{72t^{15}}{\sqrt{64t^{14} + 1}\sqrt{81t^{16} + 64t^{14} + 1}}, \\
&\quad \frac{\sqrt{64t^{14} + 1}}{\sqrt{81t^{16} + 64t^{14} + 1}} \Big).
\end{aligned}$$

On the other hand, Flc curvatures are as follows:

$$d_1(t) = \frac{\langle T'(t), D_2(t)\rangle}{\|M'(t)\|} = \frac{72t^7\left(8t^{14}+1\right)}{\sqrt{64\,s^{14}+1}\,(81t^{16}+64t^{14}+1)^{3/2}},$$

$$d_2(t) = \frac{\langle T'(t), D_1(t)\rangle}{\|M'(t)\|} = -\frac{56t^6}{\sqrt{64\,t^{14}+1}\,(81\,t^{16}+64\,t^{14}+1)},$$

$$d_3(t) = \frac{\left\langle D_2(t)', D_1(t)\right\rangle}{\|M'(t)\|} = \frac{504t^{14}}{(64\,t^{14}+1)\,(81\,t^{16}+64\,t^{14}+1)}.$$

If the radius $r = 0.25$ *is taken, the parametric equation of the tube surface* $K(t,\theta)$ *is as follows:*$(-1 \leq t \leq 1,\ \ -\pi \leq \theta \leq \pi)$

$$K(t,\theta) = \Big(t - \frac{9t^8\cos\theta}{4\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} + \frac{2t^7\sin\theta}{\sqrt{64t^{14}+1}},$$

$$t^8 - \frac{18t^{15}\cos\theta}{\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} - \frac{\sin\theta}{4\sqrt{64t^{14}+1}},$$

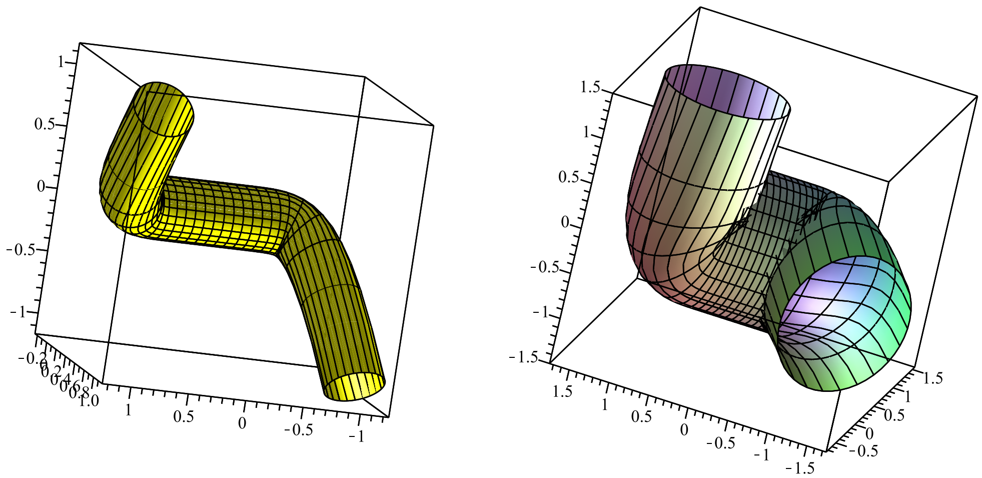
$$t^9 + \frac{\cos\theta\sqrt{64t^{14}+1}}{4\sqrt{81t^{16}+64t^{14}+1}}\Big).$$

If $\epsilon = 0.5$ *is taken, the equation of the parallel surface* $K_p(t,\theta)$ *is as follows:* $(-1 \leq t \leq 1,\ \ -\pi \leq \theta \leq \pi)$

$$K_p(t,\theta) = \Big(t - \frac{27t^8\cos\theta}{4\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} + \frac{6t^7\sin\theta}{\sqrt{64t^{14}+1}},$$

$$t^8 - \frac{54t^{15}\cos\theta}{\sqrt{64t^{14}+1}\sqrt{81t^{16}+64t^{14}+1}} - \frac{3\sin\theta}{4\sqrt{64t^{14}+1}},$$

$$t^9 + \frac{3\cos\theta\sqrt{64t^{14}+1}}{4\sqrt{81t^{16}+64t^{14}+1}}\Big).$$



(A) $K(t,\theta)$ tube surface

(B) $K_p(t,\theta)$ parallel surface

4. Conclusion

In this study, first of all, the parallel surfaces of the tube surface given with the Flc frame were defined. It was seen that the surface created by investigating the geometric features of this parallel surface was developable and minimal. The parameter curves of the parallel surface were examine. Subsequently, the tube surface and parallel surface were shown to preserve the Gaussian transform. Finally, the tube surface, which accepts a polynomial curve as its center curve, and the parallel surface of this tube surface, are given as an example, and are shown. This work can be studied in various spaces such as Minkowski space and Galilean space, and can also be repeated for higher-dimensional curves.

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References

[1] Akyiğit M., Eren K., & Kösal H.H. (2021). Tubular Surfaces with Modified Orthogonal Frame, Honam Mathematical Journal, 43(3), 453-463.

[2] Craig, T. (1883). Note on parallel surfaces. Journal Für Die Reine und Angewandte Mathematik (Crelle's journal), 94, 162-170.

[3] Çöken, A.C., Çiftci, U., & Ekici, C. (2008). On parallel timelike ruled surfaces with timelike rulings. Kuwait Journal of Science and Engineering, 35, 21-31.

[4] Dede, M. (2019). A new representation of tubular surfaces. Houston J. Math., 45, 707-720.

[5] Dede, M., Ekici, C., & Çöken, A. C. (2013). On the parallel surfaces in Galilean Space. Hacettepe Journal of Mathematics on Statistics, 42(6), 605-615.

[6] Do-Carmo, P. (1976). Differential geometry of curves and surfaces, IMPA.

[7] Eisenhart, L.P. (1909). A Treatise On The Differential Geometry Of Curves and Surfaces. Ginn and Company, Boston:New York.

[8] Eren, K., Ayvacı K. H., & S. Şenyurt.(2022). On Characterizations of Spherical Curves Using Frenet Like Curve Frame, Honam Mathematical Journal, 44, no.3 (September 1,): 391–401.

[9] Eren, K., Ayvacı K. H., & S. Senyurt.(2023). On Ruled Surfaces Constructed by the Evolution of a Polynomial Space Curve, J. of Science and Arts, 23(1),77-90.

[10] Eren K., Yıldız, Ö.G., & Akyiğit, M. (2022). Tubular surfaces associated with framed base curves in Euclidean 3-space, Math Meth App Sci., 45(18), 12110-12118.

[11] Eren K. (2021). On the Harmonic Evolute Surfaces of Tubular Surfaces in Euclidean 3-Space, Journal of Science and Arts, 2(55),449-460.

[12] Fukui, T., & Hasegawa, M. (2012). Singularities of parallel surfaces, arXix:1203.3715v1 [math.DG].

[13] Goetz, A. (1968). Introduction to Differential Geometry. Addison-Wesley Publishing Company, Canada.

[14] Kılıç, A. (1993). On the principal curvatures of parallel hypersurfaces in E^n. Erciyes Üniversitesi Fen Bilimleri Dergisi, 10, 1-4.

[15] Kızıltuğ, S. (2013). Paralel Yüzeyler ve Eğriler. Doktora tezi, Atatürk Ünviversitesi Fen Bilimleri Enstitüsü, Erzurum.

[16] Kızıltuğ, S., & Yaylı, Y. (2012). Timelike Curves on Timelike Parallel Surfaces in Minkowski 3-space. Mathematica Aeterna, 2(8), 675-687.

[17] Kızıltuğ, S., & Yaylı, Y. (2013). Spacelike Curves on Spacelike Parallel Surfaces in Minkowski 3-space. International Journal of Mathematics and Computation, 19, 0974-5718.

[18] Nizamoğlu, S. (1986). Surfaces regles parallles. Ege Univ. Fen Fak. Derg., 9 (Ser. A), 37-48.

[19] Önder, M., & Kızıltuğ, S. (2012). Bertrand and Mannheim partner D-curves on parallel surfaces in Minkowski 3-space. International Journal of Geometry, 1(2), 34-45.

[20] Savcı, Ü. Z. (2011). 3-boyutlu Öklid uzayında Paralel Regle Weingarten Yüzeyler üzerine. Doktora Tezi. Eskişehir Osmangazi Üniversitesi Fen Bilimleri Ensitüsü, 71s.

[21] Şenyurt, S., & Ayvacı, K. H. (2022). On geometry of focal surfaces due to Flc frame in Euclidean 3-space. Authorea. https://doi.org/10.22541/au.166816461.17575964/v1

[22] Şenyurt S., Eren K., & Ayvacı K. H.(2022). A Study on Inextensible Flows of Polynomial Curves with Flc Frame, Applications and Applied Mathematics: An International Journal (AAM), 17(1),123-133,.

[23] Taleshian, A. (2007). Ortogonal curvature of parallel hypersurfaces. Int. J. Contemp. Math. Sciences, 2(26), 1257-1262.

[24] Tarakcı, Ö. (2002). Sabit Sırt Uzaklıklı Hiperyüzeyler Doktora tezi Ankara Üniv. Fen Bilimleri Ens., Ankara.

[25] Ünlütürk, Y., & Ekici, C. (2013). On parallel surfaces of timelike ruled weingarten surfaces. Balkan Journal of Mathematics, 1, 72-91.

[26] Ünlütürk, Y., & Özüsağlam, E. (2012). On Parallel Surfaces in Minkowski 3-Space. TWMS. J. App. Eng. Math., 3(2), 214-22.

[27] Yaşar, A. (2011). Lorentz Uzayında Bir Hiperyüzeyin Yüksek Mertebeden Gausian Eğrilikleri. Yüksek lisans tezi. Ankara Üniversitesi Fen Bilimleri Enstitüsü, Ankara.

[28] Yoon, D.W. (2008). Some Properties of Parallel Surfaces in Euclidean 3-Spaces. Honam Mathematical J., 30(4), 637-644.

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ENERGY OF INDU-BALA PRODUCT OF GRAPHS

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ABSTRACT. The energy of a graph Γ is defined as the sum of the absolute values of its eigenvalues. In this article, we compute the energy of the Indu-Bala product of two regular graphs and establish bounds for its energy. Furthermore, we explore the concepts of equienergetic, borderenergetic, orderenergetic, and non-hyperenergetic graphs using the Indu-Bala product of two regular graphs.

Keywords: Graph energy, Indu-Bala product, Equienergetic graphs, Complement of a graph.

2020 Mathematics Subject Classification: 05C50, 05C76.

1. INTRODUCTION

Let Γ be a simple graph of order n. The degree of a vertex u_i, denoted by d_i, is defined as the number of edges incident to it. A graph Γ is said to be r-regular if and only if each vertex of Γ has degree r. The eigenvalues of the graph Γ of order n are the eigenvalues of its adjacency matrix $A(\Gamma)$, denoted by $\lambda_1, \lambda_2, \ldots, \lambda_n$.

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Let n^0, n^- and n^+ denote the number of zero, negative and positive eigenvalues of the graph Γ, respectively. The energy of a graph Γ is defined as

$$E(\Gamma) = \sum_{j=1}^{n} \mid \lambda_j \mid .$$

The line graph $L(\Gamma)$ of a graph Γ is defined as the graph whose vertex set corresponds to the edge set of Γ, where two vertices in $L(\Gamma)$ are adjacent if and only if their corresponding edges in Γ share a common vertex. The i^{th} iterated line graph of Γ, denoted by $L^i(\Gamma)$ for $i = 1, 2, \ldots$, is defined recursively as $L^i(\Gamma) = L(L^{i-1}(\Gamma))$, with $L^0(\Gamma) = \Gamma$ and $L^1(\Gamma) = L(\Gamma)$.

The concept of graph energy, which originated from Hückel molecular orbital theory, was first introduced by Gutman [6]. If two graphs of the same order have the same energy, they are called equienergetic graphs. If the energy of a graph is equal to the number of vertices n, then the graph is said to be orderenergetic [1]. If $E(\Gamma) \leq 2(n-1)$, then the graph is said to be non-hyperenergetic [17] and if $E(\Gamma) = 2(n-1)$, then Γ is said to be borderenergetic [5]. In the literature, there are various research articles that focus on equienergetic graphs. For recent papers, see [10, 11, 12, 13, 14].

Graph products such as the Cartesian product, tensor product, strong product and their corresponding energies have been well studied in the literature [2, 4, 9, 12, 14, 18]. The distance spectrum, adjacency spectrum, distance Laplacian spectrum and distance signless Laplacian spectrum of another product namely, the Indu-Bala product have been investigated in [7, 8, 16]. However, the energy of the Indu-Bala product has not yet been examined. Therefore, in this paper, we study the energy of the Indu-Bala product, which contributes to the construction of non-regular equienergetic graphs. For undefined terminology and results related to the graph spectra, we follow [3].

Definition 1.1 (Indu–Bala product). [7] *The Indu–Bala product of two graphs* Γ_1 *and* Γ_2*, denoted by* $\Gamma_1 \blacktriangledown \Gamma_2$*, is defined as follows: Let* $\Gamma_1 \vee \Gamma_2$ *denote the join of* Γ_1 *and* Γ_2*, where* $V(\Gamma_1) = \{w_1, w_2, \ldots, w_{n_1}\}$ *and* $V(\Gamma_2) = \{z_1, z_2, \ldots, z_{n_2}\}$*. Take a disjoint copy of* $\Gamma_1 \vee \Gamma_2$*, denoted by* $\Gamma_1' \vee \Gamma_2'$*, with vertex sets* $V(\Gamma_1') = \{w_1', w_2', \ldots, w_{n_1}'\}$ *and* $V(\Gamma_2') = \{z_1', z_2', \ldots, z_{n_2}'\}$*. Finally, add edges between each vertex* $z_i \in V(\Gamma_2)$ *and its corresponding copy* $z_i' \in V(\Gamma_2')$*, for all* $i = 1, 2, \ldots, n_2$*.*

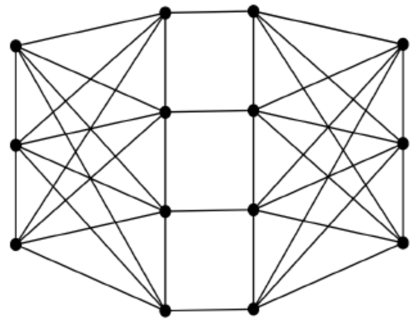

FIGURE 1. The graph $P_3 \blacktriangledown P_4$

Proposition 1.1. [8] *Let Γ_k be an r_k-regular graph of order n_k, for $k = 1, 2$. Then, the spectrum of $\Gamma_1 \blacktriangledown \Gamma_2$ is as follows:*

(a) $\lambda_k(\Gamma_1)$, with multiplicity 2 for $k = 2, 3, \ldots, n_1$;

(b) $\lambda_k(\Gamma_2) + 1$ for $k = 2, 3, \ldots, n_2$;

(c) $\lambda_k(\Gamma_2) - 1$ for $k = 2, 3, \ldots, n_2$;

(d) $\frac{(r_1+r_2+1)\pm\sqrt{(r_1+r_2+1)^2-4(r_1(r_2+1)-n_1n_2)}}{2}$ and $\frac{(r_1+r_2-1)\pm\sqrt{(r_1+r_2-1)^2-4(r_1(r_2-1)-n_1n_2)}}{2}$.

Proposition 1.2. [11] *Let a graph Γ have n vertices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\sum_{k=1}^{n} \mid \lambda_k + 1 \mid = n + E(\Gamma) - 2n^- + 2 \sum_{\lambda_k \in (-1,0)} (\lambda_k + 1).$$

Proposition 1.3. [15] *Let a graph Γ have n vertices with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then*

$$\sum_{k=1}^{n} \mid \lambda_k + 2 \mid = 2n + E(\Gamma) - 4n^- + 2 \sum_{\lambda_k \in (-2,0)} (\lambda_k + 2).$$

2. Energy of Indu-Bala product of graphs

Lemma 2.1. *Let a graph Γ have n vertices with eigenvalues $\lambda_n \leqslant \lambda_{n-1} \leqslant \cdots \leqslant \lambda_1$. Then, for $0 \leqslant p < \lambda_1$,*

$$\sum_{k=1}^{n} \mid \lambda_k - p \mid = E(\Gamma) + np - 2pn^+ - 2 \sum_{\lambda_k \in (0,p)} (\lambda_k - p).$$

Proof. Define $n_\lambda(I)$ as the count of eigenvalues of Γ within the interval I.

Let us compute $\sum\limits_{k=1}^{n} \mid \lambda_k - p \mid$,

$$\begin{aligned}\sum_{k=1}^{n} \mid \lambda_k - p \mid &= \sum_{\lambda_k \leqslant p} (-\lambda_k + p) + \sum_{\lambda_k > p} (\lambda_k - p) \\ &= \sum_{\lambda_k \leqslant p} -\lambda_k + pn_\lambda[\lambda_n, p] + \sum_{\lambda_k > p} \lambda_k - pn_\lambda(p, \lambda_1] \\ &= pn_\lambda[\lambda_n, p] - pn_\lambda(p, \lambda_1] + \sum_{\lambda_k \leqslant 0} \mid \lambda_k \mid + \sum_{\lambda_k \in (0,p]} -\lambda_k \\ &\quad + \sum_{\lambda_k > p} \lambda_k, \end{aligned} \tag{2.1}$$

The $E(\Gamma)$ can be expressed as,

$$E(\Gamma) = \sum_{k=1}^{n} \mid \lambda_k \mid = \sum_{\lambda_k \leq 0} \mid \lambda_k \mid + \sum_{\lambda_k \in (0,p]} \lambda_k + \sum_{\lambda_k > p} \lambda_k \tag{2.2}$$

The order n can be expressed as,

$$n = n_\lambda(0, p] + n_\lambda(p, \lambda_1] + n^0 + n^- \tag{2.3}$$

or,

$$n = n_\lambda[\lambda_n, p] + n_\lambda(p, \lambda_1]. \tag{2.4}$$

By equalities 2.2 and 2.4, equality 2.1 becomes,

$$\begin{aligned}\sum_{k=1}^{n} \mid \lambda_k - p \mid &= p(n - n_\lambda(p, \lambda_1]) - pn_\lambda(p, \lambda_1] + E(\Gamma) - 2 \sum_{\lambda_k \in (0,p]} \lambda_k \\ &= np - 2pn_\lambda(p, \lambda_1] + E(\Gamma) - 2 \sum_{\lambda_k \in (0,p]} \lambda_k \\ &= E(\Gamma) + np - 2pn^+ + 2pn_\lambda(0, p] \\ &\quad -2 \sum_{\lambda_k \in (0,p]} (\lambda_k - p) - 2pn_\lambda(0, p] \quad \text{by the equality 2.3} \\ \sum_{k=1}^{n} \mid \lambda_k - p \mid &= E(\Gamma) + np - 2pn^+ - 2 \sum_{\lambda_k \in (0,p)} (\lambda_k - p). \end{aligned}$$

□

Let ξ be the absolute sum of the eigenvalues mentioned in the case (d) of Proposition 1.1.

Theorem 2.1. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$, then the energy of Indu-Bala product is*

$$\begin{aligned} E(\Gamma_1 \blacktriangledown \Gamma_2) &= 2(E(\Gamma_1) + E(\Gamma_2)) + 2n_2^0 - 2(r_1 + r_2) + 2 \sum_{\lambda_i(\Gamma_2)\in(-1,0)} (\lambda_i(\Gamma_2) + 1) \\ &\quad -2 \sum_{\lambda_i(\Gamma_2)\in(0,1)} (\lambda_i(\Gamma_2) - 1) + \xi. \end{aligned}$$

Proof. Proposition 1.1 provides the eigenvalues of Indu-Bala product of Γ_k; $k = 1, 2$. Therefore,

$$\begin{aligned} E(\Gamma_1 \blacktriangledown \Gamma_2) &= 2\sum_{i=2}^{n_1} | \lambda_i(\Gamma_1) | + \sum_{i=2}^{n_2} | \lambda_i(\Gamma_2) + 1 | + \sum_{i=2}^{n_2} | \lambda_i(\Gamma_2) - 1 | + \xi \\ &= 2\sum_{i=1}^{n_1} | \lambda_i(\Gamma_1) | - 2r_1 + \sum_{i=1}^{n_2} | (\lambda_i(\Gamma_2) + 1) | - (r_2 + 1) \\ &\quad + \sum_{i=1}^{n_2} | (\lambda_i(\Gamma_2) - 1) | - (r_2 - 1) + \xi. \end{aligned}$$

By using Lemma 2.1 and Proposition 1.2, we have

$$\begin{aligned} E(\Gamma_1 \blacktriangledown \Gamma_2) &= 2E(\Gamma_1) - 2r_1 + E(\Gamma_2) + n_2 - 2n_2^- + 2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) - r_2 \\ &\quad -1 + E(\Gamma_2) + n_2 - 2n_2^+ - 2 \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) - (r_2 - 1) + \xi \\ &= 2E(\Gamma_1) - 2r_1 + 2E(\Gamma_2) + 2n_2 - 2(n_2^- + n_2^+) - 2r_2 \\ &\quad +2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) - 2 \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) + \xi \qquad (2.5) \\ &= 2(E(\Gamma_1) + E(\Gamma_2)) - 2(r_1 + r_2) + 2n_2^0 + 2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2) + 1) \\ &\quad -2 \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2) - 1) + \xi. \end{aligned}$$

□

Corollary 2.1. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then,*

$$\begin{aligned} 2E(\Gamma_1) + 2E(\Gamma_2) - 2(r_1 + r_2) + \xi \leqslant E(\Gamma_1 \blacktriangledown \Gamma_2) \\ < 2E(\Gamma_1) + 2E(\Gamma_2) + 2n_2 + \xi. \end{aligned}$$

Equality holds at the left side if and only if there is no eigenvalues in the interval $(-1, 1)$.

Proof. For upper bound, it is observed from the equation 2.5 that

$$n_2^- - \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) > 0 \text{ and } n_2^+ - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) > 0$$

Also, if we can eliminate the values r_1 and r_2 from equation 2.5 as both are positive, we get

$$E(\Gamma_1 \blacktriangledown \Gamma_2) < 2E(\Gamma_1) + 2E(\Gamma_2) + 2n_2 + \xi.$$

For lower bound, it is easy to observe from Theorem 2.1 that

$$\sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) > 0 \text{and} - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) > 0,$$

also $n_2^0 \geq 0$, on removing these values from Theorem 2.1, we obtain,

$$2E(\Gamma_1) + 2E(\Gamma_2) - 2(r_1 + r_2) + \xi < E(\Gamma_1 \blacktriangledown \Gamma_2).$$

The equality on the left side is derived from the following fact,

$$\sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) = 0, \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) = 0 \text{ and } n_2^0 = 0$$

if and only if Γ_2 has no eigenvalues in the interval $(-1, 1)$. □

There are numerous equienergetic graphs with the same regularity and same order, one can find them in the recent articles [11, 12, 13, 14]. With the help of these graphs and Indu-Bala product, one can easily construct non-regular equienergetic graphs.

Corollary 2.2. *Let $H_i; i = 1, 2$ be two r-regular graphs of same order n. Then $H_i \blacktriangledown \Gamma_2; i = 1, 2$ are equienergetic graphs if and only if $H_i; i = 1, 2$ are equienergetic.*

Proof. Proof follows from Theorem 2.1 that $H_i \blacktriangledown \Gamma_2; i = 1, 2$ are equienergetic graphs if and only if

$$\begin{aligned} &2(E(H_1) + E(\Gamma_2)) + 2n_2^0 - 2(r + r_2) + 2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) \\ &- \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) + \xi = 2(E(H_2) + E(\Gamma_2)) + 2n_2^0 - 2(r + r_2) \\ &+2 \sum_{\lambda_k(\Gamma_2)\in(-1,0)} (\lambda_k(\Gamma_2)+1) - \sum_{\lambda_k(\Gamma_2)\in(0,1)} (\lambda_k(\Gamma_2)-1) + \xi. \end{aligned}$$

On both sides, the terms of Γ_2 are common. Therefore,

$$E(H_1) = E(H_2).$$

□

Example 2.1. *The regular graphs* $K_{n,n}\Box K_{n-1}$ *and* $K_{n-1,n-1}\Box K_n$ *are non-isomorphic having the degree* $2n-2$ *and order* $2n^2-2n$*, where* $\Box$ *denotes the Cartesian product. For all* $n\geq 5$ *and* $k\geq 0$*, these graphs* $L^k(K_{n,n}\Box K_{n-1})$ *and* $L^k(K_{n-1,n-1}\Box K_n)$ *are equienergetic* [14]. *By Corollary 2.2,* $L^k(K_{n,n}\Box K_{n-1})\blacktriangledown\Gamma_2$ *and* $L^k(K_{n-1,n-1}\Box K_n)\blacktriangledown\Gamma_2$ *are equienergetic, non-regular graphs.*

The following finding presents a large collection of non-regular equienergetic graphs.

Proposition 2.1. *Let* $H_i; i=1,2$ *be two* $r(\geq 3)$*-regular graphs of same order* n*. Let* Γ_2 *be any graph. Then* $L^k(H_i)\blacktriangledown\Gamma_2; i=1,2$ *are equienergetic graphs.*

Proof. If $H_i; i=1,2$ denote $r(\geq 3)$-regular graphs with order n. Then the graphs $L^k(H_i); i=1,2$ and $k\geq 2$ are equienergetic graphs of same degree by Theorem 4.1 of [13]. Therefore, by this observation and Corollary 2.2 completes the proof. □

Corollary 2.3. *Let the order of an* r_k*-regular graph* Γ_k *be* n_k*, where* $k=1,2$*. Then* $\Gamma_1\blacktriangledown\Gamma_2$ *is non-hyperenergetic if* $E(\Gamma_1)+E(\Gamma_2)\leqslant 2n_1+n_2-1-\frac{\xi}{2}$.

Proof. The two graphs Γ_1 and Γ_2 of order n_1 and n_2 then the order of $\Gamma_1\blacktriangledown\Gamma_2$ is $2(n_1+n_2)$. If $E(\Gamma_1)+E(\Gamma_2)\leqslant 2n_1+n_2-1-\frac{\xi}{2}$, then by Corollary 2.1, we have following

$$E(\Gamma_1\blacktriangledown\Gamma_2)<2(E(\Gamma_1)+E(\Gamma_2))+2n_2+\xi\leqslant 2(2(n_1+n_2)-1).$$

This shows that, the graph $\Gamma_1\blacktriangledown\Gamma_2$ is non-hyperenergetic. □

Corollary 2.4. *Let the order of an* r_k*-regular graph* Γ_k *be* n_k*, where* $k=1,2$*. Then* $\Gamma_1\blacktriangledown\Gamma_2$ *is borderenergetic if and only if*

$$\begin{aligned}E(\Gamma_1)+E(\Gamma_2)=2(n_1+n_2)+(r_1+r_2)-n_2^0-\sum_{\lambda_k(\Gamma_2)\in(-1,0)}(\lambda_k(\Gamma_2)+1)\\+\sum_{\lambda_k(\Gamma_2)\in(0,1)}(\lambda_k(\Gamma_2)-1)-\frac{\xi}{2}-1.\end{aligned}$$

Specifically, if $\lambda_k(\Gamma_2)\notin(-1,1)$*, then* $\Gamma_1\blacktriangledown\Gamma_2$ *is borderenergetic if and only if* $E(\Gamma_1)+E(\Gamma_2)=2(n_1+n_2)+(r_1+r_2)-1-\frac{\xi}{2}$.

Proof. By the definition of borderenergetic graph and Theorem 2.1 together provide the proof. □

Corollary 2.5. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then $\Gamma_1 \blacktriangledown \Gamma_2$ is orderenergetic if and only if*

$$E(\Gamma_1) + E(\Gamma_2) = (n_1 + n_2) + (r_1 + r_2) - n_2^0 - \sum_{\lambda_k(\Gamma_2) \in (-1,0)} (\lambda_k(\Gamma_2) + 1) + \sum_{\lambda_k(\Gamma_2) \in (0,1)} (\lambda_k(\Gamma_2) - 1) - \frac{\xi}{2}.$$

Specifically if $\lambda_k(\Gamma_2) \notin (-1, 1)$ then, $\Gamma_1 \blacktriangledown \Gamma_2$ is orderenergetic if and only if $E(\Gamma_1) + E(\Gamma_2) = 2(n_1 + n_2) + (r_1 + r_2) - \frac{\xi}{2}$.

Proof. By the definition of orderenergetic graph and Theorem 2.1 together provide the proof. □

Theorem 2.2. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then,*

$$\begin{aligned} E(\overline{\Gamma}_1 \blacktriangledown \overline{\Gamma}_2) &= 2(E(\Gamma_1) + E(\Gamma_2)) + 2(n_1 + n_2) - 2(r_1 + r_2) - 4(n_1^- + n_2^-) - 4 \\ &\quad + 4 \sum_{\lambda_k(\Gamma_1) \in (-1,0)} (\lambda_k(\Gamma_1) + 1) + 2 \sum_{\lambda_k(\Gamma_2) \in (-2,0)} (\lambda_k(\Gamma_2) + 2) + \xi_1. \end{aligned}$$

Proof. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ are the eigenvalues of any regular graph Γ, then the eigenvalues of complement of Γ are $n - 1 - \lambda_1, -(\lambda_2 + 1), -(\lambda_3 + 1), \cdots, -(\lambda_n + 1)$. From Proposition 1.1, the eigenvalues of Indu-Bala product $\overline{\Gamma_1} \blacktriangledown \overline{\Gamma_2}$ are as follows:

(a) $-(\lambda_k(\Gamma_1) + 1)$, with multiplicity 2 for $k = 2, 3, \cdots, n_1$;

(b) $-\lambda_k(\Gamma_2)$ for $k = 2, 3, \cdots, n_2$;

(c) $-(\lambda_k(\Gamma_2) + 2)$ for $k = 2, 3, \cdots, n_2$;

(d) $\frac{(n_1+n_2)-(r_1+r_2+1) \pm \sqrt{((n_1+n_2)-(r_1+r_2+1))^2 - 4((n_1-1-r_1)(n_2-r_2)-n_1n_2)}}{2}$ and $\frac{(n_1+n_2)-(r_1+r_2+3) \pm \sqrt{((n_1+n_2)-(r_1+r_2+3))^2 - 4((n_1-1-r_1)(n_2-r_2-2)-n_1n_2)}}{2}$

Here, we denote the absolute sum of the all eigenvalues in the (d) case as ξ_1

$$\begin{aligned} &E(\overline{\Gamma_1} \blacktriangledown \overline{\Gamma_2}) \\ &= 2\sum_{k=2}^{n_1} | -\lambda_k(\Gamma_1) - 1 | + \sum_{k=2}^{n_2} | -\lambda_k(\Gamma_2) | + \sum_{k=2}^{n_2} | -\lambda_k(\Gamma_2) - 2 | + \xi_1 \\ &= 2\sum_{k=1}^{n_1} | \lambda_k(\Gamma_1) + 1 | - 2(r_1 + 1) + \sum_{k=1}^{n_2} | \lambda_k(\Gamma_2) | - r_2 \\ &\quad + \sum_{k=1}^{n_2} | \lambda_k(\Gamma_2) + 2 | - (r_2 + 2) + \xi_1. \end{aligned}$$

Using Propositions 1.2 and 1.3, we get $E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})$

$$\begin{aligned}
&= 2E(\Gamma_1) + 2n_1 - 4n_1^- + 4\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) - 2(r_1+1)\\
&\quad +E(\Gamma_2) - r_2 + E(\Gamma_2) + 2n_2 - 4n_2^-\\
&\quad +2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) - (r_2+2) + \xi_1\\
&= 2E(\Gamma_1) + 2E(\Gamma_2) + 4\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) - 4(n_1^- + n_2^-) - 4\\
&\quad +2(n_1+n_2) - 2(r_1+r_2) + 2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) + \xi_1\\
&\quad = 2(E(\Gamma_1)+E(\Gamma_2)) + 2(n_1+n_2) - 2(r_1+r_2) - 4(n_1^- + n_2^-) - 4\\
&\quad +4\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) + 2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) + \xi_1.
\end{aligned}$$

□

Corollary 2.6. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then,*

$$\begin{aligned}
2(E(\Gamma_1)+E(\Gamma_2)) - 4(n_1^- + n_2^-) - 2(r_1+r_2) - 4 + \xi_1 &\leq E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})\\
&< 2(E(\Gamma_1)+E(\Gamma_2)) + 2(n_1+n_2) + \xi_1.
\end{aligned}$$

Equality holds at the left side if and only if Γ_1 has no eigenvalues in the interval $(-1,0)$ and Γ_2 has no eigenvalues in the interval $(-2,0)$.

Proof. For upper bound, it can be seen from Theorem 2.2

$$n_1^- - \sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) > 0 \text{ and } 2n_2^- - \sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) > 0$$

Along these, if we can eliminate the values 4, r_1, r_2 from $E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})$ in Theorem 2.2 as these are positive, we obtain,

$$E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}) < 2E(\Gamma_1) + 2E(\Gamma_2) + 2(n_1+n_2) + \xi_1.$$

For lower bound,

$$\sum_{\lambda_k(\Gamma_1)\in(-1,0)}(\lambda_k(\Gamma_1)+1) > 0 \text{ and } \sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2) > 0$$

and $n_1, n_2 \geq 0$, on removing these values from Theorem 2.2, we obtain, $2(E(\Gamma_1)+E(\Gamma_2)) - 4(n_1^- + n_2^-) - 2(r_1+r_2) - 4 + \xi_1 < E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2})$.

The equality holds at the left side by the following fact

$$\sum_{\lambda_k(\Gamma_1)\in(-1,0)} (\lambda_k(\Gamma_1)+1) = 0 \text{ and } \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2)+2) = 0$$

if and only if Γ_1 has no eigenvalues in the interval $(-1, 0)$ and Γ_2 has no eigenvalues in the interval $(-2, 0)$. □

Corollary 2.7. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then, $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is non-hyperenergetic if $E(\Gamma_1) + E(\Gamma_2) \leqslant (n_1 + n_2) - 1 - \frac{\xi_1}{2}$.*

Proof. If $\overline{\Gamma_1}$ and $\overline{\Gamma_2}$ are graphs of order n_1 and n_2, then order of $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is $2(n_1 + n_2)$. If $E(\Gamma_1) + E(\Gamma_2) \leqslant (n_1 + n_2) - 1 - \frac{\xi_1}{2}$ and by Corollary 2.6, we have the following equation.

$$\text{i.e. } E(\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}) < 2(E(\Gamma_1) + E(\Gamma_2)) + 2(n_1 + n_2) + \xi_1 \leqslant 2(2(n_1 + n_2) - 1).$$

This shows that, the graph $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is non-hyperenergetic. □

Corollary 2.8. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then, $\overline{G_1}\blacktriangledown\overline{\Gamma_2}$ is borderenergetic if and only if*

$$E(\Gamma_1) + E(\Gamma_2) = (n_1 + n_2) + 2(n_1^- + n_2^-) + (r_1 + r_2) + 1$$
$$-2\sum_{\lambda_k(\Gamma_1)\in(-1,0)} (\lambda_k(\Gamma_1)+1) - \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2)+2) - \frac{\xi_1}{2}$$

Specifically, if Γ_1 and Γ_2 contains no eigenvalues in the interval $(-1, 0)$ and $(-2, 0)$ respectively, then $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is borderenergetic if and only if $E(\Gamma_1) + E(\Gamma_2) = 2(n_1 + n_2) + 2(n_1^- + n_2^-) + (r_1 + r_2) + 1 - \frac{\xi_1}{2}$.

Proof. The definition of borderenergetic and Theorem 2.2 together provide the proof. □

Corollary 2.9. *Let the order of an r_k-regular graph Γ_k be n_k, where $k = 1, 2$. Then $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is orderenergetic if and only if*

$$E(\Gamma_1) + E(\Gamma_2) = 2(n_1^- + n_2^-) + (r_1 + r_2) + 2$$
$$-2\sum_{\lambda_k(\Gamma_1)\in(-1,0)} (\lambda_k(\Gamma_1)+1) - \sum_{\lambda_k(\Gamma_2)\in(-2,0)} (\lambda_k(\Gamma_2)+2) - \frac{\xi_1}{2}.$$

Specifically, if Γ_1 and Γ_2 contains no eigenvalues in the intervals $(-1, 0)$ and $(-2, 0)$ respectively, then $\overline{\Gamma_1}\blacktriangledown\overline{\Gamma_2}$ is orderenergetic if and only if $E(\Gamma_1)+E(\Gamma_2) = 2(n_1+n_2)+2(n_1^- + n_2^-)+(r_1 + r_2) - \frac{\xi_1}{2}$.

Proof. The definition of orderenergetic and Theorem 2.2 together provide the proof. □

Corollary 2.10. *Let the order of an r-regular graph H_k; $k = 1,2$ be n, with no eigenvalues in the interval $(-1,0)$. Then $\overline{H_k}\blacktriangledown\overline{\Gamma_2}; k = 1,2$ are equienergetic graphs if and only if $H_k; k = 1,2$ are equienergetic with same number of negative eigenvalues.*

Proof. Proof follows from Theorem 2.2 that $\overline{H_i}\blacktriangledown\overline{\Gamma_2}; i = 1,2$ are equienergetic graphs if and only if

$$\begin{aligned}
&2(E(H_1)+E(\Gamma_2))+2(n_1+n_2)-2(r+r_2)-4(n_1^-+n_2^-)-4\\
&+4\sum_{\lambda_k(H_1)\in(-1,0)}(\lambda_k(H_1)+1)+2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2)+\xi_1 \quad = \quad 2(E(H_2)+E(\Gamma_2))\\
&\qquad +2(n_1+n_2)-2(r+r_2)-4(n_1^{*-}+n_2^-)-4\\
&+4\sum_{\lambda_k(H_2)\in(-1,0)}(\lambda_k(H_2)+1)+2\sum_{\lambda_k(\Gamma_2)\in(-2,0)}(\lambda_k(\Gamma_2)+2)+\xi_1.
\end{aligned}$$

Here, n_1^{*-} denotes the number negative eigenvalues in H_2.
On both sides, the terms of Γ_2 are common and also, H_1 and H_2 have same regularity. Therefore,

$$E(H_1)-2n_1^- = E(H_2)-2n_1^{*-}.$$

□

Example 2.2. *Let us take the graphs in Example 2.1. These are integral graphs, which means no eigenvalues in $(-1,0)$. These graphs posses same count of negative eigenvalues. Therefore, by Corollary 2.10, $\overline{L^k(K_{n,n}\square K_{n-1})}\blacktriangledown\overline{\Gamma_2}$ and $\overline{L^k(K_{n-1,n-1}\square K_n)}\blacktriangledown\overline{\Gamma_2}$ are equienergetic graphs.*

The following finding presents another large collection of non-regular equienergetic graphs.

Proposition 2.2. *Let the order of an $r(\geq 3)$-regular graph H_k; $k = 1,2$ be n and Γ_2 be any graph. Then $\overline{L^k(H_i)}\blacktriangledown\overline{\Gamma_2}$; $i = 1,2$ and $k \geq 2$ are equienergetic graphs.*

Proof. If $H_i; i = 1,2$ denote $r(\geq 3)$-regular graphs with order n. Then by Theorem 4.1 of [13], the graphs $L^k(H_i); i = 1,2$ and $k \geq 2$ are equienergetic graphs of same degree. In addition these have all negative eigenvalues equal to -2. Therefore, by this observation and Corollary 2.10 completes the proof. □

3. Conclusion

In this paper, we calculate the energy of the Indu-Bala product two regular graphs. Furthermore, we investigate the properties such as equienergetic, borderenergetic, orderenergetic and non-hyperenergetic characteristics using the Indu-Bala product. Further, one can study the Indu-Bala product of two non-regular graphs.

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References

[1] Akbari, S., Ghahremani, M., Gutman, I. and Koorepazan–Moftakhar, F. (2020). Orderenergetic graphs. MATCH Communications in Mathematical and in Computer Chemistry, 84, 325–334. https://doi.org/10.2298/aadm201227016a.

[2] Bonifácio, A. S., Vinagre, C. T. M., & de Abreu, N. M. M. (2008). Constructing pairs of equienergetic and non-cospectral graphs. Applied Mathematics Letters, 21(4), 338–341. https://doi.org/10.1016/j.aml.2007.04.002.

[3] Cvetković, D., Rowlinson, P. & Simić, S. (2009). An Introduction to the Theory of Graph Spectra, Cambridge University Press, Cambridge.

[4] Germina, K. A., Shahul H. & Thomas, Z. (2011). On products and line graphs of signed graphs, their eigenvalues and energy. Linear Algebra and its Applications, 435(10), 2432–2450 https://doi.org/10.1016/j.laa.2010.10.026.

[5] Gong, S., Li, X., Xu, G., Gutman, I. & Furtula, B. (2015). Borderenergetic graphs. MATCH Communications in Mathematical and in Computer Chemistry, 74, 321–332.

[6] Gutman, I. (1978). The energy of a graph. Ber. Math. Stat. Sekt. Forschungsz. Graz., 103, 1–22.

[7] Indulal, G. & Balakrishnan, R. (2016). Distance spectrum of Indu–Bala product of graphs. AKCE International Journal of Graphs and Combinatorics, 13, 230–234. https://doi.org/10.1016/j.akcej.2016.06.012.

[8] Patil, S. & Mathapati, M. (2019). Spectra of Indu–Bala product of graphs and some new pairs of cospectral graphs. Discrete Mathematics, Algorithms and Applications, 11(5), 1–9. https://doi.org/10.1142/S1793830919500563.

[9] Ramane, H. S., Ashoka, K., Parvathalu, B. & Patil, D. (2021). On A-energy and S-energy of certain class of graphs. Acta Universitatis Sapientiae, Mathematica, 13, 195–219. https://doi.org10.2478/ausi-2021-0009.

[10] Ramane, H. S., Parvathalu, B. & Ashoka, K. (2022). Energy of strong double graphs. Journal of Analysis, 30, 1033–1043. https://doi.org/10.1007/s41478-022-00391-4.

[11] Ramane, H. S., Parvathalu, B. & Ashoka, K. (2019). Energy of extended bipartite double graphs. MATCH Communications in Mathematical and in Computer Chemistry, 87, 653–660. (2022). https://doi.org/10.46793/match.87-3.653.

[12] Ramane, H. S., Parvathalu, B., Patil, D. & Ashoka, K. (2019). Graphs equienergetic with their complements. MATCH Communications in Mathematical and in Computer Chemistry, 82(2), 471–480.

[13] Ramane, H. S., Parvathalu B., Patil, D. & Ashoka, K., Iterated line graphs with only negative eigenvalues −2, their complements and energy. Available at arXiv: https://doi.org/10.48550/arXiv.2205.02276.

[14] Ramane, H. S., Patil, D., Ashoka, K. & Parvathalu, B. (2021). Equienergetic graphs using cartesian product and generalized composition. Sarajevo Journal of Mathematics, 17(1), 7–21.

[15] Ramane, H. S., Parvathalu, B., Ashoka, K. & Pirzada, S. (2024). On families of graphs which are both adjacency equienergetic and distance equienergetic. Indian Journal of Pure and Applied Mathematics, 55, 198–209. https://doi.org/10.1007/s13226-022-00355-1.

[16] Subarsha, B. (2022). The spectrum & metric dimension of Indu–Bala product of graphs. Discrete Mathematics, Algorithms and Applications, 14. https://doi.org/10.1142/S1793830922500379.

[17] Walikar, H. B., Gutman, I., Hampiholi, P. R. & Ramane, H. S. (2001). Non–hyperenergetic graphs. Graph Theory Notes New York, 41, 14–16.

[18] Veninstine, V. J., Xavier, P. & Afzala, R. J. (2022). Energy of Cartesian product graph networks. Przegl Adelektrotechniczny, 98, 28–33. https://doi.org/10.15199/48.2022.08.06.

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GE-ALGEBRAS WITH NORMS

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Abstract. In this paper, we introduce and study the concept of normed GE-algebras, an extension of GE-algebras equipped with a GE-norm, which provides a framework to measure the magnitude of algebraic elements. We define the magnitude function and explore its properties in the context of GE-algebras. Through theorems and propositions, we examine the behavior of sequences in these normed structures, demonstrating convergence properties, quasi-metrics, and the relationship between norms and algebraic operations. We also establish the connection between normed GE-algebras and their product spaces, as well as the implications for convergent sequences and limit uniqueness. Finally, we generalize these results to mappings between normed GE-algebras and investigate the implications of GE-morphisms in preserving convergence behavior.

Keywords: GE-norm, Normed GE-algebra, Magnitude, Convergent, Limit.
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1. Introduction

In the 1950s, Hilbert algebras were introduced by L. Henkin and T. Skolem as a means to investigate non-classical logics, particularly intuitionistic logic. As demonstrated by A. Diego, these algebras belong to the category of locally finite varieties, a fact highlighted in [6]. Over time, a community of scholars developed the theory of Hilbert algebras, as evidenced by works such as [4, 5, 7]. In the broader scope of algebraic structures, the process of generalization is of utmost importance. Y. B. Jun et al. introduced the concept of

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BH-algebras as a generalization of BCH/BCI/BCK-algebras and investigated its important properties in [9]. R. H. Abass introduced the notions of norm and distance in BH-algebras and given some basic properties in normed BH-algebras in [1].

The introduction of GE-algebras, proposed by R. K. Bandaru et al. as an extension of Hilbert algebras, marked a significant step in this direction. This advancement led to the examination of various properties, as explored in [2]. The evolution of GE-algebras was greatly influenced by filter theory. In light of this, R. K. Bandaru et al. introduced the concept of belligerent GE-filters in GE-algebras, closely investigating its attributes as documented in [3]. Generalized algebraic structures, such as GE-algebras, offer a broad framework to study a variety of algebraic and topological properties.

The concept of norms has a rich history in mathematics, originating in the study of vector spaces and Banach algebras, where norms quantify the size of elements and induce metric spaces [14]. In logical algebras, norms have been adapted to capture algebraic properties, as seen in normed BCK/BCI-algebras [8], where norms relate to implication operations, and in MV-algebras, where norms support quantitative semantics [11]. Unlike these structures, normed GE-algebras, introduced in this paper, define a GE-norm tailored to the non-commutative binary operation of GE-algebras, inducing quasi-metric spaces rather than metric spaces. This generalization extends the applicability of norms to non-linear algebraic systems, offering a novel framework for studying convergence and topological properties in generalized algebraic settings.

In this context, normed GE-algebras represent an important class that combines the algebraic properties of GE-algebras with a GE-norm, enabling the measurement of the magnitude of elements. This paper aims to extend the classical understanding of algebraic norms by introducing the concept of a GE-norm, defined as a real-valued mapping that satisfies specific properties akin to a norm in conventional algebraic systems. We begin by formally defining the notion of a GE-norm and explore its compatibility with the underlying operations of the GE-algebra. Following this, we investigate the properties of the magnitude function derived from the norm and establish a series of results on its behavior. Notably, we prove that normed GE-algebras induce quasi-metric spaces and that these spaces generate a T_0-topology. In subsequent sections, we delve into the properties of convergent sequences in normed GE-algebras, proving the uniqueness of limits and characterizing the boundedness of certain subsequences. We also establish several results concerning the preservation of normed structures under GE-morphisms, culminating in a product theorem for GE-algebras.

This work contributes to the ongoing development of generalized algebraic systems, providing both theoretical insights and practical tools for further exploration of algebraic norms, convergence, and topological spaces in GE-algebras.

2. Preliminaries

Definition 2.1 ([2]). *A GE-algebra is a non-empty set X with a constant 1 and a binary operation "$*$" satisfying the following axioms:*

(GE1) $\mathfrak{a} * \mathfrak{a} = 1$,

(GE2) $1 * \mathfrak{a} = \mathfrak{a}$,

(GE3) $\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) = \mathfrak{a} * (\mathfrak{b} * (\mathfrak{a} * \mathfrak{c}))$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$.

In a GE-algebra X, a binary relation "$\leq_X$" is defined by

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,(\mathfrak{a} \leq_X \mathfrak{b} \;\Leftrightarrow\; \mathfrak{a} * \mathfrak{b} = 1)\,. \tag{2.1}$$

Definition 2.2 ([2, 3]). *A GE-algebra X is said to be*

- *transitive if it satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X)\,(\mathfrak{a} * \mathfrak{b} \leq_X (\mathfrak{c} * \mathfrak{a}) * (\mathfrak{c} * \mathfrak{b}))\,. \tag{2.2}$$

- *commutative if it satisfies:*

$$(\forall \mathfrak{a}, \mathfrak{b} \in X)\,((\mathfrak{a} * \mathfrak{b}) * \mathfrak{b} = (\mathfrak{b} * \mathfrak{a}) * \mathfrak{a})\,. \tag{2.3}$$

Proposition 2.1 ([2]). *Every GE-algebra X satisfies the following items.*

$$\mathfrak{a} * 1 = 1. \tag{2.4}$$

$$\mathfrak{a} * (\mathfrak{a} * \mathfrak{b}) = \mathfrak{a} * \mathfrak{b}. \tag{2.5}$$

$$\mathfrak{a} \leq_X \mathfrak{b} * \mathfrak{a}. \tag{2.6}$$

$$\mathfrak{a} * (\mathfrak{b} * \mathfrak{c}) \leq_X \mathfrak{b} * (\mathfrak{a} * \mathfrak{c}). \tag{2.7}$$

$$1 \leq_X \mathfrak{a} \;\Rightarrow\; \mathfrak{a} = 1. \tag{2.8}$$

$$\mathfrak{a} \leq_X (\mathfrak{b} * \mathfrak{a}) * \mathfrak{a}. \tag{2.9}$$

$$\mathfrak{a} \leq_X (\mathfrak{a} * \mathfrak{b}) * \mathfrak{b}. \tag{2.10}$$

$$\mathfrak{a} \leq_X \mathfrak{b} * \mathfrak{c} \;\Leftrightarrow\; \mathfrak{b} \leq_X \mathfrak{a} * \mathfrak{c}. \tag{2.11}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$*. If* X *is transitive, then*

$$\mathfrak{a} \leq_X \mathfrak{b} \;\Rightarrow\; \mathfrak{c} * \mathfrak{a} \leq_X \mathfrak{c} * \mathfrak{b},\; \mathfrak{b} * \mathfrak{c} \leq_X \mathfrak{a} * \mathfrak{c}. \tag{2.12}$$

$$\mathfrak{a} * \mathfrak{b} \leq_X (\mathfrak{b} * \mathfrak{c}) * (\mathfrak{a} * \mathfrak{c}). \tag{2.13}$$

$$\mathfrak{a} \leq_X \mathfrak{b},\, \mathfrak{b} \leq_X \mathfrak{c} \;\Rightarrow\; \mathfrak{a} \leq_X \mathfrak{c}. \tag{2.14}$$

for all $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in X$*.*

Definition 2.3 ([12])**.** *Let* $(X, *_X, 1_X)$ *and* $(Y, *_Y, 1_Y)$ *be GE-algebras. A mapping* $f : X \to Y$ *is called a GE-morphism if it satisfies:*

$$(\forall \varrho_1, \varrho_2 \in X)(f(\varrho_1 *_X \varrho_2) = f(\varrho_1) *_Y f(\varrho_2)). \tag{2.15}$$

Let $\mathbb{X}_\alpha := \{(X_\alpha, *_\alpha, 1_\alpha) \mid \alpha \in \Lambda\}$ be a family of GE-algebras where Λ is an index set. Let $\prod \mathbb{X}_\alpha$ be the set of all mappings $\eth : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha$ with $\eth(\alpha) \in X_\alpha$, that is,

$$\prod \mathbb{X}_\alpha := \left\{ \eth : \Lambda \to \bigcup_{\alpha\in\Lambda} X_\alpha \mid \eth(\alpha) \in X_\alpha, \alpha \in \Lambda \right\}. \tag{2.16}$$

We define a binary operation $\circledast$ on $\prod \mathbb{X}_\alpha$ and the constant $\mathbf{1}$ by

$$\left(\forall \eth, f \in \prod \mathbb{X}_\alpha\right) ((\eth \circledast f)(\alpha) = \eth(\alpha) *_\alpha f(\alpha)) \tag{2.17}$$

and $\mathbf{1}(\alpha) = 1_\alpha$, respectively, for every $\alpha \in \Lambda$. It is routine to verify that $(\prod \mathbb{X}_\alpha, \circledast, \mathbf{1})$ is a GE-algebra, which is called the *product GE-algebra* (see [3]).

3. Normed GE-algebras

In what follows, let $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{R}$ be a GE-algebra and the set of all real numbers, respectively, unless otherwise specified. In the absence of ambiguity, the GE-algebra $\mathbb{X} := (X, *, 1_X)$ can simply be represented by $\mathbb{X}$.

Definition 3.1. *A GE-norm on* $\mathbb{X} := (X, *, 1_X)$ *is defined to be a mapping* $||\cdot|| : X \to \mathbb{R}$ *that satisfies:*

$$(\forall \varrho \in X)\left(||\varrho|| \geq 0\right), \tag{3.18}$$

$$(\forall \varrho \in X)\left(||\varrho|| = 0 \;\Leftrightarrow\; \varrho = 1_X\right), \tag{3.19}$$

$$(\forall \varrho, \varsigma, \varpi \in X)\left(||\varrho * \varpi|| \leq ||\varrho * \varsigma|| + ||\varsigma * \varpi||\right). \tag{3.20}$$

The GE-norm defined above shares similarities with classical norms, such as those in vector spaces or Banach algebras, where non-negativity and zero norm at the identity (conditions (3.18) and (3.19)) ensure a measure of magnitude [14]. However, it differs significantly due to the non-linear, non-commutative structure of GE-algebras. Unlike classical norms, which induce symmetric metrics, the GE-norm's triangle-like inequality (condition (3.20)) is tailored to the binary operation "$*$", leading to a quasi-metric space (Example 3.3). This formulation is chosen to align with the GE-algebra's axioms (GE1–GE3) and partial order $\leq_X$, ensuring compatibility with algebraic operations and enabling the study of convergence in non-commutative settings.

A *normed GE-algebra* is a GE-algebra $\mathbb{X} := (X, *, 1_X)$ equipped with a GE-norm $||\cdot|| : X \to \mathbb{R}$ and it is denoted by $(\mathbb{X}, ||\cdot||)$.

Given a GE-algebra $\mathbb{X} := (X, *, 1_X)$, if there exists a function $||\cdot||$ mapping elements of X to non-negative real numbers satisfying the conditions (3.19) and (3.20), then $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra.

Example 3.1. *For every GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, define a mapping:*

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise}, \end{cases}$$

where ϱ_0 *is a positive real number. Then* $||\cdot||$ *is a GE-norm on* $\mathbb{X} := (X, *, 1_X)$*, and so* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra.*

In normed GE-algebras, the "GE-norm" often provides a way to measure the "magnitude" of elements in a way that is compatible with the algebraic operation "$*$".

