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EDITORIAL

BAYRAM ŞAHIN *

Our **International Journal of Maps in Mathematics** journal has completed its 7th year and we are here with the first issue of the 8th year. Our journal continues to develop and gain recognition. After MathSci.net and Scopus databases, **International Journal of Maps in Mathematics** journal has started to be indexed by **EBSCO** database, which is a leading provider of research databases, e-journal and e-package subscription management, book collection development and acquisition management for universities, colleges, hospitals, corporations, governments, K12 schools and public libraries worldwide, as well as a major provider of library technology, e-books and clinical decision solutions. This important development will contribute significantly to further recognition of **International Journal of Maps in Mathematics** journal. We would like to thank our authors, editors, technical editors and our readers who contributed to the development and quality of our journal

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SHAPE STABILITY OF A QUADRATURE SURFACE PROBLEM IN INFINITE RIEMANNIAN MANIFOLDS

ABABACAR SADIKHE DJITE * AND DIARAF SECK

Abstract. In this paper, we revisit a quadradure surface problem in shape optimization. With tools from infinite-dimensional Riemannian geometry, we give simple control over how an optimal shape can be characterized. The framework of the infinite-dimensional Riemannian manifold is essential in the control of optimal geometric shape. The covariant derivative plays a key role in calculating and analyzing the qualitative properties of the shape hessian. Control only depends on the mean curvature of the domain, which is a minimum or a critical point. In the two-dimensional case, Gauss-Bonnet's theorem gives a control within the framework of the algorithm for the minimum.

Keywords: Stability, quadrature surface, shape optimization, Riemannian manifold, Gauss-Bonnet theorem

2010 Mathematics Subject Classification: Primary 49Q10, Secondary 53B20.

1. Introduction

The search for the notion of quadratures made a prodigious leap forward (1669-1704) thanks to Leibniz and Newton who, with the infinitesimal calculus, made the link between quadrature and derivative. An interesting reminder could be to explain the link with shape optimization. Regarding, a bounded domain $\Omega \subset \mathbb{R}^N$ with regular boundary, for instance $\mathcal{C}^2$, μ a signed measure compactly supported in Ω, it is well known there is a measure σ called a balayage measure carried by the surface $\partial\Omega$ and having the same potential as μ outside $\bar{\Omega}$,

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see for instance [19], [22] for more details about this topic. And in this case, by a classical approximation technique, one has the following relation:

$$\int_{\partial\Omega} h d\sigma = \langle h, \mu \rangle \qquad \forall h \in \mathcal{H}(\bar{\Omega}) \tag{1.1}$$

where $\mathcal{H}(\bar{\Omega})$ denotes the set of harmonic functions in a neighborhood of $\bar{\Omega}$. And we say that $\partial\Omega$ is a quadrature surface with respect to μ if (1.1) is satisfied.

This notion is closely linked with the overdetermined Cauchy elliptic problem. And one can claim that $\partial\Omega$ is a quadrature surface if and only if there is a solution to the following overdetermined Cauchy problem

$$\begin{cases} -\Delta u_\Omega &= \mu \quad \text{in} \quad \Omega, \\ u_\Omega &= 0 \quad \text{on} \quad \partial\Omega, \\ -\frac{\partial u_\Omega}{\partial \vec{\nu}} &= 1 \quad \text{on} \quad \partial\Omega. \end{cases} \tag{1.2}$$

The above quadrature surface free boundary problem has some physical motivations and can be related to many areas such as free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems etc. It has been intensively studied at least during the last forty years, see for example [33], [15] and the references therein for more details. Among these works, some authors have established an intimate link between the existence of quadrature surfaces and the solution of free boundary problems governed by overdetermined partial differential equations, see for instance [17], [32], [31], [13] and references therein.

The quadrature surface problem (1.2) can be tackled by a shape optimization approach when μ is regular enough, for instance by taking it in $L^2(\Omega), supp(\mu) \subset \Omega$. Fore more details see for instance [3] and [13].

Before proceeding further, let us remind that in optimisation or in the study of minimal action, one of the essential questions is the characterization of an optimum if it exists. When one is in a differentiable environment, i.e. if the objective function is differentiable as well as its constraints, if any, the first derivative and second one (hessian) play a fundamental role. In finite dimensions, the characterization results are very well known even when we are in Banach spaces.

On the other hand, when we have to deal with admissible sets of regular openings of $\mathbb{R}^N, N \geq 2$ containing the optimum to be characterized, the question is to be treated in a more delicate way. Indeed, if we consider a shape optimization problem where the variable is a regular open subset of class $\mathcal{C}^2$ and in which a boundary value problem of partial differential equations is posed, there is the computation of the second derivative. Added to this, the equivalence of norms is to be handled if any exist. In this paper, we aim at studying these issues of characterization of critical or optimal domains in the case where the minimum of the considered shape functional exists, in infinite dimensional Riemannian structures. To do so, it is crucial to find a space of forms and associated metrics.

Finding a shape space and an associated metric is a challenging task and different approaches lead to various models. One possible approach is to do as in [25], [24]. These authors proposed, a survey of various suitable inner products is given, e.g., the curvature weighted metric and the Sobolev metric. There are various types of metrics on shape spaces, e.g., inner metrics [4], [24] like the Sobolev metrics, outer metrics [6], [20], [24], metamorphosis metrics [35], the Wasserstein or Monge-Kantorovic metric on the shape space of probability measures [2], [7], the Weil-Petersson metric [21], current metrics [14], and metrics based on elastic deformations [16], [26]. However, it is a challenging task to model both, the shape space and the associated metric. There does not exist a common shape space or shape metric suitable for all applications. The suitability of an approach depends on the requirements in a given situation. In recent works, it has been shown that PDE constrained shape optimization problems can be embedded in the framework of optimization on shape spaces. E.g., in [28], shape optimization is considered as optimization on a Riemannian shape manifold, the manifold of smooth shapes. Moreover, an inner product, which is called Steklov- Poincaré metric, for the application of finite element (FE) methods, is proposed in [29].

As pointed out in [27], shape optimization can be viewed as optimization on Riemannian shape manifolds and the resulting optimization methods can be constructed and analyzed within this framework. This combines algorithmic ideas from [1] with the Riemannian geometrical point of view established in [4]
In [25], [24], a geometric structure of two-dimensional $\mathcal{C}^\infty$ shapes was introduced and subsequently generalized to shapes in higher dimensions in [23], [4], [5]. Essentially, closed curves (and closed higher-dimensional surfaces) are identified with mappings of the unit sphere to any shape under consideration. In two dimensions, this can be naturally motivated by Riemannian mapping theorem. In this work, we focus on two-dimensional shapes as subsets. And considering [3], [13], we think that it is possible to write our work in high dimensions and even if Ω is an open set with boundary of a compact $N-$dimensional Riemannian manifolds noted $\mathcal{M}$.

One of our main question is the following:

Is it possible to express the Hessian of a shape functional to get sufficient conditions so that the critical domain of the functional J assumes its minimum? To answer this question, we study the positiveness of the quadratic form of the functional J which is related to the quadrature surface that is nothing but the following free boundary problem

$$\begin{cases} -\Delta u_\Omega = f \;\text{ in } \Omega \\ u_\Omega = 0 \;\text{ on } \partial\Omega \\ -\frac{\partial u_\Omega}{\partial \vec{\nu}} = k \;\text{ on } \partial\Omega \end{cases}$$

k is a positive constant, and $f \in L^2(\Omega), suppf \subset \Omega,\ \vec{\nu}$ is the exterior unit normal vector. The above quadrature surface can be formulated as the following shape optimization problem:

$$\min_{\Omega \subset \mathbb{R}^2} J(\Omega)$$

under the following partial differential equations contraints

$$\begin{cases} -\Delta u_\Omega = f \;\text{ in } \Omega \\ u_\Omega = 0 \;\text{ on } \partial\Omega \end{cases}$$

where

$$J(\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 dx + \frac{k^2}{2}|\Omega| \tag{1.3}$$

is a real valued shape differentiable objective function, with $|\Omega| = \displaystyle\int_\Omega dx$.

In [3], [13], there are all details on existence results of quadrature surface by using shape optimization tools.

And the second question is the computation problem of the Hessian in the infinite Riemannian framework and how it can be related to the second shape derivative to deduce qualitative properties when the minimum of a regular enough shape functional exists or when Ω is a critical point the latter property means, that the first derivative of $J(\Omega)$ is equal to zero.

The paper is organized as follows:

In section 2, we give a brief survey, based on works in [25], [24], about the characterization of the tangent space in a framework of Riemannian manifolds of infinite dimensions.

Section 3 deals with the optimality condition of first order for the shape optimization and the computation of the covariant derivative. The latter plays a key role in our final result. We shall give a direct way to compute it which appears as a simplified expression.

In section 4, we shall recall some technical but classical computations of shape second derivative and establish a result (stated as a proposition) giving the expression of the quadratic form associated to the quadrature surface problem.

Section 5 which contains our main contributions, is devoted to the positiveness of the shape hessian in a Riemannian point of view of infinite dimensions. And, we shall propose a simple control which allows to get key information on the optimal shape domains when the latter are strict local minimum or critical points of the considered shape functional.

2. Characterization of tangent space at a point of B_e

The aim is to analyze the correlation of the Riemannian geometry on infinite dimensional maniolds B_e with shape optimization.
The authors would like to stress, what follows has been already done in pioneering works, see [25], [24], [23]. We only reproduce some fundamental steps related to our work.

Let Ω be a simply connected and compact subset of $\mathbb{R}^2$ with $\Omega \neq \emptyset$ and $\mathcal{C}^\infty$ boundary $\partial\Omega$. As is always the case in shape optimization, the boundary of the shape is all that matters. Thus we can identify the set of all shapes with the set of all those boundaries.

Let $Emb(\mathbb{S}^1, \mathbb{R}^2)$ be the set of all smooth embeddings on $\mathbb{S}^1$ in the plan $\mathbb{R}^2$, its elements are the injective mappings $c : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$. Let $Diff(\mathbb{S}^1)$ stands for the set of all $\mathcal{C}^\infty$ diffeomorphism on $\mathbb{S}^1$ which acts diferentiably on $Emb(\mathbb{S}^1, \mathbb{R}^2)$. Let us consider B_e as the quotient

$$Emb(\mathbb{S}^1, \mathbb{R}^2)/Diff(\mathbb{S}^1).$$

In terms of sets, we have

$$B_e(\mathbb{S}^1, \mathbb{R}^2) := \{\ [c]\ /\ c \in Emb\} \text{ where } [c] := \{c' \in Emb\ /\ c' \sim c\}. \tag{2.4}$$

To characterize the tangent space to B_e we start with the characterization of the tangent space to Emb denoted T_cEmb and the tangent space to the orbit of c by $Diff(\mathbb{S}^1)$ at c denoted by $T_c(Diff(\mathbb{S}^1).c)$. Thus the tangent space to B_e is then identified with a suplementary subspace of $T_c(Diff(\mathbb{S}^1).c)$ in T_cEmb.

Proposition 2.1. *Let $c \in Emb$, then the tangent space at c to Emb is given by:*

$$T_c Emb = \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^2). \tag{2.5}$$

Proof. Let $h \in T_c Emb$, then h is obtained by looking at a path of embeddings which passes through c. Let $c : I \times \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ be an embedding path such that $c(t,\theta) = c(\theta) + th(\theta)$ where $h : \mathbb{S}^1 \longrightarrow \mathbb{R}^2$ is $\mathcal{C}^\infty$, we have : $\frac{d}{dt}_{|t=0} c(t,\theta) = h(\theta)$. Since $c(t,\theta)$ is an embedding path then $c(t,\theta)$ is an immersion, thus

$$T_c Emb = Im(T_0 c(t,\theta)) = \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R}^2). \tag{2.6}$$

□

Proposition 2.2. *The tangent space to the orbit of c by $Diff(\mathbb{S}^1)$, is the subspace of $T_c Emb$ formed by vectors $m(\theta)$ of the type $c_\theta(\theta) = c'(\theta)$ times a function.*

Proof. We have $Diff(\mathbb{S}^1).c \subset Emb$ because these are all the bijective reparametrizations of the same curve $c(\theta)$ therefore $T_c(Diff(\mathbb{S}^1).c) \subset T_c Emb$. Let $m \in T_c(Diff(\mathbb{S}^1).c)$ then m is obtained by looking at a family of parametrizations $c(t,\theta) := c(\phi(t,\theta))$ of the curve $c(\theta)$ where

$$\phi(t,.) : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$$

is a diffeomorphism of $\mathbb{S}^1$ and t is the parameter of the variation of the reparametrization $\phi(t,s)$ of $\mathbb{S}^1$. We have $\frac{d}{dt}_{|t=0} c(t,\theta) = c'(\theta) \frac{d}{dt}_{|t=0} \phi(0,\theta)$ since, $c(t,\theta)$ is a parametrization of the curve $c(\theta)$ so it is an immersion. Thus we have

$$T_c(Diff(\mathbb{S}^1).c) = Im(T_0 c(t,\theta)) = c'(\theta) \frac{d}{dt}_{|t=0} \phi(0,\theta). \tag{2.7}$$

□

Remark 2.1. *The choice of the supplementary must abide by the action of $Diff(\mathbb{S}^1)$ i.e we choose a supplementary of $T_c(Diff(\mathbb{S}^1).c)$ in $T_c Emb$ stable by the action of $Diff(\mathbb{S}^1)$. For that it suffices to define a metric on Emb for which $Diff(\mathbb{S}^1)$ acts isometrically and define the supplementary of $T_c(Diff(\mathbb{S}^1).c)$ as its orthogonal with respect to this metric.*

Definition 2.1. *Let G^0 be metric invariant by the action of $Diff(\mathbb{S}^1)$ on the manifold $Emb(\mathbb{S}^1, \mathbb{R}^2)$, defined by the application:*

$$\begin{array}{rccl} G^0 & : & T_c Emb \times T_c Emb & \rightarrow & \mathbb{R} \\ & & (h,m) & \mapsto & \int_{\mathbb{S}^1} \langle h(\theta), m(\theta) \rangle |c'(\theta)| d\theta \end{array}$$

where $\langle h(\theta), m(\theta) \rangle$ is the ordinary scalar product of $h(\theta)$ and $m(\theta)$ in $\mathbb{R}^2$.

Proposition 2.3. *Let $c \in B_e$ then T_cB_e is colinear to the outer unit normal of Ω where Ω is a simply connected and compact subset of $\mathbb{R}^2$ and $\vec{\nu}$ is the outer unit normal of the domain Ω . In other words*

$$T_cB_e \simeq \{h \mid h = \alpha\vec{\nu}, \alpha \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})\}.$$

Proof. From the results shown above the orthogonal of $T_c(Diff(\mathbb{S}^1).c)$ in T_cEmb is the set of $h(\theta)$ in T_cEmb which are orthogonal for the metric G^0 to all $m(\theta) = \frac{d}{dt}_{|t=0}\phi(0,\theta)c'(\theta)$ this means that $h(\theta)$ must be perpendicular to $c'(\theta)$. So $h(\theta) = \alpha(\theta)\vec{\nu}(\theta)$ where $\alpha(\theta) \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})$. Therefore we have

$$T_cB_e \simeq \{h|h = \alpha\vec{\nu},\ \alpha \in \mathcal{C}^\infty(\mathbb{S}^1, \mathbb{R})\}$$

where $\vec{\nu}$ is the outer unit normal of the form Ω defined at the boundary by $\partial\Omega = c$ such that $\vec{\nu}(\theta) \perp c'(\theta)$ for all $\theta \in S^1$ and c' defines the circumferential derivative. □

Now let us consider the following terminology:

$$ds = |c_\theta| d\theta \qquad \text{arc length.}$$

Definition 2.2. *A Sobolev-type metric on the manifold $B_e(\mathbb{S}^1, \mathbb{R}^2)$ is map:*

$$\begin{array}{rcccl} G^A & : & T_cB_e \times T_cB_e & \to & \mathbb{R} \\ & & (h,m) & \mapsto & \displaystyle\int_{\mathbb{S}^1} (1 + AK_c^2(\theta))\langle h(\theta), m(\theta)\rangle |c'(\theta)| d\theta \end{array}$$

where K_c is the curvature of c and A a positive real.

Remark 2.2.
(1) *By setting $h = \alpha\vec{\nu}$, $m = \beta\vec{\nu}$ and by parametrizing $c(s)$ by arc length we have*

$$G^A(h,m) = \int_{\partial\Omega} (1 + AK_c^2(\theta))\alpha\beta ds.$$

(2) *If $A > 0$, G^A is a Riemannian metric.*

3. Optimality condition of first order and covariant derivative

The shape optimization problem that we have, consists in finding the solution of the following optimization problem:

$$\min_{\Omega} J(\Omega),$$

where

$$J(\Omega) = -\frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 dx + \frac{k^2}{2}|\Omega|$$

is a shape functional. We seek the shape derivative associated with the functional $J(\Omega)$ following the direction of the vector field $V : \mathbb{R}^2 \to \mathbb{R}^2$, $\mathcal{C}^\infty$ class :

$$dJ(\Omega)[V] = \int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}} \right)^2 \right) \langle V, \vec{\nu} \rangle d\sigma.$$

If $V_{|\partial\Omega} = \alpha \vec{\nu}$ we can still write

$$dJ(\Omega)[V] = \int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}} \right)^2 \right) \alpha d\sigma. \tag{3.8}$$

It should noted that there is a link between the shape derivative of J and the gradient in Riemannian structures see [27] and [37]. To illustrate our claim, let us consider the Sobolev metric G^A to ease the understanding of the computations. We think that it is quite possible to generalize this study in higher dimensions and even with other metrics.
Our purpose is to calculate the gradient of $J : B_e \to \mathbb{R}$ then we have :

$$dJ(\Omega)[V] = G^A(gradJ(\Omega), V) \tag{3.9}$$

if $V_{|\partial\Omega} = h$ we have

$$\begin{aligned} dJ_c(h) &= G^A(gradJ(\Omega), h) \\ dJ_c(h) &= \int_{\partial\Omega} \left(1 + AK_c^2\right) gradJ\alpha. \end{aligned}$$

But from (3.9),

$$dJ_c(h) = \int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}} \right)^2 \right) \alpha d\sigma$$

and thus

$$\int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}} \right)^2 \right) \alpha d\sigma = \int_{\partial\Omega} \left(1 + AK_c^2\right) gradJ\alpha d\sigma$$

so that

$$gradJ = \frac{1}{1 + AK_c^2} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial \vec{\nu}} \right)^2 \right).$$

The next step is to compute the explicit form of the covariant derivative $\nabla_h m \in T_c B_e$ with $h, m \in T_c B_e$.

Definition 3.1. *Let $\mathcal{M}$ be a set. A chart of $\mathcal{M}$ is a triplet $(\mathcal{U}, \psi, \mathcal{E})$ where $\mathcal{U}$ is a subset of $\mathcal{M}$, $\mathcal{E}$ is a Banach space, and ψ is a bijection from $\mathcal{U}$ to an open set $\mathcal{E}$. We say that two charts $(\mathcal{U}_1, \psi_1, \mathcal{E}_1)$ and $(\mathcal{U}_2, \psi_2, \mathcal{E}_2)$ are $\mathcal{C}^r$-compatible if:*

- *$\psi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$(respectively $\psi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$) is an open set in $\mathcal{E}_1$ (respectively $\mathcal{E}_2$).*

- *The map* $\psi_1 \circ \psi_2^{-1}$ *(respectively* $\psi_2 \circ \psi_1^{-1}$*) of* $\psi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ *in* $\psi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$ *(respectively* $\psi_1(\mathcal{U}_1 \cap \mathcal{U}_2)$ *in* $\psi_2(\mathcal{U}_1 \cap \mathcal{U}_2)$ *is of class* $\mathcal{C}^r$.

A $\mathcal{C}^r$*-atlas of* $\mathcal{M}$ *is a set of charts that are two by two* $\mathcal{C}^r$*-compatible and whose domains cover* $\mathcal{M}$*. Two atlases are* $\mathcal{C}^r$*-equivalent if their union is still a* $\mathcal{C}^r$*-atlas. A Banach manifold of class* $\mathcal{C}^r$ *is a set* $\mathcal{M}$ *equipped with a class of equivalence of* $\mathcal{C}^r$*-atlases.*

Definition 3.2. *Let* $\mathcal{M}$ *be a real (smooth) Banach manifold and* g *be a section of the bundle* $T\mathcal{M}^* \times T\mathcal{M}^*$ *of symmetric bilinear forms on* $T\mathcal{M}$*. We say that* g *is a weak Riemannian metric on* $\mathcal{M}$ *if and only if, for every* $p \in \mathcal{M}$, g_p *is a positive definite bilinear map on* $T_p\mathcal{M}$*, ie. if and only if:*

- $g_p(X, X) \geq 0$ *, for all* $X \in T_p\mathcal{M}$,
- $g_p(X, X) = 0$ *if and only if* $X = 0$.

Definition 3.3. *Let* $\mathcal{M}$ *be a real (smooth) Banach manifold and* g *be a section of the bundle* $T\mathcal{M}^* \times T\mathcal{M}^*$ *of symmetric bilinear forms on* $T\mathcal{M}$*. We say that* g *is a strongly Riemannian metric on* $\mathcal{M}$ *if, for every* $p \in \mathcal{M}$*, the injection from* $T_p\mathcal{M}$ *to* $T_p\mathcal{M}^*$ *defined by*

$$\tilde{g}_p : \left| \begin{array}{ccc} T_p\mathcal{M} & \longrightarrow & T_p\mathcal{M}^* \\ X & \mapsto & \{i_X g_p : Y \mapsto g_p(X, Y)\} \end{array} \right.$$

induces an isomorphism between $T_p\mathcal{M}$ *and* $T_p\mathcal{M}^*$.

Proposition 3.1. *Given a Banach manifold* $\mathcal{M}$ *equipped with a strongly Riemannian metric* g*, there exists a unique linear connection* ∇ *on the tangent bundle* $T\mathcal{M}^*$ *preserving* g *and having zero torsion. It is called the Levi-Civita connection of* g.

For the proof of this proposition, see Proposition A.2.6 in [36].

The following results (Propositions 3.2; 3.3 and Theorem 3.1) have been already established in a pioneering work, see [27]. We only bring a new proof and additional details in the computations of the covariant derivative. In the last part of the paper containing our main contributions, the covariant derivative plays a key role in the study of the positiveness of the quadratic form. We shall come back to this fact.

Proposition 3.2. *Let* $\Omega \subset \mathbb{R}^2$ *where* Ω *is a simply connected and compact subset of* $\mathbb{R}^2$ *and be at least of class* $\mathcal{C}^2$*, and* $\vec{\nu}$ *is the outer unit normal of the domain* Ω *and* $V, W, Z \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ *vector fields which are orthogonal to the boundaries i.e*

$$V_{|\partial\Omega} = \alpha\vec{\nu}$$

with $\alpha := \langle V_{|\partial\Omega}, \vec{\nu}\rangle$,

$$W_{|\partial\Omega} = \beta\vec{\nu}$$

with $\beta := \langle W_{|\partial\Omega}, \vec{\nu}\rangle$ *and*

$$Z_{|\partial\Omega} = \gamma\vec{\nu}$$

with $\gamma := \langle Z_{|\partial\Omega}, \vec{\nu}\rangle$ *such that* $V_{|\partial\Omega} = h := \alpha\vec{\nu}$, $W_{|\partial\Omega} = m =: \beta\vec{\nu}$ *and* $Z_{|\partial\Omega} = l := \gamma\vec{\nu}$ *belongs to the tangent space of* B_e. *Then the shape derivative* $h(G^A(m,l))$ *associated with the Riemannian metric* G^A *can be expressed as follows:*

$$\begin{aligned} h(G^A(m,l)) &= \int_{\partial\Omega}\left(2AK_c^3\alpha\beta\gamma + (1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha\right)ds \\ &+ \int_{\partial\Omega}\left((1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha + K_c(1+AK_c^2)\alpha\beta\gamma\right)ds. \end{aligned}$$

Proof. We set

$$F(c_t(\theta)) = (1+AK_c^2(\theta))\langle m(\theta), l(\theta)\rangle,$$

so that $G^A(m,l) = \int_{\mathbb{S}^1} F(c_t(\theta))|c_t'(\theta)|d\theta$. Then we calculate the following expression

$$h(G^A(m,l)) = \frac{d}{dt}_{|t=0}\left(\int_{\mathbb{S}^1} F(c_t(\theta))|c_t'(\theta)|d\theta\right)[V],$$

where $c_t(\theta)$ denotes a family of (parameterized) curves with $c_0(\theta) = c(\theta)$ and $c_t'(\theta)$ denotes the derivative with respect to θ of the curve $c_t : \theta \longrightarrow c_t(\theta)$. We have

$$\begin{aligned} h(G^A(m,l)) &= \int_{\mathbb{S}^1}\left(\tfrac{\partial[(1+AK_c^2)\beta\gamma]}{\partial\vec{\nu}}\alpha|c_t'(\theta)| + \tfrac{\partial(|c_t'(\theta)|)}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha)\right)d\theta \\ &= \int_{\mathbb{S}^1}\left(2AK_c(\tfrac{\partial K_c}{\partial\vec{\nu}})\alpha\beta\gamma + (1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha + (1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha\right)d\theta \\ &+ \int_{\mathbb{S}^1}\frac{\partial|c_t'(\theta)|}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha d\theta. \end{aligned}$$

Now let us calculate $\frac{\partial K_c}{\partial\vec{\nu}}$. We have

$$\frac{\partial K_c}{\partial\vec{\nu}} = \frac{\langle\vec{\nu}, c_\theta\rangle}{|c_\theta|^2}K_\theta + \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|}K^2 + \frac{1}{|c_\theta|}\left(\frac{1}{|c_\theta|}\left(\frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta\right)_\theta.$$

Then we have $\langle\vec{\nu}, c_\theta\rangle = 0$ because $\vec{\nu} \perp c_\theta$ and moreover,

$$\begin{aligned} \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|} &= \langle\vec{\nu}, \frac{ic_\theta}{|c_\theta|}\rangle \\ \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|} &= \langle\vec{\nu}, \vec{\nu}\rangle \\ \frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|} &= \|\vec{\nu}\|^2 = 1. \end{aligned}$$

Hence, we obtain that:

$$\frac{\partial K_c}{\partial \vec{\nu}} = K_c^2 + \frac{1}{|c_\theta|}\left(\frac{1}{|c_\theta|}\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta\right)_\theta .$$

Let us compute step by step the above last term in the right hand side.

First, we have

$$\begin{aligned}\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta &= \frac{\partial}{\partial\theta}\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)\\ &= \frac{\frac{\partial}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle|c_\theta| - \frac{\partial|c_\theta|}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|^2}\\ &= \frac{\frac{\partial}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle|c_\theta|}{|c_\theta|^2}\\ &= \frac{\frac{\partial}{\partial\theta}\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\end{aligned}$$

$$\begin{aligned}\left(\frac{\langle \vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta &= \frac{\langle \frac{\partial\vec{\nu}}{\partial\theta}, ic_\theta\rangle + \langle \vec{\nu}, i\frac{\partial c_\theta}{\partial\theta}\rangle}{|c_\theta|}\\ &= \frac{\langle \vec{\nu}'(\theta), ic_\theta\rangle + \langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\ &= \frac{\langle \vec{\nu}'(\theta), ic_\theta\rangle}{|c_\theta|} + \frac{\langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\ &= \langle \vec{\nu}'(\theta), \frac{ic_\theta}{|c_\theta|}\rangle + \frac{\langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\ &= \langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle + \frac{\langle \vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}.\end{aligned}$$

Note that $\|\vec{\nu}\|^2 = 1$ which is nothing $\langle \vec{\nu}, \vec{\nu}\rangle = 1$. Therefore, by differentiation, we get

$$\begin{aligned}\langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle + \langle \vec{\nu}(\theta), \vec{\nu}'(\theta)\rangle &= 0\\ 2\langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle &= 0\\ \langle \vec{\nu}'(\theta), \vec{\nu}(\theta)\rangle &= 0.\end{aligned}$$

Indeed, proceeding further the computation, we have

$$\begin{aligned}\left(\frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta &= \frac{\langle\vec{\nu}(\theta), ic_{\theta\theta}\rangle}{|c_\theta|}\\ &= \left\langle\vec{\nu}(\theta), \frac{ic_{\theta\theta}}{|c_\theta|}\right\rangle\\ &= \left\langle\vec{\nu}(\theta), \frac{-K_c|c_\theta|c_\theta}{|c_\theta|}\right\rangle\\ &= \langle\vec{\nu}(\theta), -K_c c_\theta\rangle\\ &= -K_c\langle\vec{\nu}(\theta), c_\theta\rangle = 0.\end{aligned}$$

Finally, from all the above steps, we have $\frac{1}{|c_\theta|}\left(\frac{1}{|c_\theta|}\left(\frac{\langle\vec{\nu}, ic_\theta\rangle}{|c_\theta|}\right)_\theta\right)_\theta = 0$ and we get

$$\frac{\partial K_c}{\partial\vec{\nu}} = K_c^2.$$

Therefore, we have

$$\begin{aligned}h(G^A(m,l)) &= \int_{\partial\Omega}\left(2AK_c(\tfrac{\partial K_c}{\partial\vec{\nu}})\alpha\beta\gamma + (1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha + (1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha\right)d\theta\\ &\quad+\int_{\partial\Omega}\frac{\partial|c_t'(\theta)|}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha d\theta\\ &= \int_{\partial\Omega}\left(2AK_c\times K_c^2\alpha\beta\gamma + (1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha + (1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha\right)d\theta\\ &\quad+\int_{\partial\Omega}\frac{\partial|c_t'(\theta)|}{\partial\vec{\nu}}(1+AK_c^2)\beta\gamma\alpha d\theta. \end{aligned}\tag{3.10}$$

Let us calculate now the following expression

$$\frac{\partial(|c_t'(\theta)|)}{\partial\vec{\nu}}.$$

To do this we parametrize $c(\theta)$ by arc length i.e $|c'(\theta)| = 1$. Since

$$\langle c'(\theta), c'(\theta)\rangle = 1$$

and differentiating it, we have

$$\langle c''(\theta), c'(\theta)\rangle = 0.$$

Then $c''(\theta) = c_{\theta\theta}(\theta)$ is proportional to $\vec{\nu}(s)$ so $c''(\theta) = K_c(\theta)\vec{\nu}(\theta)$ (this is the definition of the curvature of the curve c).

Let us compute now $\frac{d}{dt}(|c_t'(\theta)|)$ at $t=0$, where

$$\begin{aligned} |c_t'(\theta)| &= |\frac{d}{d\theta}(c(\theta)+t\vec{\nu}(\theta))| \\ &= |c'(\theta)+t\vec{\nu}'(\theta)| \\ &= (|c'(\theta)|^2+t^2|\vec{\nu}'(\theta)|^2+2t\big\langle c'(\theta),\vec{\nu}'(\theta)\big\rangle)^{\frac{1}{2}} \end{aligned} \tag{3.11}$$

From the Taylor's expansion of the previous expression in t, we see that

$$\frac{d}{dt}_{|t=0}|c_t'(\theta)| = \big\langle c'(\theta),\vec{\nu}'(\theta)\big\rangle$$

and since

$$\big\langle c'(\theta),\vec{\nu}(\theta)\big\rangle = 0$$

by differentiating we have

$$\big\langle c'(\theta),\vec{\nu}'(\theta)\big\rangle = -\big\langle c''(\theta),\vec{\nu}(\theta)\big\rangle = K_c,$$

and hence $\frac{d}{dt}(|c_t'(\theta)|) = K_c$.

One can conclude that

$$\begin{aligned} h(G^A(m,l)) &= \int_{\partial\Omega}\Big(2AK_c^3\alpha\beta\gamma+(1+AK_c^2)\tfrac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha\Big)\,ds \\ &+ \int_{\partial\Omega}\Big((1+AK_c^2)\tfrac{\partial\gamma}{\partial\vec{\nu}}\beta\alpha+K_c(1+AK_c^2)\alpha\beta\gamma\Big)\,ds. \end{aligned}$$

□

Proposition 3.3. *Let $\Omega\subset\mathbb{R}^2$ where Ω is a simply connected and compact subset of $\mathbb{R}^2$ and be at least of class $\mathcal{C}^2$, and $\vec{\nu}$ is the outer unit normal of the domain Ω and $V,W,Z\in\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ vector fields which are orthogonal to the boundaries i.e*

$$V_{|\partial\Omega} = \alpha\vec{\nu}$$

with $\alpha := \big\langle V_{|\partial\Omega},\vec{\nu}\big\rangle$,

$$W_{|\partial\Omega} = \beta\vec{\nu}$$

with $\beta := \big\langle W_{|\partial\Omega},\vec{\nu}\big\rangle$ and

$$Z_{|\partial\Omega} = \gamma\vec{\nu}$$

with $\gamma := \big\langle Z_{|\partial\Omega},\vec{\nu}\big\rangle$ such that $V_{|\partial\Omega} = h := \alpha\vec{\nu}$, $W_{|\partial\Omega} = m =: \beta\vec{\nu}$ and $Z_{|\partial\Omega} = l := \gamma\vec{\nu}$ belongs to the tangent space of B_e. Then the expression $G^A(\nabla_h m,l)+G^A(m,\nabla_h l)$ associated with the Riemannian metric G^A for all $l\in T_C B_e$, can be expressed as follows:

$$G^A(\nabla_h m,l)+G^A(m,\nabla_h l) = \int_{\partial\Omega}(1+AK_c^2)\,(\nabla_V W\gamma+\beta\nabla_V Z)\,ds.$$

Proof. By using the definition of the Riemannian metric G^A we have

$$\begin{aligned} G^A(\nabla_h m, l) &= \int_{\partial\Omega} (1 + AK_c^2) \nabla_h m \gamma \\ &= \int_{\partial\Omega} (1 + AK_c^2) \nabla_V W \gamma \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} G^A(m, \nabla_h l) &= \int_{\partial\Omega} (1 + AK_c^2) \beta \nabla_h l \\ &= \int_{\partial\Omega} (1 + AK_c^2) \beta \nabla_V Z. \end{aligned} \tag{3.13}$$

Therefore, by adding up the equations (3.12) and (3.13) we have

$$\begin{aligned} G^A(\nabla_h m, l) + G^A(m, \nabla_h l) &= \int_{\partial\Omega} (1 + AK_c^2) \nabla_V W \gamma + \int_{\partial\Omega} (1 + AK_c^2) \beta \nabla_V Z \\ &= \int_{\partial\Omega} (1 + AK_c^2) \left(\nabla_V W \gamma + \beta \nabla_V Z \right) ds. \end{aligned}$$

□

Remark 3.1. *We would like to find a linear connection that preserves the Riemannian metric G^A and if such a connection exists, it is such that $hG^A(m, l) = G^A(\nabla_h m, l) + G^A(m, \nabla_h l)$. By using the Propositions 3.2 and 3.3 we have*

$$\begin{aligned} \int_{\partial\Omega} (1 + AK_c^2) \left(\nabla_V W \gamma + \beta \nabla_V Z \right) ds &= \int_{\partial\Omega} \left(2AK_c^3 \alpha\beta\gamma + (1 + AK_c^2) \tfrac{\partial \beta}{\partial \vec{\nu}} \gamma\alpha \right) ds \\ &+ \int_{\partial\Omega} \left((1 + AK_c^2) \tfrac{\partial \gamma}{\partial \vec{\nu}} \beta\alpha + K_c \alpha\beta\gamma + AK_c^3 \alpha\beta\gamma \right) ds. \\ &= \int_{\partial\Omega} \left(3AK_c^3 + K_c \right) \alpha\beta\gamma + (1 + AK_c^2) \frac{\partial \beta}{\partial \vec{\nu}} \gamma\alpha \\ &+ (1 + AK_c^2) \frac{\partial \gamma}{\partial \vec{\nu}} \beta\alpha ds. \end{aligned}$$

From which, we get a simplified expression:

$$\begin{aligned} \nabla_V W \gamma + \beta \nabla_V Z &= \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \alpha\beta\gamma + \frac{\partial \beta}{\partial \nu} \gamma\alpha + \frac{\partial \gamma}{\partial \nu} \beta\alpha \\ \nabla_V W \gamma &= \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \alpha\beta\gamma + \frac{\partial \beta}{\partial \vec{\nu}} \gamma\alpha + \frac{\partial \gamma}{\partial \vec{\nu}} \beta\alpha - \beta \nabla_V Z. \end{aligned}$$

By pointing out that

$$\nabla_V Z = \frac{\partial \gamma}{\partial \vec{\nu}} \alpha,$$

we have

$$\begin{aligned}\nabla_V W\gamma &= \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta\gamma+\frac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha+\beta\frac{\partial\gamma}{\partial\vec{\nu}}\alpha-\beta\frac{\partial\gamma}{\partial\vec{\nu}}\alpha\\ &= \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta\gamma+\frac{\partial\beta}{\partial\vec{\nu}}\gamma\alpha.\end{aligned} \tag{3.14}$$

Finally, we have

$$\nabla_V W = \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta+\frac{\partial\beta}{\partial\vec{\nu}}\alpha. \tag{3.15}$$

Now let us verify that the connection ∇ is linear

- Let $V, W, Z \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ be vector fields which are orthogonal to the boundary of Ω, we have

$$\begin{aligned}\nabla_V(W+Z) := \nabla_h(m+l) &= \frac{\partial(\beta+\gamma)}{\partial\vec{\nu}}\alpha+\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha(\beta+\gamma)\\ &= \langle D_V(W+Z),\vec{\nu}\rangle+\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\langle V,\vec{\nu}\rangle\langle W+Z,\vec{\nu}\rangle\\ &= \langle D_VW,\vec{\nu}\rangle+\langle D_VZ,\vec{\nu}\rangle+\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle\\ &+ \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\langle V,\vec{\nu}\rangle\langle Z,\vec{\nu}\rangle\\ &= \left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\beta+\frac{\partial\beta}{\partial\vec{\nu}}\alpha+\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\gamma+\frac{\partial\gamma}{\partial\vec{\nu}}\alpha\\ \nabla_V(W+Z) &= \nabla_VW+\nabla_VZ.\end{aligned} \tag{3.16}$$

- Let f be a scalar field and $V, W \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ be vector fields which are orthogonal to the boundary of Ω, we have

$$\begin{aligned}\nabla_{fV}W &= \nabla_{fh}m\\ \nabla_{fV}W &= f\frac{\partial\beta}{\partial\vec{\nu}}\alpha\\ \nabla_{fV}W &= f\nabla_VW.\end{aligned} \tag{3.17}$$

From the equations (3.16) and (3.17) one can conclude that ∇ is linear.

Remark 3.2. *The map*

$$\xi : \left|\begin{array}{ccc} T_cB_e & \longrightarrow & T_cB_e^* \\ h & \mapsto & \{i_hG^A : m\mapsto G^A(h,m)\}\end{array}\right.$$

is linear, injective and also surjective. Indeed given G^A as defined above, for any $y \in T_cB_e^*$, we want to find a vector $h \in T_cB_e$ such that

$$\begin{aligned} \xi(h) &= y \\ &\Longleftrightarrow i_hG^A = y \\ &\Longleftrightarrow i_hG^A(m) = y(m), \qquad \forall m \in T_cB_e \\ &\Longleftrightarrow G^A(h,m) = y(m), \qquad \forall m \in T_cB_e. \end{aligned} \tag{3.18}$$

From the equation (3.18) we can write

$$\begin{aligned} G^A(h,m) &= y(m) = \int_{\partial\Omega}(1+AK_c^2).\frac{y(m)}{|\partial\Omega|(1+AK_c^2)}ds \\ G^A(h,m) &= \int_{\partial\Omega}(1+AK_c^2).\frac{1}{1+AK_c^2}.\frac{1}{|\partial\Omega|}yds \end{aligned} \tag{3.19}$$

Using the definition above, we have

$$G^A(h,m) = \int_{\partial\Omega}(1+AK_c^2)\langle h,m\rangle ds. \tag{3.20}$$

Then from the equations (3.19) and (3.20) we can show that

$$\langle h,m\rangle = \frac{y(m)}{|\partial\Omega|(1+AK_c^2)} \qquad \forall m \in T_cB_e. \tag{3.21}$$

Does there exist a $h = \alpha\vec{\nu}$ such that

$$\langle h,m\rangle = \frac{y(m)}{|\partial\Omega|(1+AK_c^2)} \qquad \forall m \in T_cB_e \text{ with } m := \beta\vec{\nu}\ ? \tag{3.22}$$

One can show that $y(m)$ is linear with respect to m. Indeed $y(m) = G^A(h,m)$ and G^A is linear with m. Since $y(m) \in \mathbb{R}$ then we can write $y(m) = k.m$ for $k \in T_cB_e$ with $k := \gamma\vec{\nu}$ and $\gamma \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R})$. Therefore from (3.22) we have

$$\alpha\beta = \frac{y(\beta\vec{\nu})}{|\partial\Omega|(1+AK_c^2)} \qquad \forall \beta \in \mathcal{C}^\infty(\mathbb{S}^1,\mathbb{R}) \tag{3.23}$$

$$\alpha\beta = \frac{k.\beta\vec{\nu}}{|\partial\Omega|(1+AK_c^2)} \tag{3.24}$$

$$\alpha\beta = \frac{\gamma\beta}{|\partial\Omega|(1+AK_c^2)} \tag{3.25}$$

$$\tag{3.26}$$

therefore

$$\alpha = \frac{\gamma}{|\partial\Omega|(1+AK_c^2)}$$

then

$$h = \frac{\gamma}{|\partial\Omega|(1 + AK_c^2)}$$

$$\int_{\partial\Omega} ds = |\partial\Omega|.$$

Now we can conclude that for

$$h = \frac{\gamma}{|\partial\Omega|(1 + AK_c^2)}$$

then

$$\forall y \in T_c B_e^* \qquad \exists h \in T_c B_e : \xi(h) = y.$$

Therefore the map ξ induces an isomorphism between $T_c B_e$ and $T_c B_e^*$. Consequently, from the Definition 3.3 we can check that G^A is a strongly Riemannian metric and then by using the Proposition 3.1 one can deduce that ∇ exists, it is unique and coincides with the Levi-Civita connection.

And then we are now able to claim the following theorem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ where Ω is a simply connected and compact subset of $\mathbb{R}^2$ and be at least of class $\mathcal{C}^2$, and $\vec{\nu}$ is the outer unit normal of the domain Ω and $V, W \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ vector fields which are orthogonal to the boundaries i.e*

$$V_{|\partial\Omega} = \alpha\vec{\nu}$$

with $\alpha := \langle V_{|\partial\Omega}, \vec{\nu} \rangle$ and

$$W_{|\partial\Omega} = \beta\vec{\nu}$$

with $\beta := \langle W_{|\partial\Omega}, \vec{\nu} \rangle$ such that $V_{|\partial\Omega} = h := \alpha\vec{\nu}$, $W_{|\partial\Omega} = m =: \beta\vec{\nu}$ belongs to the tangent space of B_e. Then the covariant derivative associated with the Riemannian metric G^A can be expressed as follows:

$$\begin{aligned} \nabla_V W : &= \nabla_h m = \frac{\partial \beta}{\partial \vec{\nu}}\alpha + \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \alpha\beta \\ &= \langle D_V W, \vec{\nu} \rangle + \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2} \right) \langle V, \vec{\nu} \rangle \langle W, \vec{\nu} \rangle \end{aligned}$$

where $D_V W$ is the directional derivative of the vector field W in the direction V.

Proof. It is a straight consequence of the above propositions and remarks. □

Remark 3.3. *It is quite possible to begin the proof of the above theorem by the application of shape calculus rules for volume and boundary functionals as in* [12], [34], [18] *on the following functional*

$$\int_{\partial\Omega}(1+AK_c^2)\alpha\beta d\sigma.$$

The remaining computations are mostly similar. We only underline that, at the end, it is necessary to see that the local covariant derivative is $\nabla_{X_x}Y = \frac{d}{dt_{|t=0}}(Y(x+tX(x)))$ *where* $Y = X = \vec{\nu}$ *and* $D_{\vec{\nu}}\vec{\nu} = 0$ *since* $|\vec{\nu}|^2 = 1, D_{\vec{\nu}}$ *being the Jacobian matrix.*

4. Sufficient condition for the minimality of a shape functional

In this section, assuming at first that there is at least one critical point, we shall first present the sufficient condition on the existence of a local minimum for a functional $J(\Omega)$ given as follows:

$$J(\Omega) = \int_{\Omega} f_0(u_\Omega, grad(u_\Omega)) \tag{4.27}$$

where f_0 is a function of $\mathbb{R}\times\mathbb{R}^n$ that we suppose to be smooth and u_Ω denotes a smooth solution of a boundary value problem.
And in the second part, in the case where $J(\Omega) = -\frac{1}{2}\int_{\Omega}|grad(u_\Omega)|^2dx + \frac{k^2}{2}|\Omega|$, we compute the second shape derivative.

The fundamental question is then to study the existence of the local strict minima of this functional under possible constraints that Ω is a critical point. That means that the first order derivative with respect to the domain is equal to zero at the domain Ω. We shall examine, for that, how this solution u_Ω varies when its domain of definition Ω moves.

Let us recall the classical method of studying a critical point. Let $(B, \|\ .\ \|_1)$ be a Banach space and let $E : (B, \|\ .\ \|_1) \longrightarrow \mathbb{R}$ be a function of class C^2 whose differential Df vanishes at 0. The Taylor-Young formula is then written as

$$E(u) = E(0) + D^2\ E(0)\ .\ (u,u) + o(||u||_1^2). \tag{4.28}$$

In particular, if the Hessian form $D^2E(0)$ is coercive in the norm $\|\ .\ \|_1$, then the critical point 0 is a strict local minimum of E. The fundamental difficulty in the study of critical forms is caused by the appearance of a second norm $\|\ .\ \|_2$ finer than $\|\ .\ \|_1$ $(i.e\ \ \|\ .\ \|_2 \leq C\|\ .\ \|_1)$. The Hessian form, is not in general, coercive for the norm $\|\ .\ \|_1$ but it is for the standard norm $\| \,.\, \|_2$. If these norms are not equivalent, which is generally the case, concluding that the minimum is strict is impossible, even locally for the strong norm. It is quite possible to give several examples. But let us reproduce a simple example of such a situation on the space $H_0^1(0,1)$ that was presented in the thesis [8]. Let us consider the functional E defined by

$$E(u) = \|u\|^2_{L^2(0,1)} - \|u\|^4_{H_0^1(0,1)}.$$

We can check that E is twice differentiable on $H_0^1(0,1)$. Further more, one has at 0:

$$\begin{cases} E'(0) = 0 \\ E''(0).(h,h) = 2\|h\|^2_{L^2(0,1).} \end{cases}$$

For each direction, we find that 0 is a strictly local minimum. Indeed, for all nonzero $u_0 \in H_0^1(0,1)$ and for all $t \in \mathbb{R}$, we have

$$E(tu_0) = t^2\|u_0\|^2_{L^2(0,1)} - t^4\|u_0\|^4_{H_0^1(0,1))} > 0 \; if \; t^2 < \frac{\|u_0\|^2_{L^2(0,1)}}{\|u_0\|^4_{H_0^1(0,1)}}.$$

However, 0 is not a local minimum even for the H_0^1 norm. Indeed, there is no $r > 0$ such that

$$\|u\|_{H_0^1(0,1)} < r \Longrightarrow E(u) > E(0) = 0 \;\; \text{i.e} \;\; \|u\|^2_{L^2(0,1)} > \|u\|^4_{H_0^1(0,1)},$$

since we can always build a sequence in $H_0^1(0,1)$ such that

$$\begin{cases} \|u_n\|_{H_0^1(0,1)} = r/2, \\ \|u_n\|_{L^2(0,1)} \longrightarrow 0 \text{ when } n \longrightarrow +\infty. \end{cases}$$

To solve this problem, we will use the Taylor's formula with an integral remainder, instead of (4.28) i.e

$$E(u) - E(0) = \int_0^1 (1-t)\; E''(tu)(u,u)\; dt. \tag{4.29}$$

This formula allows to express exactly the difference in energy between a critical form Ω_0 and a neighboring form Ω via an integral term that we can carefully estimate thanks to the study of the variations of the Hessian.

Theorem 4.1. *Let* $f_0 : \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}, \; (s,v) \longmapsto f_0(s,v)$ *be a function of class* $\mathcal{C}^3$ *and* f *a function in* $C^{0,\gamma}(\mathbb{R}^N, \mathbb{R}), \gamma \in (0,1)$. *Let* $L_0 = div(Agrad(.))$ *be strictly and uniformly elliptical operator with* A *in* $\mathcal{C}^2(\mathbb{R}^N, \; M_N(\mathbb{R}^N))$. *Let* E *be the defined shape functional on the class* $\mathcal{O}$ *of open class* $\mathcal{C}^{2,\gamma}$ *as*

$$J(\Omega) = \int_\Omega f_0(u_\Omega, \; grad(u_\Omega)),$$

where $M_N(\mathbb{R}^N)$ stands for the space of square matrix of order N and u_Ω is the solution of the homogeneous Dirichlet problem

$$\begin{cases} L_0 u = f \quad in \quad \Omega, \\ u = 0 \quad on \quad \partial\Omega. \end{cases}$$

Let $\Omega_0 \in \mathcal{O}$, then, there exist a real $\eta_0 > 0$ and an increasing function $\omega : (0, \eta_0] \longrightarrow (0, +\infty)$ with $\lim_{r \searrow 0} \omega(r) = 0$, which depend only on Ω_0, L_0, f_0 and f, such that for all $\eta \in (0, \eta_0]$ and for all $\theta \in \mathcal{C}^{2,\alpha}(\mathbb{R}^N, \mathbb{R}^N)$ satisfying

$$\|\theta - Id_{\mathbb{R}^N}\|_{2,\alpha} \leq \eta,$$

we have the following estimate valid for all t in $[0, 1]$,

$$\left| \frac{d^2}{dt^2} J(\Omega_t) - \frac{d^2}{dt^2}_{|t=0} J(\Omega_t) \right| \leq \omega(\eta) \| < V, \vec{\nu} > \|^2_{H^{1/2}(\partial\Omega_0)} \tag{4.30}$$

where $\Omega_t = \Phi_t(\Omega_0), t \in [0, 1]$ stands for the flow related to the vector field V.

This is exactly the Theorem 1 in [9] and for its proof the reader is invited to see this paper.

In the case where Ω_0 is a critical point for the functional J, to show that it is a strict local minimum, we have to study the positiveness of a quadratic form which we are going to denote by Q . This quadratic form is obtained by computing the second derivative of J with respect to the domain. So before proceeding, we need some hypothesis ;
let us suppose that:

- (i) - Ω is a $\mathcal{C}^2$- regular open domain.
- (ii) - $V(x;t) = \alpha(x)\vec{\nu}(x), \ \alpha \in H^{\frac{1}{2}}(\partial\Omega), \ \forall \ t \in [0, \epsilon[$.

In [10], (see also [9], [8]), the authors showed that it is not sufficient to prove that the quadratic form is positive to claim that: a critical shape is a minimum. In fact most of the time people use the Taylor Young formula to study the positiveness of the quadratic form.
For $t \in [0, \epsilon[$, $j(t) := J(\Omega_t) = J(\Omega) + t dJ(\Omega, V) + \frac{1}{2}t^2 d^2 J(\Omega, V, V) + o(t^2)$.
The quantity $o(t^2)$ is expressed with the norm of $\mathcal{C}^2$. The $H^{\frac{1}{2}}(\partial\Omega)$ norm appears in the expression of $d^2 J(\Omega, V, V)$. And these two norms are not equivalent. The quantity $o(t^2)$ is not smaller than $||V||_{H^{\frac{1}{2}}(\partial\Omega)}$, see the example in [10]. Then such an argument does not insure that the critical point is a local strict minimum.

In our study, we shall see that the main result in [10] can be satisfied in a simple way thanks to the hessian obtained via the Sobolev metric G^A in which the norm of $H^{1/2}(\partial\Omega)$ appears directly. And this overcomes the clasical issue. In fact the study of the sign of $\int_{\partial\Omega} H d\sigma$ becomes the only control from which one can get information on the optimal domain and then on the optimal shape.

Proposition 4.1.

Let Ω be a critical point for the functional J, then

$$\begin{aligned} Q(\alpha) &= d^2J(\Omega;V;V) \\ &= -(N-1)\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\Omega}|grad(\Lambda)|^2dx \\ &= -(N-1)k^2\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\partial\Omega}\alpha L\alpha d\sigma, \end{aligned}$$

where Λ is a solution of the following boundary value problem

$$\left\{ \begin{aligned} -\Delta\Lambda &= 0 \quad in \quad \Omega \\ \Lambda &= \alpha \quad on \quad \partial\Omega. \end{aligned} \right. \tag{4.31}$$

H is the mean curvature of $\partial\Omega$ and L is a pseudo differential operator which is known as the Steklov-Poincaré or capacity or Dirichlet to Neumann(see e.g [11]*) operator, defined by* $L\alpha = \dfrac{\partial\Lambda}{\partial\vec{\nu}}$.

Proof. We use the definition of the derivative with respect to the domain and we apply it to $dJ(\Omega, V)$.

Then we get

$$\begin{aligned} 2Q(\alpha) &= 2d^2J(\Omega,V,V) \\ &= \int_{\Omega}(\text{div}((k^2-|grad(u)|^2)V(x,0)))'dx + \int_{\Omega}\text{div}(V(x,0)\text{div}((k^2-|grad(u)|^2)V(x,0)))dx \\ 2Q(\alpha) &= \left[\int_{\partial\Omega} -2grad(u)grad(u')V(x,0).\vec{\nu} + \text{div}((k^2-|grad(u)|^2)V(x,0))V(x,0).\vec{\nu}\right]d\sigma. \end{aligned}$$

Since Ω is solution of the quadradure surface problem then $-\dfrac{\partial u}{\partial\vec{\nu}} = k$ on $\partial\Omega$.
By assumption, $\partial\Omega$ is of $\mathcal{C}^2$ class and since $u = 0$ on $\partial\Omega$,
we have
$grad(u) = \dfrac{\partial u}{\partial\vec{\nu}}\vec{\nu} = -k\vec{\nu}$. Hence

$$2Q(\alpha) = \left[\int_{\partial\Omega} 2kgrad(u').\vec{\nu}V(x,0).\vec{\nu} + \text{div}((k^2-|grad(u)|^2)V(x,0))V(x,0).\vec{\nu}\right]d\sigma.$$

A classical calculus in shape optimization leads to $u' = -\dfrac{\partial u}{\partial \vec{\nu}} V.\vec{\nu}$ on $\partial\Omega$.

Let us recall again that $-\dfrac{\partial u}{\partial \vec{\nu}} = k$ on $\partial\Omega$ and $V.\vec{\nu} = \alpha$. Then, we have $u' = k\alpha$ on $\partial\Omega$ and

$$\vec{\nu} grad(u') = \frac{\partial u'}{\partial \vec{\nu}} = k\frac{\partial \alpha}{\partial \vec{\nu}} = kL\alpha,$$

where L is a pseudo differential operator, defined by $L\alpha = \dfrac{\partial \Lambda}{\partial \vec{\nu}}$ and such that

$$\begin{cases} -\Delta\Lambda &= 0 \quad \text{in} \quad \Omega \\ \Lambda &= \alpha \quad \text{on} \quad \partial\Omega, \end{cases} \tag{4.32}$$

Λ is the extension of α in Ω.

Hence

$$2Q(\alpha) = \int_{\partial\Omega} (2k^2\alpha L\alpha - \text{div}((|grad(u)|^2 - k^2)\alpha.\vec{\nu})\alpha) d\sigma. \tag{4.33}$$

Let us compute now $\text{div}((|grad(u)|^2 - k^2)\alpha.\vec{\nu})$ on $\partial\Omega$. Since $|grad(u)| = k$ on $\partial\Omega$, we have

$$\text{div}((|grad(u)|^2 - k^2)\alpha.\vec{\nu}) = \alpha grad(|grad(u)|^2 - k^2).\vec{\nu} = \alpha grad(|grad(u)|^2).\vec{\nu}.$$

Since we have supposed that Ω of class $\mathcal{C}^2$, locally, $\partial\Omega$ can be described by a curve φ such that $x_N = \varphi(x'), x' \in \mathbb{R}^{N-1}$ and $D\varphi(x') = 0$. Here, $D\varphi(x')$ is the Jacobian matrix of φ.

Let us set $x_0 = (x', x_N) = (x', \varphi(x')) \in \partial\Omega$ then we have $u(x_0) = 0$.

By differentiating with respect to s_j for all $j \in \{1, \cdots, N-1\}$, we have

$$\frac{\partial u(x_0)}{\partial s_j} + \frac{\partial \varphi\ (x')}{\partial s_j}\ \frac{\partial u\ (x_0)}{\partial \vec{\nu}} = 0,$$

since $\dfrac{\partial \varphi(x')}{\partial s_j} = 0$, we get $\dfrac{\partial u(x_0)}{\partial s_j} = 0$.

Starting from the following equality

$$\frac{\partial u(x_0)}{\partial s_j} + \frac{\partial \varphi(x')}{\partial s_j}\ \frac{\partial u(x_0)}{\partial \vec{\nu}} = 0, \tag{4.34}$$

and by differentiating it with respect to s_i for all $i \in \{1, \cdots, N-1\}$, we have

$$\frac{\partial^2 u(x_0)}{\partial s_i\ \partial s_j} + \frac{\partial \varphi(x')}{\partial s_i}\ \frac{\partial^2 u(x_0)}{\partial \vec{\nu}\ \partial s_j} + \frac{\partial^2 \varphi(x')}{\partial s_i\ \partial s_j}\ \frac{\partial u(x_0)}{\partial \vec{\nu}}$$

$$+\frac{\partial \varphi(x')}{\partial s_j}\ \frac{\partial^2 u(x_0)}{\partial s_i\ \partial \vec{\nu}} + \frac{\partial \varphi(x')}{\partial s_j}\ \frac{\partial \varphi(x')}{\partial s_i}\ \frac{\partial^2 u(x_0)}{\partial \vec{\nu}^2} = 0.$$

Note that $u(x_0) = 0$ and $\dfrac{\partial u(x_0)}{\partial s_j} = 0\ \forall j \in \{1, \cdots, N-1\}$ and summing over the indices i, j, we have

$$\sum_{j=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_j^2} + (N-1)\ H\ \frac{\partial u(x_0)}{\partial \vec{\nu}} = 0. \tag{4.35}$$

Since $\dfrac{\partial u(x_0)}{\partial s_i} = 0\ \forall i \in \{1, \cdots, N-1\}$, we have also

$$\begin{aligned} grad(|grad(u)|^2(x_0)).\vec{\nu} &= \frac{\partial}{\partial \vec{\nu}}\left[\sum_{i=1}^{N-1}\left(\frac{\partial u(x_0)}{\partial s_i}\right)^2 + \left(\frac{\partial u(x_0)}{\partial \vec{\nu}}\right)^2\right] \\ &= 2\ \frac{\partial u(x_0)}{\partial \vec{\nu}}\ \frac{\partial^2 u(x_0)}{\partial \vec{\nu}^2}. \end{aligned}$$

In addition, we can remark that:

$\dfrac{\partial^2 u(x_0)}{\partial \vec{\nu}^2} = -\displaystyle\sum_{i=1}^{N-1} \frac{\partial^2 u(x_0)}{\partial s_i^2} - f$ on $\partial\Omega$.

Therefore, we have

$$\begin{aligned} grad(|grad(u)|^2(x_0)).\vec{\nu} &= 2\ \frac{\partial u(x_0)}{\partial \vec{\nu}}\left(-\sum_{i=1}^{N-1}\frac{\partial^2 u(x_0)}{\partial s_i^2} - f\right) \\ &= 2\ \frac{\partial u(x_0)}{\partial \vec{\nu}}\left((N-1)\ H\ \frac{\partial u(x_0)}{\partial \vec{\nu}} - f\right). \end{aligned}$$

When the support of the function f is in Ω, then $f = 0$ on $\partial\Omega$.

Finally we have

$$\begin{aligned} 2Q(\alpha) &= \int_{\partial\Omega} 2k^2\alpha L\alpha - 2(N-1)H\alpha^2\ \left(\frac{\partial u(x_0)}{\partial \vec{\nu}}\right)^2 d\sigma \\ &= \int_{\partial\Omega} 2k^2\alpha L\alpha - 2k^2(N-1)H\alpha^2 d\sigma. \end{aligned} \tag{4.36}$$

And by the Green's formula we get

$$\int_{\partial\Omega} \alpha L\alpha d\sigma = \int_{\Omega} |grad(\Lambda)|^2 dx. \tag{4.37}$$

□

5. Positiveness of the quadratic form in the infinite Riemannian point of view

Definition 5.1. *Let $J : \Omega \to \mathbb{R}$ be an functional. One defines the hessian Riemannian shape as follows:*

$$HessJ(\Omega)[V] := \nabla_V gradJ$$

where ∇_V denotes the derivative following the vector field V.

Theorem 5.1. *The hessian Riemannian shape defined by the Riemannian metric G^A verifies the following condition:*

$$G^A(HessJ(\Omega)[V], W) = d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W].$$

Proof. Our purpose is to show that

$$G^A(HessJ(\Omega)[V],W) = d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W].$$

So let us use the compatibility of the metric G^A with the Levi-Civita connection. We have

$$\begin{aligned} V.G^A(gradJ,W) &= G^A(gradJ,\nabla_V W) + G^A(\nabla_V gradJ,W), \\ G^A(\nabla_V gradJ,W) &= V.G^A(gradJ,W) - G^A(gradJ,\nabla_V W). \end{aligned}$$

Since $G^A(HessJ(\Omega)[V],W) = G^A(\nabla_V gradJ,W)$, we have

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= V.G^A(gradJ,W) - G^A(gradJ,\nabla_V W), \\ G^A(HessJ(\Omega)[V],W) &= V.(WJ) - (\nabla_V W).J), \\ G^A(HessJ(\Omega)[V],W) &= d(dJ(\Omega)[W])[V] - dJ(\Omega)[\nabla_V W] \end{aligned}$$

where $V,W \in \mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R}^2)$ are vector fields normal to the boundary $\partial\Omega$ and $d(dJ(\Omega)[W])[V]$ defines the standard Hessian shape. □

Remark 5.1. *In our quadrature surface case, for $W = m\vec{\nu}$ and $V = h\vec{\nu}$, we have*

$$dJ(\Omega)[V] = \int_{\partial\Omega} \left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\alpha d\sigma$$

and then,

$$d\left(dJ(\Omega)[W]\right)[V] = d\left(\int_{\partial\Omega}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) m d\sigma\right)[V].$$

Setting

$$\psi := k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2,$$

then

$$\psi_t := k^2 - \left(\frac{\partial u_{\Omega_t}}{\partial\vec{\nu}}\right)^2,$$

we have

$$d\left(dJ(\Omega)[W]\right)[V] \;=\; d\left(\int_{\partial\Omega_t}\psi_t m d\sigma\right)[h]. \tag{5.38}$$

This is nothing but

$$\begin{aligned} d\left(dJ(\Omega)[W]\right)[V] &= \int_{\partial\Omega}\frac{\partial\psi_t}{\partial t}_{|t=0} m d\sigma + \int_{\partial\Omega}\frac{\partial(\psi m)}{\partial\vec{\nu}} h d\sigma + \int_{\partial\Omega} K_c\psi m h d\sigma \\ &= \int_{\partial\Omega}\left[\frac{\partial\psi_t}{\partial t}_{|t=0} m + \left(\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi\right) m h + \psi\frac{\partial m}{\partial\vec{\nu}} h\right] d\sigma. \end{aligned}$$

Let us compute

$$\begin{aligned}
\frac{\partial \psi_t}{\partial t}_{|t=0} m &= \frac{\partial \left[k^2 - \left(\frac{\partial u_{\Omega_t}}{\partial \vec{\nu}_t}\right)^2\right]}{\partial t}_{|t=0} m,\\
\frac{\partial \psi_t}{\partial t}_{|t=0} m &= -2m\left[\frac{\partial (grad(u_{\Omega_t})).\vec{\nu}}{\partial t}_{|t=0} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} + D_{\vec{\nu}} V\right)\right],\\
\frac{\partial \psi_t}{\partial t}_{|t=0} m &= -2m\left[grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} + D_{\vec{\nu}} V\right)\right],
\end{aligned} \tag{5.39}$$

where $D^2 u_\Omega$ *is the hessian matrix and* $D_{\vec{\nu}}$ *the jacobian matrix of* $\vec{\nu}$ *.*

Let us calculate now the following expression: $\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0}$.