By the *magnitude* of a normed GE-algebra $(\mathbb{X}, ||\cdot||)$, we mean a real-valued function $\eth$ on $X \times X$ defined as follows:

$$(\forall \varrho, \varsigma \in X)\,(\eth(\varrho, \varsigma) = ||\varrho * \varsigma||)\,. \tag{3.21}$$

We say $\eth(\varrho, \varsigma)$ is the magnitude of (ϱ, ς).

Proposition 3.1. *The magnitude* $\eth : X \times X \to \mathbb{R}$ *of* $(\mathbb{X}, ||\cdot||)$ *has the following assertions:*

$$\eth(\varrho, \varsigma) \geq 0,\ \eth(\varrho, \varrho) = 0 = \eth(\varrho, 1_X), \tag{3.22}$$

$$\eth \text{ satisfies the triangle inequality,} \tag{3.23}$$

$$\eth(1_X, \varrho) = 0 \ \Rightarrow \ \varrho = 1_X, \tag{3.24}$$

$$\varrho \leq_X \varsigma \ \Rightarrow \ \eth(1_X, \varsigma) \leq \eth(1_X, \varrho), \tag{3.25}$$

$$\eth(\varrho, \varsigma) \leq \eth(1_X, \varsigma), \tag{3.26}$$

$$\eth(\varsigma, \varrho * \varpi) \leq \eth(\varrho, \varsigma * \varpi), \tag{3.27}$$

$$\eth(\varsigma * \varrho, \varrho) \leq \eth(1_X, \varrho), \tag{3.28}$$

$$\eth(\varrho * \varsigma, \varsigma) \leq \eth(1_X, \varrho), \tag{3.29}$$

for all $\varrho, \varsigma, \varpi \in X$.

Proof. Let $\varrho, \varsigma, \varpi \in X$. Then (3.22) and (3.23) are clear by (3.18), (3.19) and (3.19). The combination of (GE2) and (3.19) induces (3.24). Let $\varrho, \varsigma \in X$ be such that $\varrho \leq_X \varsigma$. Then $\varrho * \varsigma = 1$, and so

$$\begin{aligned} \eth(1_X, \varsigma) &\overset{(3.21)}{=} ||1_X * \varsigma|| \overset{(3.20)}{\leq} ||1_X * \varrho|| + ||\varrho * \varsigma|| = ||1_X * \varrho|| + ||1|| \\ &\overset{(3.19)}{=} ||1_X * \varrho|| + 0 = ||1_X * \varrho|| \overset{(3.21)}{=} \eth(1_X, \varrho). \end{aligned}$$

Hence (3.25) is valid. By the combination of (GE2), (2.6) and (3.25), we have (3.26). Using (GE2), (2.7) and (3.25), we get (3.27), (3.28) and (3.29). □

Proposition 3.2. *If* $\mathbb{X} := (X, *, 1_X)$ *is transitive, then the magnitude* $\eth : X \times X \to \mathbb{R}$ *of* $(\mathbb{X}, ||\cdot||)$ *satisfies:*

$$(\forall \varrho, \varsigma, \varpi \in X) \left(\eth(\varsigma * \varpi, \varrho * \varpi) \leq \eth(\varrho, \varsigma) \right). \tag{3.30}$$

Proof. Using (GE2), (2.13) and (3.25), we obtain (3.30). □

The following example shows that any magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ does not satisfy the following.

$$(\forall \varrho, \varsigma \in X) \left(\eth(\varrho, \varsigma) = 0 = \eth(\varsigma, \varrho) \ \Rightarrow \ \varrho = \varsigma \right). \tag{3.31}$$

Example 3.2. *Consider a non-commutative GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, where* $X = \{1_X, \ell_1, \ell_2, \ell_3, \ell_4\}$ *and a binary operation "$*$" is given in the following table:*

$*$	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4
1_X	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4
ℓ_1	1_X	1_X	ℓ_2	ℓ_3	1_X
ℓ_2	1_X	ℓ_4	1_X	1_X	ℓ_4
ℓ_3	1_X	ℓ_1	1_X	1_X	ℓ_1
ℓ_4	1_X	1_X	ℓ_2	ℓ_3	1_X

Define a mapping:

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise}, \end{cases}$$

where ϱ_0 *is a positive real number. Then* $||\cdot||$ *is a GE-norm on* $\mathbb{X} := (X, *, 1_X)$*, and so* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra. We can observe that* $\eth(\ell_2, \ell_3) = ||\ell_2 * \ell_3|| = ||1_X|| = 0$ *and* $\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||1_X|| = 0$*. Therefore* $\eth(\ell_2, \ell_3) = 0 = \eth(\ell_3, \ell_2)$*. But* $\ell_2 \neq \ell_3$*. Hence* (3.31) *is not valid.*

Theorem 3.1. *If* $\mathbb{X} := (X, *, 1_X)$ *is a commutative GE-algebra, then its magnitude* $\eth : X \times X \to \mathbb{R}$ *satisfies* (3.31)*.*

Proof. Let $\mathbb{X} := (X, *, 1_X)$ be a commutative GE-algebra. Then $(X, \leq_X)$ is antisymmetric. Let $\varrho, \varsigma \in X$ be such that $\eth(\varrho, \varsigma) = 0 = \eth(\varsigma, \varrho)$. Then $||\varrho * \varsigma|| = 0$ and $||\varsigma * \varrho|| = 0$, which imply from (3.19) that $\varrho * \varsigma = 1$ and $\varsigma * \varrho = 1$, i.e., $\varrho \leq_X \varsigma$ and $\varsigma \leq_X \varrho$. Hence $\varrho = \varsigma$, and so (3.31) is valid. □

The following example shows that any magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ does not satisfy the following.

$$(\forall \varrho, \varsigma \in X)\, (\eth(\varrho, \varsigma) = \eth(\varsigma, \varrho))\,. \tag{3.32}$$

Example 3.3. *Consider a non-commutative GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, where* $X = \{1_X, \ell_1, \ell_2, \ell_3\}$ *and a binary operation "$*$" is given in the following table:*

$*$	1_X	ℓ_1	ℓ_2	ℓ_3
1_X	1_X	ℓ_1	ℓ_2	ℓ_3
ℓ_1	1_X	1_X	1_X	1_X
ℓ_2	1_X	ℓ_1	1_X	1_X
ℓ_3	1_X	ℓ_1	ℓ_2	1_X

Define a mapping:

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise,} \end{cases}$$

where ϱ_0 *is a positive real number. Then* $||\cdot||$ *is a GE-norm on* $\mathbb{X} := (X, *, 1_X)$*, and so* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra. We can observe that* $\eth(\ell_2, \ell_3) = ||\ell_2 * \ell_3|| = ||1_X|| = 0$ *and* $\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||\ell_2|| = \varrho_0$*. Therefore* $\eth(\ell_2, \ell_3) \neq \eth(\ell_3, \ell_2)$*. Hence* (3.32) *is not valid.*

Example 3.3 is indicating that the magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ cannot be a metric on X, that is, $(X, \eth)$ is not a metric space. But we know that the magnitude $\eth : X \times X \to \mathbb{R}$ of $(\mathbb{X}, ||\cdot||)$ is a quasi metric on X, and thus $(X, \eth)$ is a quasi metric space which generates a T_0-space on X. For the quasi metric $\eth$ on X, we define new real-valued mappings $\eth^{-1}$ and $\eth^{\vee}$ on $X \times X$ as follows:

$$\eth^{-} : X \times X \to \mathbb{R},\ (\varrho, \varsigma) \mapsto \eth(\varsigma, \varrho). \tag{3.33}$$

$$\eth^{\vee} : X \times X \to \mathbb{R},\ (\varrho, \varsigma) \mapsto \max\{\eth(\varrho, \varsigma), \eth^{-}(\varrho, \varsigma)\}. \tag{3.34}$$

It is clear that $\eth^{-}$ and $\eth^{\vee}$ are quasi metrices on X.

The following example illustrates the quasi metrices $\eth^{-}$ and $\eth^{\vee}$ on X.

Example 3.4. *Consider the normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *in Example 3.3. Then*

$$\begin{aligned} X \times X = \{&(1_X, 1_X), (1_X, \ell_1), (1_X, \ell_2), (1_X, \ell_3), (\ell_1, 1_X), (\ell_1, \ell_1), \\ &(\ell_1, \ell_2), (\ell_1, \ell_3), (\ell_2, 1_X), (\ell_2, \ell_1), (\ell_2, \ell_2), (\ell_2, \ell_3), \\ &(\ell_3, 1_X), (\ell_3, \ell_1), (\ell_3, \ell_2), (\ell_3, \ell_3)\} \end{aligned}$$

and the binary operation "$\circledast$" on $X \times X$ *is given by Table 3.1.*

Table 3.1. Tabular representation for the operation "$\circledast$" on $X \times X$

$\circledast$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
$(1_X, \ell_2)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
$(1_X, \ell_3)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_2, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_3, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_3, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$

The quasi metrices $\eth^-$ and $\eth^\vee$ on X appear as follows.

$$\eth^-(\varrho, \varsigma) = \begin{cases} 0 & \text{if } (\varrho, \varsigma) \in (X \times X) \setminus A, \\ \varrho_0 & \text{if } (\varrho, \varsigma) \in A, \end{cases}$$

and

$$\eth^\vee(\varrho, \varsigma) = \begin{cases} 0 & \text{if } (\varrho, \varsigma) \in B, \\ \varrho_0 & \text{if } (\varrho, \varsigma) \in (X \times X) \setminus B, \end{cases}$$

where $A = \{(\ell_1, 1_X), (\ell_1, \ell_2), (\ell_1, \ell_3), (\ell_2, 1_X), (\ell_2, \ell_3), (\ell_3, 1_X)\}$ *and*

$$B = \{(1_X, 1_X), (\ell_1, \ell_1), (\ell_2, \ell_2), (\ell_3, \ell_3)\}.$$

Table 3.1 (continued)

$\circledast$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
$(1_X, 1_X)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
$(1_X, \ell_1)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
$(1_X, \ell_2)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
$(1_X, \ell_3)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	$(\ell_1, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
(ℓ_2, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_2, ℓ_2)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_2, ℓ_3)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	$(\ell_1, 1_X)$
$(\ell_3, 1_X)$	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	(ℓ_1, ℓ_3)
(ℓ_3, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_3, ℓ_2)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	$(\ell_1, 1_X)$	$(\ell_1, 1_X)$
(ℓ_3, ℓ_3)	$(\ell_1, 1_X)$	(ℓ_1, ℓ_1)	(ℓ_1, ℓ_2)	$(\ell_1, 1_X)$

Theorem 3.2. *Let $f : X \to Y$ be an onto GE-morphism from a GE-algebra $\mathbb{X} := (X, *, 1_X)$ to a GE-algebra $\mathbb{Y} := (Y, *, 1_Y)$. If $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra, then so is $(\mathbb{Y}, ||\cdot||)$.*

Proof. Assume that $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra. Since f is onto, $f^{-1}(\hbar) \neq \emptyset$ for every $\hbar \in Y$. So we can take $||\hbar|| = \inf_{\varrho \in f^{-1}(\hbar)} ||\varrho||$. It is clear that $||\hbar|| \geq 0$. If $||\hbar|| = 0$, then $\inf_{\varrho \in f^{-1}(\hbar)} ||\varrho|| = 0$, and so there exists $\varrho \in X$ such that $||\varrho|| = 0$. Hence $\varrho = 1_X$ which implies that $\hbar = f(\varrho) = f(1_X) = 1_Y$. If $\hbar = 1_Y$, then $||\hbar|| = \inf_{\varrho \in f^{-1}(\hbar)} ||\varrho|| \overset{(3.25)}{=} ||1_X|| = 0$ since $1_X \in f^{-1}(1_Y)$. Let $\hbar, \jmath, \wp \in Y$. Then there exist $\varrho, \varsigma, \varpi \in X$ such that $f(\varrho) = \hbar$, $f(\varsigma) = \jmath$

Table 3.1 (continued)

$\circledast$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	(ℓ_2, ℓ_3)
$(1_X, 1_X)$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	(ℓ_2, ℓ_3)
$(1_X, \ell_1)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
$(1_X, \ell_2)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
$(1_X, \ell_3)$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	$(\ell_2, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_2, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_3, 1_X)$	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	(ℓ_2, ℓ_3)
(ℓ_3, ℓ_1)	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
(ℓ_3, ℓ_2)	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	$(\ell_2, 1_X)$	$(\ell_2, 1_X)$
(ℓ_3, ℓ_3)	$(\ell_2, 1_X)$	(ℓ_2, ℓ_1)	(ℓ_2, ℓ_2)	$(\ell_2, 1_X)$

and $f(\varpi) = \wp$. Hence

$$
\begin{aligned}
||\hbar * \wp|| &= \inf_{u \in f^{-1}(\hbar * \wp)} ||u|| = \inf_{u \in f^{-1}(\hbar) * f^{-1}(\wp)} ||u|| = \inf_{\substack{\varrho \in f^{-1}(\hbar), \\ \varpi \in f^{-1}(\wp)}} ||\varrho * \varpi|| \\
&\overset{(3.20)}{\leq} \inf_{\substack{\varrho \in f^{-1}(\hbar), \\ \varsigma \in f^{-1}(\jmath)}} ||\varrho * \varsigma|| + \inf_{\substack{\varsigma \in f^{-1}(\jmath), \\ \varpi \in f^{-1}(\wp)}} ||\varsigma * \varpi|| \\
&= \inf_{v \in f^{-1}(\hbar) * f^{-1}(\jmath)} ||v|| + \inf_{w \in f^{-1}(\jmath) * f^{-1}(\wp)} ||w|| \\
&= \inf_{v \in f^{-1}(\hbar * \jmath)} ||v|| + \inf_{w \in f^{-1}(\jmath * \wp)} ||w|| \\
&= ||\hbar * \jmath|| + ||\jmath * \wp||.
\end{aligned}
$$

Hence $(\mathbb{Y}, ||\cdot||)$ is a normed GE-algebra. □

Theorem 3.3. *Let $f : X \to Y$ be a one-to-one GE-morphism from a GE-algebra $\mathbb{X} := (X, *, 1_X)$ to a GE-algebra*
$\mathbb{Y} := (Y, *, 1_Y)$. If $(\mathbb{Y}, ||\cdot||)$ is a normed GE-algebra, then so is $(\mathbb{X}, ||\cdot||)$.

Table 3.1 (continued)

$\circledast$	$(\ell_3, 1_X)$	(ℓ_3, ℓ_1)	(ℓ_3, ℓ_2)	(ℓ_3, ℓ_3)
$(1_X, 1_X)$	$(\ell_3, 1_X)$	(ℓ_3, ℓ_1)	(ℓ_3, ℓ_2)	(ℓ_3, ℓ_3)
$(1_X, \ell_1)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$
$(1_X, \ell_2)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$	$(\ell_3, 1_X)$
$(1_X, \ell_3)$	$(\ell_3, 1_X)$	(ℓ_3, ℓ_1)	(ℓ_3, ℓ_2)	$(\ell_3, 1_X)$
$(\ell_1, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_1, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_1, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_2, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_2, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_2, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$
$(\ell_3, 1_X)$	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, \ell_3)$
(ℓ_3, ℓ_1)	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_2)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, 1_X)$	$(1_X, 1_X)$
(ℓ_3, ℓ_3)	$(1_X, 1_X)$	$(1_X, \ell_1)$	$(1_X, \ell_2)$	$(1_X, 1_X)$

Proof. Assume that $(\mathbb{Y}, ||\cdot||)$ is a normed GE-algebra. For every $\varrho \in X$, let $||\varrho|| = ||f(\varrho)||$. Then $||\varrho|| = ||f(\varrho)|| \overset{(3.18)}{\geq} 0$ and

$$||\varrho|| = 0 \ \Leftrightarrow \ ||f(\varrho)|| = 0 \ \overset{(3.19)}{\Leftrightarrow} \ f(\varrho) = 1_X = f(1_X) \ \Leftrightarrow \ \varrho = 1_X$$

since f is a one-to-one GE-morphism. For every $\varrho, \varsigma, \varpi \in X$, we get

$$\begin{aligned} ||\varrho * \varpi|| &= ||f(\varrho * \varpi)|| = ||f(\varrho) * f(\varpi)|| \\ &\overset{(3.20)}{\leq} ||f(\varrho) * f(\varsigma)|| + ||f(\varsigma) * f(\varpi)|| \\ &= ||f(\varrho * \varsigma)|| + ||f(\varsigma * \varpi)|| \\ &= ||\varrho * \varsigma|| + ||\varsigma * \varpi||. \end{aligned}$$

Therefore $(\mathbb{X}, ||\cdot||)$ is a normed GE-algebra. □

Theorem 3.4. *Let $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$ be GE-algebras and consider the product GE-algebra $\mathbb{X} \times \mathbb{Y} := (X \times Y, \circledast, \mathbf{1})$ of $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$. Then*

$\mathbb{X} \times \mathbb{Y}$ *is a normed GE-algebra if and only if* $\mathbb{X} := (X, *, 1_X)$ *and* $\mathbb{Y} := (Y, *, 1_Y)$ *are normed GE-algebras.*

Proof. Assume that $\mathbb{X} \times \mathbb{Y}$ is a normed GE-algebra and consider the projection $f_X : X \times Y \to X$ and $f_Y : X \times Y \to Y$. Then f_X and f_Y are onto GE-morphisms. Hence $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$ are normed GE-algebras by Theorem 3.2.

Conversely, suppose that $\mathbb{X} := (X, *, 1_X)$ and $\mathbb{Y} := (Y, *, 1_Y)$ are normed GE-algebras. If $\hbar \in X \times Y$, then $\hbar = (\varrho_\hbar, \varsigma_\hbar)$ for some $\varrho_\hbar \in X$ and $\varsigma_\hbar \in Y$. Define $||\hbar|| = ||\varrho_\hbar|| + ||\varsigma_\hbar||$. Then $||\hbar|| = ||\varrho_\hbar|| + ||\varsigma_\hbar|| \geq 0$ and

$$\begin{aligned} ||\hbar|| = 0 \ &\Leftrightarrow \ ||\varrho_\hbar|| + ||\varsigma_\hbar|| = 0 \ \Leftrightarrow \ ||\varrho_\hbar|| = 0 = ||\varsigma_\hbar|| \\ &\overset{(3.19)}{\Leftrightarrow} \ \varrho_\hbar = 1_X \text{ and } \varsigma_\hbar = 1_Y \\ &\Leftrightarrow \ \hbar = (\varrho_\hbar, \varsigma_\hbar) = (1_X, 1_Y) = \mathbf{1}. \end{aligned}$$

Let $\hbar := (\varrho_\hbar, \varsigma_\hbar)$, $\jmath := (\varrho_\jmath, \varsigma_\jmath)$, $\wp := (\varrho_\wp, \varsigma_\wp) \in X \times Y$. Then

$$\begin{aligned} ||\hbar \circledast \wp|| &= ||(\varrho_\hbar * \varrho_\wp, \varsigma_\hbar * \varsigma_\wp)|| \\ &= ||\varrho_\hbar * \varrho_\wp|| + ||\varsigma_\hbar * \varsigma_\wp|| \\ &\overset{(3.20)}{\leq} (||\varrho_\hbar * \varrho_\jmath|| + ||\varrho_\jmath * \varrho_\wp||) + (||\varsigma_\hbar * \varsigma_\jmath|| + ||\varsigma_\jmath * \varsigma_\wp||) \\ &= (||\varrho_\hbar * \varrho_\jmath|| + ||\varsigma_\hbar * \varsigma_\jmath||) + (||\varrho_\jmath * \varrho_\wp|| + ||\varsigma_\jmath * \varsigma_\wp||) \\ &= ||(\varrho_\hbar * \varrho_\jmath, \varsigma_\hbar * \varsigma_\jmath)|| + ||(\varrho_\jmath * \varrho_\wp, \varsigma_\jmath * \varsigma_\wp)|| \\ &= ||\hbar \circledast \jmath|| + ||\jmath \circledast \wp||. \end{aligned}$$

Therefore $\mathbb{X} \times \mathbb{Y}$ is a normed GE-algebra. □

Definition 3.2. *Let* $(\mathbb{X}, ||\cdot||)$ *be a normed GE-algebra and consider a sequence* $\{\hbar_n\}$ *in* X. *Then* $\{\hbar_n\}$ *is said to be convergent in* X *if there exists a number* $\hbar_0$ *in* X *such that for every* $\varepsilon > 0$ *(no matter how small), there exists a natural number* k_0 *such that the magnitude for* $(\hbar_n, \hbar_0)$ *and* $(\hbar_0, \hbar_n)$ *is less than* ε *for all* $n \geq k_0$*, that is, it can be written as:*

$$\lim_{n\to\infty} \hbar_n = \hbar_0 \text{ if and only if for every } \varepsilon > 0 \text{ there exists } k_0 \in \mathbb{N} \text{ such that}$$
$$n \geq k_0 \ \Rightarrow \ \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon.$$

In this case, we say that $\hbar_0$ *is the limit of* $\{\hbar_n\}$.

Theorem 3.5. *Let* $\mathbb{X} := (X, *, 1_X)$ *be a commutative GE-algebra. In a normed GE-algebra* $(\mathbb{X}, ||\cdot||)$*, a convergent sequence cannot have two different limits, that is, If a sequence* $\{\hbar_n\}$ *converges to a limit* $\hbar_0$*, then that limit is unique.*

Proof. Let $\{\hbar_n\}$ be a convergent sequence in X, and let $\hbar_0$ and $\jmath_0$ be two limits of $\{\hbar_n\}$. Then for every $\varepsilon > 0$, there exists a natural number k_0 such that $\eth(\hbar_n, \hbar_0) < \frac{\varepsilon}{2}$, $\eth(\hbar_0, \hbar_n) < \frac{\varepsilon}{2}$, $\eth(\hbar_n, \jmath_0) < \frac{\varepsilon}{2}$ and $\eth(\hbar_0, \jmath_n) < \frac{\varepsilon}{2}$ for all $n \geq k_0$. Hence

$$\begin{aligned}\eth(\hbar_0, \jmath_0) &\overset{(3.21)}{=} ||\hbar_0 * \jmath_0|| \overset{(3.20)}{\leq} ||\hbar_0 * \hbar_n|| + ||\hbar_n * \jmath_0|| \\ &\overset{(3.21)}{=} \eth(\hbar_0, \hbar_n) + \eth(\hbar_n, \jmath_0) < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

By the similarly way, we have $\eth(\jmath_0, \hbar_0) \leq \varepsilon$. Since ε is arbitrary, it follows that $\eth(\hbar_0, \jmath_0) = 0 = \eth(\jmath_0, \hbar_0)$. Using Theorem 3.1, we conclude that $\hbar_0 = \jmath_0$. Therefore $\{\hbar_n\}$ has a unique limit. □

Theorem 3.6. *In a normed GE-algebra* $(\mathbb{X}, ||\cdot||)$*, every convergent sequence* $\{\hbar_n\}$ *in* X *satisfies:*

$$(\forall \varepsilon > 0)(\exists k_0 \in \mathbb{N})\,(n, m \geq k_0 \;\Rightarrow\; \eth(\hbar_n, \hbar_m) < \varepsilon \text{ and } \eth(\hbar_m, \hbar_n) < \varepsilon)\,. \tag{3.35}$$

Proof. Let $\mathbb{X} := \langle X, *, 1_X\rangle$ be a normed GE-algebra with GE-norm $\|\cdot\|$, and let $\eth(\varrho, \varsigma) = \|\varrho * \varsigma\|$ be the magnitude function. Suppose $\{\hbar_n\}$ is a sequence in X that converges to $\hbar_0$ in X. By definition 3.2, for every $\varepsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$,

$$\eth(\hbar_n, \hbar_0) = \|\hbar_n * \hbar_0\| < \varepsilon \quad \text{and} \quad \eth(\hbar_0, \hbar_n) = \|\hbar_0 * \hbar_n\| < \varepsilon.$$

To prove that $\{\hbar_n\}$ satisfies condition (3.35), fix $\varepsilon > 0$. Since $\{\hbar_n\}$ converges to $\hbar_0$, there exists $k_0 \in \mathbb{N}$ such that for all $n \geq k_0$,

$$\eth(\hbar_n, \hbar_0) < \frac{\varepsilon}{2} \quad \text{and} \quad \eth(\hbar_0, \hbar_n) < \frac{\varepsilon}{2}.$$

We need to show that for all $n, m \geq k_0$, $\eth(\hbar_n, \hbar_m) < \varepsilon$ and $\eth(\hbar_m, \hbar_n) < \varepsilon$. Consider $\eth(\hbar_n, \hbar_m) = \|\hbar_n * \hbar_m\|$. By the triangle-like inequality of the GE-norm (Definition 3.1, condition (3.20)), for any $\varrho, \varsigma, \varpi \in X$,

$$\|\varrho * \varpi\| \leq \|\varrho * \varsigma\| + \|\varsigma * \varpi\|.$$

Set $\varrho = \hbar_n$, $\varpi = \hbar_m$, and $\varsigma = \hbar_0$. Then,

$$\|\hbar_n * \hbar_m\| \leq \|\hbar_n * \hbar_0\| + \|\hbar_0 * \hbar_m\|,$$

i.e.,

$$\eth(\hbar_n, \hbar_m) \leq \eth(\hbar_n, \hbar_0) + \eth(\hbar_0, \hbar_m).$$

Since $n, m \geq k_0$, we have:

$$\eth(\hbar_n, \hbar_0) = \|\hbar_n * \hbar_0\| < \frac{\varepsilon}{2}, \quad \eth(\hbar_0, \hbar_m) = \|\hbar_0 * \hbar_m\| < \frac{\varepsilon}{2}.$$

Thus,

$$\eth(\hbar_n, \hbar_m) \leq \eth(\hbar_n, \hbar_0) + \eth(\hbar_0, \hbar_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Similarly, we can show that $\eth(\hbar_m, \hbar_n) < \varepsilon$. □

The converse of Theorem 3.6 is not valid as seen in the following example.

Example 3.5. *(i) For the normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *in Example 3.2, we can observe that if*

$$\hbar_n = \begin{cases} \ell_1 & \text{if } n \textit{ is odd}, \\ \ell_4 & \text{if } n \textit{ is even}, \end{cases}$$

then the sequence $\{\hbar_n\}$ *in* X *satisfies* (3.35)*. If we take* $\varepsilon > 0$ *such that* $\varrho_0 \geq \varepsilon$*, then*

$$\eth(\hbar_7, \ell_2) = ||\ell_1 * \ell_2|| = ||\ell_2|| = \varrho_0 \not< \varepsilon$$

and/or $\eth(\ell_2, \hbar_7) = ||\ell_2 * \ell_1|| = ||\ell_4|| = \varrho_0 \not< \varepsilon$*. Hence* $\{\hbar_n\}$ *is not convergent.*

(ii) Let $(0, 1] \subseteq \mathbb{R}$ *and define a binary operation "$*$" on* $(0, 1]$ *as follows:*

$$\varrho * \varsigma = \begin{cases} \varsigma & \text{if } \varrho = 1, \\ 1 & \text{otherwise.} \end{cases}$$

Then $((0, 1], *, 1)$ *is a GE-algebra. If we take a sequence* $\{\frac{1}{n+1}\}_{n \in \mathbb{N}}$*, then it satisfies* (3.35) *but does not converge in* $(0, 1]$*.*

Theorem 3.7. *Let* $\{\hbar_n\}$ *be a sequence in a normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *where* $\mathbb{X} := (X, *, 1_X)$ *is a commutative GE-algebra. Then it is convergent if and only if all of its non-trivial subsequences converge.*

Proof. Assume that $\{\hbar_n\}$ is a convergent sequence in $(\mathbb{X}, ||\cdot||)$ and let $\hbar_0$ be its limit. For every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$n \geq k_0 \;\Rightarrow\; \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon.$$

Let $\{\hbar_{\phi(n)}\}$ be a non-trivial subsequence of $\{\hbar_n\}$. If $n \geq k_0$, then $\phi(n) \geq n \geq k_0$, and so $\eth(\hbar_{\phi(n)}, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \hbar_{\phi(n)}) < \varepsilon$. This shows that $\{\hbar_{\phi(n)}\}$ is convergent.

Conversely, suppose that all of non-trivial subsequences of $\{\hbar_n\}$ converge. If $\{\hbar_n\}$ is not convergent, then there are at least two non-trivial subsequences, say $\{\hbar_{\phi(n)}\}$ and $\{\hbar_{\phi(m)}\}$, with different limits $\hbar_0$ and $\jmath_0$, respectively. This is a contradiction by Theorem 3.5, and thus $\{\hbar_n\}$ is a convergent sequence in $(\mathbb{X}, ||\cdot||)$. □

Theorem 3.8. *Let $\{\hbar_n\}$ be a sequence in $(\mathbb{X}, ||\cdot||)$. If $\hbar_0$ is a limit of $\{\hbar_n\}$, then 1_X is a limit of the sequences $\{\hbar_n * \hbar_0\}$ and $\{\hbar_0 * \hbar_n\}$.*

Proof. If $\hbar_0$ is a limit of $\{\hbar_n\}$, then for every $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that

$$n \geq k_0 \;\Rightarrow\; \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon.$$

Hence $\eth(\hbar_n * \hbar_0, 1_X) \overset{(3.22)}{=} 0 < \varepsilon$ and

$$\begin{aligned}\eth(1_X, \hbar_n * \hbar_0) &\overset{(3.21)}{=} ||1_X * (\hbar_n * \hbar_0)|| \overset{(GE2)}{=} ||\hbar_n * \hbar_0|| \\ &\overset{(3.21)}{=} \eth(\hbar_n, \hbar_0) < \varepsilon.\end{aligned}$$

Therefore 1_X is a limit of $\{\hbar_n * \hbar_0\}$. Similarly, $\{\hbar_0 * \hbar_n\}$ has a limit 1_X. □

Theorem 3.9. *Let $\{\hbar_n\}$ be a sequence in a normed GE-algebra $(\mathbb{X}, ||\cdot||)$. If $\hbar_0$ is a limit of $\{\hbar_n\}$, then $\{\eth(\hbar_n, \jmath_0)\}$ and $\{\eth(\jmath_0, \hbar_n)\}$ are bounded above for all $\jmath_0 \in X$.*

Proof. Assume that $\{\hbar_n\}$ converges to $\hbar_0$. By the definition of convergence, for every $\varepsilon > 0$ there exists a natural number k_0 such that $\eth(\hbar_n, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \hbar_n) < \varepsilon$ for all $n \geq k_0$. It follows from (3.20) that

$$\eth(\hbar_n, \jmath_0) \leq \eth(\hbar_n, \hbar_0) + \eth(\hbar_0, \jmath_0) < \varepsilon + \eth(\hbar_0, \jmath_0)$$

and $\eth(\jmath_0, \hbar_n) \leq \eth(\jmath_0, \hbar_0) + \eth(\hbar_0, \hbar_n) < \eth(\jmath_0, \hbar_0) + \varepsilon$. If $n < k_0$, then $\eth(\hbar_n, \jmath_0) = ||\hbar_n * \jmath_0|| \leq M$ and $\eth(\jmath_0, \hbar_n) = ||\jmath_0 * \hbar_n|| \leq M$ where

$$M := \max\{||\hbar_n * \jmath_0||, ||\jmath_0 * \hbar_n||\}.$$

This completes the proof. □

Let $\eth$ be the magnitude of a normed GE-algebra $(\mathbb{X}, ||\cdot||)$. Consider the following:

$$(\forall \varrho, \varsigma, \varpi \in X)\left(\varrho \leq_X \varsigma \;\Rightarrow \left\{ \begin{array}{l} \eth(\varrho, \varpi) \leq \eth(\varsigma, \varpi) \\ \eth(\varpi, \varsigma) \leq \eth(\varpi, \varrho) \end{array} \right. \right). \tag{3.36}$$

The following example shows that (3.36) is not valid in general.

Example 3.6. *Consider a GE-algebra* $\mathbb{X} := (X, *, 1_X)$*, where* $X = \{1_X, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5\}$ *and a binary operation "$*$" is given in the following table:*

$*$	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
1_X	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5
ℓ_1	1_X	1_X	1_X	ℓ_3	ℓ_3	ℓ_5
ℓ_2	1_X	ℓ_1	1_X	ℓ_4	ℓ_4	ℓ_5
ℓ_3	1_X	1_X	ℓ_2	1_X	1_X	ℓ_5
ℓ_4	1_X	1_X	1_X	1_X	1_X	ℓ_5
ℓ_5	1_X	ℓ_1	ℓ_2	ℓ_3	ℓ_4	1_X

Define a norm $||\cdot||$ *on* $\mathbb{X} := (X, *, 1_X)$ *as follows:*

$$||\cdot|| : X \to \mathbb{R},\ \varrho \mapsto \begin{cases} 0 & \text{if } \varrho = 1_X, \\ \varrho_0 & \text{otherwise}, \end{cases}$$

where ϱ_0 *is a positive real number. Then* $(\mathbb{X}, ||\cdot||)$ *is a normed GE-algebra. Note that* $\ell_3 * \ell_1 = 1_X$ *and* $\ell_4 * \ell_2 = 1_X$*, i.e.,* $\ell_3 \leq_X \ell_1$ *and* $\ell_4 \leq_X \ell_2$*. We can observe that*

$$\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||\ell_2|| = \varrho_0 \not\leq 0 = ||1_X|| = ||\ell_1 * \ell_2|| = \eth(\ell_1, \ell_2)$$

and

$$\eth(\ell_3, \ell_2) = ||\ell_3 * \ell_2|| = ||\ell_2|| = \varrho_0 \not\leq 0 = ||1_X|| = ||\ell_3 * \ell_4|| = \eth(\ell_3, \ell_4).$$

We now discuss the squeeze theorem for convergence sequences.

Theorem 3.10. *Assume that every magnitude* $\eth$ *of a normed GE-algebra* $(\mathbb{X}, ||\cdot||)$ *satisfies* (3.36)*. Let* $\{\hbar_n\}$*,* $\{\jmath_n\}$ *and* $\{\wp_n\}$ *be sequences in* $(\mathbb{X}, ||\cdot||)$ *such that* $\{\jmath_n\}$ *is trapped between* $\{\hbar_n\}$ *and* $\{\wp_n\}$ *for a sufficiently large* n*, that is, there exists a natural number* k_0 *such that* $\hbar_n \leq_X \jmath_n \leq_X \wp_n$ *for all* $n > k_0$*. If* $\{\hbar_n\}$ *and* $\{\wp_n\}$ *converge to* $\hbar_0$*, then* $\{\jmath_n\}$ *also converges to* $\hbar_0$*.*

Proof. If $\{\hbar_n\}$ and $\{\wp_n\}$ converge to $\hbar_0$, then for every $\varepsilon > 0$ there exist natural numbers $k_\hbar$ and $k_\jmath$ such that

$$n \geq k_\hbar \ \Rightarrow\ \eth(\hbar_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \hbar_n) < \varepsilon$$

and

$$n \geq k_\jmath \ \Rightarrow\ \eth(\wp_n, \hbar_0) < \varepsilon \text{ and } \eth(\hbar_0, \wp_n) < \varepsilon.$$

Using (3.36), we have

$$\eth(\hbar_n, \hbar_0) \leq \eth(\jmath_n, \hbar_0) \leq \eth(\wp_n, \hbar_0)$$

and

$$\eth(\wp_0, \hbar_n) \leq \eth(\jmath_0, \hbar_n) \leq \eth(\hbar_0, \hbar_n)$$

for all $k_0 := \max\{k_\hbar, k_\jmath\}$. It follows that if $n \geq k_0$, then $\eth(\jmath_n, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \jmath_n) < \varepsilon$. Thus $\{\jmath_n\}$ converges to $\hbar_0$. □

Theorem 3.11. *Let f be a GE-morphism from a GE-algebra $\mathbb{X} := (X, *_X, 1_X)$ to a GE-algebra $\mathbb{Y} := (Y, *_Y, 1_Y)$. Assume that $||\varrho|| = ||f(\varrho)||$ for all $\varrho \in X$. Then a sequence $\{\hbar_n\}$ in $(\mathbb{X}, ||\cdot||)$ converges to $\hbar_0$ if and only if the sequence $\{f(\hbar_n)\}$ in $(\mathbb{Y}, ||\cdot||)$ converges to $f(\hbar_0)$.*

Proof. Assume that a sequence $\{\hbar_n\}$ in $(\mathbb{X}, ||\cdot||)$ converges to $\hbar_0$. Then for every $\varepsilon > 0$, there exists a natural number k_0 such that $\eth(\hbar_n, \hbar_0) < \varepsilon$ and $\eth(\hbar_0, \hbar_n) < \varepsilon$ for all $n \geq k_0$. Using (2.15) and (3.21), we have

$$\begin{aligned}\eth(f(\hbar_n), f(\hbar_0)) &= ||f(\hbar_n) *_Y f(\hbar_0)|| = ||f(\hbar_n *_X \hbar_0)|| \\ &= ||\hbar_n *_X \hbar_0|| = \eth(\hbar_n, \hbar_0) < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\eth(f(\hbar_0), f(\hbar_n)) &= ||f(\hbar_0) *_Y f(\hbar_n)|| = ||f(\hbar_0 *_X \hbar_n)|| \\ &= ||\hbar_0 *_X \hbar_n|| = \eth(\hbar_0, \hbar_n) < \varepsilon\end{aligned}$$

Therefore the sequence $\{f(\hbar_n)\}$ converges to $f(\hbar_0)$.

Conversely, suppose that the sequence $\{f(\hbar_n)\}$ in $(\mathbb{Y}, ||\cdot||)$ converges to $f(\hbar_0)$. For every $\varepsilon > 0$ there exists a natural number k_0 such that $\eth(f(\hbar_n), f(\hbar_0)) < \varepsilon$ and $\eth(f(\hbar_0), f(\hbar_n)) < \varepsilon$ for all $n \geq k_0$. It follows that

$$\begin{aligned}\eth(\hbar_n, \hbar_0) &= ||\hbar_n *_X \hbar_0|| = ||f(\hbar_n *_X \hbar_0)|| \\ &= ||f(\hbar_n) *_Y f(\hbar_0)|| = \eth(f(\hbar_n), f(\hbar_0)) < \varepsilon\end{aligned}$$

and

$$\begin{aligned}\eth(\hbar_0, \hbar_n) &= ||\hbar_0 *_X \hbar_n|| = ||f(\hbar_0 *_X \hbar_n)|| \\ &= ||f(\hbar_0) *_Y f(\hbar_n)|| = \eth(f(\hbar_0), f(\hbar_n)) < \varepsilon\end{aligned}$$

for all $n \geq k_0$. Consequently, $\{\hbar_n\}$ converges to $\hbar_0$. □

4. Conclusion

This paper introduces normed GE-algebras, equipping GE-algebras with a GE-norm to measure element magnitudes. We defined a magnitude function $\eth(\varrho, \varsigma) = \|\varrho * \varsigma\|$ that induces a quasi-metric space, generating a T_0-topology (Theorem 3.1, Example 3.3). Key results include the Cauchy-like property of convergent sequences (Theorem 3.6), preservation of normed structures under GE-morphisms (Theorem 3.2), and properties of product spaces (Theorem 3.4). These findings establish normed GE-algebras as a robust framework for studying convergence and topological properties in generalized algebraic systems. The significance of this work lies in bridging algebraic and geometric concepts, enabling the analysis of non-commutative structures in a topological context. The quasi-metric and T_0-topology support applications in functional analysis, modeling asymmetric distances, and in mathematical logic, quantifying logical distances in non-classical logics [13]. The GE-morphism and product theorems facilitate the study of complex algebraic systems. Future work includes exploring additional topological properties, such as compactness or connectedness, in the T_0-topology. Extending GE-norms to BCK/BCI-algebras or residuated lattices could broaden their scope [8]. Applications in functional analysis (e.g., asymmetric function spaces) and topology (e.g., non-Hausdorff spaces) are promising. Open problems, such as characterizing complete normed GE-algebras, encourage further interdisciplinary research.

Normed GE-algebras offer promising applications across several mathematical disciplines. In *functional analysis*, the quasi-metric spaces induced by GE-norms (Example 3.3) provide a framework for studying function spaces with asymmetric distances, which are relevant in asymmetric functional analysis [10]. These spaces can model non-reversible processes or directed convergence, extending traditional Banach space techniques. In *topology*, the T_0-topology generated by normed GE-algebras facilitates the study of non-Hausdorff topological spaces, which are prevalent in computational topology and data analysis. This topology supports the analysis of convergence properties in generalized settings. In *mathematical logic*, normed GE-algebras, as extensions of Hilbert algebras linked to intuitionistic logic, enable quantitative semantics where the GE-norm measures the "distance" between logical propositions [13]. This can enhance reasoning frameworks in non-classical logics, such as those used in artificial intelligence and formal verification. These applications underscore the versatility of normed GE-algebras and pave the way for future interdisciplinary research.

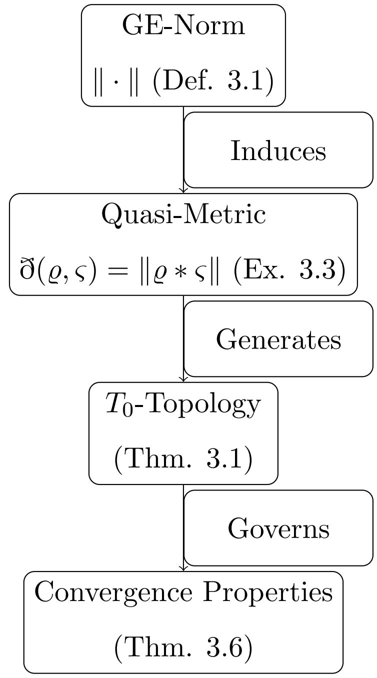

FIGURE 1. Flowchart illustrating the relationships between GE-norms, quasi-metrics, T_0-topology, and convergence properties in normed GE-algebras. The GE-norm induces a quasi-metric, which generates a T_0-topology, governing sequence convergence (e.g., Cauchy-like property in Theorem 3.6).

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References

[1] Abass, R. H. (2018). On the normed BH-algebras. Int. J. Pure Appl. Math., 119(10), 339–348.

[2] Bandaru, R. K., Borumand Saeid, A., & Jun, Y. B. (2021). On GE-algebras. Bull. Sect. Log., 50(1), 81–96. https://doi.org/10.18778/0138-0680.2020.20

[3] Bandaru, R. K., Borumand Saeid, A., & Jun, Y. B. (2022). Belligerent GE-filter in GE-algebras. J. Indones. Math. Soc., 28(1), 31–43. https://doi.org/10.22342/jims.28.1.1056.31-43

[4] Celani, S. (2002). A note on homomorphisms of Hilbert algebras. Int. J. Math. Math. Sci., 29(1), 55–61. DOI:10.1155/S0161171202011134

[5] Celani, S. (2002). Hilbert algebras with supremum. Algebr. Univ., 67, 237–255 DOI10.1007/s00012-012-0178-z

[6] Diego, A. (1966). Sur les algebres de Hilbert, Collection de Logique Mathematique, Edition Hermann, Serie A, XXI.

[7] Dudek, W. A. (1999). On ideals in Hilbert algebras. Acta Univ. Palacki. Olomuc, Fac. rer.nat. ser. Math., 38(1), 31–34. http://eudml.org/doc/23677

[8] Iorgulescu, A. (2008). *Algebras of Logic as BCK-algebras*. Editura ASE.

[9] Jun, Y. B., Roh, E. H. and Kim, H. S. (1998). On BH-algebras. Sci. Math., 1(3), 347-354.

[10] Kelley, J. L. (1955). *General Topology*. Springer.

[11] Mundici, D. (1998). *MV-algebras and their applications*. Springer.

[12] Rezaei, A., Bandaru, R. K., Borumand Saeid, A., & Jun, Y. B. (2021). Prominent GE-filters and GE-morphisms in GE-algebras. Afr. Mat. 32, 1121–1136. https://doi.org/10.1007/s13370-021-00886-6

[13] Rasiowa, H. (1974). *An Algebraic Approach to Non-Classical Logics*. North-Holland.

[14] Rudin, W. (1991). *Functional Analysis*. McGraw-Hill.

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A NEW TYPE OF IRRESOLUTE FUNCTION VIA δgp-OPEN SET

JAGADEESH B. TORANAGATTI * AND SAEID JAFARI

Abstract. In this article, a new class of complete continuity called complete δgp-irresolute is introduced. Properties and characterizations of completely δgp-irresolute functions are investigated.

Keywords: Completely δgp-irresolute, Strongly δgp-normal, Countable δgp-compact, δgp-Lindelöf.

2020 Mathematics Subject Classification: 54C08, 54C10.

1. Introduction

Many researchers have examined and analyzed various forms of continuity in academic literature. In general topology, continuity remains a vital and foundational concept in mathematics. In 1972, Crossley and Hildebrand [7] introduced the concept of irresoluteness. In 1999, Arokiarani et al. [3] studied gp-irresolute functions, followed by Balasubramanian and Sarada [5] in 2012, who explored the properties of gpr-irresolute functions. Over time, several variants of irresolute functions have been introduced. Recently, J. B. Toranagatti proposed and investigated δgp-continuity [23] as a broader interpretation of continuity. This research aims to introduce and investigate a completely δgp-irresolute function, which serves as a more robust variant of the existing gpr-irresolute function.

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2. Preliminaries

Throughout this paper ($\mathcal{M}$, τ), ($\mathcal{N}$, γ) and ($\mathcal{P}$, η) (or simply $\mathcal{M}$, $\mathcal{N}$ and $\mathcal{P}$) represent non-empty topological spaces on which no separation axioms are assumed unless otherwise stated. For a subset K of $\mathcal{M}$, $\acute{c}$(K) and $\acute{i}$(K) denote the closure of K and the interior of K, respectively.

Definition 2.1. *A set $J \subseteq \mathcal{M}$ is called:*
(i) regular closed [22] *if $\acute{c}(\acute{i}(J))$=J,*
(ii) pre-closed [18] *$\acute{c}(\acute{i}(J)) \subseteq J$.*

Definition 2.2. *A set $J \subseteq \mathcal{M}$ is called δ-closed* [28] *if $J = \acute{c}_\delta(J)$ where $\acute{c}_\delta(J) = \{b \in \mathcal{M}$: $\acute{i}(\acute{c}(U)) \cap J = \emptyset$, $U \in \Im$ and $b \in U\}$.*

Definition 2.3. *A set $J \subseteq \mathcal{M}$ is called δgp-closed* [6] *(resp., gp-closed* [17] *and gpr-closed* [10]*) if $p\acute{c}(J) \subseteq H$ whenever $J \subseteq H$ and H is δ-open (resp., open, regular open) in $\mathcal{M}$.*

Their complements are the open sets that are related to the previously listed closed sets. $\delta\mathbb{O}(\mathcal{M})$ is the collection of all δ-open sets in ($\mathcal{M}$, τ). The families of open sets, pre-open sets, regular open sets, gp-open sets, gpr-open sets, and δgp-open sets are denoted as $\mathbb{O}(\mathcal{M})$, $\mathbb{PO}(\mathcal{M})$, $\mathbb{RO}(\mathcal{M})$, $\mathbb{GPO}(\mathcal{M})$, $\mathbb{GPRO}(\mathcal{M})$ and $\delta\mathbb{GPO}(\mathcal{M})$ correspondingly.

Definition 2.4. *A function ℓ: ($\mathcal{M}$, $\Im$) $\to$ ($\mathcal{N}$, γ) is called:*
(i) R-maps [12] *if $\ell^{-1}(K) \in \mathbb{RO}(\mathcal{M})$ for every $K \in \mathbb{RO}(\mathcal{N})$;*
(ii) completely continuous [4] *if $\ell^{-1}(K) \in \mathbb{RO}(\mathcal{M})$ for every $K \in \gamma$;*
(iii) completely preirresolute [14] *(resp., completely gp-irresolute* [14] *and completely gpr-irresolute) if $\ell^{-1}(K) \in \mathbb{RO}(\mathcal{M})$ for every $K \in \mathbb{PO}(\mathcal{N})$ (resp., $K \in \mathbb{GPO}(\mathcal{N})$ and $K \in \mathbb{GPRO}(\mathcal{N})$);*
(iv) δgp-irresolute [23] *if $\ell^{-1}(K) \in \delta\mathbb{GPO}(\mathcal{M})$ for every $K \in \delta\mathbb{GPO}(\mathcal{N})$;*
(v) δgp-continuous [23] *if $\ell^{-1}(K) \in \delta\mathbb{GPO}(\mathcal{M})$ for every $K \in \gamma$;*
(vi) pre δgp-continuous [23] *if $\ell^{-1}(K) \in \delta\mathbb{GPO}(\mathcal{M})$ for every $K \in \mathbb{PO}(\mathcal{N})$;*
(vii) gpr-irresolute [5] *if $\ell^{-1}(K) \in \delta\mathbb{GPR}(\mathcal{M})$ for every $K \in \delta\mathbb{GPR}(\mathcal{N})$.*

Definition 2.5. *A space ($\mathcal{M}$, $\Im$) is called:*
(i) δgp-additive [24] *if $\delta\mathbb{GPC}(\mathcal{M})$ is closed under arbitrary intersections;*
(ii) $T_{\delta gp}$-space [6] *if $\delta\mathbb{GPC}(\mathcal{M}) = \mathbb{C}(\mathcal{M})$;*
(iii) preregular $T_{1/2}$-space [10] *if $\mathbb{GPRC}(\mathcal{M}) = \mathbb{PC}(\mathcal{M})$;*
(iv) locally indiscrete [13] *if $\Im = \mathbb{RO}(\mathcal{M})$.*

3. Completely δgp-irresolute Functions

Definition 3.1. *A function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is called as completely δgp-irresolute (briefly, c.δgp-i.) if for every point b in $\mathcal{M}$ and for any δgp-open set H that includes $\ell(b)$, there exists a δ-open set G around b such that $\ell(G) \subseteq H$.*

Theorem 3.1. *The following conditions are equivalent for a function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$:*
(i) ℓ is c.δgp-i.;
(ii) For each $q \in \mathcal{M}$ and each $D \in \delta\mathbb{GPC}(\mathcal{N}, \ell(q))$, there exists a $C \in \mathbb{RO}(\mathcal{M}, q)$ such that $\ell(C) \subseteq D$.

Proof. (i) → (ii): Let $q \in \mathcal{M}$ and $D \in \delta\mathbb{GPC}(\mathcal{N}, \ell(q))$.
$\overset{(i)}{\Longrightarrow}(\exists\ J \in \delta\mathbb{O}(\mathcal{M}, q))(\ell(J) \subset D)$.
Now, $J \in \delta\mathbb{O}(\mathcal{M}, q) \implies (\exists\ C \in \mathbb{RO}(\mathcal{M}, q)(C \subset J))$.
Therefore, $(\exists\ C \in \mathbb{RO}(\mathcal{M}, q))\ (\ell(C) \subset \ell(J) \subset D)$.
(ii) → (i): Obvious. □

Theorem 3.2. *The following conditions are identical for a function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$:*
(i) ℓ is c.δgp-i.;
(ii) For each $q \in \mathcal{M}$ and each $G \in \delta\mathbb{GPC}(\mathcal{N})$ where $\ell(q) \notin G$, there exists an $H \in \delta\mathbb{C}(\mathcal{M})$ such that $q \notin H$ and $\ell^{-1}(G) \subseteq H$;
(iii) For each $q \in \mathcal{M}$ and each $G \in \delta\mathbb{GPC}(\mathcal{N})$ where $\ell(q) \notin G$, there exists an $H \in \mathbb{RC}(\mathcal{M})$ such that $q \notin H$ and $\ell^{-1}(G) \subseteq H$;
(iv) For every $q \in \mathcal{M}$ and each $N \in \delta\mathbb{GPC}(\mathcal{N},\ell(q))$, there exists a $G \in \mathbb{O}(\mathcal{M}, q)$ such that $\ell(\acute{i}(\acute{c}(G)) \subseteq N$;
(v) For every $q \in \mathcal{M}$ and each $H \in \delta\mathbb{GPC}(\mathcal{N},\ell(q))$, there exists a $G \in \mathbb{O}(\mathcal{M}, q)$ such that $\ell(sc(G)) \subseteq H$.

Proof. Obvious. □

Remark 3.1. *We can generate the following diagram for the function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ by using Definitions 2.4 and 3.1.*

$$\begin{array}{ccccccccccc} c.gpr.i. & \to & c.\delta gp.i. & \to & c.gp.i. & \to & c.p.i. & \to & c.c. & \to & R\text{-}m. \\ \downarrow & & \downarrow & & & & & & & & \\ gpr.i. & \to & \delta gp.i. & \to & p.\delta gp.c. & \to & \delta gp.c. & & & & \end{array}$$

Notations:

c.gpr.i.: completely gpr-irresolute, **c.δgp.i.:** completely δgp-irresolute, **c.gp.i.:** completely gp-irresolute, **c.p.i.:** completely pre-irresolute, **c.c.:** completely continuous, **R-m.:** R-maps, **gpr.i.:** gpr-irresolute, **δgp.i.:** δgp-irresolute, **p.δgp.c.:** pre δgp-continuous, **δgp.c.:** δgp-continuous.

None of the implications in above diagram is reversible as shown in the following examples.

Example 3.1. *Let $\mathcal{M} = \eta = \{u_1, u_2, u_3, u_4\}$, $\tau = \{\mathcal{M}, \emptyset, \{u_1\},\{u_2\},\{u_1, u_2\},\{u_1, u_2, u_3\}\}$ and $\gamma = \{\eta, \emptyset, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$. Then:*

(i) The identity function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{M}, \gamma)$ is δgp-irresolute, but it is not completely δgp-irresolute.

(ii) Let us define ℓ: $(\mathcal{M}, \Im) \to (\mathcal{M}, \gamma)$ by $\ell(u_1) = u_1$, $\ell(u_2) = u_3 = \ell(u_3)$ and $\ell(u_4) = u_4$, In this case, ℓ is δgp-i. but not gpr-i., since $\{u_1, u_2\} \in \mathbb{GPRC}(\mathcal{M}, \gamma)$ implies that $\ell^{-1}(\{u_1, u_2\})=\{u_1\} \notin \mathbb{GPRC}(\mathcal{M}, \Im)$.

Example 3.2. *Consider $\mathcal{M} = \{u_1, u_2, u_3, u_4\}$ with the topologies*
$\Im = \{\emptyset, \mathcal{M}, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$ and
$\gamma = \{\emptyset, \mathcal{M}, \{u_1\},\{u_2\},\{u_1, u_2\},\{u_1, u_2, u_3\}\}$.
Let the function ℓ: $(\mathcal{M}, \Im) \to (\mathcal{M}, \gamma)$ be defined by $\ell(u_1) = u_2 = \ell(u_3)$, and $\ell(u_2) = u_4$ with $\ell(u_4)= u_4$. In this scenario, ℓ is c.gp-i. but not c.δgp-i.. This is evident as $\{u_4\} \in \delta\mathbb{GPC}((M, \gamma)$ leads to the conclusion that $\ell^{-1}(\{u_4\})) = \{u_2, u_4\} \notin \mathbb{RO}(\mathcal{M}, \Im)$.

Example 3.3. *Consider $(\mathcal{M}, \Im)$ as in Example 3.2. We define the function ℓ: $(\mathcal{M}, \Im) \to (M, \Im)$ by specifying $\ell(u_1)= u_2$, $\ell(u_3)= u_2$, and $\ell(u_2)= u_4$ with $\ell(u_4) = u_4$. In this context, ℓ is c.δgp-i., but it is not c.gpr-i.. This is because $\{u_1, u_4\} \in \mathbb{GPRO}(\mathcal{M}, \sigma)$ leads to the conclusion that $\ell^{-1}(\{u_1, u_4\}) = \{u_2, u_4\} \notin \mathbb{RO}(\mathcal{M}, \Im)$.*

Theorem 3.3. *For any $J \subseteq \mathcal{M}$, the following are the same where $(\mathcal{M}, \Im)$ is locally indiscrete space* [25].

(i) J is gp-closed;

(ii) J is δgp-closed;

(iii) J is gpr-closed.

As a consequence of Theorem 3.3, we can state the following theorem.

Theorem 3.4. *The statements that follow are interchangeable for ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ where $(\mathcal{N}, \gamma)$ is locally indiscrete space:*

(i) ℓ is c.gp-i.;

(ii) ℓ is c.δgp-i.;

(iii) ℓ is c.gpr-i..

Theorem 3.5. *The statements that follow are interchangeable for ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ where $(\mathcal{N}, \gamma)$ is $T_{\delta gp}$-space:*

(i) ℓ is c.c.;

(ii) ℓ is c.p-i.;

(iii) ℓ is c.gp-i.;

(iv) ℓ is c.δgp-i..

Theorem 3.6. *The statements that follow are interchangeable for ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ where $(\mathcal{N}, \gamma)$ is preregular $T_{1/2}$-space:*

(i) ℓ is c.p-i.;

(ii) ℓ is c.gp-i.;

(iii) ℓ is c.δgp-i.;

(iv) ℓ is c.gpr-i..

Definition 3.2. [11] *Every gpr-closed set for a space $(\mathcal{M}, \Im)$ is closed if and only if $\tau_g^*= \tau$ where $\tau_g^*=\{L \subseteq \mathcal{M}$: gprcl($\mathcal{M}$-L)=($\mathcal{M}$-L)$\}$.*

Theorem 3.7. *If $\gamma_g^*= \gamma$ in $(\mathcal{N}, \gamma)$. Then, the assertions that follow are the same:*

(i) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gpr-i.;

(ii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i.;

(iii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gp-i.;

(iv) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.p-i.;

(v) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.c..

Theorem 3.8. *If $\gamma_g^*= \gamma$ in $(\mathcal{N}, \gamma)$ and $(\mathcal{N}, \gamma)$ is locally indiscrete. Then, the assertions that follow are the same:*

(i) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gpr-i.;

(ii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i.;

(iii) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.gp-i.;

(iv) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.p-i.;

(v) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.c.;

(vi) ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is R-map.

Theorem 3.9. *Let ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ be such that the space $(\mathcal{N}, \gamma)$ is δgp-additive. The statement that follow are interchangeable:*

(i) ℓ is c.δgp-$\acute{i}$.;

(ii) ℓ^{-1} (δgp-$\acute{i}(R)) \subseteq \acute{i}_\delta(\ell^{-1}(R))$ for each $R \subseteq \mathcal{N}$;

(iii) ℓ $(\acute{c}_\delta(S)) \subseteq \delta$gp-$\acute{c}(\ell(S))$ for each $S \subseteq \mathcal{M}$;

(iv) $\acute{c}_\delta(\ \ell^{-1}(R)) \subseteq \ell^{-1}(\delta$gp-$\acute{c}(R))$ for each $R \subseteq \mathcal{N}$;

(v) $\ell^{-1}(B) \in \delta\mathbb{C}(\mathcal{M})$ for each $B \in \delta\mathbb{GPC}(\mathcal{N})$;

(vi) $\ell^{-1}(A) \in \delta\mathbb{O}(\mathcal{M})$ for each $A \in \delta\mathbb{GPO}(\mathcal{N})$;

(vii) $\ell^{-1}(A) \in \mathbb{RO}(\mathcal{M})$ for each $A \in \delta\mathbb{GPO}(\mathcal{N})$;

(viii) $\ell^{-1}(B) \in \mathbb{RC}(\mathcal{M})$ for each $B \in \delta\mathbb{GPC}(\mathcal{N})$.

Proof. (i) $\Longrightarrow$ (ii): Let R $\subseteq \mathcal{N}$ and x $\in \ell^{-1}(\delta\text{gp-}\acute{i}(\text{R}))$.

$$\text{b} \in \ell^{-1}\ (\delta\text{gp-}\acute{i}(\text{R}))\ \Longrightarrow\ \delta\text{gp-}\acute{i}(\text{R}) \in \delta\mathbb{GPO}(\mathcal{N}, \ell(\text{b}))$$
$$\overset{(i)}{\Longrightarrow} (\exists\ \text{S} \in \mathbb{RO}(\mathcal{M}, \text{q})\ (\ell(\text{S}) \subseteq \delta\text{gp-}\acute{i}(\text{R}) \subset \text{R})$$
$$\Longrightarrow\ (\exists\ \text{S} \in \mathbb{RO}(\mathcal{M}, \text{q}))(\text{S} \subseteq \ell^{-1}(\text{R}))\ \Longrightarrow\ \text{q} \in \acute{i}_\delta(\ell^{-1}(\text{R})).$$

(ii) $\Longrightarrow$ (iii) : Let S $\subseteq \mathcal{M}$.

$$\text{S} \subseteq \mathcal{M}\ \Longrightarrow\ \ell(\text{S}) \subseteq \mathcal{N}\ \Longrightarrow\ \mathcal{N} \setminus \ell(\text{S}) \subseteq \mathcal{N} \overset{(ii)}{\Longrightarrow} \ell^{-1}\ [\delta\text{gp-}\acute{i}(\mathcal{N}\setminus\text{S})] \subseteq \acute{i}_\delta(\ell^{-1}(\mathcal{N}\setminus\ell(\text{S})))$$
$$\Longrightarrow\ \mathcal{M} \setminus \ell^{-1}\ (\delta\text{gp-}\acute{c}(\ell(\text{S})) \subseteq \mathcal{M} \setminus \acute{c}_\delta(\ell^{-1}(\ell(\text{S}))))$$
$$\Longrightarrow\ \acute{c}_\delta\ (\text{S}) \subset \acute{c}_\delta\ ((\ell^{-1}(\ell(\text{S}))) \subseteq \ell^{-1}(\delta\text{gp-}\acute{c}(\ell(\text{S})))$$
$$\Longrightarrow\ \ell(\acute{c}_\delta(\text{S})) \subseteq \delta\text{gp-}\acute{c}(\ell(\text{S})).$$

(iii) $\Longrightarrow$ (iv): Let R $\subseteq \mathcal{N}$.

$$\text{R} \subseteq \mathcal{N}\ \Longrightarrow\ \ell^{-1}(\text{R}) \subseteq \mathcal{M} \overset{(iii)}{\Longrightarrow} \ell(\acute{c}_\delta\ (\ell^{-1}(\ell(\text{R}))) \subseteq \delta\text{gp-}\acute{c}(\ell^{-1}(\text{R})) \subseteq \delta\text{gp-}\acute{c}(\text{R})$$
$$\Longrightarrow\ \acute{c}_\delta\ ((\ell^{-1}(\text{R})) \subseteq \ell^{-1}(\delta\text{gp-}\acute{c}(\text{R})).$$

(iv) $\Longrightarrow$ (v): Let H $\in \delta\mathbb{GPC}(\mathcal{N})$.

$$\text{H} \in \delta\mathbb{GPC}(\mathcal{N})\ \Longrightarrow\ \text{H} = \delta\text{gp-}\acute{c}(\text{H})$$
$$\overset{(iv)}{\Longrightarrow} \acute{c}_\delta((\ell^{-1}(\text{H})) \subseteq \ell^{-1}(\delta\text{gp-}\acute{c}(\text{H})) = \ell^{-1}(\text{H})$$
$$\Longrightarrow\ \ell^{-1}(\text{H}) = \acute{c}_\delta\ (\ell^{-1}(\text{H}))\ \Longrightarrow\ \ell^{-1}(\text{H}) \in \delta\mathbb{C}(\text{M}).$$

(i) $\Longrightarrow$ (vi): Obvious.

(viii) $\Longleftrightarrow$ (vii) $\Longrightarrow$ (vi) : Obvious.

(vi) $\Longrightarrow$ (i): Let K $\in \delta\mathbb{GPO}(\mathcal{N})$ and q $\in \ell^{-1}(\text{K})$.

$(\mathrm{K} \in \delta\mathbb{GPO}(\mathcal{N})\ (\mathrm{q} \in \ell^{-1}(\mathrm{K})) \implies \mathrm{K} \in \delta\mathbb{GPO}(\mathcal{N},\, \ell(\mathrm{q}))$

$\xRightarrow{\text{(vi)}} (\mathrm{L}\colon = \ell^{-1}(\mathrm{K}) \in \delta\mathbb{O}(\mathcal{M},\, \mathrm{q}))\ (\ell(\mathrm{L}) \subseteq \mathrm{K}).$ □

Theorem 3.10. *The following assertions are identical for a bijection ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$:*

(i) ℓ is c.δgp-i.;

(ii) δgp-$\acute{i}(\ell(H)) \subseteq \ell(\acute{i}_\delta(H))$ for each $H \subseteq \mathcal{M}$.

Proof. (i) $\implies$ (ii) Let $\mathrm{H} \subseteq \mathcal{M}$.

$\mathrm{H} \subseteq \mathcal{M} \implies \mathcal{M} \setminus \mathrm{H} \subseteq \mathcal{M}$

$$\left.\begin{array}{r} \xRightarrow{\text{(i)}} \ell[(\mathcal{M}\setminus\acute{i}_\delta(\mathrm{H}))] = \ell[\acute{c}_\delta(\mathcal{M} \setminus\mathrm{H})] \subseteq \delta\text{gp-}\acute{c}(\ell[\mathcal{M}/\mathrm{H}]) \\ \ell \text{ is bijection} \end{array}\right\} \implies$$

$$\implies \mathcal{N} \setminus\ell[\acute{i}_\delta(\mathrm{H})] \subseteq \mathcal{N} \setminus \delta\text{gp-}\acute{i}(\ell[\mathrm{H}])$$

$\implies \delta\text{gp-}\acute{i}(\ell[\mathrm{H}]) \subseteq \ell(\acute{i}_\delta[\mathrm{H}]).$

(ii) $\implies$ (i) : Let $\mathrm{K} \subseteq \mathcal{M}$.

$$\left.\begin{array}{r} \mathrm{K} \subseteq \mathcal{M} \implies \mathcal{M}\setminus\mathrm{K}\subseteq\mathcal{M} \xRightarrow{\text{(ii)}} \delta\text{gp-}\acute{i}(\ell[\mathcal{M}\setminus\mathrm{K}]) \subseteq \ell[\acute{i}_\delta(\mathcal{M}\setminus\mathrm{K})] \\ \ell \text{ is bijection} \end{array}\right\} \implies$$

$$\implies \mathcal{N}\setminus \delta\text{gp-}\acute{c}(\ell[\mathrm{K}]) \subseteq \mathcal{N} \setminus \ell[\acute{c}_\delta(\mathrm{K})]$$

$$\implies \ell(\acute{c}_\delta(\mathrm{K})) \subseteq \delta\text{gp-}\acute{c}(\ell(\mathrm{K})).$$ □

Lemma 3.1. *Let $\mathcal{N} \subset \mathcal{M}$ and $\mathcal{N} \in \mathbb{O}\ (\mathcal{M})$. The following hold* [15].

(i) $K \in \mathbb{RO}(\mathcal{M}) \implies Y \cap K \in \mathbb{RO}(\mathcal{N},\, \tau_\mathcal{N})$.

(ii) $H \in \mathbb{RO}(\mathcal{N},\, \tau_\mathcal{N}) \implies (\exists$ a $K \in \mathbb{RO}(\mathcal{M})$ such that $H = \mathcal{N} \cap K$.

where $\tau_\mathcal{N}$=$\{\mathcal{N} \cap G \mid G \in \mathbb{O}(\mathcal{M})\}$.

Theorem 3.11. *If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i. and $K \in \Im$, then the restriction $\ell/_K : K \to \mathcal{N}$ is c.δgp-i..*

Proof. Let $\mathrm{J} \in \delta\mathbb{GPO}(\mathcal{N})$.

$$\left.\begin{array}{r} \mathrm{J} \in \delta\mathbb{GPO}(\mathcal{N}) \xRightarrow{\ell \text{ is c.}\delta\text{gp.i.}} \ell^{-1}(\mathrm{J}) \in \mathbb{RO}(\mathcal{M}) \\ \mathrm{K} \in \Im \end{array}\right\} \implies$$

$\xRightarrow{\text{lemma 3.1}} (\ell/_K)^{-1}(\mathrm{J}) = \ell^{-1}(\mathrm{J}) \cap \mathrm{K} \in \mathbb{RO}(\mathrm{K}).$ □

Lemma 3.2. *Let $\mathcal{N} \subseteq \mathcal{M}$ and $\mathcal{N} \in \mathbb{PO}(\mathcal{M})$. Then, $\mathcal{N} \cap K \in \mathbb{RO}(\mathcal{N})$ for each $K \in \mathbb{RO}(\mathcal{M})$* [2].

Theorem 3.12. *If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \Re)$ is a c.δgp-i. and $K \in \mathbb{PO}(\mathcal{M})$, then $\ell/_k : K \to \mathcal{N}$ is c. δgp-i..*

Proof. This can be inferred from Lemma 3.2. □

Theorem 3.13.

(i) If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is δgp-i. and $\acute{k}$: $(\mathcal{N}, \gamma) \to (\mathcal{P}, \sigma)$ is δgp-i., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \to (P, \sigma)$ is also c.δgp-i.

(ii) If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i. and $\acute{k}$: $(\mathcal{N}, \gamma) \to (\mathcal{P}, \sigma)$ is c.δgp-i., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \to (P, \sigma)$ remains c.δgp-i.

(iii) If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is an R-map and $\acute{k}$: $(\mathcal{N}, \gamma) \to (\mathcal{P}, \sigma)$ is c.δgp-i., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \to (P, \sigma)$ is c.δgp-i.

(iv) If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i. and $\acute{k}$: $(\mathcal{N}, \gamma) \to (\mathcal{P}, \sigma)$ is δgp-c., then the composition $\acute{k}o\ell$:$(\mathcal{M}, \Im) \to (\mathcal{P}, \sigma)$ is also c.c..

Proof. Straightforward. □

Definition 3.3. *If J, H $\in \mathbb{RO}(\mathcal{M})$ (resp., $\delta\mathbb{GPO}(\mathcal{M})$) cannot be found such that $J \cap H = \emptyset$ and $J \cup H = \mathcal{M}$, then a space $(\mathcal{M}, \Im)$ is referred to as almost connected* [8] *(resp., δgp-connected* [24]*).*

Theorem 3.14. *If ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is a surjective function that is c.δgp-i. and $(\mathcal{M}, \Im)$ is almost connected, then $(\mathcal{N}, \gamma)$ is δgp-connected.*

Proof. Let us consider that $(\mathcal{N}, \gamma)$ is not δgp-connected.

$(\mathcal{N}, \gamma)$ is not δgp-connected $\Longrightarrow$

$$\left.\begin{array}{r}(\exists\ \mathrm{C},\ \mathrm{D} \in \delta\mathbb{GPO}(\mathcal{N})\setminus\{\emptyset\})(\mathrm{C}\cap\mathrm{D} = \emptyset)(\mathrm{C}\cup\mathrm{D} = \mathcal{N}) \\ \ell \text{ is c.}\delta\text{gp-i. surjection}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\ell^{-1}(C), \ell^{-1}(D) \in \mathbb{RO}(\mathrm{M}) \setminus\{\emptyset\})(\ell^{-1}(\mathrm{C}\cap\mathrm{D}) = \ell^{-1}(\emptyset))(\ell^{-1}(\mathrm{C}\cup\mathrm{D}) = \ell^{-1}(\mathcal{N}))$

$\Longrightarrow$ $(\ell^{-1}(\mathrm{C}), \ell^{-1}(\mathrm{D}) \in \mathbb{RO}(\mathrm{M})\setminus\{\emptyset\})(\ell^{-1}(\mathrm{C}) \cap \ell^{-1}(\mathrm{D}) = \emptyset)(\ell^{-1}(\mathrm{C}) \cup \ell^{-1}(\mathrm{D}) = \mathcal{M})$.

This $(\mathcal{M}, \Im)$ is not almost connected. □

Definition 3.4.

(i) If each regular open cover of a space $(\mathcal{M}, \Im)$ has a finite subcover, then the space is said

to as nearly compact (briefly, n.c.) [20]*;*

(ii) Every countable cover of a space $(\mathcal{M}, \Im)$ *by regular open sets that has a finite subcover is called nearly countably compact (briefly, n.c.c.)* [9].

(iii) If there is a countable subcover for each cover of $\mathcal{M}$ *by regular open sets, then the space* $(\mathcal{M}, \Im)$ *is referred to as nearly Lindelöf(briefly, n.L.)* [8].

(iv) If each δgp*-open cover of a space* $(\mathcal{M}, \Im)$ *has a finite subcover, then the space is said to as* δgp*-compact (briefly,* $\delta gp.c.$*)* [26]*;*

(v) Every countable cover of a space $(\mathcal{M}, \Im)$ *by* δgp*-open sets that has a finite subcover is called countably* δgp*-compact (briefly, c.*δgp*-c.);*

(vi) If there is a countable subcover for each cover of $\mathcal{M}$ *by* δgp*-open sets, then the space* $(\mathcal{M}, \Im)$ *is referred to as* δgp*-Lindelöf(briefly,* $\delta gp.L.$*)*

Theorem 3.15. *Let* ℓ*:* $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ *be a c.* δgp*-i. surjection, then the following hold:*
(i) If $(\mathcal{M}, \Im)$ *is n.c., then* $(\mathcal{N}, \gamma)$ *is* δgp*-c.;*
(ii) If $(\mathcal{M}, \Im)$ *is n.L., then* $(\mathcal{N}, \gamma)$ *is* δgp*-L.;*
(iii)If $(\mathcal{M}, \Im)$ *is n.c.c., then* $(\mathcal{N}, \gamma)$ *is c.*δgp*-c..*

Proof. (i) Let $\mathcal{M}$ be n.c. and $\mathcal{A}$ be an δgp-open cover of $\mathcal{N}$.
$(\mathcal{A} \subset \delta\mathbb{GPO}(\mathcal{N}))\ (\mathcal{N} = \cup\ \mathcal{A}) \xRightarrow{\ell \text{ is c.}\delta\text{ gp.i}}$

$$\left.\begin{array}{r}(\text{K}:=\{\ell^{-1}(\text{J})|\ \text{J}\in\mathcal{A}\}\subset\mathbb{RO}(\mathcal{M}))(\mathcal{M}=\cup\ \text{K})\\ \mathcal{M}\text{ is nearly compact}\end{array}\right\}\Longrightarrow(\exists K^*\subset K)(|\ K^*\ |<\aleph_0(\mathcal{M}=\cup K^*))$$

$\xRightarrow{\ell \text{ is surjective}}$ (K: $= (\ell(\text{K}^*)\subset\ell(\text{K})=\mathcal{A})(|\ell(\text{B}^*)| < \aleph_0)$ $(\text{K} = \ell(\mathcal{M}) = \ell(\cup\ \text{B}^*) = \cup_{N\in N^*}\ \ell(\text{K}))$.

(ii) Let $\mathcal{M}$ be n.L. and $\mathcal{A}$ be an δgp-open cover of $\mathcal{N}$.
$(\mathcal{A} \subset \delta\mathbb{GPO}(\mathcal{N}))(|\mathcal{A}| \leq \aleph_0(\mathcal{N} = \cup\ \mathcal{A})) \xRightarrow{\ell \text{ is c.}\delta\text{ gp.i.}}$

$$\Longrightarrow\ \left.\begin{array}{r}\left(K:=\{\ell^{-1}(J)\mid J\in\mathcal{A}\}\subseteq\text{RO}(\mathcal{M})\right)(\mathcal{M}=\cup K)\\ \mathcal{M}\text{ is n.c.}\end{array}\right\}\Longrightarrow$$

$$\Longrightarrow\ \left.\begin{array}{r}\left(\exists K^*\subseteq K\quad|\ K\ |<\aleph_0\ \ \wedge\ \ \mathcal{M}=\cup K^*\right)\\ \ell\text{ is surjective}\end{array}\right\}\Longrightarrow$$

$$\Longrightarrow\ (\ell(\text{K}^*)\subset\ell(\text{K})=\mathcal{A})(|\ell(\text{K}^*)\ |<\aleph_0)(\mathcal{N}=\ell(\mathcal{M})=\ell(\cup\ \text{K}^*)=\text{U}_{\mathcal{N}\in\mathcal{N}^*}\ \ell(\text{K})).$$

(iii) Let $\mathcal{M}$ be n.c.c. and $\mathcal{A}$ be an δgp-open countable cover of $\mathcal{N}$.

$(\mathcal{A} \subset \delta\mathbb{GPO}(\mathcal{N}))(|\mathcal{A}| \leq \aleph_0)(\mathcal{N} = \cup\ \mathcal{A}) \xRightarrow{\ell \text{ is c.}\delta \text{ gp.i.}}$

$$\left.\begin{array}{r}\Longrightarrow \ \left(K := \{\ell^{-1}(J) \mid J \in \mathcal{A}\} \subseteq \mathrm{RO}(\mathcal{M})\right) \ (\mathcal{M} = \cup K) \\ \mathcal{M} \text{ is n.c.c.}\end{array}\right\} \Longrightarrow$$

$$\left.\begin{array}{r}\Longrightarrow \ \left(\exists K^* \subseteq K \ \text{ with } |K^*| < \aleph_0 \ \wedge \ \mathcal{M} = \cup K^*\right) \\ \ell \text{ is surjective}\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow (\ell\ (\mathrm{K}^*) \subset \ell(\mathrm{K}) = \mathcal{A})(|\ \ell(\mathrm{K}^*)| < \aleph_0)(\mathrm{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{K}^*)) = \cup_{K \in K^*}\ \ell(\mathrm{K}). \qquad \square$$

Definition 3.5.

(i) If there is a finite subcover for each regular closed (resp., δgp-closed) cover of a space ($\mathcal{M}$, $\Im$) then the space is said to be S-closed [27] *(resp., δgp-closed compact).*

(ii) If every countable cover of $\mathcal{M}$ by regular closed (resp., δgp-closed) sets has a finite subcover, then the space ($\mathcal{M}$, $\Im$) is called countably S-closed compact [1]*(resp., countably δgp-closed compact).*

(iii) If any cover of $\mathcal{M}$ by regular closed (resp., δgp-closed) sets admits a countable subcover, then the space ($\mathcal{M}$, $\Im$) is called S-Lindelöf [16] *(resp., δgp-closed Lindelöf).*

Theorem 3.16. *Let ℓ: ($\mathcal{M}$, $\Im$) → ($\mathcal{N}$, γ) be a c.δgp-i. surjection. The following is true:*

(i) If ($\mathcal{M}$, $\Im$) is S-closed, then ($\mathcal{N}$, γ) is δgp-closed compact.

(ii) If ($\mathcal{M}$, $\Im$) is S-Lindelöf, then ($\mathcal{N}$, γ) is δgp-closed Lindelöf.

(iii) If ($\mathcal{M}$, $\Im$) is countably S-closed compact, then ($\mathcal{N}$, γ) is countably δgp-closed compact.

Proof. (i) Let ($\mathcal{M}$, $\Im$) be S-closed and compact and $\mathcal{A}$ be an δgp-closed cover of ($\mathcal{N}$, γ).

$$\left.\begin{array}{r}(\mathcal{A} \subset \delta\mathbb{GPC}(\mathcal{N}))(\mathcal{N} = \cup\ \mathcal{A}) \xRightarrow{\ell \text{ is c.}\delta \text{ gp.i.}} (\mathrm{H}{:} = \{\ell^{-1}(\mathrm{K})|\mathrm{K} \in \mathcal{A}\} \subset \mathbb{RC}(\mathcal{M}))(\mathcal{M} = \cup\ \mathrm{H}) \\ \mathcal{M} \text{ is S-closed}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow (\exists\ \mathcal{N}^* \subset \mathrm{H})\ (|\mathrm{H}^*| < \aleph_0)\ (\mathcal{M} = \cup\ \mathrm{H}) \xRightarrow{\ell \text{ is surjective}}$

$\Longrightarrow (\mathrm{H} := (\ell(\mathrm{H}^*) \subset \ell(\mathrm{H}){=}\mathcal{A})(|\ell(\mathrm{H}^*)| < \aleph_0)(\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup\ \mathrm{H}^*) = \cup\ _{H \in H^*}\ \ell(\mathrm{H})).$

(ii) Let ($\mathcal{M}$, $\Im$) be S - Lindelöf and $\mathcal{A}$ be an δgp-closed countable cover of $\mathcal{N}$.

$(\mathcal{A} \subset \delta\mathbb{GPC}(\mathrm{Q}))\ (|\ \mathcal{A}\ | \leq \aleph_0)(\mathcal{N} = \cup\ \mathcal{A}) \xRightarrow{\ell \text{ is c.}\delta \text{ gp.i.}}$

$$\left.\begin{array}{r}\Longrightarrow \ \left(B := \{\ell^{-1}[A] \mid A \in \mathcal{A}\} \subseteq \delta RC(\mathcal{M})\right) (\mathcal{M} = \cup B) \\ \mathcal{M} \text{ is Lindelöf closed}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ ($\exists$ B$^*\subset$ B)($|$B$^*| < \aleph_0$)($\mathcal{M} = \cup$ B*)

$\xrightarrow{\ell \text{ is surjective}}$ (ℓ[B*] $\subset \ell$[B] $= \mathcal{A}$) ($|\ell$[B*]$| \leq \aleph_0$)

($\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup$ B*)) $= \cup_{B\in B^*}$ ℓ(B).

(iii) Let ($\mathcal{M}$, $\Im$) be countable S-closed compact and $\mathcal{A}$ be an δgp-closed countable cover of $\mathcal{N}$.

($\mathcal{A} \subset \delta\mathbb{GPC}(\mathcal{N})$) ($|\mathcal{A}| \leq \aleph_0$)($\mathcal{N} = \cup$ $\mathcal{A}$) $\xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}}$

$$\left.\begin{array}{r}(\text{J:=}\{\ell^{-1}\ (\text{A}) \mid \text{A} \in \mathcal{A}\} \subset \mathbb{RC}(\mathcal{M}))(|\text{J}| \leq \aleph_0)(\mathcal{M} = \cup\ \text{J})\\ \mathcal{M} \text{ is countable S-closed compact})\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow \left.\begin{array}{r}\left(\exists J^* \subseteq J \text{ with } |J^*| < \aleph_0 \quad \wedge \quad M = \cup J^*\right)\\ \ell \text{ is surjective}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ (ℓ(J*) $\subset \ell$(J) $= \mathcal{A}$) ($|$ ℓ(J*) $| < \aleph_0$)($\mathcal{N} = \ell(\mathcal{M}) = \ell(\cup$ J*) $= \cup_{J\in J^*}$ ℓ(J). $\square$

Definition 3.6. *A space ($\mathcal{M}$, $\Im$) is defined as almost regular* [19] *(or strongly δgp-regular) if for any $L \in \mathbb{RC}(\mathcal{M})$ (or $\delta\mathbb{GPC}(\mathcal{M})$ and any point $q \in \mathcal{M}\backslash L$, there exist C, $D \in \Im$ (or $\delta\mathbb{GPO}(\mathcal{M})$) such that $q \in C$, $L \subseteq D$ and $C \cap D = \emptyset$).*

Example 3.4. *Consider $\mathcal{M} = \{u_1, u_2, u_3, u_4, u_5\}$ with the topology $\Im = \{\emptyset, \mathcal{M}, \{u_1, u_2\}, \{u_3, u_4\}, \{u_1, u_2, u_3, u_4\}\}$. Then, ($\mathcal{M}$, $\Im$) is strongly δgp-regular*

Theorem 3.17. *If ℓ: ($\mathcal{M}$, $\Im$) $\rightarrow$ ($\mathcal{N}$, $\Re$) c.δgp-i. δgp-open bijection.*
If ($\mathcal{M}$, $\Im$) is an almost regular, then ($\mathcal{N}$, γ) is strongly δgp-regular.

Proof. Let F $\in \delta\mathbb{GPC}(\mathcal{N})$ and ℓ(r) $=$ s $\notin$ F.

$$\left.\begin{array}{r}\ell(\text{r})\text{=s} \notin \text{F} \in \delta\mathbb{GPC}(\mathcal{N}) \xrightarrow{\ell \text{ is c.}\delta\text{gp.i.}} \text{r} \notin \ell^{-1}\ (\text{F}) \in \mathbb{RC}(\mathcal{M})\\ \mathcal{M} \text{ is almost regular}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ ($\exists$ U, V $\in \delta\mathbb{GPO}(\mathcal{M})$) (r $\in$ U) (ℓ^{-1}(F) $\subset$ V)(U$\cap$V$= \emptyset$)

$\xrightarrow{\ell \text{ is } \delta\text{gp-open bijection}}$ (ℓ(U), ℓ(V) $\in \delta\mathbb{GPO}(\mathcal{N})$)(s $= \ell$(r) $\in \ell$(U))(F$\subset\ell$(V))(ℓ(U) $\cap$ ℓ(V) $= \emptyset$).

$\square$

Definition 3.7. *A space ($\mathcal{M}$, $\Im$) is defined as follows:*

(a) Almost normal: [21] *For each $G \in C(\mathcal{M})$ and each $H \in \mathbb{RC}(M)$ such that $G \cap H = \emptyset$, there exist J, $K \in \Im$ such that $J \cap K = \emptyset$, $G \subseteq J$ and $H \subseteq K$.*

(b) Strongly δgp-normal: For any pair G, $H \in \delta\mathbb{GPC}(\mathcal{M})$ such that $G \cap H = \emptyset$, there exist J, $K \in \delta\mathbb{GPO}(\mathcal{M})$ such that $J \cap K = \emptyset$, $G \subseteq J$ and $H \subseteq K$.

Example 3.5. *Consider $\mathcal{M}$ =$\{u_1, u_2, u_3, u_4\}$ with the topology $\Im$ =$\{\emptyset, \mathcal{M}, \{u_1\}, \{u_2\}, \{u_1, u_2\}, \{u_1, u_3\}, \{u_1, u_2, u_3\}\}$. Then, $(\mathcal{M}, \Im)$ is strongly δgp-normal.*

Theorem 3.18. *If $(\mathcal{M}, \Im)$ is an almost normal space then $(\mathcal{N}, \gamma)$ is strongly δgp-normal whenever $\ell:(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c. δgp-i. and δgp-open bijection.*

Proof. Let C, D $\in \delta\mathbb{GPC}(\mathcal{N})$ and C $\cap$ D $= \emptyset$.

$$\left.\begin{array}{r}(\mathrm{C}, \mathrm{D} \in \delta\mathbb{GPC}(\mathcal{N}))\ (\mathrm{C} \cap \mathrm{D}) = \emptyset \\ \ell \text{ is c.}\delta\text{gp.i.}\end{array}\right\} \Longrightarrow (\ell^{-1}(C), \ell^{-1}(D) \in \mathbb{RC}(\mathcal{M}))(\ell^{-1}(C \cap D) = \ell^{-1}(\emptyset))$$

$$\left.\begin{array}{r}\Longrightarrow \Big(\ell^{-1}(C),\ \ell^{-1}(D) \in \mathrm{RC}(\mathcal{M}) \ \wedge \ \ell^{-1}(C) \cap \ell^{-1}(D) = \emptyset\Big) \\ \mathrm{RC}(\mathcal{M}) \subseteq C(\mathcal{M})\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow (\ell^{-1}(\mathrm{C}) \in \mathrm{C}(\mathcal{M}))\ (\ \ell^{-1}(\mathrm{D}) \in \mathbb{RC}(\mathcal{M}))\ (\ell^{-1}(\mathrm{C}) \cap \ell^{-1}(\mathrm{D}) = \emptyset)$$

$$\xRightarrow{(\mathcal{M},\ \Im) \text{ is almost normal}}$$

$$\left.\begin{array}{r}\Longrightarrow \Big(\exists U, V \in \delta\mathrm{GPO}(\mathcal{M})\ :\ \ell^{-1}(C) \subseteq U,\ \ell^{-1}(D) \subseteq V,\ U \cap V = \emptyset\Big) \\ \ell \text{ is a } \delta\text{-gp-open bijection}\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow (\ell(\mathrm{U}), \ell(\mathrm{V}) \in \delta\mathbb{GPO}(\mathcal{N}))(\mathrm{C} \subseteq \ell(\mathrm{U}))(\mathrm{D} \subseteq \ell(\mathrm{V}))\ (\ell(\mathrm{U}) \cap \ell(\mathrm{V}) = \emptyset.$$

□

Definition 3.8. *A space $(\mathcal{M}, \Im)$ is said to be δgp-T_1* [25] *(resp., r-T_1* [8]*) if for each r, s (r $\neq$ s) $\in \mathcal{M}$, there exist K_1 and $K_2 \in \delta\mathbb{GPO}(\mathcal{M})$ (resp., $\mathbb{O}(\mathcal{M})$) $r \in K_1$, $s \in K_2$, $r \notin K_2$ and $s \notin K_1$.*

Theorem 3.19. *If $\ell:(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ is c.δgp-i. injection and $(\mathcal{N}, \gamma)$ is δgp-T_1, then $(\mathcal{M}, \Im)$ is r-T_1.*

Proof. Let r, s $\in \mathcal{M}$ and r $\neq$ s.

$$\left.\begin{array}{r}((\mathrm{r}, \mathrm{s}) \in \mathcal{M})(\mathrm{r} \neq \mathrm{s}) \xRightarrow{\ell \text{ is injective}} \ell(\mathrm{r}) \neq \ell(\mathrm{s}) \\ (\mathcal{N}, \gamma) \text{ is } \delta\text{gp - T}_1\end{array}\right\} \Longrightarrow$$

$$\Longrightarrow (\exists\ \mathrm{U} \in \delta\mathbb{GPO}(\mathcal{N}, \ell(\mathrm{r}) \text{ and } \mathrm{V} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{s})))\ (\ell(\mathrm{r}) \notin \mathrm{V})\ (\ell(\mathrm{s}) \notin \mathrm{U})$$

$$\xRightarrow{\ell \text{ is c.}\delta\text{gp.i.}} (\ell^{-1}(\mathrm{U}) \in \mathbb{RO}(\mathcal{M}, \mathrm{r}))\ (\ell^{-1}(\mathrm{V}) \in \mathbb{RO}(\mathcal{M}, \mathrm{s}))(\mathrm{r} \notin \ell^{-1}(\mathrm{V}))\ (\mathrm{s} \notin \ell^{-1}(\mathrm{U})).$$

□

Definition 3.9. *A space* $(\mathcal{M}, \Im)$ *is said to be* δ*gp-Hausdorff* [24] *(resp.* r-T_2 [8]*) for each* p, q $(p \neq q) \in \mathcal{M}$*, there exist* $\in J$, K $\delta\mathbb{GPO}(\mathcal{M})$ *(resp.,* $\mathbb{RO}(\mathcal{M})$*) such that* $p \in J$, $q \in K$ *and* $J \cap K = \emptyset$.

Theorem 3.20. *If* ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ *is c.*δ*gp-i. injection and* $(\mathcal{N}, \gamma)$ *is* δ*gp-Hausdorff, then* $(\mathcal{M}, \Im)$ *is* r-T_2.

Proof. Let r, s $\in \mathcal{M}$ and r $\neq$ s.

$$\left.\begin{array}{r}(\mathrm{r},\ \mathrm{s}) \in \mathcal{M}\times\mathcal{M})(\mathrm{r} \neq \mathrm{s}) \xRightarrow{\ell \text{ is injective}} \ell(\mathrm{r}) \neq \ell(\mathrm{s}) \\ (\mathcal{N},\ \gamma) \text{ is } \delta\text{gp-Hausdorff}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\exists\ \mathrm{A} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{r}))\ (\exists\ \mathrm{B} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{s}))\ (\ \mathrm{A} \cap \mathrm{B} = \emptyset)$

$\xRightarrow{\ell \text{ is c.}\delta\text{gp.i.}}$ $(\ell^{-1}(\mathrm{A}) \in \mathbb{RO}\ (\mathcal{M}, \mathrm{r}))\ (\ell^{-1}(\mathrm{B}) \in \mathbb{RO}(\mathcal{N}, \mathrm{s}))(\ell^{-1}(\mathrm{A}) \cap \ell^{-1}(\mathrm{B}) = \emptyset)$. □

Theorem 3.21. *Let* $(\mathcal{N}, \gamma)$ *be* δ*gp-Hausdorff space. If* ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ *and* $\acute{k}$: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ *are c.*δ*gp-i.e, then* $L = \{q \mid \ell(q) = \acute{k}(q)\}$ *is* δ*-closed in* $\mathcal{M}$.

Proof. Suppose that q $\notin$ L.

$$\left.\begin{array}{r}\mathrm{q} \notin \mathrm{L} \implies \ell(\mathrm{q}) \neq \acute{\mathrm{k}}(\mathrm{q}) \\ (\mathcal{N},\ \gamma) \text{ is } \delta\text{gp-Hausdorff}\end{array}\right\} \Longrightarrow$$

$\Longrightarrow$ $(\exists\ \mathrm{G} \in \delta\mathbb{GPO}(\mathrm{N}, \ell(\mathrm{q})))(\exists\ \mathrm{H} \in \delta\mathbb{GPO}(\mathcal{N}, \acute{\mathrm{k}}(\mathrm{q})))(\mathrm{G} \cap \mathrm{H} = \emptyset)$

$\xRightarrow{\ell \text{ and } \acute{\mathrm{k}} \text{ are c.}\delta\text{gp.i.}}$

$(\ell^{-1}(\mathrm{G}) \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))(\acute{\mathrm{k}}^{-1}(\mathrm{H}) \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))(\ell^{-1}(\mathrm{G} \cap \mathrm{H}) = \emptyset)(\acute{\mathrm{k}}^{-1}(\mathrm{G} \cap \mathrm{H}) = \emptyset)$

$\Longrightarrow$ $(\mathrm{U} := \ell^{-1}(\mathrm{G}) \cap \acute{\mathrm{k}}^{-1}(\mathrm{H}) \in \mathbb{RO}(\mathcal{M}, \mathrm{q}))\ (\mathrm{U} \cap \mathrm{L} = \emptyset) \Longrightarrow \mathrm{q} \notin \acute{c}_\delta(\mathrm{L})$.

Then, L is δ-closed in $\mathcal{M}$. □

Theorem 3.22. *Let* $(\mathcal{N}, \gamma)$ *be* δ*gp-Hausdorff space. If* ℓ: $(\mathcal{M}, \Im) \to (\mathcal{N}, \gamma)$ *is c.*δ*gp-i., then* $K = \{(p, q) \mid \ell(p) = \ell(q)\}$ *is* δ*-closed in* $\mathcal{M}\times\mathcal{M}$.

Proof. Let (p, q) $\notin$ K.

$$\left.\begin{array}{r}(\mathrm{p},\ \mathrm{q}) \notin \mathrm{K} \implies \ell(\mathrm{p}) \neq \ell(\mathrm{q}) \\ (\mathcal{N},\ \gamma) \text{ is } \delta\text{gp-Hausdorff}\end{array}\right\} \Longrightarrow$$

$(\exists\ \mathrm{G} \in \delta\mathbb{GPO}(\mathcal{N}, \ell(\mathrm{p})))(\exists\ \mathrm{H} \in \delta\mathbb{GPO}\ (\mathcal{N}, \ell(\mathrm{q})))(\mathrm{G} \cap \mathrm{H} = \emptyset)$

$\xRightarrow{\ell \text{ is c.}\delta\text{gp.i.}}$ $(\ell^{-1}(\mathrm{G}) \in \mathbb{RO}\ (\mathcal{M}, \mathrm{p}))\ (\ell^{-1}(\mathrm{H}) \in \mathbb{RO}\ (\mathcal{M}, \mathrm{q}))(\ell^{-1}(\mathrm{G}) \cap \ell^{-1}(\mathrm{H}) = \emptyset)$

$\Longrightarrow$ $(\mathrm{U} := \ell^{-1}(\mathrm{G}) \times \ell^{-1}(\mathrm{H}) \in \mathbb{RO}(\mathcal{M}\times\mathcal{M}, (\mathrm{p}, \mathrm{q}))(\mathrm{U} \cap \mathrm{K} = \emptyset)$

$\Longrightarrow$ $(\mathrm{p}, \mathrm{q}) \notin \acute{c}_\delta(\mathrm{K})$

Then, $\mathcal{M}$ is δ-closed in $\mathcal{M}\times\mathcal{M}$. □

4. Conclusion

In this research paper, we have defined completely δgp-irresolute functions, strongly δgp-regular space, and strongly δgp-normal space in topological spaces with an example and give the proof of the theorems based on their properties. We are interested in extending our research work to convergence in bitopological spaces and nano topological spaces. In addition, we plan to find some interesting concepts in bitopological spaces.

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References

[1] Abd EI-Monsef, M. E., El-Deep, S. N., & Mohmoud, R. A. (1983). β-open sets and β-continuous mappings. Bulletin of the Faculty of Science, Assiut University, Section C, 12, 77-90.

[2] Allam, A. A., Zaharan, A. M., & Hasanein, I. A. (1987). On almost continuous, δ-continuous and set connected mappings. Indian Journal of Pure and Applied Mathematics, 18(11), 99-996.

[3] Arokiarani, I., Balachandran, K., & Dontchev, J., (1999). Some characterizations of gp-irresolute and gp-continuous maps between topological spaces. Memoirs of the Faculty of Science, Kochi University. Series A, Mathematics, 20, 93-104.

[4] Arya, S. P., & Gupta, R. (1974). On strongly continuous functions. Kyungpook Mathematical Journal, 14,131-143.

[5] Balasubramanian, S., & Sarada, M. L. (2012). Almost contra-gpr-continuity. International Journal of Mathematical Engineering and Science, 6(1), 1-8.

[6] Benchalli S. S., & Toranagatti J. B. (2016). Delta generalized preclosed sets in topological spaces. International Journal of Contemporary Mathematical Sciences, 11, 281–292.

[7] Crossley, S. G., & Hildebrand, S. K. (1972). Semi-topological properties. Fundamenta Mathematicae, 74, 233-254.

[8] Ekici, E. (2005). Generalization of perfectly continuous, regular set-connected and clopen functions. Acta Mathematica Hungarica, 107(3), 193-206.

[9] Ergun, N. (1980). On nearly paracompact spaces. Istanbul University Science Faculty the Journal of Mathematics Physics and Astronomy, 45, 65-87.

[10] Gnanambal, Y. (1997). On generalized pre-regular closed sets in topological spaces. Indian Journal of Pure and Applied Mathematics, 28, 351-360.

[11] Gnanambal, Y., & Balachandran, K. (1999). On gpr-continuous functions in topological spaces. Indian Journal of Pure and Applied Mathematics, 30, 581-593.

[12] Jankovic, D. S. (1985). A note on mappings of extremally disconnected spaces. Acta Mathematica Hungarica, 46(1-2), 83-92.

[13] Jankovic, D. S. (1983). On locally irreducible spaces. Annales de la Societe Scientifique de Bruxelles, Seris 1-Sciences Mathematiques Astronomiques et Physiques, 97, 59-72.

[14] Navalagi, G. B. (2011). Completely preirresolute functions completely gp-irresolute functions. International Journal of Mathematics and Computing Applications, 3, 55-65.

[15] Long, P. E., & Herington, L. L. (1978). Basic properties of regular-closed functions. Rendiconti del Circolo Matematico di Palermo, 27, 20-28.

[16] Maio, G. Di. (1984). S-closed spaces, S-sets and S-continuous functions. Accademia delle Scienze di Torino, 118, 125-134.

[17] Maki, H., Umehara, J., & Noiri, T. (1996). Every topological space is pre-$T_{1/2}$. Memoirs of the Faculty of Science, Kochi University. Series A, Mathematics, 17, 33-42.

[18] Mashhour, A. S., Abd El-Monsef, M. E., & EL-Deeb S. N. (1982). On precontinuous and weak precontinuous mappings. Proceedings of the Mathematical and Physical Society of Egypt, 53, 47-53.

[19] Singal, M. K., & Arya, S. P. (1969). On almost regular spaces. Glasnik Matematički, 4(24), 89-99.

[20] Singal, M. K., Singal, A. R., & Mathur, A. (1969). On nearly-compact spaces. Bollettino dell'Unione Matematica Italiana, 4(2), 702-710.

[21] Singal M. K., & Arya, S. P. (1970). Almost normal and almost completely regular spaces. Glasnik Matematički, 5(25), 141-152.

[22] Stone, M. H. (1937). Applications of the theory of Boolean rings to general topology. Transactions of the American Mathematical Society, 41, 373-381.

[23] Toranagatti, J. B. (2017). Delta generalized pre-continuous functions in topological spaces. International Journal of Pure and Applied Mathematics, 116, 829-843.

[24] Toranagatti, J. B. (2018). On contra delta generalized pre-continuous functions. International Journal of Scientific Research in Mathematical and Statistical Sciences, 5, 283-288.

[25] Toranagatti, J. B. (2020). On almost contra δgp-continuous functions in topological spaces. Malaya Journal of Matematik, 8, 1213-1218.

[26] Toranagatti, J. B. (2020). A new class of continuous functions via δgp-open sets in topological spaces. The Aligarh Bulletin of Mathematics, 39(2), 103-117.

[27] Thompson, T. (1976). S-closed spaces. Proceedings of the American Mathematical Society, 60, 335-338.

[28] Velicko, N. V. (1968). H-closed topological spaces. American Mathematical Society Translations, Series 2, 78(2), 103-118.

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$\widetilde{L}_r-$BIHARMONIC NULL HYPERSURFACES IN GENERALIZED ROBERTSON-WALKER SPACETIMES

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(DEDICATED TO THE MEMORY OF PROF. DR. KRISHAN LAL DUGGAL (1929 - 2022))

Abstract. In this paper, we derive $\widetilde{L}_r-$biharmonic equations for null hypersurfaces M in Generalized Robertson-Walker (GRW) spacetimes using linearized operators $\widetilde{L}_r$ $(0 \leq r \leq \dim(M))$ built uniquely from the rigged structure given by a timelike closed and conformal rigging vector field ζ. After providing a characterization for $\widetilde{L}_r-$harmonic null hypersurfaces we study $\widetilde{L}_r-$biharmonic null hypersurfaces for $r = 0$ and $r = 1$ in low dimensions: null surfaces and $3-$dimensional null hypersurfaces.

Keywords:Null hypersurface, $\widetilde{L}_r-$biharmonic, GRW spacetimes, Rigging vector field.

2020 Mathematics Subject Classification: Primary: 53C40, Secondary: 53C42, 53C50.

1. Introduction

Consider an isometric immersion $\psi : M^n \to \mathbb{E}^m$ from a Riemannian manifold M^n into the Euclidean space $\mathbb{E}^m$. Denote by H and Δ the mean curvature vector field of M^n and the Laplace operator of M^n with respect to the induced Riemannian metric of $\mathbb{E}^m$. From the Beltrami's formula $\Delta\psi = nH$ we see that M is minimal in $\mathbb{E}^m$ if and only if its coordinate functions are harmonic. Observe that $\Delta^2\psi = n\Delta H$. Manifolds with $\Delta H = 0$, or equivalently $\Delta^2\psi = 0$ are called biharmonic. Obviously, minimal submanifolds (i.e $H = 0$) are biharmonic. The question that arises is whether the class of biharmonic submanifolds is reduced to that of minimal submanifolds. Several authors have proved it in some cases (cf. [1, 16, 18, 20, 22] and notes in the report [14]).

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A well-known Bang-Yen Chen's conjecture says : Any biharmonic submanifold in pseudo-Euclidean space $\mathbb{E}_s^{n+p}$ is minimal. But in contrast to the Euclidean case ($s = 0$, where the conjecture is not entirely solved), the conjecture generally fails for submanifolds in a pseudo-Euclidean space. B.-Y. Chen and S. Ishikawa [13] gave examples of nonminimal biharmonic (also called proper biharmonic) space-like surfaces with constant mean curvature in pseudo-Euclidean spaces $\mathbb{E}_s^4$ ($s = 1, 2$) and proper biharmonic surfaces of signature $(1,1)$ in $\mathbb{E}_s^4$ ($s = 1, 2, 3$) in [15]. Furthermore, in case of hypersurfaces, Chen has found a good relation between the finite type hypersurfaces and biharmonic ones [17, Chapter 11].

The Laplacian operator Δ involved in the biharmonicity can be seen as the first one of a sequence of n operators $L_0 = \Delta, L_1, \ldots, L_{n-1}$, where L_r stands for the linearized operator of the first variation of the $(r+1)-$th mean curvature arising from normal variations of the hypersurface. They act on smooth functions by $L_r(f) = \mathrm{tr}(T_r \circ \nabla^2 f)$, where T_r is the $r-$th Newton transformation associated with the shape operator of the hypersurface, and $\nabla^2 f$ is the self-adjoint linear operator metrically equivalent to the Hessian of f. With this extension of the Laplace operator $\Delta = L_0$ and inspired by the Chen's conjecture, it appears natural to generalize the definition of biharmonic hypersurfaces replacing Δ by the L_r. Along these lines, the L_r-conjecture has been formulated (cf. [5]) as follows:

L_r-**Conjecture 1.1** : Every Euclidean hypersurface $\psi : M^n \to \mathbb{R}^{n+1}$ satisfying the condition $L_r^2\psi = 0$ for some r, $0 \leq r \leq n-1$ has zero $(r+1)-$th mean curvature (equivalently, $(r+1)-$minimal).

This L_r-conjecture has been generalized (cf. [6]) for hypersurfaces of simply connected space forms as follows :

L_r-**Conjecture 1.2** : Let $\psi : M^n \to Q^{n+1}(c)$ be a hypersurface immersed into a simply connected space form $Q^{n+1}(c)$. If M is L_r-biharmonic then H_{r+1} is zero.

Recently, L_r-biharmonic hypersurfaces have been considered when the target space is pseudo-Riemannian and scrutinized by several authors [3, 19, 27, 28, 26] and references therein. In particular, it is shown in [27, Theorem 1.1] that on any L_k-biharmonic spacelike hypersurfaces in $\mathbb{E}_1^4$ with mutually distinct principal curvatures, if the $k-$th mean curvature H_k is constant then the same is for H_{k+1}. It is worth mentioning that all the hypersurfaces involved in the above quoted works are either spacelike or timelike, hence nondegenerate. To fill the gap, the present work focuses on L_r-biharmonic null (degenerate) hypersurfaces in generalized Robertson-Walker (GRW) spacetimes. As it is predictible due to the extra difficulties presented by the singularities of null hypersurfaces, our following results provide (partial) characterizations of such L_r-biharmonic null hypersurfaces, involving sometimes auxilliary screen foliations.

Theorem 1.1. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a timelike closed and conformal rigging vector field* ζ. *Then* M *is* $\widetilde{L}_r-$*harmonic for some* $0 \leq r < n$ *if and only if one of the following holds :*

(a) M *is* $r-$*maximal;*

(b) M *is* $(r+1)-$*maximal and* ζ *is parallele along* $M \subset \mathbb{R}^{n+2}_1$.

Theorem 1.2. *Let* $n \in \{1, 2\}$ *be integer,*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a non unit timelike closed and conformal rigging vector field* ζ.

(1) *For* $c = 0$, M *is biharmonic (i.e* $\widetilde{L}_0-$*biharmonic) if and only if it is totally geodesic, i.e null hyperplane. In particular the null mean curvature* H *vanishes.*

(2) *For* $c \neq 0$, *if* M *is biharmonic then the null mean curvature* H *is leafwise constant along the screen foliation induced by* ζ, *but not on the whole* M.

The following is a null version of the result in [27, Theorem 1.1] for $r = 1$ in generalized Robertson-Walker spaces.

Theorem 1.3. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

be a connected isometric immersion of a null hypersurface in a GRW spacetime $\overline{M}_1^{n+2}(c)$ *where* $m = n+2+c^2$, $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a non unit timelike closed and conformal rigging vector field* ζ. *Then,*

(1) *For* $n = 1$, $\psi : M^2 \longrightarrow \overline{M}_1^3(c) \subseteq \mathbb{R}^{3+c^2}_{1+t}$ *is* $\widetilde{L}_1-$*biharmonic.*

(2) *For* $n = 2$, *if* M^3 *is* $\widetilde{L}_1-$*biharmonic and the null mean curvature function* $H := \overset{\star}{H}_1$ *is leafwise constant in the screen foliation* $\mathscr{F}$ *induced by* ζ *then the same is for the*

second order mean curvature $\overset{\star}{H}_2$. *Moreover, if* $\overset{\star}{H}_2$ *is constant on the whole null hypersurface* M^3 *then this constant is zero and* M *is* $2-$*maximal.*

Throughout the paper, all geometric objects (manifolds, metrics, connections, maps,...) are smooth. The Lie algebra of vector fields on a manifold N is denoted by $\mathfrak{X}(N)$.

2. Null hypersurfaces and rigged structures

A hypersurface M of a Lorentzian manifold $(\overline{M}, g)$ is null if the metric tensor is degenerate on it, i.e the induced structure from the Lorentzian ambient manifold is degenerate.

A rigging for a null hypersurface M is a vector field ζ defined in some open neighbourhood of M such that $\zeta_p \notin T_pM$ for all $p \in M$. If ζ is defined only over M, then we call it a restricted rigging. If a rigging exists, then we can take the unique null vector field $\xi \in \mathfrak{X}(M)$ such that $g(\zeta, \xi) = 1$ (called rigged vector field) and the (screen) distribution given by $\mathcal{S}_p = \zeta_p^{\perp} \cap T_pM$ for all $p \in M$. We can also define the rigged metric as the Riemannian metric on M given by $\widetilde{g} = g + \omega \otimes \omega$, where $\omega = i^*\alpha$, α is the g-metrically equivalent one-form to ζ and $i : M \to \overline{M}$ is the canonical inclusion map. The rigged vector field ξ is unitary and orthogonal to $\mathcal{S}$ with respect to $\widetilde{g}$. Moreover, ω is $\widetilde{g}$-metrically equivalent to ξ, and is called the rigged one-form. The vector field $N = \zeta - \frac{1}{2}g(\zeta, \zeta)\xi$ is the unique null vector field defined on M, orthogonal to the screen distribution $\mathcal{S}$ and such that $g(N, \xi) = 1$.