We have

$$\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} = -grad_\Gamma(V.\vec{\nu}) - (D_{\vec{\nu}_0}.\vec{\nu})V.\vec{\nu} \quad on \quad \Gamma,$$

where $grad_\Gamma$ *is the tangential gradient,* $\Gamma = \partial\Omega$ *and* $\vec{\nu}_0 = \vec{\nu}$ *hence*

$$\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} = -grad_\Gamma(V.\vec{\nu}) - (D_{\vec{\nu}}.\vec{\nu})V.\vec{\nu} \quad on \quad \Gamma.$$

Since $D_{\vec{\nu}}.\vec{\nu} \equiv 0$, *then*

$$\frac{\partial \vec{\nu}_t}{\partial t}_{|t=0} = -grad_\Gamma(V.\vec{\nu}) \quad on \quad \Gamma.$$

So

$$\frac{\partial \psi_t}{\partial t}_{|t=0} m = -2m\left[grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}.V\right)\right].$$

And finally, we get

$$\begin{aligned}
d\left(dJ(\Omega)[W]\right)[V] = \int_{\partial\Omega} \Big[&-2m\Big(grad(u'_\Omega)\cdot\vec{\nu} + D^2 u_\Omega V\cdot\vec{\nu}\\
&+ grad(u_\Omega)\cdot\big(-grad_\Gamma(V\cdot\vec{\nu}) + D_{\vec{\nu}} V\big)\Big)\\
&+ \left(\frac{\partial \psi}{\partial \vec{\nu}} + K_c\psi\right) mh + \psi\frac{\partial m}{\partial \vec{\nu}} h\Big] d\sigma,
\end{aligned}$$

$$\begin{aligned}
d\left(dJ(\Omega)[W]\right)[V] = \int_{\partial\Omega} \Big[&-2\langle W,\vec{\nu}\rangle\Big(grad(u'_\Omega)\cdot\vec{\nu} + D^2 u_\Omega V\cdot\vec{\nu}\\
&+ grad(u_\Omega)\cdot\big(-grad_\Gamma(V\cdot\vec{\nu}) + D_{\vec{\nu}} V\big)\Big)\\
&+ \left(\frac{\partial \psi}{\partial \vec{\nu}} + K_c\psi\right)\langle W,\vec{\nu}\rangle\langle V,\vec{\nu}\rangle + \psi\langle D_V W,\vec{\nu}\rangle\Big] d\sigma.
\end{aligned}$$

On the one hand, having the following Riemannian hessian formula

$$G^A\left(HessJ(\Omega)[V], W\right) = d\left(dJ(\Omega)[W]\right)[V] - dJ(\Omega)[\nabla_V W] \tag{5.40}$$

it is possible to bring additional details on its computation.

Proposition 5.1. *We have*

$$\begin{aligned} G^A(HessJ(\Omega)[V], W) &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \left(grad\left(u'_\Omega\right).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad\left(u_\Omega\right).\left(-grad_\Gamma(V.\vec{\nu})\right.\right.\right. \\ &+ \left.\left.\left. D_{\vec{\nu}}V\right)\right)\right] d\sigma \\ &+ \int_{\partial\Omega} \left[\frac{\partial}{\partial\vec{\nu}}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) + K_c\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &- \left.\frac{3AK_c^3 + K_c}{1 + AK_c^2}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.41}$$

Proof.

$$\begin{aligned} G^A\left(HessJ(\Omega)[V], W\right) &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \big(grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}V\right)\big)\right. \\ &+ \left.\left(\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi\right)\langle W, \vec{\nu}\rangle\langle V, \vec{\nu}\rangle + \psi\langle D_V W, \vec{\nu}\rangle\right] d\sigma \\ &- \int_{\partial\Omega} \psi\langle\nabla_V W, \vec{\nu}\rangle d\sigma, \\ &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \big(grad(u'_\Omega).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}V\right)\big)\right. \\ &+ \left.\left(\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi\right)\langle W, \vec{\nu}\rangle\langle V, \vec{\nu}\rangle + \psi\langle D_V W, \vec{\nu}\rangle\right] d\sigma \\ &- \int_{\partial\Omega} \psi\left[\langle D_V W, \vec{\nu}\rangle + \left(\frac{3AK_c^3 + K_c}{1 + AK_c^2}\right)\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle\right] d\sigma, \\ &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \left(grad\left(u'_\Omega\right).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad\left(u_\Omega\right).\left(-grad_\Gamma\left(V.\vec{\nu}\right)\right.\right.\right. \\ &+ \left.\left.\left. D_{\vec{\nu}}V\right)\right)\right] d\sigma \\ &+ \int_{\partial\Omega} \left[\frac{\partial\psi}{\partial\vec{\nu}} + K_c\psi - \psi K_c\left(\frac{3AK_c^2 + 1}{1 + AK_c^2}\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma. \end{aligned}$$

Replacing ψ by its expression, we have

$$\begin{aligned} G^A(HessJ(\Omega)[V], W) &= \int_{\partial\Omega} \left[-2\langle W, \vec{\nu}\rangle \left(grad\left(u'_\Omega\right).\vec{\nu} + D^2 u_\Omega V.\vec{\nu} + grad\left(u_\Omega\right).\left(-grad_\Gamma\left(V.\vec{\nu}\right)\right.\right.\right. \\ &+ \left.\left.\left. D_{\vec{\nu}}V\right)\right)\right] d\sigma \\ &+ \int_{\partial\Omega} \left[\frac{\partial}{\partial\vec{\nu}}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) + K_c\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &- \left.\frac{3AK_c^3 + K_c}{1 + AK_c^2}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.42}$$

□

On the other hand, let us compute $G^A(HessJ(\Omega)[V],W)$ by using directly the Sobolev-type metric G^A. Then we have the following proposition.

Proposition 5.2.

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &+ \left. K_c\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.43}$$

Proof.

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left(1+AK_c^2\right)HessJ(\Omega)[V]W, \\ &= \int_{\partial\Omega}\left(1+AK_c^2\right)\nabla_V gradJ(\Omega)W, \\ &= \int_{\partial\Omega}\left(1+AK_c^2\right)\nabla_h gradJ(\Omega)m. \end{aligned}$$

Since $gradJ(\Omega)=\frac{1}{1+AK_c^2}\psi$, we have

$$\begin{aligned} \nabla_h gradJ(\Omega) &= \frac{\partial}{\partial\vec{\nu}}\left(gradJ(\Omega)\right)\alpha+\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)gradJ(\Omega)\alpha, \\ &= \frac{\partial}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\psi\right)\alpha+\frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha, \\ &= \frac{\partial}{\partial\vec{\nu}}\left[(1+AK_c^2)^{-1}\right]\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha+\frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha, \\ &= -2AK_c\frac{\partial K_c}{\partial\vec{\nu}}\left(1+AK_c^2\right)^{-2}\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha \\ &+ \frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha. \end{aligned}$$

Note that $\frac{\partial K_c}{\partial\vec{\nu}}=K_c^2$, which implies that:

$$\begin{aligned} \nabla_h gradJ(\Omega) &= \frac{-2AK_c^3}{(1+AK_c^2)^2}\psi\alpha+\frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha \\ &+ \frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha. \end{aligned} \tag{5.44}$$

Then, coming back to our hessian computation, we have

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left(1+AK_c^2\right)\left[\frac{-2AK_c^3}{(1+AK_c^2)^2}\psi\alpha + \frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1+AK_c^2}\right)\alpha\right. \\ &+ \left.\frac{1}{1+AK_c^2}\psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\right]\beta d\sigma, \\ &= \int_{\partial\Omega}\left[\frac{-2AK_c^3}{1+AK_c^2}\psi\alpha + \frac{\partial\psi}{\partial\vec{\nu}}\alpha + \psi\left(\frac{3AK_c^3+K_c}{1+AK_c^2}\right)\alpha\right]\beta d\sigma, \\ &= \int_{\partial\Omega}\left[\frac{\partial\psi}{\partial\vec{\nu}} + \psi\left(\frac{AK_c^3+K_c}{1+AK_c^2}\right)\right]\alpha\beta d\sigma, \\ &= \int_{\partial\Omega}\left[\frac{\partial\psi}{\partial\vec{\nu}} + \psi K_c\left(\frac{1+AK_c^2}{1+AK_c^2}\right)\right]\alpha\beta d\sigma. \end{aligned} \tag{5.45}$$

Replacing ψ by its expression, we have

$$\begin{aligned} G^A(HessJ(\Omega)[V],W) &= \int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right. \\ &+ \left.K_c\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.46}$$

□

Remark 5.2. *Let us note first that there is a symmetry relation with respect to the hessian which is in the case of our considered Riemannian structure a self adjoint operator with respect to the metric* G^A.

And the second fact is that it is important to underline that the formulas (5.42) *obtained from the formula in Theorem 5.1 and* (5.46) *computed by a direct method with the metric* G^A *in two different ways, have to give the same expression even if* Ω *is not a critical point. And then from these computations, one deduces that*

$$\begin{aligned} &\int_{\partial\Omega}\left[-2\langle W,\vec{\nu}\rangle\left(grad(u'_\Omega).\vec{\nu} + D^2u_\Omega V.\vec{\nu} + grad(u_\Omega).\left(-grad_\Gamma(V.\vec{\nu}) + D_{\vec{\nu}}V\right)\right)\right]d\sigma \\ &\qquad = \int_{\partial\Omega}\frac{3AK_c^3+K_c}{1+AK_c^2}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\langle V,\vec{\nu}\rangle\langle W,\vec{\nu}\rangle d\sigma. \end{aligned} \tag{5.47}$$

Remark 5.3. *In this remark, we compute* $G^A(V, HessJ(\Omega)[W])$ *to show the symmetry relation with respect to the hessian with the computation of the direct method with the metric*

G^A.

$$G^A(V, HessJ(\Omega)[W]) = \int_{\partial\Omega} \left(1 + AK_c^2\right) HessJ(\Omega)[W]V, \\ = \int_{\partial\Omega} \left(1 + AK_c^2\right) \nabla_W gradJ(\Omega)V, \\ = \int_{\partial\Omega} \left(1 + AK_c^2\right) \nabla_m gradJ(\Omega)h \tag{5.48}$$

where $V = h = \alpha\vec{\nu}$ *and* $W = m = \beta\vec{\nu}$. *Since* $gradJ(\Omega) = \frac{1}{1+AK_c^2}\psi$, *we have*

$$\nabla_m gradJ(\Omega) = \frac{\partial}{\partial\vec{\nu}}\left(gradJ(\Omega)\right)\beta + \left(\frac{3Ak_c^3 + K_c}{1 + AK_c^2}\right) gradJ(\Omega)\beta, \\ = \frac{\partial}{\partial\vec{\nu}}\left(\frac{1}{1 + AK_c^2}\psi\right)\beta + \frac{1}{1 + AK_c^2}\psi\left(\frac{3Ak_c^3 + K_c}{1 + AK_c^2}\right)\beta.$$

As previously, by the same computations, we get

$$\nabla_m gradJ(\Omega) = \frac{-2AK_c^3}{(1 + AK_c^2)^2}\psi\beta + \frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1 + AK_c^2}\right)\beta + \frac{1}{1 + AK_c^2}\psi\left(\frac{3Ak_c^3 + K_c}{1 + AK_c^2}\right)\beta.$$

And finally, we have

$$G^A(HessJ(\Omega)[W], V) = \int_{\partial\Omega} \left(1 + AK_c^2\right)\left[\frac{-2AK_c^3}{(1 + AK_c^2)^2}\psi\beta + \frac{\partial\psi}{\partial\vec{\nu}}\left(\frac{1}{1 + AK_c^2}\right)\beta \right. \\ + \left. \frac{1}{1 + AK_c^2}\psi\left(\frac{3AK_c^3 + K_c}{1 + AK_c^2}\right)\beta\right]\alpha d\sigma, \\ = \int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right) + K_c\left(k^2 - \left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\langle V, \vec{\nu}\rangle\langle W, \vec{\nu}\rangle d\sigma.$$

Let us have a look at the two formulas of the second derivation when $V = W = \alpha\vec{\nu}$. On the one hand, by Proposition 4.1, we get

$$Q(\alpha) = d^2J(\Omega; V; V), \\ = -(N-1)\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_\Omega |grad(\Lambda)|^2 dx, \\ = -(N-1)k^2\int_{\partial\Omega} H\alpha^2 d\sigma + k^2\int_{\partial\Omega}\alpha L\alpha d\sigma. \tag{5.49}$$

On the other hand by Theorem 5.1, we have

$$G^A\left(HessJ(\Omega)[V], W\right) = d\left(dJ(\Omega)[W]\right)[V] - dJ(\Omega)[\nabla_V W]. \tag{5.50}$$

Then for $V = W$ we derive

$$d\left(dJ(\Omega)[V]\right)[V] = d^2J(\Omega; V; V) = G^A\left(HessJ(\Omega)[V], V\right) + dJ(\Omega)[\nabla_V V]. \tag{5.51}$$

- If the quadrature surface problem has a solution Ω, then
  $d\left(dJ(\Omega)[V]\right)[V] = G^A\left(HessJ(\Omega)[V], V\right).$

- In previous works, the second author studied the stability and positiveness of the quadratic form, see [30] for more details. He established a proposition similar to Proposition 4.1 and gave necessary and sufficient qualitative properties in the theoretical point of view.

The one obtained involves the study of a generalized spectral Steklov problem that is reminded in the following corollary.

Corollary 5.1. *Let us consider the following generalized spectral Steklov problem:*

$$\begin{aligned} \Delta\phi_n &= 0 \ \ in \ \ \Omega\backslash K \\ \phi_n &= 0 \ \ on \ \ \partial K \\ (L+(N-1)HI)\phi_n &= (\frac{1}{\mu_n} - \|H^-\|_\infty)\phi_n \ \ on \ \ \partial\Omega, \end{aligned}$$

where I is the identity map, H is the mean curvature of Ω, K is a compact regular enough subset of Ω, $H^- = max\{-H, 0\}$ and μ_n is a decreasing sequence of eigenvalues depending also on H which goes to 0. And one must have the sign of the first eigenvalue

$$\lambda_0 := \frac{1}{\mu_0} - \|H^-\|_\infty = \inf\left\{(N-1)\int Hv^2 d\sigma + \int_{\Omega\backslash K} |grad(\Lambda)|^2 dx, v \in H^{1/2}(\partial\Omega), \int_{\partial\Omega} v^2 d\sigma = 1\right\},$$

where

$$\begin{aligned} \Delta\Lambda &= 0 \ \ in \ \ \Omega\backslash K \\ \Lambda &= 0 \ \ on \ \ \partial K \\ \frac{\partial\Lambda}{\partial\vec{\nu}} &= v \ \ on \ \ \partial\Omega. \end{aligned}$$

And the minimum is reached for ϕ_0 satisfying

$$\begin{aligned} \Delta\phi_0 &= 0 \ \ in \ \ \Omega\backslash K \\ \phi_0 &= 0 \ \ on \ \ \partial K \\ (L+(N-1)HI)\phi_0 &= \lambda_0\phi_0 \ \ on \ \ \partial\Omega. \end{aligned}$$

From our work we can deduce the following conclusions as a corollary.

Corollary 5.2.

- *What is obtained with the Riemannian hessian formula is easier to derive simple control for the characterization of the optimal shape in a number of ways.*

- *In the case of minimum,* $G^A\left(HessJ(\Omega)[V],V\right)\geq 0$. *And this inequality is equivalent to* $\displaystyle\int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\alpha^2 d\sigma\geq 0,\ \forall\alpha\in\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R})\cap H^{1/2}(\partial\Omega)$.
This is reduced to $\displaystyle\int_{\partial\Omega}\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)d\sigma\geq 0$.
One can deduce also another control, since

$$\int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\alpha^2 d\sigma=-2k^2(N-1)\int_{\partial\Omega}H\alpha^2 d\sigma=-2k^2(N-1)\int_{\partial\Omega}K_c\alpha^2 d\sigma.$$

Before proceeding further, let us underline that in two dimension $H=K_c$. *And knowing that* $\alpha\in\mathcal{C}^\infty(\mathbb{R}^2,\mathbb{R})\cap H^{1/2}(\partial\Omega)$, *the control becomes* $\displaystyle\int_{\partial\Omega}K_c d\sigma=2\pi\chi(\partial\Omega)\leq 0$ *where* $\chi(\partial\Omega)$ *is the Euler- Poincaré characteristic. And from this, we can deduce that by Gauss- Bonnet theorem the control is done on the Euler-Poincaré characteristic. And from this, we have key information to set up algorithm in order to get a good approximation of the optimal shape.*

- *Now, when* Ω *is only a critical point, to get a strict local minimum, we need the following sufficient condition:*

$$\int_{\partial\Omega}\left[\frac{\partial}{\partial\vec{\nu}}\left(k^2-\left(\frac{\partial u_\Omega}{\partial\vec{\nu}}\right)^2\right)\right]\alpha^2 d\sigma=-2k^2(N-1)\int_{\partial\Omega}K_c\alpha^2 d\sigma\geq C_0\|\alpha\|^2, C_0>0.$$

One can say also that there is $x_0\in\partial\Omega, -2k^2(N-1)K_c(x_0)\displaystyle\int_{\partial\Omega}\alpha^2 d\sigma\geq C_0\|\alpha\|^2$. *And if* $K_c(x_0)<0$, *then* Ω *is a strict local minimum. If the Euler-Poincaré characteristic is positive, then there is not a minimum.*

Conflicts of interests. The authors declare that there is no conflict of interests.

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ON $\mathcal{I}$ CONCURRENT CLASS OF SEQUENCES

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Abstract. In this paper, we demonstrate the $\mathcal{I}$-concurrent relation between sequences and the equivalence relations produced from it. A few unique features of these equivalence classes are investigated. Finally, we show that the collection of all such equivalence classes of all $\mathcal{I}$-convergent sequences under the $\mathcal{I}$-concurrent relation generates a metric space that is isometric with the set of all constant sequences.

Keywords: $\mathcal{I}$-convergence, $\mathcal{I}$-Cauchy, $\mathcal{I}$-concurrent relation.

2010 Mathematics Subject Classification: 40A35, 40D25.

1. Introduction

Natural density, additionally referred to as asymptotic density, is a fundamental notion in number theory and analysis that measures how large a subset of natural number is relative to the set of all natural numbers. For $M \subseteq \mathbb{N}$, the natural density of M is denoted by $\delta(M)$ and is defined as

$$\delta(M) = \lim_{n\to\infty} \frac{|k \leq n : k \in M|}{n}.$$

This notion holds significance not just in pure mathematics but also in disciplines such as statistical mechanics, probability theory and computer science, where understanding the distribution of numbers may reveal patterns and behaviors inside complex systems. Steinhaus

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[18] and Fast [7] in 1951, developed the idea of statistical convergence independently by implementing the idea of natural density (also known as asymptotic density). The statistical convergence of a sequence $\langle x_n \rangle$ to x_0 is attained if, for any $\varepsilon > 0$, the set $\{k \in \mathbb{N} : d(x_k, x_0) \geq \varepsilon\}$ has a density of zero. Numerous mathematicians, such as Fridy [8, 9], Salat [15], Rath and Tripathy [14], Bal [2], Sarkar et al. [17] have conducted extensive research on this convergence. In 2000, Kostyrko et al. [11] raised the concept of $\mathcal{I}$-convergence, while $\mathcal{I}$ Cauchy sequences was initially defined by Nabiev et al. [13]. $\mathcal{I}$-convergence is an extension of statistical convergence depending on the ideal's ($\mathcal{I}$'s) framework, where $\mathcal{I}$ is a family of subsets of the set of natural numbers. Although there have been a lot of generalizations of statistical convergence, we found $\mathcal{I}$-convergence the most interesting one, where $\mathcal{I}$ is an ideal. In the recent literature, there have been several publications on $\mathcal{I}$-convergence [3, 4, 5, 6, 10, 12, 3, 16], including some outstanding contributions by Bal [1].

In this study, we seek to establish a relationship between two sequences of the same nature by means of $\mathcal{I}$-convergence. In order to accomplish this, we introduce the $\mathcal{I}$-concurrent relation, which establishes an equivalence relation on the collection of all sequences in a metric space. Also, the collection of all equivalence classes produced by that equivalence relation on the set of all $\mathcal{I}$ -convergent sequences constructs a metric space.

2. Preliminaries

Prior to studying $\mathcal{I}$ concurrent sequences in depth, it is important to provide some basic definitions and notions. In this section, we briefly discuss the fundamental instruments and mathematical concepts required to comprehend the key findings.

Definition 2.1. [13] *A family $\mathcal{I}$ of subsets of a non empty set X is called an ideal if and only if $\emptyset \in \mathcal{I}$, $\mathcal{I}$ is closed under finite union and $\mathcal{I}$ is closed under subset. Also a family $\mathcal{F}$ of subsets of a non empty set X is called a filter if and only if $\emptyset \notin \mathcal{F}$, $\mathcal{F}$ is closed under finite intersection and $\mathcal{F}$ is closed under superset.*

If $X \notin \mathcal{I}$ and $\mathcal{I} \neq \emptyset$, then the ideal $\mathcal{I}$ is considered as an non-trivial ideal. If $\mathcal{I}$ is an ideal, then the collection $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X \setminus M : M \in \mathcal{I}\}$ is a filter and called the dual filter of the ideal $\mathcal{I}$. If $\mathcal{I}$ is a non-trivial ideal which contains every singleton subset of X, then $\mathcal{I}$ is considered to be an admissible ideal. '$\mathcal{I}$' will represent an admissible ideal throughout the paper.

Definition 2.2. [13] *Let $\mathcal{I}$ be an admissible ideal defined on the set $\mathbb{N}$ of natural numbers and (X, d) be a metric space. For a sequence $\langle x_n \rangle$, if for each $\varepsilon > 0$,*

$$\{k \in \mathbb{N} : d(x_k, x_0) \geq \varepsilon\} \in \mathcal{I},$$

then $\langle x_n \rangle$ is considered to be $\mathcal{I}$-convergent to x_0.

Definition 2.3. [13] *Let $\mathcal{I}$ be an admissible ideal defined on the set $\mathbb{N}$ of natural numbers and (X, d) be a metric space. For a sequence $\langle x_n \rangle$, if for every $\varepsilon > 0$ there exists a $m \in \mathbb{N}$ such that*

$$\{n \in \mathbb{N} : d\left(x_n, x_m\right) \geq \varepsilon\} \in \mathcal{I},$$

then $\langle x_n \rangle$ is considered to be an $\mathcal{I}$-Cauchy sequence in X.

3. On $\mathcal{I}$ Concurrent Sequences

Using the concept of $\mathcal{I}$-convergence, we want to create an equivalence relation on the set of all sequences that will separate the sequence space into disjoint equivalence classes. These classes of sequences will have sequences that are similar in nature, making it easier to study each one freely.

Theorem 3.1. *If the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfy the $\mathcal{I}$-Cauchy criteria in a metric space (X, d), then the sequences $\langle z_n = d(x_n, y_n) : n \in \mathbb{N} \rangle$ will satisfy the $\mathcal{I}$-Cauchy criteria in a metric space (X, d_1) where $d_1(a, b) = |a - b|$.*

Proof. Since $\langle x_n \rangle$ and $\langle y_n \rangle$ satisfies $\mathcal{I}$-Cauchy criteria, therefore, $A_1 = \{n \in \mathbb{N} : d\left(x_n, x_{m_1}\right) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$, for some $m_1 \in \mathbb{N}$ and $A_2 = \{n \in \mathbb{N} : d\left(y_n, y_{m_2}\right) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$, for some $m_2 \in \mathbb{N}$.

Let $m = max\{m_1, m_2\}$. Also, $(A_1 \cap A_2) \in \mathcal{F}(\mathcal{I})$ and $\phi \notin \mathcal{F}(\mathcal{I})$, so $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$, we have

$d_1(z_n, z_m) \leq d(x_n, x_m) + d(y_n, y_m) < \varepsilon$,

i.e., $\{n \in \mathbb{N} : d_1(z_n, z_m) < \varepsilon\} \supseteq (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I})$.

Therefore, $\langle z_n \rangle$ fulfills $\mathcal{I}$-Cauchy criteria in the metric space (X, d_1). □

Definition 3.1. *A sequence $\langle x_n \rangle$ is said to be $\mathcal{I}$ concurrent to another sequence $\langle y_n \rangle$ if the sequence $\langle z_n \rangle = \langle d(x_n, y_n) \rangle$ is such that $z_n \xrightarrow{\mathcal{I}\text{-}lim} 0$. That is, $\{n \in \mathbb{N} : z_n = d(x_n, y_n) \geq \varepsilon\} \in \mathcal{I}$.*

Example 3.1. *Let $X = \{0, 1\}$, equipped with the discrete metric $\sigma(x, y) = \begin{cases} 0, & if\ x = y, \\ 1, & otherwise, \end{cases}$. Consider the ideal $\mathcal{I}_\delta = \{A : \delta(A) = 0\}$ and the sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ where*

$$a_n = \begin{cases} 0, & for\ n = k^2, k \in \mathbb{N}, \\ 1, & otherwise, \end{cases} \quad b_n = 1\ for\ all\ n \in \mathbb{N}\ and\ c_n = \begin{cases} 1, & for\ n = k^2, k \in \mathbb{N}, \\ 0, & otherwise. \end{cases}$$

For every $\varepsilon > 0$, $\{n \in \mathbb{N} : \sigma(a_n, b_n) \geq \varepsilon\} = \{n = k^2 : k \in \mathbb{N}\} \in \mathcal{I}_\delta$, $\{n \in \mathbb{N} : \sigma(b_n, c_n) \geq \varepsilon\} = \{n \neq k^2 : k \in \mathbb{N}\} \notin \mathcal{I}_\delta$ *and* $\{n \in \mathbb{N} : \sigma(c_n, a_n) \geq \varepsilon\} = \mathbb{N} \notin \mathcal{I}_\delta$. *Thus,* $\{a_n\}$ *and* $\{b_n\}$ *are* $\mathcal{I}$ *concurrent to each other; whereas* $\{c_n\}$ *is* $\mathcal{I}$ *concurrent neither to* $\{a_n\}$ *nor to* $\{b_n\}$.

Theorem 3.2. *For two* $\mathcal{I}$ *concurrent sequences* $\langle x_n \rangle$ *and* $\langle y_n \rangle$*, if one sequence is* $\mathcal{I}$ *Cauchy, then the other also satisfies the* $\mathcal{I}$*-Cauchy criteria.*

Proof. Let $\langle x_n \rangle$ satisfy the $\mathcal{I}$ Cauchy criteria. Therefore, $A_1 = \{n \in \mathbb{N} : d(x_n, x_m) < \frac{\varepsilon}{3}\} \in \mathcal{F}(\mathcal{I})$. Since $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$ concurrent, therefore $A_2 = \{n \in \mathbb{N} : d(x_n, y_n) < \frac{\varepsilon}{3}\} \in \mathcal{F}(\mathcal{I})$.

Since $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$ there exists a $m \in (A_1 \cap A_2)$ so that

$$d(y_n, y_m) \leq d(y_n, x_n) + d(x_n, x_m) + d(x_m, y_m) < \varepsilon,$$

$$\text{i.e., } \{n \in \mathbb{N} : d(y_n, y_m) < \varepsilon\} \in \mathcal{F}(\mathcal{I}) \text{ as } (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I}).$$

Therefore, $\langle y_n \rangle$ also satisfies $\mathcal{I}$-Cauchy criteria. □

Example 3.2. *Two non* $\mathcal{I}$*-Cauchy sequences can be* $\mathcal{I}$*-Concurrent to each other. Let* $X = [0, 2]$ *and* $d(a, b) = |(a - b)|$ *for all* $a, b \in X$*. Then* (X, d) *forms a metric space. Also consider the sequences* $\langle x_n \rangle$ *and* $\langle y_n \rangle$ *of* (X, d) *where*

$$x_n = \frac{[1 + (-1)^{n+1}]}{2}$$

and

$$y_n = \begin{cases} 1, & for\ n\ is\ odd, \\ \frac{1}{n}, & for\ n\ is\ even. \end{cases}$$

Now take $\mathcal{I} = \mathcal{I}_{fin}$*, the ideal containing all finite subsets of* $\mathbb{N}$*, then neither* $\langle x_n \rangle$ *nor* $\langle y_n \rangle$ *satisfy* $\mathcal{I}$*-Cauchy criteria.*

But $\langle z_n \rangle = \langle d(x_n, y_n) \rangle$ *where* $d(x_n, y_n) = \begin{cases} 0, & for\ n\ is\ odd \\ \frac{1}{n}, & for\ n\ is\ even \end{cases}$ *is* $\mathcal{I}$*-convergent to* 0.

Since, $\{n \in \mathbb{N} : z_n \geq \varepsilon\} \in \mathcal{I}$

Therefore, $\langle x_n \rangle$ *and* $\langle y_n \rangle$ *are not* $\mathcal{I}$*-Cauchy but* $\mathcal{I}$*-concurrent sequences.*

Example 3.3. *Again, if two sequences are* $\mathcal{I}$*-Cauchy sequences, it does not imply that they are* $\mathcal{I}$*-concurrent. For example,* *Let $X = [0, 2]$ and $d(a, b) = |(a - b)|$ for all $a, b \in X$. Then (X, d) forms a metric space. Also consider the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ of (X, d) where*

$$x_n = \begin{cases} 2, & for\ n = k^2, k \in \mathbb{N}, \\ 1 + \frac{1}{n}, & otherwise \end{cases}$$

and

$$y_n = \begin{cases} 2, & for\ n = k^2, k \in \mathbb{N}, \\ \frac{1}{n}, & otherwise. \end{cases}$$

Now, if we take $I = I_\delta$, the class of all subsets of $\mathbb{N}$ whose natural density is 0, then $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-Cauchy sequences.

But $\langle z_n \rangle = \langle d(x_n, y_n) \rangle$ where $d(x_n, y_n) = \begin{cases} 0, & for\ n = k^2, k \in \mathbb{N}, \\ 1, & otherwise. \end{cases}$ *is not $\mathcal{I}$-convergent to* 0.

That is, $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-Cauchy but not $\mathcal{I}$-concurrent to each other.

Theorem 3.3. *Two sequences are $\mathcal{I}$ convergent to the same limit if and only if they are $\mathcal{I}$ concurrent sequences, one of them being $\mathcal{I}$ convergent.*

Proof. Let $\langle x_n \rangle$ be $\mathcal{I}$-convergent to the limit ℓ. Therefore, $A_1 = \{n \in \mathbb{N} : d(x_n, \ell) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$.

Also, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be $\mathcal{I}$-concurrent. Therefore, $A_2 = \{n \in \mathbb{N} : d(x_n, y_n) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$.

Since $(A_1 \cap A_2) \neq \phi$ and for all $n \in (A_1 \cap A_2)$ we have

$$d(y_n, \ell) \leq d(y_n, x_n) + d(x_n, \ell) < \varepsilon,$$

$$\text{i.e., } \{n \in \mathbb{N} : d(y_n, \ell) < \varepsilon\} \in \mathcal{F}(\mathcal{I}) \text{ as } (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I}).$$

Therefore, $\langle y_n \rangle$ is also $\mathcal{I}$-convergent to the same limit ℓ.

Conversely, let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two $\mathcal{I}$ convergent sequences converging to the same limit ℓ. That is, $B_1 = \{n \in \mathbb{N} : d(x_n, \ell) < \frac{\varepsilon}{2}\} \in \mathcal{F}(\mathcal{I})$ and $B_2 = \{n \in \mathbb{N} : d(y_n, \ell) < \frac{\varepsilon}{2}\} \in \mathcal{F}(I)$. So $\forall\ n \in (A_1 \cap A_2) \subset \mathbb{N}$ we have

$$d(x_n, y_n) \leq d(x_n, \ell) + d(y_n, \ell) < \varepsilon,$$

$$\text{i.e., } \{n \in \mathbb{N} : d(x_n, y_n) < \varepsilon\} \supseteq (A_1 \cap A_2) \in \mathcal{F}(\mathcal{I}).$$

Therefore, $\langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-concurrent sequences. □

Theorem 3.4. *Let S_X be the collection of all sequences on the metric space (X, d). Then the $\mathcal{I}$-concurrent relation $\approx_{\mathcal{I}\text{-}d}$ forms an equivalence relation on S_X.*

Proof. Since for any $\langle x_n \rangle \in S_X$, $d(x_n, x_n) = 0, \forall n \in \mathbb{N}$. Therefore, $\langle d(x_n, x_n) \rangle$ is $\mathcal{I}$-convergent to 0. So every sequence is $\mathcal{I}$-concurrent to itself. That is, the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ is a reflexive relation on S_X.

Since for any $\langle x_n \rangle, \langle y_n \rangle \in S_X$, $d(x_n, y_n) = d(y_n, x_n), \forall n \in \mathbb{N}$. Therefore, if $\langle x_n \rangle$ is $\mathcal{I}$-concurrent to $\langle y_n \rangle$ then $\langle y_n \rangle$ is also $\mathcal{I}$-concurrent to $\langle x_n \rangle$. That is, the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ is a symmetric relation on S_X.

Let $\langle x_n \rangle, \langle y_n \rangle, \langle v_n \rangle \in S_X$, Now, $d(x_n, v_n) \leq d(x_n, y_n) + d(y_n, v_n), \forall n \in \mathbb{N}$. It implies that if $\langle d(x_n, y_n) \rangle$ and $\langle d(y_n, v_n) \rangle$ are $\mathcal{I}$-convergent to 0, then $\langle d(x_n, v_n) \rangle$ is also $\mathcal{I}$-convergent to 0.

So if $(x_n) \approx_{\mathcal{I}\text{-}d} (y_n)$ and $(y_n) \approx_{\mathcal{I}\text{-}d} (v_n) \implies (x_n) \approx_{\mathcal{I}\text{-}d} (v_n)$, i.e., the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ is a transitive relation on S_X.

$\therefore$ the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$ forms an equivalence relation on S_X. □

Corollary 3.1. *The set S_X of all sequences of the space (X, d) splits into disjoint equivalent classes under the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$, so that all the sequences of one class are*

(i) Either $\mathcal{I}$-convergent to the same limit or is not $\mathcal{I}$-convergent.

(ii) Either $\mathcal{I}$-Cauchy sequences or none of them are $\mathcal{I}$-Cauchy.

Proof. This can be easily verified from the previous theorems. □

Let C_X be the collection of all $\mathcal{I}$-Convergent sequences of the metric space (X, d). Also let $\langle \boldsymbol{x_n} \rangle^* = \{\langle y_n \rangle \in C_X : \langle y_n \rangle \approx_{\mathcal{I}\text{-}d} \langle x_n \rangle\}$, where $\langle \boldsymbol{x_n} \rangle^*$ denotes the equivalence class of $\langle x_n \rangle$ under the $\mathcal{I}$-concurrent relation $(\approx_{\mathcal{I}\text{-}d})$.

We define $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = d'(\langle x_n \rangle, \langle y_n \rangle) = \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n)$

Let $\boldsymbol{C_X}^*$ denote the set of all equivalence classes $\langle \boldsymbol{x_n} \rangle^*$, where $\langle x_n \rangle \in C_X$.

Theorem 3.5. *The set of all equivalence classes $\boldsymbol{C_X}^*$ forms a metric space with the metric σ such that $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n)$.*

Proof. Let $\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*, \langle \boldsymbol{z_n} \rangle^* \in \boldsymbol{C_X}^*$. So there is a $\langle x_n \rangle \in \langle \boldsymbol{x_n} \rangle^*, \langle y_n \rangle \in \langle \boldsymbol{y_n} \rangle^*, \langle z_n \rangle \in \langle \boldsymbol{z_n} \rangle^*$.

Now since d is a metric, $\forall n \in \mathbb{N}$.

$d(x_n, y_n) \geq 0 \implies \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) \geq 0$

$\implies \sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) \geq 0$ for all $\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*, \in \boldsymbol{C_X}^*$

Now $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = 0$

$\iff \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) = 0$

$\iff \langle x_n \rangle$ and $\langle y_n \rangle$ are $\mathcal{I}$-concurrent to each other. So they belong to the same equivalence class.

That is, $\langle \boldsymbol{x_n} \rangle^* = \langle \boldsymbol{y_n} \rangle^*$.

Again, $\sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) = \mathcal{I}\text{-}lim_{n\to\infty} d(y_n, x_n) = \sigma(\langle \boldsymbol{y_n} \rangle^*, \langle \boldsymbol{x_n} \rangle^*)$ (Since d is symmetric).

$\therefore \sigma$ is symmetric.

Since d is metric, we have $d(x_n, y_n) \le d(x_n, z_n) + d(z_n, y_n)$, $\forall n \in \mathbb{N}$.

$\therefore \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, y_n) \le \mathcal{I}\text{-}lim_{n\to\infty} d(x_n, z_n) + \mathcal{I}\text{-}lim_{n\to\infty} d(z_n, y_n)$.

$\implies \sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) \le \sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{z_n} \rangle^*) + \sigma(\langle \boldsymbol{z_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*)$

Hence $(\boldsymbol{C_X}^*, \sigma)$ forms a metric space. □

Theorem 3.6. *Let* $C'_X = \{\{x_n : x_n = x, \forall n \in \mathbb{N}\} : x \in X\}$. *Then* (C'_X, d') *and* $(\boldsymbol{C_X}^*, \sigma)$ *are isometry.*

Proof. Let $f : C'_X \to \boldsymbol{C_X}^*$ is a function defined by $f(\langle x_n \rangle) = \langle \boldsymbol{x_n} \rangle^*$ where $\langle \boldsymbol{x_n} \rangle^* = \{\{z_n\} \in C_X : \langle z_n \rangle \approx_{\mathcal{I}\text{-}d} \langle x_n \rangle\}$. Now $\sigma(f(\langle x_n \rangle), f(\langle y_n \rangle)) = \sigma(\langle \boldsymbol{x_n} \rangle^*, \langle \boldsymbol{y_n} \rangle^*) = d'(\langle x_n \rangle, \langle y_n \rangle)$. Therefore, f is an isometry.

Again $f(\langle x_n \rangle) = f(\langle y_n \rangle) \implies \langle x_n \rangle = \langle y_n \rangle$ and for any equivalence class $\langle \boldsymbol{w_n} \rangle^* \in \boldsymbol{C_X}^*$ there exist a constant sequences $\langle w_n \rangle \in C'_X$, i.e., f is a bijective mapping. Hence (C'_X, d') and $(\boldsymbol{C_X}^*, \sigma)$ are isometry. □

4. Conclusion

An equivalence relation that splits the sequence space into disjoint equivalence classes has been discovered on the set of all sequences. Sequences in these categories are of the same kind with respect to $\mathcal{I}$-convergence and $\mathcal{I}$-Cauchy criteria. Further, a metric space is obtained for the collection of all these equivalence classes. It is possible to study the classes of point-wise convergence, uniform convergence, etc. independently if this idea is extended to the sequences of functions.

Declaration on Data Availability and Financial Support:

No data set has been produced or analyzed for this article. Therefore, the sharing of data is not relevant here. The individuals who wrote it don't own any proprietary or financial stake in any of the content covered in this piece of writing.

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PROPERTIES OF DIVISOR PRIME GRAPH

SANJOY KALITA AND MRIDUL DUTTA [2,*]

Abstract. Number theory is a mathematical discipline that uses concepts from graph theory. Recently, various graphs have been defined in relation to various number theoretic functions. One such graph is the divisor prime graph, which is associated with the positive divisors of a positive integer. Let n be a positive integer and $D(n)$ be the set of all positive divisors of n. The *divisor prime graph* $PG_D(n)$ is defined as a graph whose vertex set is $D(n)$ and any two vertices x and y are adjacent in $PG_D(n)$ iff $\gcd(x, y) = 1$. In this study, families of divisor prime graphs for different positive integers are investigated, along with their graph theoretic characteristics such as adjacency, diameter, radius, clique number, chromatic number, planarity, connectivity, independence number and density.

Keywords: Divisor, Prime factor, Greatest common divisor, Connectedness, Diameter, Girth, Radius, Isomorphism, Planar graph.

2010 Mathematics Subject Classification: 05C38, 05C90, 05C4.

1. Introduction

In 2000, Singh and Santhosh [15] introduced the idea of divisor graphs. A divisor graph G is an ordered pair (V, E) where V is a subset of $\mathbb{Z}$ and $uv \in E$ if and only if either $u|v$ or $v|u$ for all $u \neq v$. Many authors had studied an alternative construction of graphs by associating with algorithmic approach on MV-algebras[9], Zero divisor graphs[1, 2], total graphs, prime graphs[14]. Any graph isomorphic to a divisor graph is also called a divisor graph. Additionally, they have pointed out some of the graphs those which are divisor

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graphs and also those which are not. Chartrand et al. [6] studied many more additional properties of divisor graph. Le Anh Vinh [17] also established the existence of a divisor graph of order n and size m for every pair of $m, n \in \mathbb{Z}$ with $0 \leq m \leq n$. Christopher Frayer [8] conducted research on the necessary conditions for a Cartesian product graph to be a divisor graph. Yu-ping Tsao [16] has examined several properties of $D([n])$ and its complement, including the vertex-chromatic number, the clique number, the cover number, and independence number, where $[n] = \{i : 1 \leq i \leq n,\ n \in N\}$. Nathanson [13] introduced the concepts and the notion of congruences from number theory in Graph Theory. He initiated the new way for the emergence of a new class of graphs, namely, arithmetic graphs. An arithmetic graph is one in which any two vertices a and b are adjacent if and only if $a + b \equiv c(mod\ n)$ where $c \in S$, a pre-assigned subset of V. Its vertex set V is the set of the first n positive integers $1, 2, 3, \ldots, n$. Let $(G,\ \cdot)$ be a finite group and $S \subseteq G$ such that $s^{-1} \in S$ for all $s \in S$. S is called symmetric subset of G. A Cayley graph $C(G, S)$ is the graph in which the vertex set $V = G$ and the edge set $E = \{(a, b) : a^{-1}b \in S\ or\ b^{-1}a \in S,\ \forall a,\ b \in G\}$. If $(G,\ \cdot) = (\mathbb{Z}_n, +)$ and the symmetric set S is associated with some arithmetic function, then such Cayley graphs are called arithmetic Cayley graphs. Dejter and Giudici [7], Berrizabeitia and Giudici [3] and others have studied the cycle structure of Cayley graphs associated with certain arithmetic functions. The circumference and girth of the arithmetic Cayley graphs are investigated by Madhavi and Maheswari [11], associated with the Euler totient function $\phi(n)$, and divisor function $d(n)$. The cycle structure of these graphs has many applications in engineering and communication networks. Chalapati, Madhavi and Venkataramana [5] studied the enumeration of triangles in these graphs. The Divisor Prime graph was a novel idea developed by S. M. Nair and J. S. Kumar[12], who also looked into its structural characteristics. They integrated the concepts of prime graphs and divisor function graphs in this follow-up. In that study, maximum and minimum degrees, a null graph, an Euler graph, a cycle, a complete graph, and a bipartite graph are examined for a divisor prime graph.

In this paper, we have studied the properties of the divisor prime graph, its diameter, girth, colorability, planarity, density, etc. Any number theory or graph theory terms can be looked up in [4, 10], or any other standard literature.

2. Preliminaries

A *graph* is a set of objects represented graphically, with links connecting some pairs of objects. The points that represent the connected objects are referred to as *vertices or points*, and the links that join the vertices are called *edges or lines*. The majority of the definitions we have included here come from scholarly articles and standard literature.

A *graph* G is a pair of set $G = (V, E)$, where V is a set of all vertices or points and E is a set of all edges or lines, connecting the vertices.

A graph is called *connected* if there is a path between every pair of vertices, otherwise, it is a *disconnected* graph.

In a graph, two vertices are said to be *adjacent* if they are connected by a common edge and two edges are said to be adjacent, if there is a common vertex between the two edges.

The *degree* of vertex in a graph is defined as the number of edges incident to the vertex, say v, or the number of vertices that is adjacent to the vertex v. It is denoted by $deg(v)$. The minimum and maximum degrees of a graph G is denoted by $\delta(G)$ and $\Delta(G)$.

If all the vertices in the graph have the same degree, then the graph is called a *regular* graph. If k is the degree of the vertex, then the graph is called a k-regular graph. A connected 2-regular graph is also called a cycle graph.

A graph is said to be *complete* if each and every vertex is connected to each other. A complete graph of n vertices (i.e K_n) is a $(n-1)$- regular graph. A graph $G = (V, E)$ whose vertices can be partitioned into two disjoint and independent sets $V = V_1 \cup V_2$ such that every edge of E connects a vertex in V_1 to a vertex in V_2 is called a *bipartite* graph. A bipartite graph in which each vertex of the first set is connected to every vertex of the second set is called a *complete bipartite* graph. A *star* graph is a complete bipartite graph of the form $K_{1,n-1}$ with n-vertices, i.e., one set will have only one vertex and all the remaining vertices belong to the other set, and all these vertices are adjacent to that single vertex and not to each other. A star graph with n vertices is denoted by S_n. A graph where the degree of all its vertices is 0 is called a *null* graph and a graph where there is only one point (thus degree=0) is called a *trivial* graph.

A *walk* of a graph is an alternating sequence of points and lines beginning and ending with points, where each line is incident with the two points immediately preceding and following it. If all the lines of a walk are distinct, then it is called a *trail* and if all the points are distinct, then it is called a *path*.

The *distance* between the two vertices is the length of a geodesic between that pair of vertices. Distance between a pair of vertices u and v is denoted by $d(u, v)$. The maximum distance of a vertex, say v, from all the other vertex is called the *eccentricity* of a vertex. It is denoted by $e(v)$. The minimum eccentricity of all the vertices in the graph is considered the *radius* of the graph. It is denoted by $r(G)$. The maximum eccentricity of all the vertices in the graph is considered the *diameter* of the graph. It is denoted by $d(G)$.

Simple graphs G and H are called *isomorphic* if there is a bijection f from the vertices of G to the vertices of H such that (v, w) is an edge in G if and only if $(f(v), f(w))$ is an edge of H.

A simple, connected graph is called *planar* if there is a way to draw it on a plane so that no edges cross. Such a drawing is called an *embedding* of the graph in the plane.

The *Girth* of a simple graph is the shortest cycle contained in the graph and if there is no cycle in the graph then its girth is undefined. A complete subgraph in a graph is often called a *clique*. A clique having n number of vertices is called $n - clique$. The size of the largest clique of a graph G is called the *clique number* of G. It is denoted by $cl(G)$.

A subset I of V is an independent set of a graph $G = (V, E)$ if the vertices in I are not adjacent to each other. The independence number $\beta_0(G)$ is the size of a largest independent set in G.

The *divisor function* or *Tau function*, is a number-theoretic function that counts the positive divisors of an integer n. It is represented by the symbol $\tau(n)$. In the prime factorization of n, it can be written as the product of one and the exponent of each prime factor. The Tau function can be found mathematically for a positive integer n with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are distinct prime numbers and $a_1, a_2, \ldots, a_k$ are positive integers representing the exponents. Then $\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1)$.

Numerous branches of number theory, such as the study of perfect numbers, integer sequences, and cryptography, employ the Tau function. In conclusion, the Tau function counts the number of prime factors, while prime factorization is the process of breaking down a positive integer into its prime factors.

3. Properties of Divisor Prime Graph

The divisior prime graph presents a pictorial view of the relation between the positive divisors of a natural number n. We expect that the investigation of the theoretical properties of these graphs can help to determine some number theoretic properties of these numbers.

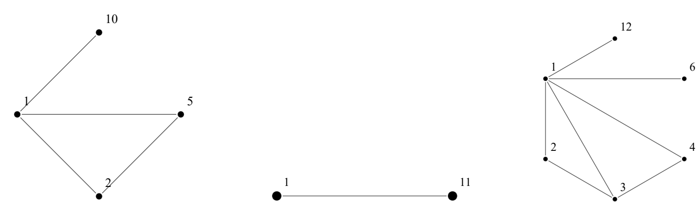
In this section, we discuss some properties of the divisor prime graph $PG_D(n)$ for $n \in \mathbb{Z}^+$ like diameter, girth, radius, clique number, planarity, etc.

Let us start the with the formal definition of the *divisor prime graph* PG_{D_n}.

Definition 3.1. (Divisor Prime Graph)[12]

Let $n \in \mathbb{Z}^+$ *and* $D(n) = \{m \in \mathbb{Z}^+ : m|n\}$. *The divisor prime graph* $PG_D(n)$ *is defined as a graph with the vertex set* $D(n)$ *and any two vertices* x *and* y *are adjacent in* $PG_D(n)$ *if* $\gcd(x, y) = 1$.

Example 3.1. *The divisor prime graphs for* $n = 10, 11, 12$ *are shown in figure* **1**.



(a) (b) (c)

FIGURE 1. (a) $PG_D(10)$ (b) $PG_D(11)$ (c) $PG_D(12)$

Theorem 3.1. [12] *For all* $n \in \mathbb{Z}^+$, $PG_D(n)$ *is connected.*

Proof. Since for every $n \in \mathbb{Z}^+$, $1 \in D(n)$ and $\gcd(1, d) = 1$ for all $d \in D(n)$, so the vertex 1 is adjacent to every vertex d in $D(n)$. Hence $PG_D(n)$ is connected for all n. □

Theorem 3.2. [12] *For all* $n \in \mathbb{Z}^+$,

$$\Delta(PG_D(n)) = \tau(n) - 1 \quad \textit{and} \quad \delta(PG_D(n)) = 1.$$

Moreover, $\Delta(PG_D(n)) = \delta(PG_D(n))$ *if* $n = 1$ *or* n *is a prime.*

Proof. For every $n \in \mathbb{Z}^+$, let $\tau(n)$ is the number of divisor of n. The vertex 1 is adjacent to all the vertices $d \neq 1$ of $PG_D(n)$ and we have $deg(1) = \tau(n) - 1$, which is maximum possible degree for any graph with $\tau(n)$ vertices. Thus $\Delta(PG_D(n)) = \tau(n) - 1$.

Also $\gcd(n, d) \neq 1$ for all divisors $d \neq 1$ of n and $\gcd(n, 1) = 1$, so n is adjacent only to 1 and since $PG_D(n)$ is connected so there is no isolated vertex. Thus $\delta(PG_D(n)) = 1$.

Since for n prime $PG_D(n) \cong K_2$, thus $\Delta(PG_D(n)) = \delta(PG_D(n))$. □

Theorem 3.3. [12] $PG_D(n)$ *is non-eulerian for all* n.

Theorem 3.4. *For all* $n \in \mathbb{Z}^+$, $Diam(PG_D(n)) \leq 2$.

Proof. If $n = 1$ then $PG_D(n) \cong K_1$ and $Diam(PG_D(n)) = 0$.
If n is prime then $PG_D(n) \cong K_2$ and $Diam(PG_D(n)) = 1$.
It is clear from theorem **3.1** that for composite n, any two non adjacent vertices u and v in $PG_D(n)$, $u - 1 - v$ is always the shortest $u - v$-path. So $Diam(PG_D(n)) = 2$.
So we can conclude that $Diam(PG_D(n)) \leq 2$. □

Theorem 3.5. *For all* $n \in \mathbb{Z}^+$, $rad(PG_D(n)) = 1$.

Proof. Since $PG_D(n)$ is connected and $Diam(PG_D(n)) \leq 2$ by theorems **3.1** and **3.4**, the ecentricity $1 \leq e(v) \leq 2$, $\forall\ v \in V(PG_D(n))$, therefore $rad(PG_D(n)) = min\{e(v) : v \in V(PG_D(n))\} = 1$. □

Theorem 3.6. *For all* $n \in \mathbb{Z}^+$, *let* k *be the number of distinct prime divisors of* n, *then* $cl(PG_D(n)) = k + 1$.

Proof. Let $p_i,\ i = 1, 2, \ldots, k$ be the distinct prime divisors of n. Since for distinct i and j, $\gcd(p_i, p_j) = 1$, so p_i *adjacent* p_j . Thus the vertices $1, p_1, p_2, \ldots, p_k$ induced a complete subgraph of order $k + 1$.

Let $v = p_ip_j$ be a vertex of $PG_D(n)$, then $\gcd(v, p_i) \neq 1$, $\gcd(v, p_j) \neq 1$ but $\gcd(v, p_r) = 1$, $\forall r \neq i, j$ which implies that the vertex v adjacent to all p_r, $r \neq i, j$. So the vertices $1, p_1, p_2, \ldots, p_{i-1}, p_{i+1}, \ldots, p_{j-1}, p_{j+1}, \ldots, p_k, v$ will induce a complete subgraph of order k. That is any set of vertices more than $k + 1$ cannot induce a complete graph. Thus the complete subgraph induced by the vertices $1, p_1, p_2, \ldots, p_k$ is the maximal clique in $PG_D(n)$. Hence $cl(PG_D(n)) = k + 1$. □

Theorem 3.7. *For all* $n \in \mathbb{Z}^+$, $girth(PG_D(n)) = 3$ *or* ∞.

Proof. If $n = 1$ then $girth(PG_D(n)) = \infty$.
If n is prime then $PG_D(n) \cong K_2$ and $girth(PG_D(n)) = \infty$.
If $n = p^k$ where $k \in \mathbb{Z}^+$, then $PG_D(n) \cong K_{1,\tau(n)-1}$ and so $girth(PG_D(n)) = \infty$.
If n has more than one prime divisors, it is clear from theorem **3.6** that $PG_D(n)$ always contain a cycle of length 3. So $girth(PG_D(n)) = 3$.
So we can conclude that $girth(PG_D(n)) = 3$ *or* ∞. □

We obtained some counter examples to the statement in *Theorem 2.5* given by Nair and Kapur in [12]. So we modified the theorem and provided a proof and an example supporting our result.

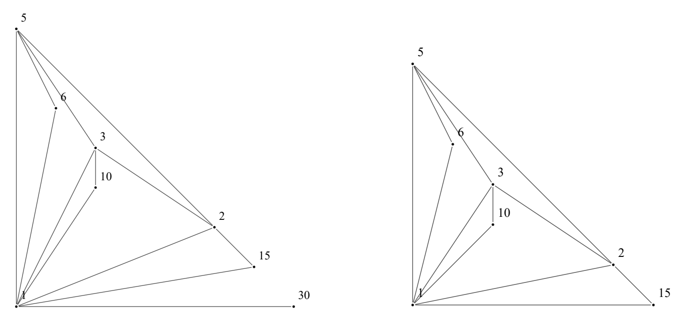
Theorem 3.8. *For any $k \in \mathbb{Z}^+$, if $n = p_1p_2\cdots p_k$, then $PG_D(n) - n$ can not be a complete graph.*

Proof. For $n = p_1p_2\cdots p_k$,

$D(n) = \{1, p_1, p_2, \cdots, p_k, p_1p_2, \cdots p_1p_k, p_1p_2 \ldots, p_{k-1}, p_1p_2, p_1p_2 \ldots p_k\}$.

So $V(PG_D(n)) - n = \{1, p_1, p_2, \cdots, p_k, p_1p_2, \cdots p_1p_k, p_1p_2 \ldots, p_{k-1}, \cdots p_2, p_2 \ldots p_k\}$.

The vertices $1, p_1, p_2, \cdots, p_k$ will induce a complete subgraph. But the vertex set of the graph contains more vertices in the form of product of primes. These vertices are adjacent to vertex 1 as well as to some of the vertices of $p_1, p_2, \cdots, p_k$ but not to all. Hence $PG_D(n) - n$ is not a complete graph. □



(a) (b)

FIGURE 2. (a) $PG_D(30)$ (b) $PG_D(30) - 30$

Following the theorems **3.6 and 3.8** finally we can give the following result.

Theorem 3.9. *Let $n_1 = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $n_2 = q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$, then $PG_D(n_1) \cong PG_D(n_2)$.*

Proof. Since $n_1 = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $n_2 = q_1^{\alpha_1}q_2^{\alpha_2}\cdots q_k^{\alpha_k}$, so

$$\tau(n_1) = \tau(n_2) = (\alpha_1 + 1)(\alpha_2 + 1)\cdots(\alpha_k + 1).$$

That is $|V(PG_D(n_1))| = |V(PG_D(n_2))|$.

Let us consider the mapping $f : D(n_1) \to D(n_2)$ defined by

$$f(p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}) = q_1^{r_1}q_2^{r_2}\cdots q_k^{r_k}$$

where $0 \leq r_i \leq \alpha_i$ for each $i = 1, 2, \ldots, k$.

So f is a one-one correspondence from $D(n_1)$ onto $D(n_2)$, *i.e.* f is a one-one correspondence from $V(PG_D(n_1))$ onto $V(PG_D(n_2))$.

It is now sufficient to prove that f preserves the adjacency from $PG_D(n_1)$ to $PG_D(n_2)$ *i.e.* $(a, b) \in E(PG_D(n_1)) \Leftrightarrow (f(a), f(b)) \in E(PG_D(n_2))$.

If possible, let $(a, b) \in E(PG_D(n_1))$ but $(f(a), f(b)) \notin E(PG_D(n_2))$.

Then there exists at least one q_i such that $q_i | \gcd(f(a), f(b))$

$\Rightarrow q_i | f(a)$ *and* $q_i | f(b)$

$\Rightarrow \exists\ p_i$ *such that* $f(p_i) = q_i$ *and* $p_i | a$ *and* $p_i | b$

$\Rightarrow\ p_i | \gcd(a, b)$

which is a contradiction to the fact that $(a, b) \in E(PG_D(n_1))$.

Thus $(a, b) \in E(PG_D(n_1)) \Leftrightarrow (f(a), f(b)) \in E(PG_D(n_2))$ for all $a, b \in V(PG_D(n_1))$. Hence $PG_D(n_1) \cong PG_D(n_2)$. □

Theorem 3.10. *Let $n \in \mathbb{Z}^+$ then $PG_D(n)$ is planar if n is any one of the following form $p^k,\ p^k q,\ p^2q^2$ or pqr, where $p,\ q,\ r$ are primes and k is nonzero positive integer.*

Proof. If $n = 1$ then $PG_D(n)$ is trivial and so is planar.

If n is prime then $PG_D(n) \cong K_2$ and $PG_D(n)$ is planar.

If $n = p^k$ where $k \in \mathbb{Z}^+$, then $PG_D(n) \cong K_{1,\tau(n)-1}$ and so $PG_D(n)$ is planar as shown in figure **3(a)**.

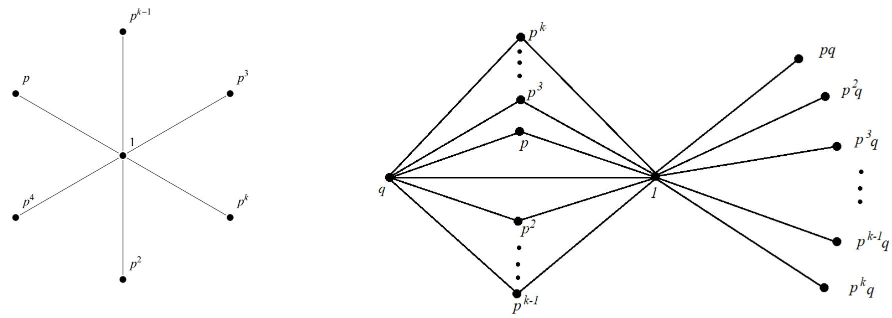
If $n = p^k q$, the vertices are $1, p, p^2, p^3, \cdots p^{k-1}, p^k, q, pq, p^2q, p^3q, \cdots p^{k-1}q, p^kq$ and the graph $PG_D(n)$ which is clearly a planar graph as shown in figure **3(b)**.

If $n = p^2q^2$, $1, p, p^2, q, q^2, pq, p^2q, pq^2, p^2q^2$ are the only vertices and from the figure **4(a)** it is clear that the graph $PG_D(n)$ is planar graph.

If $n = pqr$, the vertices are $1,\ p,\ q,\ r,\ pq,\ qr,\ pr,$ *and* pqr and the graph $PG_D(n)$ which is clearly a planar graph as shown in figure **4(b)**.

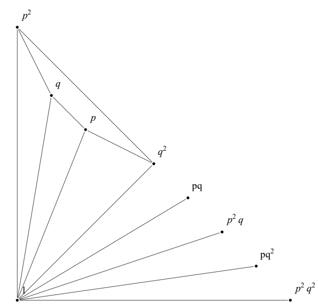
If $n = p^3q^2$, then the vertices $1, p, p^2, p^3, q, q^2$ together give $K_{3,3}$ as a induced subgraph of $PG_D(p^3q^2)$ because of which we can conclude that the graph $PG_D(p^3q^2)$ is not a planar graph. Thus for all $n = p^iq^j\ \forall i,\ j > 2$ the graph $PG_D(p^iq^j)$ is not a planar graph.

If $n = p^2qr$, then the vertices $1, p, p^2, q, r$ together give the induce subgraph K_5 of $PG_D(p^2qr)$ and so the graph $PG_D(p^2qr)$ is not a planar graph. Thus for all $n = p^iq^jr^k\ \forall i,\ j,\ k > 1$ the graph $PG_D(p^iq^jr^k)$ is not a planar graph.



(a) (b)

FIGURE 3. (a) $PG_D(p^k)$ (b) $PG_D(p^kq)$



(a) (b)

FIGURE 4. (a) $PG_D(p^2q^2)$ (b) $PG_D(pqr)$

It is clear from theorem **3.6** that if n has more than 3 distinct prime factors than $PG_D(n)$ has a clique of order greater than 4. Which implies that $PG_D(n)$ for n with more than 3 prime factors is not planar.

Hence we can conclude that $PG_D(n)$ is planar only for the values $n = 1,\ p^k,\ p^kq,\ p^2q^2$ and pqr, where $p,\ q,\ r$ are distinct primes and k is a nonzero positive integer. □

Theorem 3.11. *For any* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, $\chi(PG_D(n)) = k + 1$ *where* $\alpha_i \geq 0$.

Proof. We know that for any graph G, $cl(G) \leq \chi(G)$. From theorem **3.6** it is clear that $k + 1 = cl(PG_D(n)) \leq \chi(PG_D(n))$.

Let us assign $k + 1$ colors to the vertices $1, p_1, p_2, \cdots p_k$ of the maximal clique. Any of the remaining vertices, say d, will have at least one of the primes $p_1, p_2, \cdots p_k$ as a factor, say p_i. Then $\gcd(d, p_i) = p_i$ and so $(d, p_i) \notin E(PG_D(n))$. Thus d can be assigned with the same color assigned to p_i. Let d_i and d_j be any two vertices such that $p_i|d_i$ and $p_j|d_j$,then they can be assigned the colors of p_i and p_j respectively. Now let d be a vertex such that $p_ip_j|d$ then d is not adjacent to any of p_i, p_j, d_i and d_j, so we can assign any one color from these vertices to the vertex d. Proceeding with the same argument we can have a proper $(k+1)$- coloring of $PG_D(n)$. Thus $\chi(PG_D(n)) \leq k+1$. Hence we can conclude that $\chi(PG_D(n)) = k+1$. □

Theorem 3.12. *Let $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $\alpha_r = max\{\alpha_i : 1 \leq i \leq k\}$. Then*

$$\beta_0(PG_D(n)) = (\tau(p_r^{\alpha_r}) - 1)(\tau(n/p_r^{\alpha_r})).$$

Proof. Here $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ and $\alpha_r = max\{\alpha_i : 1 \leq i \leq k\}$.
It is easy to see that each of the sets $I_i = \{p_i^s d : 1 \leq s \leq \alpha_i, d|\Pi_{j\neq i}p_j\}$ is an independent set of $PG_D(n)$. The number of elements in I_i is given by

$$|I_i| = \tau(n) - (\tau(n/p_i^{\alpha_i})) = (\tau(p_i^{\alpha_i}) - 1)\tau(n/p_i^{\alpha_i}).$$

To prove the result it is sufficient to show that

$$|I_r| = max\{|I_i| : 1 \leq i \leq k\} = (\tau(p_r^{\alpha_r}) - 1)\tau(n/p_r^{\alpha_r}).$$

Now for $1 \leq i \leq k$ and $i \neq r$, $\tau(p_i^{\alpha_i}) < \tau(p_r^{\alpha_r})$

$\Rightarrow \tau(p_i^{\alpha_i}n') < \tau(p_r^{\alpha_r}n')$, whhere $n' = \Pi_{j\neq i,r}p_j$

$$\Rightarrow \tau(n) - \tau(p_i^{\alpha_i}n') > \tau(n) - \tau(p_r^{\alpha_r}n')$$

$$\Rightarrow \tau(n) - \tau(n/p_r^{\alpha_r}) > \tau(n) - \tau(n/p_i^{\alpha_i})$$

$$\Rightarrow (\tau(p_r^{\alpha_r}) - 1)\tau(n/p_r^{\alpha_r}) > (\tau(p_i^{\alpha_i}) - 1)\tau(n/p_i^{\alpha_i})$$

$\Rightarrow |I_r| > |I_i|$ for all $1 \leq i \leq k$ and $i \neq r$

$$\Rightarrow |I_r| = max\{|I_i| : 1 \leq i \leq k\}$$

Hence $\beta_0(PG_D(n)) = (\tau(p_r^{\alpha_r}) - 1)\tau(n/p_r^{\alpha_r})$. □

Theorem 3.13. *$PG_D(n)$ is non-hamiltonian for all n.*

Proof. $PG_D(n)$ is non-hamiltonian as $deg(n) = 1$ in $PG_D(n)$ and there can not exists any hamiltonian cycle in $PG_D(n)$. □

A subset D of V is a dominating set for a graph $G = (V, E)$ if every vertex in $V - D$ is adjacent to at least one member of D. The domination number $\gamma(G)$ is the size of a smallest dominating set for G.

Theorem 3.14. *Domination number of $PG_D(n)$ is 1 for all n.*

Proof. It is clear from the definition that for the set $D = 1 \subset V(PG_D(n))$, every vertex $v \in V(PG_D(n)) - 1$ is adjacent to 1. That is 1 is a dominating set for $PG_D(n)$ and is the smallest dominating set. Hence $\gamma(PG_D(n)) = 1$. □

A subset C of V is a dominating cut vertex set for a graph $G = (V, E)$ if $G - C$ is either a disconnected graph or a trivial graph. The point connectivity $\kappa(G)$ is the size of a smallest cut vertex set for G. A subset C' of E is a cut set for a graph $G = (V, E)$ if $G - C'$ is a disconnected graph. The line connectivity $\lambda(G)$ is the size of a smallest cut set for G.

Theorem 3.15. *Both point and line connectivity of $PG_D(n)$ is 1 for all n.*

Proof. It is clear from the definition of $PG_D(n)$ that $PG_D(n) - 1$ is a disconnected graph, so $\kappa(PG_D(n)) = 1$. Also $PG_D(n) - (1, n)$ is always disconnected, so $\lambda(PG_D(n)) = 1$. □

Theorem 3.16. *For all prime p, $Den(PG_D(p)) = 1$.*

Proof. Since for every prime p, $PG_D(p) \cong K_2$ so $Den(PG_D(p)) = 1$. □

Theorem 3.17. *For all prime p and $k \in \mathbb{Z},\ k \geq 0$, $Den(PG_D(p^k)) = \dfrac{2}{k+1}$.*

Proof. For $n = p^k$, $PG_D(p^k)$ has $k + 1$ vertices and k edges. So $Den(PG_D(p^k)) = \dfrac{k}{{}^{k+1}C_2} = \dfrac{k}{\{(k+1)k\}/2} = \dfrac{2}{k+1}$. □

Theorem 3.18. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ then $Den(PG_D(n)) = \dfrac{\sum_{d_i|n} \tau(\frac{n}{d_i})}{2 \cdot {}^{\tau(n)}C_2}$.*

Proof. For each divisor d_i of n, $deg(d_i) = \tau(\frac{n}{d_i})$. So $o(E(PG_D(n))) = \dfrac{1}{2} \displaystyle\sum_{d_i|n} \tau\left(\frac{n}{d_i}\right)$.

Hence $Den(PG_D(n)) = \dfrac{\sum_{d_i|n} \tau(\frac{n}{d_i})}{2 \cdot {}^{\tau(n)}C_2}$. □

4. Conclusion

In this study, we have investigated the nature and characteristics of the divisor prime graph. The adjacency, diameter, radius, clique number, chromatic number, planarity, connectivity, independence number, and domination number features of the divisor prime graph have also been investigated. Since this is a preliminary investigation of divisor prime graphs, the reader may be thinking about various issues. Studying the energy and distance eigenvalues of the divisor prime graph can reveal some potential problems.

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STUDY OF BI-F-HARMONIC CURVES ALONG RIEMANNIAN MAP

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Abstract. In this paper, we study bi-f-harmonic curves and helices along the Riemannian map. We find that, if a totally umbilical Riemannian map takes a horizontal bi-f-harmonic curve to bi-f-harmonic curve, then the map is totally geodesic. Then, we discuss the mean curvature vector field for horizontal bi-harmonic curves through Riemannian maps. In addition, we obtain the condition for the curvature of helix along isotropic Riemannian map.

Keywords: Bi-f-harmonic curve, bi-harmonic curve, helix, Riemannian map, totally geodesic Riemannian map.

2010 Mathematics Subject Classification: 53B20, 53C42, 53C43, 58E20.

1. Introduction

In 1964, J. Eells and J. H. Sampson [5], introduced the concept of bi-harmonic maps by generalizing the harmonic maps. Harmonic maps are the generalization of geodesics, minimal surfaces and harmonic functions. Harmonic maps have important applications in different fields of mathematics and physics with nonlinear partial differential equations. A harmonic map $\alpha : (M, g_M) \to (N, g_N)$ between the Riemannian manifolds (M, g_M) and (N, g_N) is a critical point of the energy functional,

$$E(\alpha) = \frac{1}{2}\int_{\Gamma_M} |d\alpha|^2 v_{g_M},$$

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where Γ_M is some compact domain of M and $\tau(\alpha) = Trace_{g_M}\nabla d\alpha$ is tension field of α. The harmonic map equation is an Euler-Lagrange equation of the functional $\tau(\varphi) \equiv Trace_{g_M}\nabla d\varphi = 0$, where $\tau(\varphi) = Trace_{g_M}\nabla d\varphi$ is a tension field of φ [5]. A bi-harmonic map α between the Riemannian manifolds (M, g_M) and (N, g_N) is a critical point of the bi-energy functional, $E_2(\alpha) = \frac{1}{2}\int_{\Gamma_M} |\tau(\alpha)|^2 v_{g_M}$, where Γ_M is a compact domain of M. The bi-harmonic map equation is an Euler-Lagrange equation of the functional,

$$\tau_2(\alpha) \equiv Trace_{g_M}(\nabla^\alpha\nabla^\alpha - \nabla^\alpha_{\nabla^M})\tau(\alpha) - Trace_{g_M}R^N(d\alpha, \tau(\alpha))d\alpha = 0,$$

where $R^N = [\nabla^N_X, \nabla^N_Y]Z - \nabla^N_{[X,Y]}Z$, is a Riemann curvature tensor (N, g_N) [10]. In 1991 [4], the author introduced the bi-harmonic submanifolds of Euclidean space and stated a conjecture " any bi-harmonic submanifold of Euclidean space is harmonic, thus minimal". If the definition of bi-harmonic maps for Riemannian immersion in Euclidean space is used, then the Chen's definition of a bi-harmonic submanifold coincides with the definition given by the bi-energy functional.

Bi-f-harmonic maps are the generalization of harmonic maps, f-harmonic maps and bi-harmonic maps. There are two methods to formalize the link between bi-harmonic maps and f-harmonic maps. For the first type of formalization, the authors extended the bi-energy functional in [20, 26] to the bi-f-energy functional and obtained bi-f-harmonic maps. For the second formalization, the f-energy functional is extended to the f-bi-energy functional. In [13], the author introduced the f-bi-harmonic maps by generalizing the bi-harmonic maps. A smooth map between Riemannian manifolds is an f-bi-harmonic map if it is a critical point of the f-bi-energy function defined by the integral of f-times the square norm of the tension field, where f is a smooth function on the domain.

In 1992 [7], the author introduced the Riemannian maps between Riemannian manifolds. Isometric immersions and Riemannian submersions are particular cases of Riemannian maps. The theory of isometric immersions is one of the active research areas in differential geometry [1, 2, 3]. In [6], authors studied the characterization of submanifold by taking the hyperelastic curves along an immersion. The basic properties of Riemannian submersions were studied in [8, 15]. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a smooth map between Riemannian manifolds such that $0 < rank\vartheta < min\{m, n\}$, where $dimM = m$ and $dimN = n$. Then kernel space $(Ker\vartheta_*)$ of differential map ϑ_* and g_M-orthogonal component $((Ker\vartheta_*)^\perp)$ at a point $p \in M$, are known as horizontal and vertical spaces, respectively. Thus, the tangent space T_pM of M at point p can be decomposed as $T_pM = Ker\vartheta_{*p} \oplus (Ker\vartheta_{*p})^\perp$. The range of

ϑ_* and g_2-orthogonal component at $F(p)$ on N, are denoted by $range\vartheta_*$ and $(range\vartheta_*)^\perp$, respectively. Hence, the tangent space at $F(p)$ on N, follows the decomposition

$$T_{F(p)}N = \text{Range}\vartheta_{*p} \oplus (\text{Range}\vartheta_{*p})^\perp.$$

A Riemannian map at a point $p \in M$ is a horizontal restriction

$$\vartheta^h_{*p} : \left((Ker\vartheta_{*p})^\perp, g_M(p)|_{(Ker\vartheta_{*p})^\perp}\right) \to (range\vartheta_{*p}),$$

of smooth map $\vartheta : (M, g_M) \to (N, g_N)$, such that $g_M(\vartheta_* S, \vartheta_* K) = g_N(S, K)$, where S and K are smooth sections of $\Gamma(Ker\vartheta_{*p})^\perp$ [7]. In [11, 12, 14, 18, 19, 22, 25], authors studied various types of curves such as circles, hyperelastic curves and proper curves with various maps such as immersion, embedding, Riemannian map and Clairaut Riemannian map.

We organize our paper as follows: Section 2 of this paper contains basic concepts about bi-f-harmonic curves and Riemannian maps. In section 3, we study the bi-f-harmonic curves and bi-harmonic curves through the Riemannian maps. We show that, if a totally umbilical Riemannian map takes a horizontal bi-f-harmonic curve to bi-f-harmonic curves, then the map is totally geodesic. In the same section, conditions for the mean curvature vector field are obtained by taking horizontal bi-harmonic curves through Riemannian maps. In the final section, we study helix along the Riemannian maps.

2. Preliminaries

A bi-f-harmonic map $\alpha : (M, g_M) \to (N, g_N)$ between Riemannian manifolds (M, g_M) and (N, g_N) is a critical point of bi-f-energy functional, $E_{2,f}(\alpha) = \frac{1}{2}\int_{\Gamma_M} |\tau_f(\alpha)|^2 v_{g_M}$, where Γ_M is a compact domain of M and an Euler-Lagrange equation of the functional is defined by

$$\tau^2_f(\alpha) \equiv fJ^\alpha(\tau_f(\alpha)) - \nabla^\alpha_{grad\alpha}\tau_f(\alpha) = 0,$$

where $\tau_f(\alpha)$ is the $f-$tension field of α and J^α is the Jacobi operator of the map defined by $J^\alpha(X) = -(Trace_{g_M}\nabla^\alpha\nabla^\alpha X - \nabla^\alpha_{\nabla^M}X - R^N(d\alpha, X)d\alpha)$ [17, 20]. A curve $\alpha : I \to M$ on (M, g_M) is a bi-f-harmonic curve if and only if α satisfies the condition

$$\begin{aligned}(ff''' + f'f'')X_1 + (3ff'' + 2f'^2)\nabla_{X_1}X_1 + 4ff'\nabla^2_{X_1}X_1 \\ + f^2\nabla^3_{X_1}X_1 + f^2R(\nabla_{X_1}X_1, X_1)X_1 = 0,\end{aligned} \tag{2.1}$$

where $f : I \to (0, \infty)$ is a smooth function, ∇ is a Levi-Civita connection and R is a Riemannian curvature tensor on M.

Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then a curve α on M is a horizontal curve if $\dot{\alpha}(t) \in (ker\vartheta_*)^\perp$ for every $t \in I$. If ∇^N is the Levi-Civita connection on (N, g_N) and $p_2 = \vartheta(p_1) \in N$, then the second fundamental form of ϑ is given by

$$(\nabla\vartheta_*)(X,Y) = \overset{N}{\nabla}{}^{\vartheta}_X \vartheta_*(Y) - \vartheta_*(\nabla^M_X Y), \quad \forall X, Y \in \Gamma(TM), \tag{2.2}$$

where $\overset{N}{\nabla}{}^{\vartheta}$ is the pullback connection of ∇^N [16]. The second fundamental form of a Riemannian map is symmetric and has no components in $range\vartheta_*$, that is $(\nabla\vartheta_*)(X,Y) \in (range\vartheta_*)^\perp$, $\forall\ X, Y \in \Gamma((ker\vartheta_*)^\perp)$ [23]. The scalar product of the second fundamental form is

$$g_N((\nabla\vartheta_*)(X,Y), \vartheta_*(Z)) = 0, \tag{2.3}$$

for all $X, Y, Z \in \Gamma((ker\vartheta_*)^\perp)$. Now, if $X, Y \in \Gamma((kerV_*)^\perp)$ and $V \in \Gamma((range\vartheta_*)^\perp)$, then

$$\nabla^N_{\vartheta_*(X)} V = -S_V \vartheta_*(X) + \nabla^{F^\perp}_X V, \tag{2.4}$$

where $S_V\vartheta_*(X)$ is the tangential component of $\nabla^N_{\vartheta_*(X)}V$. Since $(\nabla\vartheta_*)$ is symmetric and S_V is a symmetric linear transformation of $range\vartheta_*$, therefore

$$g_N(S_V\vartheta_*(X), \vartheta_*(Y)) = g_N(V, (\nabla\vartheta_*)(X,Y)). \tag{2.5}$$

From equations (2.2) and (2.4), we get

$$\begin{aligned} R^N(\vartheta_*(X), \vartheta_*(Y))\vartheta_*(Z) &= -S_{(\nabla\vartheta_*)(Y,Z)}\vartheta_* X + S_{(\nabla\vartheta_*)(X,Z)}\vartheta_* Y \\ &+ \vartheta_*(R^M(X,Y)Z) + (\tilde{\nabla}_X(\nabla\vartheta_*))(Y,Z) - (\tilde{\nabla}_Y(\nabla\vartheta_*))(X,Z), \end{aligned} \tag{2.6}$$

where $\tilde{\nabla}$ is the covariant derivative of the second fundamental form. The covariant derivative of $\nabla\vartheta_*$ and S are, respectively

$$(\tilde{\nabla}_X(\nabla\vartheta_*))(Y,Z) = \nabla^{\vartheta\perp}_X(\nabla\vartheta_*)(Y,Z) - (\nabla\vartheta_*)(\nabla^M_X Y, Z) - (\nabla\vartheta_*)(Y, \nabla^M_X Z), \tag{2.7}$$

and

$$(\tilde{\nabla}_X S)_V\vartheta_*(Y) = \vartheta_*(\nabla^M_X {}^*\vartheta_*(S_V\vartheta_*(Y))) - S_{\nabla^{\vartheta\perp}_X V}\vartheta_*(Y) - S_V Q \overset{N}{\nabla}{}^{\vartheta}_X \vartheta_*(Y), \tag{2.8}$$

where Q is a projection morphism on $range\vartheta_*$ and ${}^*\vartheta_*$ is an adjoint map of ϑ_*. From (2.7) and (2.8), we obtain

$$g_N((\tilde{\nabla}_X(\nabla\vartheta_*))(Y,Z), V) = g_N((\tilde{\nabla}_X S)_V\vartheta_*(Y), \vartheta_*(Z)). \tag{2.9}$$

Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then ϑ is an umbilical Riemannian map if and only if

$$(\nabla\vartheta_*)(X, Y) = g_M(X, Y)H_2, \tag{2.10}$$

where $X, Y \in \Gamma((ker\vartheta_*)^{\perp})$ and H_2 is non zero vector field on $(range\vartheta_*)^{\perp}$ [21]. The Riemannian map $\vartheta : (M, g_M) \to (N, g_N)$ is h-isotropic at $p \in M$ if

$$\mu(X) = \frac{||(\nabla\vartheta_*)(X, X)||}{||\vartheta_* X||^2}. \tag{2.11}$$

If the map is h-isotropic at every point, then the map is called h-isotropic. The map ϑ is h-isotropic at $p \in M$ if and only if $\nabla\vartheta_*$ satisfies the condition

$$g_N((\nabla\vartheta_*)(X, X), (\nabla\vartheta_*)(X, Y)) = 0, \tag{2.12}$$

for all orthogonal pair $X, Y \in \Gamma((ker\vartheta_*)^{\perp})$.

3. Characterization of bi-f-harmonic curves

Let $\alpha : I \to M$ be a curve in an m-dimensional Riemannian manifold M with an orthonormal frame $\{W_0, W_1, ...W_{m_1-1}\}$ in ΓTM_1, where $W_0 = T$, $W_1 = N$ and $W_2 = U$ are the unit tangent vector, the unit normal vector and the unit binormal vector of α, respectively. Then the Frenet equations are given by

$$\nabla_T W_j = -\kappa_j W_{j-1} + \kappa_{j+1} W_{j+1}, \qquad 0 \le j \le m-1, \tag{3.13}$$

where $\kappa_0 = \kappa_m = 0$, $\kappa_1 = \kappa = ||\nabla_T T||$ is curvature and $\tau = \kappa_2 = -\langle \nabla_T W_1, W_2 \rangle$ is torsion of α on M, respectively.

Next, we introduce the concept horizontal bi-f-harmonic curve

Definition 3.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). Then a horizontal curve on M with (2.1) is said to be a horizontal bi-f-harmonic curve on M.*

Lemma 3.1. *: Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a curve on N, where α is a horizontal curve* *on M, then*

$$(i)\qquad \bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) = -(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + \vartheta_*(\nabla^3_{X_1}X_1)$$
$$-\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$-S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) + \nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)$$
$$+ (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1,X_1) + (\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1), \tag{3.14}$$

$$(ii)\qquad \bar{R}(\vartheta_*(\nabla_{X_1}X_1),\vartheta_*(X_1))\vartheta_*(X_1) = -S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1)$$
$$+S_{(\nabla\vartheta_*)(\nabla_{X_1}X_1,X_1)}\vartheta_*(X_1) + \vartheta_*(R(\nabla_{X_1}X_1,X_1)X_1)$$
$$+(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1,X_1) - (\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(\nabla_{X_1}X_1,X_1), \tag{3.15}$$

where ∇ and $\bar{\nabla}$ are the Levi-Civita connections of M and N.

Proof. Let α be a horizontal curve with curvature κ on Riemannian manifold M and $\bar{\alpha} = \vartheta\circ\alpha$ is a curve with curvature $\bar{\kappa}$ on N. Then a vector field $\vartheta_*(X_1)$ along $\bar{\alpha}$ is defined by

$$\vartheta_*(X_1) = \vartheta_{*\alpha}X_1, \tag{3.16}$$

for all vector field $X_1(s) = X_1$ along $\alpha(s) = \alpha$.

(i) From (2.2) and (2.4), we have

$$\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1) = \bar{\nabla}_{\vartheta_*(X_1)}((\nabla\vartheta_*)(X_1,X_1) + \vartheta_*(\nabla_{X_1}X_1)) \tag{3.17}$$
$$= -S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + \nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)$$
$$+(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1) + \vartheta_*(\nabla^2_{X_1}X_1). \tag{3.18}$$

Taking covariant derivative of (3.18) and using (2.2) and (2.4), we get the required condition. (ii) From (2.6) and (2.2), we get the required equation. □

Lemma 3.2. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta\circ\alpha$ is a bi-f-harmonic curve on N, where α is a horizontal curve on M, then $(\nabla\vartheta_*)(X_1, U_1) = 0$ and*

$$ff''' + f'f'' - 3\kappa\kappa' f^2 - 4ff'\kappa^2 = 4ff'||(\nabla\vartheta_*)(X_1,X_1)||^2 + \frac{3}{2}f^2\nabla^{\vartheta\perp}_{X_1}||(\nabla\vartheta_*)(X_1,X_1)||^2.$$

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map such that α is a horizontal curve on M and $\bar{\alpha}$ is a bi-f-harmonic curve on N, then we have

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = 0. \tag{3.19}$$

From Lemma 3.1 and (3.19), we have

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)(\nabla\vartheta_*)(X_1, X_1) + (3ff'' + 2f'^2)\vartheta_*(\nabla_{X_1}X_1)$$
$$+f^2(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) - f^2S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) + f^2\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1)$$
$$-4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 4ff'\nabla^{F\perp}_{X_1}(\nabla\vartheta_*)(X_1, X_1) + 4ff'(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1)$$
$$+4ff'\vartheta_*(\nabla^2_{X_1}X_1) - f^2(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))$$
$$-f^2\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - f^2S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+f^2(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1)$$
$$+f^2S_{(\nabla\vartheta_*)(\nabla_{X_1}X_1,X_1)}\vartheta_*(X_1) + f^2\vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1)$$
$$+f^2(\bar{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1, X_1) - f^2(\bar{\nabla}_{X_1}(\nabla\vartheta_*))(\nabla_{X_1}X_1, X_1) = 0. \tag{3.20}$$

The rangeϑ_*, component of (3.20) is

$$f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}F_*(X_1))) - f^2S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+f^2\vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1) + (3ff'' + 2f'^2)\vartheta_*(\nabla_{X_1}X_1) + (ff''' + f'f'')\vartheta_*(X_1)$$
$$-4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 4ff'\vartheta_*(\nabla^2_{X_1}X_1) - f^2S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = 0. \tag{3.21}$$

From (2.8) and (2.7), we get

$$\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) = (\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^F_{X_1}\vartheta_*(X_1), \tag{3.22}$$

and

$$(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, X_1) = \nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, X_1) - 2(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1). \tag{3.23}$$

Substituting (3.22) and (3.23) in (3.21), we have

$$f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2f^2 S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$-f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^F_{X_1}\vartheta_*(X_1) + f^2\vartheta_*(R(\nabla_{X_1}X_1,X_1)X_1) + (3ff''+2f'^2)\vartheta_*(\nabla_{X_1}X_1)$$
$$+(ff'''+f'f'')\vartheta_*(X_1) - 4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 4ff'\vartheta_*(\nabla^2_{X_1}X_1)$$
$$-f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = 0. \tag{3.24}$$

Using (2.7) in (3.24), we obtain

$$f^2\vartheta_*(\nabla^3_{X_1}X_1) - f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2f^2 S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1)$$
$$-4f^2 S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) - f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^\vartheta_{X_1}\vartheta_*(X_1) + f^2\vartheta_*(R(\nabla_{X_1}X_1,X_1)X_1)$$
$$+(3ff''+2f'^2)\vartheta_*(\nabla_{X_1}X_1) + (ff'''+f'f'')\vartheta_*(X_1) - 4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$+4ff'\vartheta_*(\nabla^2_{X_1}X_1) - f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = 0. \tag{3.25}$$

Using Serret-Frenet equations of α in (3.25), we have

$$(ff'''+f'f''-3\kappa\kappa' f^2 - 4ff'\kappa^2)\vartheta_*(X_1) + (\kappa'' f^2 - \kappa^3 f^2 - \kappa\tau^2 f^2 + 3ff''\kappa$$
$$+2f'^2\kappa + 4ff'\kappa')\vartheta_*(W_1) + f^2\vartheta_*(R(\kappa W_1,X_1)X_1) - 4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)$$
$$-f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1) + (2\kappa'\tau f^2 + \kappa\tau' f^2 + 4ff'\kappa\tau)\vartheta_*(U_1) + \kappa\tau f^2\kappa_3\vartheta_*(W_3)$$
$$-f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2f^2 S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1)$$
$$-4f^2 S_{(\nabla\vartheta_*)(X_1,\kappa W_1)}\vartheta_*(X_1) - f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^\vartheta_{X_1}\vartheta_*(X_1) = 0. \tag{3.26}$$

Taking inner product of (3.26) with $\vartheta_*(X_1)$, we get

$$ff'''+f'f''-3\kappa\kappa' f^2 - 4ff'\kappa^2 - 6\kappa f^2 g_N((\nabla\vartheta_*)(X_1,W_1),(\nabla\vartheta_*)(X_1,X_1))$$
$$-4ff'g_N((\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1)) = 2f^2 g_N(S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1))$$
$$-4f^2\kappa g_N(S_{(\nabla\vartheta_*)(X_1,W_1)}\vartheta_*(X_1),\vartheta_*(X_1)) + f^2 g_N((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1),\vartheta_*(X_1)). \tag{3.27}$$

From equations (2.5), (2.7) and (2.9), we have

$$ff'''+f'f''-3\kappa\kappa' f^2 - 4ff'\kappa^2 = 4ff'g_N((\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1))$$
$$+3f^2 g_N(\nabla^{F\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1),(\nabla\vartheta_*)(X_1,X_1))$$
$$= 4ff'||(\nabla\vartheta_*)(X_1,X_1)||^2 + \tfrac{3}{2}f^2\nabla^{\vartheta\perp}_{X_1}||(\nabla\vartheta_*)(X_1,X_1)||^2. \tag{3.28}$$

The $(range\vartheta_*)^\perp$, component of (3.20) is

$$(3ff''+2f'^2)(\nabla\vartheta_*)(X_1,X_1)+4ff'\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)+4ff'(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)$$
$$+f^2\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)+f^2(\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1)+f^2(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1,X_1)$$
$$-f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))+f^2(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_*,X_*)$$
$$-f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(\nabla_{X_1}X_1,X_1)=0. \tag{3.29}$$

Also from (2.7), we get

$$(\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1,X_1)=(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)+4\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)$$
$$-2(\nabla\vartheta_*)(\nabla_{X_1}X_1,\nabla_{X_1}X_1)-2(\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1)). \tag{3.30}$$

Substituting (3.30) in (3.29) and using (2.7), we obtain

$$(3ff''+2f'^2)(\nabla\vartheta_*)(X_1,X_1)+4ff'(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)+12ff'(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)$$
$$+4f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,\nabla_{X_1}X_1)+3f^2(\nabla\vartheta_*)(\nabla_{X_1}X_1,\nabla_{X_1}X_1)+4f^2(\nabla\vartheta_*)(X_1,\nabla^2_{X_1}X_1)$$
$$-f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))+f^2(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)$$
$$+f^2(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1,X_1)=0. \tag{3.31}$$

Using Frenet equations in (3.31), we get

$$(12\kappa ff'+4\kappa'f^2)(\nabla\vartheta_*)(X_1,W_1)+4\kappa\tau f^2(\nabla\vartheta_*)(X_1,U_1)+4\kappa f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,W_1)$$
$$+3\kappa^2f^2(\nabla\vartheta_*)(W_1,W_1)+\kappa f^2(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1,X_1)+4ff'(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)$$
$$=f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))-f^2(\tilde{\nabla}^2{}_{X_1}(\nabla\vartheta_*))(X_1,X_1)$$
$$+(4\kappa^2f^2-3ff''-2f'^2)(\nabla\vartheta_*)(X_1,X_1). \tag{3.32}$$

Replacing U_1 with $-U_1$ in equation (3.32), we have

$$(12\kappa ff'+4\kappa'f^2)(\nabla\vartheta_*)(X_1,W_1)-4\kappa\tau f^2(\nabla\vartheta_*)(X_1,U_1)+4\kappa f^2(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,W_1)$$
$$+3\kappa^2f^2(\nabla\vartheta_*)(W_1,W_1)+\kappa f^2(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1,X_1)+4ff'(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)$$
$$=f^2(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)))-f^2(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)$$
$$+(4\kappa^2f^2-3ff''-2f'^2)(\nabla\vartheta_*)(X_1,X_1). \tag{3.33}$$

Subtracting equation (3.33) from equation (3.32), we have

$$(\nabla\vartheta_*)(X_1,U_1)=0. \tag{3.34}$$

□

Theorem 3.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally geodesic Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-f-harmonic curve on N and α is a horizontal curve with curvature κ on M, then*

$$ff''' + f'f'' - 3\kappa\kappa' f^2 - 4ff'\kappa^2 = 0. \tag{3.35}$$

Proof. Using the fact that ϑ is a totally geodesic Riemannian map in equation (3.28), we get the required condition. □

Corollary 3.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be an isotropic Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-f-harmonic curve on N, where α is horizontal curve with curvature κ and constant f on M, then α is a curve of constant curvature on M.*

Proof. Since ϑ is an isotropic Riemannian map, therefore from (3.28), we have

$$ff''' + f'f'' - 3\kappa\kappa' f^2 - 4ff'\kappa^2 = 4ff'||(\nabla\vartheta_*)(X_1, X_1)||^2 \tag{3.36}$$

Also, f is a constant, therefore from (3.36), we get $\kappa = C(constant)$. □

Theorem 3.2. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If ϑ is a totally umbilical Riemannian map taking a horizontal bi-f-harmonic curve α on M to a bi-f-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N, then ϑ is a totally geodesic Riemannian map.*
Conversely, a totally geodesic Riemannian map takes a horizontal bi-f-harmonic curve α on M to a bi-f-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N.

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally umbilical Riemannian map taking a horizontal bi-f-harmonic curve α on M to a bi-f-harmonic curve $\bar{\alpha} = \vartheta \circ \alpha$ on N, then from (3.31), we have

$$\begin{aligned} f^2(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) &= (3ff'' + 2f'^2)H_2 + 4ff'\nabla^{\vartheta\perp}_{X_1}H_2 \\ & f^2(\nabla^{\vartheta\perp}_{X_1})^2 H_2 - \kappa^2 f^2 H_2 + f^2(\tilde{\nabla}_{\nabla_{X_1}X_1}(\nabla\vartheta_*))(X_1, X_1). \end{aligned} \tag{3.37}$$

Substituting (3.37) in (3.20), we have

$$\begin{aligned} f^2\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) &= -4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) \\ & -f^2 S_{\nabla^{\vartheta\perp}_{X_1}H_2}\vartheta_*(X_1) - \kappa f^2||H_2||^2\vartheta_*(W_1). \end{aligned} \tag{3.38}$$

Since $S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\nabla_{X_1}X_1) = \kappa||H_2||^2\vartheta_*(W_1)$, therefore from (3.22) and (3.38), we have

$$f^2(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + 2f^2 S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + f^2 S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1)$$
$$+4ff'S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + \kappa f^2||H_2||^2\vartheta_*(W_1) = 0, \tag{3.39}$$

where

$$(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) = \frac{1}{2}\nabla^{\vartheta\perp}_{X_1}||H_2||^2\vartheta_*(X_1),$$

$$S_{\nabla^{\vartheta\perp}_{X_1}H_2}\vartheta_*(X_1) = \frac{1}{2}\nabla^{\vartheta\perp}_{X_1}||H_2||^2\vartheta_*(X_1),$$

and

$$S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) = ||H_2||^2\vartheta_*(X_1).$$

Thus from (3.39), we have

$$\frac{3}{2}f^2(\nabla^{\vartheta\perp}_{X_1}||H_2||^2)\vartheta_*(X_1) + 2f^2||H_2||^2\vartheta_*(\nabla_{X_1}X_1) + 4ff'||H_2||^2\vartheta_*(X_1) = 0. \tag{3.40}$$

Taking the inner product of (3.40) with $\vartheta_*(\nabla_{X_1}X_1)$, we have

$$||H_2|| = 0 \implies H_2 = 0. \tag{3.41}$$

Hence ϑ is a totally geodesic Riemannian map.

Conversely, suppose that ϑ is a totally geodesic Riemannian map, then we have

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = (ff''' + f'f'')\vartheta_*(X_1)$$
$$+(3ff'' + 2f'^2)\nabla_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\nabla^2_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\nabla^3_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2R(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1). \tag{3.42}$$

Since α is a horizontal bi-f-harmonic curve on M, therefore

$$(ff''' + f'f'')\vartheta_*(X_1) + (3ff'' + 2f'^2)\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) + 4ff'\bar{\nabla}^2_{\vartheta_*(X_1)}\vartheta_*(X_1)$$
$$+f^2\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + f^2\bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = 0. \tag{3.43}$$

Hence $\bar{\alpha} = \vartheta \circ \alpha$ is a bi-f-harmonic curve on N. □

3.1. **Characterization of bi-harmonic curves.** A bi-harmonic curve (bi-1-harmonic curve) is a special case of bi-f-harmonic curve for $f=1$. Let $\vartheta:(M,g_M)\to(N,g_N)$ be a Riemannian map between Riemannian manifolds (M,g_M) and (N,g_N) such that $\bar{\alpha}$ is a bi-harmonic curve on N, then

$$\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1)+\bar{R}(\vartheta_*(\nabla_{X_1}X_1),\vartheta_*(X_1))\vartheta_*(X_1)=0.$$

Taking $f=1$ in (3.26) and (3.32), we have

$$\begin{aligned}&-3\kappa\kappa'\vartheta_*(X_1)+\big(\kappa''-\kappa^3-\kappa\tau^2\big)\vartheta_*(W_1)+\vartheta_*(R(\kappa W_1,X_1)X_1)\\&-S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1)+(2\kappa'\tau+\kappa\tau')\vartheta_*(U_1)+\kappa\tau\kappa_3\vartheta_*(W_3)\\&-(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1)-2S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1)\\&-4S_{(\nabla\vartheta_*)(X_1,\kappa W_1)}\vartheta_*(X_1)-S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1)=0,\end{aligned}\tag{3.44}$$

and

$$\begin{aligned}&4\kappa'(\nabla\vartheta_*)(X_1,W_1)+4\kappa\tau(\nabla\vartheta_*)(X_1,U_1)+4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,W_1)\\&+3\kappa^2(\nabla\vartheta_*)(W_1,W_1)+\kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1,X_1)=4\kappa^2(\nabla\vartheta_*)(X_1,X_1)\\&-(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1,X_1)+(\nabla\vartheta_*)(X_1,{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))),\end{aligned}\tag{3.45}$$

respectively.

Theorem 3.3. *Let $\vartheta:(M,g_M)\to(N,g_N)$ be a totally umbilical Riemannian map between Riemannian manifolds (M,g_M) and (N,g_N). If $\bar{\alpha}=\vartheta\circ\alpha$ is a bi-harmonic curve on N and α is a horizontal bi-harmonic curve on M, then the mean curvature vector field satisfies the relations*

$$(\nabla^{\vartheta\perp}_{X_1})^2H_2=||H_2||^2H_2+\kappa^2H_2,\tag{3.46}$$

and

$$2\kappa||H_2||^2=\kappa''-\kappa^3-\kappa\tau^2+\kappa g_M(R(W_1,X_1)X_1,W_1).\tag{3.47}$$

Conversely, let $\vartheta:(M,g_M)\to(N,g_N)$ be a totally umbilical Riemannian map and mean curvature vector field satisfies the following conditions

$$(\nabla^{\vartheta\perp}_{X_1})^2H_2=||H_2||^2H_2+\kappa^2H_2,\nabla^{F\perp}_{W_1}H=2||H_2||^2\vartheta_*W_1,\tag{3.48}$$

and $||H_2||^2=constant$. Then ϑ maps a horizontal bi-harmonic curve α on M to a bi-harmonic curve $\bar{\alpha}=\vartheta\circ\alpha$ on N.

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a totally umbilical bi-harmonic Riemannian map between M and N, then from (3.45), we have

$$4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) - \kappa^2 H_2 + \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$= -(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))). \tag{3.49}$$

Replacing W_1 with $-W_1$ in equation (3.49), we get

$$-4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) - \kappa^2 H_2 - \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1)$$
$$= -(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))). \tag{3.50}$$

Subtracting (3.49) from (3.50), we obtain

$$4\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + \kappa(\tilde{\nabla}_{W_1}(\nabla\vartheta_*))(X_1, X_1) = 0. \tag{3.51}$$

From equations (3.49) and (3.51), we get

$$(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) - (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - \kappa^2 H_2 = 0. \tag{3.52}$$

From (2.5) and (2.10), we have

$$(\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) = (\nabla^{\vartheta\perp}_{X_1})^2 H_2, \tag{3.53}$$

and

$$(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) = ||H_2||^2 H_2. \tag{3.54}$$

Equations (3.52), (3.53) and (3.54), gives the first condition.

Now, taking the inner product of (3.44) with $\vartheta_*(W_1)$, we have

$$g_N((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(W_1)) + g_N(4S_{(\nabla\vartheta_*)(X_1,\kappa W_1)}\vartheta_*(X_1), \vartheta_*(W_1))$$
$$-g_N(\vartheta_*(R(\kappa W_1, X_1)X_1), \vartheta_*(W_1)) + g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1), \vartheta_*(W_1))$$
$$-\kappa'' + \kappa^3 + \kappa\tau^2 + g_N(2S_{(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(W_1))$$
$$+g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1), \vartheta_*(W_1)) = 0. \tag{3.55}$$

Since ϑ is a totally umbilical Riemannian map, therefore

$$\begin{cases} g_N((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(W_1)) = 0, \\ g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(\kappa W_1), \vartheta_*(W_1)) = \kappa||H_2||^2, \\ g_N(S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1), \vartheta_*(W_1)) = \kappa||H_2||^2. \end{cases} \tag{3.56}$$

From equation (3.55) and (3.56), we get the required condition.

Conversely, suppose that ϑ is a totally umbilical Riemannian map, then for a curve $\bar{\alpha} = \vartheta \circ \alpha$ on N, where α is a curve on M, we have

$$\begin{aligned}
\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = -||H_2||^2 H_2 \\
-\vartheta * (\nabla_{X_1}{}^*\vartheta_*(||H_2||^2\vartheta_*(X_1))) - \tfrac{1}{2}(\nabla^{\vartheta\perp}_{X_1}||H_2||^2)\vartheta_*(X_1) \\
+(\nabla^{\vartheta\perp}_{X_1})^2 H_2 - \kappa^2 H_2 - \kappa||H_2||^2\vartheta_*(W_1) + \kappa\nabla^{F\perp}_{W_1}H_2 \\
+\vartheta_*(\nabla^3_{X_1}X_1) + \vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1).
\end{aligned} \tag{3.57}$$

Taking $||H||^2 = constant$ in equation (3.57), we have

$$\begin{aligned}
\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) = -||H_2||^2 H_2 \\
+(\nabla^{\vartheta\perp}_{X_1})^2 H_2 - \kappa^2 H_2 - 2\kappa||H_2||^2\vartheta_*(W_1) + \kappa\nabla^{F\perp}_{W_1}H_2 \\
+\vartheta_*(\nabla^3_{X_1}X_1) + \vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1).
\end{aligned} \tag{3.58}$$

Using equation (3.48) in (3.58), we get

$$\begin{aligned}
\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \bar{R}(\vartheta_*(\nabla_{X_1}X_1), \vartheta_*(X_1))\vartheta_*(X_1) \\
= \vartheta_*(\nabla^3_{X_1}X_1) + \vartheta_*(R(\nabla_{X_1}X_1, X_1)X_1).
\end{aligned} \tag{3.59}$$

Hence, from equation (3.59), we can say that the curve $\bar{\alpha} = \vartheta \circ \alpha$ on N is bi-harmonic curve on N iff α is a horizontal bi-harmonic curve on M. □

4. Helices along the Riemannian map

A regular curve $\alpha = \alpha(s)$ parametrized by arc length s is an ordinary helix if their exist unit vector fields W_1 and U_1 along α and constants κ and τ ($\kappa, \tau \geq 0$) such that

$$\begin{cases}
\nabla_{X_1}X_1 = \kappa W_1, \\
\nabla_{X_1}W_1 = -\kappa X_1 + \tau U_1, \\
\nabla_{X_1}U_1 = -\tau W_1,
\end{cases} \tag{4.60}$$

where κ is known as the curvature of the helix and τ is known as the torsion of the helix [9]. If $\tau = 0$, then α reduces to the circle and if both $\kappa = 0$ and $\tau = 0$, then α reduces to the geodesic. Hence for a proper ordinary helix κ and τ both are positive constants.

Theorem 4.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be an Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha} = \vartheta \circ \alpha$ is a helix on N, where α is a horizontal curve on M, then $(\nabla\vartheta_*)(X_1, U_1) = 0$ and $\nabla^{\vartheta\perp}_{X_1}||(\nabla\vartheta_*)(X_1, X_1)||^2 + 2\kappa\kappa' = 0$.*

Proof. Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map such that α is a horizontal curve on M and $\bar{\alpha} = \vartheta \circ \alpha$ is a helix on N, then

$$\bar{\nabla}^3_{\vartheta_*(X_1)}\vartheta_*(X_1) + \lambda^2\bar{\nabla}_{\vartheta_*(X_1)}\vartheta_*(X_1) = 0. \tag{4.61}$$

From Lemma 3.1 and (4.61), we have

$$\begin{aligned} -(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - \vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) \\ -S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) - S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) \\ +\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) + (\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) \\ +\vartheta_*(\nabla^3_{X_1}X_1) + \lambda^2\vartheta_*(\nabla_{X_1}X_1) = 0. \end{aligned} \tag{4.62}$$

The $range\vartheta_*$ and $(range\vartheta_*)^\perp$, components of (4.62) are

$$\begin{aligned} -\vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) - S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) \\ -S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) + \vartheta_*(\nabla^3_{X_1}X_1) + \lambda^2\vartheta_*(\nabla_{X_1}X_1) = 0, \end{aligned} \tag{4.63}$$

and

$$\begin{aligned} -(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) \\ +\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) + (\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0, \end{aligned} \tag{4.64}$$

respectively. From (2.8) and (2.7), we get

$$\begin{aligned} \vartheta_*(\nabla_{X_1}{}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) = (\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) \\ +S_{\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) + S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^F_{X_1}\vartheta_*(X_1), \end{aligned} \tag{4.65}$$

and

$$\begin{aligned} (\nabla^{\vartheta\perp}_{X_1})^2(\nabla\vartheta_*)(X_1, X_1) = (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1) + 4\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) \\ -2(\nabla\vartheta_*)(\nabla_{X_1}X_1, \nabla_{X_1}X_1) - 2(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1). \end{aligned} \tag{4.66}$$

Substituting (4.66) in (4.64), we get

$$\begin{aligned}
&-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)\\
&+5\nabla^{\vartheta\perp}_{X_1}(\nabla\vartheta_*)(X_1, \nabla_{X_1}X_1) - 2(\nabla\vartheta_*)(\nabla_{X_1}X_1, \nabla_{X_1}X_1)\\
&-(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0. \qquad (4.67)
\end{aligned}$$

Using (2.7) in (4.67), we obtain

$$\begin{aligned}
&-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)\\
&+5(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, \nabla_{X_1}X_1) + 3(\nabla\vartheta_*)(\nabla_{X_1}X_1, \nabla_{X_1}X_1)\\
&+4(\nabla\vartheta_*)(X_1, \nabla^2_{X_1}X_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0. \qquad (4.68)
\end{aligned}$$

Using Serret-Frenet equations in (4.68), we get

$$\begin{aligned}
&-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)\\
&+5\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + 3\kappa^2(\nabla\vartheta_*)(W_1, W_1)\\
&+4\kappa'(\nabla\vartheta_*)(X_1, W_1) - 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)\\
&+4\kappa\tau(\nabla\vartheta_*)(X_1, U_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0. \qquad (4.69)
\end{aligned}$$

From (4.69), we have

$$\begin{aligned}
&-\tfrac{1}{4\kappa\tau}\Big\{ - (\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)\\
&+5\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + 3\kappa^2(\nabla\vartheta_*)(W_1, W_1)\\
&+4\kappa'(\nabla\vartheta_*)(X_1, W_1) - 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)\\
&+\lambda^2(\nabla\vartheta_*)(X_1, X_1)\Big\} = (\nabla\vartheta_*)(X_1, U_1). \qquad (4.70)
\end{aligned}$$

Changing U_1 into $-U_1$ in (4.69), we get

$$\begin{aligned}
&-(\nabla\vartheta_*)(X_1, {}^*\vartheta_*(S_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1))) + (\tilde{\nabla}^2_{X_1}(\nabla\vartheta_*))(X_1, X_1)\\
&+5\kappa(\tilde{\nabla}_{X_1}(\nabla\vartheta_*))(X_1, W_1) + 3\kappa^2(\nabla\vartheta_*)(W_1, W_1)\\
&+4\kappa'(\nabla\vartheta_*)(X_1, W_1) - 4\kappa^2(\nabla\vartheta_*)(X_1, X_1)\\
&+4\kappa\tau(\nabla\vartheta_*)(X_1, U_1) + \lambda^2(\nabla\vartheta_*)(X_1, X_1) = 0, \qquad (4.71)
\end{aligned}$$

and then subtracting from (4.69), we have

$$(\nabla\vartheta_*)(X_1, U_1) = 0. \tag{4.72}$$

Now, for second condition substituting (4.65) and (4.66) in (4.63), we have

$$\begin{aligned}&-(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2S_{\left(\tilde{\nabla}_{X_1}(\nabla\vartheta_*)\right)(X_1,X_1)}\vartheta_*(X_1),\\ &-5S_{(\nabla\vartheta_*)(X_1,\nabla_{X_1}X_1)}\vartheta_*(X_1) - S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1),\\ &\vartheta_*(\nabla^3_{X_1}X_1) + \lambda^2\vartheta_*(\nabla_{X_1}X_1) = 0.\end{aligned} \tag{4.73}$$

Using Frenet-Serret equations in (4.73), we get

$$\begin{aligned}&-(\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1) - 2S_{\left(\tilde{\nabla}_{X_1}(\nabla\vartheta_*)\right)(X_1,X_1)}\vartheta_*(X_1)\\ &-5\kappa S_{(\nabla\vartheta_*)(X_1,W_1)}\vartheta_*(X_1) - S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1)\\ &+(\kappa'' - \kappa^3 - \kappa\tau^2)\vartheta_*(W_1) + \kappa\tau\kappa_3\vartheta_*(W_3) + \lambda^2\kappa\vartheta_*(W_1)\\ &-3\kappa\kappa'\vartheta_*(X_1) + (2\kappa'\tau + \kappa\tau')\vartheta_*(U_1) = 0.\end{aligned} \tag{4.74}$$

Taking the inner product of equation (4.74) with $\vartheta_*(X_1)$, we have

$$\begin{aligned}&3\kappa\kappa' + g_N\Big((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(X_1)\Big)\\ &+2g_N\Big(S_{\left(\tilde{\nabla}_{X_1}(\nabla\vartheta_*)\right)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(X_1)\Big) + 5g_N\Big(\kappa S_{(\nabla\vartheta_*)(X_1,W_1)}\vartheta_*(X_1), \vartheta_*(X_1)\Big)\\ &+g_N\Big(S_{(\nabla\vartheta_*)(X_1,X_1)}Q\bar{\nabla}^{\vartheta}_{X_1}\vartheta_*(X_1), \vartheta_*(X_1)\Big) = 0.\end{aligned} \tag{4.75}$$

Using (2.5) and equation (4.66) in (4.75), we get

$$\begin{aligned}&\nabla^{\vartheta\perp}_{X_1}g_N\Big((\nabla\vartheta_*)(X_1,X_1), (\nabla\vartheta_*)(X_1,X_1)\Big) + 2\kappa g_N\Big((\nabla\vartheta_*)(X_1,W_1), (\nabla\vartheta_*)(X_1,X_1)\Big)\\ &+3\kappa\kappa' + g_N\Big((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(X_1)\Big) = 0.\end{aligned} \tag{4.76}$$

Using (2.5) and equation (4.66) in (2.9), we obtain

$$\begin{aligned}g_N\Big((\tilde{\nabla}_{X_1}S)_{(\nabla\vartheta_*)(X_1,X_1)}\vartheta_*(X_1), \vartheta_*(X_1)\Big) &= -2\kappa g_N\Big((\nabla\vartheta_*)(X_1,W_1), (\nabla\vartheta_*)(X_1,X_1)\Big)\\ &-\tfrac{1}{2}\nabla^{\vartheta\perp}_{X_1}g_N\Big((\nabla\vartheta_*)(X_1,X_1), (\nabla\vartheta_*)(X_1,X_1)\Big).\end{aligned} \tag{4.77}$$

Equation (4.76) and (4.77) together provides the required condition. □

Corollary 4.1. *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a isotropic Riemannian map between Riemannian manifolds (M, g_M) and (N, g_N). If $\bar{\alpha}(s) = \vartheta \circ \alpha(s)$ is a helix on N, where α is a horizontal curve on M, then curvature of α is constant.*

Proof. Taking $||(\nabla\vartheta_*)(X_1, X_1)||^2 = constant$, in a Theorem 4.1, we get $\kappa = constant$. □

Theorem 4.2. [24] *Let $\vartheta : (M, g_M) \to (N, g_N)$ be a Riemannian map between Riemannian manifolds (M, g_M), $dimM \geq 2$ and (N, g_N). Then ϑ maps a horizontal helix α on M to a helix $\bar{\alpha} = \vartheta \circ \alpha$ on N iff ϑ is totally umbilical and the mean curvature vector field H satisfies the following equation*

$$(\nabla^{\vartheta\perp}_{X_1})^2 H_2 = -\tau^2 H_2.$$

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PRINCIPAL NORMAL INDICATRIX (N) OF CURVES ACCORDING TO THEIR ALTERNATIVE FRAMES IN EUCLIDEAN 3-SPACE

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Abstract. In this paper, we define a new family of curves called principal normal indicatrix (briefly, PNI) of space curves with unit speed in 3-dimensional Euclidean space. During the definition, we use alternative frames and we give some conditions for space curves to be general helix, slant helix, plane curve or involute curve.

Keywords: Spherical indicatrix, Alternative frame

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1. Introduction

Curve theory is a fascinating area of differential geometry and therefore, attracts many researchers. Curve theory investigates the properties and classifications of curves. On the other hand, curves contain some special curves within themselves. Special curves are such curves that satisfy certain conditions or exhibit interesting geometric behaviors. Special curves are studied in different spaces and different frames and are closely related to many application areas such as physics, engineering, computer aided design, robotics and medicine. Helices, involute-evolute, Bertrand and Mannheim curve pairs are some examples of well-known special curves.

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In curve theory, one of the most attractive special curves are combined curves. If there exists a mathematical relationship between two or more curves, these curves are called combined curves. Spherical indicatrix of curves are also combined curves in curve theory. Izumiya and Takeuchi [1] defined a new kind of slant helix in Euclidean 3-space. They showed that γ is a slant helix iff the geodesic curvature of spherical image of principal normal indicatrix (N) of a space curve γ is $\sigma = \frac{\kappa^2}{(\kappa^2+\tau^2)^{3/2}}\left(\frac{\tau}{\kappa}\right)'$. Kula and Yaylı [2] studied the spherical indicatrix curves of slant helices. They showed that their spherical indicatrices were spherical helices. They [3] also gave the characterizations of slant helices via certain differential equations verified for each one of spherical indicatrix in Euclidean 3-space. In [4], Uzunoğlu et al. studied a curve whose spherical images (the tangent and binormal indicatrices) are spherical slant helices by using alternative frame and called it as a C−slant helix.

In this paper, we firstly define the principal normal indicatrix (PNI) of space curves with unit speed in 3-dimensional Euclidean space by using alternative frame. Then, we investigate the geometric properties of PNI of space curves and give some relationships between these space curves and special well-known curves such as general helix, slant helix, plane curves and involute curves.

2. Preliminaries

A regular curve $\gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^3$ has three orthonormal vectors denoted by $\vec{T}(s)$, $\vec{N}(s)$ and $\vec{B}(s)$ which are the tangent, the principal normal and the binormal unit vectors, respectively. The set $\left\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\right\}$ is called Frenet frame of γ and the unit vectors are calculated by $\vec{T}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|}$, $\vec{B}(s) = \frac{\gamma'(s)\times\gamma''(s)}{\|\gamma'(s)\times\gamma''(s)\|}$, $\vec{N}(s) = \vec{B}(s) \times \vec{T}(s)$. The orthonormal frame $\left\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\right\}$ has the Frenet-Serret formulas as

$$\begin{aligned} \vec{T}'(s) &= v\kappa(s)\vec{N}(s), \\ \vec{N}'(s) &= -v\left(\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s)\right), \\ \vec{B}'(s) &= -v\tau(s)\vec{N}(s), \end{aligned} \tag{2.1}$$

where $v = \|\gamma'(s)\|$, $\kappa(s)$ is the first and $\tau(s)$ is the second curvature functions of γ. Besides, the second curvature $\tau(s)$ is also known as torsion. The curvature functions are calculated as $\kappa(s) = \frac{\|\gamma'(s)\times\gamma''(s)\|}{\|\gamma'(s)\|^3}$, $\tau(s) = \frac{\langle\vec{\gamma}'(s)\times\vec{\gamma}''(s),\gamma'''(s)\rangle}{\|\gamma'(s)\times\gamma''(s)\|^2}$. A Darboux vector is the angular velocity vector $\vec{\omega}$ of the Frenet frame of a point moving with unit speed along a curve which enables us to interpret the curvature and torsion geometrically. The Darboux vector $\vec{\omega}$ is defined by

$\vec{\omega} = \tau\vec{T} + \kappa\vec{B}$ [5] and the unit Darboux vector is given by

$$\vec{W} = \frac{\vec{\omega}}{\|\vec{\omega}\|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}\left(\tau\vec{T} + \kappa\vec{B}\right). \tag{2.2}$$

Because $\vec{W} \perp \vec{N}$, a new unit vector $\vec{C}$ is obtained as $\vec{C} = \vec{W} \times \vec{N}$. In this case, a new orthonormal frame can be constructed along γ. This frame is called as alternative frame and is denoted by $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$. The derivative formulas of the frame $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ is presented by

$$\begin{bmatrix} \vec{N}' \\ \vec{C}' \\ \vec{W}' \end{bmatrix} = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{bmatrix} \begin{bmatrix} \vec{N} \\ \vec{C} \\ \vec{W} \end{bmatrix}, \tag{2.3}$$

where

$$f = \sqrt{\kappa^2+\tau^2}, \;\; g = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'. \tag{2.4}$$

The curve γ is a slant helix iff g/f is constant [4]. Moreover, the relationship between these frames is as follows:

$$\begin{cases} \vec{C} = -\kappa^*\vec{T} + \tau^*\vec{B}, \;\; \vec{W} = \tau^*\vec{T} + \kappa^*\vec{B}, \\ \vec{T} = -\kappa^*\vec{C} + \tau^*\vec{W}, \;\; \vec{B} = \tau^*\vec{C} + \kappa^*\vec{W}, \end{cases} \tag{2.5}$$

where $\kappa^* = \kappa/f$ and $\tau^* = \tau/f$ [6].

3. Principal Normal Indicatrix (N) of Curves

In this section, we firstly define a new spherical image by translating the unit principal normal vector $\vec{N}$ of a space curve with unit speed to the center of the unit sphere. Then, we will give the following definitions, theorems and propositions by using similar ideas in [2, 4, 7].

Definition 3.1. *Let $\gamma : I \subset \mathbb{R}^3$ and $\gamma_N : I \subset \mathbb{R}^3 \to S_0^2$ be unit speed curves. Then γ_N is called principal normal indicatrix (PNI) of the curve γ and satisfies the equation as*

$$\vec{\gamma}_N(s_N) = \vec{N}(s)$$

where S_0^2 denotes a unit sphere and, s and s_N are arc length parameters of γ and γ_N, respectively.

Now, let calculate the ratio ds/ds_N.

$$s_N = \int_0^s \left\|\vec{\gamma}_N'\right\| du = \int_0^s \left\|f\vec{C}\right\| du = \int_0^s |f|\, du$$

The differential of s_N yields

$$ds_N = f ds.$$

where $f > 0$. Then, we get

$$\frac{ds}{ds_N} = \frac{1}{f}.$$

Theorem 3.1. *Let γ_N be the PNI of a unit speed curve γ in Euclidean 3-space. Then the Frenet frame of γ_N is computed with regards to the alternative frame of γ as follows:*

$$\begin{aligned} \vec{T}_N &= \vec{C}, \\ \vec{N}_N &= -m\vec{N} + n\vec{W}, \\ \vec{B}_N &= n\vec{N} + m\vec{W} \end{aligned} \tag{3.6}$$

where $m = \frac{f}{\sqrt{f^2+g^2}}$ and $n = \frac{g}{\sqrt{f^2+g^2}}$, and the sets $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ and $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ are alternative frame of γ and Frenet frame of γ_N, respectively.

Proof. The derivative of $\vec{\gamma}_N(s_N) = \vec{N}(s)$ with respect to s is obtained as

$$\frac{d\vec{\gamma}_N}{ds} = \vec{\gamma}_N' = f\vec{C} \tag{3.7}$$

From here, we have

$$\vec{T}_N = \frac{d\vec{\gamma}_N}{ds_N} = \frac{d\vec{\gamma}_N}{ds}\frac{ds}{ds_N} = f\vec{C}\frac{1}{f} = \vec{C}$$

On the other hand, the second derivative of γ_N with respect to s yields

$$\vec{\gamma}_N'' = -f^2\vec{N} + f'\vec{C} + fg\vec{W}. \tag{3.8}$$

The cross product of (3.7) and (3.8) gives Binormal vector of γ_N as

$$\vec{B}_N = \frac{\vec{\gamma}_N' \times \vec{\gamma}_N''}{\left\|\vec{\gamma}_N' \times \vec{\gamma}_N''\right\|} = \frac{g}{\sqrt{f^2+g^2}}\vec{N} + \frac{f}{\sqrt{f^2+g^2}}\vec{W}. \tag{3.9}$$

From (3.7) and (3.9), we get

$$\vec{N}_N = \vec{B}_N \times \vec{T}_N = -\frac{f}{\sqrt{f^2+g^2}}\vec{N} + \frac{g}{\sqrt{f^2+g^2}}\vec{W}. \tag{3.10}$$

Thus, we complete the proof. □

We should note that since $f > 0$, we obtain $m > 0$. Then, the following theorem is given:

Theorem 3.2. *Let γ_N be the PNI of a unit speed curve γ. Then the following invariants are given:*

$$\kappa_N = \frac{1}{m}, \tau_N = \frac{m}{f}\left(\frac{g}{f}\right)' \tag{3.11}$$

where κ_N and τ_N are curvature and torsion of γ_N, respectively and $\kappa_N > 0$.

Proof. From the cross product of (3.7) and (3.8), we get

$$\left\|\vec{\gamma}_N' \times \vec{\gamma}_N''\right\| = f^2\sqrt{f^2+g^2}.$$

Since the curvature function of γ_N is computed by $\kappa_N = \frac{\|\vec{\gamma}_N' \times \vec{\gamma}_N''\|}{\left\|\vec{\gamma}_N'\right\|^3}$, we have

$$\kappa_N = \frac{1}{m}.$$

The torsion of γ_N is also computed by $\tau_N = \frac{\langle \vec{\gamma}_N' \times \vec{\gamma}_N'', \gamma_N''' \rangle}{\left\|\vec{\gamma}_N' \times \vec{\gamma}_N''\right\|^2}$. Then, we get

$$\tau_N = \frac{m}{f}\left(\frac{g}{f}\right)'.$$

Lastly, since $m > 0$, then $\kappa_N > 0$. □

Theorem 3.3. *The unit speed curve γ is a general helix iff the curve γ_N is a circle or a part of a circle.*

Proof. Let γ be a general helix. Then from (2.4), we have that τ/κ is constant, i.e., $g = 0$. Thus, we have $\kappa_N = 1$ and $\tau_N = 0$ which means γ_N is a circle or a part of a circle. Conversely, assume that γ_N is a circle or a part of a circle. Then, κ_N is a non-zero constant and $\tau_N = 0$. Since $f > 0$ and $m > 0$, we get $g = 0$. Therefore, γ is a general helix. □

Theorem 3.4. *The unit speed curve γ is a slant helix iff the curve γ_N is planar.*

Proof. Let γ be a slant helix. We know that the curve γ is a slant helix iff g/f is constant [4]. Then, from (3.11), we obtain $\tau_N = 0$ which means γ_N is planar. Conversely, assume that γ_N is planar. Thus, we get $\tau_N = 0$ and it implies that g/f is constant. □

Now we will take $\mu = \frac{1}{f(f^2+g^2)^{3/2}}\left(\frac{g}{f}\right)'$ to simplify the equations further. Then, we can give the following theorem:

Theorem 3.5. *Let γ_N be the PNI of a unit speed curve γ. Then the Darboux-like vector of Frenet frame $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ is given by*

$$\vec{\omega}_N = \frac{n}{m}\vec{N} + \frac{f\mu}{m^2}\vec{C} + \vec{W}.$$

Proof. We know that the Darboux vector of the Frenet frame $\left\{\vec{T}, \vec{N}, \vec{B}\right\}$ is given by $\vec{\omega} = \tau\vec{T} + \kappa\vec{B}$. Similarly, the Darboux vector of the Frenet frame $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ is given by $\vec{\omega}_N = \tau_N\vec{T}_N + \kappa_N\vec{B}_N$. Substituting the equalities in (3.6) and (3.11) into $\vec{\omega}_N$, we get $\vec{\omega}_N = \frac{g}{f}\vec{N} + \frac{1}{\sqrt{f^2+g^2}}\left(\frac{g}{f}\right)'\vec{C} + \vec{W}$. Since $m = \frac{f}{\sqrt{f^2+g^2}}$ and $n = \frac{g}{\sqrt{f^2+g^2}}$, we see that $\frac{g}{f} = \frac{n}{m}$ and

$m^2+n^2=1$. Taking $\frac{1}{f(f^2+g^2)^{3/2}}\left(\frac{g}{f}\right)'=\mu$, we write $\frac{1}{\sqrt{f^2+g^2}}\left(\frac{g}{f}\right)'=\frac{f\mu}{m^2}$. Then, we complete the proof. □

From (3.6), we see that there exists an orthonormal frame such as $\left\{\vec{N}_N,\vec{C}_N,\vec{W}_N\right\}$. In this case, we give the following corollary:

Corollary 3.1. *Let γ_N be the PNI of a unit speed curve γ in Euclidean 3-space. Then the alternative frame of γ_N is computed with regards to the alternative frame of γ as follows:*

$$\begin{aligned}\vec{N}_N&=-m\vec{N}+n\vec{W},\\ \vec{C}_N&=\tfrac{\mu nf}{\sqrt{m^2+\mu^2f^2}}\vec{N}-\tfrac{m}{\sqrt{m^2+\mu^2f^2}}\vec{C}+\tfrac{\mu mf}{\sqrt{m^2+\mu^2f^2}}\vec{W},\\ \vec{W}_N&=\tfrac{mn}{\sqrt{m^2+\mu^2f^2}}\vec{N}+\tfrac{\mu f}{\sqrt{m^2+\mu^2f^2}}\vec{C}+\tfrac{m^2}{\sqrt{m^2+\mu^2f^2}}\vec{W},\end{aligned}\tag{3.12}$$

where the sets $\left\{\vec{N},\vec{C},\vec{W}\right\}$ and $\left\{\vec{N}_N,\vec{C}_N,\vec{W}_N\right\}$ are alternative frames of γ and γ_N, respectively.