Moreover, we have the following decompositions :

$$T_p\overline{M} = T_pM \oplus span(N_p), \quad T_pM = span\{\xi_p\} \oplus \mathcal{S}_p \tag{2.1}$$

for all $p \in M$.

The rigging technique presents two main advantages. The first one is that all the geometric objects defined above from the rigging are tuned together in a way that allows linking properties of the null hypersurface with properties of the ambient space. The second one is the presence of the Riemannian rigged metric $\widetilde{g}$, which geometry is reasonably well coupled with the ambient geometry in most cases and it allows us to use Riemannian tools for the study of the null hypersurface [23].

We get from decompositions (2.1)

$$\overline{\nabla}_U V = \nabla_U V + B(U, V)N, \qquad \overline{\nabla}_U N = -A(U) + \tau(U)N \tag{2.2}$$

where $\overline{\nabla}$, ∇ are the Levi-Civita connection of $\overline{M}$ and the induced (projected) connection on M, respectively. The induced connection ∇ is torsion free but, in general, is not metric, which makes it less useful in the theory. The second fundamental form B, the one-form τ (also called rotation one form) and the screen second fundamental form C are given by

$$B(U,V) = -g(\nabla_U \xi, V), \qquad \tau(U) = -g(\nabla_U \xi, \zeta),$$

$$C(U,V) = -g(\overline{\nabla}_U N, P(V)) = -g(\overline{\nabla}_U \zeta, P(V)),$$

for all $U, V \in \mathfrak{X}(M)$, where $P : TM \to \mathcal{S}$ is the canonical projection associated to the second decomposition in (2.1). The vector field $\overline{\nabla}_U \xi = \nabla_U \xi$ is tangent to the null hypersurface M and can be decomposed as

$$\overline{\nabla}_U \xi = -\tau(U)\xi - \overset{\star}{A}(U),$$

where $\overset{\star}{A}(U) \in \mathcal{S}$. The endomorphism $\overset{\star}{A}$ is the shape operator of $\mathcal{S}$ and satisfies

$$B(U,V) = g(\overset{\star}{A}(U), V) = g(U, \overset{\star}{A}(V)), \qquad B(\xi, U) = 0.$$

Some useful identities in the theory are the following:

$$-2C(U,X) = d\omega(U,X) + (L_\zeta g)(U,X) + g(\zeta,\zeta)B(U,X), \tag{2.3}$$

the Gauss-Codazzi equation

$$\begin{aligned} g(R_{UV}W,\xi) = {} & g(\left(\nabla_U \overset{\star}{A}\right)(V), W) - g(\left(\nabla_V \overset{\star}{A}\right)(U), W) \\ & + \tau(U)g(\overset{\star}{A}(V), W) - \tau(V)g(\overset{\star}{A}(U), W), \end{aligned} \tag{2.4}$$

$$(L_\xi \widetilde{g})(X,Y) = -2B(X,Y) \tag{2.5}$$

for all $U, V, W \in \mathfrak{X}(M)$,$X, Y \in \mathcal{S}$, and the Raychaudhuri equation[9] :

$$\overline{Ric}(\xi,\xi) = \xi(H) + \tau(\xi)H - \| \overset{\star}{A} \|^2,$$

where H denotes the (non-normalized) null mean curvature of the null hypersurface given by

$$H_p = \sum_{i=1}^{n} B(e_i, e_i),$$

with $\{e_1, \ldots, e_n\}$ an orthonormal basis in $\mathcal{S}_p$. In particular, $H = -\widetilde{\mathrm{div}}\xi$.

If $B = 0$, then it is said that M is totally geodesic and if $B = \rho g$ for certain $\rho \in C^\infty(M)$, then M is totally umbilical. Observe that these definitions do not depend on the chosen rigging, although the tensors B, τ and C do depend. Throughout, the Levi-Civita connection on the normalized rigged structure $(M, \widetilde{g})$ will be denoted $\widetilde{\nabla}$ and we have for all $X, Y, Z \in \mathcal{S}$

$$C(\xi, X) = -\tau(X) - \widetilde{g}(\widetilde{\nabla}_\xi \xi, X), \qquad \widetilde{\nabla}_X Y = \overset{\star}{\nabla}_X Y - \widetilde{g}(\widetilde{\nabla}_X \xi, Y)\xi,$$

being $\overset{\star}{\nabla}$ the connection on the screen bundle $\mathcal{S}$. In particular

$$\widetilde{g}(\widetilde{\nabla}_X Y, Z) = g(\nabla_X Y, Z) = g(\overline{\nabla}_X Y, Z) \quad \forall X, Y, Z \in \mathcal{S}.$$

From now on, we assume $\overline{M}$ to be a generalized Robertson-Walker (GRW) spacetime of constant sectional curvature $c \in \{-1, 0, 1\}$, which will be denoted $\overline{M}_1^{n+2}(c)$ throughout. It is known that such spacetime admits timelike closed and conformal vector field, say ζ. We have

$$\overline{M}_1^{n+2}(c) = (I \times_f F, \overline{g}), \quad \overline{g} = -dt^2 + f^2(t) g_F$$

where f (the warping function) is a smooth positive function on I, and the fiber (F, g_F) is an $(n+1)-$dimensional Riemannian manifold of constant sectional curvature c_F [29]. So, the target space $\overline{M}_1^{n+2}(c)$ of immersion is locally isometric to one of the modele spaces : a de Sitter spacetime $\mathbb{S}_1^{n+2}$ of curvature $c = 1$, the Lorentz-Minkowski spacetime $\mathbb{R}_1^{n+2}$ when $c = 0$ or the anti de Sitter spacetime $\mathbb{H}_1^{n+2}$ (actually the universal covering of this pseudohyperbolic space $\mathbb{H}_1^{n+2}$) of curvature $c = -1$. Hence, we consider the following orientable isometric immersion

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^m$$

of the null hypersurface in $\overline{M}_1^{n+2}(c)$ where $m = n+2+c^2$ and $t = c(c-1)/2$ with $c = 1, 0, -1$.

Due to the causal character (spacelike or null) of tangent vectors to a null hypersurface in Lorentzian space, the induced singular metric on the null hypersurface has signature $(0, n)$. So the timelike concircular vector field ζ can act as rigging vector field for M. The closed and conformal vector field ζ has the outstanding property that there exists a smooth function $\sigma \in C^\infty(\overline{M})$ (the conformal factor) such that $\overline{\nabla}_U \zeta = \sigma U$ for all $U \in \mathfrak{X}(\overline{M})$. In particular $L_\zeta \overline{g} = 2\sigma \overline{g}$. For a closed and conformal rigging, the rotation $1-$form vanishes identically $(\tau = 0)$ and ξ is $g-$geodesic. Moreover, due to the closedness of ζ, $\widetilde{\nabla}_U \xi = -\overset{\star}{A}(U)$ and

$$\widetilde{\nabla}_U V = \nabla_U V + [B(U, V) - C(U, PV)]\, \xi, \tag{2.6}$$

for all $U, V \in \mathfrak{X}(M)$. Also, using (2.3) we derive the following useful relation linking the shape operators A and $\overset{\star}{A}$.

$$A = -\frac{1}{2}\lambda \overset{\star}{A} - \sigma P, \tag{2.7}$$

where $\lambda = \overline{g}(\zeta, \zeta)$ denotes the length function of ζ.

For the closed rigging ζ, the screen distribution $\mathcal{S}_p = \zeta_p^{\perp} \cap T_pM$ is integrable and gives rise to a foliation $\mathscr{F}$ on the null hypersurface. Moreover, we have shown in [11, Lemma 7] that the conformal factor σ and the length function λ are constant through the (screen) leaves $\mathscr{F}_p$, $p \in M$. In other words,

$$X \cdot \sigma = 0 \quad \text{and} \quad X \cdot \lambda = 0$$

for all $X \in \mathcal{S}$.

3. Rigged linearized operators $\widetilde{L}_r$ and technical lemmas

The shape operator $\overset{\star}{A}$ is self-adjoint and satisfies $\overset{\star}{A}\xi = 0$. Its $n+1$ real valued eigenfunctions $\overset{\star}{k}_0 = 0, \overset{\star}{k}_1, \ldots, \overset{\star}{k}_n$ are the screen principal curvatures and we let $(X_0 = \xi, X_1, \ldots, X_n)$ denote a $\widetilde{g}$−orthonormal basis of eigenvector fields of $\overset{\star}{A}$, with $\mathrm{span}(X_1, \ldots, Xn) = \mathcal{S}$. For $0 \le r \le n$, the $r - th$ null mean curvature $\overset{\star}{H}_r$ of the null hypersurface with respect to the shape operator $\overset{\star}{A}$ is given by

$$\binom{n+1}{r} \overset{\star}{H}_r = \sum_{0 \le i_1 < \cdots < i_r \le n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r} \qquad \text{and} \qquad \overset{\star}{H}_0 = 1,$$

and the null hypersurface is said to be r−maximal if $\overset{\star}{H}_r = 0$ identically on M. The following notations will be in use :

$$\overset{\star}{S}_r = \sum_{0 \le i_1 < \cdots < i_r \le n} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r}, \qquad \overset{\star}{S}{}^{\alpha}_r = \sum_{\substack{0 \le i_1 < \cdots < i_r \le n \\ i_1, \cdots, i_r \neq \alpha}} \overset{\star}{k}_{i_1} \cdots \overset{\star}{k}_{i_r} .$$

In particular $\overset{\star}{S}_0 = 1$ and $\overset{\star}{S}_1 = H$ (the null mean curvature).

For $0 \le r \le n+1$, the $r-th$ Newton transformation $\overset{\star}{T}_r$ with respect to the shape operator $\overset{\star}{A}$ is the $\mathrm{End}(\Gamma(TM))$ element given by

$$\overset{\star}{T}_r = \sum_{a=0}^{r} (-1)^a \overset{\star}{S}_a \overset{\star}{A}{}^{r-a} .$$

Inductively,

$$\overset{\star}{T}_0 = I \;\; \text{and} \;\; \overset{\star}{T}_r = (-1)^r \overset{\star}{S}_r I + \overset{\star}{A} \circ \overset{\star}{T}_{r-1},$$

where I denotes the identity of $\Gamma(TM)$ and $\overset{\star}{T}_{n+1}=0$ (follows Cayley-Hamilton's theorem). By algebraic computations, one shows the following.

Proposition 3.1 ([9]).

(1) $\overset{\star}{T}_r$ *is self-adjoint and commute with* $\overset{\star}{A}$ *for any* r*;*
(2) $\overset{\star}{T}_r X_\alpha = (-1)^r \overset{\star\alpha}{S_r} X_\alpha$ *(*for a fixed α);
(3) $\mathrm{tr}(\overset{\star}{T}_r) = (-1)^r(n+1-r)\overset{\star}{S}_r$;
(4) $\mathrm{tr}\left(\overset{\star}{A}\circ\overset{\star}{T}_{r-1}\right) = (-1)^{r-1} r \overset{\star}{S}_r$;
(5) $\mathrm{tr}\left(\overset{\star}{A}{}^2\circ\overset{\star}{T}_{r-1}\right) = (-1)^{r-1}\left(\overset{\star}{S}_1\overset{\star}{S}_r -(r+1)\overset{\star}{S}_{r+1}\right)$;
(6) $\mathrm{tr}(\overset{\star}{T}_{r-1}\circ\nabla_X\overset{\star}{A}) = (-1)^{r-1}X\cdot\overset{\star}{S}_r$.

Also, for the last item in Proposition 3.1, replacing ∇ by $\widetilde{\nabla}$, it is easy to show by a straightforward computation that

$$\mathrm{tr}(\overset{\star}{T}_{r-1}\circ\widetilde{\nabla}_X\overset{\star}{A}) = (-1)^{r-1}X\cdot\overset{\star}{S}_r\ . \tag{3.8}$$

We recall the following from [9, Remark 3, Page 68].

Theorem 3.1. *Let* (M^{n+1},ζ) *be a normalized null hypersurface of a Lorentzian space form* $(\overline{M}{}_1^{n+2}(c),\bar{g})$ *with rigged vector field* ξ *and* $\tau=0$*. Then,*

$$\xi\cdot\overset{\star}{S}_r = (-1)^{r-1}tr\left(\overset{\star}{A}{}^2\circ\overset{\star}{T}_{r-1}\right)\overset{Prop.\ 3.1\ (5)}{=}\left(\overset{\star}{S}_1\overset{\star}{S}_r -(r+1)\overset{\star}{S}_{r+1}\right). \tag{3.9}$$

Consequently, if $\overset{\star}{S}_r=0$ *for some* $r=1,...,n$*, then* $\overset{\star}{S}_k=0$ *for all* $k\geq r$*.*

For each $0\leq r\leq n$, the divergence of the operator $\overset{\star}{T}_r$: $\mathfrak{X}(M)\longrightarrow\mathfrak{X}(M)$ with respect to the rigged connection $\widetilde{\nabla}$ is the vector field $\mathrm{div}^{\widetilde{\nabla}}(\overset{\star}{\mathrm{T}}_\mathrm{r})\in\mathfrak{X}(\mathrm{M})$ defined as the trace of $\widetilde{\nabla}\,\overset{\star}{T}_r$, that is

$$\mathrm{div}^{\widetilde{\nabla}}(\overset{\star}{\mathrm{T}}_\mathrm{r}) = \left(\widetilde{\nabla}_\xi\,\overset{\star}{\mathrm{T}}_\mathrm{r}\right)(\xi) + \sum_{\mathrm{i}=1}^{\mathrm{n}}\left(\widetilde{\nabla}_{\mathrm{X_i}}\,\overset{\star}{\mathrm{T}}_\mathrm{r}\right)(\mathrm{X_i}).$$

Using the iterative formula $\overset{\star}{T}_r=(-1)^r\overset{\star}{S}_r I+\overset{\star}{A}\circ\overset{\star}{T}_{r-1}$, we have

$$\mathrm{div}^{\widetilde{\nabla}}\,\overset{\star}{\mathrm{T}}_\mathrm{r} = (-1)^\mathrm{r}\mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{S}}_\mathrm{r}\,\mathrm{I}\right) + \mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{A}}\circ\overset{\star}{\mathrm{T}}_{\mathrm{r}-1}\right).$$

But

$$\mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{S}}_\mathrm{r}\,\mathrm{I}\right) = \sum_{\alpha=0}^{n}\left(\widetilde{\nabla}_{X_\alpha}\overset{\star}{S}_r\,I\right)X_\alpha = \sum_\alpha\left[\widetilde{\nabla}_{X_\alpha}(\overset{\star}{S}_r\,X_\alpha) - \overset{\star}{S}_r\left(\widetilde{\nabla}_{X_\alpha}X_\alpha\right)\right]$$

$$= \sum_{\alpha}(X_\alpha\cdot \overset{\star}{S}_r)X_\alpha = \widetilde{\nabla}\overset{\star}{S}_r .$$

On the other side,

$$\begin{aligned}\operatorname{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{A}}\circ\overset{\star}{\mathrm{T}}_{\mathrm{r-1}}\right) &= \sum_{\alpha}\left(\widetilde{\nabla}_{X_\alpha}\left(\overset{\star}{A}\circ\overset{\star}{T}_{r-1}\right)X_\alpha\right)\\ &= \sum_{\alpha}\left[\widetilde{\nabla}_{X_\alpha}\left(\overset{\star}{A}\circ\overset{\star}{T}_{r-1}(X_\alpha)\right)-\left(\overset{\star}{A}\circ\overset{\star}{T}_{r-1}\right)\left(\widetilde{\nabla}_{X_\alpha}X_\alpha\right)\right]\\ &= \sum_{\alpha}\left(\widetilde{\nabla}_{X_\alpha}\overset{\star}{A}\right)\left(\overset{\star}{T}_{r-1}X_\alpha\right)+\overset{\star}{A}\left(\operatorname{div}^{\widetilde{\nabla}}\overset{\star}{\mathrm{T}}_{\mathrm{r-1}}\right).\end{aligned}$$

So, for all $U\in\mathfrak{X}(M)$,

$$\begin{aligned}\widetilde{g}\left(\operatorname{div}^{\widetilde{\nabla}}\overset{\star}{\mathrm{T}}_{\mathrm{r}},\mathrm{U}\right) &= \widetilde{g}\left(\operatorname{div}^{\widetilde{\nabla}}\overset{\star}{\mathrm{T}}_{\mathrm{r-1}},\overset{\star}{\mathrm{A}}\,\mathrm{U}\right)+\sum_{\alpha}\widetilde{g}\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\widetilde{\nabla}_{X_\alpha}\overset{\star}{A}\right)U\right)\\ &\quad+(-1)^rU\cdot\overset{\star}{S}_r .\end{aligned}\tag{3.10}$$

We compute $\sum_{\alpha}\widetilde{g}\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\widetilde{\nabla}_{X_\alpha}\overset{\star}{A}\right)U\right)$ using curvature relations. Before proceeding we note the following covariant derivative identity which is established by a direct computation. For all linear operator $T:\mathfrak{X}(M)\to\mathfrak{X}(M)$ and $U,V\in\mathfrak{X}(M)$,

$$\begin{aligned}\left(\widetilde{\nabla}_UT\right)(V) &= (\nabla_UT)(V)+[B(U,TV)\xi-B(U,V)T\xi]\\ &\quad-\frac{1}{2}\Big([\langle AU,TV\rangle+\langle U,A(TV)\rangle]\,\xi-[\langle AU,V\rangle+\langle U,AV\rangle]\,T\xi\Big).\end{aligned}\tag{3.11}$$

Applying (3.11) with $T=\overset{\star}{A}$ and using the fact that $\overset{\star}{A}\xi=0$ and $\overline{\nabla}\zeta=\sigma\otimes I$ we get :

$$\left(\widetilde{\nabla}_U\overset{\star}{A}\right)(V)=\left(\nabla_U\overset{\star}{A}\right)(V)+\left[\langle\overset{\star}{A}U,\overset{\star}{A}V\rangle-\langle AU,\overset{\star}{A}V\rangle\right]\xi.\tag{3.12}$$

So, for each $0\le\alpha\le n$,

$$\begin{aligned}\widetilde{g}\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\widetilde{\nabla}_{X_\alpha}\overset{\star}{A}\right)U\right) &= \widetilde{g}\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\nabla_{X_\alpha}\overset{\star}{A}\right)U\right)\\ &\quad+\left[g(\overset{\star}{A}X_\alpha,\overset{\star}{A}U)-g(AX_\alpha,\overset{\star}{A}U)\right]\times\widetilde{g}(X_{X_\alpha},\overset{\star}{T}_{r-1}\xi).\end{aligned}$$

Using item (ii) in Proposition 3.1 and (2.7) we see that the last term in above equality vanishes. Hence, in closed and conformal setting,

$$\begin{aligned}\widetilde{g}\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\widetilde{\nabla}_{X_\alpha}\overset{\star}{A}\right)U\right) &= \widetilde{g}\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\nabla_{X_\alpha}\overset{\star}{A}\right)U\right)\\ &= g\left(\overset{\star}{T}_{r-1}X_\alpha,\left(\nabla_{X_\alpha}\overset{\star}{A}\right)U\right)+\omega\left(\overset{\star}{T}_{r-1}X_\alpha\right)\omega\left(\left(\nabla_{X_\alpha}\overset{\star}{A}\right)U\right).\end{aligned}$$

We show that the last term vanishes. Indeed,

$$\begin{aligned} \omega\left(\overset{\star}{T}_{r-1} X_\alpha\right) &= \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \xi\right) = (-1)^{r-1} \overset{\star}{S}{}^{\alpha}_{r-1} \widetilde{g}(X_\alpha, \xi) \\ &= \begin{cases} 0 & \text{if} \quad \alpha \neq 0 \\ (-1)^{r-1} \overset{\star}{S}{}^{0}_{r-1} = (-1)^{r-1} \overset{\star}{S}_{r-1} & \text{if} \quad \alpha = 0, \end{cases} \end{aligned} \tag{3.13}$$

where we use the fact that $\overset{\star}{S}{}^{0}_{r-1} = \overset{\star}{S}_{r-1}$ due to $\overset{\star}{k}_0 = 0$. From (3.13) we need to compute the second factor just for $\alpha = 0$.

$$\begin{aligned} \omega\left(\left(\nabla_{X_\alpha} \overset{\star}{A}\right) U\right) &= \widetilde{g}\left((\nabla_\xi \overset{\star}{A})(U), \xi\right) \\ &= \widetilde{g}\left(\nabla_\xi (\overset{\star}{A} U) - \overset{\star}{A} (\nabla_\xi U), \xi\right) = \widetilde{g}\left(\nabla_\xi (\overset{\star}{A} U), \xi\right) \\ &= \widetilde{g}\left(\overset{\star}{\nabla}_\xi (\overset{\star}{A} U) + C(\xi, \overset{\star}{A} U)\xi, \xi\right) = C(\xi, \overset{\star}{A} U) \overset{(2.7)}{=} 0. \end{aligned}$$

Hence, for $0 \leq \alpha \leq n$,

$$\widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right) U\right) = g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_{X_\alpha} \overset{\star}{A}\right) U\right). \tag{3.14}$$

Now, Gauss-Codazzi equation (2.4) with $\tau = 0$ provides

$$\overline{g}\left(\overline{R}(U, V)W, \xi\right) = g\left((\nabla_U \overset{\star}{A})V, W\right) - g\left((\nabla_V \overset{\star}{A})U, W\right),$$

for all $U, V, V \in \mathfrak{X}(M)$, where we make use of the identity

$$\left(\nabla_U B\right)(V, W) = g\left((\nabla_U \overset{\star}{A})V, W\right) + \omega(W) g(\overset{\star}{A} U, \overset{\star}{A} V).$$

Hence, (3.14) becomes

$$\begin{aligned} \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right) U\right) &= \overline{g}\left(\overline{R}(X_\alpha, U) \overset{\star}{T}_{r-1} X_\alpha, \xi\right) \\ &\quad + g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\nabla_U \overset{\star}{A}\right)(X_\alpha)\right). \end{aligned}$$

From (3.12), the following equation holds

$$\left(\nabla_U \overset{\star}{A}\right)(X_\alpha) = \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha) - \left[g(\overset{\star}{A} U, \overset{\star}{A} X_\alpha) - g(AU, \overset{\star}{A} X_\alpha)\right]\xi$$

and we get

$$\widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right) U\right) = \overline{g}\left(\overline{R}(X_\alpha, U) \overset{\star}{T}_{r-1} X_\alpha, \xi\right) + g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right).$$

But

$$g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right) = \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right) - \omega(\overset{\star}{T}_{r-1} X_\alpha)\omega\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right). \quad (3.15)$$

Due to the relation $\overset{\star}{T}_{r-1} X_\alpha = (-1)^{r-1} \overset{\star\alpha}{S}_{r-1} X_\alpha$, we see that for $\alpha \neq 0$, $\omega(\overset{\star}{T}_{r-1} X_\alpha) = 0$. Also, for $\alpha = 0$,

$$\omega\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(\xi)\right) = \widetilde{g}\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(\xi), \xi\right) = \widetilde{g}\left(\widetilde{\nabla}_U(\overset{\star}{A}\,\xi) - \overset{\star}{A}\,(\widetilde{\nabla}_U\xi), \xi\right) = -\widetilde{g}\left(\overset{\star}{A}\,(\widetilde{\nabla}_U\xi), \xi\right) = 0,$$

hence the product $\omega(\overset{\star}{T}_{r-1} X_\alpha)\omega\left(\left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right)$ in (3.15)vanishes identically and we get

$$g\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right) = \widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_U \overset{\star}{A}\right)(X_\alpha)\right).$$

Therefore, for $0 \leq \alpha \leq n$,

$$\widetilde{g}\left(\overset{\star}{T}_{r-1} X_\alpha, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{A}\right)(U)\right) = \overline{g}\left(\overline{R}\,(X_\alpha, U)\, \overset{\star}{T}_{r-1} X_\alpha, \xi\right) + \widetilde{g}\left(X_\alpha, \left(\overset{\star}{T}_{r-1} \circ \left(\widetilde{\nabla}_U \overset{\star}{A}\right)\right)(X_\alpha)\right).$$

Returning back to (3.10) we have

$$\begin{aligned} \widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}}\, \overset{\star}{\mathrm{T}}_{\mathrm{r}}, \mathrm{U}\right) &= \widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}}\, \overset{\star}{\mathrm{T}}_{\mathrm{r-1}}, \overset{\star}{\mathrm{A}}\,\mathrm{U}\right) + \sum_\alpha \overline{g}\left(\overline{R}(X_\alpha, U)\, \overset{\star}{T}_{r-1} X_\alpha, \xi\right) \\ &\quad + \sum_\alpha \widetilde{g}\left(\left(\overset{\star}{T}_{r-1} \circ(\widetilde{\nabla}_U \overset{\star}{A})\right) X_\alpha, X_\alpha\right) + (-1)^r U\cdot \overset{\star}{S}_r \\ &= \widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}}\, \overset{\star}{\mathrm{T}}_{\mathrm{r-1}}, \overset{\star}{\mathrm{A}}\,\mathrm{U}\right) + \sum_\alpha \overline{g}\left(\overline{R}(X_\alpha, \xi)\, \overset{\star}{T}_{r-2} X_\alpha, \overset{\star}{A} U\right) \\ &\quad + \overline{g}\left(\overline{R}(X_\alpha, \xi)\, \overset{\star}{T}_{r-1} X_\alpha, U\right). \end{aligned}$$

By iterating this process, we get the following.

Lemma 3.1.

$$\widetilde{g}\left(\mathrm{div}^{\widetilde{\nabla}}\, \overset{\star}{\mathrm{T}}_{\mathrm{r}}, \mathrm{U}\right) = \sum_{i=0}^{r-1}\sum_{\alpha=0}^{n} \overline{g}\left(\overline{R}(X_\alpha, \xi)\, \overset{\star}{T}_i X_\alpha, \overset{\star}{A}{}^{r-1-i}\, U\right) \quad (3.16)$$

Corollary 3.1. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_{1+t}^{n+2+c^2}$$

be a isometric immersion of a null hypersurface in $\overline{M}_1^{n+2}(c)$ *where* $t = c(c-1)/2$ *with* $c = 1, 0, -1$, *furnished with a closed and conformal rigging vector field* ζ. *Then, for all* $f \in C^\infty(M)$.

$$\mathrm{div}^{\widetilde{\nabla}}\, \overset{\star}{\mathrm{T}}_{\mathrm{r}} = 0 \quad \textit{and} \quad \mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) = tr\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}} \circ \widetilde{\nabla}^2\mathrm{f}\right). \quad (3.17)$$

Proof. When the ambient Lorentzian manifold $\overline{M}^{n+2}$ has constant sectional curvature c, we have for a fixed r, each $i = 0 \dots r-1$ and $\alpha = 0, \dots, n$ the term $\overline{g}\left(\overline{R}(X_\alpha, \xi)\overset{\star}{T}_i X_\alpha, \overset{\star}{A}^{r-1-i} U\right)$ in (3.16) vanishes identically. So $\mathrm{div}^{\widetilde{\nabla}} \overset{\star}{\mathrm{T}}_{\mathrm{r}} = 0$. By definition,

$$\mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) = \mathrm{tr}\left(\widetilde{\nabla}\, \overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) = \sum_{\alpha=0}^{\mathrm{n}} \widetilde{\mathrm{g}}\left(\widetilde{\nabla}_{\mathrm{X}_\alpha}(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\widetilde{\nabla}\mathrm{f}), \mathrm{X}_\alpha\right),$$

and

$$\widetilde{\nabla}_{X_\alpha}(\overset{\star}{T}_r \widetilde{\nabla} f) = \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{T}_r\right)\widetilde{\nabla} f + \overset{\star}{T}_r \left(\widetilde{\nabla}_{X_\alpha}\widetilde{\nabla} f\right).$$

So,

$$\begin{aligned}
\mathrm{div}^{\widetilde{\nabla}}\left(\overset{\star}{\mathrm{T}}_{\mathrm{r}}\, \widetilde{\nabla}\mathrm{f}\right) &= \sum_{\alpha=0}^{n} \widetilde{g}\left(\widetilde{\nabla} f, \left(\widetilde{\nabla}_{X_\alpha} \overset{\star}{T}_r\right)(X_\alpha)\right) + \sum_{\alpha=0}^{n} \widetilde{g}\left(\overset{\star}{T}_r \left(\widetilde{\nabla}_{X_\alpha}\widetilde{\nabla} f\right), X_\alpha\right) \\
&= \widetilde{g}\left(\widetilde{\nabla} f, \mathrm{div}^{\widetilde{\nabla}}(\overset{\star}{\mathrm{T}}_{\mathrm{r}})\right) + \mathrm{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 f\right)
\end{aligned}$$

and the second claim in (3.17) follows from $\mathrm{div}^{\widetilde{\nabla}}(\overset{\star}{\mathrm{T}}_{\mathrm{r}}) = 0$. □

For the sake of comparison, note that in [9] using the projected (induced) connection ∇ we established the following.

Proposition 3.2. [9, Proposition 3] $\forall X \in \mathfrak{X}(M)$,

$$\begin{aligned}
g(div^{\nabla} \overset{\star}{T}_r, U) &= \sum_{a=0}^{r-1}\sum_{i=1}^{n} \bar{g}\left(\bar{R}(X_i, \xi)\overset{\star}{T}_a X_i, \overset{\star}{A}_\xi^{r-1-a} U\right) \\
&\quad + \sum_{a=0}^{r-1}\left(\tau(\overset{\star}{A}_\xi^{r-1-a} U) tr(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a) - \tau(P(\overset{\star}{A}_\xi \circ \overset{\star}{T}_a U))\right) \\
&\quad + (-1)^r \omega(U)\left(\sum_{i=1}^{n} \overset{\star}{S}{}^i_{r-1} \overset{\star}{k}{}^2_i - \xi(\overset{\star}{S}_r)\right).
\end{aligned} \tag{3.18}$$

Taking $r = 2$ and $U = \xi$ in (3.18) leads to

$$\begin{aligned}
0 = g(div^{\nabla} \overset{\star}{T}_2, \xi) &= \sum_{i=1}^{n} \overline{g}\left(\overline{R}(X_i, \xi)\overset{\star}{T}_i X_i, \xi\right) + \tau(\xi)\mathrm{tr}(\overset{\star}{A} \circ \overset{\star}{T}_1) \\
&\quad - \tau(\overset{\star}{A} \circ \overset{\star}{T}_1\, \xi) + \sum_{i=1}^{n} \overset{\star}{S}{}^i_1 \overset{\star}{k}{}^2_i - \xi \cdot \overset{\star}{S}_2 \\
&= \sum_{i=1}^{n} \overset{\star}{S}{}^i_1\, \overline{K}_\xi(\Pi_i) + \tau(\xi)\mathrm{tr}(\overset{\star}{A} \circ \overset{\star}{T}_1) \\
&\quad - \tau(\overset{\star}{A} \circ \overset{\star}{T}_1\, \xi) + \sum_{i=1}^{n} \overset{\star}{S}{}^i_1 \overset{\star}{k}{}^2_i - \xi \cdot \overset{\star}{S}_2
\end{aligned}$$

where $\overline{K}_\xi(\Pi_i) = \dfrac{\overline{g}(\overline{R}(\xi, X_i)X_i, \xi)}{\overline{g}(X_i, X_i)} = \overline{g}(\overline{R}(\xi, X_i)X_i, \xi)$ stands for the null sectional curvature of the null plane $\Pi_i = \mathrm{span}(X_i, \xi)$. But $\overset{\star}{A} \circ \overset{\star}{T}_1 \xi = 0$ and $\mathrm{tr}(\overset{\star}{A} \circ \overset{\star}{T}_1) = -2 \overset{\star}{S}_2$, so

$$\sum_{i=1}^{n} \overset{\star i}{S}_1 \overline{K}_\xi(\Pi_i) = \xi\cdot \overset{\star}{S}_2 + 2\tau(\xi) \overset{\star}{S}_2 - \sum_{i=1}^{n} \overset{\star i}{S}_1 \overset{\star}{k}_i^2 . \tag{3.19}$$

Therefore, we can state the following.

Lemma 3.2. *Let*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c)$$

be a isometric immersion of a null hypersurface in a space $\overline{M}_1^{n+2}(c)$ of constant curvature c, furnished with a conformal rigging vector field ζ. Then

$$\xi\cdot \overset{\star}{S}_2 = \sum_{i=1}^{n} \overset{\star i}{S}_1 \overset{\star}{k}_i^2 . \tag{3.20}$$

In particular, for $n = 2$

$$\xi\cdot \overset{\star}{S}_2 = \overset{\star}{S}_1 \overset{\star}{S}_2 . \tag{3.21}$$

Proof. For constant sectional curvature, $\overline{K}_\xi(\Pi_i) = 0$, $i = 1, \dots, n$ and since $\tau(\xi) = 0$, we obtain (3.20) from (3.19). □

Now, for $n = 2$,

$$\sum_{i=1}^{2} \overset{\star i}{S}_1 \overset{\star}{k}_i^2 = \overset{\star}{k}_2 \overset{\star}{k}_1^2 + \overset{\star}{k}_1 \overset{\star}{k}_2^2 = \overset{\star}{k}_1 \overset{\star}{k}_2 (\overset{\star}{k}_1 + \overset{\star}{k}_2) = \overset{\star}{S}_1 \overset{\star}{S}_2 .$$

For each Newton transformation $\overset{\star}{T}_r$, we can consider the second-order linear differential operator $\widetilde{L}_r : C^\infty(M) \to C^\infty(M)$ given by

$$\widetilde{L}_r(f) = \mathrm{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 f\right) \tag{3.22}$$

where $\widetilde{\nabla}^2 f := \widetilde{\nabla}\widetilde{\nabla} f$ stands for the $\widetilde{g}$−dual of the Hessian $\widetilde{Hess}\ f$ of f with respect to $\widetilde{g}$ on M. Observe that when $r = 0$, $\widetilde{L}_0 = \widetilde{\Delta}$ is nothing but the Laplacian operator on the Riemannian rigged structure $(M, \widetilde{g})$. Also, the second-order linear differential operator $\widetilde{L}_r$ defined here in (3.22) is different from $L_r(f) = \mathrm{tr}\left(\overset{\star}{T}_r \circ \nabla(\widetilde{\nabla} f)\right)$ as defined in [25] where a hybrid use of the (projected) induced connection ∇ and the rigged Levi-Civita connection $\widetilde{\nabla}$ on $(M, \widetilde{g})$ is made. But these two connections do not coincide in general. Indeed, the equality $\widetilde{\nabla} = \nabla$ holds if and only if $B = C$ and $\tau = 0$ (cf. [10, Theorem 4.1]).

From (3.22) and (3.17) and using divergence properties, we get

Lemma 3.3. *For all* $f, h \in C^\infty(M)$,

$$\widetilde{L}_r(fh) = f\widetilde{L}_r(h) + h\widetilde{L}_r(f) + 2\widetilde{g}\left(\widetilde{\nabla} f, \overset{\star}{T}_r \; \widetilde{\nabla} h\right). \tag{3.23}$$

For the following orientable isometric immersion

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}^m_{1+t}$$

of the null hypersurface in $\overline{M}_1^{n+2}(c)$ where $m = n+2+c^2$ and $t = c(c-1)/2$ with $c = 1, 0, -1$, we will calculate $\widetilde{L}_r$ acting on the coordinate components of the immersion ψ, i.e a function given by $\left\langle \psi, a \right\rangle$ where $a \in \mathbb{R}^m_{1+t}$ is an arbitrary fixed vector. We let $\overset{0}{\nabla}$ and $\overline{\nabla}$ denote the Levi-Civita connections on $\mathbb{R}^{n+2+c^2}_{1+t}$ and $\overline{M}_1^{n+2}(c)$, respectively. For all $U, V \in \mathfrak{X}(M)$,

$$\overset{0}{\nabla}_U \; V = \overline{\nabla}_U V - c\overline{g}(U, V)\psi$$

which, by use of (2.6) gives

$$\overset{0}{\nabla}_U \; V = \widetilde{\nabla}_U V + B(U, V)(N - \xi) + g(AU, V)\xi - cg(U, V)\psi. \tag{3.24}$$

In particular, for all $U \in \mathfrak{X}(M)$,

$$\overset{0}{\nabla}_U \xi = \overline{\nabla}_U \xi = \widetilde{\nabla}_U \xi = - \overset{\star}{A} U,$$

Lemma 3.4. *Set* $h = \left\langle \psi, a \right\rangle$, $a \in \mathbb{R}^{n+2+c^2}_{1+t}$ *with* $c = -1, 0, 1$ *and* $\lambda = \langle \zeta, \zeta \rangle$. *Then,*

$$\widetilde{\nabla} h = a^T - \langle a, N - \xi \rangle \xi = a - \langle a, N - \xi \rangle \xi - \langle a, \xi \rangle N - c\langle a, \psi \rangle \psi; \tag{3.25}$$

$$\begin{aligned} \widetilde{L}_r h \;\; &= \;\; (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1) \overset{\star}{S}_{r+1}\right] \langle \xi, a \rangle + (-1)^r (r+1) \overset{\star}{S}_{r+1} \langle \zeta, a \rangle \\ &\quad + (-1)^{r+1}(n-r)c \overset{\star}{S}_r \langle \psi, a \rangle, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \widetilde{L}_r \psi \;\; &= \;\; (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1) \overset{\star}{S}_{r+1}\right] \xi + (-1)^r (r+1) \overset{\star}{S}_{r+1} \zeta \\ &\quad + (-1)^{r+1}(n-r)c \overset{\star}{S}_r \psi. \end{aligned} \tag{3.27}$$

Proof. The function h is smooth on M and for all $X \in \mathfrak{X}(M)$,

$$\widetilde{g}(X, \widetilde{\nabla} h) = X \cdot h = X \cdot \langle \psi, a \rangle = \left\langle \overset{0}{\nabla}_X \; \psi, a \right\rangle = \langle X, a \rangle.$$

But

$$a = a^T + \langle \xi, a\rangle N + c\langle \psi, a\rangle \psi, \tag{3.28}$$

where $a^T \in \mathfrak{X}(M)$ is the tangential component of the vector a projected on M in the direction span(N, ψ). So, noting that $\omega(a^T) = \langle a, N\rangle$,

$$\begin{aligned}\widetilde{g}(X, \widetilde{\nabla} h) &= \left\langle X, a^T + \langle \xi, a\rangle N + c\langle \psi, a\rangle \psi \right\rangle = g(X, a^T) + \langle \xi, a\rangle \widetilde{g}(\xi, X)\\ &= \widetilde{g}(X, a^T) - \omega(X)\omega(a^T) + \langle \xi, a\rangle \widetilde{g}(\xi, X) = \widetilde{g}\left(X, a^T - \langle a, N-\xi\rangle \xi\right),\end{aligned}$$

and we get $\widetilde{\nabla} h = a^T - \langle a, N-\xi\rangle\xi$ and the last equality in (3.25) follows from (3.28).

Further, note that

$$\overset{0}{\nabla}_U N = -AU - c\omega(U)\psi \quad \overset{0}{\nabla}_U \xi = -\overset{\star}{A}U \quad \text{and} \quad \overset{0}{\nabla}_U \psi = U,$$

hence, a straightforward computation using (3.25) leads to

$$\begin{aligned}\overset{0}{\nabla}_U \widetilde{\nabla} h &= -c\langle \psi, a\rangle PU + \langle AU - \overset{\star}{A}U, a\rangle \xi + \langle N - \xi, a\rangle \overset{\star}{A}U\\ &\quad + \langle \overset{\star}{A}U, a\rangle N + \langle \xi, a\rangle AU - c\langle PU, a\rangle \psi.\end{aligned} \tag{3.29}$$

On the other hand, applying (3.24) with $V = \widetilde{\nabla} h$ leads to

$$\overset{0}{\nabla}_U \widetilde{\nabla} h = \widetilde{\nabla}_U \widetilde{\nabla} h + \langle a, \overset{\star}{A}U\rangle (N - \xi) + \langle a, AU\rangle \xi - c\langle PU, a\rangle \psi. \tag{3.30}$$

Therefore, using (3.29), (3.30) and (2.7) we get

$$\widetilde{\nabla}_U \widetilde{\nabla} h = \left\langle N - \frac{1}{2}(2+\lambda)\xi, a\right\rangle \overset{\star}{A}U - \langle \sigma\xi + c\psi, a\rangle PU, \tag{3.31}$$

which in terms of ζ reads

$$\widetilde{\nabla}_U \widetilde{\nabla} h = \left\langle \zeta - (1+\lambda)\xi, a\right\rangle \overset{\star}{A}U - \langle \sigma\xi + c\psi, a\rangle PU. \tag{3.32}$$

It follows from (3.32) that

$$\begin{aligned}\widetilde{L}_r h &= \operatorname{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}^2 h\right) = \sum_\alpha \widetilde{g}\left(\overset{\star}{T}_r\left(\widetilde{\nabla}_{X_\alpha} \widetilde{\nabla} h\right), X_\alpha\right)\\ &= \sum_\alpha \left[\langle \zeta - (1+\lambda)\xi, a\rangle\, \widetilde{g}(\overset{\star}{T}_r \overset{\star}{A} X_\alpha, X_\alpha) - \langle \sigma\xi + c\psi, a\rangle \widetilde{g}(\overset{\star}{T}_r\, PX_\alpha, X_\alpha)\right]\\ &= \langle \zeta - (1+\lambda)\xi, a\rangle \operatorname{tr}(\overset{\star}{A} \circ \overset{\star}{T}_r) - \langle \sigma\xi + c\psi, a\rangle \left(\operatorname{tr}(\overset{\star}{T}_r) - (-1)^r \overset{\star}{S}_r\right)\\ &= (-1)^r (r+1) \langle \zeta - (1+\lambda)\xi, a\rangle \overset{\star}{S}_{r+1} + (-1)^r (n-r) \langle \sigma\xi + c\psi, a\rangle \overset{\star}{S}_r .\end{aligned}$$

Therefore,

$$\begin{aligned}\widetilde{L}_r h &= (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\right]\langle \xi, a\rangle + (-1)^r (r+1)\overset{\star}{S}_{r+1}\langle \zeta, a\rangle \\ &\quad + (-1)^{r+1}(n-r)c\overset{\star}{S}_r \langle \psi, a\rangle,\end{aligned}$$

which is (3.26). Extend $\widetilde{L}_r$ to the $\mathbb{R}^m_t$−valued function ψ by setting

$$\widetilde{L}_r\psi = \left(\widetilde{L}_r\psi_1, \ldots, \widetilde{L}_r\psi_m\right)$$

where $\psi_i = \varepsilon_i\langle\psi, e_i\rangle$ and $(e_1, \ldots, e_m)$ stands for an orthonormal basis of $\mathbb{R}^m_{1+t}$ with $m = n+2+c^2$, $t = c(c-1)/2$ and $\varepsilon_i = \langle e_i, e_i\rangle = \pm 1$. We have

$$\begin{aligned}\widetilde{L}_r\psi &= \sum_{i=1}^m \varepsilon_i \widetilde{L}_r\langle\psi, e_i\rangle e_i \\ &= (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\right]\sum_{i=1}^m \varepsilon_i\langle\xi, e_i\rangle e_i \\ &\quad + (-1)^r(r+1)\overset{\star}{S}_{r+1}\sum_{i=1}^m \varepsilon_i\langle\zeta, e_i\rangle e_i + (-1)^{r+1}(n-r)c\overset{\star}{S}_r\sum_{i=1}^m \varepsilon_i\langle\psi, e_i\rangle e_i, \\ &= (-1)^{r+1}\left[(n-r)\sigma \overset{\star}{S}_r + (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\right]\xi \\ &\quad + (-1)^r(r+1)\overset{\star}{S}_{r+1}\zeta + (-1)^{r+1}(n-r)c\overset{\star}{S}_r\psi,\end{aligned}$$

which completes the proof. □

Remark 3.1. *Due to $\overset{\star}{S}_{n+1} = 0$, we see from above expression (3.27) that $\widetilde{L}_n\psi = 0$ and that (M, ζ) is (trivially) $\widetilde{L}_n$−harmonic.*

Lemma 3.5. *Let $a \in \mathbb{R}^m_t$ be a fixed constant vector and $U \in \mathfrak{X}(M)$. Then,*

$$\widetilde{\nabla}\langle\xi, a\rangle = -\overset{\star}{A}\, a^T, \tag{3.33}$$

where $a^T = a - \langle a, \xi\rangle N - c\langle\psi, a\rangle\psi$.

$$\begin{aligned}\widetilde{\nabla}_U\widetilde{\nabla}\langle\xi, a\rangle &= -\left(\widetilde{\nabla}_{a^T}\overset{\star}{A}\right)U - \left[\langle \overset{\star}{A}{}^2 U, a^T\rangle + \langle \frac{1}{2}\overset{\star}{A}{}^2 U + \sigma\overset{\star}{A}U, a^T\rangle\right]\xi \\ &\quad + \langle\xi, a\rangle\left(\frac{1}{2}\overset{\star}{A}{}^2 U + \sigma\overset{\star}{A}U\right) + c\langle\psi, a\rangle\overset{\star}{A}U;\end{aligned} \tag{3.34}$$

$$\begin{aligned}\widetilde{L}_r\langle\xi, a\rangle &= (-1)^{r+1}\langle\widetilde{\nabla}\overset{\star}{S}_{r+1}, a\rangle + (-1)^r(r+1)c\overset{\star}{S}_{r+1}\langle\psi, a\rangle \\ &\quad + (-1)^{r+1}\left(\left[\frac{1}{2}\lambda\overset{\star}{S}_1 - (r+1)\sigma\right]\overset{\star}{S}_{r+1} + \frac{1}{2}(r+2)\lambda\overset{\star}{S}_{r+2} - \xi\cdot\overset{\star}{S}_{r+1}\right)\langle\xi, a\rangle\end{aligned} \tag{3.35}$$

and

$$\begin{aligned}\widetilde{L}_r\xi &= (-1)^{r+1}\widetilde{\nabla}\overset{\star}{S}_{r+1} +(-1)^r(r+1)c\,\overset{\star}{S}_{r+1}\,\psi \\ &\quad +(-1)^{r+1}\left(\left[\frac{1}{2}\lambda\,\overset{\star}{S}_1 -(r+1)\sigma\right]\overset{\star}{S}_{r+1} +\frac{1}{2}(r+2)\lambda\,\overset{\star}{S}_{r+2} -\xi\cdot\overset{\star}{S}_{r+1}\right)\xi. \quad (3.36)\end{aligned}$$

Proof. Set $\nu = \langle\xi, a\rangle$. For $U \in \mathfrak{X}(M)$,

$$\begin{aligned}\widetilde{g}\left(\widetilde{\nabla}\nu, U\right) &= U\cdot\nu = U\cdot\langle\xi, a\rangle = \langle\overset{0}{\nabla}_U\,\xi, a\rangle = \langle-\overset{\star}{A}U, a\rangle \\ &= \langle-\overset{\star}{A}U, a^T\rangle = \langle U, -\overset{\star}{A}\,a^T\rangle = \widetilde{g}(U, -\overset{\star}{A}\,a^T).\end{aligned}$$

Therefore, $\widetilde{\nabla}\langle\xi, a\rangle = -\overset{\star}{A}\,a^T$. Using this expression, we get by direct computation that for all U, $W \in \mathfrak{X}(M)$,

$$\begin{aligned}\langle\overset{0}{\nabla}_U\,\widetilde{\nabla}\nu, W\rangle &= -\left\langle a^T + \langle a,\xi\rangle N, \left(\nabla_U\,\overset{\star}{A}\right)W\right\rangle \\ &\quad -\langle\overset{\star}{A}{}^2U, W\rangle\omega(a^T) + c\langle\overset{\star}{A}U, W\rangle\langle\psi, a\rangle\end{aligned}$$

It is easy to check that if $T \in End(TM)$ is a self-adjoint operator with respect to g then

$$\begin{aligned}\left\langle(\nabla_U T)V, W\right\rangle &= \left\langle V, (\nabla_U T)W\right\rangle + \omega(V)B(U, TW) \\ &\quad -\omega(TV)B(U,W) - \omega(W)B(U,TV) + \omega(TW)B(U,V).\end{aligned}$$

Applying this for $T = \overset{\star}{A}$ leads to

$$\begin{aligned}\langle\overset{0}{\nabla}_U\,\widetilde{\nabla}\nu, W\rangle &= -\left(\left\langle\left(\nabla_U\,\overset{\star}{A}\right)a^T, W\right\rangle + \omega(W)B(U\,\overset{\star}{A}\;a^T) - \omega(a^T)B(U, \overset{\star}{A}W)\right. \\ &\quad \left.+\langle a,\xi\rangle\left\langle\left(\nabla_U\,\overset{\star}{A}\right)W, N\right\rangle\right) - \langle\overset{\star}{A}{}^2U, W\rangle\omega(a^T) + c\langle\psi, a\rangle\langle\overset{\star}{A}U, W\rangle.\end{aligned}$$

But $\left\langle\left(\nabla_U\,\overset{\star}{A}\right)W, N\right\rangle = \langle\overset{\star}{A}AU, W\rangle$ and due to (2.7), we get

$$\left\langle\left(\nabla_U\,\overset{\star}{A}\right)W, N\right\rangle = \left\langle -\frac{1}{2}\lambda\,\overset{\star}{A}{}^2\,U - \sigma\,\overset{\star}{A}\,U, W\right\rangle$$

where $\lambda = \langle\zeta,\zeta\rangle$. Also, by Gauss-Codazzi equation with $\tau = 0$, the following equation holds,

$$\langle\overline{R}(U,V)\xi, W\rangle = -\langle\overline{R}(U,V)W,\xi\rangle = \langle(\nabla_V\,\overset{\star}{A})U - (\nabla_U\,\overset{\star}{A})V, W\rangle,$$

and since the ambient space has constant sectional curvature c, the left hand side vanishes, which leads to $\langle(\nabla_V\,\overset{\star}{A})U, W\rangle = \langle(\nabla_U\,\overset{\star}{A})V, W\rangle$. Therefore, $\left\langle\left(\nabla_U\,\overset{\star}{A}\right)a^T, W\right\rangle =$

$\left\langle \left(\nabla_{a^T} \overset{\star}{A}\right) U, W \right\rangle$ and

$$\langle \overset{0}{\nabla}_U \widetilde{\nabla}\nu, W\rangle = \Big\langle -\left(\nabla_{a^T}\overset{\star}{A}\right) U - \Big\langle \overset{\star}{A}^2 U, a^T\Big\rangle N + \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right)\langle \xi, a\rangle + c\langle\psi, a\rangle \overset{\star}{A} U, W\Big\rangle.$$

and this leads to

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla}\nu &= -\left(\nabla_{a^T}\overset{\star}{A}\right) U - \Big\langle \overset{\star}{A}^2 U, a^T\Big\rangle N + \langle \xi, a\rangle \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right) \\ &\quad + c\langle\psi, a\rangle \overset{\star}{A} U + \beta(U)\xi + \gamma(U)\psi. \end{aligned} \tag{3.37}$$

Taking respectively ξ and ψ components both side leads to $\beta(U) = 0$ and $\gamma(U) = c\left\langle \overset{\star}{A} U, a^T\right\rangle$. Hence,

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla}\nu &= -\left(\nabla_{a^T}\overset{\star}{A}\right) U - \Big\langle \overset{\star}{A}^2 U, a^T\Big\rangle N + \langle \xi, a\rangle \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right) \\ &\quad + c\langle\psi, a\rangle \overset{\star}{A} U + c\left\langle \overset{\star}{A} U, a^T\right\rangle \psi. \end{aligned} \tag{3.38}$$

Computing the same term $\overset{0}{\nabla}_U \widetilde{\nabla}\nu$ using the right hand side of (3.24), we get

$$\begin{aligned} \overset{0}{\nabla}_U \widetilde{\nabla}\nu &= \widetilde{\nabla}_U \widetilde{\nabla}\nu - \left\langle \overset{\star}{A}^2 U, a^T\right\rangle (N - \xi) \\ &\quad - \left\langle -\frac{1}{2}\lambda \overset{\star}{A}^2 U - \sigma \overset{\star}{A} U, a^T\right\rangle \xi + c\left\langle \overset{\star}{A} U, a^T\right\rangle \psi. \end{aligned} \tag{3.39}$$

By comparing (3.38)‘and (3.39) and using (2.7) we get,

$$\begin{aligned} \widetilde{\nabla}_U \widetilde{\nabla}\nu &= -(\nabla_{a^T}\overset{\star}{A})U - \left[\left\langle \overset{\star}{A}^2 U, a^T\right\rangle + \left\langle \frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U, a^T\right\rangle\right]\xi \\ &\quad + \langle \xi, a\rangle \left(\frac{1}{2}\lambda \overset{\star}{A}^2 U + \sigma \overset{\star}{A} U\right) + c\langle\psi, a\rangle \overset{\star}{A} U. \end{aligned} \tag{3.40}$$

Finally, taking into account that

$$(\nabla_{a^T} \overset{\star}{A})U = (\widetilde{\nabla}_{a^T} \overset{\star}{A})U - [\langle \overset{\star}{A}^2 U - \overset{\star}{A} AU, a^T\rangle\rangle]\xi,$$

we get the desired relation (3.34). Now,

$$\begin{aligned} \widetilde{L}_r\langle \xi, a\rangle &= \operatorname{tr}(\overset{\star}{T}_r \widetilde{\nabla}^2\nu) \\ &\overset{3.34}{=} -\operatorname{tr}\left(\overset{\star}{T}_r \circ \widetilde{\nabla}_{a^T} \overset{\star}{A}\right) - 0 + \langle \xi, a\rangle \left(\frac{1}{2}\lambda \operatorname{tr}(\overset{\star}{T}_r \circ \overset{\star}{A}^2) + \sigma \operatorname{tr}(\overset{\star}{T}_r \circ \overset{\star}{A})\right) + \\ &\quad c\langle\psi, a\rangle \operatorname{tr}\left(\overset{\star}{T}_r \circ \overset{\star}{A}\right), \end{aligned}$$

and (3.35) is straightforward from Proposition 3.1. The last claim (3.36) follows from

$$\widetilde{L}_r\xi = \sum_{i=1}^{m} \varepsilon_i \Big(\widetilde{L}_r\langle \xi, e_i\rangle\Big) e_i$$

where we use (3.35) componentwise. □

Before the next statement, we recall the following from [11, Lemma 4 (4)], where ζ is a closed and conformal vector field.

$$\overline{Ric}(U,\zeta) = -(n+1)U\cdot\sigma, \tag{3.41}$$

forall $U \in \mathfrak{X}(M)$. Since our ambient space $\overline{M}^{n+2}(c)$ has constant sectional curvature c, it follows from (3.42) that

$$U\cdot\sigma = -c\omega(U) \quad \text{for all} \quad U \in \mathfrak{X}(M). \tag{3.42}$$

Taking $U = \xi$ provides

$$\xi\cdot\sigma = -c. \tag{3.43}$$

It turns out that

$$\widetilde{\nabla}\sigma = (\xi\cdot\sigma)\xi = -c\xi. \tag{3.44}$$

Furthermore, for $U \in \mathfrak{X}(M)$,

$$\widetilde{\nabla}_U\widetilde{\nabla}\sigma \;\;=\;\; \widetilde{\nabla}_U(-c\xi) = c\overset{\star}{A}U, \tag{3.45}$$

and we get

$$\widetilde{L}_r\sigma = tr\left(\overset{\star}{T}_r\ (\widetilde{\nabla}_U\widetilde{\nabla}\sigma)\right) = (-1)^r(r+1)c\overset{\star}{S}_{r+1}\ . \tag{3.46}$$

As for σ, the function $\lambda = \langle\zeta,\zeta\rangle$ is (screen) leafwise constant and $\overline{\nabla}\lambda = 2\sigma\zeta$. Therefore,

$$\widetilde{\nabla}\lambda = 2\sigma\xi. \tag{3.47}$$

Hence, for all $U \in \mathfrak{X}(M)$,

$$\widetilde{\nabla}_U\widetilde{\nabla}\lambda = -2c\omega(U)\xi - 2\sigma\overset{\star}{A}U, \tag{3.48}$$

and

$$\widetilde{L}_r\lambda = \mathrm{tr}\left(\overset{\star}{T}_r\ (\widetilde{\nabla}_U\widetilde{\nabla}\lambda)\right) = (-1)^{r+1}\left(2c\overset{\star}{S}_r + 2(r+1)\sigma\overset{\star}{S}_{r+1}\right). \tag{3.49}$$

Following the same steps as above for the function $\nu = \langle\xi, a\rangle$, we establish the following.

Lemma 3.6. *Let $a \in \mathbb{R}^m_t$ be a fixed constant vector and $U \in \mathfrak{X}(M)$. Then,*

$$\widetilde{\nabla}\langle \zeta, a\rangle = \sigma a^T + \langle \sigma(\xi - N) - c\psi, a\rangle\, \xi, \tag{3.50}$$

or equivalently

$$\widetilde{\nabla}\langle \zeta, a\rangle = \sigma a + \langle \sigma(\xi - N) - c\psi, a\rangle\, \xi - \sigma\langle \xi, a\rangle N - c\sigma\langle a, \psi\rangle\psi; \tag{3.51}$$

$$\begin{aligned}\widetilde{\nabla}_U\widetilde{\nabla}\langle \zeta, a\rangle &= -c\omega(U)a^T - \sigma\left(\sigma\langle a, \xi\rangle + c\langle a, \psi\rangle\right) PU \\ &\quad - [(\lambda+1)\sigma\langle a, \xi\rangle - \sigma\langle a, \zeta\rangle - c\langle a, \psi\rangle]\, \overset{\star}{A}\, U \\ &\quad - c\left\langle \frac{1}{2}(2+\lambda)\omega(U)\xi - \omega(U)\zeta + U, a\right\rangle \xi;\end{aligned}$$

$$\begin{aligned}\widetilde{L}_r\langle \zeta, a\rangle &= (-1)^{r+1}\left[\left((n-r)\sigma^2 + 2c\right) \overset{\star}{S}_r + (r+1)(\lambda+1)\sigma\, \overset{\star}{S}_{r+1}\right] \langle \xi, a\rangle \\ &\quad + (-1)^{r+1}\left[(n-r)c\sigma\, \overset{\star}{S}_r - (r+1)c\, \overset{\star}{S}_{r+1}\right] \langle \psi, a\rangle \\ &\quad + (-1)^r (r+1)\sigma\, \overset{\star}{S}_{r+1}\, \langle \zeta, a\rangle;\end{aligned}$$

and

$$\begin{aligned}\widetilde{L}_r\zeta &= (-1)^{r+1}\left[\left((n-r)\sigma^2 + 2c\right) \overset{\star}{S}_r + (r+1)(\lambda+1)\sigma\, \overset{\star}{S}_{r+1}\right] \xi \\ &\quad + (-1)^{r+1}\left[(n-r)c\sigma\, \overset{\star}{S}_r - (r+1)c\, \overset{\star}{S}_{r+1}\right] \psi \\ &\quad + (-1)^r (r+1)\sigma\, \overset{\star}{S}_{r+1}\, \zeta.\end{aligned} \tag{3.52}$$

Now, we compute $\widetilde{L}_r^2\psi$. Starting from (3.26),

$$\begin{aligned}\widetilde{L}_r^2\langle \psi, a\rangle &= (-1)^r(r+1)\widetilde{L}_r\left(\overset{\star}{S}_{r+1}\, \langle \zeta, a\rangle\right) + (-1)^{r+1}c(n-r)\widetilde{L}_r\left(\overset{\star}{S}_r\, \langle \psi, a\rangle\right) \\ &\quad + (-1)^{r+1}(r+1)\widetilde{L}_r\left((\lambda+1)\, \overset{\star}{S}_{r+1}\, \langle \xi, a\rangle\right) + (-1)^{r+1}(n-r)\widetilde{L}_r\left(\sigma\, \overset{\star}{S}_r\, \langle \xi, a\rangle\right).\end{aligned}$$

We compute each term using Lemma 3.3, (3.33) (3.36), (3.50), (3.52), (3.44), (3.46), (3.47) and (3.49):

$$\begin{aligned}(-1)^r(r+1)\widetilde{L}_r\left(\overset{\star}{S}_{r+1}\, \langle \zeta, a\rangle\right) &= 2(-1)^r(r+1)\sigma\left\langle \overset{\star}{T}_r\, \widetilde{\nabla}\, \overset{\star}{S}_{r+1}, a\right\rangle \\ &\quad + (r+1)\left[(-1)^r\widetilde{L}_r\, \overset{\star}{S}_{r+1} + (r+1)\sigma\, \overset{\star}{S}{}^2_{r+1}\right] \langle \zeta, a\rangle \\ &\quad - (r+1)\Big[\left((n-r)\sigma^2 + 2c\right) \overset{\star}{S}_r \overset{\star}{S}_{r+1} \\ &\quad + (r+1)(\lambda+1)\sigma\, \overset{\star}{S}{}^2_{r+1}\Big]\langle \xi, a\rangle \\ &\quad - (r+1)\Big[(n-r)c\sigma\, \overset{\star}{S}_r \overset{\star}{S}_{r+1} - (r+1)c\, \overset{\star}{S}{}^2_{r+1}\end{aligned}$$

$$+2c \overset{\star}{S}_r (\xi\cdot \overset{\star}{S}_{r+1})\Big]\langle\psi, a\rangle;$$

$$\begin{aligned}(-1)^{r+1}c(n-r)\widetilde{L}_r\Big(\overset{\star}{S}_r\langle\psi,a\rangle\Big) &= (-1)^{r+1}(n-r)c\Big[\widetilde{L}_r\overset{\star}{S}_r+(-1)^{r+1}(n-r)c\overset{\star}{S}{}_r^2\Big]\langle\psi,a\rangle\\ &+(n-r)c\Big[(n-r)\sigma\overset{\star}{S}{}_r^2+(r+1)(\lambda+1)\overset{\star}{S}_r\overset{\star}{S}_{r+1}\Big]\langle\xi,a\rangle\\ &-(n-r)(r+1)c\overset{\star}{S}_r\overset{\star}{S}_{r+1}\langle\zeta,a\rangle\\ &+2(-1)^{r+1}(n-r)c\Big\langle\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_r,a\Big\rangle;\end{aligned}$$

$$\begin{aligned}(-1)^{r+1}(r+1)\widetilde{L}_r\Big((\lambda+1)\overset{\star}{S}_{r+1}\langle\xi,a\rangle\Big) &= (r+1)(\lambda+1)\overset{\star}{S}_{r+1}\Big\langle\widetilde{\nabla}\overset{\star}{S}_{r+1},a\Big\rangle\\ &+2(-1)^r(r+1)(\lambda+1)\Big\langle\overset{\star}{T}_r\circ\overset{\star}{A}\widetilde{\nabla}\overset{\star}{S}_{r+1},a\Big\rangle\\ &-(r+1)^2(\lambda+1)c\overset{\star}{S}{}_{r+1}^2\langle\psi,a\rangle\\ &+\Big[\Big(\frac{1}{2}\lambda(\lambda+1)(r+1)\overset{\star}{S}_1\\ &-(r+1)^2(\lambda+1)\sigma\Big)\overset{\star}{S}{}_{r+1}^2\\ &+\frac{1}{2}\lambda(\lambda+1)(r+1)(r+2)\overset{\star}{S}_{r+1}\overset{\star}{S}_{r+1}\\ &-(r+1)(\lambda+1)\overset{\star}{S}_{r+1}(\xi\cdot\overset{\star}{S}_{r+1})\\ &+(-1)^{r+1}(r+1)(\lambda+1)\widetilde{L}_r\overset{\star}{S}_{r+1}\\ &+2(r+1)c\overset{\star}{S}_r\overset{\star}{S}_{r+1}+2(r+1)^2\sigma\overset{\star}{S}{}_{r+1}^2\\ &-4(r+1)\sigma\overset{\star}{S}_r(\xi\cdot\overset{\star}{S}_{r+1})\Big]\langle\xi,a\rangle;\end{aligned}$$

$$\begin{aligned}(-1)^{r+1}(n-r)\widetilde{L}_r\Big(\sigma\overset{\star}{S}_r\langle\xi,a\rangle\Big) &= (n-r)\sigma\overset{\star}{S}_r\Big\langle\widetilde{\nabla}\overset{\star}{S}_{r+1},a\Big\rangle\\ &+2(-1)^r(n-r)\sigma\Big\langle\overset{\star}{T}_r\circ\overset{\star}{A}\widetilde{\nabla}\overset{\star}{S}_r,a\Big\rangle\\ &-(n-r)(r+1)\sigma c\overset{\star}{S}_r\overset{\star}{S}_{r+1}\langle\psi,a\rangle\\ &+(n-r)\Big[(-1)^{r+1}\sigma\widetilde{L}_r\overset{\star}{S}_r\\ &+(n-r)\Big(\frac{1}{2}\lambda\sigma\overset{\star}{S}_1-(r+1)(c+\sigma^2)\Big)\overset{\star}{S}_r\overset{\star}{S}_{r+1}\\ &+2(n-r)c\overset{\star}{S}_r(\xi\cdot\overset{\star}{S}_r)+\frac{1}{2}\lambda(n-r)(r+2)\sigma\overset{\star}{S}_r\overset{\star}{S}_{r+2}\\ &-(n-r)\sigma\overset{\star}{S}_r(\xi\cdot\overset{\star}{S}_{r+1})\Big]\langle\xi,a\rangle.\end{aligned}$$

Putting all the above together, we get the following.

Proposition 3.3. *Let*

$$\psi: M^{n+1}\longrightarrow\overline{M}_1^{n+2}(c)\subseteq\mathbb{R}^m_{1+t}$$

be a isometric immersion of a null hypersurface in the Robertson-Walker space $\overline{M}_1^{n+2}(c)$ *where* $m=n+2+c^2$, $t=c(c-1)/2$ *with* $c=1,0,-1$, *furnished with a timelike closed and* *conformal rigging vector field* ζ. *If* $\lambda = \langle \zeta, \zeta \rangle$ *denotes the squared length function of* ζ *and* σ *its conformal factor, Then,*

$$\begin{aligned}
\widetilde{L}_r^2 \langle \psi, a \rangle &= \Big[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + (n-r)\sigma \Big] \Big\langle \widetilde{\nabla} \overset{\star}{S}_{r+1}, a \Big\rangle \\
&+2(-1)^r (r+1)(\lambda+1) \Big\langle (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_{r+1}, a \Big\rangle \\
&+2(-1)^r (n-r)\sigma \Big\langle (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_r, a \Big\rangle \\
&+2(-1)^r (r+1)\sigma \Big\langle \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_{r+1}, a \Big\rangle \\
&+2(-1)^{r+1} (n-r) c \Big\langle \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_r, a \Big\rangle \\
&+\Lambda_r^{\xi} \langle \xi, a \rangle + \Lambda_r^{\zeta} \langle \zeta, a \rangle + \Lambda_r^{\psi} \langle \psi, a \rangle
\end{aligned} \tag{3.53}$$

for a fixed $a \in \mathbb{R}^m_{1+t}$*; and*

$$\begin{aligned}
\widetilde{L}_r^2 \psi &= \Big[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + (n-r)\sigma \Big] \widetilde{\nabla} \overset{\star}{S}_{r+1} \\
&+2(-1)^r (r+1)(\lambda+1) (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_{r+1} \\
&+2(-1)^r (n-r)\sigma (\overset{\star}{T}_r \circ \overset{\star}{A}) \widetilde{\nabla} \overset{\star}{S}_r \\
&+2(-1)^r (r+1)\sigma \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_{r+1} \\
&+2(-1)^{r+1} (n-r) c \overset{\star}{T}_r \widetilde{\nabla} \overset{\star}{S}_r \\
&+\Lambda_r^{\xi} \xi + \Lambda_r^{\zeta} \zeta + \Lambda_r^{\psi} \psi;
\end{aligned} \tag{3.54}$$

with Λ_r^{ξ}, Λ_r^{ζ} *and* Λ_r^{ψ} *as follows :*

$$\begin{aligned}
\Lambda_r^{\xi} &= (-1)^{r+1}(r+1)(\lambda+1) \widetilde{L}_r \overset{\star}{S}_{r+1} + (-1)^{r+1} \sigma (n-r) \widetilde{L}_r \overset{\star}{S}_r \\
&+(r+1)\lambda \left(\frac{1}{2}(\lambda+1) \overset{\star}{S}_1 - 2(r+1)\sigma \right) \overset{\star}{S}{}^2_{r+1} + c(n-r)^2 \sigma \overset{\star}{S}{}^2_r \\
&+(n-r) \left(\frac{1}{2}\lambda\sigma \overset{\star}{S}_1 + (r+1)(c\lambda - 2\sigma^2) \right) \overset{\star}{S}_r \overset{\star}{S}_{r+1} \\
&+\frac{1}{2}(r+1)(r+2)\lambda(\lambda+1) \overset{\star}{S}_{r+1} \overset{\star}{S}_{r+2} \\
&+\frac{1}{2}(r+2)(n-r)\lambda\sigma \overset{\star}{S}_r \overset{\star}{S}_{r+2} + 2(n-r)c \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_r) \\
&-\Big[(r+1)(\lambda+1) \overset{\star}{S}_{r+1} + \sigma(n+3r+4) \overset{\star}{S}_r \Big] (\xi \cdot \overset{\star}{S}_{r+1});
\end{aligned} \tag{3.55}$$

$$\Lambda_r^{\zeta} = (r+1) \Big[(-1)^r \widetilde{L}_r \overset{\star}{S}_{r+1} + (r+1)\sigma \overset{\star}{S}{}^2_{r+1} - (n-r)c \overset{\star}{S}_r \overset{\star}{S}_{r+1} \Big] \tag{3.56}$$

and

$$\begin{aligned}
\Lambda_r^{\psi} &= c\Big[(-1)^{r+1}(n-r) \widetilde{L}_r \overset{\star}{S}_r + (n-r)^2 c \overset{\star}{S}{}^2_r - (r+1)^2 \lambda \overset{\star}{S}{}^2_{r+1} \\
&-2(r+1)(n-r)\sigma \overset{\star}{S}_r \overset{\star}{S}_{r+1} - 2(r+1) \overset{\star}{S}_r (\xi \cdot \overset{\star}{S}_{r+1}) \Big].
\end{aligned} \tag{3.57}$$

Remark 3.2. *Observe that*

$$\widetilde{\nabla}\overset{\star}{S}_r = P\widetilde{\nabla}\overset{\star}{S}_r + (\xi\cdot \overset{\star}{S}_r)\xi, \quad \overset{\star}{T}_r \widetilde{\nabla}\overset{\star}{S}_r = P\left[\overset{\star}{T}_r \widetilde{\nabla}\overset{\star}{S}_r\right] + (-1)^r \overset{\star}{S}_r (\xi\cdot \overset{\star}{S}_r)\xi$$

and similar formulas for $\widetilde{\nabla}\overset{\star}{S}_{r+1}$ *and* $\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_{r+1}$. *So we get the following useful equivalent formula for (3.54)*

$$\begin{aligned}
\widetilde{L}_r^2\psi &= \left[(r+1)(\lambda+1)\overset{\star}{S}_{r+1} + (n-r)\sigma\right] P\widetilde{\nabla}\overset{\star}{S}_{r+1} \\
&\quad +2(-1)^r(r+1)(\lambda+1)(\overset{\star}{T}_r\circ\overset{\star}{A})\widetilde{\nabla}\overset{\star}{S}_{r+1} \\
&\quad +2(-1)^r(n-r)\sigma(\overset{\star}{T}_r\circ\overset{\star}{A})\widetilde{\nabla}\overset{\star}{S}_r \\
&\quad +2(-1)^r(r+1)\sigma P\,\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_{r+1} \\
&\quad +2(-1)^{r+1}(n-r)cP\,\overset{\star}{T}_r\widetilde{\nabla}\overset{\star}{S}_r \\
&\quad +\overset{\star}{\Lambda}{}_r^{\xi}\,\xi + \Lambda_r^{\zeta}\zeta + \Lambda_r^{\psi}\psi;
\end{aligned} \tag{3.58}$$

with

$$\begin{aligned}
\overset{\star}{\Lambda}{}_r^{\xi} &= (-1)^{r+1}(r+1)(\lambda+1)\widetilde{L}_r\overset{\star}{S}_{r+1} + (-1)^{r+1}\sigma(n-r)\widetilde{L}_r\overset{\star}{S}_r \\
&\quad +(r+1)\lambda\left(\frac{1}{2}(\lambda+1)\overset{\star}{S}_1 - 2(r+1)\sigma\right)\overset{\star}{S}{}^2_{r+1} + c(n-r)^2\sigma\,\overset{\star}{S}{}^2_r \\
&\quad +(n-r)\left(\frac{1}{2}\lambda\sigma\overset{\star}{S}_1 + (r+1)(c\lambda-2\sigma^2)\right)\overset{\star}{S}_r\overset{\star}{S}_{r+1} \\
&\quad +\frac{1}{2}(r+1)(r+2)\lambda(\lambda+1)\overset{\star}{S}_{r+1}\overset{\star}{S}_{r+2} \\
&\quad +\frac{1}{2}(r+2)(n-r)\lambda\sigma\overset{\star}{S}_r\overset{\star}{S}_{r+2} - 2(r+1)\sigma\overset{\star}{S}_r(\xi\cdot\overset{\star}{S}_{r+1}).
\end{aligned} \tag{3.59}$$

Definition 3.1. *A connected isometric immersion*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_q^m$$

of a null hypersurface in $\overline{M}_1^{n+2}(c)$ *furnished with a rigging vector field* ζ *is said to be* $\widetilde{L}_r-$*biharmonic if the position vector field* ψ *satisfies the condition* $\widetilde{L}_r^2\psi = 0$.

Remark 3.3. *Based on (3.58),(3.59), (3.56), (3.57) and Theorem 3.1, a* $r-$*maximal null hypersurface*

$$\psi : M^{n+1} \longrightarrow \overline{M}_1^{n+2}(c) \subseteq \mathbb{R}_q^m$$

is biharmonic. For this, we fix that proper $\widetilde{L}_r-$*biharmonic null hypersurfaces are* $\widetilde{L}_r-$ *biharmonic, but not* $r-$*maximal.*

4. Examples

Example 4.1 (Null cone torus). *Let $n \geq m \geq 2$ be integers. Consider*

$$M = \{x \in \mathbb{L}^{n+3} \mid -x_0^2 + x_1^2 + \cdots + x_{m+1}^2 = 0, \quad x_{m+2}^2 + \cdots + x_{n+2}^2 = 1\} \cap \{x_0 > 0\}.$$

It is easy to see that $M = \Lambda_0^{m+1} \times \mathbb{S}^{n-m}$ is a null hypersurface of the De Sitter spacetime $\mathbb{S}_1^{n+2}$ given by the product of the lightcone Λ_0^{m+1} of dimension $m+1$ with the $n-m$ standard sphere $\mathbb{S}^{n-m}$ (a null cone torus). A timelike closed and conformal rigging for M is given by

$$\zeta = \partial_0 + x_0 x,$$

with (null) rigged vector field

$$\xi = -\frac{1}{x_0} \cdot \Big(x_0, x_1, \ldots, x_{m+1}, 0, \ldots, 0 \Big).$$

Then the shape operator is

$$\overset{\star}{A} \simeq \left[\begin{array}{c|c|c} 0 & \cdots \quad \cdots & 0 \\ \vdots & \frac{1}{x_0} I_m & 0 \\ \hline \vdots & 0 & 0_{n-m} \\ 0 & & \end{array} \right],$$

and we get that

$$\overset{\star}{H}_r = \begin{cases} \binom{n+1}{r}^{-1} \binom{m}{r} \cdot \dfrac{1}{(x_0)^r} & \text{if} \quad 0 \leq r \leq m \\ \\ 0 & \text{if} \quad m+1 \leq r \leq n+1 \end{cases} \tag{4.60}$$

Based on Remark 3.3, we see that $M = \Lambda_0^{m+1} \times \mathbb{S}^{n-m}$ is $\widetilde{L}_k-$biharmonic for $m+1 \leq k \leq n+1$.

Example 4.2 (Null cone cylinder). *Let $1 \leq m \leq n-1$ be integers, and*

$$M = \{x \in \mathbb{L}^{n+2} \mid -x_0^2 + x_1^2 + \cdots + x_{m+1}^2 = 0, \quad x_0 > 0\}.$$

This null cone cylinder $\Lambda_0^{m+1} \times \mathbb{R}^{n-m}$ is a null hypersurface in $\mathbb{L}^{n+2}$, for which a natural timelike closed and conformal rigging is given by the constant vector field

$$\zeta = \partial_1$$

with corresponding rigged vector field

$$\xi = -\frac{1}{x_0}\cdot\Big(x_0, x_1, \ldots, x_{m+1}, 0, \ldots, 0\Big).$$

Similar computations as in above Example 4.1, show that the hight order mean curvatures are given as in (4.60) and $\Lambda_0^{m+1}\times\mathbb{R}^{n-m}$ *is* $\widetilde{L}_k-$*biharmonic for* $m+1\leq k\leq n+1$.

5. Proofs of main results

5.1. **Proof of Theorem 1.1.** The L_r-harmonicity condition reads

$$\begin{aligned} 0=\widetilde{L}_r\psi &= (-1)^{r+1}\Big[(n-r)\sigma \overset{\star}{S}_r + (r+1)\lambda \overset{\star}{S}_{r+1}\Big]\xi + \Big[(-1)^r(r+1)\overset{\star}{S}_{r+1}\Big]\zeta \\ &\quad + (-1)^{r+1}\Big[c(n-r)\overset{\star}{S}_r\Big]\psi. \end{aligned}$$

This is equivalent to

$$\overset{\star}{S}_{r+1}=0,\quad \sigma\overset{\star}{S}_r=0 \quad \text{and} \quad c\overset{\star}{S}_r=0.$$

Obviously, due to Theorem 3.1, if $\overset{\star}{S}_r=0$ the above system is satisfied. Assume $\overset{\star}{S}_r\neq 0$. Then, $\overset{\star}{S}_{r+1}=0$ and $\sigma=0$ and the latter implies $c=0$ due to (3.43). □

5.2. **Proof of Theorem 1.2.** We prove cases $n=1$ and $n=2$ separately.

- Case $n=1$.

  From (3.54) with $n=1$ and $r=0$,

$$\widetilde{L}_0^2\psi = \Big[(\lambda+1)\overset{\star}{S}_1 + 3\sigma\Big]P\widetilde{\nabla}\overset{\star}{S}_1 + 2(\lambda+1)\overset{\star}{A}\widetilde{\nabla}\overset{\star}{S}_1 + \overset{\star}{\Lambda}{}_0^{\xi}\,\xi + \Lambda_0^{\zeta}\zeta + \Lambda_0^{\psi}\psi,$$

  with

$$\begin{aligned} \overset{\star}{\Lambda}{}_0^{\xi} &= -(\lambda+1)\widetilde{\Delta}\overset{\star}{S}_1 + \frac{\lambda}{2}\Big[(\lambda+1)\overset{\star}{S}_1 - 3\sigma\Big]\overset{\star}{S}{}_1^2 \\ &\quad + (c\lambda - 2\sigma^2)\overset{\star}{S}_1 - 2\sigma(\xi\cdot\overset{\star}{S}_1) + c\sigma, \end{aligned} \tag{5.61}$$

$$\Lambda_0^{\zeta} = \widetilde{\Delta}\overset{\star}{S}_1 + \sigma\overset{\star}{S}{}_1^2 - c\overset{\star}{S}_1 \tag{5.62}$$

  and

$$\Lambda_0^{\psi} = c\Big[c - \lambda\overset{\star}{S}{}_1^2 - 2\sigma\overset{\star}{S}_1 - 2(\xi\cdot\overset{\star}{S}_1)\Big] \tag{5.63}$$

where we used $\overset{\star}{S}_2 = 0$. Therefore, the condition $\widetilde{L}_0^2\psi = 0$ is equivalent to

$$\overset{\star}{A}P\widetilde{\nabla}\overset{\star}{S}_1 = -\frac{(\lambda+1)\overset{\star}{S}_1 + 3\sigma}{2(\lambda+1)}P\widetilde{\nabla}\overset{\star}{S}_1 \tag{5.64}$$

$$\widetilde{\Delta}\overset{\star}{S}_1 + \sigma\overset{\star}{S}_1^2 - c\overset{\star}{S}_1 = 0 \tag{5.65}$$

$$c\Big[c - \lambda\overset{\star}{S}_1^2 - 2\sigma\overset{\star}{S}_1 - 2(\xi\cdot\overset{\star}{S}_1)\Big] = 0 \tag{5.66}$$

$$-(\lambda+1)\widetilde{\Delta}\overset{\star}{S}_1 + \frac{\lambda}{2}\Big[(\lambda+1)\overset{\star}{S}_1 - 3\sigma\Big]\overset{\star}{S}_1^2 + (c\lambda - 2\sigma^2)\overset{\star}{S}_1 - 2\sigma(\xi\cdot\overset{\star}{S}_1) + c\sigma = 0. \tag{5.67}$$

Assume $P\widetilde{\nabla}\overset{\star}{S}_1 \neq 0$. Then, we see that $P\widetilde{\nabla}\overset{\star}{S}_1$ is an eigenvector field of $\overset{\star}{A}$ with eigenfunction (a screen principal curvature)

$$\overset{\star}{k} = -\frac{(\lambda+1)\overset{\star}{S}_1 + 3\sigma}{2(\lambda+1)}.$$

Since the null surface M is $2-$dimensional, it follows that $\overset{\star}{k} = 0$ or $\overset{\star}{k} = \overset{\star}{S}_1$. But each of the two cases implies $\overset{\star}{S}_1 = \overset{\star}{S}_1(\sigma,\lambda)$ which leads to a condradiction since σ and λ are leafwise constant. We conclude that $P\widetilde{\nabla}\overset{\star}{S}_1 = 0$ and $\overset{\star}{S}_1$ is leafwise constant. Observe that by the Raychaudhuri equation (2), if $\overset{\star}{S}_1$ is constant on the whole M, this constant is zero. But the case $c \neq 0$ implies $\overset{\star}{S}_1 \neq 0$. Indeed, $\overset{\star}{S}_1 = 0$ in (5.67) leads to $\sigma = 0$ on M and $c = -\xi\cdot\sigma = 0$ which is a contradiction. Hence, for $c \neq 0$, $\overset{\star}{S}_1$ is not constant on the whole M. To go further, let (ξ, X) be a local $\widetilde{g}-$orthonormal basis of M. Since $\widetilde{\nabla}\overset{\star}{S}_1 = (\xi\cdot\overset{\star}{S}_1)\xi = \overset{\star}{S}_1^2\,\xi$ we get

$$\widetilde{\Delta}\overset{\star}{S}_1 = \widetilde{g}(\widetilde{\nabla}_\xi\widetilde{\nabla}\overset{\star}{S}_1, \xi) + \widetilde{g}(\widetilde{\nabla}_X\widetilde{\nabla}\overset{\star}{S}_1, X = \widetilde{g}(\widetilde{\nabla}_\xi(\overset{\star}{S}_1^2\,\xi), \xi) + \widetilde{g}(\widetilde{\nabla}_X(\overset{\star}{S}_1^2\,\xi), X) = \overset{\star}{S}_1^3. \tag{5.68}$$

Consider the case where $c = 0$ and assume $\overset{\star}{S}_1 \neq 0$. From (5.65) and (5.68) we get $(\overset{\star}{S}_1 + \sigma)\overset{\star}{S}_1^2 = 0$. Therefore $\overset{\star}{S}_1 = -\sigma$. Then we get

$$\sigma^2 = \overset{\star}{S}_1^2 = \xi\cdot\overset{\star}{S}_1 = -\xi\cdot\sigma = c = 0.$$

Therefore, $\sigma = 0$ on M and $\overset{\star}{S}_1 = -\sigma = 0$ which is a contradiction.

- Case $n = 2$.

With $r = 0$, equation (3.54) reads

$$\widetilde{L}_0^2\psi = \Big[(\lambda+1)\overset{\star}{S}_1 + 4\sigma\Big]P\widetilde{\nabla}\overset{\star}{S}_1 + 2(\lambda+1)\overset{\star}{A}\widetilde{\nabla}\overset{\star}{S}_1 + \overset{\star}{\Lambda}{}_0^{\xi}\,\xi + \Lambda_0^{\zeta}\zeta + \Lambda_0^{\psi}\psi, \tag{5.69}$$

with

$$\begin{aligned}\overset{\star}{\Lambda}{}^{\xi}_0 &= -(\lambda+1)\widetilde{\Delta}\overset{\star}{S}_1 + \lambda\Big[\frac{1}{2}(\lambda+1)\overset{\star}{S}_1 - 2\sigma\Big]\overset{\star}{S}{}_1^2 + 2\Big[\frac{1}{2}\lambda\sigma\overset{\star}{S}_1 + c\lambda - 2\sigma^2\Big]\overset{\star}{S}_1 \\ &\quad + \lambda(\lambda+1)\overset{\star}{S}_1\overset{\star}{S}_2 + 2\lambda\sigma\overset{\star}{S}_2 + 4c\sigma - 2\sigma(\xi\cdot\overset{\star}{S}_1). \end{aligned} \tag{5.70}$$

$$\Lambda^{\zeta}_0 = \widetilde{\Delta}\overset{\star}{S}_1 + \sigma\overset{\star}{S}{}_1^2 - 2c\overset{\star}{S}_1, \tag{5.71}$$

and

$$\Lambda^{\psi}_0 = c\Big[4c - \lambda\overset{\star}{S}{}_1^2 - 4\sigma\overset{\star}{S}_1 - 2(\xi\cdot\overset{\star}{S}_1)\Big]. \tag{5.72}$$

Therefore, the biharmonicity condition amounts to

$$\overset{\star}{A}P\widetilde{\nabla}\overset{\star}{S}_1 = -\frac{(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)}P\widetilde{\nabla}\overset{\star}{S}_1, \quad \overset{\star}{\Lambda}{}^{\xi}_0 = 0, \quad \Lambda^{\zeta}_0 = 0 \quad \text{and} \quad \Lambda^{\psi}_0 = 0. \tag{5.73}$$

Assume $P\widetilde{\nabla}\overset{\star}{S}_1 \neq 0$. Then we see from the first equation in (5.73) that

$$\overset{\star}{k}_1 = -\frac{(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)}$$

is a screen principal curvature. Also, it is easy to see that the screen shape operator is (with $\overset{\star}{k}_0 = 0$),

$$\overset{\star}{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overset{\star}{k}_1 & 0 \\ 0 & 0 & \overset{\star}{k}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)} & 0 \\ 0 & 0 & \frac{3(\lambda+1)\overset{\star}{S}_1 + 4\sigma}{2(\lambda+1)} \end{bmatrix}.$$

From Raychaudury equation (2) and due to $\tau(\xi) = 0$ and $\overline{Ric}(\xi,\xi) = 0$, we have

$$\xi\cdot\overset{\star}{S}_1 = \frac{1}{2(\lambda+1)^2}\Big[5(\lambda+1)^2\overset{\star}{S}{}_1^2 + 16(\lambda+1)\sigma\overset{\star}{S}_1 + 16\sigma^2\Big]. \tag{5.74}$$

Now, we treat the cases $c = 0$ and $c \neq 0$ separately.

Assume $c \neq 0$. Eq. (5.74) in the last equation in (5.73) yields

$$(\lambda+5)\overset{\star}{S}_1^2+\frac{4\sigma(\lambda+5)}{\lambda+1}\overset{\star}{S}_1+\frac{16\sigma^2}{(\lambda+1)^2}-4c=0.$$

But, $\lambda+5\not\equiv 0$, otherwise we get $c=0$ from (3.43) and (3.47) which is a contradiction. So, $\overset{\star}{S}_1=\overset{\star}{S}_1(\lambda,\sigma)$. Therefore, since λ and σ are (screen) leafwise constant, the same is for $\overset{\star}{S}_1$ and we get $P\widetilde{\nabla}\overset{\star}{S}_1=0$ which is a contradiction.

Assume now that $c=0$. It follows from the third equation in (5.73) that

$$\widetilde{\Delta}\overset{\star}{S}_1=-\sigma\overset{\star}{S}_1^2. \tag{5.75}$$

Also,

$$\overset{\star}{S}_2=\frac{1}{2}\left(\overset{\star}{S}_1^2-\xi\cdot\overset{\star}{S}_1\right)\overset{(5.74)}{=}-\frac{3}{4}\overset{\star}{S}_1^2-\frac{4\sigma}{\lambda+1}\overset{\star}{S}_1-\frac{4\sigma^2}{(\lambda+1)^2}. \tag{5.76}$$

Therefore, by replacing the expressions (5.75), (5.76) and (5.74) in the second equation in (5.73) we get

$$\frac{1}{4}\lambda(\lambda+1)\overset{\star}{S}_1^3+\frac{1}{2}(11\lambda+8)\sigma\overset{\star}{S}_1^2+\frac{4\sigma^2}{\lambda+1}(4\lambda+5)\overset{\star}{S}_1+\frac{8\sigma^3}{(\lambda+1)^2}(\lambda^2+2\lambda+2)=0$$

which is polynomial in $\overset{\star}{S}_1$ with degree 3 since $\lambda(\lambda+1)\neq 0$. Therefore, $\overset{\star}{S}_1=\overset{\star}{S}_1(\lambda,\sigma)$ which implies again a contradiction $P\widetilde{\nabla}\overset{\star}{S}_1=0$ since λ and σ are (screen) leafwise constant. Finally, we conclude that $P\widetilde{\nabla}\overset{\star}{S}_1=0$ and $\overset{\star}{S}_1$ is (screen) leafwise constant. Now we are interested in knowing whether $\overset{\star}{S}_1$ can be globally constant over the whole hypersurface M, in which case this constant would necessarily be zero. For this, observe that due to (5.72) and the last equation in (5.73), $c\neq 0$ implies $\|\overset{\star}{A}\|^2=\xi\cdot\overset{\star}{S}_1\neq 0$ and the answer is negative. It remains to analyze the case where $c=0$. Use (5.71) and the third equation in (5.73) to get

$$\widetilde{\Delta}\overset{\star}{S}_1=-\sigma\overset{\star}{S}_1^2. \tag{5.77}$$

Also, $0=c=-\xi\cdot\sigma$ and being leafwise constant, we see that σ restricts to a constant over the whole M. Assume this constant to be zero. From the second equation in (5.73) and (5.70) we get

$$\overset{\star}{S}_1(\overset{\star}{S}_1^2+2\overset{\star}{S}_2)=0. \tag{5.78}$$

In this relation, assume $\overset{\star}{S}_2\neq 0$, then by Theorem 3.1, $\overset{\star}{S}_1\neq 0$ and we get

$$\frac{1}{2}\left(\overset{\star}{S}_1^2-\xi\cdot\overset{\star}{S}_1\right)=\overset{\star}{S}_2=-\frac{1}{2}\overset{\star}{S}_1^2,$$

i.e $\xi\cdot \overset{\star}{S}_1 = 2 \overset{\star}{S}{}_1^2$. Before we go further, we note the following. Choose a local $\widetilde{g}$−orthonormal frame (X_0, X_1, X_2) consisting of eigenvectors of $\overset{\star}{A}$ such that $X_0 = \xi$ and $X_1, X_2 \in \Gamma(\mathcal{S})$. Then, by a straightforward computation, $\widetilde{\Delta}\overset{\star}{S}_1 = \xi\cdot(\xi\cdot \overset{\star}{S}_1) - (\xi\cdot \overset{\star}{S}_1)\overset{\star}{S}_1$. Therefore, $0 = \widetilde{\Delta}\overset{\star}{S}_1 = 4 \overset{\star}{S}{}_1^3 - 2 \overset{\star}{S}{}_1^3$, thus, $\overset{\star}{S}_1 = 0$: a contradiction. So, in (5.78), we have $\overset{\star}{S}_2 = 0$ and consequently $\overset{\star}{S}_1 = 0$. Now, assume that σ restricts on M to a non zero constant. Substituting (5.77) and $\overset{\star}{S}_2$ in the second equation in (5.73) yields

$$\lambda(\lambda+1) \overset{\star}{S}{}_1^3 + (\lambda+1)\sigma \overset{\star}{S}{}_1^2 - 4\sigma^2 \overset{\star}{S}_1 - \left[\lambda(\lambda+1) \overset{\star}{S}_1 + (\lambda+2)\sigma\right](\xi\cdot \overset{\star}{S}_1) = 0.$$

Taking again derivative with respect to ξ both side leads to

$$\begin{aligned}\lambda(\lambda+1)(\xi\cdot \overset{\star}{S}_1)^2 + \left[-2\lambda(\lambda+1) \overset{\star}{S}{}_1^2 + (3\lambda+2)\sigma \overset{\star}{S}_1 + 6\sigma^2\right](\xi\cdot \overset{\star}{S}_1)& \\ - \left[(5\lambda+2) \overset{\star}{S}_1 + \lambda^2 + \sigma\lambda + 4\sigma\right]\sigma \overset{\star}{S}{}_1^2 = 0& \qquad (5.79)\end{aligned}$$

Observe that since $\xi\cdot \overset{\star}{S}_1 = 0$ implies $\overset{\star}{S}_1 = 0$, we infer that $\xi\cdot \overset{\star}{S}_1$ is solution of Eq. (5.79). Consequently, we have

$$\xi\cdot \overset{\star}{S}_1 = 0 \quad \text{or} \quad \begin{cases} \xi\cdot \overset{\star}{S}_1 = \dfrac{2\lambda(\lambda+1) \overset{\star}{S}{}_1^2 - (3\lambda+2)\sigma \overset{\star}{S}_1 - 6\sigma^2}{\lambda(\lambda+1)} \\ \left[(5\lambda+2) \overset{\star}{S}_1 + \lambda^2 + \sigma\lambda + 4\sigma\right]\sigma \overset{\star}{S}{}_1^2 = 0 \end{cases}. \qquad (5.80)$$

Observe that $\overset{\star}{S}_1 = 0$ is incompatible with the second system in (5.80) as it implies $\sigma = 0$ which is a contradiction. So, for this system, $\overset{\star}{S}_1 \neq 0$ and we get

$$(5\lambda+2) \overset{\star}{S}_1 + \lambda^2 + \sigma\lambda + 4\sigma = 0.$$

But $5\lambda + 2 \not\equiv 0$, otherwise $2\sigma = \xi\cdot\lambda = 0$ and $\sigma = 0$, a contradiction. Therefore,

$$\overset{\star}{S}_1 = -\frac{\lambda^2 + \sigma\lambda + 4\sigma}{5\lambda+2}, \qquad (5.81)$$

from which we get

$$\xi\cdot \overset{\star}{S}_1 = \frac{-2\sigma}{(5\lambda+2)^2}\left[5\lambda^2 + 4\lambda - 18\sigma\right]. \qquad (5.82)$$

Replacing (5.81) in the first equation of the system in (5.80) yields

$$\xi\cdot \overset{\star}{S}_1 = 2\left(\frac{\lambda^2+\sigma\lambda+4\sigma}{5\lambda+2}\right)^2 + \frac{3\lambda+2}{\lambda(\lambda+1)}\frac{\lambda^2+\sigma\lambda+4\sigma}{5\lambda+2}\sigma - \frac{6\sigma^2}{\lambda(\lambda+1)}. \qquad (5.83)$$

From (5.82) and (5.83) we see that $\lambda = \lambda(\sigma) = \text{const}$ i.e $0 = \xi \cdot \lambda = 2\sigma$ i.e $\sigma = 0$ and this is a contradiction. So the second expression of $\xi \cdot \overset{\star}{S}_1$ is not admissible and we conclude that $\xi \cdot \overset{\star}{S}_1 = 0$ is the only one solution, and this implies $\overset{\star}{S}_1 = 0$ and the proof is complete. □

5.3. **Proof of Theorem 1.3.** From (3.54) with $n = 1$ and $r = 1$,

$$\widetilde{L}_1^2\psi = 2(\lambda+1)\overset{\star}{S}_2 P\widetilde{\nabla}\overset{\star}{S}_2 - 4(\lambda+1)(\overset{\star}{T}_1 \circ \overset{\star}{A})\widetilde{\nabla}\overset{\star}{S}_2 - 4\sigma P\overset{\star}{T}_1 \widetilde{\nabla}\overset{\star}{S}_2 + \overset{\star\xi}{\Lambda}_1 \xi + \Lambda_1^\zeta \zeta + \Lambda_1^\psi \psi, \quad (5.84)$$

with

$$\overset{\star\xi}{\Lambda}_1 = 2(\lambda+1)\widetilde{L}_1\overset{\star}{S}_2 + 2\lambda\Big(\frac{1}{2}(\lambda+1)\overset{\star}{S}_1 - 4\sigma\Big)\overset{\star}{S}{}_2^2 + 3\lambda(\lambda+1)\overset{\star}{S}_2\overset{\star}{S}_3 - 4\sigma\overset{\star}{S}_1 (\xi \cdot \overset{\star}{S}_2)$$

$$\Lambda_1^\zeta = 2\Big[-\widetilde{L}_1\overset{\star}{S}_2 + 2\sigma\overset{\star}{S}_2\Big], \quad \text{and } \Lambda_1^\psi = -4c\Big[\lambda\overset{\star}{S}{}_2^2 + \overset{\star}{S}_1 (\xi \cdot \overset{\star}{S}_2)\Big].$$

But for the null surface M^2, we have $\overset{\star}{S}_2 = \overset{\star}{k}_0\overset{\star}{k}_1 = 0$. So, $\widetilde{L}_1^2\psi = 0$ and M^2 is $\widetilde{L}_1-$biharmonic and item (1) is proved.

Let $n = 2$ and $r = 1$ in (3.54). We treat separately the cases $\sigma = 0$ and $\sigma \neq 0$.

- For $\sigma = 0$ we see that $c = 0$ and $\lambda = \text{cste}$. So,

$$\widetilde{L}_1^2\psi = 2(\lambda+1)\overset{\star}{S}_2 P\widetilde{\nabla}\overset{\star}{S}_2 - 4(\lambda+1)(-\overset{\star}{S}_1\overset{\star}{A} + \overset{\star}{A}{}^2)P\widetilde{\nabla}\overset{\star}{S}_2 + \overset{\star\xi}{\Lambda}_1 \xi + \Lambda_1^\zeta\zeta + \Lambda_1^\psi\psi, \quad (5.85)$$

with

$$\overset{\star\xi}{\Lambda}_1 = \lambda(\lambda+1)\overset{\star}{S}_1\overset{\star}{S}_2, \quad \Lambda_1^\zeta = \widetilde{L}_1\overset{\star}{S}_2 \quad \text{and} \quad \Lambda_1^\psi = 0, \quad (5.86)$$

where we used $\overset{\star}{S}_3 = 0$. From the $\widetilde{L}_1-$biharmonicity condition, the first equality in (5.86) yields $\overset{\star}{S}_1\overset{\star}{S}_2 = 0$ which implies $\overset{\star}{S}_2 = 0$. Indeed, if $\overset{\star}{S}_1 \neq 0$ then $\overset{\star}{S}_2 = 0$. Now, by Theorem 3.1 $\overset{\star}{S}_1 = 0$ implies $\overset{\star}{S}_2 = 0$.

- For $\sigma \neq 0$,

$$\begin{aligned}\widetilde{L}_1^2\psi &= \Big[2(\lambda+1)\overset{\star}{S}_2 + 4\sigma\overset{\star}{S}_1 + \sigma\Big]P\widetilde{\nabla}\overset{\star}{S}_2 + 4(\lambda+1)\left[\left(\overset{\star}{S}_1 - \frac{\sigma}{\lambda+1}\right)\overset{\star}{A} - \overset{\star}{A}{}^2\right]P\widetilde{\nabla}\overset{\star}{S}_2 \\ &\quad - 2c\overset{\star}{S}_1 P\widetilde{\nabla}\overset{\star}{S}_1 + 2\sigma\left[\left(\overset{\star}{S}_1 + \frac{c}{\sigma}\right)\overset{\star}{A} - \overset{\star}{A}{}^2\right]P\widetilde{\nabla}\overset{\star}{S}_1 + \overset{\star\xi}{\Lambda}_1 \xi + \Lambda_1^\zeta\zeta + \Lambda_1^\psi\psi.\end{aligned}$$

Assume $\overset{\star}{S}_1$ is $\mathscr{F}-$leafwise constant. Set

$$\overset{\star}{D} = \left(\overset{\star}{S}_1 - \frac{\sigma}{\lambda+1}\right)\overset{\star}{A} - \overset{\star}{A}{}^2.$$

The $\widetilde{L}_1-$bihamonicity condition implies

$$\overset{\star}{D}\ P\widetilde{\nabla}\overset{\star}{S}_2 = -\left[\frac{1}{2}\ \overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}(4\ \overset{\star}{S}_1 + 1)\right] P\widetilde{\nabla}\overset{\star}{S}_2\ . \tag{5.87}$$

Observe that ξ is also an eigenvector field of $\overset{\star}{D}$ associated to the eigenvalue $\overset{\star}{\lambda}_0 = 0$. Also, $\overset{\star}{D}$ is diagonalizable and

$$\mathrm{trace}(\overset{\star}{D}) = 2\ \overset{\star}{S}_2 - \frac{\sigma}{\lambda+1}\ \overset{\star}{S}_1\ .$$

Assume $P\widetilde{\nabla}\overset{\star}{S}_2 \neq 0$. It follows from (5.87) that

$$\overset{\star}{\lambda}_1 = -\left[\frac{1}{2}\ \overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}(4\ \overset{\star}{S}_1 + 1)\right]$$

is an eigenfunction for $\overset{\star}{D}$. Observe that $\overset{\star}{\lambda}_1 \neq 0$. Otherwise, $\overset{\star}{S}_2 = \dfrac{-\sigma}{2(\lambda+1)}(4\ \overset{\star}{S}_1 + 1)$ which implies $P\widetilde{\nabla}\ \overset{\star}{S}_2 = 0$ and this is a contradiction. We find that the third eigenfunction of $\overset{\star}{D}$ is

$$\overset{\star}{\lambda}_2 = \mathrm{trace}(\overset{\star}{D}) - \overset{\star}{\lambda}_1 = \frac{5}{2}\ \overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}.$$

Without losing generality we can choose a local $\widetilde{g}-$orthonormal frame field consisting of eigenvector fields of $\overset{\star}{D}$ such that

$$X_0 = \xi, \quad X_1 = \frac{P\widetilde{\nabla}\overset{\star}{S}_2}{\left\|P\widetilde{\nabla}\overset{\star}{S}_2\right\|} \in \Gamma(\mathcal{S}) \quad \text{and} \quad X_2 \in \Gamma(\mathcal{S}).$$

In this local frame, $\overset{\star}{D}$ takes the form

$$\overset{\star}{D} = \begin{bmatrix} \overset{\star}{\lambda}_0 & 0 & 0 \\ 0 & \overset{\star}{\lambda}_1 & 0 \\ 0 & 0 & \overset{\star}{\lambda}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\left[\frac{1}{2}\ \overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)}(4\ \overset{\star}{S}_1 + 1)\right] & 0 \\ 0 & 0 & \frac{5}{2}\ \overset{\star}{S}_2 + \frac{\sigma}{4(\lambda+1)} \end{bmatrix}$$

Taking into account the ξ, ζ and ψ components we also derive the following equations :

$$\begin{aligned} \overset{\star}{\Lambda}{}_1^{\xi} &= 2(\lambda+1)\widetilde{L}_1\ \overset{\star}{S}_2 + \sigma\widetilde{L}_1\ \overset{\star}{S}_1 + 2\lambda\left[\frac{1}{2}(\lambda+1)\ \overset{\star}{S}_1 - 4\sigma\right]\overset{\star}{S}_2 + c\sigma\ \overset{\star}{S}{}_1^2 \\ &\quad + \left[\frac{1}{2}\lambda\sigma\ \overset{\star}{S}_1 + 2c(c\lambda - 2\sigma^2)\right]\overset{\star}{S}_1\overset{\star}{S}_2 - 4\sigma\ \overset{\star}{S}_1\ (\xi\cdot\ \overset{\star}{S}_2) = 0; \end{aligned} \tag{5.88}$$

$$2\Lambda_1^\zeta = -\widetilde{L}_1 \overset{\star}{S}_2 + 2\sigma \overset{\star}{S}{}_2^2 - c \overset{\star}{S}_1 \overset{\star}{S}_2 = 0; \tag{5.89}$$

$$\Lambda_1^\psi = c\left[\widetilde{L}_1 \overset{\star}{S}_1 + c \overset{\star}{S}_1 + c \overset{\star}{S}_1 - 4\lambda \overset{\star}{S}{}_2^2 - 4\sigma \overset{\star}{S}_1 \overset{\star}{S}_2 - 4 \overset{\star}{S}_1 (\xi\cdot \overset{\star}{S}_2)\right] = 0. \tag{5.90}$$

Observe that since $\overset{\star}{S}_3 = 0$, we have $\xi\cdot \overset{\star}{S}_2 = \overset{\star}{S}_1 \overset{\star}{S}_2$. Let us compute $\widetilde{L}_1 \overset{\star}{S}_1$.

$$\widetilde{L}_1 \overset{\star}{S}_1 = \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_\xi \widetilde{\nabla} \overset{\star}{S}_1, \xi\right) + \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X_1} \widetilde{\nabla} \overset{\star}{S}_1, X_1\right) + \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X2} \widetilde{\nabla} \overset{\star}{S}_1, X_2\right)$$

where $\widetilde{\nabla} \overset{\star}{S}_1 = (\xi\cdot \overset{\star}{S}_1)\xi = (\overset{\star}{S}{}_1^2 - 2 \overset{\star}{S}_2)\xi$. Computing each term leads to

$$\widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_\xi \widetilde{\nabla} \overset{\star}{S}_1, \xi\right) = -2 \overset{\star}{S}{}_1^4 + 6 \overset{\star}{S}{}_1^2 \overset{\star}{S}_2;$$

$$\widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X_1} \widetilde{\nabla} \overset{\star}{S}_1, X_1\right) = \widetilde{g}\left(\overset{\star}{T}_1 \circ \widetilde{\nabla}_{X2} \widetilde{\nabla} \overset{\star}{S}_1, X_2\right) = \overset{\star}{S}_2 (\overset{\star}{S}{}_1^2 - 2 \overset{\star}{S}_2).$$

So,

$$\widetilde{L}_1 \overset{\star}{S}_1 = 8 \overset{\star}{S}{}_1^2 \overset{\star}{S}_2 - 2 \overset{\star}{S}{}_1^4 - 4 \overset{\star}{S}{}_2^2. \tag{5.91}$$

Assume $c \neq 0$. From (5.90) and (5.91),

$$-4(\lambda+1) \overset{\star}{S}{}_2^2 + [4 \overset{\star}{S}{}_1^2 - 4\sigma \overset{\star}{S}_1] \overset{\star}{S}_2 - 2 \overset{\star}{S}{}_1^4 + c \overset{\star}{S}_1 = 0.$$

Hence, since $\lambda + 1 \neq 0$ we see that $\overset{\star}{S}_2 = \overset{\star}{S}_2 (\overset{\star}{S}_1, \lambda, \sigma)$ and this implies $P\widetilde{\nabla} \overset{\star}{S}_2 = 0$ which is a contradiction.

Assume $c = 0$. We get from (5.89), $\widetilde{L}_1 \overset{\star}{S}_2 = 2\sigma \overset{\star}{S}{}_2^2$ with σ constant on M. Using (5.88), we derive

$$4\lambda \overset{\star}{S}{}_2^2 + \left[(4 + \frac{1}{2}\lambda)\sigma \overset{\star}{S}{}_1^2 + (\lambda^2 + \lambda - 4\sigma^2) \overset{\star}{S}_1 - 4\sigma\lambda\right] \overset{\star}{S}_2 - 2\sigma \overset{\star}{S}{}_1^4 = 0.$$

But $\lambda < 0$ since ζ is timelike. So, $\overset{\star}{S}_2 = \overset{\star}{S}_2 (\overset{\star}{S}_1, \lambda, \sigma)$ and this implies $P\widetilde{\nabla} \overset{\star}{S}_2 = 0$ which is again a contradiction.

Finally, we conclude that $P\widetilde{\nabla} \overset{\star}{S}_2 = 0$ i.e $\overset{\star}{S}_2$ is leafwise constant in the screen foliation $\mathscr{F}$.

Assume that $\overset{\star}{S}_2$ and hence $\overset{\star}{H}_2$ is constant on the whole null hypersurface M^3. Then $0 = \xi\cdot \overset{\star}{S}_2 = \overset{\star}{S}_1 \overset{\star}{S}_2$ and this implies again $\overset{\star}{S}_2 = 0$ as shown in previous argument above. □

Discussion. Consider the case where the rigging is a unit timelike vector field, i.e $\lambda = \langle\zeta, \zeta\rangle = -1$. Due to $\widetilde{\nabla}\lambda = 2\sigma\xi$ and $\xi \cdot \sigma = -c$, we get $\sigma = 0$ on the null hypersurface M and $c = 0$. Hence, when the rigging ζ is a timelike unit closed and conformal vector field, the target space of immersion is necessarily Minkowskian, and ζ is a Killing vector field in a neighbourhood of the null hypersurface. Moreover,

$$\widetilde{L}_r^2\psi = \left[(-1)^r(r+1)\widetilde{L}_r \overset{\star}{S}_{r+1}\right]\zeta.$$

Consequently, the null hypersurface connected isometric immersion $\psi : M^{n+1} \longrightarrow \mathbb{R}_1^{n+2}$ furnished with a timelike unit closed and conformal vecor field (a Killing rigging) ζ is $r-$biharmonic if and only if $\widetilde{L}_r\overset{\star}{S}_{r+1}= 0$.

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References

[1] Akutagawa, K., & Maeta, S. (2013). Biharmonic properly immersed submanifolds in Euclidean spaces. *Geometriae Dedicata*, 164, 351–355.

[2] Aminian, M. (2020). L_k-biharmonic hypersurfaces in space forms with three distinct principal curvatures. *Communications of the Korean Mathematical Society*, 35(4), 1221–1244.