Proof. The norm of $\vec{\omega}_N$ in Theorem 3.5 is obtained as $\|\vec{\omega}_N\|=\frac{\sqrt{m^2+\mu^2f^2}}{m^2}$. So, we get the Darboux unit vector $\vec{W}_N=\frac{\vec{\omega}_N}{\|\vec{\omega}_N\|}$. Since $\vec{C}_N=\vec{W}_N\times\vec{N}_N$, we satisfy the desired result. □

Theorem 3.6. *Let γ_N be the PNI of a unit speed curve γ. Then the following invariants are given:*

$$f_N=\sqrt{\frac{1}{m^2}+\frac{\mu m^4}{f^3}},\, g_N=\frac{f^3}{f^3+\mu m^2}\left(\frac{\mu m^5}{f^3}\right)'$$

where f_N and g_N are curvatures of γ_N, and $f_N>0$.

Proof. From (2.4), we can write $f_N=\sqrt{\kappa_N^2+\tau_N^2}$ and $g_N=\frac{\kappa_N^2}{\kappa_N^2+\tau_N^2}\left(\frac{\tau_N}{\kappa_N}\right)'$. Substituting $\kappa_N=\frac{1}{m}$, $\tau_N=\frac{m^2}{f}\left(\frac{g}{f}\right)'$ in (3.11) into f_N and g_N, we get (3.13). □

Theorem 3.7. *Let that $\bar{\gamma}$ be an involute of the unit speed curve γ and the set $\left\{\vec{\bar{T}},\vec{\bar{N}},\vec{\bar{B}}\right\}$ denotes the Frenet frame of $\bar{\gamma}$. Then the relationship between the Frenet frames $\left\{\vec{T}_N,\vec{N}_N,\vec{B}_N\right\}$ and $\left\{\vec{\bar{T}},\vec{\bar{N}},\vec{\bar{B}}\right\}$ of γ_N and $\bar{\gamma}$, respectively is computed as*

$$\begin{cases}\vec{T}_N=\vec{\bar{N}},\\ \vec{N}_N=-m\vec{\bar{T}}+n\vec{\bar{B}},\\ \vec{B}_N=n\vec{\bar{T}}+m\vec{\bar{B}}.\end{cases}\tag{3.13}$$

Proof. Since $\bar{\gamma}$ is an involute of the unit speed curve γ, we know that

$$\begin{aligned}\vec{\bar{T}}&=\vec{N},\\ \vec{\bar{N}}&=-\tfrac{\kappa}{f}\vec{T}+\tfrac{\tau}{f}\vec{B},\\ \vec{\bar{B}}&=\tfrac{\tau}{f}\vec{T}+\tfrac{\kappa}{f}\vec{B}\end{aligned}\tag{3.14}$$

where $f=\sqrt{\kappa^2+\tau^2}$ [2, 5, 6]. From (2.4) and (2.5), we can rewrite (3.14) as follows:

$$\vec{\bar{T}}=\vec{N},\quad \vec{\bar{N}}=\vec{C},\quad \vec{\bar{B}}=\vec{W}. \tag{3.15}$$

Substituting the last three equalities into (3.6), we complete the proof. □

Remark 3.1. *The alternative frame* $\left\{\vec{N},\vec{C},\vec{W}\right\}$ *of the unit speed curve* γ *and the Frenet frame* $\left\{\vec{\bar{T}},\vec{\bar{N}},\vec{\bar{B}}\right\}$ *of its involute* $\bar{\gamma}$ *coincide with each other at any moment.*

Proposition 3.1. *Let* $\bar{\kappa}$ *and* $\bar{\tau}$ *be the curvature and torsion functions of involute* $\bar{\gamma}$ *of the unit speed curve* γ*, respectively. Then, PNI* γ_N *of the curve* γ *is an involute of* $\bar{\gamma}$ *iff* $f=\bar{\kappa}$ *and* $g=\bar{\tau}$*.*

Proof. If γ_N is an involute of the curve $\bar{\gamma}$, we get

$$\begin{aligned}
&\vec{T}_N=\vec{\bar{N}},\\
&\vec{N}_N=-\tfrac{\bar{\kappa}}{\bar{f}}\vec{\bar{T}}+\tfrac{\bar{\tau}}{\bar{f}}\vec{\bar{B}},\\
&\vec{B}_N=\tfrac{\bar{\tau}}{\bar{f}}\vec{\bar{T}}+\tfrac{\bar{\kappa}}{\bar{f}}\vec{\bar{B}}
\end{aligned} \tag{3.16}$$

where $\bar{f}=\sqrt{\bar{\kappa}^2+\bar{\tau}^2}$ [2, 5, 6]. When we compare (3.16) with (3.13), we complete the proof. □

3.1. Relationships Between Principal Normal Indicatrix (N) and Tangent Indicatrix (T) of Curves.

Definition 3.2. *Let* $\gamma: I\subset\mathbb{R}^3$ *and* $\gamma_T: I\subset\mathbb{R}^3\to S_0^2$ *be unit speed curves in Euclidean 3-space. Then* γ_T *is called tangent indicatrix of the curve* γ *and satisfies the equation as*

$$\vec{\gamma}_T(s_T)=\vec{T}(s)$$

where S_0^2 *denotes a unit sphere and,* s *and* s_T *are arc length parameters of* γ *and* γ_T*, respectively.*

Now, let calculate the ratio ds/ds_T.

$$s_T=\int_0^s\left\|\vec{\gamma}_T'\right\|du=\int_0^s\left\|\kappa\vec{N}\right\|du=\int_0^s|\kappa|\,du$$

The differential of s_T gives

$$ds_T=\kappa ds.$$

where $\kappa>0$. Then, we get

$$\frac{ds}{ds_T}=\frac{1}{\kappa}.$$

Theorem 3.8. *Let γ_T be the tangent indicatrix of a unit speed curve γ in Euclidean 3-space. Then the Frenet frame $\left\{\vec{T}_T, \vec{N}_T, \vec{B}_T\right\}$ of γ_T is computed with regards to the alternative frame $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ of γ as follows:*

$$\begin{aligned} \vec{T}_T &= \vec{N}, \\ \vec{N}_T &= -\tfrac{\kappa}{f}\vec{T} + \tfrac{\tau}{f}\vec{B}, \\ \vec{B}_T &= \tfrac{\tau}{f}\vec{T} + \tfrac{\kappa}{f}\vec{B} \end{aligned} \tag{3.17}$$

where $f = \sqrt{\kappa^2 + \tau^2}$.

Proof. The derivative of $\vec{\gamma}_T(s_T) = \vec{T}(s)$ is obtained as

$$\frac{d\vec{\gamma}_T}{ds} = \vec{\gamma}_T' = \kappa\vec{N}.$$

From here, we have

$$\vec{T}_T = \frac{d\vec{\gamma}_T}{ds_T} = \frac{d\vec{\gamma}_T}{ds}\frac{ds}{ds_T} = \kappa\vec{N}\frac{1}{\kappa} = \vec{N}$$

The second derivative of γ_T also gives

$$\vec{\gamma}_T'' = -\kappa^2\vec{T} + \kappa'\vec{N} + \kappa\tau\vec{B}.$$

The cross product of $\vec{\gamma}_T'$ and $\vec{\gamma}_T''$ gives binormal vector of γ_T as

$$\vec{B}_T = \frac{\vec{\gamma}_T' \times \vec{\gamma}_T''}{\left\|\vec{\gamma}_T' \times \vec{\gamma}_T''\right\|} = \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\vec{T} + \frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\vec{B}.$$

Thus, we have

$$\vec{N}_T = \vec{B}_T \times \vec{T}_T = -\frac{\kappa}{\sqrt{\kappa^2+\tau^2}}\vec{T} + \frac{\tau}{\sqrt{\kappa^2+\tau^2}}\vec{B}.$$

Thus, we complete the proof. □

From (2.5), we can rewrite the equations in (3.17) as follows:

$$\vec{T}_T = \vec{N}, \quad \vec{N}_T = \vec{C}, \quad \vec{B}_T = \vec{W}. \tag{3.18}$$

Therefore, from (3.15), the following corollary is given:

Corollary 3.2. *The Frenet frame $\left\{\vec{\bar{T}}, \vec{\bar{N}}, \vec{\bar{B}}\right\}$ of involute curve $\bar{\gamma}$ of the unit speed curve γ coincide with the Frenet frame $\left\{\vec{T}_T, \vec{N}_T, \vec{B}_T\right\}$ of tangent indicatrix γ_T of the curve γ.*

Corollary 3.3. *The tangent indicatrix curve γ_T is an involute of the unit speed curve γ.*

Proof. From (3.17) the proof is satisfied clearly. □

Now, from (3.6) and (3.17), we can give the following proposition:

Proposition 3.2. *Let that γ_N and γ_T be principal normal and tangent indicatrices of the unit speed curve γ in Euclidean 3-space with regards to the Frenet frames $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ and $\left\{\vec{T}_T, \vec{N}_T, \vec{B}_T\right\}$, respectively. Then, we have*

$$\begin{aligned} \vec{T}_N &= \vec{N}_T, \\ \vec{N}_N &= -m\vec{T}_T + n\vec{B}_T, \\ \vec{B}_N &= n\vec{T}_T + m\vec{B}_T \end{aligned} \tag{3.19}$$

where $m = \frac{f}{\sqrt{f^2+g^2}}$ and $n = \frac{g}{\sqrt{f^2+g^2}}$.

Proposition 3.3. *PNI γ_N of the unit speed curve γ is an involute of the tangent indicatrix γ_T of the curve γ iff $f = \kappa$ and $g = \tau$.*

3.2. **Relationships Between Principal Normal Indicatrix (N) and Binormal Indicatrix (B) of Curves.**

Definition 3.3. *Let $\gamma : I \subset \mathbb{R}^3$ and $\gamma_B : I \subset \mathbb{R}^3 \to S_0^2$ be unit speed curves in Euclidean 3-space. Then γ_B is called binormal indicatrix of the curve γ and satisfies the equation as*

$$\vec{\gamma}_B(s_B) = \vec{B}(s)$$

where S_0^2 denotes a unit sphere and, s and s_B are arc length parameters of γ and γ_B, respectively.

Now, let calculate the ratio ds/ds_B.

$$s_B = \int_0^s \left\|\vec{\gamma}_B'\right\| du = \int_0^s \left\|-\tau\vec{N}\right\| du = \int_0^s |\tau|\, du$$

The differential of s_B gives

$$ds_B = \tau ds.$$

where $\tau > 0$. Then, we get

$$\frac{ds}{ds_B} = \frac{1}{\tau}.$$

Theorem 3.9. *Let γ_B be the binormal indicatrix of a unit speed curve γ in Euclidean 3-space. Then the Frenet frame $\left\{\vec{T}_B, \vec{N}_B, \vec{B}_B\right\}$ of γ_T is computed with regards to the alternative frame $\left\{\vec{N}, \vec{C}, \vec{W}\right\}$ of γ as follows:*

$$\begin{aligned} \vec{T}_B &= -\vec{N}, \\ \vec{N}_B &= \tfrac{\kappa}{f}\vec{T} - \tfrac{\tau}{f}\vec{B}, \\ \vec{B}_B &= \tfrac{\tau}{f}\vec{T} + \tfrac{\kappa}{f}\vec{B} \end{aligned} \tag{3.20}$$

where $f = \sqrt{\kappa^2 + \tau^2}$.

Proof. The derivative of $\vec{\gamma}_B(s_B) = \vec{B}(s)$ is obtained as

$$\frac{d\vec{\gamma}_B}{ds} = \vec{\gamma}_B' = -\tau\vec{N}.$$

From here, we have

$$\vec{T}_B = \frac{d\vec{\gamma}_B}{ds_B} = \frac{d\vec{\gamma}_B}{ds}\frac{ds}{ds_B} = -\tau\vec{N}\frac{1}{\tau} = -\vec{N}$$

The second derivative of γ_B also gives

$$\vec{\gamma}_B'' = \kappa\tau\vec{T} - \tau'\vec{N} - \tau^2\vec{B}.$$

The cross product of $\vec{\gamma}_B'$ and $\vec{\gamma}_B''$ gives binormal vector of γ_B as

$$\vec{B}_B = \frac{\vec{\gamma}_B' \times \vec{\gamma}_B''}{\left\|\vec{\gamma}_B' \times \vec{\gamma}_B''\right\|} = \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}\vec{T} + \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}\vec{B}.$$

Thus, we have

$$\vec{N}_B = \vec{B}_B \times \vec{T}_B = \frac{\kappa}{\sqrt{\kappa^2 + \tau^2}}\vec{T} - \frac{\tau}{\sqrt{\kappa^2 + \tau^2}}\vec{B}.$$

Thus, we complete the proof. □

From (2.5), we can rewrite the equations in (3.20) as follows:

$$\vec{T}_B = -\vec{N}, \quad \vec{N}_B = -\vec{C}, \quad \vec{B}_B = \vec{W}. \tag{3.21}$$

Therefore, from (3.15), the following corollary is given:

Corollary 3.4. *The Frenet frame* $\left\{\vec{\bar{T}}, \vec{\bar{N}}, \vec{\bar{B}}\right\}$ *of involute curve* $\bar{\gamma}$ *of the unit speed curve* γ *coincide with the Frenet frame* $\left\{\vec{T}_B, \vec{N}_B, \vec{B}_B\right\}$ *of binormal indicatrix* γ_B *of the curve* γ*, i.e.,* $\vec{\bar{T}} = -\vec{T}_B, \quad \vec{\bar{N}} = -\vec{N}_B, \quad \vec{\bar{B}} = \vec{B}_B$.

Corollary 3.5. *The binormal indicatrix curve* γ_B *is an involute of the unit speed curve* γ.

Proof. From (3.20) the proof is satisfied clearly. □

Now, from (3.6) and (3.20), we can give the following proposition:

Proposition 3.4. *Let that* γ_N *and* γ_B *be principal normal and binormal indicatrices of the unit speed curve* γ *in Euclidean 3-space with regards to the Frenet frames* $\left\{\vec{T}_N, \vec{N}_N, \vec{B}_N\right\}$ *and* $\left\{\vec{T}_B, \vec{N}_B, \vec{B}_B\right\}$*, respectively. Then, we have*

$$\begin{aligned} \vec{T}_N &= -\vec{N}_B, \\ \vec{N}_N &= m\vec{T}_B + n\vec{B}_B, \\ \vec{B}_N &= -n\vec{T}_B + m\vec{B}_B \end{aligned} \tag{3.22}$$

where $m = \frac{f}{\sqrt{f^2+g^2}}$ *and* $n = \frac{g}{\sqrt{f^2+g^2}}$.

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NULL HYPERSURFACE NORMALIZED BY THE STRUCTURE VECTOR FIELD IN A PARASASAKIAN MANIFOLD

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Abstract. We examine the geometry of a null hypersurface M of a para-Sasakian manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ transversal to the structure vector field K. The later is then a rigging ζ for M, and M is called K-normalized null hypersurface. We characterize the geometry of such a null hypersurface and prove under some conditions that there exist leaves of an integrable distribution of the screen distribution admitting an almost para complex structure. Also, we derive certain non-existence results and discuss some properties of semi-symmetric(resp. locally symmetric) K-normalized null hypersurfaces of para-Sasakian manifolds, for instance, we demonstrate that any para-Sasakian manifold admitting a semi-symmetric totally geodesic K-normalized null hypersurface is of constant negative curvature along the null hypersurface.

Keywords: Almost paracontact manifold, Para-Sasakian manifold, K-Normalized null hypersurface, Rigging vector field, Structure vector field.

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1. Introduction

A submanifold M of a semi-Riemannian manifold is null if the induced metric tensor is degenerate on M. Null hypersurfaces are specifically essential because of their applications

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in physics and mainly in general relativity. The principal differences between null and non-degenerate hypersurfaces stand up because of the absence of natural projections on the former. This prevents the usual geometric objects from being induced on null hypersurfaces.

The rigging technique introduced in [6] has shown to be an efficient tool to study a null hypersurface. Briefly, the main idea consists of choosing a vector field ζ, called a rigging, such that $\zeta_p \notin T_pM$ for all $p \in M$. From this unique arbitrary choice, we derive all the geometric objects needed to handle a null hypersurface: a null section of $\mathrm{Rad}(TM)$, a screen distribution in TM, a transversal null section, and all the associated tensors.

Several authors have studied the geometry of null submanifolds of para-Sasakian manifolds tangent to the structure vector field [1, 9,]. In [8], the authors considered the case where the null hypersurface is transversal to the structure vector field K. The later is then a rigging ζ for M, and M will be called K-normalized null hypersurface in this work. The question now arise of knowing wether it is always possible to select a structure vector field with specific geometric properties (closedness, quasi-conformality, etc.) but also with prescribed geometric properties for the null hypersurface (curvature condition, umbilicity, geodesibility, etc.). The goal of this paper is to provide a few answers to the above questions by studying the geometry of K-normalized null hypersurfaces in para-Sasakian manifolds.

The organization of this paper is the following. Section 2 contains all the preliminaries needed. In Section 3, we give an example, characterize the underlined null hypersurface (Theorem 3.1), and prove under some condition that there exist leaves of an integrable distribution of the screen distribution admitting an almost para-complex structure (Theorem 3.2). We establish sufficient conditions to guarantee that the Ricci type tensor Ric is an induced symmetric Ricci tensor of M (Theorem 3.3). We also show that there is no screen invariant K-normalized null hypersurface in an almost para-contact metric manifold $\overline{M}$ (Theorem 3.4) , and we establish obstruction results involving the geometric conditions on the structure vector field (Theorem 3.5, (Theorem 3.6 and Theorem 3.7). In Section 4, we discuss some properties of a semi-symmetric (resp. locally symmetric) normalized null hypersurfaces of para-Sasakian manifolds. We show that a K-normalized null hypersurface is totally geodesic if and only if it is locally symmetric (Theorem 4.1) and that any para-Sasakian manifold admitting a semi-symmetric totally geodesic K-normalized null hypersurface is of constant negative curvature along the null hypersurface (Theorem 4.2).

2. Preliminaries

In this section, we give a brief review about rigging tectnique and Para-sasakian manifolds.

2.1. **Rigging technique for null hypersurface.** Let $(\overline{M}^{n+2}, \overline{g})$ be a Lorentzian manifold and (M, g) a null hypersurface of $(\overline{M}, \overline{g})$. Due to Gutiérrez and Olea; see [6]. A *rigging* for M is a vector field ζ defined on some open set of $\overline{M}$ containing M such that for each $p \in M$ $\zeta_p \notin T_pM$. Given a rigging ζ for M, we set $\overline{\omega} = \overline{g}(\zeta, \cdot)$, $\omega = i^*\overline{\omega}$, $\breve{g} = \overline{g} + \overline{\omega} \otimes \overline{\omega}$ and $\widetilde{g} = i^*\breve{g}$, where $i : M \hookrightarrow \overline{M}$ is the canonical inclusion map. It is well known that $\widetilde{g}$ is a Riemannian metric on M. The *rigged vector field* on M is the unique null vector field ξ given by $\tilde{g}(\xi, .) = \omega$ and it satisfies $\overline{g}(\zeta, \xi) = 1$. A rigging ζ defines a screen distribution $\mathscr{S}(\zeta)$ given by $\mathscr{S}(\zeta) = TM \cap \zeta^{\perp} = \ker \omega$. The null transversal vector field on M is

$$N = \zeta - \frac{1}{2}\overline{g}(\zeta, \zeta)\xi, \tag{2.1}$$

which is the unique null vector field such that $\overline{g}(N, \xi) = 1$. Moreover, it is worth noting that $T\overline{M}$ admits the following splitting

$$\begin{aligned} T\overline{M}|_M &= TM \oplus span(N) \\ &= \{\mathscr{S}(\zeta) \oplus span(\xi)\} \oplus span(N). \end{aligned} \tag{2.2}$$

According to the decomposition (2.2), the Gauss and Weingarten equations of M and $\mathscr{S}(\zeta)$ are the following (see[4, p. 82-85]):

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \qquad \nabla_X PY = \overset{\star}{\nabla}_X PY + C(X, PY)\xi, \tag{2.3}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \qquad \nabla_X \xi = -\overset{\star}{A}_\xi X - \tau(X)\xi, \quad \tau(X) = \overline{g}(\overline{\nabla}_X N, \xi), \tag{2.4}$$

$\forall X, Y$ tangent to M. Here, ∇ and $\overset{\star}{\nabla}$ are induced linear connction on TM and $\mathscr{S}(TM)$, respectively, B and C are the second fundamental forms on TM and $\mathscr{S}(\zeta)$ respectively. Moreover, A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathscr{S}(TM)$, respectively, connected with the second fundamental forms by $B(X, Y) = g(\overset{\star}{A}_\xi X, Y)$ and $C(X, PY) = g(A_N X, PY)$, and τ is a 1-form on TM. The induced linear connection ∇ is not a metric connection. In fact, using the fact that $\overline{\nabla}\overline{g} = 0$, we have

$$(\nabla_X g)(Y, Z) = B(X, Y)\omega(Z) + B(X, Z)\omega(Y), (2.16) \tag{2.5}$$

$\forall X, Y, Z \in \Gamma(TM)$. Also C is not symmetric since

$$C(X, Y) - C(Y, X) = g(\nabla_X Y - \nabla_Y X, N) = \omega([X, Y]), \forall X, Y \in \mathscr{S}(\zeta). \tag{2.6}$$

Let us denote by $\overline{R}$ and R the Riemannian curvature tensors of $\overline{\nabla}$ and ∇, respectively. Using (2.3) and (2.4), we get the so called Gauss-Codazzi equations [4]

$$\overline{g}(\overline{R}(X,Y)Z,\xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z), \quad (2.7)$$

$$\overline{g}(\overline{R}(X,Y)Z,PW) = \overline{g}(R(X,Y)Z,PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW), \quad (2.8)$$

$$\overline{g}(\overline{R}(X,Y)\xi,N) = \overline{g}(R(X,Y)\xi,N) = C(Y,\overset{\star}{A}_\xi X) - C(\overset{\star}{A}_\xi Y,X) - 2d\tau(X,Y), \quad (2.9)$$

$\forall X,Y,Z$ and $W \in \Gamma(TM)$.

We say that the rigging vector field ζ has a quasi-conformal screen distribution if there exists ϕ and σ in $C^\infty(M)$ such that

$$A_N X = \rho \overset{\star}{A}_\xi X + \sigma PX, \quad (2.10)$$

for any $X \in \Gamma(TM)$. For $\sigma = 0$, we simply say that ζ has conformal screen distribution. We say that the rigging vector field is distinguished if the one-form τ vanishes. A null hypersuface M is said to be totally umbilical (resp. totally geodesic) in $\overline{M}$ if there exists a smooth function k on M such that

$$B(X,Y) = kg(X,Y) \quad (2.11)$$

(resp. B vanishes identically on M). Remembering that $\overset{\star}{A}_\xi \xi = 0$, M is totally umbilical (resp. totally geodesic) in $\overline{M}$ if $\overset{\star}{A}_\xi X = kX$ for any $X \in \Gamma(\mathscr{S}(TM))$ (resp. $\overset{\star}{A}_\xi = 0$).

Also the screen distribution $\mathscr{S}(\zeta)$ is totally umbilical (resp. totally geodesic) in M if there is a smooth function λ such that $C(X,PY) = \lambda g(X,Y)$ for all $X,Y \in \Gamma(TM)$ (resp. C vanishes identically)([4],[2]).

2.2. **Para-Sasakian Manifolds.** A $(2n+1)$ dimensional manifold $\overline{M}^{2n+1}$ is said to be an almost paracontact metric manifold, if it admits a tensor field $\overline{\phi}$ of type $(1,1)$, a structure vector field K, a 1-form $\overline{\eta}$ and a pseudo-Riemannian metric $\overline{g}$ satisfying the following conditions[7][11]:

$$\overline{\phi}^2 = I - \overline{\eta} \otimes K, \quad \overline{\eta}(K) = 1, \quad \overline{\phi}(K) = 0, \quad \overline{\eta} \circ \overline{\phi} = 0 \quad (2.12)$$

$$\overline{g}(\overline{\phi}X,\overline{\phi}Y) = -\overline{g}(X,Y) + \overline{\eta}(X)\overline{\eta}(Y), X,Y \in \Gamma(T\bar{M}), \quad (2.13)$$

where I denotes the identity transformation. From (2.13), we deduce

$$\overline{g}(\overline{X},\overline{\phi}Y) = -\overline{g}(\overline{\phi}X,Y) \quad (2.14)$$

$$\overline{g}(X,K) = \overline{\eta}(X), \quad (2.15)$$

for $X, Y \in \Gamma(TM)$. From (2.15), we get

$$\overline{g}(K, K) = 1.$$

An almost para-contact metric manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ is called a para-Sasakian manifold if [11]

$$(\overline{\nabla}_X \overline{\phi})Y = -\bar{g}(X, Y)K + \overline{\eta}(Y)X, \ \forall X, Y \in \Gamma(T\bar{M}). \tag{2.16}$$

From (2.16), we have

$$\overline{\nabla}_X K = -\overline{\phi}X. \tag{2.17}$$

Example 2.1. [1] *Let $\overline{M} = \mathbb{R}^{2n+1}$ be the $(2n+1)$-dimensional real space with standard coordinate system $(x_1, y_1, x_2, y_2, ..., x_n, y_n, z)$. Defining*

$$\begin{aligned}
&\bar{\phi}\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}, \quad \bar{\phi}\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \quad \bar{\phi}\frac{\partial}{\partial z} = 0,\\
&\overline{\eta} = dz, \quad K = \frac{\partial}{\partial z},\\
&\bar{g} = \overline{\eta} \otimes \overline{\eta} + \sum_{\alpha=1}^{n} dx_\alpha \otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha \otimes dy_\alpha,
\end{aligned}$$

where $\alpha = 1, 2, ..., n$. The set $(\mathbb{R}^{2n+1}_n, \overline{\phi}, K, \eta, \overline{g})$ is an almost paracontact metric manifold.

If the paraholomorphic sectional curvature denoted by c is constant on the para-Sasakian manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$, then the later is a para-Sasakian space form. Moreover, the curvature tensor $\overline{R}$ of $\overline{M}$ satisfies [9, Theorem 2.2]

$$\begin{aligned}
\overline{g}(\overline{R}(X, Y)Z, W) = {} & \frac{c-3}{4}\{\overline{g}(Y, Z)\overline{g}(X, W) - \overline{g}(X, Z)\overline{g}(Y, W)\}\\
& + \frac{(c+1)}{4}\Big\{\overline{\eta}(X)\overline{\eta}(Z)\overline{g}(Y, W) - \overline{\eta}(Y)\overline{\eta}(Z)\overline{g}(X, W)\\
& + \overline{g}(X, Z)\overline{\eta}(Y)\overline{\eta}(W) - \overline{g}(Y, Z)\overline{\eta}(X)\overline{\eta}(W)\\
& + \overline{g}(\overline{\phi}Y, Z)\overline{g}(\overline{\phi}X, W) - \overline{g}(\overline{\phi}X, Z)\overline{g}(\overline{\phi}Y, W) - 2\overline{g}(\overline{\phi}X, Y)\overline{g}(\overline{\phi}Z, W)\Big\},
\end{aligned} \tag{2.18}$$

$\forall\ X, Y, Z \in \Gamma(TM)$. We refer to $\overline{M}(c)$ as a para-Sasakian space form.

3. Normalized null hypersurfaces of a para-Sasakian manifold

Let (M, ζ) be a normalized null hypersurface of a para-Sasakian manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$. K has the following pointwise decomposition along M:

$$K = K_{\mathscr{S}} + \gamma\xi + \beta N,$$

where γ and β are smooth functions on $\overline{M}$ defined by $\beta = \overline{\eta}(\xi)$, $\gamma = \overline{\eta}(N)$, and $K_{\mathscr{S}} \in \Gamma(\mathscr{S}(\zeta))$. Consider a global vector field U on $\mathscr{S}(\zeta)$ and its corresponding 1-form μ defined by

$$U = -\overline{\phi}\xi, \ \mu(X) = \overline{g}(X, U), \forall X \in \Gamma(TM). \tag{3.19}$$

From (2.2), we have

$$\overline{\phi}X = \phi X + \mu(X)N, \forall X \in \Gamma(TM), \tag{3.20}$$

where ϕ is a (1, 1)-tensor field on M. From (2.17) and (3.20), we obtain the following result, which is similar to the one given in [3, Proposition 3.1]

Proposition 3.1. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a null hypersurface of $\overline{M}$. Then, $\forall X \in \Gamma(TM)$ we get*

$$\begin{aligned} \overset{\star}{\nabla}_X K_{\mathscr{S}} &= \gamma \overset{\star}{A}_\xi X + \beta A_N X - P(\phi(X)), \\ X \cdot \gamma &= \gamma\tau(X) - C(X, K_{\mathscr{S}}) - \omega(\phi(X)), \\ X \cdot \beta &= -\beta\tau(X) - B(X, K_{\mathscr{S}}) - \mu(X), \end{aligned} \tag{3.21}$$

where P is the projection morphism of $\Gamma(TM)$ onto $\Gamma(\mathscr{S}(\zeta))$ associated to the decomposition(2.2).

Proof. On one hand, $\forall X \in \Gamma(TM)$, we get

$$\begin{aligned} \overline{\nabla}_X K &\overset{(2.17)}{=} -\overline{\phi}X \\ &= -\phi X - \mu(X)N = -P(\phi X) - \omega(\phi X)\xi - \mu(X))N. \end{aligned}$$

On the other hand, $\forall X \in \Gamma(TM)$, we get

$$\begin{aligned} \overline{\nabla}_X K = \overset{\star}{\nabla}_X K_{\mathscr{S}} &- \gamma \overset{\star}{A}_\xi X - \beta A_N X \\ &+ \left(C(X, K_{\mathscr{S}}) + X \cdot \gamma - \gamma\tau(X)\right)\xi \\ &+ \left(X \cdot \beta + \beta\tau(X) + B(X, K_{\mathscr{S}})\right)N. \end{aligned}$$

Matching the tangential, radical and transversal components of the expressions above we get the result. □

Now, we suppose that the structure vector field K never belongs to the tangent space of the null hypersurface M. In this case K can be taken as a rigging ζ for M. Thus, we have the following.

Definition 3.1. *A null hypersurface M of an almost para-contact metric manifold $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ such that the structure vector field K is a rigging for M is said to be* ***K-normalized***.

This leads to the following direct consequence of our Proposition 3.1.

Corollary 3.1. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then for all $X \in \Gamma(TM)$, we have*

$$A_N X = -\frac{1}{2}\overset{\star}{A}_\xi X + P(\phi(X)), \tag{3.22}$$

$$\tau(X) = 2\omega(\phi(X)) = -\mu(X). \tag{3.23}$$

Proof. Since $K = \zeta$, then Eq.(2.1) leads to $\beta = 1$, $\gamma = \frac{1}{2}\overline{g}(\zeta, \zeta) = \frac{1}{2}$. Using this in (3.21) together with the fact that $\zeta_{\mathscr{S}} = 0$, we get the result. □

Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a (2n+1)-dimensional para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. It is worth noting that

$$\overline{\eta}(\xi) = 1, \ \overline{\eta}(N) = \frac{1}{2}. \tag{3.24}$$

Applying $\overline{\phi}$ to the first equation of (3.19), we get

$$\overline{\phi} U \overset{(2.12)}{=} -N + \frac{1}{2}\xi. \tag{3.25}$$

Also, from (2.14), it is obvious that

$$\overline{g}(\overline{\phi}N, \xi) = -\frac{1}{2}\overline{g}(\overline{\phi}\xi, \xi) = 0, \quad \overline{g}(\overline{\phi}N, N) = -\frac{1}{2}\overline{g}(\overline{\phi}\xi, N) = 0, \tag{3.26}$$

Then,

$$2\overline{\phi}(N) = -\overline{\phi}(\xi) = U \in \mathscr{S}(\zeta) \tag{3.27}$$

since the components of both $\overline{\phi}N$ and $\overline{\phi}\xi$ with respect to ξ and N vanish.

From (3.23) and (3.26), the following Corollary holds:

Corollary 3.2. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Setting $W = N - \frac{1}{2}\xi$, we have*

$$\tau(\xi) = -\mu(\xi) = 2\omega(\phi\xi) = 0, \tag{3.28}$$

$$\overline{\phi}X = P(\phi X) + \tau(X)W, \ \forall X, Y \in \Gamma(TM). \tag{3.29}$$

Definition 3.2. [1] *Let* $(\overline{M}^{2n+1}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *be an almost paracontact metric manifold and* M *a null hypersurface of* $\overline{M}$*.* M *is said to be screen invariant (resp screen semi-invariant) if* $\overline{\phi}(X)$ *(resp. both* $\overline{\phi}N$ *and* $\overline{\phi}\xi$*) belong(s) to the screen distribution for all* $X \in \mathscr{S}(\zeta)$*.*

From equation (3.27), we get the following proposition given in [8, Proposition 3.1].

Proposition 3.2. [8] *A* K*-normalized null hypersurface of an almost para-contact metric manifold* $\overline{M}$ *is rather a screen semi-invariant null hypersurface of* $\overline{M}$*.*

The following Theorem proves the converse of this proposition. Namely, a rigged screen semi-invariant null hypersurface in an almost paracontact metric manifold is transversal to the structure vector field.

Theorem 3.1. *Let* M *be a null hypersurface of an almost paracontact metric manifold* $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ *and* (M, ζ) *a normalized null hypersurface. Let* ξ *and* N_ζ *be the rigged vector field and the null transversal vector field associated to* ζ*. If* $\overline{\phi}(span\{\xi\}) = \overline{\phi}(span\{N_\zeta\})$ *then* K *is transversal to* M*. If in addition* $\overline{g}(\zeta, \zeta) = 1$ *and* $\overline{\phi}N = -\frac{1}{2}\overline{\phi}\xi$*, then* M *is a* K*-normaized null hypersurface.*

Proof. In this proof, N stands for N_ζ. Assume that $\overline{\phi}span\{\xi\} = \overline{\phi}span\{N_\zeta\}$. Then, there exists a non vanishing function θ such that $\overline{\phi}\xi = \theta\overline{\phi}N$. The inner product of this relation with respect to $\overline{\phi}\xi$ and $\overline{\phi}N$ give $(\overline{\eta}(\xi))^2 = \theta(-1+\overline{\eta}(\xi)\overline{\eta}(N))$ and $-1+\overline{\eta}(\xi)\overline{\eta}(N) = \theta(\overline{\eta}(N))^2$. Since $\theta \neq 0$, we get $\overline{\eta}(\xi) \neq 0$ and $\overline{\eta}(N) \neq 0$ and $(\overline{\eta}(\xi))^2 = (\theta\overline{\eta}(N))^2$. The later gives $\overline{\eta}(\xi) = \pm\theta\overline{\eta}(N)$. The case $\overline{\eta}(\xi) = \theta\overline{\eta}(N)$ implies that $\overline{\eta}(\xi)^2 = \theta\overline{\eta}(\xi)\overline{\eta}(N) = -\theta + \theta\overline{\eta}(\xi)\overline{\eta}(N)$, which is a contradiction. Thus $\overline{\eta}(\xi) = -\theta\overline{\eta}(N)$, from which $\overline{\eta}(\xi)^2 = -\theta\overline{\eta}(\xi)\overline{\eta}(N) = -\theta + \theta\overline{\eta}(\xi)\overline{\eta}(N)$, that is

$$\overline{\eta}(\xi)\overline{\eta}(N) = \frac{1}{2}. \tag{3.30}$$

Since $\theta = -\frac{\overline{\eta}(\xi)}{\overline{\eta}(N)} \neq 0$ and $\overline{\phi}\xi = \theta\overline{\phi}N$, it is worth noting that $\overline{\eta}(N)\overline{\phi}\xi + \overline{\eta}(\xi)\overline{\phi}N = 0$. Applying $\overline{\phi}$ to this equation to get $\overline{\eta}(N)\xi - \overline{\eta}(N)\overline{\eta}(\xi)K + \overline{\eta}(\xi)N - \overline{\eta}(\xi)\overline{\eta}(N)K = 0$. This together with (3.30) give $K = \overline{\eta}(N)\xi + \overline{\eta}(\xi)N = \gamma\xi + \beta N$. Thus K is transversal to M, which gives the first claim.

Now,

$$\overline{\phi}N = -\frac{1}{2}\overline{\phi}\xi \implies \overline{\phi}(N + \frac{1}{2}\xi) = 0 \implies \overline{\phi}\zeta = 0.$$

Operating $\overline{\phi}$ to the last equation of above relation and using the first equation of (2.13), we have $\zeta = \overline{\eta}(\zeta)K$. This leads to $\zeta = \pm K$ as $\overline{g}(\zeta,\zeta) = 1$. Therefore, K is a rigging for M and M is K-normalized. Which completes the proof.

□

Example 3.1. *Let $\overline{M} = \mathbb{R}^5$ be a 5-dimensional almost paracontact metric manifold with the structure $(\overline{\phi}, K, \overline{\eta}, \overline{g})$ given in Example 2.1.*

Consider a submanifold M of $(\mathbb{R}^5_2, \overline{\phi}, \zeta, \overline{\eta}, \overline{g})$ given by

$$M = \{(x_1, y_1, x_2, y_2, z) \in \mathbb{R}^5 | x_1 + x_2 - \sqrt{3}y_1 + z = 0\}.$$

It worth noting that TM is spanned by

$$\{V_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial z}, \quad V_2 = \frac{\partial}{\partial x_2} - \frac{\partial}{\partial z}, \quad V_3 = \frac{\partial}{\partial y_1} + \sqrt{3}\frac{\partial}{\partial z}, \quad V_4 = \frac{\partial}{\partial y_2}\}.$$

Since $K = \frac{\partial}{\partial z}$ is a spacelike vector field, then we may use it as a rigging ζ for M. Then, the corresponding rigged vector field is

$$\xi = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \sqrt{3}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial z}.$$

The associated null transversal vector field is

$$N = K - \frac{1}{2}\xi = \frac{1}{2}(-\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} - \sqrt{3}\frac{\partial}{\partial y_1} + \frac{\partial}{\partial z}).$$

The associated screen distribution is

$$\mathscr{S}(\zeta) = \{U_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \quad U_2 = \frac{\partial}{\partial x_1} + 2\frac{\partial}{\partial x_2} + \sqrt{3}\frac{\partial}{\partial y_1}, \quad U_3 = \frac{\partial}{\partial y_2}\}.$$

Next,

$$\begin{aligned} \overline{\phi}\xi &= \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + \sqrt{3}\frac{\partial}{\partial x_1} \\ &= \frac{\sqrt{3}}{3}(2U_1 + U_2) + U_3. \end{aligned}$$

From which we have $\overline{\phi}N = -\frac{1}{2}\overline{\phi}\xi \in \mathscr{S}(\zeta)$. Thus M is screen semi-invariant.

Now, using (3.19) in (2.13) leads to $\overline{g}(U,U) = 1$, thus the distribution $\overline{\phi}(\langle U \rangle)$ is nondegenerate. Then we are able to define the unique nondegenerate distribution D_0 by

Definition 3.3.

$$\mathscr{S}(\zeta) = D_0 \perp \langle U \rangle.$$

This ends in the subsequent decomposition:

$$TM = \{D_0 \perp \langle U\rangle\} \perp \langle \xi\rangle, \tag{3.31}$$

$$T\overline{M} = \{D_0 \perp \langle U\rangle\} \perp \{\langle \xi\rangle \oplus \langle N\rangle\}.$$

Proposition 3.3. [8] *D_0 is $\overline{\phi}$- invariant.*

Setting

$$D = \langle \xi\rangle \perp D_0 \quad \text{and} \quad D' = \langle U\rangle,$$

it follows that

$$TM = D \oplus D'.$$

Let M be a K-normalized null hypersurface, and S be the projection morphism of TM on D_0 with respect to the decomposition (3.31). From this, any vector field X on M is expressed as follows

$$X = SX + \omega(X)\xi + \mu(X)U. \tag{3.32}$$

Applying $\overline{\phi}$ to (3.32) and using (3.19), (3.25) and the fact $\overline{\eta}(X) = \omega(X)$, we have

$$\overline{\phi}X = \psi X - \frac{1}{2}\mu(X)\xi - \overline{\eta}(X)U + \mu(X)N = \psi X - \overline{\eta}(X)U + \mu(X)W, \tag{3.33}$$

where ψ is a globally defined tensor field of type (1, 1) on TM by

$$\psi X = \overline{\phi}SX, \forall X \in \Gamma(TM). \tag{3.34}$$

Applying $\overline{\phi}$ to (3.33) and using (2.12), (3.25), (3.19) and $\overline{\eta}(X) = \omega(X)$, we have

$$\begin{aligned} X - \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N = \overline{\phi}^2X &= \overline{\phi}(\psi X) + \frac{1}{2}\mu(X)U + \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N + \frac{1}{2}\mu(X)U \\ &= \psi^2X - \frac{1}{2}\mu(\psi X)\xi - \overline{\eta}(\psi X)U \\ &+ \mu(\psi X)N + \mu(X)U + \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N \\ &= \psi^2X + \mu(X)U + \frac{1}{2}\omega(X)\xi - \overline{\eta}(X)N. \end{aligned}$$

This leads to

$$\psi^2X = X - \omega(X)\xi - \mu(X)U, \tag{3.35}$$

which implies that

$$\psi^2X = SX.$$

Substituting (3.33) into (2.13), we have

$$-g(X,Y)+\overline{\eta}(X)\overline{\eta}(Y)=\overline{g}(\overline{\phi}X,\overline{\phi}Y)=g(\psi X,\psi Y)-\mu(X)\mu(Y)-\overline{\eta}(X)\overline{\eta}(Y).$$

This gives

$$g(\psi X,\psi Y)=-g(X,Y)+2\omega(X)\omega(Y)+\mu(X)\mu(Y). \tag{3.36}$$

Theorem 3.2. *Let $(\overline{M}^{(2n+1)},\overline{\phi},K,\overline{\eta},\overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$ such that $C(X,Y)=C(Y,X)$, $B(X,\overline{\phi}Y)=B(\overline{\phi}X,Y)$, $\forall X,Y\in\Gamma(D_0)$. Then (M_0,g,ψ) is an almost paracomplex manifold, where M_0 is a leaf of the almost paracontact complex distribution D_0.*

Proof. Under the hypothesis together with [8, Theorem 6.2], we have that D_0 is integrable. From (3.35) and (3.36), we have

$$\psi^2X=X,\ \ g(\psi X,\psi Y)=-g(X,Y)\ \forall X,Y\in\Gamma(D_0). \tag{3.37}$$

From (3.37), the claim follows. □

Proposition 3.4. *If M is a K-normalized in $(\overline{M}^{2n+1},\overline{\phi},K,\overline{\eta},\overline{g})$, then for $X,Y\in\Gamma(TM)$, we have*

$$(\nabla_X\phi)Y=-\frac{1}{2}g(X,Y)\xi+\omega(Y)X+\mu(Y)A_NX+\frac{1}{2}B(X,Y)U, \tag{3.38}$$

$$(\nabla_X\mu)(Y)=-g(X,Y)-B(X,\phi Y)-\tau(X)\mu(Y),$$

$$\nabla_XU=-X-\tau(X)U+\phi(\overset{\star}{A}_\xi X), \tag{3.39}$$

$$B(X,U)=\mu(\overset{\star}{A}_\xi X), \tag{3.40}$$

$$\overset{\star}{\nabla}_{PX}U=-PX+\tau(X)\overline{\phi}\xi+P(\phi(\overset{\star}{A}_\xi X)),$$

$$C(X,U)=\omega(\phi(\overset{\star}{A}_\xi X))-\omega(X). \tag{3.41}$$

Proof. Let $X,Y\in\Gamma(TM)$, we get

$$\begin{aligned}-g(X,Y)\zeta+\overline{\eta}(Y)X&=(\overline{\nabla}_X\overline{\phi})Y=\overline{\nabla}_X\bar{\phi}Y-\bar{\phi}(\overline{\nabla}_XY)\\&=\overline{\nabla}_X(\phi Y+\mu(Y)N)-\bar{\phi}(\nabla_XY+B(X,Y)N)\\&=\overline{\nabla}_X\phi Y+\overline{\nabla}_X\mu(Y)N-\bar{\phi}(\nabla_XY)-B(X,Y)\bar{\phi}N\\&=(\nabla_X\phi)Y+(\nabla_X\mu)(Y)N+B(X,\phi Y)N-\mu(Y)A_NX\\&\quad+\mu(Y)\tau(X)N-B(X,Y)\bar{\phi}N.\end{aligned} \tag{3.42}$$

Also,

$$\begin{aligned}\nabla_X U + B(X,U)N &= \overline{\nabla}_X U \\ &= -(\overline{\nabla}_X \overline{\phi}\xi) \\ &= -\Big((\overline{\nabla}_X \overline{\phi})\xi + \overline{\phi}(\overline{\nabla}_X \xi)\Big) \\ &= -X + \tau(X)\overline{\phi}\xi + \overline{\phi}(\overset{\star}{A}_\xi X) \\ &= -X + \tau(X)\overline{\phi}\xi + \phi(\overset{\star}{A}_\xi X) + \mu(\overset{\star}{A}_\xi X)N. \qquad (3.43)\end{aligned}$$

Next, it is worth noting that

$$\begin{aligned}\overset{\star}{\nabla}_X U + C(X,U)\xi + B(X,U)N = -PX - \omega(X)\xi + \tau(X)\overline{\phi}\xi + P(\phi(\overset{\star}{A}_\xi X)) \\ + \omega(\phi(\overset{\star}{A}_\xi X))\xi + \mu(\overset{\star}{A}_\xi X)N. \qquad (3.44)\end{aligned}$$

When we equate tangential and normal parts in (3.42)(resp, (3.43), (3.44)), we get the result. □

The following result is a direct consequence of Proposition 3.4.

Corollary 3.3. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a null hypersurface of $\overline{M}$. Then there is no K-normalization such that $\nabla\phi = 0$ or $\nabla U = 0$.*

Proof. (i) Replacing X and Y with ξ in (3.38) and X by ξ in (3.39) give $(\nabla_\xi \phi)\xi = \xi$ and $\nabla_\xi U = \xi$, which completes the proof. □

Proposition 3.5. *Let $(\overline{M}^{(2n+1)}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then $\forall X \in \Gamma(TM)$, we have*

$$\phi(A_N X) = -\phi(\overset{\star}{A}_\xi X) + 2X + 2\tau(X)U - \frac{1}{2}\omega(X)\xi.$$

Moreover, $\phi(A_N\xi)$ and ξ are linearly related.

Proof. Since $U = 2\overline{\phi}N$, we have

$$\begin{aligned}\nabla_X U + B(X,U)N = \overline{\nabla}_X U &= 2(\overline{\nabla}_X \overline{\phi}N) \\ &= 2\Big((\overline{\nabla}_X \overline{\phi})N + \overline{\phi}(\overline{\nabla}_X N)\Big) \\ &\overset{(2.3)-(2.16)}{=} -\omega(X)\xi - 2\omega(X)N + X - 2\overline{\phi}(A_N X) + 2\tau(X)\overline{\phi}N \\ &= -\omega(X)\xi + \tau(X)U - 2\phi(A_N X) - 2\mu(A_N X)N - 2\omega(X)N,\end{aligned}$$

that is

$$\nabla_X U = -\omega(X)\xi + \tau(X)U - 2\phi(A_N X) + X. \tag{3.45}$$

By equating (3.39) and (3.45), we have

$$\phi(A_N X) = -\frac{1}{2}\phi(\overset{\star}{A}_\xi X) + X + \tau(X)U - \frac{1}{2}\omega(X)\xi, \tag{3.46}$$

which gives the first claim. Now, setting $X = \xi$ in (3.46), we have $\phi(A_N\xi) = -\frac{1}{2}\xi$, which completes the proof. □

It is known that the Ricci type tensor Ric is an induced symmetric Ricci tensor of M if and only if the one form τ is closed on M [4, Theorem 3.2]. Using this, we get the following.

Theorem 3.3. *Let M be a K-normalized null hypersurface of a para-Sasakian manifold $\overline{M}$. Then Ric is an induced symmetric Ricci tensor of M if and only if $B(\phi X, Y) = B(X, \phi Y)$.*

Proof. From (3.23), we have $\tau(X) = -\mu(X)$. Differentiating this and using (3.43), we get

$$\begin{aligned} Y\tau(X) &= -\overline{g}(\overline{\nabla}_Y X, U) - \overline{g}(X, \overline{\nabla}_Y U) \\ &\overset{(2.3)-(3.43)}{=} -\mu(\nabla_Y X) + g(X,Y) + \tau(Y)\mu(X) - \overline{g}(X, \overline{\phi}(\overset{\star}{A}_\xi Y)) \\ &\overset{(3.23)(2.14)}{=} -\mu(\nabla_Y X) + g(X,Y) - \tau(Y)\tau(X) + \overline{g}(\overline{\phi}(X), \overset{\star}{A}_\xi Y) \\ &\overset{(3.20)}{=} -\mu(\nabla_Y X) + g(X,Y) - \tau(Y)\tau(X) + B(\phi(X), Y), \forall X, Y \in \Gamma(TM). \end{aligned} \tag{3.47}$$

Interchanging X and Y in (3.47), we get

$$X\tau(Y) - Y\tau(X) = -\mu([X,Y]) + B(\phi Y, X) - B(\phi X, Y). \tag{3.48}$$

On the other hand, (3.23) gives

$$\tau([X,Y]) = -\mu([X,Y]). \tag{3.49}$$

Then, by (3.48), (3.49) and the definition of $d\tau$, we have

$$\begin{aligned} 2d\tau(X,Y) &= [X\tau(Y) - Y\tau(X) - \tau([X,Y])] \\ &= B(\phi Y, X) - B(\phi X, Y). \end{aligned} \tag{3.50}$$

Thus $d\tau(X,Y) = 0$ if and only if $B(\phi X, Y) = B(X, \phi Y)$. Therefore, the claim follows from [4, Theorem 3.2]. □

Corollary 3.4. *Let M be a K-normalized null hypersurface of a para-Sasakian manifold $\overline{M}$. If M is totally geodesic, then the one form τ is closed. Moreover, Ric is an induced symmetric Ricci tensor of M.*

The presence of transversal structure vector field K in para-Sasakian manifolds prevents the existence of invariant null hypersurfaces. However, it is not the case when M is tangent to K (see [1, Theorem 10 and 14]). In the following, we obtain some non-existence results for K-normalized null hypersurfaces of a para-Sasakian manifolds.

Theorem 3.4. *There is no screen-invariant K-normalized null hypersurface in an almost para-contact metric manifold $\overline{M}$.*

Proof. If $\overline{\phi}(X) \in \mathscr{S}(\zeta)$, $\forall X \in \mathscr{S}(\zeta)$, then using (2.14) and (3.27), we will get $\overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = -\overline{g}(\xi, \overline{\phi}(\overline{\phi}\xi)) = 0$, which is absurd since from (2.13) $\overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = 1$. □

Theorem 3.5. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then,*

(i) *K cannot be screen quasi-conformal.*
(ii) *$\mathscr{S}(\zeta)$ cannot be totally umbilical in M.*
(iii) *M cannot be distinguished.*

Proof. Since from (3.28) of Corollary 3.2, $\tau(\xi) = -\mu(\xi) = 0$, then we have from (3.29) that $\overline{\phi}\xi = P(\overline{\phi}\xi)$.

(i) If K is screen quasi-conformal, then from (2.10), we will have $A_N\xi = 0$, which is a contracdition since from (3.22) $A_N\xi = P(\overline{\phi}\xi) = \overline{\phi}\xi \neq 0$. Indeed, if $\overline{\phi}\xi = 0$, then we will get from (2.13) that $0 = \overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = -g(\xi, \xi) + \overline{\eta}(\xi)\overline{\eta}(\xi) = 1$, which is absurd.

(ii) If $\mathscr{S}(\zeta)$ is totally umbilical, then using (3.41) of Proposition 3.4, we have $1 = C(\xi, U) = \gamma g(\xi, U) = 0$, which is absurd.

(iii) If $\tau(X) = 0$ $\forall X \in \mathscr{S}(\zeta)$, we will have from (3.23) that $\mu(U) = 0$ as $U = -\overline{\phi}\xi \in \mathscr{S}(\zeta)$, which is absurd since from (2.13) $1 = \overline{g}(\overline{\phi}\xi, \overline{\phi}\xi) = \mu(U)$. This completes the proof. □

Now, $\forall X \in \Gamma(TM)$, $\forall Y \in \mathscr{S}(\zeta)$, we have

$$\begin{aligned} C(X, Y) &= \overline{g}(\nabla_X Y, N) \\ &= \overline{g}(\overline{\nabla}_X Y, N) \\ &= -\overline{g}(Y, \overline{\nabla}_X N). \end{aligned} \tag{3.51}$$

From item (*iii*) of Theorem 3.5, together with (3.51) and (3.51), we have the following corollary.

Corollary 3.5. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M a K-normalized null hypersurface of $\overline{M}$. Then,*

(a) *$\mathscr{S}(\zeta)$ cannot be a parallel distribution.*

(b) *N cannot be a closed conformal vector field.*

Theorem 3.6. *Let $(\overline{M}^{2n+1}, \overline{\phi}, K, \overline{\eta}, \overline{g})(n \geq 1)$ be a para-Sasakian manifold. Then the structure vector field K cannot be a closed normalization for any null hypersurface M.*

Proof. Suppose that $K = \zeta$ is a closed normalization for any null hypersurface M, then $\overline{\eta} = \overline{g}(\zeta, .)$ is a closed 1-form on M which implies that $\mathscr{S}(\zeta)$ is integrable. This implies from (2.6) that C is symmetric on $\mathscr{S}(\zeta)$. From (3.22), we have

$$-\overline{g}(\overline{\phi}X, Y) + \frac{1}{2}B(X, Y) + C(X, Y) = 0, \ \forall X, Y \in \mathscr{S}(\zeta). \tag{3.52}$$

Using (3.52) together with the fact that B and C are symmetric, we have

$$\overline{g}(\overline{\phi}X, Y) - \overline{g}(\overline{\phi}Y, X) = 0, \ \forall X, Y \in \mathscr{S}(\zeta). \tag{3.53}$$

Since $\overline{g}(\overline{\phi}X, Y) \overset{(2.14)}{=} -\overline{g}(X, \overline{\phi}Y)$, (3.53) leads to $\overline{g}(\overline{\phi}X, Y) = 0$. This together with (3.33) give

$$\overline{g}(\psi X, Y) = 0, \ \forall X, Y \in \mathscr{S}(\zeta). \tag{3.54}$$

From (3.54) together with the fact that $\mathscr{S}(\zeta)$ is non-degenerate, we have $\psi X = 0$ for all $X \in \mathscr{S}(\zeta)$. This and (3.35), give $X = 0$ for all $X \in \mathscr{S}(\zeta)$. Which is absurd since $\mathscr{S}(\zeta)$ is of rank $2n - 1$ with $n > 1$. Thus ζ cannot be closed. □

Theorem 3.7. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian Lorentzian manifold . Then the structure vector field K cannot be a normalization for any flat null hypersurface M with parallel screen shape operator $\overset{\star}{A}_\xi$.*

Proof. Let $X \in \mathcal{S}(\zeta)$, we have

$$\begin{aligned}
\overline{g}(R(X, \xi)\xi, X) &= \overline{g}(\nabla_X \nabla_\xi \xi, X) - \overline{g}(\nabla_\xi \nabla_X \xi, X) - \overline{g}(\nabla_{[X,\xi]}\xi, X) \\
&= \tau(\xi)\overline{g}(\overset{\star}{A}_\xi (X), X) - \overline{g}(\nabla_\xi(-\tau(X)\xi - \overset{\star}{A}_\xi (X)), X) + \overline{g}(\overset{\star}{A}_\xi ([X, \xi]), X) \\
&\overset{(3.28)}{=} \overline{g}(\nabla_\xi \overset{\star}{A}_\xi (X), X) + \overline{g}(\overset{\star}{A}_\xi (\nabla_X \xi) - \overline{g}(\overset{\star}{A}_\xi (\nabla_\xi X), X) \\
&= \overline{g}((\nabla_\xi \overset{\star}{A}_\xi)(X), X) - \overline{g}(\overset{\star}{A}_\xi (X), \overset{\star}{A}_\xi (X)).
\end{aligned} \tag{3.55}$$

If M were flat and $\overset{\star}{A}_\xi$ parallel then (3.55) will imply that

$$g(\overset{\star}{A}_\xi (X), \overset{\star}{A}_\xi (X)) = 0$$

which means that M is totally geodesic as the screen distribution is positive definite. By using this we will get

$\forall X, Y \in \Gamma(TM)$,

$$\begin{aligned} 0 = R(X,Y)U &= \nabla_X \nabla_Y U - \nabla_Y \nabla_X U - \nabla_{[X,Y]} U \\ &\overset{(3.39)}{=} -\nabla_X Y - X\tau(Y)U + \tau(Y)X + \tau(Y)\tau(X)U \\ &+ \nabla_Y X + Y\tau(X)U - \tau(X)Y - \tau(X)\tau(Y)U \\ &+ [X,Y] + \tau(Y,X)U \\ &= \tau(Y)X - \tau(X)Y + \frac{1}{2} d\tau(Y,X)U \\ &\overset{3.4}{=} \tau(Y)X - \tau(X)Y. \end{aligned} \tag{3.56}$$

Setting $X = \xi$ in (3.56) and using (3.28), we will have $0 = \tau(Y)\xi \ \forall X \in \Gamma(TM)$, that is $\tau(X) = 0 \ \forall X \in \Gamma(TM)$, which contradicts item (iii) of Theorem 3.5 and completes the proof of the Theorem. □

Theorem 3.8. *Let M be a totally umbilical K-normalized null hypersurface of a para-Sasakian space form $(\overline{M}(c), \overline{\phi}, K, \overline{\eta}, \overline{g})$. Then the umbilical factor k satisfies the partial differential equation*

$$\xi(k) - k^2 - \frac{(c+1)}{8} = 0. \tag{3.57}$$

Moreover, if M is totally geodesic, then $c = -1$.

Setting $X = W = \xi$ in (2.18) together with the fact that $\overline{\eta}(\xi) = 1$, we have

$$\overline{g}(\overline{R}(\xi, Y)Z, \xi) = \frac{(c+1)}{4} \Big\{ -\overline{g}(Y,Z) + 3\mu(Z)\mu(Y) \Big\}. \tag{3.58}$$

From (2.11)-(2.5), it is worth noting that

$$\begin{aligned} (\nabla_X B)(Y,Z) &= XB(Y,Z) - B(\nabla_X Y, Z) - B(\nabla_X Z, Y) \\ &= X(k)g(Y,Z) + kXg(Y,Z) - kg(\nabla_X Y, Z) - kg(\nabla_X Z, Y) \\ &= X(k)g(Y,Z) + k(\nabla_X g)(Y,Z) \\ &= X(k)g(Y,Z) + k^2\{g(X,Y)\eta(Z) + g(X,Z)\eta(Y)\}. \end{aligned}$$

This leads to

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z)$$
$$= \{X(k) - k^2\eta(X) + k\tau(X)\}g(Y,Z) - \{Y(k) - k^2\eta(Y) + k\eta(Y)\}g(X,Z).$$

Replacing X by ξ in this equation and using (2.7) and (3.58), we have

$$\frac{(c+1)}{4}\Big\{ - g(PY,PZ) + 3\mu(PZ)\mu(PX)\Big\} = g(PY,PZ)\Big\{\xi(k) - k^2 + k\tau(\xi)\Big\}.$$

Choosing $PX = PZ = U \in \mathscr{S}(\zeta)$ together with the fact that $\mu(U) = \overline{g}(U,U) = 1$ and $\tau(\xi) = 0$, the previous equation becomes

$$\xi(k) - k^2 - \frac{(c+1)}{8} = 0.$$

Which gives item (3.57). Setting $k = 0$ in this equation, we have $c = -1$, which completes the proof.

Theorem 3.9. *There is no K-normalized null hypersurface of para-Sasakian space form $\overline{M}(c)(c \neq -1)$ such that the second fundamental form B is parallel.*

Proof. From (2.7) and (3.58), we have

$$\frac{(c+1)}{4}\Big\{ - \overline{g}(Y,Z) + 3\mu(Z)\mu(Y)\Big\} = (\nabla_\xi B)(Y,Z) - (\nabla_Y B)(\xi,Z) + \tau(\xi)B(Y,Z).$$

Being B parallel, choosing $PY = PZ = U \in \mathscr{S}(\zeta)$ together with the fact that $\tau(\xi) = 0$, $\mu(U) = 1 = g(U,U)$, the previous equation becomes $\frac{(-c-1)}{8} = 0$, that is $c = -1$, which is a contradiction. Hence, the claim holds. □

4. K-Normalized null hypersurfaces with certain symmetries

This section deals with locally symmetric and semi-symmetric K-normalized null hypersurfaces of para-Sasakian space forms.

We say that a null hypersurface M is locally symmetric [5], if the following holds

$$(\nabla_W R)(X,Y)Z = 0 \quad \forall X,Y,Z,W \in \Gamma(TM).$$

Using Lemma 3.2 in [5], $\forall X, Y, Z, W\ T \in \mathscr{S}(\zeta)$, we have,

$$\begin{aligned}\overline{g}((\overline{\nabla}_W\overline{R})(X,Y)Z,T) &= \overline{g}((\nabla_W R)(X,Y)Z,T) + (\nabla_W B)(X,Z)C(Y,T)\\ &+ B(X,Z)g((\nabla_W A_N)Y,T) - (\nabla_W B)(Y,Z)C(X,T)\\ &- B(Y,Z)g((\nabla_W A_N)X,T) - B(Y,Z)\tau(X)C(W,T)\\ &+ (\nabla_Y B)(X,Z)C(W,T) - (\nabla_X B)(Y,Z)C(W,T),\end{aligned} \tag{4.59}$$

and

$$\begin{aligned}\overline{g}((\overline{\nabla}_W\overline{R})(X,Y)Z,N) &= g((\nabla_W R)(X,Y)Z,N) + B(X,Z)g((\nabla_W(A_N Y),N)\\ &- B(Y,Z)g((\nabla_W(A_N X),N) - B(W,X)\overline{R}(N,Y,Z,N)\\ &- B(W,Y)\overline{R}(X,N,Z,N).\end{aligned} \tag{4.60}$$

Lemma 4.1. *Let $\overline{M}(c)$ be a para-Sasakian space form and $\overline{R}$ the Riemannian curvature tensor of Levi-Civita connection $\overline{\nabla}$. Then we have for any $X, Y, Z, W \in \Gamma(TM)$,*

$$(\overline{\nabla}_W\overline{R})(X,Y)Z = \frac{c+1}{4}\begin{pmatrix}\overline{g}(Y,Z)\overline{g}(X,\overline{\phi}W)\zeta - \overline{g}(X,Z)\overline{g}(Y,\overline{\phi}W)\zeta + \overline{g}(Y,Z)\overline{\eta}(X)\overline{\phi}W\\ -\overline{g}(X,Z)\overline{\eta}(Y)\overline{\phi}W + \overline{g}(Y,\overline{\phi}W)\overline{\eta}(Z)X - \overline{g}(X,\overline{\phi}W)\overline{\eta}(Z)Y\\ +\overline{g}(Z,\overline{\phi}W)\overline{\eta}(Y)X - \overline{g}(Z,\overline{\phi}W)\overline{\eta}(X)Y\end{pmatrix}. \tag{4.61}$$

Proof. Proof of the Lemma 4.1 here [9]. □

The following result is a transversal version of Theorem 4.2 of [9], where it was assumed that the structure vector field is tangent to the null hypersurface.

Theorem 4.1. *Let M be K-normalized null hypersurfaces of a para-Sasakian space form $\overline{M}(c)$. If M is locally symmetric, then $c = -1$. If $c = -1$, then M is locally symmetric if and only if it is totally geodesic.*

Proof. Let $\overline{M}(c)$ be a para-Sasakian space form and M a locally symmetric K-normalized null hypersurface of $\overline{M}(c)$. From (4.61), we have

$$\overline{g}((\overline{\nabla}_W\overline{R})(\xi,Y)\xi,N) = \frac{c+1}{4}\overline{g}(Y,\overline{\phi}W), \quad \forall W \in \Gamma(TM) \text{ and } Y \in \mathscr{S}(\zeta). \tag{4.62}$$

From (2.18), we have

$$\overline{g}(\overline{R}(\xi,N)\xi,N) \overset{(3.24)-(3.26)}{=} \frac{c-3}{4} - \frac{(c+1)}{4} = -1. \tag{4.63}$$

By taking $X = \xi$ and $Z = \xi$ in (4.60) and using (4.63)-(4.62), we obtain

$$B(W,Y) = \frac{c+1}{4}\overline{g}(Y, \overline{\phi}W), \tag{4.64}$$

for any $W \in \Gamma(TM)$ and $Y \in \mathscr{S}(\zeta)$.

Taking $Y = U$ and $W = \xi$ in this equation together with the fact that $\overline{g}(U,U) = 1$, we have $c = -1$. Hence, the first claim holds. Now, let (M,ζ) K-normalized in $\overline{M}(c)$ with $c = -1$. If M is locally symmetric, we get $B = 0$, due to (4.64). Conversely if (M,ζ) is totally geodesic, using (4.59), (4.60) and (3.48), we get

$$g((\nabla_W R)(X,Y)Z, PT) = 0 \quad \text{and} \quad g((\nabla_W R)(X,Y)Z, N) = 0.$$

Which completes the proof. □

Definition 4.1. [10] *We say that M is semi-symmetric if R satisfies $R(X,Y)R = 0$ $\forall X, Y \in \Gamma(TM)$, where $R(X,Y)$ operates on R as a derivation of the tensor algebra at each point.*

Theorem 4.2. *Let $(\overline{M}, \overline{\phi}, K, \overline{\eta}, \overline{g})$ be a para-Sasakian manifold and M be a totally geodesic K-normalized null hypersurface of $\overline{M}$. If M is semi-symmetric, then $\overline{M}$ is of constant negative curvature along the null hypersurface.*

Proof. $\forall X, Y, Z \in \Gamma(TM)$, we have

$$\begin{aligned}(\nabla_Z R)(X,Y)U - R(X,Y)Z = \nabla_Z(R(X,Y)U) - R(\nabla_Z X, Y)U - R(X, \nabla_Z Y)U \\ - R(X,Y)\nabla_Z U - R(X,Y)Z.\end{aligned} \tag{4.65}$$

But, from (3.56), we have

$$\begin{aligned}\nabla_Z R(X,Y)U &= \nabla_Z(\tau(Y)X) - \nabla_Z(\tau(X)Y) \qquad (4.66)\\ &= (Z\cdot\tau(Y))X + \tau(Y)\nabla_Z X - (Z\cdot\tau(X))Y - \tau(X)\nabla_Z Y\\ &\overset{(3.39)}{=} -\mu(\nabla_Z Y)X + g(Z,Y)X - \tau(Z)\tau(Y)X + \tau(Y)\nabla_Z X\\ &\quad + \mu(\nabla_Z X)Y - g(Z,X)Y + \tau(Z)\tau(X)Y - \tau(X)\nabla_Z Y\\ -R(\nabla_Z X, Y)U - R(X,Y)\nabla_Z U &\overset{(3.56)}{=} -\tau(Y)\nabla_Z X + \tau(\nabla_Z X)Y - \tau(\nabla_Z Y)X + \tau(X)\nabla_Z Y\\ -R(X,Y)\nabla_Z U - R(X,Y)Z &\overset{(3.39)}{=} +R(X,Y)Z + \tau(Z)\tau(Y)X - \tau(Z)\tau(X)Y - R(X,Y)Z.\end{aligned}$$

Substituting the above equations in (4.65), we have

$$(\nabla_Z R)(X,Y)U - R(X,Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.67}$$

Setting $Z = U$ in (4.67) and using (3.56), we get $\forall X, Y \in \Gamma(TM)$

$$(\nabla_U R)(X,Y)U = R(X,Y)U + g(Y,U)X - g(X,U)Y \overset{(3.23)-(3.39)}{=} 0. \tag{4.68}$$

Next, $\forall X, Y, Z, W \in \Gamma(TM)$

$$\begin{aligned}
0 &\overset{(3.56)}{=} (R(W,Z)R)(X,Y)U = R(W,Z)R(X,Y)U - R(X,Y)R(W,Z)U \\
&- R(R(W,Z)X,Y)U - R(X,R(W,Z)Y)U \\
&= \tau(Y)R(W,Z)X - \tau(X)R(W,Z)Y - \tau(Z)R(X,Y)W + \tau(W)R(X,Y)Z \\
&- \tau(Y)R(W,Z)X + \tau(R(W,Z)X)Y - \tau(R(W,Z)Y)X + \tau(X)R(W,Z)Y \\
&= -\tau(Z)R(X,Y)W + \tau(W)R(X,Y)Z + \tau(R(W,Z)X)Y - \tau(R(W,Z)Y)X \\
&\overset{(2.8)}{=} -\tau(Z)R(X,Y)W + \tau(W)R(X,Y)Z + \overline{g}(\overline{R}(W,Z)U,X)Y - \overline{g}(\overline{R}(W,Z)U,Y)X \\
&\overset{(3.43)}{=} \tau(W)\{g(Z,Y)X - g(Z,X)Y + R(X,Y)Z\} \\
&-\tau(Z)\{g(W,Y)X - g(W,X)Y + R(X,Y)W\} \\
&\overset{(4.65)}{=} \tau(W)(\nabla_W R)(X,Y)U - \tau(Z)(\nabla_Z R)(X,Y)U.
\end{aligned} \tag{4.69}$$

Setting $W = U$ in (4.69) and using (4.68), we have $(\nabla_Z R)(X,Y)U = 0$. From this and (4.67), we have $R(X,Y)Z = -g(Y,Z)X + g(X,Z)Y$, for all $X, Y, Z \in \Gamma(TM)$. From this together with (2.7) and (2.8), we have $\overline{g}(\overline{R}(X,Y)Z,W) = -\{\overline{g}(Y,Z)\overline{g}(X,W) - \overline{g}(X,Z)\overline{g}(Y,W)\}$ for all $X, Y, Z, W \in \Gamma(TM)$. Which completes the proof. □

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INVESTIGATION OF ∗-YAMABE CONFORMAL SOLITON ON LP-KENMOTSU MANIFOLDS

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Abstract. The primary objective of this paper is to examine the ∗-Yamabe conformal soliton, which exhibits a potential vector field that is torse-forming on a Lorentzian Para(LP)-Kenmotsu manifold. Next, we examine the characteristics of the scalar curvature for ∗-Yamabe conformal soliton on the LP-Kenmotsu manifold. The development of the description of the vector field in the context of the ∗-Yamabe conformal soliton has been undertaken. Furthermore, we have improved multiple applications of vector fields, specifically the formation of torse on a LP-Kenmotsu manifold, by utilizing a ∗-Yamabe conformal soliton. At last, we also provide an example for ∗-Yamabe conformal soliton on three-dimensional LP-Kenmotsu manifold.

Keywords: ∗-Yamabe conformal soliton, LP-Kenmotsu manifold, conformal killing vector field, torse forming vector field.

2010 Mathematics Subject Classification: 53B30, 53C15, 53C25.

1. Introduction

In subsequent years, para-kenmotsu manifolds gaining significant values for scholarly interest to, prompting numerous authors to elucidate the manifold's obtaining intresting properties. The introduction of LP-Kenmotsu manifolds, also referred to as Lorentzian almost paracontact metric manifolds in [12].

The idea of Ricci flow, which is an evolution equation for metrics on a Riemannian manifold

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was presented in 1982 [10]. The equation governing the Ricci flow is as follows:

$$\frac{\partial g}{\partial t} = -2S, \tag{1.1}$$

on a compact Riemannian manifold M equipped with a Riemannian metric g, where S denote the Ricci tensor on M.

A generalization of an Einstein metric is referred to as a Ricci soliton. A Ricci soliton on the manifold M is defined as a triple (g, Ω, α), where the symbol g represents a Riemannian or semi-Riemannian metric, Ω denotes a vector field known as the potential vector, and real scalar ϑ is such that

$$\pounds_\Omega g + 2S + 2\alpha g = 0, \tag{1.2}$$

where $\pounds_\Omega$ represents the Lie derivative operator acting on the vector field Ω, S denotes the Ricci tensor, g represents the Riemannian metric, Ω is a vector field, and α is a scalar. The Ricci soliton exhibits three distinct behaviors, namely shrinking, steady, and expanding, which correspond to the values of α being negative, zero, and positive, respectively. Ricci solitons have been the subject of investigation by multiple researchers, as evidenced by the works of various authors [1, 8, 16, 17, 18, 19, 24].

The Yamabe flow was first proposed by Hamilton [11] as a method for producing Yamabe metrics on compact Riemannian manifolds. The Yamabe flow is a process by which a time-dependent metric $g(\cdot, t)$ on a Riemannian or pseudo-Riemannian manifold M evolves according to a specific equation,

$$\frac{\partial}{\partial t} g(t) = -r g(t), \quad g(0) = g_0, \tag{1.3}$$

where r is the scalar curvature of the manifold M.

In the context of two-dimensional spaces, it has been established that the Yamabe flow is mathematically equivalent to the Ricci flow, as stated in [10]. The Ricci flow, characterized by the equation $\frac{\partial}{\partial t} g(t) = -2S(g(t))$, involves the Ricci tensor denoted by S. Nevertheless, it is important to acknowledge that in dimension beyond two, the Ricci and Yamabe flow have distinct characteristics. This is due to the fact that, whereas the Yamabe flow preserves the conformal class of the metric, the Ricci flow does not necessarily possess this property.

On a Riemannian or pseudo-Riemannian manifold (M, g), the definition of a Yamabe soliton may be found. This soliton corresponds to a self-similar solution of the Yamabe flow [2]

$$\frac{1}{2} \pounds_\Omega g = (r - \alpha) g, \tag{1.4}$$

where the symbol $\pounds_{\Omega}g$ represents the Lie derivative of the metric g with respect to the vector field Ω. In this context, r refers to the scalar curvature, and α is a constant. Furthermore, it is stated that a Yamabe soliton exhibits expanding, steady behavior, or shrinking depending on the sign of α, which can be positive, zero, or negative, respectively. If α is a smooth function, the equation (1.4) is referred to as an almost Yamabe soliton according to the [2]. Numerous researchers have examined the solitons on contact manifolds following the emergence of Ricci soliton as well as the Yamabe soliton [6, 7, 9, 21].

The concept of conformal Ricci soliton [20, 21] was introduced by Basu and Bhattacharyya [3] in 2015 as

$$\pounds_{\Omega}g + 2S = \left[2\alpha - \left(q + \frac{2}{n}\right)\right]g. \tag{1.5}$$

In the above context, the symbol S is employed to designate the Ricci tensor, q represents a scalar non-dynamical field that exhibits temporal variation, α symbolizes a constant, and n signifies the dimension of the manifold.

The earliest proposals for the $*$-Ricci tensor on almost Hermitian manifolds and the $*$-Ricci tensor on real hypersurfaces in non-flat complex space were put forward by Tachibana [23] and Hamada [13], respectively. In their works, the $*$-Ricci tensor is precisely defined as follows:

$$S^{*}(X,Y) = \frac{1}{2}(Tr\{\varphi \, o \, R(X,\varphi Y)\}), \tag{1.6}$$

for any vector fields X and Y defined on a manifold M^n, let φ be a $(1,1)$-tensor field and the trace operator is denoted by Tr.

If the equation $S^{*}(X,Y) = \vartheta g(X,Y) + \varrho\eta(X)\eta(Y)$ holds for all vector fields X and Y, where ϑ and ϱ are smooth functions, then the manifold is referred to as a $*$-η-Einstein manifold. Moreover, if $\varrho = 0$, that is, if $S^{*}(X,Y) = \vartheta g(X,Y)$ for every vector fields X and Y, and thus the manifold can be characterized as $*$-Einstein.

In 2014, Kaimakamis and Panagiotidou [14] proposed the notion of a $*$-Ricci soliton, which can be precisely characterized as

$$\pounds_{\Omega}g + 2R^{*} + 2\alpha g = 0, \tag{1.7}$$

where X and Y are vector fields defined on the manifold M^n, and for any constant α.

By utilizing equations (1.4), (1.5), and (1.7) as references, we proceed to establish the concept of a $*$-Yamabe conformal soliton as follows:

Definition 1.1. *A manifold (M, g) of dimension n that is Riemannian or pseudo-Riemannian is considered to admit a $*$-Yamabe conformal soliton if*

$$(\pounds_\Omega g)(X,Y) + \left[2\alpha - 2r^* - \left(q + \frac{2}{n}\right)\right] g(X,Y) = 0, \tag{1.8}$$

for any vector fields X and Y, the Lie derivative of the metric g along the vector field Ω is denoted by $\pounds_\Omega g$. Here, $r^* = Tr(S^*)$ represents the $*$-scalar curvature, and α is a constant. The classification of $*$-Yamabe conformal solitons to be as either shrinking, steady, or shrinking depending on the sign of α, which can be positive, zero, or negative, respectively. If the vector field Ω can be expressed as the gradient of a smooth function h (i.e., $\Omega = grad(h)$) on the manifold M, then the equation (1.8) is referred to as a $*$-Yamabe conformal gradient soliton.

In contrast, in the context of a Riemannian or pseudo-Riemannian manifold (M, g), a vector field υ is said to be torse-forming if it does not have any points where it vanishes [27]. Then,

$$\nabla_X \upsilon = \chi X + \varsigma(X)\upsilon, \tag{1.9}$$

where ∇ denotes the Levi-Civita connection associated with the metric g, χ is a smooth function while ς is a 1-form. Furthermore, the vector field υ is referred to as concircular (see [4, 26]) if the 1-form ς disappears in the same way, in the equation (1.9). The vector field denoted by the symbol υ is known as the concurrent [22, 28] if, in equation (1.9), the 1-form ς vanishes identically and the function χ is equal to 1. The vector field υ is referred to as recurrent if the function χ in equation (1.9) is equal to zero. Finally, when $\chi = \varsigma = 0$ in (1.9), the vector field υ is often known as a parallel vector field.

In 2017, Chen [5] introduced an innovative vector field known as the torque-vector field. A vector field υ is referred to as a torqued vector field if it satisfies equation (1.9) with $\varsigma(\upsilon) = 0$. In the present scenario, the function χ is commonly referred to as the torqued function, while the 1-form ς is regarded as the torqued form of υ.

The framework of the article is as follows:

In the second section, following a concise introduction, we have presented several essential findings that will be utilized in subsequent sections. In Section 3, we have constructed a $*$-Yamabe conformal soliton that admits a LP-Kenmotsu manifold. The properties of the soliton, specifically the Laplacian of the smooth function, have also been established. The manifold has been characterized in cases where the vector field exhibits conformal killing properties. In the subsequent section, we have presented a demonstration of various properties pertaining to vector fields on $*$-Yamabe conformal soliton. Section 4 presents a series of results that establish the existence of torse producing vector fields and $*$-Yamabe conformal solitons on manifolds with a given metric. Section 5 of this study focuses on the practical implementation of the Laplace equation within the fields of gravity and physics. Section 6 presents the construction of an exemplary scenario to provide evidence for the presence of a $*$-Yamabe conformal soliton on a three-dimensional LP-Kenmotsu manifold. Let M be an n-dimensional Lorentzian metric manifold.

2. Preliminaries

In this section, we recall some fundamental notations and formulas of almost para contact metric manifolds.

Let M be an $(2n+1)$-dimensional Lorentzian metric manifold. This means that it is endowed with a structure (M, ϕ, ξ, η, g), where ϕ is a (1,1)-type tensor field, ξ is a Reeb vector field, η is a 1-form on M and g is a Lorentzian metric tensor satisfying [25]

$$\phi^2(X) = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \eta\, o\, \phi = 0, \quad \phi\xi = 0, \tag{2.10}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.11}$$

$$g(X, \phi Y) = g(\phi X, Y), \tag{2.12}$$

$$g(X, \xi) = \eta(X), \tag{2.13}$$

for all vector fields X, Y . Then (M, ϕ, ξ, η, g) is said to be Lorentzian almost paracontact metric manifold.

Theorem 2.1. [12, 15] *A Lorentzian almost paracontact metric manifold (M, ϕ, ξ, η, g) is called Lorentzian para-Kenmotsu manifold if and only if*

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X, \tag{2.14}$$

for all vector fields $X, Y \in \Gamma(TM)$, where ∇ and $\Gamma(TM)$ denote the Levi-Civita connection and differentiable vector fields set on M respectively.

Corollary 2.1. *Let (M, ϕ, ξ, η, g) be a Lorentzian para-Kenmotsu manifold. Then, we have*

$$\nabla_X \xi = -X - \eta(X)\xi. \tag{2.15}$$

In a LP-Kenmotsu manifold, the following relations are hold,

$$\eta(R(X,Y)Z) = g(R(X,Y)Z,\xi) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.16}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.17}$$

$$R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.18}$$

where R denotes the Riemannian Curvature tensor.

$$S(X,\xi) = 2n\eta(X), \tag{2.19}$$

$$S(\phi X,\phi Y) = S(X,Y) + 2n\eta(X)\eta(Y), \tag{2.20}$$

$$(\nabla_X \eta)Y = -g(X,Y) - \eta(X)\eta(Y), \tag{2.21}$$

for any vector fields X, Y, $Z \in \Gamma(M)$.

It is now understood,

$$\pounds_\xi g(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi), \tag{2.22}$$

for any vector fields X, Y, $Z \in \Gamma(M)$.

Subsequently, by employing equations (2.15) and (2.22), we obtain

$$\pounds_\xi g(X,Y) = -2[g(X,Y) + \eta(X)\eta(Y)]. \tag{2.23}$$

Proposition 2.1. *The ∗-Ricci tensor on a (2n+1)-dimensional LP-Kenmotsu manifold is expressed as*

$$S^*(X,Y) = S(X,Y) + 2ng(X,Y). \tag{2.24}$$

Furthermore, by selecting $X = \sigma^i$ and $Y = \sigma^i$ in the aforementioned equation, where σ^i governs the elements of a local orthonormal frame and doing a summation across the range of i from 1 to $2n+1$, we may get the following result

$$r^* = r + 4n^2 + 2n, \tag{2.25}$$

where r^* represents the ∗-scalar curvature of the manifold M.

3. Main Results

Proposition 3.1. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, ξ, α), where the reeb vector field ξ, subsequently the soliton can be put into distinct categories as either expanding, steady, or shrinking, the outcome is contingent upon the inequality* $r + 4n^2 + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) \lesseqqgtr 0$.

Proof. Let us consider a manifold M of the dimension $(2n+1)$ that is equipped with a LP- Kenmotsu structure. When we substitute $\Omega = \xi$ into the equation of the $*$ - Yamabe conformal soliton (2.15) on the manifold M, we obtain

$$(\pounds_\xi g)(X,Y) + \left[2\alpha - 2r^* - \left(q + \frac{2}{2n+1}\right)\right] g(X,Y) = 0, \tag{3.26}$$

for every vector fields X and Y belonging to the set of vector fields on M.
From (2.23) and (2.25), the above equation can be rewritten

$$\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right) - 1\right] g(X,Y) - \eta(X)\eta(Y) = 0. \tag{3.27}$$

Substituting $Y = \xi$ into the above equation and referring to equation (2.10), we obtain

$$\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right] \eta(X) = 0. \tag{3.28}$$

Since $\eta(X) \neq 0$, the equation presented above can be expressed as

$$\alpha = r + 4n^2 + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right). \tag{3.29}$$

The proof is concluded. □

Corollary 3.1. *If the metric g of a flat para-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton, then, the soliton exhibits expanding, steady, or shrinking behavior depending on the expression value* $4n^2 + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) \lesseqqgtr 0$, *for the reeb vector field ξ.*

Proof. Considering the fact that the manifold has a flat, denoted by $r = 0$, we can deduce from equation (3.29) that α is equal to $4n^2 + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right)$.
Therefore, the evidence presented substantiates the claim. □

Proposition 3.2. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, Ω, α), where Ω is the gradient of a smooth function h, then the Laplacian equation satisfied by h is*

$$\triangle(h) = -(2n+1)\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right].$$

Proof. In this investigation, we will take into account a $*$-Yamabe conformal soliton (g, Ω, α) that is defined on a $(2n+1)$-dimensional LP-Kenmotsu manifold M as

$$(\pounds_{\Omega} g)(X,Y) + \left[2\alpha - 2r^{*} - \left(q + \frac{2}{2n+1}\right)\right] g(X,Y) = 0 \tag{3.30}$$

for each vector fields X, $Y \in \Gamma(M)$.

By substituting $X = Y = \sigma^{i}$ into the aforementioned equation, where σ^{i} denotes a local orthonormal frame, and a summation from $i = 1$ to $2n+1$, with the aid of equation (2.25), we obtain

$$div\Omega + (2n+1)\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right] = 0. \tag{3.31}$$

Given that the vector field Ω can be expressed as the gradient of a smooth function h on the manifold M, we can rewrite equation (3.31) as follows:

$$\triangle(h) = -(2n+1)\left[\alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right]. \tag{3.32}$$

The symbol $\triangle(h)$ denotes the Laplacian equation, which is satisfied by the function h.
The proof is concluded. □

Based on the aforementioned theorem, it is possible to assert

Remark 3.1. *Consider a $(2n+1)$-dimensional LP-Kenmotsu manifold with metric g. Let this metric satisfy the $*$-Yamabe conformal soliton (g, Ω, α). The vector field Ω exhibits solenoidal behavior if and only if the scalar curvature becomes $\alpha - 4n^2 - 2n - \frac{1}{2}(q + \frac{2}{2n+1})$.*

Proof. Given the vector field Ω is solenoidal, meaning that its divergence is zero, i.e., $div\Omega = 0$, equation (3.31) yields

$$r = \alpha - 4n^2 - 2n - \frac{1}{2}(q + \frac{2}{2n+1}). \tag{3.33}$$

On the other hand, let us consider the scalar curvature of the manifold, denoted as r. It can be expressed as $r = \alpha - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)$.
Subsequently, by referencing equation (3.31), it can be deduced that the divergence of vector field Ω is zero, thereby indicating that Ω possesses solenoidal characteristics. Therefore, the evidence provided confirms the claim. □

Definition 3.1. *If the following relation holds,*

$$(\pounds_{\Omega} g)(X,Y) = 2\Phi g(X,Y), \tag{3.34}$$

then the vector field ξ is called as a conformal Killing vector field. Where Φ represents a function of the coordinates, specifically a conformal scalar.
Furthermore, in the case where Φ is not always the same, the conformal killing vector field Ω is often denoted as proper. Furthermore, in the case where Φ remains constant, the vector field Ω is referred to as a homothetic vector field. Conversely, when the constant Φ takes on a non-zero value, Ω is characterized as a proper homothetic vector field. If Φ is equal to zero in the aforementioned equation, the symbol Ω is commonly used to denote a vector field known as a killing vector field.

Based on the aforementioned definition, it is possible to assert

Proposition 3.3. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension conforms to the $*$-Yamabe conformal soliton (g, Ω, α). Subsequently, Ω is*
(i) proper vector field if the expression $r + 4n^2 - \alpha + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ is not constant,
(ii) homothetic vector field if the expression $r + 4n^2 - \alpha + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ is constant,
(iii) proper homothetic vector field if the expression $r + 4n^2 - \alpha + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ has a non-zero value,
(iv) killing vector field if the expression $\alpha = r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$
where Ω represents a conformal Killing vector field.

Proof. Consider a $(2n+1)$-dimensional LP-Kenmotsu manifold M equipped with a $*$-Yamabe conformal soliton (g, Ω, α), where Ω represents a conformal killing vector field. When utilizing equations (1.8), (2.25), and (3.34) and substituting $Y = \xi$, the following result is obtained

$$\left[\Phi + \alpha - r - 4n^2 - 2n - \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right]\eta(X) = 0. \tag{3.35}$$

Given that $\eta(X) \neq 0$, we can deduce that

$$\Phi = r + 4n^2 - \alpha + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right). \tag{3.36}$$

The proof is concluded. □

Proposition 3.4. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension conforms to the $*$-Yamabe conformal soliton (g, Ω, α), then we have the vector Ω and its metric dual 1-form $\emptyset$ fulfills the given relation,*

$$(divH)\Omega + S(Y, \Omega) = 0,$$

and

$$\nabla_X|\Omega|^2+2g(FX,\Omega)-(\pounds_\Omega g)(X,\Omega)=0,$$

where S stands for the Ricci tensor.

Proof. By utilizing the Lie derivative property, we are able to express

$$(\pounds_\Omega g)(X,Y)=g(\nabla_X\Omega,Y)+g(\nabla_Y\Omega,X), \tag{3.37}$$

for arbitrary vector fields X and Y.

Subsequently, by utilizing (2.25) and (3.37), (1.8) can be expressed as

$$g(\nabla_X\Omega,Y)+g(\nabla_Y\Omega,X)+\left[2\alpha-2(r+4n^2+2n)-\left(q+\frac{2}{2n+1}\right)\right]g(X,Y)=0. \tag{3.38}$$

The symbol ø represents a 1-form, which possesses a metric equivalence to the vector field Ω. Consequently, it can be expressed as $ø(X)=g(X,\Omega)$, where X denotes any vector field. The mathematical representation of the exterior derivative $dø$ may be stated in the following manner

$$2(dø)(X,Y)=g(\nabla_X\Omega,Y)-g(\nabla_Y\Omega,X). \tag{3.39}$$

Given that $dø$ is skew-symmetric, we can proceed to define a tensor field H of type $(1,1)$ by

$$(dø)(X,Y)=g(X,HY), \tag{3.40}$$

then H is skew self-adjoint, it satisfies the property of $g(X,HY)=-g(HX,Y)$.

Equation (3.40) can be expressed as

$$(dø)(X,Y)=-g(HX,Y). \tag{3.41}$$

By utilizing (3.41), (3.39) can be transformed

$$g(\nabla_X\Omega,Y)-g(\nabla_Y\Omega,X)=-2g(HX,Y). \tag{3.42}$$

By combining equations (3.42) and (3.38) in simultaneously performing a common factorization of the variable Y, we obtain

$$\nabla_X\Omega=-HX-\left[\alpha-(r+4n^2+2n)-\frac{1}{2}\left(q+\frac{2}{2n+1}\right)\right]X. \tag{3.43}$$

By substituting the aforementioned equation into the expression $R(X,Y)\Omega=\nabla_X\nabla_Y\Omega-\nabla_Y\nabla_X\Omega-\nabla_{[X,Y]}\Omega$, we obtain

$$R(X,Y)\Omega=(\nabla_Y H)X-(\nabla_X H)Y. \tag{3.44}$$

Upon observing that $d\o$ is closed, we can derive

$$g(X, (\nabla_Z H)Y) + g(Y, (\nabla_X H)Z) + g(Z, (\nabla_Y H)X) = 0. \tag{3.45}$$

By taking the inner product of equation (3.44) with respect to Z, we obtain

$$g(R(X, Y)\Omega, Z) = g((\nabla_Y H)X, Z) - g((\nabla_X H)Y, Z). \tag{3.46}$$

Given that H is skew-adjoint, it follows that $\nabla_X H$ is also skew-adjoint. Subsequently, by employing (3.45), (3.46) can be expressed as

$$g(R(X, Y)\Omega, Z) = g(X, (\nabla_Z H)Y). \tag{3.47}$$

By substituting $X = Z = \sigma^i$ into the given equation, where σ^i governs the elements of a local orthonormal frame and doing a summation across the range of i from 1 to $2n + 1$, we may get the following result

$$div(H)Y - S(Y, \Omega) = 0, \tag{3.48}$$

where $divH$ refers to the a divergence of the tensor field denoted by H.

In this step, we will calculate the covariant derivative of the squared g-norm of Ω utilizing (3.43) in the following manner:

$$\nabla_X |\Omega|^2 = 2g(\nabla_X \Omega, \Omega)$$

$$= -2g(HX, \Omega) - \left[2\alpha - 2(r + 4n^2 + 2n) - \left(q + \frac{2}{2n+1}\right)\right] g(X, \Omega). \tag{3.49}$$

Once again, employing equations (1.8) and (2.25), we obtain

$$(\pounds_\Omega g)(X, Y) = -\left[2\alpha - 2r - 8n^2 - 4n - \left(q + \frac{2}{2n+1}\right)\right] g(X, Y). \tag{3.50}$$

Subsequently, by employing (3.50), (3.49) can be transformed

$$\nabla_X |\Omega|^2 + 2g(HX, \Omega) - (\pounds_\Omega g)(X, \Omega) = 0. \tag{3.51}$$

And thus, the evidence is presented. □

4. Some Results on the LP-Kenmotsu Manifold for the $*$-Yamabe conformal soliton with Torse-forming vector field

By utilizing the equation (1.9) that describes the torse forming vector field, it is possible to assert

Proposition 4.1. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, Ω, α), where a torse forming vector field υ, we can express $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)}$ and the soliton exhibits expanding, steady, shrinking according as $r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)} \lesseqgtr 0$.*

Proof. Given that (g, υ, α) be a $*$- Yamabe conformal soliton defined on a $(2n+1)$-dimensional LP-Kenmotsu manifold M, where a torse-forming vector field υ, we can derive the following equations from (1.8) and (2.25):

$$(\pounds_\upsilon g)(X,Y) + \left[2\alpha - 2r - 8n^2 - 4n - \left(p + \frac{2}{2n+1}\right)\right] g(X,Y) = 0. \tag{4.52}$$

The notation $(\pounds_\upsilon g)(X,Y)$ represents the Lie derivative of the metric g with respect to the vector field υ.

Now, utilizing (2.15), we obtain

$$\begin{aligned}(\pounds_\upsilon g)(X,Y) &= g(\nabla_X \upsilon, Y) + g(X, \nabla_Y \upsilon) \\ &= 2\chi g(X,Y) + \varsigma(X) g(\upsilon, Y) + \varsigma(Y) g(\upsilon, X),\end{aligned} \tag{4.53}$$

for every X, $Y \in M$.

Subsequently, by referring to equations (4.52) and (4.53), we obtain.

$$\left[r + 4n^2 - \alpha - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right)\right] g(X,Y) = \frac{1}{2}\Big[\varsigma(X) g(\upsilon, Y) + \varsigma(Y) g(X, \upsilon)\Big]. \tag{4.54}$$

By contracting the above equation throughout X and Y, we obtain

$$\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\left(q + \frac{2}{2n+1}\right) - \frac{\varsigma(\upsilon)}{(2n+1)}. \tag{4.55}$$

The proof is now concluded. □

Based on the aforementioned theorem, it is possible to assert that,

Corollary 4.1. *If the metric g of a LP-Kenmotsu manifold with $(2n+1)$ dimension satisfies the $*$-Yamabe conformal soliton (g, Ω, α), where a torse forming vector field υ, Consequently, if υ is*

(i) concircular, the equation becomes $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. The soliton is

either expanding, steady, shrinking depending on the value of the expression $r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqgtr 0.$

(ii) concurrent, the equation becomes $\alpha = r + 4n^2 - 1 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 - 1 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqgtr 0.$

(iii) recurrent, the equation may be written as $\alpha = r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)}$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)} \lesseqgtr 0.$

(iv) parallel, the equation becomes $\alpha = r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqgtr 0.$

(v) torqued, the equation becomes $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$. *The soliton is either expanding, steady, shrinking depending on the value of the expression* $r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) \lesseqgtr 0.$

Proof. In (4.55), if the 1-form ς disappears the same way, meaning that υ becomes concircular, then α can be expressed as $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$.

In the given equation (4.55), if the 1-form ς disappears the same way and the function χ is equal to 1, then υ becomes concurrent. Consequently, the expression for α can be simplified to $r + 4n^2 - 1 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$.

In equation (4.55), when the function χ is equal to zero, meaning that υ becomes recurrent, the expression for α is given by $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big) - \frac{\varsigma(\upsilon)}{(2n+1)}$.

When $\chi = \varsigma = 0$ in equation (4.55), which means that υ becomes parallel, the expression for α simplifies to $r + 4n^2 + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$.

Lastly, in equation (4.55), when $\varsigma(\upsilon) = 0$, In this context, the symbol υ can be interpreted as a torqued vector field. Consequently, the expression $\alpha = r + 4n^2 - \chi + 2n + \frac{1}{2}\big(q + \frac{2}{2n+1}\big)$ can be derived.

Thus, the evidence is provided. □

5. Applications of Laplace Equation in Physics and Gravity

The Laplace equation is a second-order partial differential equation that finds extensive application in physics due to its solution, known as harmonic functions. These functions arise in various contexts, including the determination of electrical, magnetic, and gravitational potentials, steady-state temperatures, and hydrodynamics problems.

- Both the real and imaginary components of a complex analytic function satisfy the Laplace equation. That is if $z = x + iy$ and $f(x, y) = u(x, y) + iv(x, y)$, the essential requirement for the function $f(z)$ to possess analyticity is that the real part u and the imaginary part v satisfy the C-R equation, $u_x = v_y$, $u_y = -v_x$, the symbols u_x and u_y represent the first partial derivatives of the function u with respect to the variables x and y, respectively and v_x and v_y represent the first partial derivatives of the function v with respect to the variables x and y, respectively. Consequently, $u_{yy} = (-v_x)_y = -(v_y)_x = -(u_{xx})$. Hence, the function u fulfills the Laplace equation.
- In the given situation, whereby a specific area demonstrates a charge density of zero, while allowing for non-zero charge densities at its limits, the electric potential V inside that zone conforms to the Laplace equation. By solving the Laplace equation, we can determine the electric potential, a crucial quantity that allows us to easily calculate the electric field using the equation $E = \nabla V$. Consequently, we can determine the force experienced by a charge using the equation $F = qE$. In the field of physics, numerous intriguing scenarios arise wherein our focus lies on the potential within regions characterized by a zero charge density. Conventional examples include both the inside and outside regions of a charged hollow sphere, as well as the exterior portion of charged metal plates. The fact that each of the situations has a unique combination of boundary conditions is one of the things that makes the Laplace equation so intriguing.

In a broad context, the gravitational and electric potentials, represented by the symbol V, adhere to Poisson's equation, which is expressed as $\nabla^2 V = L(x, y, z)$, where $L(x, y, z)$ represents the given charged density. The equation of Laplace and the equation of Poisson are two of the simplest examples of a type of partial differential equations known as elliptical PDEs.Laplace was the pioneer in introducing a multitude of intriguing mathematical techniques that have been employed to solve electrical partial differential equations (PDEs).

- In the field of electrostatics, as per Maxwell's equation, a two-dimensional electric fluid (u_1, u_2), which is not dependent on time, fulfills the following conditions:

$$\nabla \times (u_1, u_2, 0) = ((u_2)_x - (u_1)_y)\hat{k_1}$$

where $\hat{k_1}$ is the standard unit vector and

$$\nabla \cdot (u_1, u_2) = Q.$$

The symbol Q is used to denote the charge density.

The equation of Laplace is applicable to three-dimensional scenarios in the fields of electrostatics and fluid dynamics, similar to its utilization in two-dimensional contexts.

- Furthermore, it is worth noting that this phenomenon also finds applications in the field of gravity. Let $\tilde{g}$, $\tilde{\rho}$, and G denote the gravitational field, mass density, and gravitational constant, respectively. The differential expression representing Gauss's law for gravity may be stated as follows:

$$\nabla \cdot \tilde{g} = -4\pi G\tilde{\rho}.$$

Furthermore, the equation $\nabla^2 V = 4\pi G\tilde{\rho}$ represents Poisson's gravitational field equations. The physical significance of this scenario can be understood by considering Proposition 3.2 and the equation (3.32). This equation represents a Laplace equation with a potential vector field of gradient type.

The Laplace equation for gravitational fields, $\nabla^2 V = 0$, is true in empty space with $\tilde{\rho} = 0$.

6. Example

We examine the 3-dimensional manifold $M = \{(l_1, l_2, l_3) \in \mathbb{R}^3, (l_1, l_2, l_3) \neq (0,0,0)\}$, in the context of $\mathbb{R}^3$, the coordinates (l_1, l_2, l_3) are referred to as standard coordinates. The vector fields referred to in the context are

$$\sigma^1 = l_3\frac{\partial}{\partial l_1}, \qquad \sigma^2 = l_3\frac{\partial}{\partial l_2}, \qquad \sigma^3 = l_3\frac{\partial}{\partial l_3}$$

are linearly independent at every point of M.

Let g denote the Riemannian metric that is defined by

$$g(\sigma^1,\sigma^2) = g(\sigma^2,\sigma^3) = g(\sigma^3,\sigma^1) = 0,$$
$$g(\sigma^1,\sigma^1) = g(\sigma^2,\sigma^2) = 1, \;\; g(\sigma^3,\sigma^3) = -1.$$

Let 1-form η that is defined by $\eta(Z) = g(Z,\sigma^3)$, for any $Z \in \Gamma(M)$, Let $\Gamma(M)$ denote the collection of all differentiable vector fields on the manifold M and ϕ denote the $(1,1)$-tensor field that is defined by

$$\phi\sigma^1 = -\sigma^1, \;\; \phi\sigma^2 = -\sigma^2, \;\; \phi\sigma^3 = 0.$$

By utilizing the property of linearity for both ϕ and g, we can deduce that

$$\eta(\sigma^3) = -1, \;\; \phi^2 X = X + \eta(X)\sigma^3, \;\; g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y),$$

for every $X, Y \in \chi(M)$. Therefore, when $\sigma^3 = \xi$, the tuple (ϕ, ξ, η, g) establishes a Lorentzian almost paracontact metric structure on the manifold M.
Consider the Levi-Civita connection denoted by ∇, which is associated with the Riemannian metric g. Subsequently, we possess

$$[\sigma^1, \sigma^2] = 0, \quad [\sigma^1, \sigma^3] = -\sigma^1, \quad [\sigma^2, \sigma^3] = \sigma^2.$$

The connection ∇ associated with the metric g is defined in the following manner

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]),$$

which is referred to as Koszul's formula.
Using Koszul's formula, it is straightforward to compute

$$\nabla_{\sigma^1}\sigma^1 = \sigma^3, \quad \nabla_{\sigma^1}\sigma^2 = 0, \quad \nabla_{\sigma^1}\sigma^3 = -\sigma^1,$$
$$\nabla_{\sigma^2}\sigma^1 = 0, \quad \nabla_{\sigma^2}\sigma^2 = \sigma^3, \quad \nabla_{\sigma^2}\sigma^3 = -\sigma^2,$$
$$\nabla_{\sigma^3}\sigma^1 = 0, \quad \nabla_{\sigma^3}\sigma^2 = 0, \quad \nabla_{\sigma^3}\sigma^3 = 0.$$

Based on the aforementioned, it can be deduced that the manifold fulfills the equation $\nabla_X \xi = -X - \eta(X)\xi$, where $\xi = \sigma^3$. Therefore, the manifold under consideration can be classified as a LP-Kenmotsu manifold.
In addition to this, the Riemannian curvature tensor, denoted by R, can be represented as

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Hence,

$$R(\sigma^1, \sigma^2)\sigma^2 = -\sigma^1, \quad R(\sigma^1, \sigma^3)\sigma^3 = -\sigma^1, \quad R(\sigma^2, \sigma^1)\sigma^1 = -\sigma^2,$$
$$R(\sigma^2, \sigma^3)\sigma^2 = -\sigma^2, \quad R(\sigma^3, \sigma^1)\sigma^1 = -\sigma^3, \quad R(\sigma^3, \sigma^2)\sigma^2 = -\sigma^3,$$
$$R(\sigma^1, \sigma^2)\sigma^3 = 0, \quad R(\sigma^2, \sigma^3)\sigma^1 = 0, \quad R(\sigma^3, \sigma^1)\sigma^2 = 0.$$

Next, the Ricci tensor S is expressed as

$$S(\sigma^1, \sigma^1) = -2, \quad S(\sigma^2, \sigma^2) = -2, \quad S(\sigma^3, \sigma^3) = -2. \tag{6.56}$$

Furthermore, the scalar curvature transforms

$$r = \sum_{i=1}^{3} S(\sigma_i, \sigma_i) = -2. \tag{6.57}$$

By utilizing equations (2.24) and (6.56),

$$S^*(\sigma^1, \sigma^1) = 0, \quad S^*(\sigma^2, \sigma^2) = 0, \quad S^*(\sigma^3, \sigma^3) = -4. \tag{6.58}$$

Hence,

$$r^* = Tr(S^*) = 4. \tag{6.59}$$

Let us consider the potential vector field as $\Omega = 2l_1 \frac{\partial}{\partial l_1} + 2l_2 \frac{\partial}{\partial l_2} + l_3 \frac{\partial}{\partial l_3}$.
Then $(\pounds_\Omega g)(\sigma^1, \sigma^1) = -2g(\pounds_\Omega \sigma^1, \sigma^1) = 2$.
similarly, $(\pounds_\Omega g)(\sigma^1, \sigma^1) = 2, \quad (\pounds_\Omega g)(\sigma^3, \sigma^3) = 0$.
Therefore, we have

$$\sum_{i=1}^{3} (\pounds_\Omega g)(\sigma^i, \sigma^i) = 4. \tag{6.60}$$

Now putting $X = Y = \sigma^i$ in the (1.8), by performing a summation across the range of i from 1 to 3 and using equations (6.59) and (6.60), the resulting expression is derived.

$$\alpha = \frac{3q + 14}{6} \tag{6.61}$$

The aforementioned α, as defined, fulfills equation (3.31), therefore it can be deduced that g sets up an $*$-Yamabe conformal soliton on the 3-dimensional LP-Kenmotsu manifold M.

7. Conclusion

In this study, we examine intriguing findings regarding the Lorentzian Para-Kenmotsu metric, specifically its characterization as a $*$-conformal soliton with a torse forming vector field. Additionally, we provide an illustrative example of such a manifold. In this study, we also present derived outcomes concerning $*$-Yamabe conformal solitons featuring a torse-forming vector field on the respective manifold. In addition, we explore various applications of the Laplace equation in the fields of Physics and Gravity.

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STABILITY OF A MICROTEMPERATURES DAMPED POROUS-ELASTIC SYSTEM WITH NONLINEAR DISSIPATION AND NONLINEAR DISTRIBUTED DELAY

MOHAMED HOUASNI * AND ABDELKARIM KELLECHE

Abstract. In this paper, we consider a one-dimensional porous-elastic system with a nonlinear dissipation and a nonlinear distributed delay subjected to microtemperatures effects. We establish an energy decay rate by using a perturbed energy method and some properties of convex functions, but regardless of the wave speeds of the system. Our result is new and extends some previous results to nonlinearity case.

Keywords: Decay, micro-temperatures, nonlinear distributed delay, nonlinear damping, Porous elasticity.

2010 Mathematics Subject Classification: 35B40, 93D05, 93D15, 74F05.

1. Introduction

In this present work, we aim to study the following nonlinear damped porous elastic system having a nonlinear distributed delay

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x + \gamma h_1(u_t) + \int_{\tau_1}^{\tau_2} \mu(s) h_2(u_t(x,t-s))\, ds = 0, \text{ in } (0,1)\times\mathbb{R}_+, \\ J\varphi_{tt} - \delta\varphi_{xx} + k_1\omega_x + bu_x + \xi\varphi = 0, \text{ in } (0,1)\times\mathbb{R}_+, \\ \tau w_t + k_2 w + k_1\varphi_{tx} - k_3\omega_{xx} = 0, \text{ in } (0,1)\times\mathbb{R}_+. \end{cases} \tag{1.1}$$

subjected to microtemperature effects and nonlinear damping, with the mixed boundary conditions

$$u(0,t) = \varphi(1,t) = w(0,t) = u_x(1,t) = \varphi_x(0,t) = w_x(1,t) = 0. \tag{1.2}$$

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The evolution equations for one-dimensional theories of porous materials with temperature and microtemperature is given by

$$\begin{cases} \rho u_{tt} = T_x, \quad J\phi_{tt} = H_x + G, \\ \rho\eta_t = q_x, \quad \rho E_t = P_x + q - Q. \end{cases}$$

Here G is the equilibrated body force, T is the stress, H is the equilibrated stress, η is the entropy, q is the heat flux, P is the first heat flux moment, Q is the mean heat flux and E is the first moment of energy. The variables u and ϕ are, respectively, the displacement of the solid elastic material and the volume fraction. The constitutive equations are

$$\begin{cases} T = \mu u_x + b\phi + \gamma u_{tx} - \beta\theta, & H = \delta\phi_x - dw, \\ G = -bu_x - \xi\phi + m\theta - \sigma\phi_t, & \eta = \beta u_x + c\theta + m\phi, \\ q = \kappa\theta_x + \kappa_1 w, & P = -\kappa_2 w_x, \\ Q = \kappa_3 w + \kappa_4\theta_x, & \rho E = -\alpha w - d\phi_x. \end{cases}$$

Where $\rho, J, \mu, b, \gamma, \delta, d, \xi, m, \tau, \beta, c, \kappa, \kappa_1, \kappa_2, \kappa_3, \kappa_4$ and α are the constitutive coefficients whose physical meaning is well known. It is worth noting that θ and w are the temperature and microtemperatures, respectively.

The coefficients of the system, in one-dimensional case, satisfy

$$\xi > 0, \quad \delta > 0, \quad \mu > 0, \quad \rho > 0, \quad J > 0, \quad \text{and} \quad \mu\xi \geq b^2,$$

where b is a real number different from zero. On the other hand, we assume that the thermal conductivity κ and the thermal capacity c are positive, which means that thermal effects are present. While, if microtemperatures are considered, parameters α, κ_2 and κ are positive. γ and σ are nonnegative. If $\sigma > 0$ and $\gamma > 0$, it means that the system is subjected to porous dissipation and viscoelastic dissipation, respectively.

In the absence of thermal effect (i.e $\kappa = 0$), Dridi and Djebabla [17] considered the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } (0,1) \times \mathbb{R}_+, \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta - \beta\varphi_t, \text{ in } (0,1) \times \mathbb{R}_+, \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } (0,1) \times \mathbb{R}_+, \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } (0,1) \times \mathbb{R}_+, \end{cases}$$

with Neumann (on φ, θ)-Dirichlet (on u, w) boundary conditions. In which they proved that the combination of porous-viscosity and microtemperature stabilized the system exponentially regardless of the coefficients of system. In [36], Saci and Djebabla are concerned with the following system

$$\begin{cases} \rho u_{tt} = \mu u_{xx} + b\varphi_x - \gamma\theta_x, \text{ in } (0,1)\times(0,\infty), \\ J\varphi_{tt} = \delta\varphi_{xx} - bu_x - \xi\varphi - dw_x + m\theta, \text{ in } (0,1)\times(0,\infty), \\ c\theta_t = -\gamma u_{tx} - m\varphi_t - k_1 w_x, \text{ in } (0,1)\times(0,\infty), \\ \alpha w_t = k_2 w_{xx} - k_3 w - k_1\theta_x - d\varphi_{tx}, \text{ in } (0,1)\times(0,\infty), \end{cases} \tag{1.3}$$

they improved the result obtained in Dridi and Djebabla [17] by proving that a unique dissipation given by microtemperatures is sufficiently strong enough to produce exponential stability in absence of both thermal conductivity and the porous dissipation (i.e $\kappa = \beta = 0$), under a new stability number given by

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} - \frac{\gamma^2}{c\rho}.$$

By neglecting the nonlinear damping (i.e. $\gamma = 0$) and the nonlinear distributed delay, we obtain the system

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x = 0, \text{ in } (0,L)\times(0,\infty), \\ J\varphi_{tt} - \delta\varphi_{xx} + k_1\omega_x + bu_x + \xi\varphi = 0, \text{ in } (0,L)\times(0,\infty), \\ \tau w_t + k_2 w + k_1\varphi_{tx} - k_3\omega_{xx} = 0, \text{ in } (0,L)\times(0,\infty). \end{cases} \tag{1.4}$$

This system has been studied by Santos et al. [37] with fractional dissipation damping $(\sigma\varphi_t)$ in the second equaton, they concluded that the case (i.e $\sigma = 0$) is an interesting open problem. Apalara [4] provided a solution to this last by considered the system (1.4) under the following Neumann (on φ)-Dirichlet (on u, w) boundary conditions

$$u(0,t) = u(1,t) = \varphi_x(0,t) = \varphi_x(1,t) = w(0,t) = w(1,t) = 0, t > 0,$$

and established the same results of Santos et al. [37] in the absence of porous dissipation (i.e $\sigma = 0$), he showed that the unique dissipation given by microtemperature damping is strong enough to exponentially stabilize the system if and only if the wave speeds of the system are equal

$$\left(\frac{\rho}{\mu} = \frac{J}{\delta}\right).$$

Now, let us recall some results about the effect of nonlinear damping mechanisms on similar problems, Apalara [3] considered the following porous system:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & x \in (0,1), t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi + \alpha(t)g(\phi_t) = 0, & x \in (0,1), t > 0, \end{cases} \tag{1.5}$$

the term $\alpha(t)g(\phi_t)$ is the nonlinear damping, which subjected on the second equation. He established a general and an explicit decay rate result for the energy of system (1.5) with the condition of same speed of propagation, that is

$$\frac{\mu}{\rho} = \frac{\delta}{J}. \tag{1.6}$$

It is worth mentioning that in the case of $\mu = b = \xi$, the system (1.5) becomes

$$\begin{cases} \rho u_{tt} - \mu (u_x + \phi)_x = 0, & x \in (0,1), t > 0, \\ J\phi_{tt} - \delta\phi_{xx} + \mu (u_x + \phi) + \alpha(t)g(\phi_t) = 0, & x \in (0,1), t > 0, \end{cases} \tag{1.7}$$

which is a Timoshenko system with nonlinear damping. Alabau-Boussouira [2] studied (1.7) with $\alpha(t) = 1$ and proved a general semi-explicit formula for the decay rate of the energy at infinity with the condition (1.6). Mustafa and Messaoudi [28] considered (1.7) with all the coefficients $\rho = \mu = J = \delta = 1$ and obtained a general and an explicit decay result, depending on α and g.

Among the most important property of a physical system is the time delay by which the response to a subjected force is delayed in its effect (see [38]). The original study of this effect on a system was first introduced by Datko et al. [15] in 1986 when they showed that the presence of the delay may not only destabilize a system which is asymptotically stable in the absence of the delay but may also lead to ill-posedness (see also [30] and [32]). On the other hand, it has been established that voluntary introduction of delay can benefit the control (see [1]). Choucha et al. [13] considered a porous thermoelastic system with microtemperature effect, temperatures and distributed delay terms. they proved the well posedness of the system, and established an exponential stability of its solution. Moumen et al. [29] are concerned with one-dimensional porous–elastic systems with nonlinear damping, infinite memory and distributed delay terms, they proved that the solution energy has an explicit and optimal decay for the cases of equal and nonequal speeds of wave propagation. We refer the interested readers to [3, 5, 7, 10, 16, 18, 19, 20, 21, 22, 25, 27, 33, 34, 40, 41] and references therein for details discussion on the subject.

According to these observations and results above, one can ask the following questions:

1) Is it possible to stabilize system (1.5) with nonlinear damping and nonlinear distributed delay (nonlinearity case) sebjected in the first equation? If so, does the stabilization of the system depend on a relationship between the coefficients of the system?

2) What assumptions can be made about h_1 and h_2 to ensure the stabilization of the system?

In the present work, we shall give an answers to these questions by considering (1.1) under appropriate assumptions on the weight of the delay and without imposing any restrictive growth assumption on the damping term at the origin, we establish an energy decay rate by using a perturbed energy method and some properties of convex functions. These arguments of convexity were introduced and developed by Cavalcanti et al. [12], Daoulatli et al. [14], Lasiecka [23], and used by Liu and Zuazua [26] and others. We obtain our results regardless of the wave speeds χ of the system which was mentioned in [4, 37]. Our result then extends some previous results to nonlinearity case.

2. Preliminaries

In this section, we shall present some definitions and preliminaries for well study of our problem (1.1). Throughout this paper, c or c_i, $i = 1, 2$ represent a positive constant and C_p is used to denote the Poincaré-type constant.

Concerning the delay term, we introduce the following variable:

$$u_t(x, t - rs) := \vartheta(x, r, t, s),\ x \in (0, 1),\ r \in (0, 1),\ s \in (\tau_1, \tau_2),\ t > 0,$$

which satisfies

$$s\vartheta_t(x, r, t, s) + \vartheta_r(x, r, t, s) = 0,\ x \in (0, 1), r \in (0, 1), s \in (\tau_1, \tau_2), t > 0.$$

Consequently, system (1.4) becomes

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\varphi_x + \gamma h_1(u_t) + \int_{\tau_1}^{\tau_2} \mu(s) h_2(\vartheta(x, 1, t, s))ds = 0,\ \text{in } (0, 1) \times \mathbb{R}_+, \\ J\varphi_{tt} - \delta\varphi_{xx} + k_1\omega_x + bu_x + \xi\varphi = 0,\ \text{in } (0, 1) \times \mathbb{R}_+, \\ \tau w_t + k_2 w + k_1\varphi_{tx} - k_3 w_{xx} = 0,\ \text{in } (0, 1) \times \mathbb{R}_+, \\ s\vartheta_t(x, r, t, s) + \vartheta_r(x, r, t, s) = 0,\ \text{in } (0, 1) \times (0, 1) \times \mathbb{R}_+ \times (\tau_1, \tau_2), \\ u(x, 0) = u_0(x), u_t(x, 0) = u_1(x), \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), \\ u_t(x, t - \rho s) := \vartheta(x, r, t, s),\ r \in (0, 1), s \in (\tau_1, \tau_2), \\ \vartheta(x, r, 0, s) = f_0(x, -rs),\ x \in (0, 1), s \in (\tau_1, \tau_2). \end{cases} \tag{2.8}$$

With the mixed boundary conditions

$$u(0, t) = \varphi(1, t) = w(0, t) = u_x(1, t) = \varphi_x(0, t) = w_x(1, t) = 0. \tag{2.9}$$

Next, we suppose that h_1 and h_2 satisfy the following assumptions:

($\boldsymbol{A}1$). $h_1 : \mathbb{R} \to \mathbb{R}$ is a continuous and non-decreasing function with $h(0) = 0$ such that there exist positive constants k_1, k_2 and l and a convex, continuous and increasing function

$h:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2(]0,+\infty[)$ satisfying: $h''=0$ on $[0,l]$ or ($h'(0)=0$ and $h''>0$ on $(0,l]$) such that

$$\begin{aligned} &h\left(s^2+h_1^2(s)\right)\le h_1(s)\,s \quad \text{for } |s|\le l,\\ &k_1s^2\le h_1(s)\,s\le k_2s^2 \qquad \text{for } |s|>l. \end{aligned} \tag{2.10}$$

(***A***2). $h_2:\mathbb{R}\to\mathbb{R}$ is an odd non-decreasing Lipschitz function such that there exist positive constants α_1,α_2,c satisfying

$$\alpha_1 sh_2(s)\le\Gamma_2(s)\le\alpha_2 sh_1(s), \tag{2.11}$$

where $\Gamma_2(s)=\int_0^s h_2(r)dr$.

In addition, for address the case of nonlinear delay, we assume there exists a positive constant σ such that

$$\|\mu(s)\|_\infty\frac{1-\alpha_1}{\alpha_1}<\sigma \text{ and } 0<\gamma-\sigma\alpha_2(\tau_2-\tau_1)-\alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds. \tag{2.12}$$

Remark 2.1. *1) Hypothesis* $(A1)$ *implies that* $sh_1(s)>0$*, for all* $s\neq0$*.*

2) The hypothesis $(A1)$ *with* $l=1$ *was first introduced by Lasiecka and Tataru* [24]*.*

3) By the mean value Theorem for integrals and the monotonicity of h_2*, it follows that*

$$\Gamma_2(s)=\int_0^s h_2(r)dr\le sh_2(s), \tag{2.13}$$

consequently, $\alpha_1\le\alpha_2\le1$*.*

Let us now give an example for functions h_1 and h_2.

Example 2.1. *Let the function* $h_1(r)=r^\kappa, r\in(0,1]$ *(i.e* $l=1$*), and* $\kappa\geqslant1$*.* $h_1'(r)=\kappa r^{\kappa-1}$ *which is strictly positive. In the neighborhood of* 0 *, let us set the function* h *defined by*

$$h(r)=c_\kappa r^{\frac{\kappa+1}{2}},$$

where $c_\kappa=(2\kappa)^{-\frac{\kappa+1}{2}}$*. So, for* $\kappa=1, h$ *is linear on* $[0,1]$*, otherwise strictly convex on* $(0,1]$*,* $h'(0)=0$ *and* $h''>0$ *on* $(0,1]$*. In addition, we have*

$$h^{-1}(r)=2\kappa r^{\frac{2}{\kappa+1}}.$$

Now, let r *be near* 0*, (2.10) can be deduced from fact that* $r^\kappa+r^{2\kappa}\leqslant2\kappa r^2$*. Next, suppose we set the non-decreasing odd function* $h_2(r)=3^{-\kappa}r^3\,(h_2'\geqslant0)$*, then* $rh_2(r)\leqslant rh_1(r)$ *on* $(0,1]$*. Then, (2.13) follows automatically, that is* $\Gamma_2(r)=\frac{3^{-\kappa}}{4}r^4\leqslant rh_2(r)=3^{-\kappa}r^4$*, since* $\Gamma_2(r)\leqslant rh_1(r)$*, taking* $\alpha_1\leqslant\frac14$ *and* $\alpha_2\geqslant\alpha_1$*, (2.11) is deduced.*

3. Well possedness

In this section, we shall study the well-posedness of solutions to problem (1.1)-(1.2). We give existence and uniqueness results for our system using the semigroup theory. First, let us denote by $\vartheta(.)$ to $\vartheta(x,r,t,s)$, $\vartheta(1)$ to $\vartheta(x,1,t,s)$ and $\vartheta(0)$ to $\vartheta(x,0,t,s)$. Next, if we denote $U=(u,v,\varphi,\psi,w,\vartheta)^T$, where $v=u_t$, and $\psi=\varphi_t$, then, system can be rewritten as follows:

$$\begin{cases} U_t+\mathcal{A}U=0, t>0, \\ U(x,0)=U_0(x)=(u_0,u_1,\varphi_0,\varphi_1,w_0,f_0)^T. \end{cases}$$

The operators $\mathcal{A}: D(\mathcal{A})\subset\mathcal{H}\to\mathcal{H}$ is given by

$$\mathcal{A}U=\begin{pmatrix} -v \\ -\frac{\mu}{\rho}u_{xx}-\frac{b}{\rho}\varphi_x+\frac{\gamma}{\rho}h_1(u_t)+\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1))ds \\ -\psi \\ -\frac{\delta}{J}\varphi_{xx}+\frac{b}{J}u_x+\frac{\xi}{J}\varphi+\frac{k_1}{J}w_x \\ -\frac{k_3}{\tau}w_{xx}+\frac{k_2}{\tau}w+\frac{k_1}{\tau}\psi_x \\ \frac{1}{s}\vartheta_r \end{pmatrix},$$

and $\mathcal{H}$ is the energy space given by

$$\mathcal{H}=H_*^1(0,1)\times L^2(0,1)\times H_\diamond^1(0,1)\times L^2(0,1)\times H_*^1(0,1)\times L^2((0,1)\times(0,1)\times(\tau_1,\tau_2)),$$

such that

$$\begin{aligned} H_*^1(0,1) &= \left\{u\in H^1(0,1): u(0)=0\right\}, \\ H_\diamond^1(0,1) &= \left\{\varphi\in H^1(0,1): \varphi(1)=0\right\}. \end{aligned}$$

The domain of $\mathcal{A}$ is given by

$$D(\mathcal{A})=\left\{\begin{array}{l} U=(u,v,\varphi,\psi,w,\vartheta) \mid \\ u,w\in H^2(0,1)\cap H_*^1(0,1), v\in H_*^1(0,1), \\ \varphi\in H^2(0,1)\cap H_\diamond^1(0,1), \psi\in H_\diamond^1(0,1), \\ \vartheta\in H_0^1((0,1);H^1(0,1)), \\ u_x(1)=w_x(1)=\varphi_x(0)=0 \end{array}\right\}.$$

For any $U=(u,v,\varphi,\psi,w,\vartheta)^T\in\mathcal{H},\tilde{U}=(\tilde{u},\tilde{v},\tilde{\varphi},\tilde{\psi},\tilde{w},\widetilde{\vartheta})^T\in\mathcal{H}$, we equip $\mathcal{H}$ with the inner product defined by

$$\begin{aligned}\langle U,\tilde{U}\rangle_{\mathcal{H}} &= \mu\int_0^1 u_x\tilde{u}_x dx+\rho\int_0^1 v\tilde{v}dx+\xi\int_0^1\varphi\tilde{\varphi}dx+\delta\int_0^1\varphi_x\tilde{\varphi}_x dx\\ &\quad +b\int_0^1\left(u_x\tilde{\varphi}+\tilde{u}_x\varphi\right)dx+J\int_0^1\psi\tilde{\psi}dx+\tau\int_0^1 w\tilde{w}dx\\ &\quad +\sigma\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\left|\mu(s)\right|\vartheta(.)\widetilde{\vartheta}(.)dsdrdx. \qquad (3.14)\end{aligned}$$

Remark 3.1. *Under the condition $\alpha\xi>b^2$, it is easy to see that (3.14) defines an inner product.*

Note that

$$\xi\varphi^2+2bu_x\varphi+\mu u_x^2=\frac{1}{\mu}\left[(\mu\xi-b^2)\varphi^2+(\mu u_x+b\varphi)^2\right]\geq 0. \qquad (3.15)$$

For any $U,\tilde{U}\in D(\mathcal{A})$, we have

$$\begin{aligned}&\left(\mathcal{A}+m\mathcal{I}\right)U-\left(\mathcal{A}+m\mathcal{I}\right)\tilde{U}\\ &=\begin{pmatrix} -\left(v-\widetilde{v}\right)+m\left(u-\widetilde{u}\right)\\ \begin{pmatrix} -\frac{\mu}{\rho}\left(u_{xx}-\widetilde{u}_{xx}\right)-\frac{b}{\rho}\left(\varphi_x-\widetilde{\varphi}_x\right)+\frac{\gamma}{\rho}\left(h_1(v)-h_1(\widetilde{v})\right)\\ +m\left(v-\widetilde{v}\right)+\frac{1}{\rho}\int_{\tau_1}^{\tau_2}\mu(s)\left[h_2(\vartheta(1))-h_2(\widetilde{\vartheta}(1))\right]ds\end{pmatrix}\\ -\left(\psi-\widetilde{\psi}\right)+m\left(\varphi-\widetilde{\varphi}\right)\\ -\frac{\delta}{J}\left(\varphi_{xx}-\widetilde{\varphi}_{xx}\right)+\frac{b}{J}\left(u_x-\widetilde{u}_x\right)+\frac{\xi}{J}\left(\varphi-\widetilde{\varphi}\right)+\frac{k_1}{J}\left(w_x-\widetilde{w}_x\right)+m\left(\psi-\widetilde{\psi}\right)\\ -\frac{k_3}{\tau}\left(w_{xx}-\widetilde{w}_{xx}\right)+\frac{k_2}{\tau}\left(w-\widetilde{w}\right)+\frac{k_1}{\tau}\left(\psi_x-\widetilde{\psi}_x\right)+m\left(w-\widetilde{w}\right)\\ \frac{1}{s}\left(\vartheta_r-\widetilde{\vartheta}_r\right)+m\left(\vartheta-\widetilde{\vartheta}\right)\end{pmatrix},\end{aligned}$$

then, using (3.15) and the fact that h_1 is a non-decreasing function, we get for $L_{h_2}$the Lipschitz constant for h_2,

$$\begin{aligned}&\langle\left(\mathcal{A}+m\mathcal{I}\right)U-\left(\mathcal{A}+m\mathcal{I}\right)\tilde{U},\left(U-\tilde{U}\right)\rangle_{\mathcal{H}}\\ &= m\int_0^1\left(\underbrace{\mu\left(u_x-\widetilde{u}_x\right)^2+\xi\left(\varphi-\widetilde{\varphi}\right)^2+2b\left(u_x-\widetilde{u}_x\right)\left(\varphi-\widetilde{\varphi}\right)}_{\geq 0}\right)dx\\ &\quad +m\int_0^1\left(J\left(\psi-\widetilde{\psi}\right)^2+\xi\left(\varphi-\widetilde{\varphi}\right)^2+\delta\xi\left(\varphi_x-\widetilde{\varphi_x}\right)^2\right)dx\\ &\quad +\int_0^1\left(k_3\left(w_x-\widetilde{w}_x\right)^2+\left(\tau m+k_2\right)\left(w-\widetilde{w}\right)^2\right)dx\\ &\quad +\left(m-\frac{\sigma}{2}\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1\left(v-\widetilde{v}\right)^2dx\end{aligned}$$

$$
\begin{aligned}
&+\gamma \int_0^1 \underbrace{\left(h_1(v)-h_1(\widetilde{v})\right)(v-\widetilde{v})}_{\geq 0} dx \\
&+\sigma \int_0^1 (v-\widetilde{v}) \int_{\tau_1}^{\tau_2} \mu(s)\left(h_2(\vartheta(1))-h_2\left(\widetilde{\vartheta}(1)\right)\right) dsdx \\
&+\frac{\sigma}{2} \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\left(\vartheta(1)-\widetilde{\vartheta}(1)\right)^2 dsdx \\
&+\sigma m \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)\left(\vartheta-\widetilde{\vartheta}\right)^2 dsdrdx.
\end{aligned}
$$

Then, by using Young's inequality, then for small ϵ and sufficiently large m, we obtain

$$
\begin{aligned}
&\langle (\mathcal{A}+m\mathcal{I}) U-(\mathcal{A}+m\mathcal{I}) \tilde{U},\left(U-\tilde{U}\right)\rangle_{\mathcal{H}} \\
\geq\ & m \int_0^1\left(\underbrace{\mu\left(u_x-\widetilde{u}_x\right)^2+\xi(\varphi-\widetilde{\varphi})^2+2b\left(u_x-\widetilde{u}_x\right)(\varphi-\widetilde{\varphi})}_{\geq 0}\right) dx \\
&+m \int_0^1\left(J\left(\psi-\widetilde{\psi}\right)^2+\xi(\varphi-\widetilde{\varphi})^2+\delta\xi\left(\varphi_x-\widetilde{\varphi_x}\right)^2\right) dx \\
&+\int_0^1\left(k_3\left(w_x-\widetilde{w}_x\right)^2+\left(\tau m+k_2\right)(w-\widetilde{w})^2\right) dx \\
&+\left(m-\frac{\sigma}{4\epsilon}-\frac{\sigma}{2} \int_{\tau_1}^{\tau_2} \mu(s) ds\right) \int_0^1(v-\widetilde{v})^2 dx \\
&+\gamma \int_0^1 \underbrace{\left(h_1(v)-h_1(\widetilde{v})\right)(v-\widetilde{v})}_{\geq 0} dx \\
&+\left(\frac{\sigma}{2}-\epsilon L_{h_2}^2\right) \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\left(\vartheta(1)-\widetilde{\vartheta}(1)\right)^2 dsdx \\
&+\sigma m \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s)\left(\vartheta-\widetilde{\vartheta}\right)^2 dsdrdx \\
\geq\ & 0,
\end{aligned}
$$

which implies that $\mathcal{A}$ is a m-accretive operator. One can prove that $\mathcal{A}+mI$ is a maximal monotone operator. for this latter, it is sufficient to demonstrate that $R(\lambda I+\mathcal{A})=\mathcal{H}$ for a large constant λ. From the fact that $D(\mathcal{A})$ is dense in $\mathcal{H}$ (see Proposition 7.1 in [11]) and the nonlinear semigroup theory [8, 9, 39], we can give the following well-posedness result.