[3] Aminian, M., & Namjoo, M. (2021). Proper L_k-biharmonic hypersurfaces in the Euclidean sphere with two principal curvatures. *Journal of Mahani Mathematical Research Center*, 10(1), 69–78.

[4] Aminian, M., & Kashani, S. M. B. (2014). L_k-biharmonic hypersurfaces in the Euclidean space. *Taiwanese Journal of Mathematics*. https://doi.org/10.11650/tjm.18.2014.4830

[5] Aminian, M., & Kashani, S. M. B. (2015). L_k-biharmonic hypersurfaces in the Euclidean space. *Taiwanese Journal of Mathematics*, 19, 861–874.

[6] Aminian, M., & Kashani, S. M. B. (2017). L_k-biharmonic hypersurfaces in space forms. *Acta Mathematica Vietnamica*, 42, 471–490.

[7] Atindogbe, C., Gutiérrez, M., & Hounnonkpe, R. (2018). New properties on normalized null hypersurfaces. *Mediterranean Journal of Mathematics*, 15, 166.

[8] Atindogbe, C., Gutiérrez, M., & Hounnonkpe, R. (2021). Compact null hypersurfaces in Lorentzian manifolds. *Advances in Geometry*, 21(2), 251–263.

[9] Atindogbe, C., & Fosting, H. T. (2015). Newton transformations on null hypersurfaces. *Communications in Mathematics*, 23, 57–83.

[10] Atindogbe, C., Ezin, J. P., & Tossa, T. (2003). Pseudo-inversion of degenerate metrics. *International Journal of Mathematics and Mathematical Sciences*, 55, 3479–3501.

[11] Atindogbe, C., & Olea, B. (2022). Conformal vector fields and null hypersurfaces. *Results in Mathematics*, 77, 129.

[12] Chen, B.-Y. (1991). Some open problems and conjectures on submanifolds of finite type. *Soochow Journal of Mathematics*, 17(2), 169–188.

[13] Chen, B.-Y., & Ishikawa, S. (1991). Biharmonic surfaces in pseudo-Euclidean spaces. *Memoirs of the Faculty of Science, Kyushu University, Series A*, 45(2), 323–347.

[14] Chen, B.-Y. (1996). A report on submanifolds of finite type. *Soochow Journal of Mathematics*, 22, 117–337.

[15] Chen, B.-Y., & Ishikawa, S. (1998). Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces. *Kyushu Journal of Mathematics*, 52, 167–185.

[16] Chen, B.-Y., & Munteanu, M. I. (2013). Biharmonic ideal hypersurfaces in Euclidean spaces. *Differential Geometry and Its Applications*, 31, 1–16.

[17] Chen, B.-Y. (2014). *Total mean curvature and submanifolds of finite type*. World Scientific.

[18] Defever, F. (1998). Hypersurfaces of $\mathbb{E}^4$ with harmonic mean curvature vector. *Mathematische Nachrichten*, 196, 61–69.

[19] Defever, F., Kaimakamis, G., & Papantoniou, V. (2006). Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space $\mathbb{E}^4_s$. *Journal of Mathematical Analysis and Applications*, 315(1), 276–286.

[20] Dimitrić, I. (1992). Submanifolds of $\mathbb{E}^m$ with harmonic mean curvature vector. *Bulletin of the Institute of Mathematics, Academia Sinica*, 20, 53–65.

[21] Duggal, K. L., & Bejancu, A. (1996). *Lightlike submanifolds of semi-Riemannian manifolds and applications*. Kluwer Academic Publishers.

[22] Hasanis, T., & Vlachos, T. (1995). Hypersurfaces in $\mathbb{E}^4$ with harmonic mean curvature vector field. *Mathematische Nachrichten*, 172, 145–169.

[23] Gutiérrez, M., & Olea, B. (2016). Induced Riemannian structures on null hypersurfaces. *Mathematische Nachrichten*, 289, 1219–1236.

[24] Gutiérrez, M., & Olea, B. (2015). Totally umbilic null hypersurfaces in generalized Robertson-Walker spaces. *Differential Geometry and Its Applications*, 42, 15–30.

[25] Fotsing Tetsing, H., Atindogbe, C., & Ngakeu, F. (2023). Normalized null hypersurfaces in the Lorentz-Minkowski space satisfying $L_r x = Ux + b$. *Tamkang Journal of Mathematics*, 54(4), 353–378. https://doi.org/10.5556/j.tkjm.54.2023.4851

[26] Jiancheng, L., & Li, D. (2015). Classification of proper biharmonic hypersurfaces in pseudo-Riemannian space forms. *Differential Geometry and Its Applications*, 41, 110–122.

[27] Pashaie, F., & Mohammadpouri, A. (2017). L_k-biharmonic spacelike hypersurfaces in Minkowski 4-space $\mathbb{E}^4_1$. *Sahand Communications in Mathematical Analysis*, 5(1), 21–30.

[28] Pashaie, F. (2022). An extension of biconservative timelike hypersurfaces in Einstein space. *Proyecciones Journal of Mathematics*, 41(1), 335–351.

[29] Sánchez, M. (1999). On the geometry of generalized Robertson–Walker spacetimes: Curvature and Killing fields. *Journal of Geometry and Physics*, 31, 1–15.

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STATISTICAL CONVERGENCE OF DOUBLE SEQUENCES OF BI-COMPLEX NUMBERS

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Abstract. This article presents the concepts of statistically bounded, statistically bounded convergence, statistically bounded null, statistically regular convergence, statistically regular null double sequences of bi-complex numbers, statistically convergent double sequences in Pringsheim's sense, and statistically null double sequences in Pringsheim's sense. We have established that these spaces are linear, and we have demonstrated their many algebraic, topological, and geometric properties using the Euclidean norm defined on bi-complex numbers. Suitable examples have been discussed.

Keywords: Double sequences, Statistical convergence, Bi-complex numbers, $\mathbb{BC}$-convex, $\mathbb{BC}$-strictly convex, $\mathbb{BC}$- uniformly convex.

2020 Mathematics Subject Classification: 40A05, 46A45, 46B45, 30L99.

1. Introduction

The notion of convergence double sequence was first proposed by Pringsheim [14]. Bromwich [2] has some of the earliest works on double sequence spaces. The concept of regular convergence of double sequence was later introduced by Hardy [5]. Additionally, double sequences of bi-complex numbers were introduced by Kumar and Tripathy in various directions [6], [7], and [8].

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Definition 1.1. [13] *Norm (Euclidean Norm) on* $\mathbb{C}_2$ *is defined by*

$$\begin{aligned}\|\gamma\|_{\mathbb{C}_2} &= \sqrt{w^2+x^2+y^2+z^2}\\ &= \sqrt{|u_1|^2+|u_2|^2}\\ &= \sqrt{\frac{|\mu_1|^2+|\mu_2|^2}{2}}.\end{aligned}$$

$\mathbb{C}_2$ *becomes a modified Banach algebra with respect to this norm in the sense that*

$$\|\gamma\cdot s\|_{\mathbb{C}_2} \leq \sqrt{2}\|\gamma\|_{\mathbb{C}_2}\cdot\|s\|_{\mathbb{C}_2}.$$

Definition 1.2. *Three types of conjugations are defined in the bi-complex numbers (Rochon, Shapiro* [16]*) as follows,*

(1) i_1- *conjugation of bi-complex number* γ *is* $\gamma^* = \overline{u_1} + i_2\overline{u_2}$, *for all* $u_1, u_2 \in \mathbb{C}_1$ *and* $\overline{u_1}, \overline{u_2}$ *are complex conjugates of* u_1, u_2 *respectively.*
(2) i_2- *conjugation of bi-complex number* γ *is* $\tilde{\gamma} = u_1 - i_2u_2$, *for all* $u_1, u_2 \in \mathbb{C}_1$.
(3) i_1i_2- *conjugation of bi-complex number* γ *is* $\gamma' = \overline{u_1} - i_2\overline{u_2}$, *for all* $u_1, u_2 \in \mathbb{C}_1$ *and* $\overline{u_1}, \overline{u_2}$ *are complex conjugates of* u_1, u_2 *respectively.*

The concept of statistical convergence was introduced Fast [3] and reintroduced by Schoenberg [18]. It was also discussed in the work of Zygmund [20]. Subsequently, several researchers including Fridy and Orhan [4], Maddox [11], Salat [17], Mursaleen and Edely [12], Rath and Tripathy [15], Tripathy [19] and others explored this notion in various contexts".
A subset E of $\mathbb{N}$ is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n\to\infty}\frac{1}{n}\sum_{l\leq n}\chi_E(l) \text{ exists },$$

where χ_E is the characteristic function of E.
A single sequence (γ_l) is said to be statistically convergent to L if for each $\varepsilon > 0, \delta(\{l \in \mathbb{N} : \|\gamma_l - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$ and write $\gamma_l \xrightarrow{\text{stat}} L$ or $stat - \lim \gamma_l = L$. A sequence that is statistically convergent to zero is called a statistically null sequence.
The density of a subset E of $\mathbb{N}\times\mathbb{N}$ is defined as

$$\delta_2(E) = \lim_{n,k\to\infty}\frac{1}{nk}\sum_{l\leq n}\sum_{m\leq k}\chi_E(l,m) \text{ exists.}$$

A double sequence (γ_{lm}) is said to be statistically convergent to L in Pringsheim's sense if for every $\varepsilon > 0, \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$, written as $st - \lim_{l,m\to\infty}\gamma_{lm} = L$.

A double sequence (γ_{lm}) is said to be statistically null if it is statistically convergent to 0 in Pringsheim's sense.

A double sequence (γ_{lm}) is said to be statistically regular convergent if it converges in Pringsheim's sense and the following statistical limits exist

$$stat - \lim_{l\to\infty} \gamma_{lm} = P_m, \text{ exists, for each } m \in \mathbb{N},$$

and

$$stat - \lim_{m\to\infty} \gamma_{lm} = Q_l, \text{ exists, for each } l \in \mathbb{N}.$$

For regularly null sequences we have $P_m = Q_l = L = 0$, for all $l, m \in \mathbb{N}$.

Definition 1.3. *Let (γ_{lm}) and (t_{lm}) be two double sequences, then we say that $\gamma_{lm} = t_{lm}$, for all most all l and m (in short a.a.l and m) if $\delta_2(\{(l,m) : \gamma_{lm} \neq t_{lm}\}) = 0$.*

Definition 1.4. *A double sequence (γ_{lm}) of bi-complex numbers is said to be statistically divergent to ∞ if for any given $G, \delta_2(\{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} > G\}) = 0$. Similarly, statistically divergent to $-\infty$ is defined.*

Definition 1.5. *A double sequence (γ_{lm}) is said to be statistically Cauchy if for every $\varepsilon > 0$, there exists $n = n(\varepsilon)$ and $k = k(\varepsilon)$ such that $\delta_2(\{(l,m) : \|\gamma_{lm} - \gamma_{nk}\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$.*

Definition 1.6. [10] *The set of bi-complex numbers is a commutative ring. Modules over rings are defined in the same way as vector spaces are over fields. A module defined over the bi-complex number ring $\mathbb{BC}$ is known as a $\mathbb{BC}-$ module or simply module.*

Definition 1.7. [6] *A double sequence (γ_{lm}) of bi-complex numbers is called bounded, if there exists a real number $M > 0$ such that*

$$\|\gamma_{lm}\|_{\mathbb{C}_2} \leq M, \text{ for all } l, m \in \mathbb{N}.$$

and the set of all bounded double sequences of bi-complex numbers, defined by;

$${}_2\ell_\infty(\mathbb{C}_2) := \left\{ \gamma = (\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : \sup_{l,m\in\mathbb{N}} \|\gamma_{lm}\|_{\mathbb{C}_2} < \infty \right\}.$$

The sequence space ${}_2\ell_\infty(\mathbb{C}_2)$ is a normed linear space with respect to

$$\|A\| = \sup_{l,m} \|\gamma_{lm}\|_{\mathbb{C}_2}.$$

2. Definitions and Preliminaries

In this paper, the notations ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}(\mathbb{C}_2), {}_2\bar{c}_0(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2),\ {}_2\bar{c}_0^B(\mathbb{C}_2)$ are used to denote the spaces of bi-complex double sequences that are statistically bounded, statistically convergence in Pringsheim's sense, statistically null in Pringsheim's sense, statistically regularly convergent, statistically regularly null, statistically bounded convergent in Pringsheim's sense, statistically bounded null in Pringsheim's sense, respectively.

$$
\begin{aligned}
&{}_2\bar{\ell_\infty}(\mathbb{C}_2) := \{(\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : \exists\, 0 < M \in \mathbb{C}_0 : \delta_2(\{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} \geq M\}) = 0\};\\
&{}_2\bar{c}(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) :\ \text{there exists } L \in \mathbb{C}_2 \text{ such that } st - \lim_{\substack{l\to\infty\\ m\to\infty}} \gamma_{lm} = L \right\};\\
&{}_2\bar{c}_0(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\omega(\mathbb{C}_2) : st - \lim_{\substack{l\to\infty\\ m\to\infty}} \gamma_{lm} = 0 \right\};\\
&{}_2\bar{c}^R(\mathbb{C}_2) := \Bigg\{ (\gamma_{lm}) \in {}_2\bar{c}(\mathbb{C}_2) : st - \lim_{l\to\infty} \gamma_{lm} = \gamma_m,\ \text{exists for each}\\
&\qquad\qquad m \in \mathbb{N} \text{ and } st - \lim_{m\to\infty} \gamma_{lm} = \gamma_l,\ \text{exists for each } l \in \mathbb{N} \Bigg\};\\
&{}_2\bar{c}_0^R(\mathbb{C}_2) := \left\{ (\gamma_{lm}) \in {}_2\bar{c}^R(\mathbb{C}_2) : \gamma_m = \gamma_l = L = 0,\ \text{for all } l, m \in \mathbb{N} \right\};\\
&{}_2\bar{c}^B(\mathbb{C}_2) := {}_2\bar{c}(\mathbb{C}_2) \cap {}_2\ell_\infty(\mathbb{C}_2) \text{ and } {}_2\bar{c}_0^B(\mathbb{C}_2) = {}_2\bar{c}_0(\mathbb{C}_2) \cap {}_2\ell_\infty(\mathbb{C}_2).
\end{aligned}
$$

Definition 2.1. [7] *Let E be a subset of a linear space X. Then E is said to be convex (or $\mathbb{BC}-$ convex) if $(1-\lambda)(\gamma_{lm}) + \lambda(t_{lm}) \in E$ for all $(\gamma_{lm}), (t_{lm}) \in E$ and scalar $\lambda \in [0,1]$.*

Definition 2.2. [8] *A Banach space X is said to be strictly convex (or $\mathbb{BC}-$ strictly convex) if $(\gamma_{lm}), (t_{lm}) \in S_X$ with $(\gamma_{lm}) \neq (t_{lm})$ implies that $\|\lambda(\gamma_{lm}) + (1-\lambda)(t_{lm})\|_X < 1$, for all $\lambda \in (0,1)$, where S_X is unit sphere.*

Definition 2.3. [9] *A Banach space X is considered uniformly convex (or $\mathbb{BC}-$ uniformly convex) if, for any ε with $0 < \varepsilon \leq 2$, the following inequalities hold true: $\|\gamma_{lm}\|_X \leq 1, \|t_{lm}\|_X \leq 1$ and $\|(\gamma_{lm}) - (t_{lm})\|_X \geq \varepsilon$ imply that there is a $\delta = \delta(\varepsilon) > 0$ such that*

$$\left\| \frac{(\gamma_{lm}) + (t_{lm})}{2} \right\|_X \leq 1 - \delta.$$

3. Main Result

In this section, the following results are established.

Theorem 3.1. *If a double sequence* (γ_{lm}) *of bi-complex numbers* $\gamma_{lm} = u_{1lm} + i_2 u_{2lm}$ *for all* $l, m \in \mathbb{N}$ *is a statistically bounded double sequence of bi-complex numbers, then the double sequences* (u_{1lm}) *and* (u_{2lm}) *of bi-complex numbers are also statistically bounded double sequences of bi-complex numbers.*

Proof. Let (γ_{lm}) be a statistically bounded double sequence of bi-complex numbers. There exists a positive real number M, such that $\delta_2(\{(l,m) : \|\gamma_{lm}\|_{\mathbb{C}_2} \geq M\}) = 0$, which implies $\delta_2(\{(l,m) : \|u_{1lm} + i_2 u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0$ and $\delta_2(\{(l,m) : \|u_{plm}\|_{\mathbb{C}_2} \geq M\}) \leq \delta_2(\{(l,m) : \|u_{1lm} + i_2 u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0$ for $p = 1, 2$. Hence, (u_{1lm}) and (u_{2lm}) are statistically bounded double sequences of bi-complex numbers.

Conversely, let (u_{1lm}) and (u_{2lm}) are statistically bounded double sequences of bi-complex numbers. Then, without loss of generality, we can find $M > 0$, such that

$$\delta_2(\{(l,m) : \|u_{1lm}\|_{\mathbb{C}_2} \geq M\}) = 0,$$

and

$$\delta_2(\{(l,m) : \|u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0.$$

Consequently, the following inequality yields the result;

$$\begin{aligned}
&\delta_2(\{(l,m) : \|u_{1lm} + i_2 u_{2lm}\|_{\mathbb{C}_2} \geq M\}) \\
&\leq \delta_2(\{(l,m) : \|u_{1lm}\|_{\mathbb{C}_2} \geq M\}) + \delta_2(\{(l,m) : \|u_{2lm}\|_{\mathbb{C}_2} \geq M\}) = 0.
\end{aligned}$$

(By sub-additive property)

Hence, (γ_{lm}) is statistically bounded. □

We formulate the following corollaries based on the previous theorem:

Corollary 3.1. *If a double sequence* (γ_{lm}) *of bi-complex numbers, where* $\gamma_{lm} = x_{1lm} + i_1 x_{2lm} + i_2 x_{3lm} + i_1 i_2 x_{4lm}$*, is statistically bounded double sequence of bi-complex numbers, then the double sequences* (x_{plm})*,* $p = 1, 2, 3, 4.$ *of bi-complex numbers are also statistically bounded double sequences.*

Corollary 3.2. *If a double sequence* (γ_{lm}) *of bi-complex numbers, where* $\gamma_{lm} = \mu_{1lm} e_1 + \mu_{2lm} e_2$*, is statistically bounded double sequence of bi-complex numbers, then the double sequences* (μ_{1lm}) *and* (μ_{2lm}) *are also statistically bounded double sequences.*

Theorem 3.2. *If a double sequence* (γ_{lm}) *of bi-complex numbers, where* $\gamma_{lm} = u_{1lm} + i_2 u_{2lm}$ *for all* $l, m \in \mathbb{N}$ *is statistically convergent to* $\gamma = u_1 + i_2 u_2$ *with respect to the Euclidean norm on* $\mathbb{C}_2$ *if and only if* (u_{1lm}) *and* (u_{2lm}) *are statistically convergent to* u_1 *and* u_2 *respectively.*

Proof. Consider (γ_{lm}) be statistically convergent to γ. Then, by definition, for every $\varepsilon > 0$

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - \gamma\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

$$\implies \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(\gamma_{1lm} - \gamma_1) + i_2(\gamma_{2lm} - \gamma_2)\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

$$\implies \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Now, consider the set

$$A_\varepsilon = \{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}.$$

Since, $\|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon$ implies $\|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon$,
we have,

$$A_\varepsilon \subseteq \{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}$$

$$\implies \delta_2(A_\varepsilon) \leq \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Similarly, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, the double sequences (γ_{1lm}) and (γ_{2lm}) of bi-complex numbers are statistically convergent to γ_1 and γ_2 respectively. □

Theorem 3.3. *If a bounded double sequence* (γ_{lm}), *where* $\gamma_{lm} = e_1\mu_{1lm} + e_2\mu_{2lm}$ *is statistically Cauchy, then* (γ_{lm}) *is a Cauchy double sequence in* $\|\cdot\|_{\mathbb{C}_2}$.

Proof. Let (γ_{lm}) be statistically Cauchy double sequence of bi-complex numbers; then, for each $\varepsilon > 0$, there exists $n_0, k_0 \in \mathbb{N}$, such that

$$\delta_2(\{(l,m) : \|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Substituting $\gamma_{lm} = e_1\mu_{1lm} + e_2\mu_{2lm}$, we have

$$\|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} = \|e_1(\mu_{1lm} - \mu_{1n_0k_0}) + e_2(\mu_{2lm} - \mu_{2n_0k_0})\|_{\mathbb{C}_2}.$$

Using the properties of the Euclidean norm on $\mathbb{C}_2$, then

$$\|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} = \sqrt{\|\mu_{1lm} - \mu_{1n_0k_0}\|^1_{\mathbb{C}_2} + \|\mu_{2lm} - \mu_{2n_0k_0}\|^2_{\mathbb{C}_2}}.$$

Since (γ_{lm}) is a statistically Cauchy double sequence of bi-complex numbers, we have;

$$\delta_2(\{(l,m) : \|\mu_{1lm} - \mu_{1n_0k_0}\|^1_{\mathbb{C}_2} \geq \varepsilon^1\}) = 0,$$

and

$$\delta_2(\{(l,m) : \|\mu_{2lm} - \mu_{2n_0k_0}\|^2_{\mathbb{C}_2} \geq \varepsilon^2\}) = 0,$$

for some $\varepsilon_1, \varepsilon_2 > 0$, such that $\varepsilon^2 = \varepsilon_1^2 + \varepsilon_2^2$.

This implies that the statistical bounds of $(\|\mu_{1lm} - \mu_{1n_0k_0}\|_{\mathbb{C}_2})$ and $(\|\mu_{2lm} - \mu_{2n_0k_0}\|_{\mathbb{C}_2})$ are zero as $\varepsilon_1, \varepsilon_2 \to 0$.

Hence, for any $\varepsilon > 0$, we have

$$\|\gamma_{lm} - \gamma_{n_0k_0}\|_{\mathbb{C}_2} = e_1(\|\mu_{1lm} - \mu_{1n_0k_0}\|^1_{\mathbb{C}_2}) + e_2(\|\mu_{2lm} - \mu_{2n_0k_0}\|^2_{\mathbb{C}_2}) \to 0.$$

Thus, (γ_{lm}) is a Cauchy double sequence of bi-complex numbers in $\|\cdot\|_{\mathbb{C}_2}$. □

Corollary 3.3. *If a double sequence (γ_{lm}) of bi-complex numbers, where $\gamma_{lm} = e_1\mu_{1lm} + e_2\mu_{2lm}$ is statistically convergent, then (γ_{lm}) is a Cauchy sequence in $\|\cdot\|_{\mathbb{C}_2}$.*

Theorem 3.4. *Let (γ_{lm}) be a statistically convergent double sequence of bi-complex numbers to L. If $(t_{lm}) \in [(\gamma_{lm})]$, then (t_{lm}) is also statistically convergent to L in $\|\cdot\|_{\mathbb{C}_2}$*

Proof. Since (γ_{lm}) is statistically convergent double sequence of bi-complex numbers to L, by definition, for every $\varepsilon > 0$;

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Given that $(t_{lm}) \in [(\gamma_{lm})]$, we have:

$$\|\gamma_{lm} - t_{lm}\|_{\mathbb{C}_2} = 0, \text{ for all } l, m \in \mathbb{N}.$$

Now,

$$\|t_{lm} - L\|_{\mathbb{C}_2} \leq \|\gamma_{lm} - L\|_{\mathbb{C}_2} + \|t_{lm} - \gamma_{lm}\|_{\mathbb{C}_2}.$$

Substituting $\|\gamma_{lm} - t_{lm}\|_{\mathbb{C}_2} = 0$, we get

$$\|t_{lm} - L\|_{\mathbb{C}_2} \leq \|\gamma_{lm} - L\|_{\mathbb{C}_2}.$$

Since (γ_{lm}) is statistically convergent to L, for every $\varepsilon > 0$:

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2}\}) = 0.$$

It follows that:

$$\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|t_{lm}-L\|_{\mathbb{C}_2}\geq\varepsilon\})\leq\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{lm}-L\|_{\mathbb{C}_2}\geq\varepsilon\})=0.$$

Hence, the double sequence of bi-complex numbers (t_{lm}) is statistically convergent to L in $\|\cdot\|_{\mathbb{C}_2}$. □

The decomposition theorem for statistically bounded sequences of bi-complex numbers for single sequences was demonstrated by Bera and Tripathy [1].
The decomposition theorem for double sequences of bi-complex numbers is as follows.

Theorem 3.5. *A bounded double sequence (s_{lm}) of bi-complex numbers and a statistically null double sequence (t_{lm}) of bi-complex numbers exist if a double sequence (γ_{lm}) of bi-complex numbers is statistically bounded. This means that $(\gamma_{lm})=(s_{lm})+(t_{lm})$.*

Proof. Let (γ_{lm}), where $\gamma_{lm}=\mu_{1lm}e_1+\mu_{2lm}e_2$, be a statistically bounded double sequence. Then $\delta_2(B)=0$, where $B=\{(l,m):\|\gamma_{lm}\|_{\mathbb{C}_2}\geq M\}$.
Define the double sequences (s_{lm}) and (t_{lm}) as follows:

$$s_{lm}=\begin{cases}\gamma_{lm}, & \text{if } k\in B^c;\\ \theta, & \text{otherwise .}\end{cases}$$

$$t_{lm}=\begin{cases}\theta, & \text{if } k\in B^c;\\ \gamma_{lm}, & \text{otherwise .}\end{cases}$$

From the above construction of (s_{lm}) and (t_{lm}), we have

$$(\gamma_{lm})=(s_{lm})+(t_{lm}),$$

where $(s_{lm})\in {}_2\ell_\infty(\mathbb{C}_2)$ and $(t_{lm})\in {}_2\bar{c}_0(\mathbb{C}_2)$. □

We state the following theorem without a proof that can be established by standard techniques.

Theorem 3.6. *Let (γ_{lm}) be a double sequence of bi-complex numbers and $L,L'\in\mathbb{C}_2$. If $st_2-lim\|\gamma_{lm}\|_{\mathbb{C}_2}=L$. and $st_2-lim\|\gamma_{lm}\|_{\mathbb{C}_2}=L'$, then $L=L'$.*

Theorem 3.7. *A double sequence (γ_{lm}) of bi-complex numbers is statistically convergent if and only if (γ_{lm}) is statistically Cauchy.*

Proof. Let (γ_{lm}) be statistically convergent to a number $L \in \mathbb{C}_2$. Then for every $\varepsilon > 0$, the set

$$\{(l,m), l \leq n, m \leq k : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}$$

has double natural density zero. Choose two numbers p and q such that $\|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon$. Now let

$$A = \{(l,m), l \leq n, m \leq k : \|\gamma_{lm} - \gamma_{pq}\|_{\mathbb{C}_2} \geq \varepsilon\},$$

$$B = \{(l,m), l \leq n, m \leq k : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\},$$

$$C = \{(l,m), l = p \leq n, m = q \leq k : \|\gamma_{pq} - L\|_{\mathbb{C}_2} \geq \varepsilon\}.$$

Then $A \subseteq B \cup C$, and therefore $\delta_2(A) \leq \delta_2(B) + \delta_2(C) = 0$. Hence, (γ_{lm}) is statistically Cauchy.

Conversely, assume that (γ_{lm}) is statistically Cauchy but not statistically convergent. This implies that there does not exist a unique $L \in \mathbb{C}_2$ such that $\|\gamma_{lm} - L\|_{\mathbb{C}_2} \to 0$, in the sense of statistical convergence. Instead, there must exist two distinct points $L_1, L_2 \in \mathbb{C}_2$ and some $\varepsilon > 0$, such that the sets

$$B_1 = \{(l,m) : \|\gamma_{lm} - L_1\|_{\mathbb{C}_2} < \varepsilon\} \text{ and } B_2 = \{(l,m) : \|\gamma_{lm} - L_2\|_{\mathbb{C}_2} < \varepsilon\}$$

both have double natural density greater than zero: $\delta_2(B_1) > 0$ and $\delta_2(B_2) > 0$.

Since $L_1 \neq L_2$, the distance between these two points is positive:

$$\|L_1 - L_2\|_{\mathbb{C}_2} = \delta > 0.$$

For $(l,m) \in B_1 \cap B_2$, we have

$$\|\gamma_{lm} - L_1\|_{\mathbb{C}_2} < \varepsilon, \ \|\gamma_{lm} - L_2\|_{\mathbb{C}_2} < \varepsilon.$$

By the triangle inequality

$$\|L_1 - L_2\|_{\mathbb{C}_2} \leq \|\gamma_{lm} - L_1\|_{\mathbb{C}_2} + \|\gamma_{lm} - L_2\|_{\mathbb{C}_2}.$$

Substituting the bounds for $\|\gamma_{lm} - L_1\|_{\mathbb{C}_2}$ and $\|\gamma_{lm} - L_2\|_{\mathbb{C}_2}$, we get

$$\|L_1 - L_2\|_{\mathbb{C}_2} < \varepsilon + \varepsilon = 2\varepsilon.$$

Since $L_1 \neq L_2$, their distance $\|L_1 - L_2\|_{\mathbb{C}_2} = \delta > 0$.

Choose $\varepsilon > 0$, such that $2\varepsilon < \delta$.

This creates a contradiction because the inequality $\|L_1 - L_2\|_{\mathbb{C}_2} \leq 2\varepsilon$ can not hold when $2\varepsilon < \delta$.

The assumption that (γ_{lm}) is statistically Cauchy but not statistically convergent leads to a contradiction.

Therefore, if (γ_{lm}) is statistically Cauchy, it must also be statistically convergent to a unique limit $L \in \mathbb{C}_2$. □

Theorem 3.8. *Let (γ_{lm}) and (t_{lm}) be double sequences of bi-complex numbers. If (t_{lm}) is a convergent double sequence such that $\gamma_{lm} \neq t_{lm}$ for all l and m, then (γ_{lm}) is statistically convergent.*

Proof. Suppose that $\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \gamma_{lm} \neq t_{lm}\}) = 0$ and $\lim_{l,m\to\infty} \|t_{lm}\|_{\mathbb{C}_2} = L$. Then for every $\varepsilon > 0$,

$$\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\} \subseteq \{(l,m) \in \mathbb{N} \times \mathbb{N} : \gamma_{lm} \neq t_{lm}\}.$$

Therefore,

$$\begin{aligned} &\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) \\ &\subseteq \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) + \delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \gamma_{lm} \neq t_{lm}\}). \end{aligned} \tag{3.1}$$

Since, $\lim_{l,m\to\infty} \|t_{lm}\|_{\mathbb{C}_2} = L$, the set $\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}$ contains finite number of integers. Hence,

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|t_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Using the inequality Eq. (3.1), we get

$$\delta_2(\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|\gamma_{lm} - L\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

for every $\varepsilon > 0$. Consequently, $st - \lim_{l,m\to\infty} \|(\gamma_{lm})\|_{\mathbb{C}_2} = L$. □

Corollary 3.4. *Let (γ_{lm}) be a statistically Cauchy sequence. Then there exists a convergent double sequence (t_{lm}) of bi-complex numbers such that $\gamma_{lm} = t_{lm}$, for almost all l and m.*

The following two theorems, Theorems 3.9 and 3.10, are stated without proof, as they can be established using standard techniques.

Theorem 3.9. *Let the double sequences (γ_{lm}) and (t_{lm}) of bi-complex numbers and $L, L' \in \mathbb{C}_2$ and $\alpha \in \mathbb{C}_2 - \mathcal{O}_2$. If $st_2 - \lim \|(\gamma_{lm})\|_{\mathbb{C}_2} = L$ and $st_2 - \lim \|(t_{lm})\|_{\mathbb{C}_2} = L'$. Then*

(1) $st_2 - \lim \|(\gamma_{lm} + t_{lm})\|_{\mathbb{C}_2} = L + L'$

(2) $st_2 - \lim \|\alpha \cdot (\gamma_{lm})\|_{\mathbb{C}_2} = \|\alpha\|_{\mathbb{C}_2} \cdot L$

Theorem 3.10. *A double sequence (γ_{lm}) of bi-complex numbers is statistically convergent to a bi-complex numbers L if and only if there exists a subset $K = \{(n,k) \subset \mathbb{N}\times\mathbb{N} : n,k = 1,2,...\}$ such that $\delta_2(K) = 1$ and*

$$\lim_{n,k} \gamma_{l_n m_k} = L.$$

Theorem 3.11. *If (γ_{lm}), where $\gamma_{lm} = u_{1lm} + i_2 u_{2lm}$ is statistically convergent to $\gamma = u_1 + i_2 u_2$ with respect to the Euclidean norm on $\mathbb{C}_2$ if and only if (u_{1lm}) and (u_{2lm}) are statistically convergent to u_1 and u_2 respectively.*

Proof. Consider (γ_{lm}) be statistically convergent to γ. Then for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{lm} - \gamma\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

$$\Longrightarrow \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{1lm} - \gamma_1) + i_2(\gamma_{2lm} - \gamma_2)\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0$$

$$\Longrightarrow \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Now,

$$\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\} \subseteq \{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}.$$

Thus, we have

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) \leq \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} + \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Similarly, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

Hence, the double sequences (γ_{1lm}) and (γ_{2lm}) are statistically convergent to γ_1 and γ_2, respectively.

Conversely, let (u_{1lm}) and (u_{2lm}) be statistically convergent to u_1 and u_2 respectively.

Then, for every $\varepsilon > 0$,

$$\delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{1lm} - \gamma_1\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0. \;\&\; \delta_2(\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|\gamma_{2lm} - \gamma_2\|_{\mathbb{C}_2} \geq \varepsilon\}) = 0.$$

We have

$$\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{1lm}-\gamma_1\|_{\mathbb{C}_2}\geq\varepsilon\}\cup\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{2lm}-\gamma_2\|_{\mathbb{C}_2}\geq\varepsilon\}$$
$$=\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|(\gamma_{1lm}-\gamma_1)+i_2(\gamma_{2lm}-\gamma_2)\|_{\mathbb{C}_2}\geq\varepsilon\}.$$

Thus,

$$\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|(\gamma_{1lm}-\gamma_1)+i_2(\gamma_{2lm}-\gamma_2)\|_{\mathbb{C}_2}\geq\varepsilon\})$$
$$\leq\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{1lm}-\gamma_1\|_{\mathbb{C}_2}\geq\varepsilon\}+\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{2lm}-\gamma_2\|_{\mathbb{C}_2}\geq\varepsilon\})$$
$$\text{(by subadditive property)}$$
$$\implies\delta_2(\{(l,m)\in\mathbb{N}\times\mathbb{N}:\|\gamma_{lm}-\gamma\|_{\mathbb{C}_2}\geq\varepsilon\})=0,\ \text{for every }\varepsilon>0.$$

Hence, (γ_{lm}) is a statistically convergent to γ with respect to the Euclidean norm on $\mathbb{C}_2$. □

We establish the following results based on the apparent proof.

Corollary 3.5. *If the double sequence of bi-complex numbers (γ_{lm}), where $\gamma_{lm}=u_{1lm}+i_2u_{2lm}$ is statistically convergent to $\gamma=u_1+i_2u_2=\mu_1e_1+\mu_2e_2$ with respect to Euclidean norm on $\mathbb{C}_2$ if and only if (μ_{1lm}) and (μ_{2lm}) are statistically convergent to μ_1 and μ_2 respectively.*

Corollary 3.6. *If double sequences (μ_{1lm}) and (μ_{2lm}) are statistically convergent to $L\in\mathbb{C}_2$, then double sequence of bi-complex numbers (γ_{lm}) is statistically convergent to L with respect to Euclidean norm on $\mathbb{C}_2$.*

Theorem 3.12. *Define the function $d_{{}_2\ell_\infty^-(\mathbb{C}_2)}$ by*

$$d_{{}_2\ell_\infty^-(\mathbb{C}_2)}:{}_2\ell_\infty^-(\mathbb{C}_2)\times{}_2\ell_\infty^-(\mathbb{C}_2)\to[0,\infty),(\gamma,t)\to d_{{}_2\ell_\infty^-(\mathbb{C}_2)}(\gamma,t)=\sup_{l,m\in\mathbb{N}}\{\|\gamma_{lm}-t_{lm}\|_{\mathbb{C}_2}\},$$

where $\gamma=(\gamma_{lm}),t=(t_{lm})\in{}_2\ell_\infty^-(\mathbb{C}_2)$. Then $({}_2\ell_\infty^-(\mathbb{C}_2),d_{{}_2\ell_\infty^-(\mathbb{C}_2)})$ is a complete metric space.

Proof. The proof is trivial from Theorem 9 [7]. □

Remark 3.1. *If (γ_{lm}) be statistically convergent to $L\in\mathbb{C}_2$ with respect to the Euclidean norm on $\mathbb{C}_2$, then*

(1) *(γ_{lm}^*) is statistically convergent to γ^* with respect to Euclidean norm on $\mathbb{C}_2$ and converse is also true.*
(2) *$(\widetilde{\gamma_{lm}})$ is statistically convergent to $\widetilde{\gamma}$ with respect to Euclidean norm on $\mathbb{C}_2$ and converse is also true.*

(3) (γ'_{lm}) *is statistically convergent to* γ' *with respect to Euclidean norm on* $\mathbb{C}_2$ *and converse is also true.*

Remark 3.2. *If* (γ_{lm}) *be statistically convergent with respect to the Euclidean norm on* $\mathbb{C}_2$*, then*

(1) $(|\gamma_{lm}|^2_{i_1}) = (\gamma_{lm} \cdot \widetilde{\gamma_{lm}})$ *is also statistically convergent with respect to Euclidean norm on* $\mathbb{C}_2$*.*

(2) $(|\gamma_{lm}|^2_{i_2}) = (\gamma_{lm} \cdot \gamma^*_{lm})$ *is also statistically convergent with respect to Euclidean norm on* $\mathbb{C}_2$*.*

(3) $(|\gamma_{lm}|^2_{i_1 i_2}) = (\gamma_{lm} \cdot \gamma'_{lm})$ *is also statistically convergent with respect to Euclidean norm on* $\mathbb{C}_2$*.*

Theorem 3.13. *The sets* ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}(\mathbb{C}_2), {}_2\bar{c}_0(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}^R_0(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}^B_0(\mathbb{C}_2)$ *are* $\mathbb{BC}$*-module.*

Proof. We prove that the set ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ is a $\mathbb{BC}$-module. The proofs for the other sets follow analogously based on their respective definitions.

Let, $(\gamma_{lm}), (t_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$. By definition of vector addition,

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0,$$

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(t_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Consider the sum $(\gamma_{lm}) + (t_{lm})$, using the triangle inequality for the norm $\|\cdot\|_{\mathbb{C}_2}$, we have;

$$\|(\gamma_{lm}) + (t_{lm})\|_{\mathbb{C}_2} \leq \|(\gamma_{lm})\|_{\mathbb{C}_2} + \|(t_{lm})\|_{\mathbb{C}_2}.$$

Now, analyze the density condition for $(\gamma_{lm}) + (t_{lm})$;

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(\gamma_{lm}) + (t_{lm})\|_{\mathbb{C}_2} \geq M\}|.$$

By subadditivity of the density measure, this is bounded by

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| + \lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(t_{lm})\|_{\mathbb{C}_2} \geq M\}|.$$

Since, both terms on the right-hand side are zero by assumption, we conclude.

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) \in \mathbb{N} \times \mathbb{N} : \|(\gamma_{lm}) + (t_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Thus, $(\gamma_{lm}) + (t_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$, showing closure under addition Let $a \in \mathbb{C}_2$ and $(\gamma_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$. By definition of scalar multiplication,

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

For the scalar product $a \cdot (\gamma_{lm})$, using the property of the norm

$$\|a \cdot (\gamma_{lm})\|_{\mathbb{C}_2} = |a|_{\mathbb{C}_2} \cdot \|(\gamma_{lm})\|_{\mathbb{C}_2},$$

where $|a|_{\mathbb{C}_2}$ is the modulus of a in $\mathbb{C}_2$.

Now analyze the density condition for $a \cdot (\gamma_{lm})$;

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|a \cdot (\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}|.$$

This is equivalent to;

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq \frac{M}{|a|_{\mathbb{C}_2}}\}|.$$

Since, the right-hand side is zero by the assumption that $(\gamma_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$, we conclude

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|a \cdot (\gamma_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Thus, $a\cdot(\gamma_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$, showing closure under scalar multiplication. Since ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ satisfies closure under addition and scalar multiplication, it is a $\mathbb{BC}$-module.

Similarly, using analogous arguments, the other sets can be shown to be $\mathbb{BC}$-modules. □

Theorem 3.14. *The classes of the double sequences* ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ *of bi-complex numbers are* $\mathbb{BC}$*-convex.*

Proof. We first prove the $\mathbb{BC}$-convexity for ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$. The other classes can be established similarly. Let $(\gamma_{lm}), (t_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$. Then there exist constants $M_1, M_2 > 0$ such that

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M_1\}| = 0,$$

$$\lim_{l,m\to\infty} \frac{1}{lm}|\{(l,m) \in \mathbb{N}\times\mathbb{N} : \|(t_{lm})\|_{\mathbb{C}_2} \geq M_2\}| = 0.$$

Define $M = max\{M_1, M_2\}$. For $0 \leq \lambda \leq 1$, consider the convex combination

$$(\delta_{lm}) = \lambda(\gamma_{lm}) + (1-\lambda)(t_{lm}).$$

Using the triangle inequality, we have

$$\|(\delta_{lm})\|_{\mathbb{C}_2} \leq \lambda\|(\gamma_{lm})\|_{\mathbb{C}_2} + (1-\lambda)\|(t_{lm})\|_{\mathbb{C}_2}.$$

For $(l, m) \in \mathbb{N} \times \mathbb{N}$ such that $\|(\delta_{lm})\|_{\mathbb{C}_2} \geq M$, at least one of $\|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M_1$ or $\|(t_{lm})\|_{\mathbb{C}_2} \geq M_2$, must hold.
Thus,

$$|\{(l,m) : \|(\delta_{lm})\|_{\mathbb{C}_2} \geq M\}| \leq |\{(l,m) : \|(\gamma_{lm})\|_{\mathbb{C}_2} \geq M_1\}| + |\{(l,m) : \|(t_{lm})\|_{\mathbb{C}_2} \geq M_2\}|.$$

Dividing by lm and taking the limit as $l, m \to \infty$.

$$\lim_{l,m\to\infty} \frac{1}{lm} |\{(l,m) : \|(\delta_{lm})\|_{\mathbb{C}_2} \geq M\}| = 0.$$

Hence, $(\delta_{lm}) \in {}_2\bar{\ell_\infty}(\mathbb{C}_2)$. Proving ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ is $\mathbb{BC}$-convex. Similarly, the other cases can be established. □

Remark 3.3. *The classes of the double sequences* ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ *of bi-complex numbers are not* $\mathbb{BC}$*-strictly convex.*

This follows from the following example for the case ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$. The other classes can be established similarly.

Example 3.1. *Let the double sequences* (γ_{lm}) & (t_{lm}) *of bi-complex numbers defined by*

$$(\gamma_{lm}) = \begin{pmatrix} (\frac{1}{2} - \frac{\sqrt{3}}{2}i_1)e_1 + (\frac{1}{2} + \frac{\sqrt{3}}{2}i_1)e_2 & (\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i_1)e_1 + (\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}i_1)e_1 + (\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i_1)e_2 & (\frac{1}{3} + \frac{2\sqrt{2}}{3}i_1)e_1 + (\frac{1}{3} - \frac{2\sqrt{2}}{3}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and

$$(t_{lm}) = \begin{pmatrix} \theta & (\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i_1)e_1 + (\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}}i_1)e_1 + (\frac{1}{\sqrt{5}} + \frac{2}{\sqrt{5}}i_1)e_2 & (\frac{1}{3} + \frac{2\sqrt{2}}{3}i_1)e_1 + (\frac{1}{3} - \frac{2\sqrt{2}}{3}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then, $\|(\gamma_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = \|(t_{lm})\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)} = 1$ *and*

$$\left\|\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)(\gamma_{lm})+\left\{1-\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)\right\}(t_{lm})\right\|_{{}_2\bar{\ell_\infty}(\mathbb{C}_2)}$$

$$=\sup_{l,m\in\mathbb{N}}\left\|\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)\gamma_{lm}+\left\{1-\left(\frac{1}{2}e_1+\frac{1}{2}e_2\right)\right\}t_{lm}\right\|_{\mathbb{C}_2}$$

$$=\sup_{l,m\in\mathbb{N}}\Bigg[\Bigg\|(\frac{1}{4}-\frac{\sqrt{3}}{4}i_1)e_1+(\frac{1}{4}+\frac{\sqrt{3}}{4}i_1)e_2,(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_2,\theta,\theta,...,$$

$$(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_2,(\frac{1}{6}+\frac{2\sqrt{2}}{6}i_1)e_1+(\frac{1}{6}-\frac{2\sqrt{2}}{6}i_1)e_2,\theta,\theta,...,$$

$$\theta,\theta,...,$$

$$+\Bigg\{\theta,(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2,\theta,\theta,...,$$

$$(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_2,(\frac{1}{3}+\frac{2\sqrt{2}}{3}i_1)e_1+(\frac{1}{3}-\frac{2\sqrt{2}}{3}i_1)e_2,\theta,\theta,...,$$

$$\theta,\theta,...,$$

$$-\Bigg(\theta,(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_2,\theta,\theta,...,$$

$$(\frac{1}{2\sqrt{5}}-\frac{2}{2\sqrt{5}}i_1)e_1+(\frac{1}{2\sqrt{5}}+\frac{2}{2\sqrt{5}}i_1)e_2,(\frac{1}{6}+\frac{2\sqrt{2}}{6}i_1)e_1+(\frac{1}{6}-\frac{2\sqrt{2}}{6}i_1)e_2,\theta,\theta,...,$$

$$\theta,\theta,...,\Bigg)\Bigg\}\Bigg\|_k\Bigg]$$

$=\sup_{l,m\in\mathbb{N}}\left\{\frac{1}{2},1,\theta\right\}=1.$ *for* $\lambda=(\frac{1}{2}e_1+\frac{1}{2}e_2)\in\mathbb{C}_2$.
Hence, ${}_2\bar{\ell_\infty}(\mathbb{C}_2)$ *is not* $\mathbb{BC}$*-strictly convex.*

Remark 3.4. *The classes of the double sequences* ${}_2\bar{\ell_\infty}(\mathbb{C}_2), {}_2\bar{c}^R(\mathbb{C}_2), {}_2\bar{c}_0^R(\mathbb{C}_2), {}_2\bar{c}^B(\mathbb{C}_2), {}_2\bar{c}_0^B(\mathbb{C}_2)$ *of bi-complex numbers are not* $\mathbb{BC}$*-uniformly convex.*

This follows from the following Example.

Example 3.2. *Let the double sequences* (γ_{lm}) & (t_{lm}) *of bi-complex numbers defined by*

$$(\gamma_{lm})=\begin{pmatrix}(\frac{1}{2}-\frac{\sqrt{3}}{2}i_1)e_1+(\frac{1}{2}+\frac{\sqrt{3}}{2}i_1)e_2 & (\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & \theta & \theta & \dots\\ (\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & (\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots\\ \theta & \theta & \theta & \theta & \dots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},$$

and

$$(t_{lm}) = \begin{pmatrix} (-\frac{1}{2}+\frac{\sqrt{3}}{2}i_1)e_1+(-\frac{1}{2}-\frac{\sqrt{3}}{2}i_1)e_2 & (\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & \theta & \theta & \dots \\ (\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1)e_1+(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1)e_2 & (\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1)e_1+(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1)e_2 & \theta & \theta & \dots \\ \theta & \theta & \theta & \theta & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Then $\|(\gamma_{lm})\|_{2\bar{\ell}_\infty(\mathbb{C}_2)} = \|(t_{lm})\|_{2\bar{\ell}_\infty(\mathbb{C}_2)} = 1$ *and*

$$\begin{aligned} \|(\gamma_{lm})-(t_{lm})\|_{2\bar{\ell}_\infty(\mathbb{C}_2)} &= \sup_{l,m\in\mathbb{N}}\{\|\gamma_{lm}-t_{lm}\|_{\mathbb{C}_2} : l,m\in\mathbb{N}\} \\ &= \sup_{l,m\in\mathbb{N}}\left\{\left\|(\frac{2}{2}-\frac{2\sqrt{3}}{2}i_1)e_1+(\frac{2}{2}+\frac{2\sqrt{3}}{2}i_1)e_2\right\|_{\mathbb{C}_2}\right\} \\ &= 2. \end{aligned}$$

and $\varepsilon \le \|(\gamma_{lm})-(t_{lm})\|_{2\bar{\ell}_\infty(\mathbb{C}_2)} = 2.$

On the other hand,

$$\begin{aligned} \left\|\frac{(\gamma_{lm})+(t_{lm})}{2}\right\|_{2\bar{\ell}_\infty(\mathbb{C}_2)} &= \sup_{l,m\in\mathbb{N}}\left\|\frac{\gamma_{lm}+t_{lm}}{2}\right\|_{\mathbb{C}_2} \\ &= \sup_{l,m\in\mathbb{N}}\left\{\left\|\left(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_1+\left(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_2\right\|_{\mathbb{C}_2},\right. \\ &\left\|\left(\frac{1}{\sqrt{3}}+\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_1+\left(\frac{1}{\sqrt{3}}-\frac{\sqrt{2}}{\sqrt{3}}i_1\right)e_2\right\|_{\mathbb{C}_2}, \\ &\left.\left\|\left(\frac{1}{\sqrt{5}}+\frac{2}{\sqrt{5}}i_1\right)e_1+\left(\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}i_1\right)e_2\right\|_{\mathbb{C}_2}\right\} \\ &= 1. \end{aligned}$$

Thus, there does not exist $\delta(\varepsilon)$ *such that*

$$\left\|\frac{(\gamma_{lm})+(t_{lm})}{2}\right\|_{2\bar{\ell}_\infty(\mathbb{C}_2)} \le 1-\delta.$$

Therefore, we have ${}_2\bar{\ell}_\infty(\mathbb{C}_2)$ *is not* $\mathbb{BC}$*-uniformly convex.*

Similarly, other classes can also be proved.

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References

[1] Bera S. & Tripathy, B.C. (2023). Statistical bounded sequences of bi-complex numbers. Probl. Anal. Issues Anal., 30(2), 3-16.

[2] Bromwich, T.J. (1965). An Introduction to the Theory of Infinite Series. Macmillan, New York.

[3] Fast, H. (1951). Sur la convergence statistique. Colloq. Math., 2, 241-244.

[4] Fridy, J. A. & Orhan, C. (1997). Statistical limit superior and limit inferior. Proc. Amer. Math. Soc., 125(12), 3625-3631.

[5] Hardy, G.H. (1917). On the convergence of certain multiple series. Proc. Camb. Phil. Soc., 6, 86-95.

[6] Kumar, S. & Tripathy, B.C. (2024). Double Sequences of Bi-complex Numbers. Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci., 94(4), 463–469.

[7] Kumar, S. & Tripathy, B.C. (2024). Almost Convergent Double Sequences of Bi-complex Numbers. Filomat, 38(11), 3957–3970.

[8] Kumar, S. & Tripathy, B.C. (2024). Difference Double Sequences of Bi-complex Numbers. Ann. Acad. Rom. Sci. Ser. Math. Appl., 16(2), 135–149.

[9] Kumar, S. & Tripathy, B.C. (2025). Some Geometric Properties of Double Sequences of Bi-complex Numbers. Natl. Acad. Sci. Lett., https://doi.org/10.1007/s40009-025-01640-3

[10] Lavoie, R. G., Marchildon, L. & D Rochon, D. (2010). Finite-dimensional bicomplex Hilbert spaces. Adv. Appl. Clifford Algebras, 21(3), 561–581.

[11] Maddox, I. J. (1989). A Tauberion condition for statistical convergence. Math. Proc. Comb. Phil. Soc., 272-280.

[12] Mursaleen, M. & Edely, O. H. H. (2003). Statistical convergence of double sequences. J. Math. Anal. Appl., 288(1), 223-231.

[13] Price, G.B. (1991). An Introduction to Multicomplex Spaces and Function, M. Dekker, New York.

[14] Pringsheim, A. (1900). Zur Theorie der zweifach unendlichen Zahlenfolgen. Math. Ann., 53(3), 289-321.

[15] Rath, D. & Tripathy, B.C. (1996). Matrix maps on sequence spaces associated with sets of integers. Indian J. Pure Appl. Math., 27(2), 197-206.

[16] Rochon, D. & Shapiro, M. (2004). On algebraic properties of bi-complex and hyperbolic numbers. Anal. Univ. Oradea, fasc. Math. Monthly, 11, 71-110.

[17] Salat, T. (1980). On statistically convergent sequences of real numbers. Math. Slovaca, 30(2), 139-150.

[18] Schoenberg, I. J. (1959). The integrability of certain functions and related summability methods. Amer. Math. Monthy, 66, 361-375.

[19] Tripathy, B. C. (2003). Statistically convergent double sequences. Tamkang J. Math., 34(3), 2031-237.

[20] Zygmund, A. (1993). Trigonometric Series. vol II Cambridge University Press.

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FAULHABER-TYPE FORMULAS FOR THE SUMS OF POWERS OF ARITHMETIC SEQUENCES

MORGAN SCHREFFLER AND TÜRKAY YOLCU *

Abstract. In this article, we derive explicit formulas for computing the sums of powers in arithmetic sequences. We begin with a historical odyssey, tracing the contributions of some of the world's most influential mathematicians whose work has shaped and inspired our approach. We then present two distinct Faulhaber-type formulas–one involving Bernoulli numbers and closely resembling the classical formula for sums of powers of integers. To establish these results, we employ two different techniques: the first is based on the principle of invariance, while the second uses the differencing operator applied to polynomials. Although the methods differ in form, we emphasize that they share the similar computational complexity, a point we demonstrate with illustrative examples at the end.

Keywords: Faulhaber-type formulas, Bernoulli numbers, Principle of invariance, Differencing operator.