Proposition 3.1. *Assume (***A**1*)-(***A**2*) hold and let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C\left(\mathbb{R}_+, \mathcal{H}\right)$ of problem (2.8). Moreover, if $U_0 \in D(\mathcal{A})$, then*

$$
U \in C\left(\mathbb{R}_+, D(\mathcal{A})\right) \cap C^1\left(\mathbb{R}_+, \mathcal{H}\right).
$$

Now, let us recall few of some known algebraic and integral inequalities.

Lemma 3.1. *([11], Hölder's Inequality) Let $1 \leq p \leq \infty$, assume that $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |fg|\, dx \leq \|f\|_p \|g\|_q . \tag{3.16}$$

Lemma 3.2. [11] *(Poincaré's inequality) Suppose I is a bounded interval. Then there exists a constant C (depending on $|I| < \infty$) such that*

$$\|u\|_{W^{1,p}(I)} \leq C \|u'\|_{L^P(I)}, \text{ for all } u \in W_0^{1,p}(I). \tag{3.17}$$

Lemma 3.3. *([11], Cauchy-Schwarz Inequality) Every inner product satisfies the Cauchy-Schwarz inequality*

$$\langle x_1, x_2 \rangle \leq \|x_1\| \|x_2\| . \tag{3.18}$$

The equality sign holds if and only if x_1 and x_2 are dependent.

Lemma 3.4. [11]*(Young's Inequality) For all $a, b \in \mathbb{R}^+$, we have*

$$ab \leq \epsilon a^2 + \frac{b^2}{4\epsilon}, \tag{3.19}$$

where ϵ is any positive constant.

Next, Let us denote by h^* the conjugate function in the sense of Young of a convex function h (see [6], p. 64), that is,

$$h^*(p) = \sup_{t \in \mathbb{R}_+} (pt - h(t)).$$

Assume that $h'' > 0$, then for $p \geq 0$ a given number, h^* is the Legendre transform of h (see Liu and Zuazua [26]), which is given by

$$h^*(p) := p\left[h'\right]^{-1}(p) - h(\left[h'\right]^{-1}(p)), \tag{3.20}$$

and which satisfies the following inequality

Lemma 3.5. [31]*(Young's Inequality for the convex functions) Let h a convex function, h^* its conjugate in the sense of Young, we have*

$$px \leq h(x) + h^*(p) \quad \forall p, x \geq 0. \tag{3.21}$$

Remark 3.2. *Thanks to (3.20), along with (2.11), we write*

$$\begin{aligned} h^*(h_2(\vartheta(1))) &= \vartheta(1)h_2(\vartheta(1) - h(\vartheta(1)) \\ &\leq (1-\alpha_1)\vartheta(1)h_2(\vartheta(1). \end{aligned} \tag{3.22}$$

Next, for $\epsilon_0 > 0$ we define the functions J and K as below

$$J(t) := \begin{cases} t, & \text{if } h'' = 0 \text{ on } [0,l] \\ th'(\epsilon_0 t), & \text{if } h'(0) = 0 \text{ and } h'' > 0 \text{ on } (0,l] \end{cases} \tag{3.23}$$

and

$$K(t) = \int_t^1 \frac{1}{J(s)}\, ds \tag{3.24}$$

respectively.

Remark 3.3. *The relation (3.20) and the fact that $h(0) = 0$ and $(h')^{-1}$, h are increasing functions yield*

$$h^*(p) \leq p\left[h'\right]^{-1}(p) \quad \forall p \geq 0. \tag{3.25}$$

4. General decay

In this section, we give some lemmas allow us to prove the stability result o the solution. We define the functional energy of solutions of problem (1.1)-(1.2) as follows:

$$\begin{aligned} E(t) \; : &= \frac{1}{2}\int_0^1 \left(\rho {u_t}^2 + J\varphi_t^2 + \delta\varphi_x^2 + \underbrace{\xi\varphi^2 + 2bu_x\varphi + \mu u_x^2}_{\geq 0} + \tau w^2 \right) dx \\ &+ \frac{\sigma}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(.))dsdrdx. \end{aligned} \tag{4.26}$$

Lemma 4.1. *Let (u, φ, w) be a solution of (2.8)-(2.9). Then the energy functional $E(t)$, satisfies*

$$\begin{aligned} \mathrm{E}'(t) &\leq -k_2\int_0^1 w^2 dx - k_3\int_0^1 w_x^2 dx \\ &\quad -c\left[\int_0^1 u_t h_1(u_t)\, dx + \int_0^1\int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\right] \\ &\leq 0. \end{aligned} \tag{4.27}$$

Proof. We multiply $(2.8)_1$ by u_t, $(2.8)_2$ by φ_t, and $(2.8)_3$ by w and then integrate over $(0,1)$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_0^1\left[\rho {u_t}^2+J\varphi_t^2+\delta\varphi_x^2+\underbrace{\xi\varphi^2+2bu_x\varphi+\mu u_x^2}_{\geq 0}+\tau w^2\right]dx \qquad (4.28)\\
&+\gamma\int_0^1 u_t h_1(u_t)\,dx+\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)u_t h_2(\vartheta(1))dsdx\\
= &-k_2\int_0^1 w^2dx-k_3\int_0^1 w_x^2dx.
\end{aligned}$$

Now, by multiplying the fourth equation in (2.8) by $\sigma\,|\mu(s)|\,h_2(\vartheta(x,r,t,s))$, and integrating over $(0,1)\times(\tau_1,\tau_2)\times(0,1)$, using integration by parts, the definition of Γ_2, and the boundary conditions, gives

$$\begin{aligned}
\sigma\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 s\mu(s)\vartheta_t h_2(\vartheta(.))drdsdx &= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 \mu(s)\vartheta_r h_2(\vartheta(.))drdsdx\\
\sigma\frac{d}{dt}\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1 s\mu(s)\Gamma_2(\vartheta(.))drdsdx &= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\int_0^1\frac{\partial}{\partial r}\Gamma_2(\vartheta(.))drdsdx\\
&= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(1))dsdx\\
&\quad+\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(0))dsdx\\
&= -\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(1))dsdx\\
&\quad+\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(u_t)\,dsdx. \qquad (4.29)
\end{aligned}$$

The combination of (4.28)- (4.29) gives us

$$\begin{aligned}
E'(t) &= -\gamma\int_0^1 u_t h_1(u_t)\,dx-\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)u_t h_2(\vartheta(1))dsdx\\
&\quad-\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(\vartheta(1))dsdx-k_3\int_0^1 w_x^2dx\\
&\quad+\sigma\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(u_t)\,dsdx-k_2\int_0^1 w^2dx. \qquad (4.30)
\end{aligned}$$

So, by return to the convex conjugate of Γ_2, taking $p=h_2(\vartheta(1))$ and $x=u_t$, we get

$$u_t h_2(\vartheta(1)) \leq \Gamma_2^*(h_2(\vartheta(1)))+\Gamma_2(u_t), \qquad (4.31)$$

$$\text{where, } \Gamma_2^*(h_2(\vartheta(1))) = \vartheta(1)h_2(\vartheta(1))-\Gamma_2(\vartheta(1)). \qquad (4.32)$$

Using (4.30)- (4.32) and $(A2)$, we obtain

$$
\begin{aligned}
E'(t) \leq & -\gamma \int_0^1 u_t h_1(u_t)\,dx + \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\
& + (\sigma+1)\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\Gamma_2(u_t)\,dsdx - (\sigma+1)\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\Gamma_2(\vartheta(1))dsdx \\
& -k_2 \int_0^1 w^2 dx - k_3 \int_0^1 w_x^2 dx \\
\leq & -\left[\gamma - \alpha_2\sigma(\tau_2-\tau_1) - \alpha_2 \int_{\tau_1}^{\tau_2} \mu(s)ds\right] \int_0^1 u_t h_1(u_t)\,dx \\
& - \int_0^1 \int_{\tau_1}^{\tau_2} [\frac{\alpha_1\sigma}{\|\mu(s)\|_\infty} - (1-\alpha_1)]\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\
& -k_2 \int_0^1 w^2 dx - k_3 \int_0^1 w_x^2 dx.
\end{aligned}
$$

Finally, by using (2.12) we obtain (4.27). □

Lemma 4.2. *Let* (u,φ,w) *be a solution of (2.8)-(2.9). Then, the functional*

$$
I_1(t) = \frac{d}{4}\int_0^1 u_t u dx - \tau \int_0^1 w\left(\int_0^x u_t(y)dy\right)dx, t \geq 0,
$$

satisfies, for $\eta, d > 0$ *and* $\forall t \geq 0$

$$
\begin{aligned}
I_1'(t) \leq & \left(c\eta - \frac{\mu d}{8\rho}\right)\int_0^1 u_x^2 dx + d\int_0^1 u_t^2 dx + (3c_1 + 2c\eta)\int_0^1 w^2 dx \\
& + c_1 \int_0^1 \left(\varphi_t^2 + \varphi^2 + w_x^2\right) dx + \frac{c}{\eta}\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\
& + \frac{c}{2\eta}\int_0^1 h_1^2(u_t)\,dx.
\end{aligned} \tag{4.33}
$$

Proof. Differentiating $I_1(t)$ and integrating by parts, we get

$$
\begin{aligned}
I_1'(t) = & \frac{d}{4}\int_0^1 u_t^2 dx - \frac{\mu d}{4\rho}\int_0^1 u_x^2 dx - \frac{bd}{4\rho}\int_0^1 \varphi u_x dx \\
& - \frac{\gamma d}{4\rho}\int_0^1 h_1(u_t)\,u dx - \frac{d}{4\rho}\int_0^1 u \int_{\tau_1}^{\tau_2} \mu(s)h_2(\vartheta(1))dsdx \\
& + k_3 \int_0^1 w_x u_t dx + k_2 \int_0^1 w\left(\int_0^x u_t(y)dy\right)dx - k_1 \int_0^1 \varphi_t u_t dx \\
& + \frac{\tau\mu}{\rho}\int_0^1 w u_x dx + \frac{\tau b}{\rho}\int_0^1 w\varphi dx \\
& + \frac{\tau\gamma}{\rho}\int_0^1 w\left(\int_0^x h_1(u_t)\,dy\right)dx + \frac{\tau}{\rho}\int_0^1 w \int_{\tau_1}^{\tau_2} \mu(s)\left(\int_0^x h_2(\vartheta(y,1))dy\right)dsdx.
\end{aligned} \tag{4.34}
$$

By using Cauchy-Schwarz inequality (3.18), we obtain

$$\begin{aligned}
\left(\int_0^x u_t(y)dy\right)^2 &\leq \left(\int_0^1 u_t\ dx\right)^2 \leq \int_0^1 u_t^2\ dx,\\
\left(\int_0^x h_1\left(u_t\right)dy\right)^2 &\leq \left(\int_0^1 h_1\left(u_t\right)\ dx\right)^2\ dx \leq \int_0^1 h_1^2\left(u_t\right)\ dx,\\
\left(\int_0^x h_2(\vartheta(y,1))dy\right)^2 &\leq \left(\int_0^1 h_2(\vartheta(1))\ dx\right)^2\ dx \leq \int_0^1 h_2^2(\vartheta(1))dx.
\end{aligned}$$

Next, On account of (2.13), (3.21), and (3.22), we obtain

$$h_2^2(\vartheta(x,1,t,s)) \leq 2\vartheta(x,1,t,s)h_2(\vartheta(x,1,t,s)). \tag{4.35}$$

Then, using Young's inequality (3.19), Cauchy–Schwarz inequality and (4.35), we get

$$-\frac{bd}{4\rho}\int_0^1 \varphi u_x dx \leq \frac{d\mu}{16\rho}\int_0^1 u_x^2 dx + \frac{c_1}{2}\int_0^1 \varphi^2 dx, \tag{4.36}$$

$$\frac{\tau\mu}{\rho}\int_0^1 w u_x dx \leq \frac{d\mu}{16\rho}\int_0^1 u_x^2 dx + c_1\int_0^1 w^2 dx, \tag{4.37}$$

$$k_2\int_0^1 w\left(\int_0^x u_t(y)dy\right)dx \leq \frac{d}{4}\int_0^1 u_t^2 dx + c_1\int_0^1 w^2 dx, \tag{4.38}$$

$$-k_1\int_0^1 \varphi_t u_t dx \leq \frac{d}{4}\int_0^1 u_t^2 dx + c_1\int_0^1 \varphi_t^2 dx, \tag{4.39}$$

$$k_3\int_0^1 w_x u_t dx \leq \frac{d}{4}\int_0^1 u_t^2 dx + c_1\int_0^1 w_x^2 dx, \tag{4.40}$$

$$\left|-\frac{\gamma d}{4\rho}\int_0^1 h_1\left(u_t\right)u dx\right| \leq \frac{c}{4\eta}\int_0^1 h_1^2\left(u_t\right)dx + c\eta\int_0^1 u_x^2 dx, \tag{4.41}$$

$$\begin{aligned}
\left|-\frac{d}{4\rho}\int_0^1 u\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1)dsdx\right| &\leq \eta c_p\int_{\tau_1}^{\tau_2}\mu(s)ds\int_0^1 u_x^2 dx\\
&\quad +\frac{1}{4\eta}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)h_2^2(\vartheta(1))dsdx,\\
&\leq +\frac{c}{2\eta}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\
&\quad +c\eta\int_0^1 u_x^2 dx,
\end{aligned} \tag{4.42}$$

$$\frac{\tau b}{\rho}\int_0^1 w\varphi dx \leq c_1\int_0^1 w^2 dx + \frac{c_1}{2}\int_0^1 \varphi^2 dx, \tag{4.43}$$

$$\frac{\tau\gamma}{\rho}\int_0^1 w\left(\int_0^x h_1\left(u_t\right)dy\right)dx \leq \frac{c}{4\eta}\int_0^1 h_1^2\left(u_t\right)dx + c\eta\int_0^1 w^2 dx, \tag{4.44}$$

$$\begin{aligned}
\frac{\tau}{\rho}\int_0^1 w\int_{\tau_1}^{\tau_2}\mu(s)\left(\int_0^x h_2(\vartheta(y,1))dy\right)dsdx &\leq \frac{c}{2\eta}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\
&\quad +c\eta\int_0^1 w^2 dx.
\end{aligned} \tag{4.45}$$

By substituting (4.36)-(4.45) into (4.34), we get (4.33). □

Lemma 4.3. *Let* (u, φ, w) *be a solution of (2.8)-(2.9). Then the functional*

$$I_2(t) := J\int_0^1 \varphi_t\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi(y)dy\right)dx, t \geq 0,$$

satisfies, for any $\varepsilon_1 > 0$, *the following estimate*

$$\begin{aligned} I_2'(t) \quad \leq \quad & -\frac{\delta}{2}\int_0^1 \varphi_x^2 dx - \lambda\int_0^1 \varphi^2 dx + \varepsilon_1\int_0^1 u_t^2 dx + c_2\int_0^1 w_x^2 dx \\ & +\left(J + \frac{3c_2}{\varepsilon_1}\right)\int_0^1 \varphi_t^2 dx + \varepsilon_1\int_0^1 h_1^2(u_t)\,dx \\ & +2\varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx, \end{aligned} \tag{4.46}$$

where $\lambda = \left(\xi - \frac{b^2}{\mu}\right)$.

Proof. By differentiating $I_2(t)$, we obtain

$$\begin{aligned} I_2'(t) \quad = \quad & J\int_0^1 \varphi_{tt}\varphi dx + J\int_0^1 \varphi_t^2 dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)dy\right)dx \\ & -\frac{b\rho}{\mu}\int_0^1 u_{tt}\left(\int_0^x \varphi(y)dy\right)dx. \end{aligned}$$

Next, using integrating by parts together with the boundary conditions, we get

$$\begin{aligned} I_2'(t) \quad = \quad & -\delta\int_0^1 \varphi_x^2 dx - \left(\xi - \frac{b^2}{\mu}\right)\int_0^1 \varphi^2 dx + J\int_0^1 \varphi_t^2 dx \\ & -k_1\int_0^1 w_x\varphi dx - \frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)dy\right)dx \\ & -\frac{\gamma_1 b}{\mu}\int_0^1 h_1(u_t)\left(\int_0^x \varphi(y)dy\right)dx \\ & -\frac{\gamma_2 b}{\mu}\int_0^1\left(\int_0^x \varphi(y)dy\right)\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1))dsdx. \end{aligned}$$

Thanks to Young's, Poincaré (3.17) and Cauchy-Schwarz's inequalities and (4.35), we obtain

$$\begin{aligned} -k_1\int_0^1 w_x\varphi dx \quad & \leq \quad \frac{\delta}{2}\int_0^1 \varphi_x^2 dx + c_2\int_0^1 w_x^2 dx, \\ -\frac{b\rho}{\mu}\int_0^1 u_t\left(\int_0^x \varphi_t(y)dy\right)dx \quad & \leq \quad \varepsilon_1\int_0^1 u_t^2 dx + \frac{c_2}{\varepsilon_1}\int_0^1 \varphi_t^2 dx, \end{aligned}$$

$$\begin{aligned}
-\frac{\gamma_1 b}{\mu}\int_0^1 h_1(u_t)\left(\int_0^x \varphi_t(y)dy\right)dx &\leq \varepsilon_1\int_0^1 h_1^2(u_t)\,dx+\frac{c_2}{\varepsilon_1}\int_0^1 \varphi_t^2 dx,\\
-\frac{\gamma_1 b}{\mu}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)h_2(\vartheta(1))\left(\int_0^x \varphi_t(y)dy\right)dsdx &\leq \varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)h_2^2(\vartheta(1))dsdx\\
&\quad+\frac{c_2}{\varepsilon_1}\int_0^1\varphi_t^2dx\\
&\leq 2\varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\
&\quad+\frac{c_2}{\varepsilon_1}\int_0^1\varphi_t^2dx.
\end{aligned}$$

Then, we find that

$$\begin{aligned}
I_2'(t) &\leq -\frac{\delta}{2}\int_0^1\varphi_x^2dx-\lambda\int_0^1\varphi^2dx+\varepsilon_1\int_0^1u_t^2dx+c_2\int_0^1w_x^2dx\\
&\quad+\left(J+\frac{c_2}{\varepsilon_1}\right)\int_0^1\varphi_t^2dx+\varepsilon_1\int_0^1h_1^2(u_t)\,dx\\
&\quad+2\varepsilon_1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx,
\end{aligned}$$

which is (4.46) with $\lambda=\xi-\frac{b^2}{\mu}>0$. □

Lemma 4.4. *Let (u,φ,w) be a solution of (2.8)-(2.9). Then, the functional*

$$I_3(t)=-\tau\int_0^1 w\left(\int_0^x\varphi_t(y)dy\right)dx,$$

satisfies, for any $\varepsilon_2,\varepsilon_3,\varepsilon_4>0$, the following estimate

$$\begin{aligned}
I_3'(t) &\leq (-k_1+2\varepsilon_2)\int_0^1\varphi_t^2dx+\varepsilon_2C_p\int_0^1u_x^2dx+\varepsilon_4\int_0^1\varphi^2dx\\
&\quad+c_2\left(\frac{(b\tau)^2}{4\varepsilon_2}+\frac{\delta^2}{4\varepsilon_3}+\frac{\xi^2}{4\varepsilon_4}+\frac{k_2^2}{4\varepsilon_2}+k_1\right)\int_0^1w^2dx\\
&\quad+\varepsilon_3\int_0^1\varphi_x^2dx.
\end{aligned}\tag{4.47}$$

Proof. By differentiating $I_3(t)$, integrating by parts and using (2.8), we obtain

$$\begin{aligned}
I_3'(t) &= -Jk_1\int_0^1\varphi_t^2dx+k_1\int_0^1w^2dx+\tau b\int_0^1wudx\\
&\quad-Jk_3\int_0^1w_x\varphi_tdx-\xi\int_0^1w\left(\int_0^x\varphi(y)dy\right)dx\\
&\quad+Jk_2\int_0^1w\left(\int_0^x\varphi_t(y)dy\right)dx-\delta\int_0^1w\varphi_xdx.
\end{aligned}\tag{4.48}$$

Using Young's, Cauchy-Schwarz's and Poincaré inequalities, we find

$$\begin{aligned}
\tau b\int_0^1 wudx &\le \varepsilon_2 C_p\int_0^1 u_x^2dx+\frac{(\tau b)^2}{4\varepsilon_2}\int_0^1 w^2dx, &(4.49)\\
-\delta\int_0^1 w\varphi_xdx &\le \varepsilon_3\int_0^1 \varphi_x^2dx+\frac{\delta^2}{4\varepsilon_3}\int_0^1 w^2dx, &(4.50)\\
-\xi\int_0^1 w\left(\int_0^x \varphi(y)dy\right)dx &\le \varepsilon_4\int_0^1 \varphi^2dx+\frac{\xi^2}{4\varepsilon_4}\int_0^1 w^2dx, &(4.51)\\
Jk_2\int_0^1 w\left(\int_0^x \varphi_t(y)dy\right)dx &\le \varepsilon_2\int_0^1 \varphi_t^2dx+\frac{k_2^2}{4\varepsilon_2}\int_0^1 w^2dx, &(4.52)\\
-Jk_3\int_0^1 w_x\varphi_tdx &\le \varepsilon_2\int_0^1 \varphi_t^2dx+\frac{k_3^2}{4\varepsilon_2}\int_0^1 w_x^2dx. &(4.53)
\end{aligned}$$

Estimate (4.47) follows by substituting (4.49)-(4.53) into(4.48). □

Lemma 4.5. *Let (u,φ,w) be a solution of (2.8)-(2.9), then, the functional*

$$I_4(t)=\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-sr}\Gamma_2(\vartheta(.))dsdrdx, \tag{4.54}$$

satisfies the estimate

$$\begin{aligned}
I_4'(t) \le\ & -\alpha_1 e^{-\tau_2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx\\
&+\alpha_2\left(\int_{\tau_1}^{\tau_2}\mu(s)ds\right)\int_0^1 u_th_1(u_t)dx\\
&-e^{-\tau_2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(1))dsdrdx. \qquad (4.55)
\end{aligned}$$

Proof. Differentiating I_4, using the fourth equation in (2.8), $(A2)$, and the fact that $\vartheta(0)=u_t$, we obtain

$$\begin{aligned}
I_4'(t) &= \int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\vartheta_t(.)e^{-sr}h_2(\vartheta(.))dsdrdx\\
&= -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta_r(.)e^{-sr}h_2(\vartheta(.))dsdrdx\\
&= -\int_0^1\int_{\tau_1}^{\tau_2}\int_0^1\mu(s)\frac{d}{dr}\left[e^{-sr}\Gamma_2(\vartheta(.))\right]drdsdx\\
&\quad -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-sr}\Gamma_2(\vartheta(.))dsdrdx\\
&= -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\mu(s)\Gamma_2(\vartheta(1))dsdx+\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\Gamma_2(u_t)dsdx
\end{aligned}$$

$$\begin{aligned}
& -\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)e^{-sr}\Gamma_2(\vartheta(.))dsdrdx \\
\leq\ & -\alpha_1 e^{-\tau_2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\vartheta(1)h_2(\vartheta(1))dsdx \\
& -e^{-\tau_2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\Gamma_2(\vartheta(.))dsdrdx \\
& +\alpha_2\int_{\tau_1}^{\tau_2}\mu(s)ds\int_0^1 u_t h_1(u_t)\,dx.
\end{aligned}$$

Using the fact that $-e^{-sr}\leq -e^{-s}$ for all $r\in[0,1]$, we then obtain (4.55). □

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + \sum_{i=1}^{3} N_i I_i(t) + I_4(t), \tag{4.56}$$

here, N, N_1, N_2 and N_3 are positive constants.

Lemma 4.6. *Let (u,φ,w) be a solution of (2.8)-(2.9). Then, there exist two positive constants μ_1 and μ_2 such that the Lyapunov functional (4.56) satisfies*

$$\mu_1 \mathrm{E}(t) \leq \mathcal{L}(t) \leq \mu_2 \mathrm{E}(t), \forall t\geq 0. \tag{4.57}$$

Proof. From (4.56), we have

$$\begin{aligned}
|\mathcal{L}(t)-NE(t)| \leq\ & \frac{dN_1}{4}\int_0^1 |u_t u|\,dx + \tau N_1\int_0^1 \left|w\left(\int_0^x u_t(y)dy\right)\right|dx \\
& +JN_2\int_0^1 |\varphi_t\varphi|\,dx + \frac{b\rho N_2}{\mu}\int_0^1\left|u_t\left(\int_0^x \varphi(y)dy\right)\right|dx \\
& +N_3\tau J\int_0^1\left|w\left(\int_0^x \varphi_t(y)dy\right)\right|dx \\
& +\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\left|se^{-sr}\right|\Gamma_2(\vartheta(.))dsdrdx.
\end{aligned}$$

By using Young's, Poincaré and Cauchy-Schwarz inequalities, we obtain

$$|\mathcal{L}(t)-NE(t)| \leq \gamma E(t),$$

which yields

$$(N-\gamma)E(t) \leq \mathcal{L}(t) \leq (N+\gamma)E(t), \tag{4.58}$$

this completes the proof. □

Theorem 4.1. *Let* $(\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, q_0, f_0)^T \in \mathcal{H}$ *be given. Assume that* $A1 - A2$ *are satisfied, then there exist* $c_1, c_2, c_3 > 0$ *for which the (weak) solution of problem (2.8)-(2.9) satisfies*

$$E(t) \leq c_1 K^{-1}(c_2 t + c_3), \ \forall t \geq 0. \tag{4.59}$$

Proof. By differentiating equation (4.56), then recalling Eqs. (4.27), (4.33), (4.46), (4.47) and (4.55), we have

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & (-k_3 N + c_1 N_1 + c_2 N_2) \int_0^1 w_x^2 dx + (-k_2 N + (3c_1 + 2c\eta) N_1) \int_0^1 w^2 dx \\
&+ N_3 c_2 \left(\frac{(b\tau)^2}{4\varepsilon_2} + \frac{\delta^2}{4\varepsilon_3} + \frac{\xi^2}{4\varepsilon_4} + \frac{k_2^2}{4\varepsilon_2} + k_1 \right) \int_0^1 w^2 dx \\
&+(dN_1 + N_2\varepsilon_1) \int_0^1 u_t^2 dx + \left(\alpha_2 \int_{\tau_1}^{\tau_2} \mu(s) ds + \frac{c}{4\eta} - N c_0 \right) \int_0^1 u_t h_1(u_t) dx \\
&+ \left(\left(c\eta - \frac{\mu d}{8\rho} \right) N_1 + \varepsilon_2 C_p N_3 \right) \int_0^1 u_x^2 dx + \left(\frac{N_1 c}{2\eta} + N_2 \varepsilon_1 \right) \int_0^1 h_1^2(u_t) dx \\
&+(-\lambda N_2 + c_1 N_1 + N_3 \varepsilon_4) \int_0^1 \varphi^2 dx + \left(-\frac{\delta}{2} N_2 + N_3 \varepsilon_3 \right) \int_0^1 \varphi_x^2 dx \\
&+ \left(N_1 c_1 + \left(J + \frac{3c_2}{\varepsilon_1} \right) N_2 + N_3(-k_1 + 2\varepsilon_2) \right) \int_0^1 \varphi_t^2 dx \\
&+ \left(\left(2\varepsilon_1 N_2 + \frac{N_1 c}{\eta} \right) - N c_0 - \alpha_1 e^{-\tau_2} \right) \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) \vartheta(1) h_2(\vartheta(1)) ds dx \\
&- e^{-\tau_2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) \Gamma_2(\vartheta(.)) ds dr dx.
\end{aligned} \tag{4.60}$$

At this point, we set $\varepsilon_1 = 1$, $\varepsilon_2 = \frac{1}{N_3}$ and choose η small enough so that

$$\eta \leq \frac{\mu d}{8c\rho}.$$

Next, take N_1 large enough so that,

$$\left(c\eta - \frac{\mu d}{8\rho} \right) N_1 + C_p < 0.$$

Let us fix N_1 and select $\varepsilon_3 = \varepsilon_4 = \frac{1}{N_3}$, choose N_2 large enough so that

$$-\frac{\delta}{2} N_2 + 1 < 0 \text{ and } -\lambda N_2 + c_1 N_1 + 1 < 0.$$

Fix N_2 and select ε_2 so small that

$$\varepsilon_2 < \frac{k_1}{2},$$

choose N_3 large enough so that

$$N_1 c_1 + (J + 3c_2) N_2 + N_3(-k_1 + 2\varepsilon_2) < 0 .$$

Finally, we choose N large enough so that

$$\begin{array}{l}
-k_3 N + c_1 N_1 + c_2 N_2 < 0, \\
-k_2 N + (3c_1 + 2c\eta) N_1 + N_3 c_2 \left(\frac{(b\tau)^2}{4\varepsilon_2} + \frac{\delta^2}{4\varepsilon_3} + \frac{\xi^2}{4\varepsilon_4} + \frac{k_2^2}{4\varepsilon_2} + k_1 \right) < 0, \\
\left(2\varepsilon_1 N_2 + \frac{N_1 c}{\eta} \right) - N c_0 - \alpha_1 e^{-\tau_2} < 0, \\
\alpha_2 \int_{\tau_1}^{\tau_2} \mu(s) ds + \frac{c}{4\eta} - N c_0 < 0.
\end{array}$$

All these choices with the relation (4.60) leads to

$$\begin{array}{l}
\mathcal{L}'(t) \leq -\lambda_1 \int_0^1 \left(u_x^2 + u_t^2 + \varphi^2 + \varphi_x^2 + \varphi_t^2 + w^2 + \sigma \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) \Gamma_2(\vartheta(.)) ds dr \right) dx \\
-\lambda_2 \int_0^1 \left(w_x^2 + w^2 \right) dx + c \left(\int_0^1 u_t^2 dx + \int_0^1 h_1^2(u_t) dx \right), \; c, \lambda_1, \lambda_2 > 0.
\end{array} \tag{4.61}$$

On the other hand, from (4.26) and by using Young's inequality, we obtain

$$\begin{aligned}
E(t) \leq \; & \frac{1}{2} \int_0^1 \left(\rho u_t^2 + J\varphi_t^2 + (\mu + |b|) u_x^2 + \delta \varphi_x^2 + (\xi + |b|) \varphi^2 + \tau w^2 \right) dx \\
& + \frac{\sigma}{2} \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) \Gamma_2(\vartheta(.)) ds dr dx \\
& \leq \upsilon_1 \left(\int_0^1 \left(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2 + w^2 \right) dx \right) \\
& + \upsilon_1 \sigma \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) \Gamma_2(\vartheta(.)) ds dr dx, \upsilon_1 > 0,
\end{aligned}$$

which implies that

$$\begin{aligned}
& - \int_0^1 \left(u_t^2 + \varphi_t^2 + u_x^2 + \varphi_x^2 + \varphi^2 + w^2 + \sigma \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) \Gamma_2(\vartheta(.)) ds dr \right) dx \\
\leq \; & -\upsilon_2 E(t), \upsilon_2 > 0.
\end{aligned} \tag{4.62}$$

The combination of (4.61) and (4.62) gives

$$\begin{aligned}
\mathcal{L}'(t) \; \leq \; & -c_1 \int_0^1 \left(u_t^2 + \varphi_t^2 + w^2 + (u_x + \varphi)^2 + \varphi_x^2 + \sigma \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) \Gamma_2(\vartheta(.)) ds dr \right) dx \\
& + c \left(\int_0^1 u_t^2 dx + \int_0^1 h_1^2(u_t) dx \right) \\
\leq \; & -c_1 E(t) + c \left(\int_0^1 \left(u_t^2 + h_1^2(u_t) \right) dx \right).
\end{aligned} \tag{4.63}$$

Let us define the following sets

$$\Sigma_u = \{ x \in (0,1) : |u_t(x,t)| > l \} \text{ and } \Theta_u = (0,1) \backslash \Sigma_u.$$

Now, we estimate the last term in the right-hand side of (4.63). First, we have

$$\begin{aligned}
\int_0^1 \left(u_t^2 + h_1^2(u_t) \right) dx \; = \; & \int_{\Sigma_u} \left(u_t^2 + h_1^2(u_t) \right) dx \\
& + \int_{\Theta u} \left(u_t^2 + h_1^2(u_t) \right) dx.
\end{aligned}$$

Using $\boldsymbol{A1}$ and (4.27), we easily show that

$$\begin{aligned}\int_{\Sigma_u}\left(u_t^2+h_1^2\left(u_t\right)\right)dx &\leq \left(k_1^{-1}+k_2\right)\int_{\Sigma_\psi}u_th_1\left(u_t\right)dx\\ &\leq \left(k_1^{-1}+k_2\right)\int_0^1 u_th_1\left(u_t\right)dx\\ &\leq -cE'\left(t\right).\end{aligned}\tag{4.64}$$

If $h''=0$ on $[0,l]$: This implies that there exist $k_1',k_2'>0$ such that $k_1's^2\leq h_1\left(s\right)s\leq k_2's^2$ for all $s\in\mathbb{R}_+$, and then (4.64) is also satisfied for $\left|u_t\left(x,t\right)\right|\leq l$, then on all $(0,1)$. From (4.63), (4.64), we arrive at

$$\left(\mathcal{L}(t)+cE\left(t\right)\right)'\leq -cJ\left(E\left(t\right)\right),\quad \forall t\geq 0,\tag{4.65}$$

where J is defined in (3.23).

If $h'(0)=0$ and $h''>0$ on $(0,l]$: Since h is convex and increasing, h^{-1} is concave and increasing, by using $\boldsymbol{A1}$, the reversed Jensen's inequality for concave function (see [35], p. 61), and (4.27), we obtain,

$$\begin{aligned}\int_{\Theta_\psi}\left(u_t^2+h_1^2\left(u_t\right)\right)dx &\leq \int_{\Theta_\psi}h^{-1}\left(u_th_1\left(u_t\right)\right)dx\\ &\leq \int_{\Theta_\psi}h^{-1}\left(u_th_1\left(u_t\right)\right)dx\\ &\leq \left|\Theta_\psi\right|h^{-1}\left(\int_{\Theta_\psi}\frac{1}{\left|\Theta_\psi\right|}u_th_1\left(u_t\right)dx\right)\\ &\leq ch^{-1}\left(\int_{\Theta_\psi}u_th_1\left(u_t\right)dx\right)\\ &\leq ch^{-1}\left(\int_0^1 u_th_1\left(u_t\right)\right)dx\\ &\leq ch^{-1}\left(-cE'\left(t\right)\right).\end{aligned}\tag{4.66}$$

Therefore, from (4.63), (4.64) and (4.66), we find that

$$\mathcal{L}'(t)\leq -cE\left(t\right)+ch^{-1}(-cE'\left(t\right))-cE'\left(t\right),\quad \forall t\geq 0.$$

By using Young's inequality (3.21), (3.25) and the fact that

$$E'\leq 0,\ and\ h''>0,$$

we get for $\varepsilon_0 > 0$ small enough and $c_0 > 0$ large enough,

$$\begin{aligned}
& \left[h'\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}(t)+cE(t)\right]+c_0 E(t)\right]' \\
= \;& \varepsilon_0' E(t)\, h''\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}(t)+cE(t)\right] \\
& +h'\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}'(t)+cE'(t)\right]+c_0' E(t) \\
\leq \;& -ch'\left(\varepsilon_0 E(t)\right) E(t)+ch'\left(\varepsilon_0 E(t)\right) h^{-1}(-cE'(t))+c_0' E(t) \\
\leq \;& -ch'\left(\varepsilon_0 E(t)\right) E(t)+ch^*(h'\left(\varepsilon_0 E(t)\right))-cE'(t)+c_0' E(t) \\
\leq \;& -ch'\left(\varepsilon_0 E(t)\right) E(t)+c\varepsilon_0 h'\left(\varepsilon_0 E(t)\right) E(t) \\
\leq \;& -ch'\left(\varepsilon_0 E(t)\right) E(t)=-cJ\left(E(t)\right).
\end{aligned} \tag{4.67}$$

Now, let us define the following functional:

$$\mathcal{G}(t)=\begin{cases} \mathcal{L}(t)+cE(t) & \text{if } h''=0 \text{ on } [0,l], \\ h'\left(\varepsilon_0 E(t)\right)\left[\mathcal{L}(t)+cE(t)\right]+c_0 E(t) & \text{if } h'(0)=0 \text{ and } h''>0 \text{ on } (0,l]. \end{cases}$$

Using (4.57), we have

$$\mathcal{G} \sim E,$$

and exploiting (4.65) and (4.67), we easily deduce that

$$\mathcal{G}'(t) \leq -cJ\left(E(t)\right), \quad \forall t \geq 0.$$

Next, let us set

$$\mathcal{R}(t)=\varepsilon \mathcal{G}(t),$$

where $0<\varepsilon<\bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\mathcal{G}(t) \leq \frac{1}{\bar{\varepsilon}} E(t), \quad \forall t \geq 0.$$

We also have

$$\mathcal{M} \sim E, \tag{4.68}$$

and for $t \geq 0$

$$\mathcal{M}'(t) \leq -c\varepsilon J\left(\mathcal{M}(t)\right). \tag{4.69}$$

Noting that $K'=-1/J$ (see (3.24)), we get from (4.69)

$$\mathcal{M}'(t)\, K'\left(\mathcal{M}(t)\right) \geq c\varepsilon, \quad \forall t \geq 0.$$

A simple integration over $(0,t)$ then yields

$$K\left(\mathcal{M}(t)\right) \geq K\left(\mathcal{M}(0)\right)+c\varepsilon t.$$

Then, since K^{-1} is decreasing , we deduce that

$$\begin{aligned}\mathcal{M}(t) &\leq K^{-1}\left(c\varepsilon t+K\left(\mathcal{M}(0)\right)\right)\\ &\leq K^{-1}\left(c_2 t+c_3\right).\end{aligned}$$

From this latter inequality and (4.68) we obtain easily (4.59). Then the proof is completed. □

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INEXTENSIBLE FLOWS OF CURVES WITH QUASI-FRAME IN 3-DIMENSIONAL GALILEAN SPACE G_3

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Abstract. In this study we research inextensible flows of curves in 3-dimensional Galilean space G_3 with a new aspect. For this research we use a new adapted frame which called quasi-frame in 3-dimensional Galilean space G_3. From this perspective, inextensible curve flows are examined with the help of this frame then important characterizations and results are obtained.

Keywords: Galilean space, inextensible flows of curves, quasi frame.

2010 Mathematics Subject Classification: 53A35.

1. Introduction

The theory of curves is one of the most intriguing and thoroughly studied topics in differential geometry. Additionally, curve flows, which determine the evolution of curves or surfaces, are crucial to this theory. In this case, the curve's flow can be used to analyze the change in the curve. It is argued that a curve has an inextensible flow if the arc length is preserved. In addition to structural mechanics [19], computer vision [9, 14], and computer animation [3] all use the inextensible flows of curves and surfaces. The techniques researched in this paper are produced by Gage and Hamilton [7] and Grayson [8]. Kwon and Park offered a thorough description of the differences between heat flows and inextensible flows of planar curves [12]. Furthermore, in $\mathbb{R}^3$, Kwon et al. reveals a general formulation for developable surfaces and

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inextensible flows of curves [11]. Latifi and Razavi examined inextensible flows of curves in Minkowski 3-space [13]. Inextensible flows of curves were analyzed by Ögrenmis and Yeneroğlu [15] in the three-dimensional Galilean space G_3 and by Öztekin and Gün Bozok [16] in the four-dimensional Galilean space.

In the literature, computations have often been performed using the Frenet frame. Nevertheless, in certain situations, the Frenet frame has drawbacks. For instance, it is impossible to define the Frenet frame when the second derivative is zero. An alternative frame can therefore be defined in this situation. The frame known as quasi frame or q-frame is one of these alternative frames. Using the quasi-normal vector established by Coquillart in 1987 [1], a q-frame was obtained. This frame's principal concept is that the projection and tangent vectors are multiplied to obtain the vector known as the quasi-normal vector. Using a quasi-normal vector along a space curve, Dede et al. defined a new frame known as the q-frame [2]. With the help of these definitions, the quasi frame has been examined for different curves in many different spaces [5, 6, 10, 18].

In this research paper, with the help of the quasi frame inextensible flows of curves are researched in 3-dimensional Galilean space G_3. In this regards, new characterizations and important results have been obtained for inextensible curve flows.

2. Preliminaries

The Galilean space is one of the Cayley-Klein spaces with the projective metric with the signature (0,0,+,+). In 3-dimensional Galilean space denoted as G_3 the scalar product is descripted by

$$\langle w_1, w_2\rangle = \begin{cases} x_1x_2 & , if\ x_1 \neq 0 \vee x_2 \neq 0 \\ y_1y_2 + z_1z_2 & , if\ x_1 = 0 \wedge x_2 = 0 \end{cases} \tag{2.1}$$

where $w_1 = (x_1, y_1, z_1)$ and $w_2 = (x_2, y_2, z_2)$. Consideringly for a vector $w = (x, y, z)$ the Galilean norm can be expressed by

$$\|w\| = \begin{cases} x & , \ if\, x \neq 0 \\ \sqrt{y^2 + z^2} & , if\, x = 0 \end{cases} . \tag{2.2}$$

For an admissible curve C of the class C^r $(r \geq 3)$ in G_3 the following characterization can be defined

$$r = r\left(s, y\left(s\right), z\left(s\right)\right), \tag{2.3}$$

here s is the arc length on C. Also for this curve the curvature and torsion can be represented as

$$\kappa(s)=\sqrt{y''^2+z''^2} \text{ and } \tau(s)=\frac{1}{\kappa^2(s)}\det\left(r'(s),r''(s),r'''(s)\right). \tag{2.4}$$

The orthonormal trihedron is expressed by

$$\begin{aligned} T(s) &= \left(1,y'(s),z'(s)\right), \\ N(s) &= \frac{1}{\kappa(s)}\left(0,y''(s),z''(s)\right), \\ B(s) &= \frac{1}{\kappa(s)}\left(0,-z''(s),y''(s)\right), \end{aligned} \tag{2.5}$$

where t,n,b are the tangent, principal normal and binormal vectors, respectively. Moreover, the Frenet formulas can be given by,

$$\begin{aligned} T'(s) &= \kappa(s)N(s), \\ N'(s) &= \tau(s)B(s), \\ B'(s) &= -\tau(s)N(s). \end{aligned} \tag{2.6}$$

For detailed information about Galilean space we refer to [20, 17].

In 3-dimensional Galilean space the quasi-frame, which is crucial to a variety of geometric computations is derived from Frenet-Serret frame of the curve and can be described by $\{T(s),N_q(s),B_q(s)\}$ as;

$$T=\frac{\alpha'}{\|\alpha'\|},\quad N_q=\frac{T\times z}{\|T\times z\|},\quad B_q=T\times N_q, \tag{2.7}$$

where z is the projection vector given by either $(1,0,0)$, $(0,1,0)$ or $(0,0,1)$. The parallelism respect to unit tangent vector T determines the choice of the projection vector z. Here it is selected $z=(1,0,0)$. Let $\theta(s)$ is an angle between N and N_q then the quasi-frame, known as $\{T(s),N_q(s),B_q(s)\}$ can be written

$$N_q=\cos\theta N+\sin\theta B, \tag{2.8}$$

$$B_q=-\sin\theta N+\cos\theta B, \tag{2.9}$$

and

$$N=\cos\theta N_q-\sin\theta B_q, \tag{2.10}$$

$$B=\sin\theta N_q+\cos\theta B_q. \tag{2.11}$$

Also using the equations (2.6) and (2.10), it is obtained that

$$T' = \kappa N = \kappa \cos\theta N_q - \kappa \sin\theta B_q. \tag{2.12}$$

By using the replacement $K_1 = \kappa\cos\theta$ and $K_2 = \kappa\sin\theta$, the following equation can be found,

$$T' = K_1 N_q - K_2 B_q. \tag{2.13}$$

In the same way considering the equations (2.8) and (2.9), it is determined that

$$N_q' = K_3 B_q, \quad B_q' = -K_3 N_q, \tag{2.14}$$

where $\theta' + \tau = K_3$. Therefore, the quasi-formulas are given by

$$\begin{aligned} T' &= K_1 N_q - K_2 B_q, \\ N_q' &= K_3 B_q, \\ B_q' &= -K_3 N_q. \end{aligned} \tag{2.15}$$

Consequently the quasi-curvatures K_1, K_2 and K_3 can be represented as

$$K_1 = \kappa\cos\theta, \quad K_2 = \kappa\sin\theta, \quad K_3 = \theta' + \tau, \tag{2.16}$$

where κ, τ are curvature and torsion, respectively [4].

Corollary 2.1. *Let $\alpha(s)$ be a curve in G_3. The quasi-curvatures K_1, K_2 and K_3 can be given, respectively, by* [4]

$$K_1 = g(T', N_q), K_2 = -g(T', B_q), K_3 = g(N_q', B_q) = -g(B_q', N_q). \tag{2.17}$$

Corollary 2.2. *The quasi-frame in the value of G_3, is a generalization of the Frenet frame. To be more precise, the quasi-frame and the Frenet frame are equal when K_2 equals zero* [4].

Example 2.1. *Let $\beta : I \longrightarrow G_3$ be a curve defined as*

$$\beta(s) = (s, \sin 3s, \cos 3s).$$

Then the quasi frame of β is

$$\begin{aligned} T &= (1, 3\cos 3s, -3\sin 3s), \\ N_q &= (0, -\sin 3s, -\cos 3s), \\ B_q &= (0, \cos 3s, -\sin 3s), \end{aligned}$$

and quasi-curvatures are

$$K_1 = 9 \ K_2 = 0 \ K_3 = -3.$$

3. Inextensible Flows of Curves according to Quasi-Frame in G_3

According to this research, $\beta : [0, l] \times [0, w) \longrightarrow G_3$ is considered as a one parameter family of smooth curves in 3-dimensional Galilean space G_3 where l is the arc length of the initial curve. Moreover, u is the curve parameterization variable where $0 \leq u \leq l$. If the speed of the curve β is denoted as $v = \left|\frac{\partial \beta}{\partial u}\right|$, then the arc length of β can be represented as

$$s(u) = \int_0^u \left|\frac{\partial \beta}{\partial u}\right| du\,, \tag{3.18}$$

and $\frac{\partial}{\partial s}$ can be expressed by

$$\frac{\partial}{\partial s} = \frac{1}{v}\frac{\partial}{\partial u}\,,$$

here the arc length parameter is $ds = vdu$. For any flow in G_3 the following equation can be written

$$\frac{\partial \beta}{\partial t} = f_1 T + f_2 N_q + f_3 B_q, \tag{3.19}$$

where $\{T, N_q, B_q\}$ is quasi-frame in G_3. The arc length variation is given by

$$s(u,t) = \int_0^u vdu\ .$$

In G_3, the requirement that the curve not be subject to either elongation or compression can be given as the following condition

$$\frac{\partial}{\partial t} s(u,t) = \int_0^u \frac{\partial v}{\partial t} du = 0\,, \tag{3.20}$$

for $u \in [0, l]$ [15].

Definition 3.1. *A curve evolution $\beta(u,t)$ and its flow $\frac{\partial \beta}{\partial t}$ in G_3 are called inextensible if the following equation is satisfied* [15],

$$\frac{\partial}{\partial t}\left|\frac{\partial \beta}{\partial u}\right| = 0.$$

Lemma 3.1. *Let $\frac{\partial \beta}{\partial t} = f_1 T + f_2 N_q + f_3 B_q$ be a smooth flow of the curve β where $\{T, N_q, B_q\}$ is a quasi-frame in G_3. The flow is inextensible then*

$$\frac{\partial v}{\partial t} = \frac{\partial f_1}{\partial u}. \tag{3.21}$$

Proof. Let $\frac{\partial \beta}{\partial t}$ be a smooth flow of the curve β in G_3. Using the definition of β, we reach

$$v^2 = \left\langle \frac{\partial \beta}{\partial u}, \frac{\partial \beta}{\partial u} \right\rangle. \tag{3.22}$$

Since $\dfrac{\partial}{\partial u}$ and $\dfrac{\partial}{\partial t}$ are commute, we get

$$v\frac{\partial v}{\partial t}=\left\langle \frac{\partial \beta}{\partial u},\frac{\partial}{\partial u}\left(f_1T+f_2N_q+f_3B_q\right)\right\rangle. \tag{3.23}$$

Substituting (2.15) in (3.23) we reach

$$\begin{aligned}\frac{\partial v}{\partial t} &= \left\langle T,\left(\frac{\partial f_1}{\partial u}\right)T+\left(\frac{\partial f_2}{\partial u}+f_1vK_1-f_3vK_3\right)N_q\right.\\ &+ \left.\left(\frac{\partial f_3}{\partial u}-f_1vK_2+f_2vK_3\right)B_q\right\rangle .\end{aligned}$$

If necessary calculations are done then the equation (3.21) can be obtained easily. □

Theorem 3.1. *Let* $\dfrac{\partial \beta}{\partial t}=f_1T+f_2N_q+f_3B_q$ *be a smooth flow of the curve* β *in* G_3*. The flow is inextensible if and only if*

$$\frac{\partial f_1}{\partial s}=0. \tag{3.24}$$

Proof. Considering the equations (3.20) and (3.21) we have

$$\frac{\partial}{\partial t}s\left(u,t\right)=\int_0^u\frac{\partial v}{\partial t}du=\int_0^u\frac{\partial f_1}{\partial u}=0. \tag{3.25}$$

The proof can be finished by reversing the argument to demonstrate sufficiency. Therefore, the desired result is obtained. □

We now limit ourselves to parametrized curves with arc length. In other words, $v=1$ and the local coordinate u corresponds to s which is the arc length of the curve. Therefore the following lemma can be given;

Lemma 3.2. *Let* $\dfrac{\partial \beta}{\partial t}=f_1T+f_2N_q+f_3B_q$ *be a smooth flow of the curve* β *in* G_3*. If the flow is inextensible then,*

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)N_q+\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)B_q,\\ \frac{\partial N_q}{\partial t} &= \psi B_q,\\ \frac{\partial B_q}{\partial t} &= -\psi N_q,\end{aligned} \tag{3.26}$$

where $\psi=\left\langle \frac{\partial N_q}{\partial t},B_q\right\rangle$.

Proof. Since $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial s}$ are commute we get

$$\frac{\partial T}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \beta}{\partial s}=\frac{\partial}{\partial s}\left(f_1T+f_2N_q+f_3B_q\right).$$

Thus, it can be seen that

$$\begin{aligned}\frac{\partial T}{\partial t} &= \left(\frac{\partial f_1}{\partial s}\right)T + \left(\frac{\partial f_2}{\partial s} + f_1K_1 - f_3K_3\right)N_q \\ &+ \left(\frac{\partial f_3}{\partial s} + f_2K_3 - f_1K_2\right)B_q. \end{aligned} \tag{3.27}$$

Substitute the equation (3.24) in (3.27), we find

$$\frac{\partial T}{\partial t} = \left(\frac{\partial f_2}{\partial s} + f_1K_1 - f_3K_3\right)N_q + \left(\frac{\partial f_3}{\partial s} + f_2K_3 - f_1K_2\right)B_q.$$

Let us differentiate the quasi-frame with respect to t as

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}\langle T, N_q\rangle = \left(\frac{\partial f_2}{\partial s} + f_1K_1 - f_3K_3\right) + \left\langle T, \frac{\partial N_q}{\partial t}\right\rangle, \\ 0 &= \frac{\partial}{\partial t}\langle T, B_q\rangle = \left(\frac{\partial f_3}{\partial s} + f_2K_3 - f_1K_2\right) + \left\langle T, \frac{\partial B_q}{\partial t}\right\rangle, \\ 0 &= \frac{\partial}{\partial t}\langle N_q, B_q\rangle = \left\langle \frac{\partial N_q}{\partial t}, B_q\right\rangle + \left\langle N_q, \frac{\partial B_q}{\partial t}\right\rangle. \end{aligned}$$

Considering the above equation and the following equations

$$\left\langle \frac{\partial N_q}{\partial t}, N_q\right\rangle = \left\langle \frac{\partial B_q}{\partial t}, B_q\right\rangle = 0,$$

then, we obtain

$$\begin{aligned} \frac{\partial N_q}{\partial t} &= \psi B_q, \\ \frac{\partial B_q}{\partial t} &= -\psi N_q, \end{aligned}$$

where $\psi = \left\langle \frac{\partial N_q}{\partial t}, B_q\right\rangle$. □

Theorem 3.2. *Let* $\frac{\partial \beta}{\partial t} = f_1T + f_2N_q + f_3B_q$ *be a smooth flow of the curve* β *in* G_3*. If the flow is inextensible, the following partial differential equation holds:*

$$\begin{aligned} \frac{\partial K_1}{\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial f_2}{\partial s} + f_1K_1 - f_3K_3\right) - \left(\frac{\partial f_3}{\partial s} + f_2K_3 - f_1K_2\right)K_3 - K_2\psi, \\ \frac{\partial K_2}{\partial t} &= K_1\psi - \frac{\partial}{\partial s}\left(\frac{\partial f_3}{\partial s} + f_2K_3 - f_1K_2\right)K_3 - \left(\frac{\partial f_2}{\partial s} + f_1K_1 - f_3K_3\right)K_3, \\ \frac{\partial K_3}{\partial t} &= \frac{\partial \psi}{\partial s}, \end{aligned}$$

where $\psi = \left\langle \frac{\partial N_q}{\partial t}, B_q\right\rangle$.

Proof.

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial s}\left[\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)N_q+\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)B_q\right],\\ &= \frac{\partial}{\partial s}\left[\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)\right]N_q+\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)K_3B_q\\ &+ \frac{\partial}{\partial s}\left[\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)\right]B_q\\ &- \left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)K_3N_q. \end{aligned} \tag{3.28}$$

Also, we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial T}{\partial s} &= \frac{\partial}{\partial t}\left(K_1N_q-K_2B_q\right), \qquad (3.29)\\ &= \frac{\partial K_1}{\partial t}N_q+K_1\psi B_q-\frac{\partial K_2}{\partial t}B_q+K_2\psi N_q,\end{aligned}$$

where $\psi=\left\langle \frac{\partial N_q}{\partial t},B_q\right\rangle$. Hence from (3.28) and (3.29), we get

$$\begin{aligned}\frac{\partial K_1}{\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)-\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)K_3-K_2\psi,\\ \frac{\partial K_2}{\partial t} &= K_1\psi-\frac{\partial}{\partial s}\left(\frac{\partial f_3}{\partial s}+f_2K_3-f_1K_2\right)-\left(\frac{\partial f_2}{\partial s}+f_1K_1-f_3K_3\right)K_3.\end{aligned}$$

Also,

$$\begin{aligned}\frac{\partial}{\partial s}\frac{\partial N_q}{\partial t} &= \frac{\partial}{\partial s}\left(\psi B_q\right)\\ &= \frac{\partial \psi}{\partial s}B_q-K_3\psi N_q.\end{aligned} \tag{3.30}$$

Also, we have

$$\begin{aligned}\frac{\partial}{\partial t}\frac{\partial N_q}{\partial s} &= \frac{\partial}{\partial t}\left(K_3B_q\right)\\ &= \frac{\partial K_3}{\partial t}B_q-K_3\psi N_q.\end{aligned} \tag{3.31}$$

Hence from (3.30) and (3.31), we get

$$\frac{\partial K_3}{\partial t}=\frac{\partial \psi}{\partial s}. \tag{3.32}$$

□

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STUDY OF SOME HARMONICITY PROBLEMS CONCERNING THE RESCALED SASAKI METRIC

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Abstract. This paper introduces the concept of harmonicity on the tangent bundle endowed with a rescaled Sasaki metric. Firstly, we study the harmonicity of a vector field on the tangent bundle. Secondly, we investigate the harmonicity of the composition of a vector field and mapping between Riemannian manifolds. Afterwards, we explore the harmonicity of composition between the natural projection of a Riemannian manifold and a map of this manifold to another. Finally, we investigate the harmonicity of the tangent map.

Keywords:Riemannian manifold, tangent bundle, rescaled Sasaki metric, harmonicity.

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1. Introduction

The tangent bundle of a Riemannian manifold can be endowed with Riemannian metrics defined from the Riemannian metric of the base manifold. The most famous of these is the Sasaki metric [19]. Several authors have studied the geometry of the tangent bundle endowed with the Sasaki metric (see [21, 4, 18]). Some authors have constructed other metrics on tangent bundles, which represent deformations of the Sasaki metric on tangent bundles (see [2, 11, 16]). The rescaled metric is between the deformations of the Sasaki metric on the tangent bundle, which have been studied and developed in several recent studies (see [10, 20, 22]).

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The main objective of this research is to investigate the harmonicity concerning the Rescaled Sasaki metric on the tangent bundle. After stating the introduction, we describe the preliminary results of the tangent bundle and basic properties of the Rescaled Sasaki metric. In section 3, We give certain harmonic problems of a vector field concerning this metric. (Theorem 3.2, Theorem 3.3 and Theorem 3.6). In section 4, we investigate the harmonicity of the composition of a vector field and mapping between Riemannian manifolds (Theorem 4.1 and Theorem 4.2). Next, in section 5, we explore the harmonicity of composition between the natural projection of a Riemannian manifold and a map of this manifold to another (Theorem 5.1 and Theorem 5.2). In the last section, we examine the harmonicity of the tangent map (Theorem 6.1, Theorem 6.2 and Theorem 6.3).

2. Preliminary Results

Consider the k-dimensional Riemannian manifold M^k endowed with the Riemannian metric g and the bundle projection (natural projection) $\pi : TM^k \to M^k$. The local coordinates (U, x^i), $i = 1, \ldots, k$ on M^k induces on TM^k a system of local coordinates $(\pi^{-1}(U), x^i, v^i = x^{\bar{i}})$, $i = 1, \ldots, k$ on TM^k. Denote by ∇ the Levi-Civita connection of g and by Γ_{ij}^s the Christoffel symbols of ∇. Let $\Im_0^1(M^k)$ be the module of C^∞ vector fields on M^k over the ring of real-valued C^∞ functions on M^k. There is a direct sum decomposition defined by the Levi Civita connection ∇.

$$T_{(x,v)}TM^k = V_{(x,v)}TM^k \oplus H_{(x,v)}TM^k$$

of the tangent bundle to TM^k at all $(x, v) \in TM^k$ into the vertical distribution

$$V_{(x,v)}TM^k = Ker(d\pi_{(x,v)}) = \{\alpha^i \frac{\partial}{\partial v^i}|_{(x,v)},\, \alpha^i \in \mathbb{R}\}$$

and the horizontal distribution

$$H_{(x,v)}TM^k = \{\alpha^i \frac{\partial}{\partial x^i}|_{(x,v)} - \alpha^i v^j \Gamma_{ij}^s \frac{\partial}{\partial v^s}|_{(x,v)},\, \alpha^i \in \mathbb{R}\}.$$

Given a vector field $Z = Z^i \frac{\partial}{\partial x^i}$ on M^k. The vertical and horizontal lifts of Z are defined by:

$$\begin{aligned} {}^V Z &= Z^i \frac{\partial}{\partial v^i}, \\ {}^H Z &= Z^i (\frac{\partial}{\partial x^i} - v^j \Gamma_{ij}^s \frac{\partial}{\partial v^s}). \end{aligned}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - v^j \Gamma_{ij}^k \frac{\partial}{\partial v^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial v^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=1,\ldots,k}$ is a local adapted frame on TTM^k.

Definition 2.1. [20] *Given a Riemannian manifold (M^k, g) and a strictly positive smooth functions $f : M^k \to]0; +\infty[$. The Rescaled Sasaki metric on the tangent bundle TM^k of M^k is defined by:*

$$\begin{aligned} G^f({}^HX, {}^HY)_{(x,v)} &= f(x)g_x(X, Y), \\ G^f({}^VX, {}^HY)_{(x,v)} &= G^f({}^HX, {}^VY)_{(x,v)} = 0, \\ G^f({}^VX, {}^VY)_{(x,v)} &= g_x(X, Y), \end{aligned}$$

for all vector fields X and Y on M^k and $(x, v) \in TM^k$. Note that, if $f = 1$, then G^f is the Sasaki metric[19].

Theorem 2.1. [20] *Given a Riemannian manifold (M^k, g) and the Levi-Civita connection $\widetilde{\nabla}$ of the tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. Then we have the following formulas:*

$$\begin{aligned} 1.\, \widetilde{\nabla}_{{}^HX}{}^HY &= {}^H\big(\nabla_X Y + \frac{1}{2f}(X(f)Y + Y(f)X - g(X, Y)grad f)\big) - \frac{1}{2}{}^V(R(X, Y)v), \\ 2.\, \widetilde{\nabla}_{{}^HX}{}^VY &= \frac{1}{2f}{}^H\big(R(v, Y)X\big) + {}^V(\nabla_X Y), \\ 3.\, \widetilde{\nabla}_{{}^VX}{}^HY &= \frac{1}{2f}{}^H\big(R(v, X)Y\big), \\ 4.\, \widetilde{\nabla}_{{}^VX}{}^VY &= 0, \end{aligned}$$

for all vector fields X and Y on M^k, where R is the curvature tensor of ∇ on (M^k, g).

Now, we will introduce some basic concepts concerning harmonic maps. Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds. If ψ is a critical point of the energy functional,

$$E(\psi) = \int_K e(\psi)\, v_g, \tag{2.1}$$

the map ψ is called harmonic. for all compact domain $K \subseteq M^k$. Here

$$e(\psi) := \frac{1}{2}|d\psi|^2 = \frac{1}{2}Tr_g\, h(d\psi, d\psi) \tag{2.2}$$

is the energy density of ψ, $|d\psi|$ is the Hilbert-Schmitd norm of $d(\psi)$ and v_g is the Riemannian volume form on M^k. The first variation of the energy [13] is expressed by:

$$\frac{d}{dt}E(\psi_t)\Big|_{t=0} = -\int_K h(\tau(\psi), V)v_g, \tag{2.3}$$

for all smooth 1-parameter variation $\{\psi_t\}_{t\in I}$ of ψ and $V = \frac{d}{dt}\psi_t\Big|_{t=0}$.
Then, ψ is to be harmonic if and only if $\tau(\psi) = 0$, where

$$\tau(\psi) := Tr_g \nabla d\psi, \tag{2.4}$$

is called the tension field of ψ, see [6, 7, 8, 12, 14]. Recently, numerous authors have extensively explored this topic, including its application to the tangent bundle [1, 23].

3. Harmonicity of section $X : (M^k, g) \to (TM^k, G^f)$

Lemma 3.1. [14, 15] *Consider a Riemannian manifold (M^k, g). Then the following equation holds:*

$$d_xY(X_x) = {}^HX_{(x,v)} + {}^V(\nabla_X Y)_{(x,v)} \tag{3.5}$$

for all vector fields X, Y on M^k and $(x, v) \in TM^k$, where $Y_x = v$.

Lemma 3.2. [24] *Consider a Riemannian manifold (M^k, g). Then the following equation holds:*

$$g(\bar{\Delta}Z, Z) = |\nabla Z|^2 - \frac{1}{2}\Delta|Z|^2, \tag{3.6}$$

for all vector field Z on M^k, where $\bar{\Delta}Z := -Tr_g\nabla^2 Z = -Tr_g(\nabla_*\nabla_* - \nabla_{\nabla_{**}})Z$ denotes the rough Laplacian of Z.

Lemma 3.3. [24] *Consider a Riemannian manifold (M^k, g). Then the following equation holds:*

$$\bar{\Delta}(\rho Z) = \rho\bar{\Delta}Z - (\Delta\rho)Z - 2\nabla_{grad\rho}Z, \tag{3.7}$$

for all vector field Z on M^k, where ρ is a smooth function of M^k.

Lemma 3.4. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. If Z is a vector field on M^k, its corresponding energy density is expressed by:*

$$e(Z) \;=\; \frac{kf}{2} + \frac{1}{2}|\nabla Z|^2. \tag{3.8}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k and $(x,v)\in TM^k$ such that $Z_x=v$, from (2.2), we have:

$$\begin{aligned} e(Z)_x &= \frac{1}{2}|d_xZ|^2 \\ &= \frac{1}{2}Tr_g\, G^f(dZ,dZ)_{(x,v)} \\ &= \frac{1}{2}\sum_{i=1}^{k} G^f(dZ(E_i),dZ(E_i))_{(x,v)}. \end{aligned}$$

Using (3.5), we obtain:

$$\begin{aligned} e(Z) &= \frac{1}{2}\sum_{i=1}^{k} G^f({}^HE_i+{}^V(\nabla_{E_i}Z),{}^HE_i+{}^V(\nabla_{E_i}Z)) \\ &= \frac{1}{2}\sum_{i=1}^{k}\big(G^f({}^HE_i,{}^HE_i)+G^f({}^V(\nabla_{E_i}Z),{}^V(\nabla_{E_i}Z))\big) \\ &= \frac{1}{2}\sum_{i=1}^{k}\big(fg(E_i,E_i)+g(\nabla_{E_i}Z,\nabla_{E_i}Z)\big) \\ &= \frac{kf}{2}+\frac{1}{2}|\nabla Z|^2. \end{aligned}$$

□

Theorem 3.1. *Given a Riemannian manifold (M^k,g) and its tangent bundle (TM^k,G^f) endowed with the Rescaled Sasaki metric. If Z is a vector field on M^k, its associated tension field is expressed by:*

$$\tau(Z) = {}^H\Big(\frac{2-k}{2f}grad f+\frac{1}{f}Tr_g(R(Z,\nabla_*Z)*)\Big)-{}^V\bar{\Delta}Z. \tag{3.9}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k and $(x,v)\in TM^k$ such that $(\nabla_{E_i}E_i)_x=0$ and $Z_x=v$. From (2.4) and (3.5), we have:

$$\begin{aligned} \tau(Z)_x &= \big(Tr_g\nabla dZ\big)_x \\ &= \sum_{i=1}^{k}\big(\nabla^Z_{E_i}dZ(E_i)\big)_x \\ &= \sum_{i=1}^{k}\big(\widetilde{\nabla}_{dZ(E_i)}dZ(E_i)\big)_{(x,v)} \\ &= \sum_{i=1}^{k}\big(\widetilde{\nabla}_{({}^HE_i+{}^V(\nabla_{E_i}Z))}({}^HE_i+{}^V(\nabla_{E_i}Z))\big)_{(x,v)} \\ &= \sum_{i=1}^{k}\big(\widetilde{\nabla}_{{}^HE_i}{}^HE_i+\widetilde{\nabla}_{{}^HE_i}{}^V(\nabla_{E_i}Z)+\widetilde{\nabla}_{{}^V(\nabla_{E_i}Z)}{}^HE_i+\widetilde{\nabla}_{{}^V(\nabla_{E_i}Z)}{}^V(\nabla_{E_i}Z)\big)_{(x,v)}. \end{aligned}$$

Using Theorem 2.1, we obtain:

$$\begin{aligned}\tau(Z) &= \sum_{i=1}^{k}\Big({}^{H}\big(\nabla_{E_i}E_i+\frac{1}{2f}(2E_i(f)E_i-g(E_i,E_i)gradf)\big)-\frac{1}{2}{}^{V}(R(E_i,E_i)Z)\\ &\quad+\frac{1}{2f}{}^{H}(R(Z,\nabla_{E_i}Z)E_i)+{}^{V}(\nabla_{E_i}\nabla_{E_i}Z)+\frac{1}{2f}{}^{H}(R(Z,\nabla_{E_i}Z)E_i)\\ &= \sum_{i=1}^{k}\Big(\frac{1}{2f}{}^{H}\big(2E_i(f)E_i-gradf\big)+\frac{1}{f}{}^{H}(R(Z,\nabla_{E_i}Z)E_i)+{}^{V}(\nabla_{E_i}\nabla_{E_i}Z)\Big).\end{aligned}$$

To simplify the last statement, we use the following equations:

$$\begin{aligned}\sum_{i=1}^{k}\frac{1}{f}R(Z,\nabla_{E_i}Z)E_i &= \frac{1}{f}Tr_g(R(Z,\nabla_* Z)*),\\ \sum_{i=1}^{k}\nabla_{E_i}\nabla_{E_i}Z &= -\bar{\Delta}Z.\end{aligned}$$

Then, we find (3.9). □

From Theorem 3.1 we get the following:

Theorem 3.2. *Given a Riemannian manifold (M^k,g) and its tangent bundle (TM^k,G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is a harmonic map if and only if the following conditions hold:*

$$Tr_g(R(Z,\nabla_* Z)*)=\frac{k-2}{2}gradf \quad \text{and} \quad \bar{\Delta}Z=0.$$

Corollary 3.1. *Given a Riemannian manifold (M^k,g) and its tangent bundle (TM^k,G^f) endowed with the Rescaled Sasaki metric. If $k=2$ or $f=constant$, then every vector field that is parallel on M^k is harmonic map.*

Given a compact oriented Riemannian manifold (M^k,g) and its tangent bundle (TM^k,G^f) endowed with the Rescaled Sasaki metric. Let Z a vector field on M^k. The energy $E(Z)$ of Z is defined to be the energy of the corresponding map $Z:(M^k,g)\to(TM^k,G^f)$[5]. More precisely, from (3.8), we get

$$E(Z)=\frac{k}{2}\int_{M^k}f\,v_g+\frac{1}{2}\int_{M^k}|\nabla Z|^2 v_g. \tag{3.10}$$

Definition 3.1. [5, 17] *Given a Riemannian manifold (M^k,g) and its tangent bundle (TM^k,G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is called harmonic vector field if the corresponding map $Z:(M^k,g)\to(TM^k,G^f)$ is a critical point for the energy functional E, only considering variations between maps defined by vector fields.*

Theorem 3.3. *Given a Riemannian manifold* (M^k, g) *and its tangent bundle* (TM^k, G^f) *endowed with the Rescaled Sasaki metric. Let* Z *a vector field on* M^k *and* $E : \Im^1_0(M^k) \to [0, +\infty)$ *the energy functional restricted to the space of all vector fields. Then the following condition holds:*

$$\frac{d}{dt}E(Z_t)\big|_{t=0} = \int_{M^k} g(\bar{\Delta}Z, V)v_g \tag{3.11}$$

for all smooth 1*-parameter variation* $\vartheta : M^k \times (-\epsilon, \epsilon) \to TM^k$ *of* Z *i.e.* $\vartheta(x, 0) = Z(x)$, $Z_t(x) = \vartheta(x, t) \in T_xM^k$ *for all* $(x, t) \in M^k \times (-\epsilon, \epsilon)$, $(\epsilon > 0)$ *and* $V \in \Im^1_0(M^k)$ *is the vector field on* M^k *expressed by:*

$$V(x) = \frac{d}{dt}Z_x(t)\Big|_{t=0} = \lim_{t\to 0}\frac{1}{t}(\vartheta(x, t) - \vartheta(x, 0)),$$

where $Z_x(t) = \vartheta(x, t), (x, t) \in M^k \times (-\epsilon, \epsilon)$.

Proof. Given a smooth 1-parameter variation $\vartheta : M^k \times (-\epsilon, \epsilon) \to TM^k$ of Z, such that $Z_t(x) = \vartheta(x, t) \in T_xM^k$ for all $(x, t) \in M^k \times (-\epsilon, \epsilon)$ and $\vartheta(x, 0) = Z(x)$. From (2.1), we have

$$E(Z_t) = \int_{M^k} e(Z_t)v_g.$$

Using (2.3), we get:

$$\frac{d}{dt}E(Z_t)\Big|_{t=0} = -\int_{M^k} G^f(\mathcal{V}, \tau(Z))v_g, \tag{3.12}$$

where $\mathcal{V}$ is the infinitesimal variation induced by ϑ, i.e.,

$$\mathcal{V}(x) = d_{(x,0)}\vartheta(0, \frac{d}{dt})\big|_{t=0} = \frac{d}{dt}Z_t(x)\big|_{t=0}.$$

It is well known that

$$\mathcal{V} = {}^V V \circ Z, \tag{3.13}$$

see [5, p.58]. Finally, by (3.9), (3.12) and (3.13), we find:

$$\begin{aligned}\frac{d}{dt}E(Z_t)\Big|_{t=0} &= -\int_{M^k} G^f({}^V V, \tau(Z))v_g \\ &= -\int_{M^k} G^f({}^V V, -{}^V\bar{\Delta}Z)v_g \\ &= \int_{M^k} g(\bar{\Delta}Z, V)v_g.\end{aligned}$$

□

If (M^k, g) is a non-compact oriented Riemannian manifold, then Theorem 3.3 holds. In fact, if M^k is non-compact, we can choose $V \in \Im_0^1(M^k)$ which support is contained in an open subset W in M^k whose closure is compact. Then (3.11) is as follows:

$$\frac{d}{dt}E(Z_t)\big|_{t=0} = \int_W g(\bar{\Delta}Z, V)v_g.$$

We derive from this a necessary and sufficient condition for a vector field is a harmonic vector field or harmonic map, respectively.

Theorem 3.4. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is harmonic vector field if and only if $\bar{\Delta}Z = 0$.*

Using Theorem 3.2 and Theorem 3.4, we obtain the following:

Theorem 3.5. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is harmonic map if and only if Z is harmonic vector field and*

$$Tr_g(R(Z, \nabla_* Z)*) = \frac{k-2}{2} gradf.$$

It is clear that, all parallel vector field on M^k is harmonic vector field. Conversely, we have:

Theorem 3.6. *Given a compact oriented Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. A vector field Z on M^k is harmonic vector field if and only if Z is parallel.*

Proof. We suppose that Z is a harmonic vector field on M^k, from Theorem 3.4, we find $\bar{\Delta}Z = 0$. By (3.6), we obtain $|\nabla Z|^2 = \frac{1}{2}\Delta|Z|^2$. Applying the divergence Theorem, we get

$$\int_{M^k} |\nabla Z|^2 v^g = 0.$$

Since $|\nabla Z|^2$ is a positive function, hence $\nabla Z = 0$. □

Theorem 3.7. *Given a compact oriented Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. If $k = 2$ or $f = constant$, a vector field Z on M^k is harmonic map if and only if Z is parallel.*

Example 3.1. *Let $\mathbb{R}^2$ endowed with the Riemannian metric in polar coordinate defined by:*

$$g = dr^2 + r^2 d\theta^2.$$

Relatively to the orthonormal frame

$$E_1 = \partial_r, \quad E_2 = \frac{1}{r}\partial_\theta$$

we have,

$$\nabla_{E_1}E_1 = \nabla_{E_1}E_2 = 0, \quad \nabla_{E_2}E_1 = \frac{1}{r}E_2, \quad \nabla_{E_2}E_2 = -\frac{1}{r}E_1,$$
$$R(E_1, E_2)E_1 = R(E_1, E_2)E_2 = 0.$$

Let $Z = \alpha(r)E_1$ be a vector field, where α is a smooth real function. Using simple calculations, we find:

$$Tr_g(R(Z, \nabla_* Z)*) = 0,$$
$$\bar{\Delta}E_1 = \frac{1}{r^2}E_1,$$

Using (3.7)*, we obtain:*

$$\bar{\Delta}Z = (-\alpha'' - \frac{1}{r}\alpha' + \frac{1}{r^2}\alpha)E_1.$$

i) From Theorem 3.4, we conclude that Z is harmonic vector field equivalently $\bar{\Delta}Z = 0$, then

$$-\alpha'' - \frac{1}{r}\alpha' + \frac{1}{r^2}\alpha = 0. \tag{3.14}$$

The general solution of differential equation (3.14) *is*

$$\alpha(r) = c_1 r + \frac{c_2}{r},$$

where, c_1 and c_2 be real constants.
ii) Since $k = 2$ and $Tr_g(R(Z, \nabla_* Z)*) = 0$, from Theorem 3.5, the vector fields $Z = (c_1 r + \frac{c_2}{r})E_1$ are also harmonic maps.
iii) However the vector field $Y = (r - \frac{1}{r})E_1$ is harmonic but non parallel, because $\nabla_{E_1}Y = (1 + \frac{1}{r^2})E_1 \neq 0$.

Example 3.2. *Consider $\mathbb{R}^2$ with the Riemannian metric*

$$g = e^{2y}dx^2 + dy^2.$$

Relatively to the orthonormal frame $\{E_1 = e^{-y}\partial_x, E_2 = \partial_y\}$ *with respect to* g, *we have:*

$$\nabla_{E_1}E_1 = -E_2,\ \nabla_{E_1}E_2 = E_1,\ \nabla_{E_2}E_1 = \nabla_{E_2}E_2 = 0,$$

$$R(E_1, E_2)E_1 = E_2,\ R(E_1, E_2)E_2 = -E_1.$$

Let $Z = \rho(y)E_2$ *be a vector field, where* ρ *is a smooth non-zero real function. According to simple calculations, we find:*

$$Tr_g(R(Z, \nabla_* Z)*) = -\rho^2 E_2,$$

$$\bar{\Delta}Z = (-\rho'' - \rho' + \rho)E_2.$$

i) From Theorem 3.4, we conclude that $Z = \rho(y)E_2$ *is harmonic vector field if and only if*

$$-\rho'' - \rho' + \rho = 0. \tag{3.15}$$

The general solution of differential equation (3.15) *is*

$$\rho(y) = ae^{\frac{-1-\sqrt{5}}{2}y} + be^{\frac{-1+\sqrt{5}}{2}y},$$

where a *and* b *be non-zero real constants at the same time.*
ii) Since $k = 2$ *and* $Tr_g(R(Z, \nabla_* Z)*) \neq 0$, *from Theorem 3.5, the vector fields* $Z = (ae^{\frac{-1-\sqrt{5}}{2}y} + be^{\frac{-1+\sqrt{5}}{2}y})E_2$ *are never harmonic maps.*
On the other hand,we have $\nabla_{E_1}Z = \rho\, E_1 = (ae^{\frac{-1-\sqrt{5}}{2}y} + be^{\frac{-1+\sqrt{5}}{2}y})E_1 \neq 0$, *then the vector fields* Z *is non parallel, i.e. the vector fields* Z *are harmonic vector fields but neither harmonic maps nor parallel vector fields.*

Example 3.3. *The torus* $T^2 = \mathbb{S}^1 \times \mathbb{S}^1$ *(* 2*-dimensional compact oriented Riemannian manifold) endowed with the metric:*

$$g = \frac{4}{(1 + x^2)^2}dx^2 + \frac{4}{(1 + y^2)^2}dy^2.$$

Let $Z = f_1(x, y)\partial_x + f_2(x, y)\partial_y$ *be a vector field, where* f_1 *and* f_2 *are smooth functions. According to simple calculations, we find:*

$$\nabla_{\partial_x}\partial_x = \frac{-2x}{1 + x^2}\partial_x,\ \nabla_{\partial_x}\partial_y = \nabla_{\partial_y}\partial_x = 0,\ \nabla_{\partial_y}\partial_y = \frac{-2y}{1 + y^2}\partial_y,$$

$$Tr_g(R(X, \nabla_* X)*) = 0.$$

i) From Theorem 3.6, we conclude that Z is harmonic vector field $\Leftrightarrow \nabla Z = 0$

$$\Leftrightarrow \begin{cases} \nabla_{\partial_x} Z = 0 \\ \nabla_{\partial_y} Z = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} (\frac{\partial f_1}{\partial x} - \frac{2x}{1+x^2} f_1)\partial_x + \frac{\partial f_2}{\partial x}\partial_y = 0 \\ \frac{\partial f_1}{\partial y}\partial_x + (\frac{\partial f_2}{\partial y} - \frac{2y}{1+y^2} f_2)\partial_y = 0 \end{cases}$$

We conclude that the function f_1 is only dependent on x, while the function f_2 is only dependent on y, so we find the following system:

$$\begin{cases} \frac{\partial f_1}{\partial_x} - \frac{2x}{1+x^2} f_1 = 0 \\ \frac{\partial f_2}{\partial_y} - \frac{2y}{1+y^2} f_2 = 0. \end{cases}$$

The general solution for this system is

$$f_1(x) = c_1(1+x^2), \ f_2(y) = c_2(1+y^2)$$

where, c_1 and c_2 be real constants.

ii) Since $k = 2$ and $Tr_g(R(Z, \nabla_* Z)*) = 0$, from Theorem 3.6, we deduce that $Z = c_1(1+x^2)\partial_x + c_2(1+y^2)\partial_y$ are also harmonic maps.

4. Harmonicity of the composition of a vector field and mapping between Riemannian manifolds

Lemma 4.1. [23] *Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds, a vector field Y on N^n and the smooth map ξ defined by $\xi := Y \circ \psi$. Then*

$$d\xi(X) = {}^H(d\psi(X)) + {}^V(\nabla^\psi_X \xi), \tag{4.16}$$

for all vector field X on M^k, where ∇^ψ is the pull-back connection on the pull-back bundle $\psi^{-1}TN^n$.

Theorem 4.1. *Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds and the tangent bundle (TN^n, H^f) of N^n endowed with the Rescaled Sasaki metric. Then the tension field of the map $\xi := Y \circ \psi$ is expressed by:*

$$\begin{aligned} \tau(\xi) \quad = \quad & {}^H\big(\tau(\psi) + \frac{1}{f} d\psi(grad^{M^k}(f\circ\psi)) - \frac{1}{2f}|d\psi|^2 grad^{N^n} f + \frac{1}{f} Tr_g(R^{N^n}(\xi, \nabla^\psi_* \xi) d\psi(*))\big) \\ & -{}^V\big(\Delta^\psi \xi\big), \end{aligned}$$

where $\Delta^\psi \xi := -Tr_g(\nabla^\psi)^2\xi = -Tr_g(\nabla^\psi_* \nabla^\psi_* - \nabla^\psi_{\nabla_* *})\xi$ is the rough Laplacian of ξ on $\psi^{-1}TN^n$.

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k such that $(\nabla_{E_i}E_i)_x = 0$ and $\xi(x) = (\psi(x), Y_{\psi(x)}), Y_{\psi(x)} = v \in T_{\psi(x)}N^n$.

Using (4.16), we have:

$$\begin{aligned}
\tau(\xi)_x &= (Tr_g \nabla d\xi)_x \\
&= \sum_{i=1}^{k} (\nabla^{\xi}_{E_i} d\xi(E_i))_x \\
&= \sum_{i=1}^{k} \nabla^{TN^n}_{d\xi(E_i)} d\xi(E_i))_{(\psi(x),v)} \\
&= \sum_{i=1}^{k} \big(\nabla^{TN^n}_{({}^H(d\psi(E_i))+{}^V(\nabla^{\psi}_{E_i}\xi))} ({}^H(d\psi(E_i)) + {}^V(\nabla^{\psi}_{E_i}\xi))\big)_{(\psi(x),v)} \\
&= \sum_{i=1}^{k} \big(\nabla^{TN^n}_{{}^H(d\psi(E_i))}{}^H(d\psi(E_i)) + \nabla^{TN^n}_{{}^H(d\psi(E_i))}{}^V(\nabla^{\psi}_{E_i}\xi) + \nabla^{TN^n}_{{}^V(\nabla^{\psi}_{E_i}\xi)}{}^H(d\psi(E_i)) \\
&\quad + \nabla^{TN^n}_{{}^V(\nabla^{\psi}_{E_i}\xi)}{}^V(\nabla^{\psi}_{E_i}\xi)\big)_{(\psi(x),v)}.
\end{aligned}$$

From Theorem 2.1, we obtain:

$$\begin{aligned}
\tau(\xi) &= \sum_{i=1}^{k} \Big({}^H\big(\nabla^{N^n}_{d\psi(E_i)} d\psi(E_i) + \frac{1}{2f}(2d\psi(E_i)(f)d\psi(E_i) - h(d\psi(E_i), d\psi(E_i))grad^{N^n} f)\big) \\
&\quad + \frac{1}{f}{}^H(R^{N^n}(\xi, \nabla^{\psi}_{E_i}\xi)d\psi(E_i)) + {}^V(\nabla^{N^n}_{d\psi(E_i)}\nabla^{\psi}_{E_i}\xi)\Big) \\
&= \sum_{i=1}^{k} \Big({}^H\big(\nabla^{\psi}_{E_i} d\psi(E_i) + \frac{1}{2f}(2E_i(f\circ\psi)d\psi(E_i) - h(d\psi(E_i), d\psi(E_i))grad^{N^n} f)\big) \\
&\quad + \frac{1}{f}R^{N^n}(\xi, \nabla^{\psi}_{E_i}\xi)d\psi(E_i)\big) + {}^V\big(\nabla^{\psi}_{E_i}\nabla^{\psi}_{E_i}\xi\big)\Big) \\
&= {}^H\big(\tau(\psi) + \frac{1}{f}d\psi(grad^{M^k}(f\circ\psi)) - \frac{1}{2f}|d\psi|^2 grad^{N^n} f + \frac{1}{f}Tr_g(R^{N^n}(\xi, \nabla^{\psi}_{*}\xi)d\psi(*))\big) \\
&\quad - {}^V\big(\Delta^{\psi}\xi\big).
\end{aligned}$$

□

From Theorem 4.1 we obtain:

Theorem 4.2. *Given a smooth map $\psi : (M^k, g) \to (N^n, h)$ between two Riemannian manifolds and (TN^n, H^f) be the tangent bundle of N^n endowed with the Rescaled Sasaki metric. Then the map $\xi := Y \circ \psi$ is harmonic if and only if the following conditions hold:*

$$\begin{cases}
\tau(\psi) = -\frac{1}{f}Tr_g(R^{N^n}(\xi, \nabla^{\psi}_{*}\xi)d\psi(*)) - \frac{1}{f}d\psi(grad^{M^k}(f\circ\psi)) + \frac{1}{2f}|d\psi|^2 grad^{N^n} f, \\
\Delta^{\psi}\xi = 0.
\end{cases}$$

5. HARMONICITY OF COMPOSITION BETWEEN THE NATURAL PROJECTION AND A SMOOTH MAP

Lemma 5.1. *Given a Riemannian manifold (M^k, g) and its tangent bundle (TM^k, G^f) endowed with the Rescaled Sasaki metric. Then The tension field of the canonical projection $\pi : (TM^k, G^f) \to (M^k, g)$ is expressed by:*

$$\tau(\pi) = \frac{k-2}{2f^2}(gradf) \circ \pi. \tag{5.17}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\ldots,k}$ on M^k, then $\{\frac{1}{\sqrt{f_1}}{}^HE_i, {}^VE_j\}_{i,j=1,\ldots,k}$ forms a local orthonormal frame field on the tangent bundle TM^k. From Theorem 2.1, we find:

$$\begin{aligned}
\tau(\pi)_{(x,v)} &= (Tr_{G^f}\nabla d\pi)_{(x,v)} \\
&= \sum_{i=1}^{k}\Big(\nabla_{d\pi(\frac{1}{\sqrt{f}}{}^HE_i)}d\pi(\frac{1}{\sqrt{f}}{}^HE_i) - d\pi(\widetilde{\nabla}_{\frac{1}{\sqrt{f}}{}^HE_i}\frac{1}{\sqrt{f}}{}^HE_i)\Big)_{\pi(x,v)} \\
&\quad + \sum_{j=1}^{k}\Big(\nabla_{d\pi({}^VE_j)}d\pi({}^VE_j) - d\pi(\widetilde{\nabla}_{{}^VE_j}{}^VE_j)\Big)_{\pi(x,v)}.
\end{aligned}$$

But because $d\pi({}^VZ) = 0$ and $d\pi({}^HZ) = Z \circ \pi$, for each vector field Z on M^k, we find:

$$\begin{aligned}
\tau(\pi) &= \sum_{i=1}^{k}\Big(\nabla_{(\frac{1}{\sqrt{f}}E_i\circ\pi)}(\frac{1}{\sqrt{f}}E_i\circ\pi) - d\pi\big({}^H(\nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i)\big) \\
&\quad - \frac{1}{2f}d\pi\big({}^H(2\frac{1}{\sqrt{f}}E_i(f)\frac{1}{\sqrt{f}}E_i - g(\frac{1}{\sqrt{f}}E_i, \frac{1}{\sqrt{f}}E_i)gradf)\big)\Big) \circ \pi \\
&= \sum_{i=1}^{k}\Big(\nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i - \nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i - \frac{1}{2f^2}(2E_i(f)E_i - g(E_i, E_i)gradf) \circ \pi\Big) \circ \pi \\
&= \frac{k-2}{2f^2}(gradf) \circ \pi.
\end{aligned}$$

□

Theorem 5.1. *Given a smooth map between Riemannian manifolds $\psi : (M^k, g) \to (N^n, h)$. The tension field of the map $\zeta := \psi \circ \pi$ is expressed by:*

$$\tau(\zeta) = \frac{1}{f}(\tau(\psi) + \frac{k-2}{2f}d\psi(gradf)) \circ \pi. \tag{5.18}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\ldots,k}$ on M^k, then $\{\frac{1}{\sqrt{f_1}}{}^HE_i, {}^VE_j\}_{i,j=1,\ldots,k}$ forms a local orthonormal frame field on the tangent bundle TM^k. Then the tension field of the map $\zeta := \psi \circ \pi$ is given by [6, 8]

$$\tau(\zeta) = \tau(\psi \circ \pi) = d\psi(\tau(\pi)) + Tr_{G^f}\nabla d\psi(d\pi, d\pi),$$

then, we have:

$$\begin{aligned}
Tr_{G^f}\nabla d\psi(d\pi,d\pi) &= \sum_{i=1}^{k}\Big(\nabla^{N^n}_{d\psi(d\pi(\frac{1}{\sqrt{f}}{}^HE_i))}d\psi(d\pi(\frac{1}{\sqrt{f}}{}^HE_i)) - d\psi(\nabla_{d\pi(\frac{1}{\sqrt{f}}{}^HE_i)}d\pi(\frac{1}{\sqrt{f}}{}^HE_i))\Big)\circ\pi \\
&+ \sum_{j=1}^{k}\Big(\nabla^{N^n}_{d\psi(d\pi({}^VE_j))}d\psi(d\pi({}^VE_j)) - d\psi(\nabla_{d\pi({}^VE_j)}d\pi({}^VE_j))\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\nabla^{\psi}_{(\frac{1}{\sqrt{f}}E_i)\circ\pi}d\psi((\frac{1}{\sqrt{f}}E_i)\circ\pi) - d\psi\big(\nabla_{(\frac{1}{\sqrt{f}}E_i)\circ\pi}(\frac{1}{\sqrt{f}}E_i)\circ\pi\big)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\nabla^{\psi}_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}d\psi(E_i) - d\psi\big(\nabla_{\frac{1}{\sqrt{f}}E_i}\frac{1}{\sqrt{f}}E_i\big)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\frac{1}{\sqrt{f}}E_i(\frac{1}{\sqrt{f}})d\psi(E_i) + \frac{1}{f}\nabla^{\psi}_{E_i}d\psi(E_i) \\
&- d\psi(\frac{1}{\sqrt{f}}E_i(\frac{1}{\sqrt{f}})E_i + \frac{1}{f}\nabla_{E_i}E_i)\Big)\circ\pi \\
&= \sum_{i=1}^{k}\Big(\frac{1}{f}\nabla^{\psi}_{E_i}d\psi(E_i) - \frac{1}{f}d\psi\big(\nabla_{E_i}E_i\big)\Big)\circ\pi \\
&= \frac{1}{f}\tau(\psi)\circ\pi.
\end{aligned}$$

Using (5.17), we obtain (5.18). □

Theorem 5.2. *Given a smooth map* $\psi : (M^k, g) \to (N^n, h)$ *between two Riemannian manifolds. Then the map* $\zeta := \psi\circ\pi$ *is harmonic if and only if*

$$\tau(\psi) = \frac{2-n}{2f}d\psi(grad f).$$

6. Harmonicity of the tangent map

Given Riemannian manifolds (M^k, g), (N^n, h) and their tangent bundles (TM^k, G^{f_1}), (TN^n, H^{f_2}) respectively, equipped with the Rescaled Sasaki metrics, such that f_1, f_2 are strictly positive smooth functions on M^k, N^n respectively.

Lemma 6.1. [9] *Given a smooth map* $\psi : (M^k, g) \to (N^n, h)$ *between two Riemannian manifolds and its tangent map*

$$\begin{aligned}
\Psi = d\psi : TM^k &\longrightarrow TN^n \\
(x, v) &\longmapsto (\psi(x), d\psi(v))
\end{aligned}$$

we have:

$$\begin{cases} d\Psi({}^{V}X) = {}^{V}\big(d\psi(X)\big), \\ d\Psi({}^{H}X) = {}^{H}\big(d\psi(X)\big) + {}^{V}\big(\nabla d\psi(x,v)\big), \end{cases}$$

for all vector field X on M^k.

Theorem 6.1. *Given a smooth map between Riemannian manifolds $\psi : (M^k, g) \to (N^n, h)$, then the tension field associated to the tangent map $\Psi : (TM^k, G^{f_1}) \longrightarrow (TN^n, H^{f_2})$ is given by:*

$$\begin{aligned} \tau(\Psi) &= {}^{H}\Big(\frac{1}{f_1}\tau(\psi) + \frac{k-2}{2f_1^2} d\psi(grad^{M^k} f_1) + \frac{1}{f_1 f_2} d\psi(grad^{M^k}(f_2\circ\psi)) \\ &\quad - \frac{1}{2f_1 f_2}|d\psi|^2 grad^{N^n} f_2 + \frac{1}{f_1 f_2} Tr_g(R^{N^n}(d\psi(u), \nabla d\psi(*,u))d\psi(*))\Big) \\ &\quad + {}^{V}\Big(\frac{k-2}{2f_1^2}\nabla d\psi(grad^{M^k} f_1, u) + \frac{1}{f_1} Tr_g(\nabla^{\psi}_{*}(\nabla d\psi(*,u)))\Big). \end{aligned}$$

Proof. Given a local orthonormal frame field $\{E_i\}_{i=1,\dots,k}$ on M^k such that $(\nabla^{M^k}_{E_i} E_i)_x = 0$, then $\big\{\frac{1}{\sqrt{f_1}}{}^{H}E_i, {}^{V}E_j\big\}_{i,j=1,\dots,k}$ forms a local orthonormal frame field on TM^k, we have:

$$\begin{aligned} \tau(\Psi)_{(x,v)} &= \sum_{i=1}^{k}\big(\nabla^{\Psi}_{\frac{1}{\sqrt{f_1}}{}^{H}E_i} d\Psi(\frac{1}{\sqrt{f_1}}{}^{H}E_i) - d\Psi(\nabla^{TM^k}_{\frac{1}{\sqrt{f_1}}{}^{H}E_i}\frac{1}{\sqrt{f_1}}{}^{H}E_i)\big)_{(\psi(x),d\psi(u))} \\ &\quad + \sum_{j=1}^{n}\big(\nabla^{\Psi}_{{}^{V}E_j} d\Psi({}^{V}E_j) - d\Psi(\nabla^{TM^k}_{{}^{V}E_j}{}^{V}E_j)\big)_{(\psi(x),d\psi(u))} \\ &= \Big(\sum_{i=1}^{k}\big(\frac{1}{\sqrt{f_1}}{}^{H}E_i(\frac{1}{\sqrt{f_1}})d\Psi({}^{H}E_i) + \frac{1}{f_1}\nabla^{\Psi}_{{}^{H}E_i} d\Psi({}^{H}E_i) + \sum_{j=1}^{n}\big(\nabla^{TN^n}_{d\Psi({}^{V}E_j)} d\Psi({}^{V}E_j)\big) \\ &\quad - d\Psi\big(\frac{1}{\sqrt{f_1}}{}^{H}E_i(\frac{1}{\sqrt{f_1}}){}^{H}E_i + \frac{1}{f_1}\nabla^{TM^k}_{{}^{H}E_i}{}^{H}E_i)\big)\Big)_{(\psi(x),d\psi(u))}. \end{aligned}$$

Using Theorem 2.1 and Lemma 6.1, we obtain:

$$\begin{aligned} \tau(\Psi) &= \frac{1}{f_1}\sum_{i=1}^{k}\Big(\nabla^{TN^n}_{d\Psi({}^{H}E_i)} d\Psi({}^{H}E_i) - d\Psi\big(\nabla^{TM^k}_{{}^{H}E_i}{}^{H}E_i\big)\Big) + \sum_{j=1}^{n}\big(\nabla^{TN^n}_{d\Psi({}^{V}E_j)} d\Psi({}^{V}E_j)\big) \\ &= \frac{1}{f_1}\sum_{i=1}^{k}\Big(\nabla^{TN^n}_{{}^{H}(d\psi(E_i))}{}^{H}(d\psi(E_i)) + \nabla^{TN^n}_{{}^{H}(d\psi(E_i))}{}^{V}(\nabla d\psi(E_i,u)) + \nabla^{TN^n}_{{}^{V}(\nabla d\psi(E_i,u))}{}^{H}(d\psi(E_i)) \\ &\quad + \nabla^{TN^n}_{{}^{V}(\nabla d\psi(E_i,u))}{}^{V}(\nabla d\psi(E_i,u)) - d\Psi\big({}^{H}\big(\frac{1}{2f_1}(2E_i(f_1)E_i - g(E_i,E_i)grad^{M^k} f_1)\big)\big)\Big) \\ &\quad + \sum_{j=1}^{n}\big(\nabla^{TN^n}_{{}^{V}(d\psi(E_j))}{}^{V}(d\psi(E_j))\big) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{f_1}\sum_{i=1}^{k}\Big({}^{H}\big(\nabla^{N^n}_{d\psi(E_i)}d\psi(E_i) + \frac{1}{2f_2}(2d\psi(E_i)(f_2)d\psi(E_i) - h(d\psi(E_i), d\psi(E_i))grad^{N^n}f_2)\big)\\
&\quad +\frac{1}{f_2}{}^{H}(R^{N^n}(d\psi(u), \nabla d\psi(E_i,u))d\psi(E_i)) + {}^{V}(\nabla^{N^n}_{d\psi(E_i)}\nabla d\psi(E_i,u))\Big)\\
&\quad +\frac{k-2}{2f_1^2}d\Psi\big({}^{H}\big(grad^{M^k}f_1)\big)\\
&= {}^{H}\big(\frac{1}{f_1}\tau(\psi) + \frac{k-2}{2f_1^2}d\psi(grad^{M^k}f_1) + \frac{1}{f_1f_2}d\psi(grad^{M^k}(f_2\circ\psi))\\
&\quad -\frac{1}{2f_1f_2}|d\psi|^2grad^{N^n}f_2 + \frac{1}{f_1f_2}Tr_g(R^{N^n}(d\psi(u), \nabla d\psi(*,u))d\psi(*))\big)\\
&\quad +{}^{V}\big(\frac{k-2}{2f_1^2}\nabla d\psi(grad^{M^k}f_1,u) + \frac{1}{f_1}Tr_g(\nabla^{\psi}_{*}(\nabla d\psi(*,u)))\big).
\end{aligned}
$$

□

Theorem 6.2. *Given a smooth map between Riemannian manifolds $\psi : (M^k, g) \to (N^n, h)$, then the tangent map $\Psi : (TM^k, G^{f_1}) \longrightarrow (TN^n, H^{f_2})$ is harmonic if and only if*

$$
\begin{aligned}
\tau(\psi) &= \frac{2-k}{2f_1}d\psi(grad^{M^k}f_1) - \frac{1}{f_2}d\psi(grad^{M^k}(f_2\circ\psi)) + \frac{1}{2f_2}|d\psi|^2grad^{N^n}f_2\\
&\quad -\frac{1}{f_2}Tr_g(R^{N^n}(d\psi(u), \nabla d\psi(*,u))d\psi(*))\big),
\end{aligned}
$$

and

$$
Tr_g\nabla^{\psi}_{*}(\nabla d\psi(u,*)) = \frac{2-n}{2f_1}\nabla d\psi(grad^{M^k}f_1, u).
$$

Theorem 6.3. *If $\psi : (M^k, g) \to (N^n, h)$ is totally geodesic, then the tangent map $\Psi : (TM^k, G^{f_1}) \longrightarrow (TN^n, H^{f_2})$ is harmonic if and only if*

$$
\tau(\psi) = \frac{2-k}{2f_1}d\psi(grad^{M^k}f_1) - \frac{1}{f_2}d\psi(grad^{M^k}(f_2\circ\psi)) + \frac{1}{2f_2}|d\psi|^2grad^{N^n}f_2.
$$

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EXPLORING THE RECIPROCAL FUNCTIONAL EQUATIONS: APPROXIMATIONS IN DIVERSE SPACES

IDIR SADANI *

Abstract. In this study, we explore the generalized Hyers-Ulam-Rassias stability of a specific reciprocal-type functional equation. The equation is given by

$$\Omega(2u+v)+\Omega(2u-v)=\frac{2\Omega(u)\Omega(v)\displaystyle\sum_{\substack{k=0\\ k\text{ is even}}}^{l} 2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega(v)^{\frac{l-k}{l}}}{\left(4\Omega(v)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\right)^{l}}$$

and we consider its behavior in both non-zero real and non-Archimedean spaces. Additionally, an appropriate counter-example is provided to demonstrate the failure of the stability result in the singular case.

Keywords: Reciprocal functional equation, non-Archimedean space, non-zero real space, approximations, Cauchy sequence, functional inequality, generalized Hyers-Ulam stability, convergence.

2010 Mathematics Subject Classification: 39B52, 39B72.

1. Introduction

The exploration of the stability of functional equations began with Ulam's [20] famous question at a Mathematical Colloquium held at the University of Wisconsin in 1940. In the following year, Hyers [9] presented a partial solution to Ulam's question. Subsequently, Th.M. Rassias [11], Aoki [1], J.M. Rassias [12], and Găvruţa [8] expanded and generalized Hyers's findings to include the Cauchy functional equation in various adaptations.

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In 2010, Ravi and Senthil Kumar [13] studied the stability of the reciprocal type functional equation

$$f(x+y)=\frac{f(x)f(y)}{f(x)+f(y)},$$

where $f:X\to\mathbb{R}$ is a mapping with X as the space of non-zero real numbers.

In 2014, Kim and Bodaghi [2] introduced and studied the stability of the quadratic reciprocal functional equation

$$f(2x+y)+f(2x-y)=\frac{2f(x)f(y)[4f(y)+f(x)]}{(4f(y)-f(x))^2}.$$

In 2017, Kim et al. [10] introduced and studied the stability of the reciprocal-cubic functional equation

$$c(2x+y)+c(x+2y)=\frac{9c(x)c(y)[c(x)+c(y)+2c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}(c(x)^{\frac{1}{3}}+c(y)^{\frac{1}{3}})]}{[2c(x)^{\frac{2}{3}}+2c(y)^{\frac{2}{3}}+5c(x)^{\frac{1}{3}}c(y)^{\frac{1}{3}}]^3}$$

and the reciprocal-quartic functional equation

$$q(2x+y)+q(2x-y)=\frac{2q(x)q(y)[q(x)+16q(y)+24\sqrt{q(x)q(y)}]}{[4\sqrt{q(y)}-\sqrt{q(x)}]^4}$$

in non-Archimedean fields.

In the same year, Bodaghi and Senthil Kumar [4] introduced and obtained the stability of the following reciprocal-quintic functional equation

$$q(2x+y)+q(2x-y)=\frac{4q(x)q(y)[16q(y)+40q(x)^{\frac{2}{5}}q(y)^{\frac{3}{5}}+5q(x)^{\frac{4}{5}}q(y)^{\frac{1}{5}}]}{[4q(y)^{\frac{2}{5}}-q(x)^{\frac{2}{5}}]^5}$$

and reciprocal-sextic functional equation

$$s(2x+y)+s(2x-y)=\frac{2s(x)s(y)[s(x)+60s(x)^{\frac{2}{3}}s(y)^{\frac{1}{3}}+240s(x)^{\frac{1}{3}}+64s(y)]}{[4s(y)^{\frac{1}{3}}-s(x)^{\frac{1}{3}}]^6}.$$

In 2020, Bodaghi et al [6] considered the following reciprocal-nonic functional equation

$$n(2x+y)+n(2x-y)=\frac{4n(x)n(y)}{(4n(y)^{\frac{2}{9}}-n(x)^{\frac{2}{9}})^9}\Big[256n(y)+2304n(x)^{\frac{2}{9}}n(y)^{\frac{7}{9}}+2016n(x)^{\frac{4}{9}}n(y)^{\frac{5}{9}} +336n(x)^{\frac{6}{9}}n(y)^{\frac{3}{9}}+n(x)^{\frac{8}{9}}n(y)^{\frac{1}{9}}\Big]$$

and the reciprocal-decic functional equation

$$d(2x+y)+d(2x-y)=\frac{2d(x)d(y)}{(4d(y)^{\frac{1}{5}}-d(x)^{\frac{1}{5}})^{10}}\Big[1024d(y)+11520d(x)^{\frac{1}{5}}d(y)^{\frac{4}{5}}+13440d(x)^{\frac{2}{5}}d(y)^{\frac{3}{5}} +3360d(x)^{\frac{3}{5}}d(y)^{\frac{2}{5}}+180d(x)^{\frac{4}{5}}d(y)^{\frac{1}{5}}+d(x)\Big]$$

and obtained various stability results in non-Archimedean fields and some proper examples for their non-stabilities.

The other results pertaining to the stability of different reciprocal-type functional equations can be found, for instance, in [5, 19, 14, 15, 3, 16, 17, 18].

In this study, we introduce the following $l-$power reciprocal functional equation

$$\Omega(2u+v)+\Omega(2u-v)=\frac{2\Omega(u)\Omega(v)\sum\limits_{\substack{k=0\\ k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega(v)^{\frac{l-k}{l}}}{\left(4\Omega(v)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\right)^{l}}, \tag{1.1}$$

then, we examine the general solution and its various stability results in non-zero real numbers and in non-Archimedean fields with a proper example for their non-stability.

2. General solution of (1.1)

This section provides the solution for the functional equation (1.1). Assume R^* denotes the set of non-zero real numbers.

We begin with the following lemma.

Lemma 2.1. *Let $a\in\mathbb{N}^*$. Then, we have*

$$\frac{(-1)^l(a-2)^l+(a+2)^l}{2(a)^l}=\sum_{\substack{k=0\\ k\text{ is even}}}^{l}\left(\frac{2}{a}\right)^{l-k}\binom{l}{k}. \tag{2.2}$$

Proof. Let us prove it by mathematical induction. First, for $l=0$, we get

$$\frac{(-1)^0(a-2)^0+(a+2)^0}{2a^0}=\frac{1+1}{2}=\sum_{\substack{k=0\\ k\text{ even}}}^{0}\left(\frac{2}{a}\right)^{0-k}\binom{0}{k}=\binom{0}{0}=1.$$

The statement is true for $l=0$. Next, we assume that for $l=n$, it is true, i.e.

$$\frac{(-1)^n(a-2)^n+(a+2)^n}{2a^n}=\sum_{\substack{k=0\\ k\text{ even}}}^{n}\left(\frac{2}{a}\right)^{n-k}\binom{n}{k}. \tag{2.3}$$

We must now prove that the formula holds for $l=n+1$, i.e.

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\sum_{\substack{k=0\\ k\text{ even}}}^{n+1}\left(\frac{2}{a}\right)^{n+1-k}\binom{n+1}{k}. \tag{2.4}$$

To do this, we use the binomial theorem to obtain

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\frac{(-1)^{n+1}}{2a^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}a^k(-1)^{n+1-k}2^{n+1-k}$$
$$+\frac{1}{2a^{n+1}}\sum_{k=0}^{n+1}\binom{n+1}{k}a^k2^{n+1-k}. \tag{2.5}$$

Next, by simplifications, we get

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{a^k 2^{n+1-k}}{2a^{n+1}}((-1)^{-k}+1). \tag{2.6}$$

Finally, since k is even, we obtain

$$\frac{(-1)^{n+1}(a-2)^{n+1}+(a+2)^{n+1}}{2a^{n+1}}=\sum_{k=0}^{n+1}\binom{n+1}{k}\left(\frac{2}{a}\right)^{n+1-k}. \tag{2.7}$$

The statement is true when $l=n+1$. Hence, by the principle of mathematical induction, the statement is true for all $l\geq 0$. □

Theorem 2.1. *Let $f:\mathbb{R}^*\to\mathbb{R}$ be a continuous function fulfilling the equation (1.1). Assuming $\Omega(u)\neq 0$ and $4\Omega(v)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\neq 0$ for all $u,v\in\mathbb{R}^*$. Then f takes the form*

$$\Omega(u)=\frac{c}{u^l},\ \textit{for all } u\in\mathbb{R}^*,$$

where $c\neq 0$.

Proof. Assuming $f:\mathbb{R}^*\to\mathbb{R}$ satisfies the functional equation (1.1). Substituting (u,v) by (u,u) in (1.1), yields

$$\Omega(3u)+\Omega(u)=\frac{2\Omega(u)\displaystyle\sum_{\substack{k=0\\k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}}{3^l}.$$

Setting $a=1$ in (2.2), gives

$$\sum_{\substack{k=0\\k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}=\frac{3^l+1}{2}. \tag{2.8}$$

Hence

$$\Omega(3u)=\frac{1}{3^l}\Omega(u),\quad\text{for all } u\in\mathbb{R}^*. \tag{2.9}$$

By induction, we prove that for all $k\in\mathbb{N}^*$,

$$\Omega(ku)=\frac{1}{k^l}\Omega(u). \tag{2.10}$$

Assuming this is true for $k\in\{1,2,\dots,n-1\}$ we prove it for $k=n$. To do this, replacing (u,v) with $(u,(n-2)u)$ in (1.1), we get

$$\Omega(nu)+\Omega(-(n-4)u)=\frac{2\Omega(u)\Omega((n-2)u)\displaystyle\sum_{\substack{k=0\\k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega((n-2)u)^{\frac{l-k}{l}}}{\left(4\Omega((n-2)u)^{\frac{2}{l}}-\Omega(u)^{\frac{2}{l}}\right)^l}. \tag{2.11}$$

Using the recurrence hypothesis:

$$\Omega(nu)+\frac{1}{(n-4)^l}\Omega(-u)=\frac{2\frac{1}{(n-2)^l}\Omega(u)\displaystyle\sum_{\substack{k=0\\ k\text{ is even}}}^{l}\frac{2^{l-k}}{(n-2)^{l-k}}\binom{l}{k}}{\left(4\frac{1}{(n-2)^2}-1\right)^l}$$

then,

$$\Omega(nu)+\frac{1}{(n-4)^l}\Omega(-u)=\frac{2(n-2)^l\Omega(u)\displaystyle\sum_{\substack{k=0\\ k\text{ is even}}}^{l}\frac{2^{l-k}}{(n-2)^{l-k}}\binom{l}{k}}{(-n(n-4))^l}.$$

By taking $a=n-2$ in (2.2), we get

$$(-1)^l n^l(n-4)^l\Omega(nu)+(-1)^l n^l\Omega(-u)=\left((-1)^l(n-4)^l+n^l\right)\Omega(u). \tag{2.12}$$

Now, replacing (u,v) by $(u,(n-3)u)$ in (1.1),

$$\Omega((n-1)u)+\Omega(-(n-5)u)=\frac{2\frac{1}{(n-3)^l}\Omega(u)\displaystyle\sum_{\substack{k=0\\ k\text{ is even}}}^{l}2^{l-k}\binom{l}{k}\frac{1}{(n-3)^{l-k}}}{\left(4\frac{1}{(n-3)^2}-1\right)^l}. \tag{2.13}$$

Using the recurrence hypothesis and by taking $a=n-3$ in (2.2), a simple calculation gives

$$(-1)^l(n-5)^l\Omega(u)+(-1)^l(n-1)^l\Omega(-u)=\left((-1)^l(n-5)^l+(n-1)^l\right)\Omega(u).$$

This implies that

$$(-1)^l\Omega(-u)=\Omega(u). \tag{2.14}$$

By using (2.14) in (2.12), we get

$$(-1)^l n^l(n-4)^l\Omega(nu)+n^l\Omega(u)=\left((-1)^l(n-4)^l+n^l\right)\Omega(u).$$

Then,

$$\Omega(nu)=\frac{1}{n^l}\Omega(u).$$

Thus, the formula (2.10) is true for $k=n$. Therefore Ω is of the form $\frac{c}{u^l}$. □

3. Stability of (1.1) in $\mathbb{R}^*$

For convenience, we introduce the operator $\Lambda : \mathbb{R}^* \to \mathbb{R}$ as

$$\Lambda(u,v) = \Omega(2u+v) + \Omega(2u-v) - \frac{2\Omega(u)\Omega(v) \displaystyle\sum_{\substack{k=0 \\ k \text{ is even}}}^{l} 2^{l-k}\binom{l}{k}\Omega(u)^{\frac{k}{l}}\Omega(v)^{\frac{l-k}{l}}}{\left(4\Omega(v)^{\frac{2}{l}} - \Omega(u)^{\frac{2}{l}}\right)^{l}},$$

for all $u, v \in \mathbb{R}^*$. We are now ready to present our first main result, as follows.

Theorem 3.1. *Let $Q : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}$ be a function fulfilling*

$$\sum_{s=0}^{\infty} \frac{1}{3^{ls}} Q\left(\frac{u}{3^{s+1}}, \frac{v}{3^{s+1}}\right) < \infty \tag{3.15}$$

for all $u, v \in \mathbb{R}^*$. If $\Omega : \mathbb{R}^* \to \mathbb{R}$ fulfilling

$$|\Lambda(u,v)| \le Q(u,v) \tag{3.16}$$

for all $u, v \in \mathbb{R}^*$, then there is a uniquely defined reciprocal function $G : \mathbb{R}^* \to \mathbb{R}$ that fulfilling (1.1) and the inequality

$$|\Omega(u) - G(u)| \le \sum_{s=0}^{\infty} \frac{1}{3^{ls}} Q\left(\frac{u}{3^{s+1}}, \frac{u}{3^{s+1}}\right), \text{ for all } u \in \mathbb{R}^*. \tag{3.17}$$

Proof. We substitute (u, v) by (u, u) in (3.16) and using (2.8) we get

$$\left|\Omega(3u) - \frac{\Omega(u)}{3^l}\right| \le Q(u,u) \tag{3.18}$$

for all $u \in \mathbb{R}^*$. Substituting u by $\frac{u}{3}$ in (3.18), we obtain

$$\left|\Omega(u) - \frac{1}{3^l}\Omega\left(\frac{u}{3}\right)\right| \le Q\left(\frac{u}{3}, \frac{u}{3}\right) \tag{3.19}$$

for all $u \in \mathbb{R}^*$. Now, by setting $u = \frac{u}{3}$ in (3.19), dividing by 3^l, and then adding the resulting inequality to (3.19), we obtain

$$\left|\Omega(u) - \frac{1}{3^{2l}}\Omega\left(\frac{u}{3^2}\right)\right| \le Q\left(\frac{u}{3}, \frac{u}{3}\right) + \frac{1}{3^l}Q\left(\frac{u}{3^2}, \frac{u}{3^2}\right), \text{ for all } u \in \mathbb{R}^*. \tag{3.20}$$

Similarly, by continuing this process and applying induction on a positive integer m, we obtain

$$\left|\Omega(u) - \frac{1}{3^{ml}}\Omega\left(\frac{u}{3^m}\right)\right| \le \sum_{s=0}^{m-1} \frac{1}{3^{ls}} Q\left(\frac{u}{3^{s+1}}, \frac{u}{3^{s+1}}\right), \text{ for all } u \in \mathbb{R}^*. \tag{3.21}$$

Thereafter, if we choose any integers m' and m such that $m' > m > 0$, we obtain

$$\begin{aligned}
\left|\frac{1}{3^{lm'}}\Omega\left(\frac{u}{3^{m'}}\right) - \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^{m}}\right)\right| &= \left|\frac{1}{3^{lm'}}\Omega\left(\frac{u}{3^{m'}}\right) - \frac{1}{3^{l(m'-1)}}\Omega\left(\frac{u}{3^{l(m'-1)}}\right) + \cdots \right.\\
&\quad \left. + \frac{1}{3^{l(m'-1)}}\Omega\left(\frac{u}{3^{l(m'-1)}}\right) - \cdots + \frac{1}{3^{l(m+1)}}\Omega\left(\frac{u}{3^{m+1}}\right) - \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^{m}}\right)\right|\\
&\leq \frac{1}{3^{l(m'-1)}}Q\left(\frac{u}{3^{m'}}, \frac{u}{3^{m'}}\right) + \cdots + \frac{1}{3^{lm}}Q\left(\frac{u}{3^{m+1}}, \frac{u}{3^{m+1}}\right)\\
&\leq \sum_{j=m}^{m'-1} \frac{1}{3^{lj}}Q\left(\frac{u}{3^{j+1}}, \frac{u}{3^{j+1}}\right) \qquad (3.22)
\end{aligned}$$

for all $u \in \mathbb{R}^*$. Letting $m' \to \infty$ in (3.22) and we use (3.15), the sequence $\{\frac{1}{3^{lm}}\Omega(\frac{u}{3^m})\}$ is Cauchy for each $u \in \mathbb{R}^*$. We know that $\mathbb{R}$ is Banach, we can introduce $G : \mathbb{R}^* \to \mathbb{R}$ by $g(u) = \lim_{m\to\infty} \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)$. To prove that g fulfilling (1.1), substituting (u, v) by $(3^{-m}u, 3^{-m}v)$ in (3.16) and dividing by 3^{lm}, we arrive

$$|3^{-lm}\Lambda(3^{-m}u,\ 3^{-m}v)| \leq 3^{-lm}Q(3^{-m}u,\ 3^{-m}v), \forall u, v \in \mathbb{R}^* \text{ and } m \in \mathbb{N}^*. \qquad (3.23)$$

Taking $m \to \infty$ in (3.23) and by (3.15), we find that G fulfilling (1.1) for all $u, v \in \mathbb{R}^*$. One more, setting $m \to \infty$ in (3.21), we arrive at (3.17). Now, we need to demonstrate that G is unique. Suppose $G' : \mathbb{R}^* \to \mathbb{R}$ is another reciprocal mapping that also fulfilling (1.1) and (3.17). Clearly, we have $G'(3^{-m}u) = 3^{lm}G'(u), G(3^{-m}u) = 3^{lm}G(u)$ and utilizing (3.17), we obtain

$$\begin{aligned}
|G'(u) - G(u)| &= 3^{-lm}\left|G'(3^{-m}u) - G(3^{-m}u)\right| \qquad (3.24)\\
&\leq 3^{-lm}\left(\left|G'(3^{-m}u) - \Omega(3^{-m}u)\right| + \left|\Omega(3^{-m}u) - G(3^{-m}u)\right|\right)\\
&\leq 2\sum_{j=0}^{\infty} \frac{1}{3^{l(m+j)}}Q\left(\frac{u}{3^{m+j+1}},\ \frac{u}{3^{m+j+1}}\right)\\
&\leq 2\sum_{j=m}^{\infty} \frac{1}{3^{lj}}Q\left(\frac{u}{3^{j+1}}, \frac{u}{3^{j+1}}\right)
\end{aligned}$$

for all $u \in \mathbb{R}^*$. Letting $m \to \infty$ in (3.24), we obtain the unicity of G. □

The following corollaries are immediate consequences of Theorem 3.1.

Corollary 3.1. *Let $\Omega : \mathbb{R}^* \to \mathbb{R}$ be a mapping for which there exists $\epsilon > 0$ such that*

$$|\Lambda(u, v)| \leq \epsilon$$

holds for all $u, v \in \mathbb{R}^*$. Then,

$$G(u) = \lim_{m\to\infty} \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)$$

for all $u \in \mathbb{R}^*, m \in \mathbb{N}$ *and* $G : \mathbb{R}^* \to \mathbb{R}$ *is the unique mapping satisfying the reciprocal functional equation (1.1) such that*

$$|\Omega(u) - G(u)| \leq \frac{3^l}{3^l - 1}\epsilon$$

for every $u \in \mathbb{R}^*$.

Proof. By taking $Q(u, v) = \epsilon$ in Theorem 3.1 we arrive at the desired result. □

Corollary 3.2. *Let* $\epsilon > 0$ *and* $\alpha \neq -l$ *be real numbers, and* $\Omega : \mathbb{R}^* \to \mathbb{R}$ *be a mapping satisfying the functional inequality*

$$|\Lambda(u, v)| \leq \epsilon(|u|^\alpha + |v|^\alpha)$$

for all $u, v \in \mathbb{R}^*$. *Then, there exists a unique reciprocal mapping* $G : \mathbb{R}^* \to \mathbb{R}$ *satisfying the functional equation (1.1) and*

$$|\Omega(u) - G(u)| \leq \frac{2.3^l\epsilon}{3^{\alpha+l} - 1}|u|^\alpha$$

for all $u \in \mathbb{R}^*$.

Proof. By letting $Q(u, v) = \epsilon(|u|^\alpha + |v|^\alpha)$ for all $u, v \in \mathbb{R}^*$ in Theorem 3.1 we get the desired result. □

Corollary 3.3. *Let* $\epsilon > 0$ *and* $\alpha \neq -l$ *be real numbers, and* $\Omega : \mathbb{R}^* \to \mathbb{R}$ *be a mapping satisfying*

$$|\Lambda(u, v)| \leq \epsilon(|u|^{\frac{\alpha}{2}}|v|^{\frac{\alpha}{2}} + |u|^\alpha + |v|^\alpha)$$

for all $u, v \in \mathbb{R}^*$. *Then, there exists a unique reciprocal mapping* $G : \mathbb{R}^* \to \mathbb{R}$ *satisfying the functional equation (1.1) and*

$$|\Omega(u) - G(u)| \leq \frac{\epsilon 3^{l+1}}{3^{\alpha+l} - 1}|u|^\alpha$$

for all $u \in \mathbb{R}^*$.

Proof. By taking $Q(u, v) = \epsilon(|u|^{\frac{\alpha}{2}}|v|^{\frac{\alpha}{2}} + |u|^\alpha + |v|^\alpha)$ for all $u, v \in \mathbb{R}^*$ in Theorem 3.1 we get the desired result. □

Corollary 3.4. *Let* $\Omega : \mathbb{R}^* \to \mathbb{R}$ *be a mapping and there exist* p, q *with* $p + q \neq -l$. *If there exists* $\epsilon \geq 0$ *such that*

$$|\Lambda(u, v)| \leq \epsilon|u|^p|v|^q$$

for all $u, v \in \mathbb{R}^*$, then there exists a unique reciprocal mapping $G : \mathbb{R}^* \to \mathbb{R}$ satisfying the functional equation (1.1) and

$$|\Omega(u) - G(u)| \leq \frac{3^l \epsilon}{3^{p+q+l} - 1}|u|^{p+q}$$

for all $u \in \mathbb{R}^*$.