2020 Mathematics Subject Classification: 11B25, 11B68, 11B75, 11Y35, 11Y55.

1. Introduction

According to legend, a precocious primary school student by the name of Carl Friedrich Gauss surprised his teacher by calculating

$$S = 1 + 2 + 3 + \cdots + 100,$$

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almost without effort. His trick was to consider the sum twice, once adding in ascending order, and once in descending order, then adding them side-by-side. In modern notation,

$$2S = \sum_{n=1}^{100} n + \sum_{n=1}^{100} (101 - n) = \sum_{n=1}^{100} 101 = 100 \times 101 = 10,100.$$

Once Gauss had calculated twice the sum he wanted, the only thing left was to divide his result by 2, and voila! He had obtained $S = 5,050$. Extending his technique to compute the sum of the first N positive integers, the general formula

$$\sum_{n=1}^{N} n = 1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

is easily obtained. The elegance of Gauss's technique is made more evident when we consider that it can be used to compute the sum of *any* finite arithmetic sequence $\{a, a+r, \ldots, a+Nr\}$. Indeed, we obtain

$$\sum_{n=0}^{N} (a + rn) = \frac{(2a + rN)(N+1)}{2}.$$

Despite the brilliance of young Gauss's tenacious tallying, the formulas above were known at least 2000 years before his birth! In fact, formulas for the sum of squares and the sum of cubes,

$$\sum_{n=1}^{N} n^2 = 1^2 + 2^2 + \cdots + N^2 = \frac{N(N+1)(2N+1)}{6},$$

$$\sum_{n=1}^{N} n^3 = 1^3 + 2^3 + \cdots + N^3 = \frac{N^2(N+1)^2}{4},$$

respectively, were known in antiquity [20, 27, 19, 2, 18, 17, 3]. The former was described by the legendary Greek polymath Archimedes in his work *On Conoids and Spheroids* [1, 6], and the latter is famously attributed to Nicomachus of Gerasa, another well-known Greek mathematician, who published the result in his *Introduction to Arithmetic* [2, 5]. Due to the nature of ancient Greek mathematics, these proofs are geometric in nature, and certainly valuable from both a mathematical and a historical standpoint. The interested reader can find more information in David M. Burton's historical treatise [2].

A formula for the sum of the fourth powers of the first N natural numbers, $\sum n^4$, is described by Pierre de Fermat in letters written in 1636 to Gilles Persone de Roberval and Marin Mersenne, all of whom were prominent mathematicians at the time [2]. It is possible that Fermat was under the impression that such a formula was unknown at the time [27]. Fermat's belief notwithstanding, formulas for the sums of powers of integers up to the 17th power, and even as high as the 23rd power, were known to the German mathematician Johann Faulhaber as late as 1631 [27, 8, 21]. Arguably, Faulhaber's greatest contribution was the discovery of the fact that, in the case of an odd exponent k, the sum $\sum_{n=1}^{N} n^k$ is a polynomial in terms of the variable $a = \frac{N(N+1)}{2}$ and, moreover, for odd $k > 1$ the polynomial is divisible by a^2. He also described a method to obtain a formula for the sum $\sum n^{2k}$ once the formula for the sum $\sum n^{2k+1}$ is known [27, 26, 8, 21, 23, 16]. It is not clear whether Faulhaber knew how to prove his assertions in generality; in his day, mathematical discoveries were usually kept secret, given as challenges to other mathematicians, or intentionally written in code! [27, 15, 21]

A proof of the general explicit formula for the sum $\sum n^k$, and its rigorous verification, would have to wait until 1834, with the publication of Carl Jacobi's paper *De usu legitimo formulae summatoriae Maclaurinianae* [9]. Jacobi's formula incorporates the Bernoulli numbers, which is the sequence of rational numbers $\{B_N\}_{N=0}^{\infty}$ given recursively by the formula [1]

$$B_0 = 1, \qquad \sum_{n=0}^{N} \binom{N+1}{n} B_n = 0, \quad N \geq 1. \tag{1.1}$$

The first few nonzero Bernoulli numbers are given in Table 1.1 below. Note that for all odd $N > 1$, $B_N = 0$.

Table 1.1. Bernoulli numbers.

N	0	1	2	4	6	8	10	12
B_N	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$

The numbers B_N can also be defined explicitly by the formula

$$B_N = \sum_{n=0}^{N} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{k^N}{n+1}, \quad \text{where } 0^0 \text{ and } \binom{0}{0} \text{ are both taken to be 1.}$$

Though this will not be necessary for our discussion, it is still an important observation, since any explicit formula for calculating sums of powers of integers which utilizes the Bernoulli numbers would be easily stymied by requiring the Bernoulli numbers to be calculated recursively. We will refer to the Bernoulli numbers again in Section 2.2.

The Bernoulli numbers were discovered independently by the Swiss mathematician Jakob Bernoulli, as well as the Japanese mathematician Seki Takakazu [2, 11, 10]. Specifically,

[1] There are two common conventions for defining the Bernoulli numbers, but only the sign of B_1 is affected by the choice. We use the convention in which $B_1 = -\frac{1}{2}$.

Bernoulli discovered the sequence while attempting to derive formulas for the sums of powers of integers. As an interesting side note, both Bernoulli's and Takakazu's discoveries were published posthumously, the former in 1713 and the latter a year earlier in 1712 [11, 10]. Armed with the Bernoulli numbers, Jacobi was able to rigorously verify Faulhaber's formula (often referred to as Bernoulli's formula) [9],

$$\sum_{n=1}^{N} n^k = \frac{N+1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (N+1)^{k-m} B_m. \tag{1.2}$$

Our goal is to find a Faulhaber-type formula for sums of powers of finite arithmetic sequences, i.e., sums of the form

$$S_{N,k}[a,r] := \sum_{n=0}^{N} (a+rn)^k,$$

where a and r are arbitrary real numbers, and k and N are nonnegative integers. Using the binomial theorem and changing the order of summation yields

$$\begin{aligned} S_{N,k}[a,r] &= \sum_{n=0}^{N} (a+nr)^k \\ &= \sum_{n=0}^{N} \sum_{m=0}^{k} \binom{k}{m} a^{k-m} n^m r^m \\ &= \sum_{m=0}^{k} \binom{k}{m} a^{k-m} r^m \sum_{n=0}^{N} n^m \\ &= \sum_{m=0}^{k} \binom{k}{m} a^{k-m} r^m S_{N,m}[0,1]. \end{aligned} \tag{1.3}$$

The aforementioned formula provides a straightforward recursive approach for calculating $S_{N,k}[a,r]$, contingent upon having knowledge of the sums $S_{N,m}[0,1]$. If the values of these sums are unknown, this method can become quite laborious. As an example, let us use the formula above to find closed formulas for the first few sums $S_{N,k}[a,r]$. Note that

$$S_{N,0}[0,1] = N+1,$$

and that

$$\begin{aligned} S_{N,1}[a,r] &= (N+1)a + \frac{N(N+1)}{2}r, \\ S_{N,2}[a,r] &= (N+1)a^2 + 2\frac{N(N+1)}{2}ar + \frac{N(N+1)(2N+1)}{6}r^2, \\ S_{N,3}[a,r] &= (N+1)a^3 + 3\frac{N(N+1)}{2}a^2r + 3\frac{N(N+1)(2N+1)}{6}ar^2 \\ &\quad + \frac{N^2(N+1)^2}{4}r^3. \end{aligned}$$

The next important development in our story comes courtesy of Blaise Pascal [27, 2]. In his famous work *Treatise on the Arithmetic Triangle* [12], the author outlines a formula for the sums of powers of terms in an arithmetic sequence. He illustrates his process via the example

$$P = 5^3 + 8^3 + 11^3 + 14^3 = 4712,$$

and claims that the process can be generalized. In fact, his approach is similar to the one we will begin with in Section 2, before using it to derive our explicit formula. First, Pascal showed (using techniques he had developed earlier in the paper) that

$$(t+3)^4 - t^4 = 12t^3 + 54t^2 + 108t + 81.$$

He then substituted $t = 5$, $t = 8$, $t = 11$, and $t = 14$ into this identity and added the results, noticing that the left side "telescopes." At this point, we are left with

$$\begin{aligned} 17^4 - 5^4 = 12(5^3 + 8^3 + 11^3 + 14^3) + 54(5^2 + 8^2 + 11^2 + 14^2) \\ + 108(5 + 8 + 11 + 14) + 81(1 + 1 + 1 + 1), \end{aligned}$$

from which we compute

$$17^4 - 5^4 = 12P + 54(406) + 108(38) + 81(4).$$

Finally, solving for P gives $P = 4712$ as expected.

More generally, Pascal recognized that on one hand, the sum

$$\sum_{n=1}^{N} \left[(a + (n+1)r)^{k+1} - (a + nr)^{k+1} \right]$$

"telescopes" to the expression $[a + (N+1)r]^{k+1} - a^{k+1}$, while on the other hand, it can be expanded (using the entries from his eponymous triangle) into a combination of sums with exponents lower than $k+1$. At this point, he solved for $S_{N,k}[a,r]$ in terms of the sums of lower powers, i.e., the sums $S_{n,0}[a,r]$, $S_{n,1}[a,r], \ldots$, and $S_{n,k-1}[a,r]$.

Observe that Pascal's method is recursive in nature, requiring the formulas for the sums of all lower powers in order to compute the desired sum. This makes the method easy to describe, but impractical to use directly for larger numbers of terms or for higher powers. However, this recursive approach is the first step to obtaining our Faulhaber-type formulas, which do not rely on recursion. These formulas exhibit a remarkable level of effectiveness, which is made apparent by their resilience and their ability to unify and generalize various formulas involving the summation of different powers of integers. This potent set of formulas demonstrates its versatility when applied to the computation of exceedingly high powers of arithmetic sums, as well as when dealing with a substantial number of terms within these sums of arithmetic sequences.

In Section 2, we derive an explicit formula to calculate $S_{N,k}[a,r]$ which is quite similar to Equation (1.2). We first obtain a recursive formula in the style of Pascal. Next, we develop a unique approach to proving Faulhaber's formula without induction. The crux of our proof relies on a clever argument in which we show that a particular expression, which we call $Q_{N,k}(j)$, is invariant with respect to j. It turns out that $\frac{N+1}{k+1}Q_{N,k}(0)$ and $\frac{N+1}{k+1}Q_{N,k}(k-1)$ are equal to the left- and right-hand sides of Equation (1.2), respectively, from which Faulhaber's formula follows immediately. We also provide a more traditional inductive proof of Equation (1.2). Finally, we combine Faulhaber's formula with our recursive formula, culminating in Theorem 2.3.

In Section 3, we use methods from the theory of finite differences to derive a somewhat different version of Faulhaber's formula which does not require any knowledge of the Bernoulli numbers. As in Section 2, we take inspiration from Pascal and consider a special telescoping sum; the notation of difference operators arises naturally as a result, and provides us with an alternate, yet equally robust, Faulhaber-type formula, stated explicitly in Theorem 3.1.

2. Faulhaber's Formula with Bernoulli Numbers

In what follows, N and k are always nonnegative integers, a and r are real numbers, and we use the conventions that $0^0 = 1$ and $\binom{0}{0} = 1$.

2.1. **Recursive Formulas.** Recall that in Section 1 we introduced the notation

$$S_{N,k}[a,r] = \sum_{n=0}^{N}(a+nr)^k = a^k + (a+r)^k + \cdots + (a+Nr)^k.$$

Our first theorem is a recursive formula for $S_{N,k}[a,r]$ in terms of the sums $S_{N,m}[a,r]$ with lower exponents $m = 0, 1, \ldots, k-1$. This is an important first step in finding an explicit formula for $S_{N,k}[a,r]$, and a generalization of Pascal's method, which we discussed in Section 1. Immediately thereafter, we provide as a corollary a recursive formula for the sums $S_{N,k}[0,1]$ in terms of the sums $S_{N,m}[0,1]$. We make extensive use of this corollary in our deductive proof of Faulhaber's formula for $S_{N,k}[0,1]$, as well as the Faulhaber-type formula for $S_{N,k}[a,r]$.

Theorem 2.1. *The sums $S_{N,k}[a,r]$ are given recursively in k by the formula*

$$S_{N,k}[a,r] = (N+1)a^k + \sum_{m=1}^{k} \binom{k}{m} \frac{r^m}{m+1} \left[(N+1)^{m+1}a^{k-m} - S_{N,k-m}[a,r]\right].$$

Proof. The result is trivial when $r = 0$, so we prove it for nonzero r. We begin by considering the telescoping sum

$$T(N) := \sum_{n=0}^{N} \left([a+(n+1)r]^{k+1} - [a+nr]^{k+1}\right).$$

On one hand, if we first telescope $T(N)$, then use the binomial theorem to expand the expression $[a+(N+1)r]^{k+1}$, we obtain

$$\begin{aligned} T(N) &= [a+(N+1)r]^{k+1} - a^{k+1} \\ &= \sum_{m=0}^{k+1} \left[\binom{k+1}{m}(N+1)^m r^m a^{k+1-m}\right] - a^{k+1} \\ &= \sum_{m=1}^{k+1} \binom{k+1}{m}(N+1)^m r^m a^{k+1-m} \\ &= \sum_{m=0}^{k} \binom{k+1}{m+1}(N+1)^{m+1} r^{m+1} a^{k-m} \\ &= (k+1)r \sum_{m=0}^{k} \binom{k}{m} \frac{(N+1)^{m+1} r^m}{m+1} a^{k-m}. \end{aligned}$$

On the other hand, if we first distribute the sum in n, then use the binomial theorem to expand the expression $[a+(n+1)r]^{k+1} = [(a+nr)+r]^{k+1}$, we are left with

$$\begin{aligned} T(N) &= \sum_{n=0}^{N}\left[\sum_{m=0}^{k+1}\binom{k+1}{m} r^m (a+nr)^{k+1-m}\right] - S_{N,k+1}[a,r] \\ &= \sum_{m=0}^{k+1}\left[r^m \binom{k+1}{m}\sum_{n=0}^{N}(a+nr)^{k+1-m}\right] - S_{N,k+1}[a,r] \\ &= \sum_{m=0}^{k+1}\left[\binom{k+1}{m} r^m S_{N,k+1-m}[a,r]\right] - S_{N,k+1}[a,r] \\ &= \sum_{m=1}^{k+1}\binom{k+1}{m} r^m S_{N,k+1-m}[a,r] \\ &= \sum_{m=0}^{k}\binom{k+1}{m+1} r^{m+1} S_{N,k-m}[a,r] \\ &= (k+1)r\sum_{m=0}^{k}\binom{k}{m}\frac{r^m}{m+1} S_{N,k-m}[a,r]. \end{aligned}$$

Equating the two expressions above and dividing by $(k+1)r$ gives us

$$\sum_{m=0}^{k}\binom{k}{m}\frac{(N+1)^{m+1}r^m}{m+1}a^{k-m} = \sum_{m=0}^{k}\binom{k}{m}\frac{r^m}{m+1}S_{N,k-m}[a,r],$$

from which we deduce

$$\begin{aligned} S_{N,k}[a,r] &= \sum_{m=0}^{k}\binom{k}{m}\frac{(N+1)^{m+1}r^m}{m+1}a^{k-m} - \sum_{m=1}^{k}\binom{k}{m}\frac{r^m}{m+1}S_{N,k-m}[a,r] \\ &= (N+1)a^k + \sum_{m=1}^{k}\binom{k}{m}\frac{r^m}{m+1}\left((N+1)^{m+1}a^{k-m} - S_{N,k-m}[a,r]\right), \end{aligned}$$

which is the desired result. □

Corollary 2.1. *The sums $S_{N,k}[0,1]$ are given recursively in k by the formula*

$$S_{N,k}[0,1] = \frac{(N+1)^{k+1}}{k+1} - \frac{1}{k+1}\sum_{m=0}^{k-1}\binom{k+1}{m}S_{N,m}[0,1].$$

Proof. First, consider the mth term

$$\tau_m := \binom{k}{m}\frac{r^m}{m+1}(N+1)^{m+1}a^{k-m}$$

from the sum in Theorem 2.1. Observe that when $a=0$ and $r=1$,

$$\tau_m = \frac{(N+1)^{k+1}}{k+1}\,\delta_k(m),$$

where $\delta_k(m)$ is the Kroenecker delta. If we substitute the revised τ_m back into the sum from Theorem 2.1, then channel our "inner Gauss" and rewrite the sum in *descending* order, we conclude

$$\begin{aligned}
S_{N,k}[0,1] &= \frac{(N+1)^{k+1}}{k+1} - \sum_{m=1}^{k} \binom{k}{m} \frac{1}{m+1} S_{N,k-m}[0,1] \\
&= \frac{(N+1)^{k+1}}{k+1} - \sum_{m=0}^{k-1} \binom{k}{k-m} \frac{1}{k-m+1} S_{N,m}[0,1] \\
&= \frac{(N+1)^{k+1}}{k+1} - \frac{1}{k+1} \sum_{m=0}^{k-1} \binom{k}{m} \frac{k+1}{k+1-m} S_{N,m}[0,1] \\
&= \frac{(N+1)^{k+1}}{k+1} - \frac{1}{k+1} \sum_{m=0}^{k-1} \binom{k+1}{m} S_{N,m}[0,1].
\end{aligned}$$

This is precisely the desired result. □

Before moving on to the next section and demonstrating one of our main results, it is worth noting that Theorem 2.1 remains a labor-intensive method for directly computing the sums $S_{N,k}[a,r]$, yet it yields identical outcomes to those previously observed.

$$\begin{aligned}
S_{N,1}[a,r] &= (N+1)a + \frac{r}{2}((N+1)^2 - S_{N,0}[a,r]) \\
&= (N+1)a + \frac{r}{2}((N+1)^2 - (N+1)) \\
&= (N+1)a + \frac{N(N+1)}{2} r, \\
S_{N,2}[a,r] &= (N+1)a^2 + r((N+1)^2 a - S_{N,1}[a,r]) + \frac{r^2}{3}((N+1)^3 - S_{N,0}[a,r]) \\
&= (N+1)a^2 + r\left((N+1)^2 a - (N+1)a - \frac{N(N+1)}{2} r\right) \\
&\quad + \frac{r^2}{3}((N+1)^3 - (N+1)) \\
&= (N+1)a^2 + 2\frac{N(N+1)}{2} ar + \frac{N(N+1)(2N+1)}{6} r^2,
\end{aligned}$$

and, after much simplification which we leave to the reader,

$$\begin{aligned}
S_{N,3}[a,r] &= (N+1)a^3 + \frac{3r}{2}((N+1)^2 a^2 - S_{N,2}[a,r]) \\
&\quad + r^2((N+1)^3 a - S_{N,1}[a,r]) + \frac{r^3}{4}((N+1)^4 - S_{N,0}[a,r]) \\
&= (N+1)a^3 + 3\frac{N(N+1)}{2} a^2 r + 3\frac{N(N+1)(2N+1)}{6} ar^2 \\
&\quad + \frac{N^2(N+1)^2}{4} r^3.
\end{aligned}$$

As we can see, even for small k, direct use of Theorem 2.1 to calculate a formula for $S_{N,k}[a,r]$ is quite time-consuming. However, without an efficient algorithm for generating formulas for the sums $S_{N,k}[0,1]$, it remains our only option. This is precisely the issue we address in the next section, and unsurprisingly, it is Theorem 2.1 (or rather, its corollary) which we rely on most heavily to obtain our results.

2.2. **Connections with Bernoulli numbers.** In Section 1, we introduced the Bernoulli numbers $\{B_j\}$, which are a recursively defined sequence of rational numbers. For our purposes, we shall utilize the following reformulation of Equation (1.1):

$$B_0 = 1, \quad B_{j+1} = -\frac{1}{j+2}\sum_{n=0}^{j}\binom{j+2}{n}B_n, \quad j \geq 0. \tag{2.4}$$

In order to prove the main results for this section, we first establish two intermediate lemmas. The first of these can be regarded as an alternate recursive definition of the Bernoulli numbers. The second lemma invokes the first to demonstrate the invariance under j of a particular quantity, which we refer to as $Q_{N,k}(j)$, via repeated use of Corollary 2.1. Before embarking on this quest of quantification, we introduce a bit of notation to clean up future calculations. Let j, k, and m be integers, with $0 \leq j, m < k$, and define the sums

$$\Theta_{j,k}(m) := \sum_{n=0}^{j} B_n \binom{k+1}{n}\binom{k+1-n}{m},$$

and

$$Q_{N,k}(j) := \sum_{m=0}^{j}\binom{k+1}{m}(N+1)^{k-m}B_m - \frac{1}{N+1}\sum_{m=0}^{k-j-1}\Theta_{j,k}(m)S_{N,m}[0,1].$$

Armed with this new notation, we are in position to establish the lemmas.

Lemma 2.1. *For any integers j and k satisfying $0 \leq j < k$, we have*

$$\Theta_{j,k}(k-j-1) = -(k-j)\binom{k+1}{j+1}B_{j+1}.$$

Proof. We use the well-known identities

$$\binom{n}{k} = \binom{n}{n-k}, \quad \binom{n}{k}\binom{n-k}{\ell-k} = \binom{n}{\ell}\binom{\ell}{k}, \quad \text{and} \quad \binom{n}{k+1} = \frac{n-k}{k+1}\binom{n}{k},$$

along with the recursive definition of the Bernoulli numbers expressed in Equation (2.4), to establish the equality

$$\begin{aligned}\Theta_{j,k}(k-j-1) &= \sum_{n=0}^{j}\binom{k+1}{n}\binom{k+1-n}{j+2-n}B_n \\ &= \binom{k+1}{j+2}\sum_{n=0}^{j}\binom{j+2}{n}B_n \\ &= \frac{k-j}{j+2}\binom{k+1}{j+1}\sum_{n=0}^{j}\binom{j+2}{n}B_n \\ &= -(k-j)\binom{k+1}{j+1}B_{j+1},\end{aligned}$$

which is precisely the desired result. □

Lemma 2.2. *For all integers j, $0 \leq j < k$,*

$$Q_{N,k}(j) = Q_{N,k}(j+1).$$

Proof. Before beginning, in order to prevent heads from unnecessarily spinning, we briefly outline the process. We first suitably partition $Q_{N,k}(j)$ into two terms $A(j)$ and $B(j)$. Following this, with surgical precision we will introduce an auxiliary term $C(j)$ which in turn is split into two more terms $D(j)$ and $E(j)$ so that $A(j) + D(j) = A(j+1)$ and $E(j) + B(j) - C(j) = B(j+1)$. In the end, we will have

$$\begin{aligned}Q_{N,k}(j) &= A(j) + B(j) \\ &= A(j) + C(j) + B(j) - C(j) \\ &= [A(j) + D(j)] + [E(j) + B(j) - C(j)] \\ &= A(j+1) + B(j+1) \\ &= Q_{N,k}(j+1).\end{aligned} \tag{2.5}$$

Now for the gory details. Let

$$A(j) := \sum_{m=0}^{j}\binom{k+1}{m}(N+1)^{k-m}B_m,$$

$$B(j) := -\frac{1}{N+1}\sum_{m=0}^{k-j-1}\Theta_{j,k}(m)S_{N,m}[0,1],$$

so that $Q_{N,k}(j) = A(j) + B(j)$. Next, we define $C(j)$ to be the last term in $B(j)$, which corresponds to $m = k - j - 1$. In other words,

$$C(j) := -\frac{1}{N+1}\Theta_{j,k}(k-j-1)S_{N,k-j-1}[0,1].$$

Using our result from Lemma 2.1, we reformulate $C(j)$ to be

$$C(j) = \frac{k-j}{N+1}\binom{k+1}{j+1}S_{N,k-j-1}[0,1]\,B_{j+1}. \tag{2.6}$$

Subsequently, we apply the recursive formula from Corollary 2.1 to the sum $S_{N,k-j-1}[0,1]$ and obtain

$$S_{N,k-j-1}[0,1] = \frac{1}{k-j}\left[(N+1)^{k-j} - \sum_{m=0}^{k-j-2}\binom{k-j}{m}S_{N,m}[0,1]\right]. \tag{2.7}$$

Inserting (2.7) into (2.6), we are left with the following decomposition of $C(j)$:

$$\begin{aligned} C(j) &=: D(j) + E(j) \\ &= \binom{k+1}{j+1}(N+1)^{k-(j+1)}B_{j+1} + \frac{-B_{j+1}}{N+1}\sum_{m=0}^{k-j-2}\binom{k-j}{m}\binom{k+1}{j+1}S_{N,m}[0,1]. \end{aligned}$$

Notice that adding $D(j)$ to $A(j)$ gives $A(j+1)$. Explicitly, we have

$$\begin{aligned} A(j) + D(j) &= \sum_{m=0}^{j}\binom{k+1}{m}(N+1)^{k-m}B_m + \binom{k+1}{j+1}(N+1)^{k-(j+1)}B_{j+1} \\ &= \sum_{m=0}^{j+1}\binom{k+1}{m}(N+1)^{k-m}B_m \\ &= A(j+1). \end{aligned}$$

Furthermore, adding $E(j)$ to $B(j) - C(j)$ gives $B(j+1)$. Indeed,

$$\begin{aligned} E(j) + (B(j) - C(j)) &= \frac{-1}{N+1}B_{j+1}\sum_{m=0}^{k-j-2}\binom{k-j}{m}\binom{k+1}{j+1}S_{N,m}[0,1] + \\ &\quad + \frac{-1}{N+1}\sum_{m=0}^{k-j-2}\Theta_{j,k}(m)S_{N,m}[0,1] \\ &= \frac{-1}{N+1}\sum_{m=0}^{k-j-2}\left[\binom{k+1}{j+1}\binom{k+1-(j+1)}{m}B_{j+1} + \Theta_{j,k}(m)\right]S_{N,m}[0,1] \\ &= \frac{-1}{N+1}\sum_{m=0}^{k-(j+1)-1}\Theta_{j+1,k}(m)S_{N,m}[0,1] \\ &= B(j+1). \end{aligned}$$

This completes the proof as previously outlined in Equation (2.5). ☐

We are now ready to present two different proofs of Faulhaber's eponymous Formula. The first is a direct proof which, as far as the authors are aware, is a novel approach to the problem. Thanks to the work put into Lemmas 2.1 and 2.2, it is both short and elegant. The second proof invokes the principle of strong induction on k, and relies on the recursion formula of Corollary 2.1. Faulhaber's Formula is well-known, yet is instrumental in proving the first of our two main results, a Faulhaber-type formula for $S_{N,k}[a,r]$ which utilizes the Bernoulli numbers.

Theorem 2.2 (Faulhaber's Formula)**.** *For any $k \geq 0$,*

$$S_{N,k}[0,1] = \sum_{n=0}^{N} n^k = \frac{N+1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (N+1)^{k-m} B_m. \tag{2.8}$$

Remark 2.1. *It is worth nothing that it is common to use N instead of $N+1$ in* (2.8) *when B_1 is taken to be $1/2$ instead of $-1/2$. That is, for any $k \geq 0$, if $B_1 = 1/2$, then*

$$S_{N,k}[0,1] = \sum_{n=0}^{N} n^k = \frac{1}{k+1} \sum_{m=0}^{k} \binom{k+1}{m} N^{k-m+1} B_m. \tag{2.9}$$

Direct Proof. On the one hand, the recursion formula of Corollary 2.1 is equivalent to the equation

$$Q_{N,k}(0) = \frac{k+1}{N+1} S_{N,k}[0,1].$$

On the other hand, using the recursive definition of the Bernoulli numbers from Equation (2.4), we observe that

$$\begin{aligned}
\Theta_{k-1,k}(0) &= \sum_{n=0}^{k-1} B_n \binom{k+1}{n} \binom{k+1-n}{0} \\
&= \sum_{n=0}^{k-1} B_n \binom{k+1}{n} \\
&= -(k+1) B_k \\
&= -\binom{k+1}{k} (N+1)^{k-k} B_k.
\end{aligned}$$

Consequently, after recalling that $S_{N,0}[0,1] = N+1$, we arrive at

$$\begin{aligned} Q_{N,k}(k-1) &= \sum_{m=0}^{k-1} \binom{k+1}{m} (N+1)^{k-m} B_m - \frac{1}{N+1} \sum_{m=0}^{k-(k-1)-1} \Theta_{k-1,k}(m) S_{N,m}[0,1] \\ &= \sum_{m=0}^{k-1} \binom{k+1}{m} (N+1)^{k-m} B_m - \frac{1}{N+1} \Theta_{k-1,k}(0)\, S_{N,0}[0,1] \\ &= \sum_{m=0}^{k-1} \binom{k+1}{m} (N+1)^{k-m} B_m + \binom{k+1}{k} (N+1)^{k-k} B_k \\ &= \sum_{m=0}^{k} \binom{k+1}{m} (N+1)^{k-m} B_m. \end{aligned}$$

Finally, by the invariance of $Q_{N,k}(j)$ under j, which we painstakingly established in Lemma 2.2, we conclude

$$Q_{N,k}(0) = Q_{N,k}(k-1),$$

from which the desired result follows immediately. □

And now we present our strong induction proof of Theorem 2.2.

Strong Induction Proof. In the base case where $k=0$, we know $S_{N,0}[0,1] = (N+1)$. Keeping in mind that $B_0 = 1$, the right-hand side of Equation (2.8) is

$$\frac{N+1}{0+1} \binom{0+1}{0} (N+1)^{0-0} B_0 = N+1,$$

hence the base case is proved. Next, suppose that for some $K > 0$ Equation (2.8) holds whenever $0 \le k \le K-1$. We will show that the result also holds when $k = K$. Indeed, by Corollary 2.1, we have

$$S_{N,K}[0,1] = \frac{(N+1)^{K+1}}{K+1} - \frac{1}{K+1} \sum_{m=0}^{K-1} \binom{K+1}{m} S_{N,m}[0,1]. \tag{2.10}$$

We now use the induction hypothesis, reverse the order of summation a la Gauss, and utilize the identity

$$\frac{1}{m+\ell+1} \binom{K+1}{m+\ell} \binom{m+\ell+1}{m} = \frac{1}{\ell+1} \binom{K+1}{\ell} \binom{K-\ell+1}{m},$$

to compute

$$\begin{aligned}
\sum_{m=0}^{K-1}\binom{K+1}{m}S_{N,m}[0,1] &= \sum_{m=0}^{K-1}\binom{K+1}{m}\left[\frac{N+1}{m+1}\sum_{\ell=0}^{m}\binom{m+1}{\ell}(N+1)^{m-\ell}B_\ell\right]\\
&= \sum_{m=0}^{K-1}\binom{K+1}{m}\left[\frac{N+1}{m+1}\sum_{\ell=0}^{m}\binom{m+1}{m-\ell}(N+1)^{\ell}B_{m-\ell}\right]\\
&= \sum_{\ell=0}^{K-1}(N+1)^{\ell+1}\sum_{m=\ell}^{K-1}\frac{1}{m+1}\binom{K+1}{m}\binom{m+1}{m-\ell}B_{m-\ell}\\
&= \sum_{\ell=0}^{K-1}(N+1)^{\ell+1}\sum_{m=0}^{K-\ell-1}\frac{1}{m+\ell+1}\binom{K+1}{m+\ell}\binom{m+\ell+1}{m}B_m\\
&= \sum_{\ell=0}^{K-1}\binom{K+1}{\ell}\frac{(N+1)^{\ell+1}}{\ell+1}\sum_{m=0}^{K-\ell-1}\binom{K-\ell+1}{m}B_m.
\end{aligned}$$

At this stage, we use the definition of the Bernoulli numbers from Equation (2.4) to write

$$\sum_{m=0}^{K-1}\binom{K+1}{m}S_{N,m}[0,1] = -\sum_{\ell=0}^{K-1}\binom{K+1}{\ell}\frac{(N+1)^{\ell+1}}{\ell+1}(K-\ell+1)B_{K-\ell}.$$

Additionally, multiplication by a clever choice of 1 yields

$$(N+1)^{K+1} = \binom{K+1}{K}\frac{(N+1)^{K+1}}{K+1}(K-K+1)B_{K-K},$$

which, when substituted into Equation (2.10) above, enables us to conclude

$$\begin{aligned}
S_{N,K}[0,1] &= \frac{1}{K+1}\sum_{\ell=0}^{K}\binom{K+1}{\ell}\frac{(N+1)^{\ell+1}}{\ell+1}(K-\ell+1)B_{K-\ell}\\
&= \frac{1}{K+1}\sum_{\ell=0}^{K}\binom{K+1}{K-\ell}\frac{(N+1)^{K-\ell+1}}{K-\ell+1}(\ell+1)B_{\ell}\\
&= \frac{1}{K+1}\sum_{\ell=0}^{K}\binom{K+1}{\ell}(N+1)^{K-\ell+1}B_{\ell}\\
&= \frac{N+1}{K+1}\sum_{m=0}^{K}\binom{K+1}{m}(N+1)^{K-m}B_{m}.
\end{aligned}$$

This completes the induction proof. □

Our main result for this paper is a byproduct of all we have done thus far, but especially Equation (1.3) and Theorem 2.2.

Theorem 2.3 (Faulhaber's Formula for $S_{N,k}[a,r]$). *For all $a, r \in \mathbb{R}$, we have*

$$S_{N,k}[a,r] = \sum_{m=0}^{k}\binom{k}{m}\frac{a^{k-m}r^m}{m+1}\left[\sum_{j=0}^{m}\binom{m+1}{j}(N+1)^{m-j+1}B_j\right].$$

Proof. The result follows from Equation (1.3), which states

$$S_{N,k}[a,r] = \sum_{m=0}^{k} \binom{k}{m} a^{k-m} r^m S_{N,m}[0,1],$$

along with a straightforward use of Theorem 2.2 to replace the expression $S_{N,m}[0,1]$. □

There are a number of alternative formulations and proofs of Faulhaber's formula. They are proven using tools which include, but are not limited to, generating functions, matrix techniques, Bernoulli polynomials, the Stirling numbers of the first and second kinds, and finite discrete convolutions [21, 14, 23]. In this section, we have explored Faulhaber's Formula through the use of the Bernoulli numbers. In what follows, we turn our attention to the versatile theory of finite differences, which is applied extensively in fields as far-flung as statistics, combinatorics, numerical analysis, differential equations, and even particle physics. In particular, we make use of the finite discrete difference operator, which we introduce in the next section.

3. Faulhaber's Formula via Differencing

We have already seen that the sums of powers of integers can be calculated using a telescoping sum. We formalize this notion by introducing the (forward, finite) difference operator Δ whose action on a functions f is defined by [4]

$$\Delta[f](x) = f(x+1) - f(x).$$

Further, for m a nonnegative integer, the mth order difference operator Δ^m is given recursively by

$$\Delta^0[f] = f, \qquad \Delta^{m+1}[f] = \Delta[\Delta^m[f]], \quad m \geq 1.$$

For example,

$$\begin{aligned} \Delta[f](x) &= f(x+1) - f(x) \\ \Delta^2[f](x) &= f(x+2) - 2f(x+1) + f(x) \\ \Delta^3[f](x) &= f(x+3) - 3f(x+2) + 3f(x+1) - f(x). \end{aligned}$$

In general, we can express $\Delta^m[f]$ in the following manner.

Lemma 3.1. *For any function f and integer $m \geq 0$,*

$$\Delta^m[f](x) = \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} f(x+j).$$

Proof. We use induction on m. When $m = 0$, we have $\Delta^0[f](x) = f(x)$ by definition, and

$$\sum_{j=0}^{0}(-1)^{0-j}\binom{0}{j}f(x+j) = (-1)^0\binom{0}{0}f(x+0) = f(x),$$

thus the result holds in the base case. Now, suppose the result holds for some natural number m. Recalling Pascal's identity,

$$\binom{m}{j-1} + \binom{m}{j} = \binom{m+1}{j},$$

we compute

$$\begin{aligned}
\Delta^{m+1}[f](x) &= \Delta\left[\Delta^m[f]\right](x) \\
&= \Delta\left[\sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(\cdot+j)\right](x) \\
&= \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+1+j) - \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}f(x+j) \\
&= \sum_{j=0}^{m+1}(-1)^{m+1-j)}\binom{m}{j-1}f(x+j) + \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m}{j}f(x+j) \\
&= \sum_{j=0}^{m+1}(-1)^{(m+1)-j)}\left[\binom{m}{j-1} + \binom{m}{j}\right]f(x+j) \\
&= \sum_{j=0}^{m+1}(-1)^{m+1-j}\binom{m+1}{j}f(x+j).
\end{aligned}$$

This completes the induction proof. □

Since it will be relevant later in the discussion, we also present a special case of Lemma 3.1 as a corollary.

Corollary 3.1. *If $P(x) = (a+rx)^k$, then*

$$\Delta^m[P](0) = \sum_{j=0}^{m}(-1)^{m-j}\binom{m}{j}(a+jr)^k. \tag{3.11}$$

From here on, we focus our attention to the case in which Δ is acting on a polynomial P, despite the richness of the more general theory [4]. It is not difficult to see that the operator Δ reduces the degree of a polynomial P by 1. That is to say, if P is a polynomial of degree $\deg(P)$, then $\Delta[P]$ is a polynomial of degree $\deg(P)-1$. This follows easily from the binomial theorem. Indeed,

$$(x+1)^d - x^d = \sum_{j=0}^{d-1}\binom{d}{j}x^j.$$

Furthermore, for any integer $m > \deg(P)$, we observe that $\Delta^m[P] = 0$.

One important advantage of using difference operators is it gives us the ability to write polynomials evaluated at integer inputs, say $n \geq 0$, as

$$P(n) = \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j}. \tag{3.12}$$

This special case of a far more general result (called the Gregory-Newton interpolation formula, first published by Newton in 1687 [13]) can be easily verified by a robust and interesting use of induction on n. Indeed, in the base case of $n = 0$, both sides of Equation (3.12) are equal to $P(0)$. Now assume Equation (3.12) holds true for some $n \geq 0$. Then, noticing that $P(n+1) = P(n) + \Delta[P](n)$, and appealing to Pascal's identity and the fact that $\Delta^{\deg(P)+1}[P] = 0$, we can conclude

$$\begin{aligned}
P(n+1) &= P(n) + \Delta[P](n) \\
&= \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j} + \sum_{j=0}^{\deg(P)} \Delta^{j+1}[P](0) \binom{n}{j} \\
&= \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j} + \sum_{j=1}^{\deg(P)} \Delta^{j}[P](0) \binom{n}{j-1} \\
&= P(0) + \sum_{j=1}^{\deg(P)} \Delta^j[P](0) \left[\binom{n}{j} + \binom{n}{j-1}\right] \\
&= P(0) + \sum_{j=1}^{\deg(P)} \Delta^j[P](0) \binom{n+1}{j} \\
&= \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n+1}{j}.
\end{aligned}$$

Now that Equation (3.12) has been verified, we are ready to take the plunge back into our original problem. Recall that we are seeking a way to reduce the number of computations required to evaluate the sums $S_{N,k}[a,r]$, which are merely sums of the form $\sum_{n=0}^{N} P(n)$ for a particular polynomial P. Armed with Equation (3.12), and the aptly named "hockey stick identity," that is,

$$\sum_{n=0}^{N} \binom{n}{j} = \sum_{n=j}^{N} \binom{n}{j} = \binom{N+1}{j+1}, \tag{3.13}$$

we can convert a sum $\sum_{n=0}^{N} P(n)$ with $N+1$ terms into a sum with only $\deg(P)+1$ terms. This can be quite advantageous if N is significantly greater than $\deg(P)$. Indeed, interchanging the finite sums in n and j allows us to write [4]

$$\sum_{n=0}^{N} P(n) = \sum_{n=0}^{N} \left[\sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{n}{j} \right] = \sum_{j=0}^{\deg(P)} \Delta^j[P](0) \binom{N+1}{j+1}. \tag{3.14}$$

Let us consider a simple polynomial: the monic monomial $P(n) = n^k$ of degree $k \geq 1$. In light of Corollary 3.1, it is not difficult to see that we can write

$$\Delta^j[n^k](0) = j! \left\{ {k \atop j} \right\},$$

where $\left\{ {k \atop j} \right\}$ denotes the Stirling number of the second kind given by

$$\left\{ {k \atop j} \right\} = \frac{1}{j!} \sum_{\ell=0}^{j} (-1)^{j-\ell} \binom{j}{\ell} \ell^k, \qquad j \geq 0.$$

In view of Equation (3.14) and the fact $\left\{ {k \atop 0} \right\} = 0$, we obtain the formula

$$\sum_{n=1}^{N} n^k = \sum_{j=1}^{k} j! \left\{ {k \atop j} \right\} \binom{N+1}{j+1}. \tag{3.15}$$

We now illustrate the utility of the formulas discussed above by way of a concrete example.

Example 3.1. *Let $P(n) = n^2$. We have*

$$\Delta[n^2](0) = 1! \left\{ {2 \atop 1} \right\} = 1, \quad \textit{and} \quad \Delta^2[n^2](0) = 2! \left\{ {2 \atop 2} \right\} = 2,$$

and hence, by (3.12) *we are able to write n^2 as the sum*

$$n^2 = \Delta P(0) \binom{n}{1} + \Delta^2 P(0) \binom{n}{2} = \binom{n}{1} + 2\binom{n}{2}.$$

In light of Equation (3.13)*, we have recovered the well-known formula*

$$\sum_{n=0}^{N} n^2 = \sum_{j=1}^{2} j! \left\{ {2 \atop j} \right\} \binom{N+1}{j+1} = \binom{N+1}{2} + 2\binom{N+1}{3} = \frac{N(N+1)(2N+1)}{6}.$$

We are now ready to present yet another Faulhaber-type formula, one which does not make use of the Bernoulli numbers, but makes use of the terms of the original sum $S_{N,k}[a,r]$, rather than being written in powers of a, r, and $N+1$.

Theorem 3.1 (Faulhaber-type Formula for $S_{N,k}[a,r]$)**.** *For all $a, r \in \mathbb{R}$,*

$$S_{N,k}[a,r] = \sum_{m=0}^{k} \left[\binom{N+1}{m+1} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (a+jr)^k \right]. \tag{3.16}$$

Proof. Letting $P(n) = (a + rn)^k$ in Corollary 3.1, and substituting into Equation 3.14, we get

$$S_{N,k}[a,r] = \sum_{n=0}^{N} (a+nr)^k = \sum_{n=0}^{N} P(n) = \sum_{m=0}^{k} \Delta^m[P](0) \binom{N+1}{m+1}$$
$$= \sum_{m=0}^{k} \left[\left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (a+jr)^k \right) \binom{N+1}{m+1} \right]$$

from which we get the formula in Equation 3.16, completing the proof. □

As a parting gift, we illustrate the utility of Theorems 2.3 and 3.1 with a worked example. One common aspect of the calculations which should be noted is the number of terms required to evaluate the sum. In fact, a careful inspection of the two formulas reveals that they both require that $(k+1)(k+2)/2$ terms be taken into account. In the example presented below, we have $k = 2$, and there are indeed $(3)(4)/2 = 6$ terms in both calculations. This is far better than the 101 terms it would take in order to calculate the sum directly.

Example 3.2. *Let us compute*

$$S_{100,2}[5,3] = \sum_{n=0}^{100} (5+3n)^2 = 5^2 + 8^2 + 11^2 + \cdots + 305^2.$$

Recalling the values $B_0 = 1$, $B_1 = -1/2$, and $B_2 = 1/6$, Theorem 2.3 states

$$S_{100,2}[5,3] = \sum_{m=0}^{2} \binom{2}{m} \frac{5^{2-m}3^m}{m+1} \left[\sum_{j=0}^{m} \binom{m+1}{j} 101^{m+1-j} B_j \right]$$
$$= \binom{2}{0} \frac{5^2 3^0}{1} \left[\sum_{j=0}^{0} \binom{1}{j} 101^{1-j} B_j \right] + \binom{2}{1} \frac{5^1 3^1}{2} \left[\sum_{j=0}^{1} \binom{2}{j} 101^{2-j} B_j \right]$$
$$+ \binom{2}{2} \frac{5^0 3^2}{3} \left[\sum_{j=0}^{2} \binom{3}{j} 101^{3-j} B_j \right]$$
$$= 2525 B_0 + 15 \left[\binom{2}{0} 101^2 B_0 + \binom{2}{1} 101^1 B_1 \right]$$
$$+ 3 \left[\binom{3}{0} 101^3 B_0 + \binom{3}{1} 101^2 B_1 + \binom{3}{2} 101^1 B_2 \right]$$
$$= 3(101^3) - \frac{3}{2}(101^2) - \frac{1}{2}(101)$$
$$= 3,199,175,$$

while the formula from Theorem 3.1 yields

$$\begin{aligned} S_{100,2}[5,3] &= \sum_{m=0}^{2} \left[\binom{101}{m+1} \sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} (5+3j)^2 \right] \\ &= \binom{101}{1}(-1)^0\binom{0}{0}5^2 + \binom{101}{2}\left((-1)^1\binom{1}{0}5^2 + (-1)^0\binom{1}{0}8^2\right) \\ &\quad + \binom{101}{3}\left((-1)^2\binom{2}{0}5^2 + (-1)^1\binom{2}{1}8^2 + (-1)^0\binom{2}{2}11^2\right) \\ &= 3,199,175. \end{aligned}$$

As expected, the two results above are in agreement. It is also worth noting that the second calculation from Theorem 3.1 is a more robust and practical choice, as it avoids the use of Bernoulli numbers.

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References

[1] Archimedes. (circa 225 BCE). *On Conoids and Spheroids*.

[2] Burton, D. (2010). *The History of Mathematics: An Introduction* (7th ed.). McGraw Hill.

[3] Gould, S. H. (1955). The method of Archimedes. *American Mathematical Monthly, 62*, 473–476.

[4] Brualdi, R. A. (2012). *Introductory Combinatorics*. Pearson Education.

[5] Nicomachus (of Gerasa). (1926). *Introduction to Arithmetic* (M. L. D'Ooge, F. E. Robbins, & L. C. Karpinski, Eds.). University of Michigan Press. Retrieved from https://books.google.com/books?id=Ut7uAAAAMAAJ

[6] Stein, S. (1999). *Archimedes: What did he do besides cry Eureka?*. Mathematical Association of America.

[7] Mahoney, M. S. (1994). *The mathematical career of Pierre de Fermat, 1601–1665* (2nd ed.). Princeton University Press.

[8] Faulhaber, J. (1631). *Academia Algebrae*.

[9] Jacobi, C. (1834). De usu legitimo formulae summatoriae Maclaurinianae. *Journal für die reine und angewandte Mathematik, 12*, 263–272.

[10] Smith, D. E., & Mikami, Y. (2004). *A History of Japanese Mathematics*. Dover Publications. Retrieved from https://books.google.com/books?id=pTcQsvfbSu4C

[11] Kitagawa, T. L. (2022). The origin of the Bernoulli numbers: mathematics in Basel and Edo in the early eighteenth century. *Mathematical Intelligencer, 44*(1), 46–56. https://doi.org/10.1007/s00283-021-10072-y

[12] Pascal, B. (1653). *Traité du triangle arithmétique, avec quelques autres petits traitez sur la mesme matière.* Published posthumously in 1665.

[13] Newton, I. (1687). *Principia* (Vol. 3).

[14] Merca, M. (2015). An alternative to Faulhaber's formula. *American Mathematical Monthly, 122*(6), 599–601. https://doi.org/10.4169/amer.math.monthly.122.6.599

[15] Dunham, W. (1990). *Journey Through Genius: Great Theorems of Mathematics*. Wiley. Retrieved from https://books.google.com/books?id=Jx9Z7ONLn70C

[16] Zielinski, R. (2022). Faulhaber's formula, odd Bernoulli numbers, and the method of partial sums. *Integers, 22*, Paper No. A78, 11. https://doi.org/10.1177/1471082x221076884

[17] Edwards, A. W. F. (1986). A quick route to sums of powers. *American Mathematical Monthly, 93*(6), 451–455. https://doi.org/10.2307/2323466

[18] Edwards, A. W. F. (1982). Sums of powers of integers: a little of the history. *Mathematical Gazette, 66*(435), 22–28. https://doi.org/10.2307/3617302

[19] Burrows, B. L., & Talbot, R. F. (1984). Sums of powers of integers. *American Mathematical Monthly, 91*(7), 394–403. https://doi.org/10.2307/2322985

[20] Beardon, A. F. (1996). Sums of powers of integers. *American Mathematical Monthly, 103*(3), 201–213. https://doi.org/10.2307/2975368

[21] Knuth, D. E. (1993). Johann Faulhaber and sums of powers. *Mathematics of Computation, 61*(203), 277–294. https://doi.org/10.2307/2152953

[22] Rademacher, H. (1973). *Topics in analytic number theory* (E. Grosswald, J. Lehner, & M. Newman, Eds.; Vol. 169). Springer-Verlag.

[23] Zielinski, R. (2019). Faulhaber and Bernoulli. *Fibonacci Quarterly, 57*(1), 32–34.

[24] Gessel, I. M., & Viennot, X. G. (1989). Determinants, paths, and plane partitions. 1989 preprint.

[25] Cereceda, J. L. (2015). Explicit form of the Faulhaber polynomials. *College Mathematics Journal, 46*(5), 359–363. https://doi.org/10.4169/college.math.j.46.5.359

[26] Cereceda, J. L. (2021). Bernoulli and Faulhaber. *Fibonacci Quarterly, 59*(2), 145–149.

[27] Beery, J. (2010, July). Sums of powers of positive integers. *Convergence*. https://doi.org/10.4169/loci003284

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THE EXISTENCE AND IDENTIFICATION OF STRONGLY CONNECTED EDGE DIRECTION ASSIGNMENTS IN BRIDGELESS GRAPHS AND MULTIGRAPHS

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ABSTRACT. The aim of this paper is two-fold. First, we will provide clarity on a result concerning the strong connectivity, a concept whose usefulness is readily apparent in several fields of study including social networking and transport networks, of a bridgeless connected graph achieved through the depth-first search (DFS) technique. To this end, we will demonstrate two rigorous mathematical proofs of this robust and well-known result. One proof takes the approach of seeking a contradiction by investigating the relationship between directed paths and maximal strongly connected subgraphs after the application of DFS. The other proof features a direct approach that demonstrates that for each tree edge $\{U, V\}$, there is a directed path from V to U by utilizing the fact that each edge in a connected multigraph on at least two vertices is either a bridge or is included in some cycle. Second, for a multigraph without a bridge, we provide two different proofs ensuring the existence of an assignment of edge directions that induces strong connectivity. One of these proofs utilizes the previous fact, whereas the second proof is independent of it and features a technique that focuses on collapsing entire connected multigraphs into a single vertex.

Keywords: Graph theory, Bridgeless multigraphs, Depth-first search, Strong connectivity.

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1. Introduction

Graph theory has rich history dating back to 1736, when Leonhard Euler published a paper using this versatile branch of mathematics to approach a problem concerning the seven bridges of Königsberg [8, 17]. Today, graph theory has garnered interest and found applications in other branches of mathematics as well as in several disciplines within academia. Specifically, some applications of graph theory include additive number theory [1], cryptography [28], molecular topology [5], Alzheimer's Disease [12], algebra [29], the study of DNA and biological networks [27], spectroscopy and quantum chemistry [7], chemistry [6, 16], social media and social networking [10, 24, 2, 4], blockchain technologies [22], social trust models [31], maze solving [18, 23, 25] and GPS networks [19]. In particular, we would like to examine the applications of graph theory in computer science as well as the inherent mathematical beauty therein.

In the realm of graph theory and the computational sciences, various algorithms, such as Depth-First Search (DFS), Breadth-First Search, Dijkstra's Algorithm, and Floyd-Warshall's Algorithm [11, 17, 15] play important roles in understanding graph structures. Each of these algorithms have a variety of applications. For more information pertaining to some applications of these algorithms, see [11, 21] and the references therein. Particularly, the Depth-First Search Algorithm serves as a means of graph traversal for the sake of identifying vertices and their relationships to underlying structures embedded within a given graph and an associated directed graph. This algorithm commences its journey from the *root*, an arbitrarily selected vertex from the given graph, and thoroughly explores each and every vertex as far as possible before traversing its moves backward.

Ever since a version of DFS was introduced as a means of solving mazes [18], it has been widely regarded as a versatile tool for approaching problems in both theory and practice pertaining to, for example, finding strongly connected components in a directed graph [11] and topological sorting [11]. With regard to applications in the computational sciences, DFS sees use in various areas including, but not limited to, image recognition [3] and computing search trees [30]. For further information concerning the implementation and execution of DFS in the computational sciences and the associated data structures, see [18]. Before we continue, we will revisit some graph theoretic definitions that will assist us in this article. Additionally, we will provide some motivating examples for some of the concepts introduced.

In this paper, a graph is a simple graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} \neq \emptyset$ is a nonempty set of vertices and $\mathcal{E}$ is a set of edges. A *multigraph* is a graph that can have *parallel edges* as well as *loops*, neither of which are possessed by (simple) graphs. Two vertices U and V are said to be *adjacent* if there is an edge between them. The *degree* of a vertex $V \in \mathcal{V}$ is the number of vertices to which V is adjacent and is denoted $\deg(V)$. Sometimes, we wish to assign directions to the edges of a graph. To see an example of this, let us recall the *Collatz Conjecture*. That is, let us define the function $C : \mathbb{N} \to \mathbb{N}$ given by

$$C(n) = \begin{cases} 3n+1, & \text{if } n \text{ is odd} \\ n/2, & \text{if } n \text{ is even.} \end{cases}$$

The Collatz Conjecture states that for any $n \in \mathbb{N}$, repeated applications of C will eventually result in 1. For example, let $n = 6$. Observe that

$$C^8(6) = C^7(3) = C^6(10) = C^5(5) = C^4(16) = C^3(8) = C^2(4) = C(2) = 1.$$

Now, observe that we can represent this repeated application of C as a graph whose edges represent the notion that adjacent numbers have the property that one of the numbers is the result of applying C to the other. To indicate which number is obtained from applying C to another number, we can use arrows as is done in Figure 1

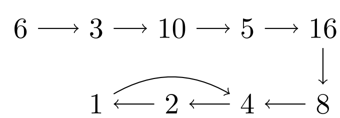

FIGURE 1. A graphical visualization of applying C to $n = 6$.

Above is an example of a directed graph. A *directed graph* (or *digraph*) is a graph whose edges are each assigned a direction. A graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ is called a *subgraph* of another graph $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2)$ if $\mathcal{V}_1 \subseteq \mathcal{V}_2$ and $\mathcal{E}_1 \subseteq \mathcal{E}_2$. If this is the case, then we can write $\mathcal{G}_1 \subseteq \mathcal{G}_2$. If we have a graph $\mathcal{G}$ along with a nonempty set $X \subseteq \mathcal{V}(\mathcal{G})$, then the subgraph of $\mathcal{G}$ *induced* by X is the graph with vertex set X and edge set consisting of all edges $\{U, V\}$ with both U and V elements of X.

Let us again consider the graph depicted in Figure 1. Suppose that each number denotes a particular building in a city. Suppose further that if two buildings are joined by an edge, then there exists a one-way, road, whose direction is dictated by the direction of the edge, connecting these buildings. For example, one could travel from building 10 to building 4 through the roads creating the sequence $10 \to 5 \to 16 \to 8 \to 4$. However, in the way that these roads are directed, one would be unable to travel from building 4 to building 10. Let us make the graph-theoretic phenomenon in this example more precise.

The *connectivity* of an undirected graph depends on its vertices' capacity to reach one another along the edges of the graph. A *walk* is a sequence of adjacent vertices and can be thought of as moving between the vertices of a graph along the edges. A *closed walk* is a walk that begins and ends on the same vertex. A *path* is a walk that does not repeat vertices. A *cycle* is a closed path of the form $\{V_1, \ldots, V_k, V_{k+1} = V_1\}$. A *chord* is an edge that is not contained within a cycle, but joins two vertices within said cycle. It is said that two vertices are *connected* if there exists a path between them. It is said that a graph $\mathcal{G}$ is *connected* if for any two vertices $u, v \in \mathcal{V}(\mathcal{G})$, u and v are connected. Furthermore, a graph is called *disconnected* if it is not connected. For a connected graph $\mathcal{G}$, an edge $e \in \mathcal{E}(\mathcal{G})$ is called a *bridge* if the graph obtained by removing the edge e from the graph G is disconnected. On the other hand, the concept of strong connectivity is applicable solely to directed graphs. In the context of a directed graph, a graph $\mathcal{G}$ is *strongly connected* if and only if there is a directed path from a vertex V to a vertex U, as well as a directed path from U to V, for each pair of vertices U and V in $\mathcal{V}(\mathcal{G})$. The study of strongly connected graphs has various applications as well as additional routes for further inquiry in, for example, disciplines concerned with the reduction of complexity in certain problems [9, 11, 13]. A *strongly connected component* of a directed graph is a subgraph that is maximal with respect to the property of being strongly connected [14]. For an example of the study of strongly connected graphs and how it pertains to social networking, see [14]. For an example of how the study of strongly connected graphs can be considered in the study of public transport networks as well as their efficiency, see [26].

Recalling the Collatz Conjecture, let us consider a directed graph whose vertices consist of all elements of $\mathbb{N}$ and whose edges are constructed in the following way. Suppose $m, n \in \mathbb{N}$ are such that $C(m) = n$. Then we construct a directed edge from m to n. Let us refer to this graph as the *Collatz Graph*. Then observe that disproving the Collatz Conjecture could be simplified to locating a directed cycle in the Collatz Graph other than the cycle $\{4, 2, 1, 4\}$. Using our new terminology, we can again consider the graph in Figure 1 as a network of buildings in a city. Recall that there exists a path from 10 to 4 but that there does not exist a path from 4 to 10. From this, we can conclude that this graph is not strongly connected. Why is this? In fact, there is a direct relationship between the existence of bridges in a graph and the existence of an edge direction assignment that makes the resulting directed graph strongly connected. Observe that in the graph in Figure 1, every edge is a bridge other than the edges of the cycle $\{4, 2, 1, 4\}$. However, if we simply add the directed edge $\{1, 6\}$, we obtain the following graph.

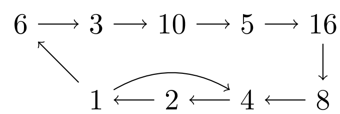

FIGURE 2. The graph in Figure 1 along with the directed edge $\{1, 6\}$.

Notice that in this graph, for each pair of vertices, there exists a directed path from one to the other. As such, we can conclude that this graph is strongly connected. Observe further that there does not exist a bridge in this graph. That is, the removal of any edge will not leave the resulting graph disconnected. It may, however, leave the resulting directed graph no longer strongly connected. In this paper, we will further explore the connection between the lack of bridges in a connected graph, and that graph's potential to have a strongly connected edge direction assignment. Thus, notice that if we think of the graph in Figure 2 as a transportation network with the directed edges representing one-way roads and the vertices representing buildings, one could travel between any two buildings. Let us wrap up our discussion of graph-theoretic terminology with a brief discussion about trees.

A *tree* is a connected graph containing no cycles. It can be observed that the number of vertices in a tree is one more than the number of edges. The converse is not necessarily true. For $j = i, \ldots, k-1$, V_j is called the parent of V_{j+1} in the DFS tree, and the edge (V_k, V_i) is called a *back-edge*. See [17, 20] for the properties of trees and rooted trees. Observe that every edge of a tree is a bridge. As an example, referring back to Figure 1, if we think of the cycle $\{4, 2, 1, 4\}$ as a single vertex $\boxed{4}$. Then the resulting graph would be as follows.

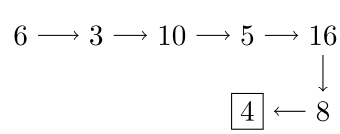

FIGURE 3. A graphical visualization of collapsing the cycle $\{4, 2, 1, 4\}$ into a single vertex $\boxed{4}$.

Observe that this graph is a tree, and that, indeed, each edge of this graph is a bridge and that this graph is not strongly connected. We will utilize this concept of collapsing subgraphs into a single vertex in a later proof. For this, see our second proof of Theorem 6.1.

This paper is organized in the following manner. In Section 2, we provide a motivating example demonstrating the usefulness of strong connectivity in network survivability and how it relates to trust networks. In Section 3, after recalling the Depth-First Search Algorithm, we discuss Theorem 3.1, concerning a technique for constructing a strongly connected edge direction assignment in a connected bridgeless graph, by presenting an example demonstrating its applicability. In Section 4, after presenting important facts concerning strong connectivity and cycles, we provide a proof of Theorem 3.1 by contradiction. In Section 5, we provide a second proof of Theorem 3.1 after proving Lemma 5.1, which states that each edge in a connected multigraph on at least two vertices is either a bridge or is included in a cycle. In Section 6, we provide two proofs of Theorem 6.1, which states that in a connected, bridgeless multigraph, there exists an edge direction assignment that makes the resulting directed graph strongly connected. One of these proofs invokes Lemma 5.1 and the other does not. Finally, we present concluding remarks in Section 7.

2. A Motivating Example of Strong Connectivity

The study of graph and multigraph connectivity can be used in a variety of real-world applications including the study of *network survivability* [20]. Let $\mathcal{G}$ be a simple, undirected, connected graph. Let $\kappa_v(\mathcal{G})$ denote the *vertex connectivity* of $\mathcal{G}$, or the smallest number of vertices whose removal from $\mathcal{G}$ can disconnect $\mathcal{G}$ or turn it into the trivial graph on a single vertex [20]. Similarly, let $\kappa_e(\mathcal{G})$ denote the *edge connectivity* of $\mathcal{G}$, or the smallest number of edges whose removal from $\mathcal{G}$ can disconnect $\mathcal{G}$ [20]. Both $\kappa_v(\mathcal{G})$ and $\kappa_e(\mathcal{G})$ are used to assess the network survivability of a network, or "the capacity of a network to retain connections among its nodes after some edges or nodes are removed" [20]. To further explore the applications of graph and multigraph connectivity, we can consider the concept of a *fault-tolerant* communications network, which is a communications network that "has at least two alternative paths between each pair of vertices" [20]. For additional details regarding network survivability and fault-tolerant communications networks, see Chapter 5 of [20].

Let us consider the following motivating example for the sake of demonstrating the applicability of network survivability.

Example 2.1. *Suppose we have a simple, undirected, connected graph $\mathcal{G}$ whose vertices represent people and whose edges represent a friendship between two people. It would be reasonable to infer, that if two vertices are connected, then the two corresponding people can have information transferred between them. Suppose $\mathcal{G}$ is the graph represented in Figure 4.*

$$A — B — C — D$$

FIGURE 4. A group of people and their corresponding friendships as a graph $\mathcal{G}$.

Observe that this is a quite fragile network, as removing B or C from the group would preclude A and D from sharing information. That is, $\kappa_v(\mathcal{G}) = 1$. Similarly, if any two people choose to end their friendship with one another, then the graph will become disconnected, and thus, there will be at least two people who cannot share information. As such, $\kappa_e(\mathcal{G}) = 1$. However, if the people in this group acknowledge this fact about their network survivability, they can act to strengthen their friend group's ability to share information by suggesting that the people represented by vertices A and D form a friendship, as shown in Figure 5.

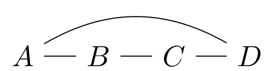

FIGURE 5. A stronger friendship network than in Figure 4.

In this graph, which we will call $\mathcal{G}'$, there is no vertex, nor edge, that can be removed to cause the resulting graph to be disconnected. Removing two vertices or edges, however, will cause the graph to be disconnected. As such, $\kappa_v(\mathcal{G}') = \kappa_e(\mathcal{G}') = 2$. We can further observe that the network represented by the graph in Figure 5 is fault-tolerant. Since there is no single edge in $\mathcal{G}'$ that disconnects the graph upon removal, we acknowledge that $\mathcal{G}'$ (and the friendship network represented therein) has stronger network survivability than $\mathcal{G}$ since no bridge exists in $\mathcal{G}'$. In fact, we observe that for a simple, undirected, connected graph $\mathcal{G}$, if $\kappa_e(\mathcal{G}) > 1$, then $\mathcal{G}$ does not contain a bridge.

Continuing with our supposition that the vertices of a graph $\mathcal{G}$ represent people and edges of $\mathcal{G}$ represent a friendship between two people, let us further assume that the friend group represented by $\mathcal{G}$ has a method of communication dependent on the trust one person has in another. That is, suppose that for any two vertices $v_1, v_2 \in \mathcal{V}(\mathcal{G})$, either the person represented by v_1 can receive information given by the person represented by v_2 or they can give information to the person represented by v_2. In this case, the graph becomes a *directed graph. Now, the friend group may be inclined to ponder whether or not they can share information throughout the network if they were to impose the trust-dependent structure described above. Asking this question would be equivalent to determining whether the graph representing the network could be made strongly connected. If $\kappa_e(\mathcal{G}) > 1$, then the answer to this question is "yes."*

3. DFS and its Relation to Strong Connectivity

One purpose of this paper is to provide two different mathematically rigorous proofs of Theorem 3.1, a well-known result, which appears in [17]. Although the result is fairly intuitive, the demonstration of this fact is quite intricate. Moving forward, we will include several figures to supplement the reader's understanding of the theorems, proofs, and applications discussed.

Before we begin, let us recall the Depth-First Search Algorithm which inductively operates on a graph with n vertices in the following manner.

(1) Start by picking any vertex from the graph. Label that vertex as V_1.
(2) Visit an adjacent vertex of the vertex labeled V_1 and label it V_2.
(3) Visit an adjacent unlabeled vertex of the vertex labeled V_2 and label it V_3.
(4) Continue this process until vertices have been labeled $V_1, V_2, \ldots, V_r$ and the vertex labeled V_r is not adjacent to an unlabeled vertex for $1 \leq r \leq n$.
(5) If $r < n$, select the largest i, $1 \leq i \leq r$, such that V_i is adjacent to an unlabeled vertex. Assign the label V_{r+1} to that vertex and return to step (4). Otherwise, if $r = n$, we are done.

Now, we can state the theorem, as referenced in [17], Theorem 5.8 on page 259.

Theorem 3.1. [17] *Suppose we apply depth-first search to a connected, bridgeless graph. If we assign directions to tree edges from lower depth-first search label to higher and to back edges from higher label to the lower, then the resulting directed graph is strongly connected.*

In order to comprehend this result, let us commence by examining an illustrative example that highlights the potency of Theorem 3.1.

Example 3.1. *Let us consider the connected graph $\mathcal{G}$ depicted in Figure 6, which does not possess a bridge.*

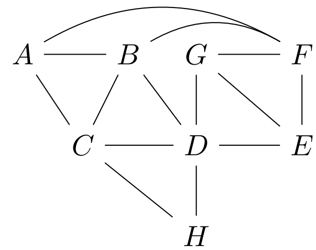

FIGURE 6. A connected graph $\mathcal{G}$ without a bridge.

Let us demonstrate the flexibility and versatility of DFS by exemplifying three distinct applications and the resulting strongly connected graphs. In any application of DFS, the designation of the root is arbitrary. As such, we could begin the algorithm on vertex A, B, C, D, or any other vertex. For the sake of simplicity and readability, we will start with A as our root in the following applications of DFS. Let us again consider the friendship and trust network as in Example 2.1. We see that through this application of DFS, we can determine a trust structure that would allow for strong connectivity, or in the example, a flow of information between all members of the friend group.

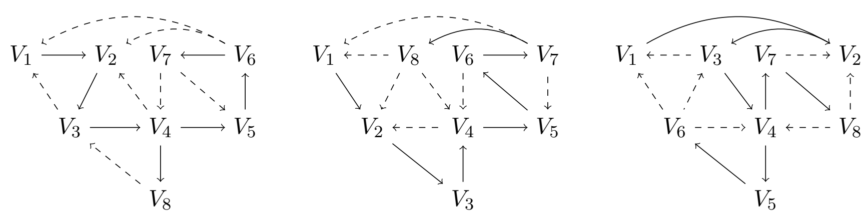

FIGURE 7. Three applications of DFS to $\mathcal{G}$ in Figure 6 with dashed back-edges.

Having applied DFS to $\mathcal{G}$, we can display the resulting strongly connected graphs using the directions assigned in Figure 7.

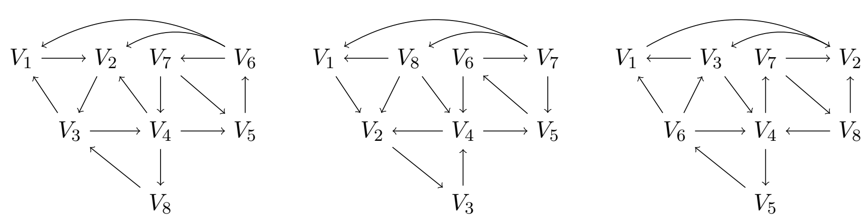

FIGURE 8. Three distinct strongly connected directed representations of $\mathcal{G}$ as in Figure 6.

We encourage the reader to ensure the strong connectivity of each graph presented in Figure 8 by confirming the existence of closed walks containing all of the vertices of $\mathcal{G}$ therein.

4. First Proof: A Contradiction

Prior to commencing our proof, we shall first revisit certain preliminary and relatively-easy-to-verify facts about strong connectivity of finite directed graphs. These simple facts are crucial for understanding the proofs we present in this article; therefore, we provide statements and straightforward examples for each fact for the sake of completeness. We encourage the reader to verify these facts for themself.

(I) A directed cycle is strongly connected.

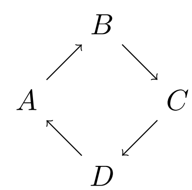

Figure 9. An illustration of Fact (I).

(II) A graph consisting solely of two directed cycles that share a common vertex (an "8" or "∞" shape) is strongly connected.

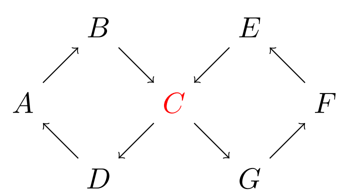

Figure 10. An illustration of Fact (II) with common vertex C.

(III) A graph consisting solely of two directed cycles that share a common directed edge is strongly connected.

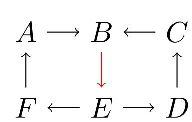

Figure 11. An illustration of Fact (III) with common directed edge $\{B, E\}$.

(IV) A graph containing two strongly connected directed subgraphs with some common vertex, or some common directed edge, is strongly connected.

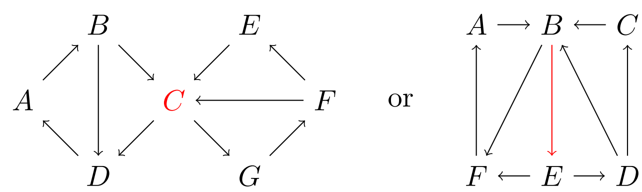

FIGURE 12. Two illustrations of Fact (IV).

Now that we have equipped ourselves with the appropriate mathematical instruments, let us discuss our first proof of Theorem 3.1.

Proof of Theorem 3.1. Let $V_1, V_2, \ldots, V_n$ be the labels of the depth-first search tree $\mathcal{T}$ where i designates the label number of vertex V_i.

Now, we will follow these steps to execute the proof.

(1) $\{V_1, V_2\}$ is a tree edge, i.e., a directed path from V_1 to V_2.

(2) Let $\{V_1, V_2, \ldots, V_{n_1-1}, V_{n_1}\}$ be a longest directed path with consecutive labels starting from V_1 in $\mathcal{T}$ as in Figure 13.

$$V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_{n_1-1} \longrightarrow V_{n_1}$$

FIGURE 13. A longest directed path in a depth-first search.

Then, we first make the following observations:

(a) $\deg(V_{n_1}) = 1$ in $\mathcal{T}$. Indeed, if $\deg(V_{n_1}) > 1$ in $\mathcal{T}$, then

$$\{V_1, V_2, \ldots, V_{n_1-1}, V_{n_1}\}$$

would not be a longest directed path in a depth-first search. Consequently, V_{n_1} is not adjacent to any vertex in $\{V_{n_1+1}, V_{n_1+2}, \ldots, V_{n-1}, V_n\}$ according to the depth-first search.

(b) $\deg(V_{n_1}) > 1$ in $\mathcal{G}$. This must be the case because otherwise, $\{V_{n_1-1}, V_{n_1}\}$ would be a bridge.

(c) Hence, by (a) and (b), there is a vertex V_{b_1} in $\{V_1, V_2, \ldots, V_{n_1-1}\}$ such that $\{V_{n_1}, V_{b_1}\}$ is a back edge, and so $\{V_{b_1}, V_{b_1+1}, \ldots, V_{n_1-1}, V_{n_1}\}$ is a directed cycle as in Figure 14. Therefore, the directed cycle

$$\{V_{b_1}, V_{b_1+1}, \ldots, V_{n_1-1}, V_{n_1}\}$$

is a strongly connected subgraph of $\mathcal{G}$.

$$V_{b_1} \longrightarrow V_{b_1+1} \longrightarrow \cdots \longrightarrow V_{n_1-1} \longrightarrow V_{n_1}$$

FIGURE 14. A directed cycle of $\mathcal{G}$.

(d) Let s_1 be the smallest number less than or equal to b_1 and m_1 be the maximum number greater than or equal to n_1 such that
   (i) $\{V_1, V_2, \ldots, V_{s_1-1}, V_{s_1}\}$ is a directed path in $\mathcal{T}$ and
   (ii) $\mathcal{G}_1 = \{V_{s_1}, V_{s_1+1}, \ldots, V_{m_1-1}, V_{m_1}\}$ is a strongly connected subgraph of $\mathcal{G}$.

$$V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_{s_1-1} \longrightarrow \{V_{s_1}, V_{s_1+1}, \ldots, V_{m_1-1}, V_{m_1}\} = \mathcal{G}_1$$

FIGURE 15. A directed path to a maximal strongly connected subgraph $\mathcal{G}_1$ of $\mathcal{G}$.

(3) We shall prove that $s_1 = 1$ and $m_1 = n$, and so the graph induced by the vertices $\{V_1, V_2, \ldots, V_n\}$ is strongly connected.

Suppose that $m_1 < n$. Then there is an integer, denoted as i_1, which represents the largest label less than or equal to m_1, such that $\{V_{i_1}, V_{m_1+1}\}$ is a tree edge. By the depth-first search, there is no vertex in $\{V_{i_1+1}, V_{i_1+2}, \ldots, V_{m_1}\}$ adjacent to any vertex in $\{V_{m_1+1}, V_{m_1+2}, \ldots, V_n\}$.

First, we will demonstrate that $i_1 \geq s_1$. That is to say, $V_{i_1} \in \mathcal{G}_1$. In fact, if $i_1 < s_1$, then no vertex in $\mathcal{G}_1$ is adjacent to any vertex in $\{V_{m_1+1}, V_{m_1+2}, \ldots, V_n\}$. Hence there is a vertex V_m in $\mathcal{G}_1$ which is adjacent to a vertex V_s in $\{V_1, V_2, \ldots V_{s_1-2}\}$. This must be the case because otherwise, $\{V_{s_1-1}, V_{s_1}\}$ would be a bridge. Using this back edge $\{V_m, V_s\}$, we have a directed path $\{V_m, V_s, V_{s+1}, \ldots, V_{s_1-1}, V_{s_1}\}$.

$$\cdots \longrightarrow V_s \longrightarrow V_{s+1} \longrightarrow \cdots \longrightarrow V_{s_1-1} \longrightarrow V_{s_1} \longrightarrow \cdots \longrightarrow V_m \longrightarrow \cdots \longrightarrow V_{m_1} \longrightarrow \cdots$$

FIGURE 16. A larger directed cycle of $\mathcal{G}$ when $i_1 < s_1$.

Since $\mathcal{G}_1$, induced by $\{V_{s_1}, V_{s_1+1}, V_{s_1+2}, \ldots, V_{m_1}\}$, is a strongly connected subgraph of $\mathcal{G}$, there is a directed path from V_{s_1} to V_m, and so we have a directed cycle from V_m to V_s to V_{s_1} to V_m as shown in Figure 16. Therefore, if we had $i_1 < s_1$, then the subgraph induced by $\{V_s, V_{s+1}, \ldots, V_{s_1-1}, V_{s_1}, V_{s_1+1}, \ldots, V_{m_1}\}$ would be strongly connected by Fact (IV). This contradicts the selection of the minimum value s_1 and the maximum value m_1 in (2d). Therefore, we know that $i_1 \geq s_1$. Let

$$\{V_{i_1}, V_{m_1+1}, V_{m_1+2}, \ldots, V_{n_2}\}$$

be a longest directed path beginning with the tree edge $\{V_{i_1}, V_{m_1+1}\}$ and with consecutive labels $m_1+1, m_1+2, \ldots, n_2$ in $\mathcal{T}$.

$$\begin{array}{ccc} V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_{s_1-1} \longrightarrow & \{V_{s_1}, \ldots, V_{i_1}, \ldots, V_{m_1}\} \\ & \downarrow \\ V_{n_2} \longleftarrow \cdots \longleftarrow V_{m_1+2} \longleftarrow & V_{m_1+1} \end{array}$$

FIGURE 17. A directed path from V_{i_1} to the vertex V_{n_2}.

Then

(a) $\deg(V_{n_2}) = 1$ in $\mathcal{T}$ by the longest property. Hence, V_{n_2} is not adjacent to any vertex in $\{V_{n_2+1}, V_{n_2+2}, \ldots, V_n\}$ by the depth-first search.

(b) $\deg(V_{n_2}) > 1$ in $\mathcal{G}$. This must be the case because otherwise, $\{V_{n_2-1}, V_{n_2}\}$ would be a bridge.

(c) Hence, there is a vertex V_{b_2} in $\{V_1, V_2, \ldots V_{n_2-1}\}$ such that $\{V_{n_2}, V_{b_2}\}$ is a back edge.

Now we consider 3 cases: $b_2 < s_1$, $s_1 \leq b_2 \leq m_1$, and $b_2 > m_1$.

Case 1: $b_2 < s_1$. If this is the case, then

$$\{V_{i_1}, V_{m_1+1}, V_{m_1+2}, \ldots V_{n_2}, V_{b_2}, V_{b_2+1}, \ldots, V_{s_1}\}$$

is a directed path. Since $\mathcal{G}_1$, the graph induced by

$$\{V_{s_1}, V_{s_1+1}, V_{s_1+2}, \ldots, V_{m_1}\}$$

is a strongly connected subgraph, there is a directed path from V_{s_1} to V_{i_1}, and so we have a directed cycle from V_{i_1} to V_{n_2} to V_{b_2} to V_{s_1} to V_{i_1}. Hence, the graph induced by

$$\{V_{b_2}, V_{b_2+1}, \ldots, V_{s_1}, \ldots, V_{i_i}, \ldots, V_{m_1}, V_{m_1+1}, V_{m_1+2}, \ldots, V_{n_2}\}$$

is a strongly connected subgraph of $\mathcal{G}$ by Fact (IV). This contradicts the selection of s_1 and m_1.

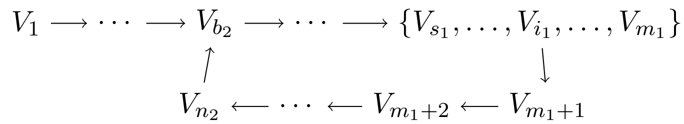

FIGURE 18. A directed path from V_{i_1} through V_{m_1+1}, V_{n_2}, V_{b_2} to V_{s_1}.

Case 2: $s_1 \leq b_2 \leq m_1$. If this is the case, then similar to Case 1, we will have a directed cycle from V_{i_1} to V_{n_2} to V_{b_2} to V_{i_1}. Hence, the graph induced by

$$\{V_{s_1}, V_{s_1+1}, V_{s_1+2}, \ldots, V_{m_1}, V_{m_1+1}, V_{m_1+2}, \ldots, V_{n_2}\}$$

is a strongly connected subgraph of $\mathcal{G}$ by Fact (IV). This contradicts the selection of s_1 and m_1.

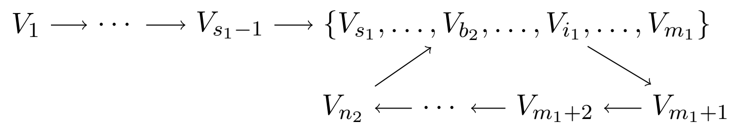

FIGURE 19. A directed cycle from V_{i_1} through V_{m_1+1}, V_{n_2}, V_{b_2} to V_{i_1}.

Case 3: $b_2 > m_1$. In this case, $\{V_{b_2}, V_{b_2+1}, \ldots, V_{n_2}, V_{b_2}\}$ becomes a directed cycle. Let s_2 be the smallest number less than or equal to b_2 and m_2 be the maximum number greater than or equal to n_2 such that $\mathcal{G}_2$, the graph induced by

$$\{V_{s_2}, V_{s_2+1}, \ldots, V_{b_2}, V_{b_2+1}, \ldots, V_{n_2}, \ldots, V_{m_2}\},$$

is a strongly connected subgraph with consecutive labels. Note that, by the selection of $\mathcal{G}_1$, $m_1 < s_2$. Also,

$$\{V_{i_1}, V_{m_1+1}, V_{m_1+2}, \ldots, V_{s_2}\}$$

is a directed path from $\mathcal{G}_1$ to $\mathcal{G}_2$. Moreover, there is no back edge from a vertex in $\mathcal{G}_2$ to any vertex with label less than s_2 by the "maximum" property. By repeating this finitely many, say r, times, we will have disjoint "maximal" strongly connected subgraphs $\mathcal{G}_j$, induced by

$$\{V_{s_j}, V_{s_j+1}, V_{s_j+2}, \ldots, V_{i_j}, \ldots, V_{m_j}\},$$

for $1 \leq j \leq r$, such that $\{V_1, V_2, \ldots, V_{s_1}\}$ is a directed path in the DFS tree $\mathcal{T}$, $\{V_{i_j}, V_{m_j+1}, V_{m_j+2}, \ldots, V_{s_{j+1}}\}$ is a directed path from $\mathcal{G}_j$ to $\mathcal{G}_{j+1}$, and $V_{m_r} = V_n$.

$$\begin{array}{l}
V_1 \longrightarrow V_2 \longrightarrow \cdots \longrightarrow V_{s_1-1} \longrightarrow \{V_{s_1}, \dots, V_{i_1}, \dots, V_{m_1}\} \ = \mathcal{G}_1 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
V_{s_2-1} \longleftarrow \cdots \longleftarrow V_{m_1+2} \longleftarrow V_{m_1+1} \\
\quad \downarrow \\
\{V_{s_2}, \dots, V_{i_2}, \dots, V_{m_2}\} \ = \mathcal{G}_2 \\
\quad \downarrow \\
V_{m_2+1} \longrightarrow \cdots \longrightarrow \{V_{s_{r-2}}, \dots, V_{i_{r-2}}, \dots, V_{m_{r-2}}\} = \mathcal{G}_{r-2} \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
V_{s_{r-1}-1} \longleftarrow \cdots \longleftarrow V_{m_{r-2}+2} \longleftarrow V_{m_{r-2}+1} \\
\quad \downarrow \\
\{V_{s_{r-1}}, , \dots, V_{i_{r-1}}, \dots, V_{m_{r-1}}\} \ = \mathcal{G}_{r-1} \\
\quad \downarrow \\
V_{m_{r-1}+1} \longrightarrow V_{m_{r-1}+2} \longrightarrow \cdots \longrightarrow V_{s_r-1} \\
\qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\
\qquad\qquad\qquad\qquad \{V_{s_r}, \dots, V_n\} \ = \mathcal{G}_r
\end{array}$$

FIGURE 20. A structure of disjoint, maximal, and strongly connected subgraphs.

Now, if no vertex in $\{V_1, V_2, \dots, V_{m_{r-1}}\}$ is adjacent to any vertex in $\mathcal{G}_r$, then the first edge $\{V_{i_{r-1}}, V_{m_{r-1}+1}\}$ in the directed path from $\mathcal{G}_{r-1}$ to $\mathcal{G}_r$ would be a bridge, which is a contradiction. If there is a vertex V_s in $\{V_1, V_2, \dots, V_{m_{r-1}}\}$ which is adjacent to a vertex V_m in $\mathcal{G}_r$, then $\{V_s, V_m\}$ is a back edge, which would contradict the "maximal" property of $\mathcal{G}_i s$. Hence r must be equal to 1 and $m_1 = n$.

Furthermore, we will prove (by contradiction) that s_1 must be equal to 1. In fact, if $s_1 > 1$, then there is a vertex V_s in $\{V_1, V_2, \dots, V_{s_1-1}\}$ and a vertex V_m in

$$\mathcal{G}_1 = \{V_{s_1}, V_{s_1+1}, \dots, V_n\}$$

such that $\{V_m, V_s\}$ is a back edge. This must be the case because otherwise, $\{V_{s_1-1}, V_{s_1}\}$ would be a bridge.

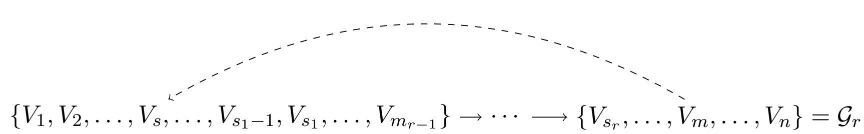

FIGURE 21. A critical back edge $\{V_m, V_s\}$.

Hence, the graph induced by

$$\{V_s, V_{s+1}, \ldots, V_{s_1}, V_{s_1+1}, \ldots, V_n\}$$

is a strongly connected subgraph of $\mathcal{G}$ by Fact (IV). This contradicts the selection of s_1. Therefore, s_1 must be equal to 1.

This completes the proof. □

5. Second Proof: A Direct Method

We will now focus our efforts on proving the following lemma, which will assist us in our second proof of Theorem 3.1.

Lemma 5.1. *Let $\mathcal{G}$ be a connected multigraph with $n > 1$ vertices. Then each edge in $\mathcal{G}$ is either a bridge or is included in a cycle.*

Proof. Let $\{U, V\}$ be an edge in $\mathcal{G}$. Now, we perform a DFS algorithm beginning with $U = V_1$ and $V = V_2$ and let $V_1, V_2, \ldots, V_n$ be the labels in the resulting tree $\mathcal{T}$. By assigning a direction for each edge in $\mathcal{T}$ from lower label to higher label, $\mathcal{T}$ becomes a rooted tree with vertex V_1 as its root and there is a (unique) directed path P_i from V_1 to any vertex V_i. Now, we divide all the V_is into two parts:

$$P_U = \{V_i \mid P_i \text{ does not include } V_2\} \quad \text{and} \quad P_V = \{V_j \mid P_j \text{ includes } V_2\}. \tag{5.1}$$

Then P_U and P_V are disjoint and $\{U, V\}$ is the only edge in $\mathcal{T}$ joining P_U and P_V. If there is no other edge in $\mathcal{G}$ connecting P_U and P_V, then $\{U, V\}$ is a bridge. If there is some edge $\{V_i, V_j\}$ in $\mathcal{G}$ where $V_i \in P_U$ and $V_j \in P_V$, then $\{U, V\}$ is included in the (undirected) cycle

$$\{V_i, \ldots, V_1 = U, V_2 = V, \ldots, V_j, V_i\}.$$

Thus, we can conclude that each edge in $\mathcal{G}$ is either a bridge or is included in a cycle. □

For an illustration of the sets P_U and P_V as used in Equation (5.1) within Lemma 5.1 as well as the cases contemplated in the Lemma, let us consider the following examples.

Example 5.1. (a) *Let $\mathcal{G}_1$ be the graph on the left depicted in Figure 22. To its right is an application of DFS to $\mathcal{G}_1$.*

FIGURE 22. An illustration of Lemma 5.1 with $\{U, V\}$ as a bridge.

In this case, we observe that $P_U = \{V_1, V_5, V_6\}$ *and* $P_V = \{V_2, V_3, V_4\}$ *and thus, we have that* $P_U \cap P_V = \emptyset$. *Furthermore, we observe that* $\{U, V\}$ *is the only edge of* $\mathcal{G}_1$ *connecting* P_U *and* P_V. *Hence,* $\{U, V\}$ *is a bridge of* $\mathcal{G}_1$.

(*b*) *Let* $\mathcal{G}_2$ *be the graph on the left depicted in Figure 23. To its right is an application of DFS.*

FIGURE 23. An illustration of Lemma 5.1 with $\{U, V\}$ as an edge of a cycle.

In this case, we observe that $P_U = \{V_1\}$ *and* $P_V = \{V_2, V_3, V_4, V_5, V_6\}$ *and thus, we have that* $P_U \cap P_V = \emptyset$. *In this case,* $\{U, V\}, \{U, C\}$, *and* $\{U, D\}$ *all join* P_U *with* P_V *in* $\mathcal{G}_2$, *and as such,* $\{U, V\}$ *is not the only edge connecting the sets. Moreover, we see that* $\{U, V\}$ *is not a bridge of* $\mathcal{G}_2$, *and thus, is included in a cycle of* $\mathcal{G}_2$.

We will now provide another proof of Theorem 3.1 using the machinery we have established in Lemma 5.1.

Proof of Theorem 3.1. Suppose depth-first search is applied to a connected graph $\mathcal{G}$ without a bridge. Let us assign a tree edge from lower to higher depth-first search number and assign a back edge from higher to lower depth-first search number.

We will prove that the resulting directed graph is strongly connected. To this end, let us assign $V_1, V_2, \ldots, V_n$ as the labels of the depth-first search tree $\mathcal{T}$ with i the label number of V_i. Then $\mathcal{T}$ is a rooted tree with V_1 as the root and there is a directed path from V_1 to any vertex V_i by DFS.

We shall prove that for any vertex V_i, there is a directed path from V_i to V_1. Doing so will ensure that $\mathcal{G}$ is strongly connected. To this end, it suffices to show that there is a directed path from V to U for each tree edge $\{U, V\}$.

Let $\{U, V\}$ be an arbitrary tree edge. Then, by Lemma 5.1, since $\mathcal{G}$ is bridgeless, the edge $\{U, V\}$ is included in some cycle $\mathcal{C}$. Let m be the lowest label among the vertices in $\mathcal{C}$ and $\mathcal{T}_{V_m}$ be the subtree of $\mathcal{T}$ formed by V_m and its descendants. To illustrate this, consider the graph on the left of Figure 24, and the designation of $\mathcal{T}_{V_m}$ after the application of DFS.

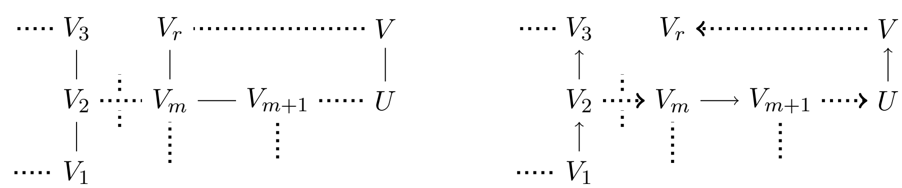

FIGURE 24. An example graph $\mathcal{G}$ and the corresponding $\mathcal{T}_{V_m}$ identified.

Then we have that V_m is the root of $\mathcal{T}_{V_m}$. If we denote the (undirected) cycle $\mathcal{C}$ by $\{U_1 = V_m, U_2, \ldots, U_s, U_1\}$ such that $U = U_i$ and $V = U_{i+1}$ for some i, then $\{U_1, U_2, \ldots, U_s\}$ is a directed path in $\mathcal{T}_{V_m}$ and (U_s, U_1) is a back edge. As such, $\{U_1, U_2, \ldots, U_s, U_1\}$ becomes a directed cycle. Moreover, (U, V) is included in the directed cycle $\mathcal{C} = \{U_1, U_2, \ldots, U_s, U_1\}$. Hence, there is a directed path from V to U. To further illustrate this, included in Figure 25 is the graph from Figure 24 with the cycle in question highlighted in red on the left as well as assigned directions such that there exists both a directed path from $U = U_i$ to $V = U_{i+1}$ and from $V = U_{i+1}$ to $U = U_i$ on the right.

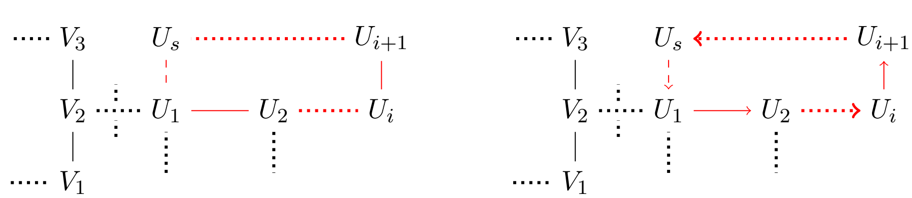

FIGURE 25. An undirected cycle $\mathcal{C}$ and directed paths from $U = U_i$ to $V = U_{i+1}$ and from $V = U_{i+1}$ to $U = U_i$.

Since $\{U, V\}$ was selected arbitrarily, this process guarantees that performing DFS in the way provided will ensure the existence of a directed path from V to U for every tree edge $\{U, V\}$. Thus, $\mathcal{G}$ becomes strongly connected. □

6. Strong Connectivity of a Bridgeless Multigraph

Let us consider and acknowledge the fact that given a bridgeless multigraph on at least two vertices, we can ensure the existence of a strongly connected graph as a result of some assignment of directions to the edges of said graph.

Theorem 6.1. *Let $\mathcal{G}$ be a connected multigraph with $n > 1$ vertices without a bridge. Then we can make $\mathcal{G}$ a strongly connected multigraph by assigning a direction to each edge in $\mathcal{G}$.*

We will prove this theorem in two ways. One of these ways will rely on Lemma 5.1, whereas the other will not.

Proof of Theorem 6.1 (Using Lemma 5.1). Pick up an edge $\{U, V\}$ in $\mathcal{G}$. Then by Lemma 5.1, $\{U, V\}$ is contained in some cycle C_1. We can assign a direction to each edge in the cycle C_1 to make it a directed cycle.

If C_1 contains all the vertices in $\mathcal{G}$, then $\mathcal{G}$ is strongly connected regardless of the directions of edges not in the cycle C_1.

If C_1 does not contain all the vertices in $\mathcal{G}$. Let $\mathcal{G}_1$ be the subgraph induced by C_1. Then there is an edge $\{V_1, V_2\}$ such that V_1 is in $\mathcal{G}_1$ and V_2 is not in $\mathcal{G}_1$. By Lemma 5.1, $\{V_1, V_2\}$ is contained in some cycle C_2. This leads us to contemplate the two following cases.

(a) If V_1 is the only common vertex of $\mathcal{G}_1$ and C_2, then we can assign a direction to each edge in the cycle C_2 to make it a directed cycle, and so the subgraph $\mathcal{G}_2$ induced by $\mathcal{G}_1 \cup C_2$ is strongly connected regardless the directions of edges not in $\mathcal{G}_1$ and C_2.

(b) If there is more than one common vertex between $\mathcal{G}_1$ and C_2, then let V_1' be the first other common vertex in the portion of C_2 formed by the directed path $\{V_1, V_2, \ldots, V_1'\}$. Then along the directed path from V_1' to V_1 in $\mathcal{G}_1$, we can assign a direction to each edge in $V_1, V_2, \ldots, V_1'$ to form a directed cycle C_2', and so $\mathcal{G}_2$, the graph induced by the vertices of $C_1 \cup C_2'$, is strongly connected regardless of the directions of edges not in the cycles C_1 and C_2'. Since $\mathcal{G}$ has finitely many vertices, after r iterations of this procedure, we will have $\mathcal{G}_r$, the graph induced by the vertices of $C_1 \cup C_2' \cup \cdots \cup C_r'$, which contains all the n vertices in $\mathcal{G}$, and so $\mathcal{G}_r = \mathcal{G}$ is strongly connected.

This concludes the proof. □

Below is a simple illustration of the central concepts of the above proof.

Example 6.1. *First, let us consider Case (a). That is, below are two cycles that share exactly one common vertex. Here, we consider the cycles*

$$C_1 = \{V_1, U, V, V_1\} \quad \textit{and} \quad C_2 = \{V_1, V_2, V_3, V_4, V_1\}.$$

Observe that, in Figure 26, the graph induced by $C_1 \cup C_2$ *is strongly connected, since it contains the closed walk*

$$W := \{V_1, U, V, V_1, V_2, V_3, V_4, V_1\}.$$

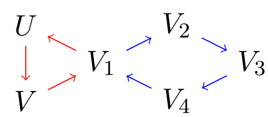

FIGURE 26. Case (a) with C_1 and C_2.

Furthermore, below are simple illustrations of an instance where Case (b) is in effect. Let $\mathcal{G}$ *be the graph in Figure 27 on the left. First, we acknowledge* $\mathcal{G}_1$*, the graph induced by*

$$C_1 = \{U, V, V_1, V_5, V_1', U\},$$

and assign directions to the edges.

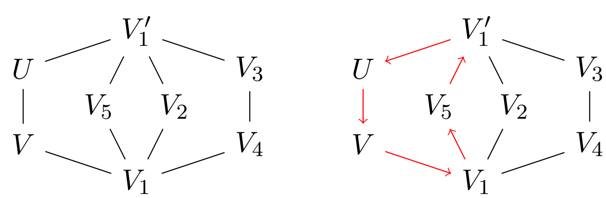

FIGURE 27. An example of Case (b) with C_1 designated.

Notice that $\mathcal{G}_1$ *is strongly connected. Now, we observe that the vertices* $V_1, V_2, V_1', V_3,$ *and* V_4 *induce another cycle. We will call this cycle* C_2*. Then, we identify the path* $\{V_1, V_2, V_1'\}$ *as the portion of* C_2 *that shares two common vertices with* C_1 *and construct the cycle* $C_2' = \{V_1, V_2, V_1', U, V, V_1\}$*. Thus, we have assigned directions to the edges in* $\mathcal{G}_2$*, the graph induced by the vertices in* $C_1 \cup C_2'$*. Observe that* $\mathcal{G}_2$ *is strongly connected. Now, we observe the cycle*

$$C_3 = \{V_1, V_4, V_3, V_1', U, V, V_1\}.$$

Next, we identify the path $\{V_1, V_4, V_3, V_1'\}$ *as the portion of* C_3 *that shares two common vertices with* $\mathcal{G}_2$ *and construct the cycle* $C_3' = \{V_1, V_4, V_3, V_1', U, V, V_1\}$.

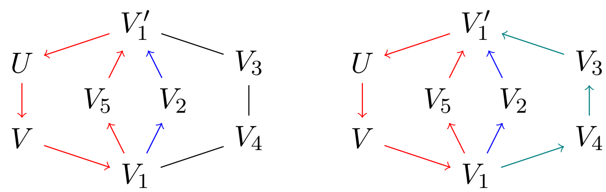

FIGURE 28. An example of Case (b) with $C_1 \cup C_2' \cup C_3'$ designated.

Now, after appending C_3', we have that each edge of the graph has been assigned a direction. Moreover, $\mathcal{G}_3$, the graph induced by $C_1 \cup C_2' \cup C_3'$, contains all of the vertices and edges of $\mathcal{G}$. Furthermore, we have that $\mathcal{G}_3$ is strongly connected and that $\mathcal{G}_3 = \mathcal{G}$. Therefore, $\mathcal{G}$ is strongly connected with the assigned directions.

Proof of Theorem 6.1 (Without using Lemma 5.1). First, we know that since $\mathcal{G}$ has no bridges, $\mathcal{G}$ is not a tree. Hence $\mathcal{G}$ contains a cycle $C_1 = \{V_1, V_2, \ldots, V_k, V_1\}$. Assigning directions

$$(V_1, V_2), \ldots, (V_{k-1}, V_k), (V_k, V_1),$$

we obtain a directed cycle C_1 and consequently achieve a strongly connected subgraph of $\mathcal{G}$. If there are any chords in C_1, we arbitrarily assign a direction to the chords in question. Moving forward, we have a strongly connected submultigraph $\mathcal{C}_1^*$ of $\mathcal{G}$ consisting of all vertices in C_1 and all the edges of $\mathcal{G}$ which have both end vertices in C_1. Now, we will employ $\mathcal{C}_1^*$ to construct a new connected multigraph $\mathcal{G}_1$ without a bridge. To this end, we will consider $\mathcal{C}_1^*$ as a single vertex, say $\overline{V}$, and all the vertices in $\mathcal{G}$ but not in $\mathcal{C}_1^*$ as the other vertices in a new graph $\mathcal{G}_1$, with vertex set $\overline{V} \cup (V(\mathcal{G}) - V(\mathcal{C}_1^*))$, and keep all the edges in $\mathcal{G}$ while collapsing all the edges in $\mathcal{C}_1^*$ into the vertex $\overline{V}$. Then, we are left with the following cases.

(a) If $\mathcal{G}_1$ contains only one vertex $\overline{V}$, then $\mathcal{C}_1^* = \mathcal{G}$, and so we are done.

(b) If $\mathcal{G}_1$ contains more than one vertex, then $\mathcal{G}_1$ is a connected multigraph with $n_1 > 1$ vertices without a bridge and $n_1 < n$. Since n is a finite number, after finitely many, say r, steps, we will have that $\mathcal{G}_r$ contains a single vertex. As a result, the entire graph $\mathcal{G}$ becomes a strongly connected multigraph.

This concludes our proof. □

Let us illustrate the above proof.

Example 6.2. *First, we will consider Case (a). Observe that the graph in Figure 29 is a cycle with a single chord, and so we can assign directions to the edges of the graph in such a* *way that the resulting graph is strongly connected. Furthermore, this allows us to immediately collapse the entire directed graph into* $\overline{V}$*, thus completing our procedure.*

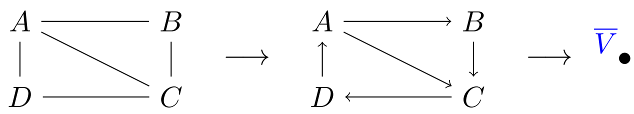

FIGURE 29. An example of Case (a).

Note that we could have assigned the cycle to be counter-clockwise and the chord in the other direction.

Now, we will contemplate an instance of Case (b). Consider the graph in Figure 30.

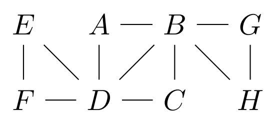

FIGURE 30. An example of a graph satisfying Case (b).

Next, we will identify cycles, direct each corresponding cycle, and collapse them one by one until we are left with a single "vertex" representing the entire graph as a strongly connected portion. First, we consider C_1*.*

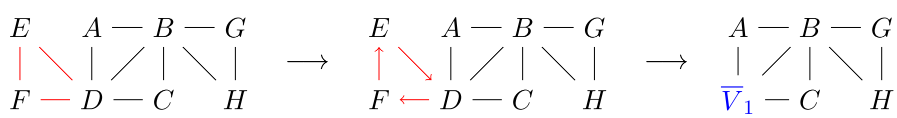

FIGURE 31. The identification of C_1, direction of $\mathcal{C}_1^*$, and collapse of $\mathcal{C}_1^*$ into $\overline{V}_1$.

Next, we turn our attention to C_2*.*

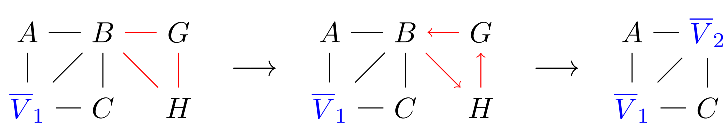

FIGURE 32. The identification of C_2, direction of $\mathcal{C}_2^*$, and collapse of $\mathcal{C}_2^*$ into $\overline{V}_2$.

Finally, we will acknowledge C_3*.*

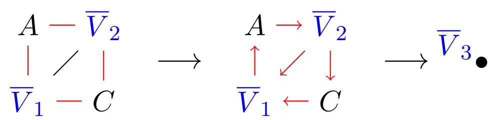

FIGURE 33. The identification of C_3, direction of $\mathcal{C}_3^*$, and collapse of $\mathcal{C}_3^*$ into $\overline{V}_3$.

Now, since each $\overline{V}_i$ *is strongly connected and we have simplified the given graph to* $\overline{V}_3$*, we are done and have a strongly connected graph.*

7. Conclusion

As mentioned in the introduction, studying strong connectivity in graphs is instrumental in various real-world applications of graph theory, including social networks and transportation systems. The ability to determine whether a graph's edges can be oriented to produce a strongly connected directed graph is particularly valuable for analyzing graph structures and the relationships they model. Furthermore, understanding exactly how to construct such an edge direction assignment could also prove to be a very useful endeavor. Within our paper, we have presented two proofs of a known result regarding the ability of the Depth-First Search Algorithm to generate a strongly connected graph from a bridgeless graph as well as two proofs of the existence of a strongly connected assignment of edge directions for a bridgeless multigraph.

For a potential application of the content of this manuscript, note that the algorithms that we provided in the proofs can be implemented into software for the purpose of utilizing computers to extrapolate information concerning computationally complex strongly connected graphs, multigraphs, and components that could potentially pertain to real-world applications.

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References

[1] Alon, N., & Erdős, P. (1985). An application of graph theory to additive number theory. *European Journal of Combinatorics, 6*(3), 201–203. `https://doi.org/10.1016/S0195-6698(85)80027-5`

[2] Xu, L., Jiang, C., Wang, J., Yuan, J., & Ren, Y. (2014). Information security in big data: Privacy and data mining. *IEEE Access, 2*, 1149–1176. `https://doi.org/10.1109/ACCESS.2014.2362522`

[3] Wu, J., & Tsai, Y. J. (2005). Real-time speed limit sign recognition based on locally adaptive thresholding and depth-first-search. *American Society for Photogrammetry and Remote Sensing, 71*(4), 405–414. https://doi.org/10.14358/PERS.71.4.405

[4] Zhang, L., & Li, K. (2021). Influence maximization based on backward reasoning in online social networks. *Mathematics, 9*(24), 3189. https://doi.org/10.3390/math9243189

[5] Amigó, J. M., Gálvez, J., & Villar, V. M. (2009). A review on molecular topology: Applying graph theory to drug discovery and design. *Naturwissenschaften, 96*, 749–761. https://doi.org/10.1007/s00114-009-0536-7

[6] Balaban, A. T. (1985). Applications of graph theory in chemistry. *Journal of Chemical Information and Computer Sciences, 25*, 334–343.

[7] Balasubramanian, K. (1985). Applications of combinatorics and graph theory to spectroscopy and quantum chemistry. *Chemistry Review, 85*, 599–618.

[8] Biggs, N. L., Lloyd, E. K., & Wilson, R. J. (1999). *Graph Theory 1736–1936* (1st ed.). Oxford, England: Clarendon Press.

[9] Biró, C., & Kusper, G. (2018). Equivalence of strongly connected graphs and black-and-white 2-SAT problems. *Miskolc Mathematical Notes, 19*(2), 755–768. https://zbmath.org/1424.94109

[10] Chakraborty, A., Dutta, T., Mondal, S., & Nath, A. (2018). Application of graph theory in social media. *International Journal of Computer Sciences and Engineering, 6*(10), 722–729.

[11] Cormen, T. H., Leiserson, C. E., Rivest, R. L., & Stein, C. (2022). *Introduction to Algorithms* (4th ed.). MA, USA: The MIT Press.

[12] delEtoile, J., & Adeli, H. (2017). Graph theory and brain connectivity in Alzheimer's disease. *The Neuroscientist, 23*(6), 616–626. https://doi.org/10.1177/1073858417702621

[13] He, H., Xu, T., Chen, J., Cui, Y., & Song, J. (2024). A granulation strategy-based algorithm for computing strongly connected components in parallel. *Mathematics, 12*(11), 1723. https://doi.org/10.3390/math12111723

[14] Dhingra, S., Dodwad, P. S., & Madan, M. (2016). Finding strongly connected components in a social network graph. *International Journal of Computer Applications, 136*(7), 1–5. https://doi.org/10.5120/ijca2016908481

[15] Ray, S. S. (2012). *Graph Theory with Algorithms and its Applications: In Applied Science and Technology.* Springer.

[16] Hansen, P. J., & Jurs, P. C. (1988). Chemical applications of graph theory. Part I. Fundamentals and topological indices. *Journal of Chemical Education, 65*(7), 574–580. https://doi.org/10.1021/ed065p574

[17] Dossey, J. A., Otto, A. D., Spence, L. E., & Vanden Eynden, C. (2018). *Discrete Mathematics* (5th ed.). NY, USA: Pearson.

[18] Even, S. (2005). *Graph Algorithms* (2nd ed.). Cambridge, England: Cambridge University Press.

[19] Even-Tzur, G. (2001). Graph theory applications to GPS networks. *GPS Solutions, 5*(1), 31–38. https://doi.org/10.1007/PL00012874

[20] Gross, J. L., Yellen, J., & Anderson, M. (2018). *Graph Theory and Its Applications* (3rd ed.). FL, USA: Chapman and Hall/CRC.

[21] Husain, Z., Al Zaabi, A., Hildmann, H., Saffre, F., Ruta, D., & Isakovic, A. F. (2022). Search and rescue in a maze-like environment with ant and Dijkstra algorithms. *Drones, 6*(10), 273. `https://doi.org/10.3390/drones6100273`

[22] Jayabalasamy, G., Pujol, C., & Latha Bhaskaran, K. (2024). Application of graph theory for blockchain technologies. *Mathematics, 12*(8), 1133. `https://doi.org/10.3390/math12081133`

[23] Kumar, N., & Kaur, S. (2019). A review of various maze solving algorithms based on graph theory. *International Journal for Scientific Research and Development, 6*, 431–434.

[24] Majeed, A., & Rauf, I. (2020). Graph theory: A comprehensive survey about graph theory applications in computer science and social networks. *Inventions, 5*(1), 10. `https://doi.org/10.3390/inventions5010010`

[25] Martín-Nieto, M., Castaño, D., Horta Muñoz, S., & Ruiz, D. (2024). Solving mazes: A new approach based on spectral graph theory. *Mathematics, 12*(15), 2305. `https://doi.org/10.3390/math12152305`

[26] Mouronte-López, M. L. (2021). Modeling the public transport networks: A study of their efficiency. *Complexity*. `https://doi.org/10.1155/2021/3280777`

[27] Muhiuddin, G., Samanta, S., Aljohani, A. F., & Alkhaibari, A. M. (2023). A study on graph centrality measures of different diseases due to DNA sequencing. *Mathematics, 11*(14), 3166. `https://doi.org/10.3390/math11143166`

[28] Nandhini, R., Maheswari, V., & Balaji, V. (2018). A graph theory approach on cryptography. *Journal of Computational Mathematica, 2*(1), 97–104. `https://doi.org/10.26524/32`

[29] Swan, R. G. (1963). An application of graph theory to algebra. *Proceedings of the American Mathematical Society, 14*(3), 367–373. `https://doi.org/10.2307/2033801`

[30] Thornton, C. J., & Boulay, B. du. (1992). *Artificial Intelligence Through Search* (1st ed.). Springer.

[31] Zhao, J., Jiang, N., Pei, K., Wen, J., Zhan, H., & Tu, Z. (2024). TPoison: Data-poisoning attack against GNN-based social trust model. *Mathematics, 12*(12), 1813. `https://doi.org/10.3390/math12121813`

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Contents