Proof. Letting $Q(u, v) = \epsilon|u|^p|v|^q$ for all $u, v \in \mathbb{R}^*$ in Theorem 3.1, we obtain the required result. □

4. Stability of (1.1) in non-Archimedean field

In this section, A and B denote a non-Archimedean field and a complete non-Archimedean field, respectively. For any non-Archimedean field A, let $A^* = A \setminus \{0\}$. Familiarity with non-Archimedean fields' properties is assumed.

The second main result can be stated as follows.

Theorem 4.1. *Let $\Upsilon : \mathbb{A}^* \times \mathbb{A}^* \to [0,\ \infty[$ be a mapping such that*

$$\lim_{m\to\infty} \left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^{m+1}},\ \frac{v}{3^{m+1}}\right) = 0, \quad \textit{for all } u, v \in \mathbb{A}^*. \tag{4.25}$$

Assuming that $g : \mathbb{A}^* \to \mathbb{B}$ is a mapping fulfilling the following

$$|\Lambda(u, v)| \leq \Upsilon(u, v), \quad \textit{for all } u, v \in \mathbb{A}^*. \tag{4.26}$$

Then, there is a uniquely defined reciprocal function $g : \mathbb{A}^* \to \mathbb{B}$ such that

$$|\Omega(u) - g(u)| \leq \max\left\{\left|\frac{1}{3^l}\right|^{k+1} \Upsilon\left(\frac{u}{3^{k+1}}, \frac{u}{3^{k+1}}\right) : k \in \mathbb{N} \cup \{0\}\right\}, \ \textit{for all } u \in \mathbb{A}^*. \tag{4.27}$$

Proof. Changing (u, v) to (u, u) in (4.26), one finds

$$\left|\Omega(u) - \frac{1}{3^l}\Omega\left(\frac{u}{3}\right)\right| \leq |3^l|\Upsilon\left(\frac{u}{3}, \frac{u}{3}\right) \tag{4.28}$$

for all $u \in \mathbb{A}^*$. Now, considering u as $\frac{u}{3^m}$ in (4.28) and multiplying by $\left|\frac{1}{3^l}\right|^m$, we get

$$\left|\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right) - \frac{1}{3^{l(m+1)}}\Omega\left(\frac{u}{3^{(m+1)}}\right)\right| \leq \left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^{m+1}}, \frac{u}{3^{m+1}}\right) \tag{4.29}$$

for all $u \in \mathbb{A}^*$. It is easy to obtain from the inequalities (4.25) and (4.29) that the sequence $\left\{\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^{lm}}\right)\right\}$ is Cauchy and converges to a well defined function g since $\mathbb{B}$ is complete. Then, put $g : \mathbb{A}^* \to \mathbb{B}$ as

$$g(u) = \lim_{m\to\infty} \frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right). \tag{4.30}$$

Furthermore, for every element $u \in \mathbb{A}^*$ and each nonnegative integers m, we have the following

$$\begin{aligned}\left|\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)-g(u)\right| &= \left|\sum_{k=0}^{m-1}\left[\frac{1}{3^{l(k+1)}}\Omega\left(\frac{u}{3^{(k+1)}}\right)-\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)\right]\right| \\ &\le \max\left\{\left|\frac{1}{3^{l(k+1)}}\right|\Omega\left(\frac{u}{3^{(k+1)}}\right)-\frac{1}{3^{lm}}\Omega\left(\frac{u}{3^m}\right)\right| : 0 \le k < m\} \qquad (4.31) \\ &\le \max\left\{\left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^{m+1}}, \frac{u}{3^{m+1}}\right) : 0 \le k < m\right\}.\end{aligned}$$

As $m \to \infty$ in the inequality (4.31) and by using (4.30), we observe that the inequality (4.27) is valid. By applying inequalities (4.25), (4.26), and (4.30), for all $u, v \in \mathbb{A}^*$, we arrive at the following

$$\begin{aligned}|\Lambda(u,v)| &= \lim_{m\to\infty}\left|\frac{1}{3^l}\right|^m \left|\Lambda\left(\frac{u}{3^m}, \frac{v}{3^m}\right)\right| \\ &\le \lim_{m\to\infty}\left|\frac{1}{3^l}\right|^m \Upsilon\left(\frac{u}{3^m}, \frac{v}{3^m}\right) \\ &= 0.\end{aligned}$$

Therefore, the mapping g fulfills (4.25), making it a reciprocal mapping. To establish the uniqueness of g, suppose that $g' : \mathbb{A}^* \to \mathbb{B}$ is another reciprocal mapping that also fulfills (4.27). Then

$$\begin{aligned}|g(u)-g'(u)| &= \lim_{n\to\infty}\left|\frac{1}{3^l}\right|^n \left|g\left(\frac{u}{3^n}\right)-g'\left(\frac{u}{3^n}\right)\right| \\ &\le \lim_{n\to\infty}\left|\frac{1}{3^l}\right|^n \max\left\{\left|g\left(\frac{u}{3^n}\right)-\Omega\left(\frac{u}{3^n}\right)\right|, \left|\Omega\left(\frac{u}{3^n}\right)-g'\left(\frac{u}{3^n}\right)\right|\right\} \\ &\le \lim_{n\to\infty}\lim_{m\to\infty}\max\left\{\max\left\{\left|\frac{1}{3^l}\right|^{k+n}\Upsilon\left(\frac{u}{3^{k+n+1}}, \frac{u}{3^{k+n+1}}\right) \quad : n \le k \le m+n\right\}\right\} \\ &= 0\end{aligned}$$

for all $u \in \mathbb{A}^*$. This shows that g is the only such mapping, thereby concluding the proof. □

As a direct consequence of Theorem 4.1, we have the following corollaries.

Corollary 4.1. *Let $\mu > 0$ be a constant. If $\Omega : \mathbb{A}^* \to \mathbb{B}$ satisfies*

$$|\Lambda(u,v)| \le \mu$$

for all $u, v \in \mathbb{A}^*$, then there exists a unique reciprocal mapping $g : \mathbb{A}^* \to \mathbb{B}$ satisfying (1.1) and

$$|\Omega(u) - g(u)| \le \mu$$

for all $u \in \mathbb{A}^*$.

Proof. Taking $\Upsilon(u,v) = \mu$ in Theorem 4.1, we get the required result. □

Corollary 4.2. *Let* $\mu \geq 0$ *and* $a \neq -l$, *be fixed constants. If* $\Omega : \mathbb{A}^* \to \mathbb{B}$ *satisfies*

$$|\Lambda(u,v)| \leq \mu(|u|^a + |v|^a)$$

for all $u, v \in \mathbb{A}^*$, *then there exists a unique reciprocal mapping* $g : \mathbb{A}^* \to \mathbb{B}$ *satisfying (1.1) and*

$$|\Omega(u) - g(u)| \leq \begin{cases} \frac{|2|\mu}{|3|^a}|u|^a, & a > -l, \\ |2|\mu|3|^l|u|^a, & a < -l, \end{cases}$$

for all $u \in \mathbb{A}^*$.

Proof. Considering $\Upsilon(u,v) = \mu(|u|^a + |v|^a)$ in Theorem 4.1, we obtain the desired result. □

Corollary 4.3. *Let* $\Omega : \mathbb{A}^* \to \mathbb{B}$ *be a mapping and let there exist real numbers* $p, q, a = p + q \neq -l$ *and* $\mu \geq 0$ *such that*

$$|\Lambda(u,v)| \leq \mu|u|^p|v|^q$$

for all $u, v \in \mathbb{A}^*$. *Then, there exists a unique reciprocal mapping* $g : \mathbb{A}^* \to \mathbb{B}$ *satisfying (1.1) and*

$$|\Omega(u) - g(u)| \leq \begin{cases} \frac{\mu}{|3|^a}|u|^a, & a > -l, \\ \mu|3|^l||u|^a, & a < -l, \end{cases}$$

for all $u \in \mathbb{A}^*$.

Proof. Letting $\Upsilon(u,v) = \mu|u|^p|v|^q$, for all $u, v \in \mathbb{A}^*$ in Theorem 4.1, we acquire the requisite result. □

Corollary 4.4. *Let* $\mu \geq 0$ *and* $a \neq -l$ *be real numbers, and* $\Omega : \mathbb{A}^* \to \mathbb{B}$ *be a mapping satisfying the functional inequality*

$$|\Lambda(u,v)| \leq \mu(|u|^{\frac{a}{2}}|v|^{\frac{a}{2}} + |u|^a + |v|^a)$$

for all $u, v \in \mathbb{A}^*$ *Then, there exists a unique reciprocal mapping* $g : \mathbb{A}^* \to \mathbb{B}$ *satisfying (1.1) and*

$$|\Omega(u) - g(u)| \leq \begin{cases} \frac{|3|\mu}{|3|^a}|u|^a, & a > -l, \\ |3|\mu|3|^l|u|^a, & a < -l, \end{cases}$$

for all $u \in \mathbb{A}^*$.

Proof. Letting $\Upsilon(u,v) = \mu(|u|^{\frac{a}{2}}|v|^{\frac{a}{2}} + |u|^a + |v|^a)$ in Theorem 4.1, the result follows directly. □

5. Counter-Examples

In this section, using the well-known counter-example provided by Gajda [7], we demonstrate that the equation (1.1) is not applicable for $\alpha = -l$ in Corollary 3.2, within the context of non-zero real numbers. Let's define the function

$$\zeta(u) = \begin{cases} \frac{c}{u^l}, & \text{for } u \in (1, \infty) \\ c, & \text{elsewhere} \end{cases} \tag{5.32}$$

where $\zeta : \mathbb{R}^* \to \mathbb{R}$. Let $\Omega : \mathbb{R}^* \to \mathbb{R}$ be a function defined as

$$\Omega(u) = \sum_{m=0}^{\infty} 3^{-lm} \zeta(3^{-m}u) \tag{5.33}$$

for all $u \in \mathbb{R}$. Assume the mapping $\Omega : \mathbb{R}^* \to \mathbb{R}$ defined in (5.33) fulfills the following inequality

$$|\Lambda(u,v)| \leq c \frac{3^{l+1}+1}{3^l - 1} \left(|u|^{-l} + |v|^{-l} \right) \tag{5.34}$$

for all $u, v \in \mathbb{R}^*$. We prove that there do not exist a reciprocal mapping $G : \mathbb{R}^* \to \mathbb{R}$ and a constant $\delta > 0$ such that

$$|\Omega(u) - G(u)| \leq \delta \, |u|^{-l} \tag{5.35}$$

for all $u \in \mathbb{R}^*$. Initially, we show that Ω fullfils (5.34). Using (5.32), we have

$$|\Omega(u)| = \left| \sum_{m=0}^{\infty} 3^{-lm} \zeta(3^{-m}u) \right| \leq \sum_{m=0}^{\infty} \frac{c}{3^{lm}} = \frac{3^l}{3^l - 1} c.$$

We can see that Ω is bounded by $\frac{c3^l}{3^l-1}$ on $\mathbb{R}$. If $|u|^{-l} + |v|^{-l} \geq 1$, then the left hand side of (5.34) is less than $\frac{c(3^{l+1}+1)}{3^l-1}$. Now, assume that $0 < |u|^{-l} + |v|^{-l} < 1$. Therefore, there exists a positive integer m such that

$$\frac{1}{3^{l(m+1)}} \leq |u|^{-l} + |v|^{-l} < \frac{1}{3^{lm}}. \tag{5.36}$$

Thus, the inequality (5.36) yields $3^{lm} \left(|u|^{-l} + |v|^{-l} \right) < 1$, or equivalently: $3^{lm} u^{-l} < 1$, $3^{lm} v^{-l} < 1$. So,

$$\frac{u^l}{3^{lm}} > 1, \quad \frac{v^l}{3^{lm}} > 1.$$

Hence, the last inequalities imply $\frac{u^l}{3^{l(m-1)}} > 3^l > 1$, $\frac{v^l}{3^{l(m-1)}} > 3^l > 1$ and thus we find

$$\frac{1}{3^{m-1}}(u) > 1, \quad \frac{1}{3^{m-1}}(v) > 1, \quad \frac{1}{3^{m-1}}(2u+v) > 1, \quad \frac{1}{3^{m-1}}(2u-v) > 1.$$

Hence, for every value of $m = 0, 1, 2, \ldots, n-1$, we get

$$\frac{1}{3^n}(u) > 1, \quad \frac{1}{3^n}(v) > 1, \quad \frac{1}{3^n}(2u+v) > 1, \quad \frac{1}{3^n}(2u-v) > 1,$$

and $\Delta(3^{-n}u, 3^{-n}v) = 0$ for $m = 0, 1, 2, \ldots, n-1$. Applying (5.32) and the definition of Ω, we get

$$\begin{aligned}
|\Delta(u,v)| &\leq \sum_{m=n}^{\infty} \frac{c}{3^{lm}} + \sum_{m=n}^{\infty} \frac{c}{3^{lm}} + \frac{3^l+1}{3^l} \sum_{m=n}^{\infty} \frac{c}{3^{lm}} \\
&\leq c \frac{3^{l+1}+1}{3^l} \cdot \frac{1}{3^{lm}} \left(1 - \frac{1}{3^l}\right)^{-1} \\
&\leq c \left(\frac{3^{l+1}+1}{3^l-1}\right) \cdot \frac{1}{3^{l(m+1)}} \\
&\leq c \left(\frac{3^{l+1}+1}{3^l-1}\right) \left(|u|^{-l} + |v|^{-l}\right)
\end{aligned}$$

for all $u, v \in \mathbb{R}^*$. This means that the inequality (5.34) holds. We claim that the $l-$power reciprocal functional equation (1.1) is not stable for $\alpha = -l$ in Corollary 3.2. Suppose that there exists a reciprocal mapping $\Omega : \mathbb{R}^* \longrightarrow \mathbb{R}$ satisfying (5.35). So, we have

$$|\Omega(u)| \leq (\delta+1)|u|^{-l}. \tag{5.37}$$

Furthermore, a positive integer m can be choosen with the condition $m\,c > \delta + 1$. If $u \in (1, 3^{m-1})$, then $3^{-n}u \in (1, \infty)$ for all $m = 0, 1, 2, \ldots, n-1$ and therefore

$$|\Omega(u)| = \sum_{m=0}^{\infty} \frac{\zeta(3^{-m}u)}{3^{lm}} \geq \sum_{m=0}^{n-1} \frac{3^{lm}c}{u^l \cdot 3^{lm}} = \frac{m\,c}{u^l} > (\delta+1)u^{-l}$$

which contradicts (5.37). Thus, the $l-$power functional equation (1.1) is not stable for $\alpha = -l$ in Corollary 3.2.

6. Conclusion

In this paper, we have successfully explored the generalized Hyers-Ulam-Rassias stability of a reciprocal-type functional equation, focusing on its behavior in non-zero real and non-Archimedean spaces with suitable counter-examples.

Through detailed analysis, we derived a general solution for the functional equation in the real number space and established the conditions for stability using various inequality techniques. Furthermore, our study extends these findings to non-Archimedean fields, highlighting the unique characteristics and behaviors of solutions in such spaces.

Further research could explore additional types of functional equations and their stability across various mathematical fields, enhancing the framework established in this study.

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ON ULTRAMETRIC PSEUDOSPECTRA OF THE DIRECT SUM OF LINEAR OPERATOR PENCILS

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Abstract. In this paper, we introduce the concepts of pseudospectra, condition pseudospectra, determinant spectra and trace pseudospectra of the direct sum of bounded linear operator pencils on ultrametric Banach spaces. We prove numerous results about them and we give some examples to illustrate our work.

Keywords: Ultrametric Banach spaces, pseudospectra, condition pseudospectra, direct sum of operators, linear operator pencils.

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1. Introduction and preliminaries

In ultrametric operator theory, Ammar et al. [1] introduced and studied the concept of pseudospectra of closed linear operators on ultrametric Banach spaces. The notion of ultrametric condition pseudospectra of bounded linear operators was introduced by the authors [2].

Throughout this paper, F is an ultrametric Banach space over an ultrametric complete valued field $\mathbb{K}$ with a non-trivial valuation $|\cdot|$, $\mathcal{L}(F)$ denotes the collection of each continuous linear operators on F and $\mathcal{M}_n(\mathbb{K})$ is the collection of any $n\times n$ matrices with coefficients in $\mathbb{K}$. If $S\in\mathcal{L}(F)$, $R(S)$ and $N(S)$ denote the range and the kernel of S respectively. Remember that, an unbounded linear operator $S\ :D(S)\subseteq F\to F$ will be called closed if for each $(x_n)\subset D(S)$ with $\lim\limits_{n\to\infty}\|x_n-x\|=0$ and $\lim\limits_{n\to\infty}\|Ax_n-y\|=0$ for some $x\in F$ and $y\in F$,

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hence $x \in D(A)$ with $y = Sx$. $\mathcal{C}(F)$ is the set of all closed linear operators on F. If $A \in \mathcal{L}(F)$ and B is an unbounded operator, hence $S + B$ is closed if and only if B is closed [4]. For more details on ultrametric pseudospectra and condition pseudospectra of linear operators, we refer to [1], [2], [6], [8], [9], [10], [11], [12], [13] and [14]. We continue by recalling some preliminaries.

Definition 1.1. [4] *Let F be a vector space over $\mathbb{K}$. A function $\|\cdot\| : F \to \mathbb{R}_+$ is an ultrametric norm if:*

(i) *For each $v \in F$, $\|v\| = 0$ if and only if $v = 0$,*
(ii) *For each $v \in F$ and $\lambda \in \mathbb{K}$, $\|\lambda v\| = |\lambda|\|v\|$,*
(iii) *For all $v, y \in F$, $\|v + y\| \leq \max(\|v\|, \|y\|)$.*

Definition 1.2. [4] *An ultrametric normed space is a pair $(F, \|\cdot\|)$ where F is a vector space over $\mathbb{K}$ and $\|\cdot\|$ is an ultrametric norm on F.*

Definition 1.3. [4] *An ultrametric Banach space is a complete ultrametric normed space.*

Proposition 1.1. [4] *The direct sum of two ultrametric Banach spaces is an ultrametric Banach space.*

Definition 1.4. [4] *An ultrametric Banach space F is said to be a free Banach space if there is a set $(v_i)_{i\in I}$ of F indexed by a set I such that all element $v \in F$ can be written uniquely as follows $v = \sum_{i\in I} \lambda_i v_i$ and $\|v\| = \sup_{i\in I} |\lambda_i|\|v_i\|$.*
The family $(v_i)_{i\in I}$ is called an orthogonal basis for F. If, for each $i \in I$, $\|v_i\| = 1$, hence $(v_i)_{i\in I}$ is called an orthonormal basis of F.

Definition 1.5. [4] *Let $S \in \mathcal{L}(F)$. The resolvent set $\rho(S)$ of S is*

$$\rho(S) = \{\lambda \in \mathbb{K} : (S - \lambda I)^{-1} \in \mathcal{L}(F)\}. \tag{1.1}$$

The spectrum $\sigma(S)$ of S is $\mathbb{K}\backslash\rho(S)$.

Lemma 1.1. [5] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$ and $C \in \mathcal{C}_c(F)$, then $S + C \in \Phi(F)$.*

Lemma 1.2. [2] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. If $S \in \Phi(F)$, hence for each $C \in \mathcal{C}_c(F)$, we get $S + C \in \Phi(F)$ and $ind(S + K) = ind(S)$.*

We have the following definition.

Definition 1.6. [1] *Let F be an ultrametric Banach space over $\mathbb{K}$, let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S)$ of S is*

$$\sigma_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \ \|(S - \lambda I)^{-1}\| > \varepsilon^{-1}\},$$

with the convention $\|(S - \lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(S)$.

Theorem 1.1. [1] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$, let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Then*

$$\sigma_\varepsilon(S) = \bigcup_{C \in \mathcal{L}(F) : \|C\| < \varepsilon} \sigma(S + C).$$

Theorem 1.2. [2] *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ and let $S \in \mathcal{L}(F)$. Then*

$$\sigma_e(S) = \bigcap_{C \in \mathcal{C}_c(F)} \sigma(S + C).$$

We generalise the Definition 3.7 of [1]: this definition remains valid for any ultrametric Banach spaces over a non-trivally complete ultrametric valued field $\mathbb{K}$ not only $\mathbb{E}_\omega$.

Definition 1.7. *Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. The essential pseudospectrum of S is*

$$\sigma_{e,\varepsilon}(S) = \mathbb{K} \backslash \{\lambda \in \mathbb{K} : S + M - \lambda I \in \Phi_0(F) \text{ for all } M \in \mathcal{L}(F), \ \|M\| < \varepsilon\},$$

where $\Phi_0(F)$ is the set of each bounded Fredholm operators on F of index 0.

We generalise the Theorem 3.8 of [1] as follows.

Theorem 1.3. *Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence*

$$\sigma_{e,\varepsilon}(S) = \bigcup_{M \in \mathcal{L}(F) : \|M\| < \varepsilon} \sigma_e(S + M).$$

We have the following:

Theorem 1.4. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence,*

$$\sigma_{e,\varepsilon}(S) = \sigma_{e,\varepsilon}(S + K) \ \text{ for each } \ K \in \mathcal{C}_c(F).$$

Proof. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, let $\lambda \notin \sigma_{e,\varepsilon}(S)$, hence for any $C \in \mathcal{L}(F)$ with $\|C\| < \varepsilon$,

$$S + C - \lambda I \in \Phi(F) \text{ and } ind(S + C - \lambda I) = 0.$$

From Lemma 1.2, for each $K \in \mathcal{C}_c(F)$ and $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, we have

$$S + C + K - \lambda I \in \Phi(F) \text{ and } ind(S + C + K - \lambda I) = 0. \tag{1.2}$$

By (1.2), we get

$$\lambda \notin \sigma_{e,\varepsilon}(S+K).$$

Then

$$\sigma_{e,\varepsilon}(S+K) \subseteq \sigma_{e,\varepsilon}(S).$$

The opposite inclusion follows from symmetry. □

Remark 1.1. *The Theorem 1.4 showed that the essential pseudospectra of bounded linear operators is invariant under perturbation of completely continuous linear operators on ultrametric Banach space over a spherically complete field* $\mathbb{K}$.

Theorem 1.5. *Let* F *be an ultrametric Banach space over a spherically complete field* $\mathbb{K}$ *such that* $\|F\| \subseteq |\mathbb{K}|$. *Let* $S \in \mathcal{L}(F)$ *and* $\varepsilon > 0$. *Then*

$$\sigma_{e,\varepsilon}(S) = \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K).$$

Proof. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, let $\lambda \notin \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K)$, hence there exists $K \in \mathcal{C}_c(F)$ such that $\lambda \notin \sigma_\varepsilon(S+K)$. By Theorem 1.1, we have $\lambda \in \rho(S+K+C)$, for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$. We have

$$S+C+K-\lambda I \in \Phi(F) \text{ and } ind(S+C+K-\lambda I) = 0. \tag{1.3}$$

By Lemma 1.1 and Lemma 1.2, we have

$$S+C-\lambda I \in \Phi(F) \text{ and } ind(S+C-\lambda I) = 0. \tag{1.4}$$

We get

$$\lambda \notin \sigma_{e,\varepsilon}(S).$$

Then,

$$\sigma_{e,\varepsilon}(S) \subseteq \bigcap_{K \in \mathcal{C}_c(F)} \sigma_\varepsilon(S+K). \tag{1.5}$$

Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(S)$. Using Theorem 1.3, we have for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, $\lambda \notin \sigma_e(S+C)$. By Theorem 1.2, there is $K \in \mathcal{C}_c(F)$ with $\lambda \notin \sigma(S+K+C)$, hence for all $C \in \mathcal{L}(F)$ such that $\|C\| < \varepsilon$, $\lambda \in \rho(S+K+C)$. Hence

$$\lambda \in \bigcap_{C \in \mathcal{L}(F) : \|C\| < \varepsilon} \rho(S+K+C). \tag{1.6}$$

From Theorem 1.1, $\lambda \notin \sigma_\varepsilon(S+K)$. Consequently,

$$\lambda \notin \bigcap_{K\in\mathcal{C}_c(F)} \sigma_\varepsilon(S+K).$$

Thus

$$\sigma_{e,\varepsilon}(S) = \bigcap_{K\in\mathcal{C}_c(F)} \sigma_\varepsilon(S+K).$$

□

We generalise the Proposition 3.13 of [1] as follows.

Proposition 1.2. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$. If $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, then*

(i) $\sigma_{e,\varepsilon}(S) \subset \sigma_\varepsilon(S)$.

(ii) *For each ε_1 and ε_2 with $0 < \varepsilon_1 < \varepsilon_2$, $\sigma_e(S) \subset \sigma_{e,\varepsilon_1}(S) \subset \sigma_{e,\varepsilon_2}(S)$.*

Similarly to the proof of Proposition 3.14 of [1], we have the following:

Proposition 1.3. *Let F be an ultrametric Banach space over a spherically complete field $\mathbb{K}$ such that $\|F\| \subseteq |\mathbb{K}|$. Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$. Hence*

$$\sigma_e(S) = \bigcap_{\varepsilon>0} \sigma_{e,\varepsilon}(S).$$

Definition 1.8. [6] *Let $S \in \mathcal{L}(F)$ and $\varepsilon > 0$, the condition pseudospectrum $\Lambda_\varepsilon(S)$ of S is*

$$\Lambda_\varepsilon(S) = \sigma(S) \cup \{\lambda \in \mathbb{K} : \|(S-\lambda I)\|\|(S-\lambda I)^{-1}\| > \frac{1}{\varepsilon}\},$$

with the convention $\|(S-\lambda I)\|\|(S-\lambda I)^{-1}\| = \infty$ if $\lambda \in \sigma(S)$.

We generalise the results of [14] as follows.

Definition 1.9. *Let $S \in \mathcal{C}(F)$, $B \in \mathcal{L}(F)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(S,B)$ of (S,B) on F is defined by*

$$\sigma_\varepsilon(S,B) = \sigma(S,B) \cup \{\lambda \in \mathbb{K} : \|(S-\lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudoresolvent $\rho_\varepsilon(S,B)$ of (S,B) is defined by

$$\rho_\varepsilon(S,B) = \rho(S,B) \cap \{\lambda \in \mathbb{K} : \|(S-\lambda B)^{-1}\| \le \varepsilon^{-1}\},$$

by convention $\|(S-\lambda B)^{-1}\| = \infty$ if $\lambda \in \sigma(S,B)$.

Proposition 1.4. *Let $S \in \mathcal{C}(F)$, $B \in \mathcal{L}(F)$ and $\varepsilon > 0$, we get*

(i) $\sigma(S,B) = \bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B)$.

(ii) *For any* ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(S,B) \subset \sigma_{\varepsilon_1}(S,B) \subset \sigma_{\varepsilon_2}(S,B)$.

Theorem 1.6. *Let* F *be an ultrametric Banach space over a spherically complete field* $\mathbb{K}$ *such that* $\|F\| \subseteq |\mathbb{K}|$, *let* $S \in \mathcal{C}(F), B \in \mathcal{L}(F)$ *and* $\varepsilon > 0$. *Hence*

$$\sigma_\varepsilon(S,B) = \bigcup_{C\in\mathcal{L}(F):\|C\|<\varepsilon} \sigma(S+C,B).$$

The condition pseudospectra of operator pencils is defined as follows:

Definition 1.10. [8] *Let* F *be an ultrametric Banach space over* $\mathbb{K}$, *let* $B, S \in \mathcal{L}(F)$ *and* $\varepsilon > 0$. *The pseudospectrum* $\sigma_\varepsilon(S,B)$ *of the linear operator pencil* (S,B) *on* F *is*

$$\Lambda_\varepsilon(S,B) = \sigma(S,B) \cup \{\lambda \in \mathbb{K} : \|(S-\lambda B)\|\|(S-\lambda B)^{-1}\| > \varepsilon^{-1}\},$$

with the convention $\|(S-\lambda B)\|\|(S-\lambda B)^{-1}\| = \infty$ *if* $\lambda \in \sigma(S,B)$.

Proposition 1.5. [8] *Let* $B, S \in \mathcal{L}(F)$ *and* $\varepsilon > 0$, *we get*

(i) $\sigma(S,B) = \bigcap_{\varepsilon>0} \Lambda_\varepsilon(S,B)$.

(ii) *For any* ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(S,B) \subset \Lambda_{\varepsilon_1}(S,B) \subset \Lambda_{\varepsilon_2}(S,B)$.

Theorem 1.7. [8] *Let* F *be an ultrametric Banach space over a spherically complete field* $\mathbb{K}$ *such that* $\|F\| \subseteq |\mathbb{K}|$, *let* $B, S \in \mathcal{L}(F)$ *and* $\varepsilon > 0$. *Hence,*

$$\Lambda_\varepsilon(S,B) = \bigcup_{C\in\mathcal{L}(F):\|C\|<\varepsilon\|S-\lambda B\|} \sigma(S+C,B).$$

2. Main results

Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$. The space $X = \oplus_{i=1}^n X_i$ endowed by for all $i \in \{1,\cdots,n\}$, $x_i \in X_i$, $\|x_1 \oplus x_2 \oplus \cdots \oplus x_n\| = \max_{i\in\{1,\cdots,n\}} \|x_i\|$ is an ultrametric Banach space over $\mathbb{K}$ [4]. One can see that if for each $i \in \{1,\cdots,n\}$, $A_i \in \mathcal{L}(X_i)$, hence $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n \in \mathcal{L}(X)$. We get the following definition.

Definition 2.1. *Let* $A_i, T_i \in \mathcal{L}(X_i)$. *The spectrum* $\sigma(A,T)$ *of* (A,T) *is given by*

$$\sigma(A,T) = \{\lambda \in \mathbb{K} : A - \lambda T \textit{ is not invertible in } \mathcal{L}(\oplus_{i=1}^n X_i)\},$$

where $A = \oplus_{i=1}^n A_i$ *and* $T = \oplus_{i=1}^n T_i$. *The resolvent set of* (A,T) *is*

$$\rho(A,T) = \{\lambda \in \mathbb{K} : (A-\lambda T)^{-1} \in \mathcal{L}(\oplus_{i=1}^n X_i)\}.$$

For $i = 2$, we obtain the following proposition.

Proposition 2.1. *Let X, Y be two ultrametric Banach spaces over $\mathbb{K}$. Let $A, S \in \mathcal{L}(X)$, $B, T \in \mathcal{L}(Y)$. The spectrum of $(A \oplus B) - \lambda(S \oplus T) \in \mathcal{L}(X \oplus Y)$ is given by*

$$\sigma(A \oplus B, S \oplus T) = \sigma(A, S) \cup \sigma(B, T).$$

Proof. If $\lambda \in \sigma(A \oplus B, S \oplus T)$, hence $(A \oplus B) - \lambda(S \oplus T)$ is not invertible, then $A - \lambda S$ is not invertible in $\mathcal{L}(X)$ or $B - \lambda T$ is not invertible in $\mathcal{L}(Y)$, consequently $\lambda \in \sigma(A, S) \cup \sigma(B, T)$. Hence $\sigma(A \oplus B, S \oplus T) \subseteq \sigma(A, S) \cup \sigma(B, T)$. Similarly, we obtain that $\sigma(A, S) \cup \sigma(B, T) \subseteq \sigma(A \oplus B, S \oplus T)$. Consequently,

$$\sigma(A \oplus B, S \oplus T) = \sigma(A, S) \cup \sigma(B, T).$$

□

More generally, one can see that.

Proposition 2.2. *Let $A_i, B_i \in \mathcal{L}(X_i)$. Set $A = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$, then*

$$\sigma(A, B) = \bigcup_{i=1}^{n} \sigma(A_i, B_i)$$

and

$$\rho(A, B) = \bigcap_{i=1}^{n} \rho(A_i, B_i).$$

Now, we define the pseudospectra of the operator pencil (A, B) on $\oplus_{i=1}^n X_i$ where $A = \oplus_{i=1}^n A_i$, $B = \oplus_{i=1}^n B_i$ and for all $i \in \{1, \cdots, n\}$, $A_i, B_i \in \mathcal{L}(X_i)$, we have the following:

Definition 2.2. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The pseudospectrum $\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ of the bounded linear operator pencil $(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$ on $\oplus_{i=1}^n X_i$ is given by*

$$\sigma_\varepsilon(A, B) = \sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) \cup \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i, B_i) : \sup_{i \in \{1, \cdots, n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\},$$

where $A = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$.

Remark 2.1. *It is easy to check that $\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcup_{i=1}^{n} \sigma_\varepsilon(A_i, B_i)$.*

Proposition 2.3. *Let $(X_i)_{1 \leq i \leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$, let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. Set $S = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$, then*

(i) $\sigma(S, B) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)$.

(ii) *If $0 < \varepsilon_1 < \varepsilon_2$, hence $\sigma(S, B) \subset \sigma_{\varepsilon_1}(S, B) \subset \sigma_{\varepsilon_2}(S, B)$.*

Proof. (i) From Definition 2.2, for each $\varepsilon > 0$, $\sigma(S,B) \subset \sigma_\varepsilon(S,B)$, hence $\sigma(S,B) \subset \bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B)$, since

$$\bigcap_{\varepsilon>0} \sigma_\varepsilon(S,B) = \sigma(S,B) \cup \bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i,B_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)^{-1}\| > \varepsilon^{-1}\}$$

and $\bigcap_{\varepsilon>0} \{\lambda \in \bigcap_{i=1}^{n} \rho(A_i,B_i) : \sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)^{-1}\| > \varepsilon^{-1}\} = \emptyset$ because of for all $i \in \{1,\cdots,n\}$, $(A_i-\lambda B_i)^{-1}$ are bounded linear operators, hence we get a contradiction, thus $\lambda \in \sigma(S,B)$.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$ and $\lambda \in \sigma_{\varepsilon_1}(S,B)$, consequently, $\sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, hence $\lambda \in \sigma_{\varepsilon_2}(S,B)$.

□

Lemma 2.1. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, hence*

$$\sup_{\lambda\in\sigma(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)} |\lambda| \le \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)} |\lambda|.$$

Proof. Set $S = \oplus_{i=1}^n A_i$ and $B = \oplus_{i=1}^n B_i$, since $\sigma(S,B) \subseteq \sigma_\varepsilon(S,B)$, we get

$$\sup_{\lambda\in\sigma(S,B)} |\lambda| \le \sup_{\lambda\in\sigma_\varepsilon(S,B)} |\lambda|.$$

□

Put $r_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \sup_{\lambda\in\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i)} |\lambda|$, we have the following:

Lemma 2.2. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, hence $r_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \sup_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i, B_i)$.*

Proof. By Remark 2.1, $\sigma_\varepsilon(\oplus_{i=1}^n A_i, \oplus_{i=1}^n B_i) = \bigcup_{i=1}^{n} \sigma_\varepsilon(A_i,B_i)$. It is easy to see that

$$r_\varepsilon(\oplus_{i=1}^n A_i, B_i) = \sup_{i\in\{1,\cdots,n\}} r_\varepsilon(A_i,B_i).$$

□

We get the following examples.

Example 2.1. *Let $(A_k)_{1\le k\le n}$ and $(B_k)_{1\le k\le n}$ be two linear operators defined on $\mathbb{K}^2$ by*

$$A_k = \begin{pmatrix} \lambda_k & 0 \\ 0 & \mu_k \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & \beta_k \end{pmatrix},$$

where $\lambda_k, \mu_k \in \mathbb{K}$, $\alpha_k, \beta_k \in \mathbb{K}\backslash\{0\}$ *for each* $k \in \{1, \cdots, n\}$. *Then*

$$\sigma(\oplus_{k=1}^{n} A_k, \oplus_{k=1}^{n} B_k) = \bigcup_{k=1}^{n} \Big\{\frac{\lambda_k}{\alpha_k}, \frac{\mu_k}{\beta_k}\Big\}$$

and

$$\sigma_\varepsilon(\oplus_{k=1}^{n} A_k, \oplus_{k=1}^{n} A_k) = \bigcup_{k=1}^{n} \Big\{\frac{\lambda_k}{\alpha_k}, \frac{\mu_k}{\beta_k}\Big\} \cup \{\lambda \in \mathbb{K} : \sup_{1 \leq k \leq n} \|(A_k - \lambda B_k)^{-1}\| > \frac{1}{\varepsilon}\}.$$

Example 2.2. *Let* F *be an ultrametric free Banach space over* $\mathbb{K}$ *with an orthogonal basis* $(e_m)_{m\in\mathbb{N}}$. *Consider the linear operators* $(A_k)_{1\leq k\leq n}$ *and* $(B_k)_{1\leq k\leq n}$ *given by for all* $x \in F$ *and for each* $k \in \{1, \cdots, n\}$, $A_k x = \lambda_k x$ *and* $B_k x = \mu_k x$ *where* $\lambda_k \in \mathbb{K}$ *and* $\mu_k \in \mathbb{K}\backslash\{0\}$. *Put* $A = \oplus_{k=1}^{n} A_k$ *and* $B = \oplus_{k=1}^{n} B_k$. *One can see that*

$$\sigma(A, B) = \bigcup_{k=1}^{n} \{\frac{\lambda_k}{\mu_k}\}$$

and for all $k \in \{1, \cdots, n\}$ *and for each* $\lambda \in \rho(A_k, B_k)$, $\|(A_k - \lambda B_k)^{-1}\| = \frac{1}{|\lambda_k - \lambda\mu_k|}$, *then*

$$\sigma_\varepsilon(A_k, B_k) = \{\frac{\lambda_k}{\mu_k}\} \cup B(\frac{\lambda_k}{\mu_k}, \frac{\varepsilon}{|\mu_k|}).$$

Consequently,

$$\sigma_\varepsilon(A, B) = \bigcup_{k=1}^{n} \{\frac{\lambda_k}{\mu_k}\} \cup \bigcup_{k=1}^{n} B(\frac{\lambda_k}{\mu_k}, \frac{\varepsilon}{|\mu_k|}).$$

Definition 2.3. *Let* $(X_i)_{1\leq i\leq n}$ *be a sequence of ultrametric Banach spaces over* $\mathbb{K}$ *and let* $A_i, B_i \in \mathcal{L}(X_i)$ *and* $\varepsilon > 0$. *The condition pseudospectrum of the bounded linear operator pencil* $(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$ *on* $\oplus_{i=1}^{n} X_i$ *is*

$$\Lambda_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\},$$

where $S = \oplus_{i=1}^{n} A_i$ *and* $B = \oplus_{i=1}^{n} B_i$. *With the convention* $\sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| = \infty$ *if* $\lambda \in \sigma(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.

Remark 2.2. *It is easy to see that* $\bigcup_{i=1}^{n} \Lambda_\varepsilon(A_i, B_i) \subset \Lambda_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.

Proposition 2.4. *Let* $A_i, B_i \in \mathcal{L}(X_i)$ *and* $\varepsilon > 0$, *then*

(i) $\sigma(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) = \bigcap_{\varepsilon>0} \Lambda_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.

(ii) *If* $0 < \varepsilon_1 < \varepsilon_2$, *then* $\sigma(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) \subset \Lambda_{\varepsilon_1}(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) \subset \Lambda_{\varepsilon_2}(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.

Proof. Put $S = \oplus_{i=1}^{n} A_i$ and $B = \oplus_{i=1}^{n} B_i$.

(i) From Definition 2.4, for each $\varepsilon > 0$, $\sigma(S, B) \subset \Lambda_\varepsilon(S, B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \Lambda_\varepsilon(S, B)$ and $\lambda \notin \sigma(S, B)$. Using $\lim_{\varepsilon\to\to 0} \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)^{-1}\| = \infty$, we get a contradiction.

(ii) If $0 < \varepsilon_1 < \varepsilon_2$ and $\lambda \in \Lambda_{\varepsilon_1}(S, B)$, thus $\sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)\| \sup_{i\in\{1,\cdots,n\}} \|(A_i-\lambda B_i)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$, then $\lambda \in \Lambda_{\varepsilon_2}(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.

□

We conclude the following:

Lemma 2.3. *Let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$, hence*

$$\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i,\oplus_{i=1}^{n} B_i)} |\lambda| \leq \sup_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^{n} A_i,\oplus_{i=1}^{n} B_i)} |\lambda|.$$

Proof. Since $\sigma(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) \subseteq \Lambda_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$, then

$$\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i,\oplus_{i=1}^{n} B_i)} |\lambda| \leq \sup_{\lambda\in\Lambda_\varepsilon(\oplus_{i=1}^{n} A_i,\oplus_{i=1}^{n} B_i)} |\lambda|.$$

□

Now, we introduce a new definition of condition pseudospectra of the direct sum of the operator pencil $(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$ as follows.

Definition 2.4. *Let $(X_i)_{1\leq i\leq n}$ be a sequence of ultrametric Banach spaces over $\mathbb{K}$ and let $A_i, B_i \in \mathcal{L}(X_i)$ and $\varepsilon > 0$. The condition pseudospectrum of the bounded linear operator pencil $(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$ on $\oplus_{i=1}^{n} X_i$ is*

$$\Lambda'_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : \sup_{i\in\{1,\cdots,n\}} \|(A_i - \lambda B_i)\|\|(A_i - \lambda B_i)^{-1}\| > \varepsilon^{-1}\},$$

where $S = \oplus_{i=1}^{n} A_i$ and $B = \oplus_{i=1}^{n} B_i$.

Remark 2.3. (i) *It is easy to see that $\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) = \bigcup_{i=1}^{n} \Lambda'_\varepsilon(A_i, B_i)$.*

(ii) $\sigma(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) = \bigcap_{\varepsilon>0} \Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.

(iii) *If $0 < \varepsilon_1 < \varepsilon_2$, then $\sigma(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) \subset \Lambda'_{\varepsilon_1}(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i) \subset \Lambda'_{\varepsilon_2}(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.*

(iv) *For any $\varepsilon > 0$, hence $\sup_{\lambda\in\sigma(\oplus_{i=1}^{n} A_i,\oplus_{i=1}^{n} B_i)} |\lambda| \leq \sup_{\lambda\in\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i,\oplus_{i=1}^{n} B_i)} |\lambda|$.*

(v) *The condition pseudospectrum $\Lambda'_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$ of the pencil $(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$ gives nice properties than $\Lambda_\varepsilon(\oplus_{i=1}^{n} A_i, \oplus_{i=1}^{n} B_i)$.*

We obtain the following examples.

Example 2.3. *Let* $(A_k)_{1\leq k\leq n}$ *and* $(B_k)_{1\leq k\leq n}$ *be two linear operators defined on* $\mathbb{K}^2$ *by*

$$A_k = \begin{pmatrix} \alpha_k & 0 \\ 0 & -\alpha_k \end{pmatrix}$$

and

$$B_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix},$$

where $\alpha_k \in \mathbb{K}$, $\lambda_k \in \mathbb{K}\backslash\{0\}$ *for any* $k \in \{1, \cdots, n\}$. *Set* $A = \oplus_{k=1}^n A_k$ *and* $B = \oplus_{k=1}^n B_k$, *then*

$$\sigma(\oplus_{k=1}^n A_k, \oplus_{k=1}^n B_k) = \bigcup_{k=1}^n \{\frac{-\alpha_k}{\lambda_k}, \frac{\alpha_k}{\lambda_k}\}$$

and

$$\Lambda_\varepsilon(A,B) = \bigcup_{k=1}^n \{\frac{-\alpha_k}{\lambda_k}, \frac{\alpha_k}{\lambda_k}\} \cup \{\lambda \in \mathbb{K} : \sup_{1\leq k\leq n} \|A_k - \lambda B_k\| \sup_{1\leq k\leq n} \|(A_k - \lambda B_k)^{-1}\| > \frac{1}{\varepsilon}\},$$

where for all $k \in \{1, \cdots, n\}$, $\|A_k - \lambda B_k\| = \max\{|\alpha_k|, |\lambda\lambda_k|\}$ *and for each* $\lambda \in \rho(A_k, B_k)$,

$$\|(A_k - \lambda B_k)^{-1}\| = \max\{\frac{|\alpha_k|}{|(\lambda\lambda_k)^2 - \alpha_k^2|}, \frac{|\lambda\lambda_k|}{|(\lambda\lambda_k)^2 - \alpha_k^2|}\}.$$

We begin with the following:

Definition 2.5. [3] *Let* $S, B \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$, *the* ε*-determinant spectrum* $d_\varepsilon(S,B)$ *of the matrix pencil* (S,B) *is*

$$d_\varepsilon(S,B) = \{\lambda \in \mathbb{K} : |\det(S - \lambda B)| \leq \varepsilon\}.$$

We get the following:

Proposition 2.5. *Let* $S, B \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$, *then*

$$d_{\varepsilon_1}(S) \cap d_{\varepsilon_2}(B) \subset d_\varepsilon(S \oplus B),$$

where $\varepsilon = \varepsilon_1\varepsilon_2$.

Proof. Let $\lambda \in d_{\varepsilon_1}(S) \cap d_{\varepsilon_2}(B)$, thus $\lambda \in d_{\varepsilon_1}(S)$ and $\lambda \in d_{\varepsilon_2}(B)$, hence $|\det(S - \lambda I)| \leq \varepsilon_1$ and $|\det(B - \lambda I)| \leq \varepsilon_2$. Consequently,

$$\begin{aligned} |\det(S \oplus B - \lambda(I \oplus I))| &= |\det(S - \lambda I)\det(B - \lambda I)| \\ &= |\det(S - \lambda I)||\det(B - \lambda I)| \\ &\leq \varepsilon_1\varepsilon_2, \end{aligned}$$

thus $\lambda \in d_\varepsilon(S \oplus B)$ where $\varepsilon = \varepsilon_1\varepsilon_2$, hence $d_{\varepsilon_1}(S) \cap d_{\varepsilon_2}(B) \subset d_\varepsilon(S \oplus B)$ where $\varepsilon = \varepsilon_1\varepsilon_2$. □

Proposition 2.6. *Let $S, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$, hence*

(i) *For all $0 < \varepsilon_1 \leq \varepsilon_2$, we have $d_{\varepsilon_1}(S \oplus B) \subseteq d_{\varepsilon_2}(S \oplus B)$,*

(ii) *For each $\mu \in \mathbb{K}$, $d_\varepsilon(\mu(I \oplus I), I \oplus I) = \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq \varepsilon^{\frac{1}{2n}}\}$,*

(iii) *If $\det(B) \neq 0$ and for each $\mu \in \mathbb{K}$, then $d_\varepsilon(\mu(I \oplus B), I \oplus B) = \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq (\frac{\varepsilon}{|\det(B)|})^{\frac{1}{2n}}\}$.*

Proof. (i) For all $0 < \varepsilon_1 \leq \varepsilon_2$, let $\lambda \in d_{\varepsilon_1}(S \oplus B)$, hence $|\det(S \oplus B - \lambda(I \oplus I))| \leq \varepsilon_1 \leq \varepsilon_2$, thus $\lambda \in d_{\varepsilon_2}(S \oplus B)$.

(ii) Let $\mu \in \mathbb{K}$ hence $|\det(\mu I - \lambda I)| = |\lambda - \mu|^n$, then

$$\begin{aligned} d_\varepsilon(\mu(I \oplus I), I \oplus I) &= \{\lambda \in \mathbb{K} : |\det(\mu(I \oplus I) - \lambda(I \oplus I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu|^{2n} \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq \varepsilon^{\frac{1}{2n}}\}. \end{aligned}$$

(iii) We have

$$\begin{aligned} d_\varepsilon(\mu(I \oplus B), I \oplus B) &= \{\lambda \in \mathbb{K} : |\det(\mu(I \oplus B) - \lambda(I \oplus B))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\det((\mu - \lambda)I)||\det((\mu - \lambda)B)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu|^{2n}|\det(B)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \mu| \leq (\frac{\varepsilon}{|\det(B)|})^{\frac{1}{2n}}\}. \end{aligned}$$

□

Example 2.4. *If*

$$S = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \text{ and } B = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p),$$

where $a, b, c, d \in \mathbb{Q}_p$. Hence for any $\varepsilon > 0$,

$$\begin{aligned} d_\varepsilon(S \oplus B) &= \{\lambda \in \mathbb{Q}_p : |\det(S \oplus B - \lambda(I \oplus I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\det(S - \lambda I)||\det(B - \lambda I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |(a - \lambda)(b - \lambda)(c - \lambda)(d - \lambda)| \leq \varepsilon\}. \end{aligned}$$

Example 2.5. *If*

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Hence for each $\varepsilon > 0$,

$$\begin{aligned} d_\varepsilon(S \oplus B) &= \{\lambda \in \mathbb{Q}_p : |\det(S \oplus B - \lambda(I \oplus I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\det(S - \lambda I)||\det(B - \lambda I)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda(1-\lambda)(2-\lambda)(1+\lambda)| \leq \varepsilon\}. \end{aligned}$$

We obtain the following proposition.

Proposition 2.7. *Let* $D_1, D_2 \in \mathcal{L}(\mathbb{Q}_p^n)$ *be two diagonal operators with for each* $i \in \{1, \cdots, n\}$, $D_1 e_i = \lambda_i e_i$ *and* $D_2 e_i = \mu_i e_i$ *with* $\lambda_i, \mu_i \in \mathbb{Q}_p$, $\lambda_i \neq \lambda_{i+1}$ *and* $\mu_i \neq \mu_{i+1}$. *Hence*

$$d_\varepsilon(D_1 \oplus D_2) = \{\lambda \in \mathbb{Q}_p : |\lambda_1 - \lambda| \cdots |\lambda_n - \lambda||\lambda - \mu_1| \cdots |\lambda - \mu_n| \leq \varepsilon\}.$$

Proof. We have

$$\text{for each } i \in \{1, \cdots, n\},\ (D_1 - \lambda)e_i = (\lambda_i - \lambda)e_i$$

and

$$\text{for each } i \in \{1, \cdots, n\},\ (D_2 - \lambda)e_i = (\mu_i - \lambda)e_i,$$

where $(e_k)_{1 \leq k \leq n}$ is a basis of $\mathbb{Q}_p^n$.

Thus $|\det(D_1 - \lambda I)| = |\lambda_1 - \lambda| \cdots |\lambda_n - \lambda|$ and $|\det(D_2 - \lambda I)| = |\mu_1 - \lambda| \cdots |\mu_n - \lambda|$. Hence for any $\varepsilon > 0$,

$$\begin{aligned} d_\varepsilon(D_1 \oplus D_2) &= \{\lambda \in \mathbb{Q}_p : |\det(D_1 \oplus D_2 - \lambda(I \oplus I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\det(D_1 - \lambda I)\det(D_2 - \lambda I)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda - \lambda_1| \cdots |\lambda - \lambda_n||\mu_1 - \lambda| \cdots |\mu_n - \lambda| \leq \varepsilon\}. \end{aligned}$$

□

We have:

Definition 2.6. [3] *Let* $S, B \in \mathcal{M}_n(\mathbb{K}), \varepsilon > 0$, *the trace pseudospectrum* $Tr_\varepsilon(S, B)$ *of the matrix pencil* (S, B) *is*

$$Tr_\varepsilon(S, B) = \sigma(S, B) \cup \{\lambda \in \mathbb{K} : |tr(S - \lambda B)| \leq \varepsilon\}.$$

As the classical setting, we have.

Proposition 2.8. *Let* $(A_i)_{1\leq i\leq n} \in \mathcal{M}_n(\mathbb{K})$, *then the trace of the diagonal block matrix* $\oplus_{i=1}^n A_i$ *is*

$$tr(\oplus_{i=1}^n A_i) = \sum_{i=1}^n tr(A_i).$$

We conclude the following:

Proposition 2.9. *Let* $(A_i)_{1\leq i\leq n} \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n > 0$. *Then*

$$\bigcap_{i=1}^n Tr_{\varepsilon_i}(A_i) \subset Tr_\varepsilon(\oplus_{i=1}^n A_i),$$

where $\varepsilon = \max\limits_{1\leq i\leq n} \varepsilon_i$.

Proof. If $\lambda \in \bigcap\limits_{i=1}^n Tr_{\varepsilon_i}(A_i)$, then for each $i \in \{1;\cdots,n\}$, $\lambda \in Tr_{\varepsilon_i}(A_i)$, hence for all $i \in \{1;\cdots,n\}$, $\lambda \in \sigma(A_i) \cup \{\lambda \in \mathbb{K} : |tr(A_i - \lambda I)| \leq \varepsilon_i\}$, since $\lambda \in \bigcap\limits_{i=1}^n \sigma(A_i) \subset \bigcup\limits_{i=1}^n \sigma(A_i)$ and from Proposition 2.8, we have

$$\begin{aligned} |tr(\oplus_{i=1}^n A_i - \lambda(\oplus_{i=1}^n I))| &= |\sum_{i=1}^n tr(A_i - \lambda I)| \\ &\leq \max_{1\leq i\leq n} |tr(A_i - \lambda I)| \\ &\leq \max_{1\leq i\leq n} \varepsilon_i, \end{aligned}$$

hence $\lambda \in Tr_\varepsilon(\oplus_{i=1}^n A_i)$, where $\varepsilon = \max\limits_{1\leq i\leq n} \varepsilon_i$.

□

Theorem 2.1. *Let* $S, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$. *Hence*

(i) *If* $0 < \varepsilon_1 \leq \varepsilon_2$, $Tr_{\varepsilon_1}(S \oplus B) \subset Tr_{\varepsilon_2}(S \oplus B)$,

(ii) *For all* $\beta \in \mathbb{K}$ *and* A *is invertible and* $Tr(A) \neq 0$, *we get*

$$Tr_\varepsilon(\beta(A \oplus A), A \oplus A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \beta| \leq \frac{\varepsilon}{|2tr(A)|}\Big\}.$$

Proof. (i) It follows from Definition 2.6.

(iii) If $\beta, \lambda \in \mathbb{K}$, hence

$$|tr(\beta(A \oplus A) - \lambda(A \oplus A))| = |\lambda - \beta||2tr(A)| \leq \varepsilon.$$

Thus

$$Tr_\varepsilon(\beta(A \oplus A), A \oplus A) = \Big\{\lambda \in \mathbb{K} : |\lambda - \beta| \leq \frac{\varepsilon}{|2tr(A)|}\Big\}.$$

□

We finish with the following example.

Example 2.6. *Let* $\varepsilon > 0$. *If* $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} 2 & 5 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_2)$. *Then*

$$Tr_\varepsilon(A \oplus B) = (\{1,3\} \cup \{1,2\}) \cup \{\lambda \in \mathbb{Q}_p : |7 - 4\lambda| \leq \varepsilon\}.$$

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ON LACUNARY $A^{\mathcal{I}}$-STATISTICAL CONVERGENCE OF FUZZY TRIPLE SEQUENCES OF ORDER γ

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Abstract. In this study, we propose the concepts of f-lacunary $A^{\mathcal{I}}$-statistical convergence of order γ and strongly f-lacunary $A^{\mathcal{I}}$-summability of order γ for triple sequences of fuzzy numbers. Additionally, we explore fundamental connections between these convergence notions. As a practical application, we apply this newly defined convergence to establish a fuzzy Korovkin-type approximation theorem concerning triple sequences of fuzzy positive linear operators. To highlight the effectiveness of our result, we provide an example that demonstrates the superiority of the established theorem over its classical counterpart.

Keywords: Fuzzy sequence, ideal, fuzzy type Korovkin-theorem, lacunary sequence, regular matrix, triple sequence, A-statistical convergence.

2010 Mathematics Subject Classification: 40G10, 41A36.

1. Introduction

The concept of statistical convergence for sequences, an extension of the usual notion of convergence, was initially introduced in [7, 39]. This concept has spurred extensive research across various spaces and has been influential in the fields of summability theory, functional analysis, and measure theory, among others (see [5, 6], [9], [14], [17], [20], [25], [26], [29]). In their 2008 study [40], Şahiner et al. investigated statistical convergence within the context of triple sequences. For a comprehensive understanding of optimal convergence in triple

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sequences, see [42] and other related sources. A significant advancement in convergence theory, including statistical convergence, was made by Kostyrko et al. [22], who introduced the notions of $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence in metric spaces using ideals based on natural numbers. Following Kostyrko et al.'s work, similar investigations have been conducted for function sequences in random 2-normed spaces [37] and other areas. Further studies have explored these concepts in metric spaces [28], 2-normed spaces [41], and for localized sequences in metric spaces [30], with additional references provided in [11, 12, 13, 19, 36, 45].

Recently, Aizpuru et al. [1] extended the concept of natural density by introducing the f-density of a subset of positive integers using an unbounded modulus function. In 2015, Bhardwaj and Dhawan [3] introduced the definitions of f-density and f-statistical convergence of order γ. Furthermore, Şengül and Et [44] advanced the field by proposing the concept of lacunary statistical convergence of order γ in 2018, employing the modulus function.

To address uncertainty and vagueness, Zadeh [46] introduced the concepts of fuzzy sets, fuzzy logic, and fuzzy numbers in 1965. Since then, fuzzy logic has found applications in various fields such as artificial intelligence, control systems, and decision-making processes. In 1986, Matloka [24] extended these ideas to sequence space theory. The concept of statistical convergence for sequences of fuzzy numbers was later explored by Savaş [34]. For additional details on fuzzy sequence spaces, see [4], [15], [18], [38], and the associated references.

Building on the previous research, we develop and examine the properties of f-lacunary $A^{\mathcal{I}}$-statistical convergence of order γ and strongly f-lacunary $A^{\mathcal{I}}$-summability of order γ for triple sequences of fuzzy numbers. We also explore the interrelationship between these newly defined concepts. Finally, we utilize lacunary triple sequences, the modulus function, and a regular matrix to establish a fuzzy Korovkin-type theorem for triple sequences of fuzzy numbers. As a result, our findings become specific cases of the results presented in [32].

2. Preliminaries

The sets of all natural numbers, all real numbers, and all complex numbers are represented by the letters $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively, throughout the text. Let $E \subseteq \mathbb{N}$ and $E(r) = \{i \in E : i \leq r\}$. Recall that the natural or asymptotic density of E is defined by $\delta(E) = \lim_{r\to\infty} \frac{|E(r)|}{r}$ if the limit exists.

$$\lim_{r\to\infty} \frac{1}{r} |\{j : j \leq r : |y_j - y| \geq \varepsilon\}| = 0, \text{ for all } \varepsilon > 0$$

indicates that the sequence (y_j) statistically converges to y [8]. Since then, the idea of ideal of subsets of $\mathbb{N}$ has been used to expand the concept of statistical convergence to include the idea of $\mathcal{I}$-convergence [22]. Let Z be a non-empty set and $\mathcal{P}(Z)$ be the family of all subsets of Z. An ideal, denoted as $\mathcal{I}$ $(\subset \mathcal{P}(Z))$ is a family of subsets of Z satisfying the following conditions: (a) $E, R \in \mathcal{I}$ imply $E \cup R \in \mathcal{I}$ (b) $R \in \mathcal{I}$, $E \subset R$ imply $E \in \mathcal{I}$, while an admissible ideal $\mathcal{I}$ of Z covers Z. If $Z \notin \mathcal{I}$, $\mathcal{I} \neq \emptyset$, the family of sets $\mathcal{F}(\mathcal{I}) = \{M \subset Z : Z \backslash M \in \mathcal{I}\}$ forms a filter of Z. By $\mathcal{I}_{fin}$ and $\mathcal{I}_\delta$, respectively, we indicate the ideal that is composed of all finite subsets and density zero subsets of $\mathbb{N}$. A sequence $a = (a_k)$ is said to be $\mathcal{I}$-convergent to $b \in \mathbb{R}$ provided for every $\varepsilon > 0$, the set $A(\varepsilon) = \{n \in \mathbb{N} : |a_k - b| \geq \varepsilon\}$ belongs to $\mathcal{I}$ [22]. When considering $\mathcal{I} = \mathcal{I}_{fin}$, $\mathcal{I}$-convergence of the sequence aligns with ordinary convergence, and when considering $\mathcal{I} = \mathcal{I}_\delta$, it aligns with statistical convergence. Furthermore, it is worth noting that [35] delves into the concept of $\mathcal{I}$-statistically convergence. A sequence (a_k) is deemed $\mathcal{I}$-statistically convergent to a if $\{n \in \mathbb{N} : 1/n \left|\{k \leq n : |a_k - a| \geq \varepsilon\}\right| \geq \delta\}$ belongs to $\mathcal{I}$ for each $\varepsilon, \delta > 0$. Then, a is the $\mathcal{I}$-statistical limit of the sequence (a_k) and $\mathcal{I}$-*st*-$\lim_{k\to\infty} a_k = a$.

The lacunary sequence $\theta = (k_r)$, $r \to \infty$, is a nonnegative integers sequence that increases where $k_0 = 0$, $h_r = (k_r - k_{r-1})$ and $h_r \to \infty$ (and $r \to \infty$). If the following limit holds for every $\varepsilon > 0$, then a sequence (y_k) is lacunary statistically convergent to $y : \lim_{r\to\infty} 1/h_r \left|\{k \in I_r : |y_k - y| \geq \varepsilon\}\right| = 0$, where $I_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. If the following limit holds for every $\varepsilon > 0$, then a sequence (y_k) is lacunary statistically convergent to y of order $\gamma : \lim_{r\to\infty} 1/h_r^\gamma \left|\{k \in I_r : |y_k - y| \geq \varepsilon\}\right| = 0$, where $(h_r^\gamma) = (h_1^\gamma, h_2^\gamma, \ldots, h_r^\gamma, \ldots)$ [43].

A modulus function $g : [0, \infty) \to [0, \infty)$ such that (i) $x = 0 \Leftrightarrow g(x) = 0$; (ii) the function g is increasing; (iii) for all $x, y \in [0, \infty)$, $g(x + y) \leq g(y) + g(x)$; (iv) the function g is continuous from the right at point 0 [31]. Therefore, the function g needs be continuous throughout the the interval $[0, \infty)$.

If the following limit holds for every $\varepsilon > 0$, a sequence (y_k), is f-lacunary statistically convergent to y of order $\gamma : \lim_{r\to\infty} 1/f(h_r^\gamma) f(\left|\{k \in I_r : |y_k - y| \geq \varepsilon\}\right|) = 0$, where $(h_r^\gamma) = (h_1^\gamma, h_2^\gamma, \ldots, h_r^\gamma, \ldots)$ [44].

Lemma 2.1 ([24])**.** $\lim_{t\to\infty} \frac{f(t)}{t} = \inf \left\{ \frac{f(t)}{t} : t > 0 \right\}$ *for any modulus function* f.

We now recall the following definitions which were given in [10, 16, 23, 24, 46].

A fuzzy number, denoted as $\tilde{a}$, is characterized as a fuzzy set of real numbers spanning the interval from $\mathbb{R}$ to $[0,1]$ and fulfilling the following properties:

F1. there is such a t in $\mathbb{R}$ such that $\tilde{a}(t)=1$ i.e., $\tilde{a}$ is normal,

F2. $\tilde{a}(t)\geq \min\{\tilde{a}(d),\tilde{a}(c)\}=\tilde{a}(d)\wedge\tilde{a}(c)$ where $c<t<d$, that is $\tilde{a}$ is fuzzy convex

F3. $\tilde{a}$ is upper semi continuous,

F4. $supp(\tilde{a})=\overline{\{t\in\mathbb{R}:\tilde{a}(t)>0\}}$ is compact.

Also, for $\alpha\in(0,1]$, the α-level cut of $\tilde{a}$ can be defined as $[\tilde{a}]_\alpha=\{t\in\mathbb{R}:\tilde{a}(t)\geq\alpha\}=[\tilde{a}^-_\alpha,\tilde{a}^+_\alpha]$, the lower and upper boundaries of the α-level cut of the fuzzy number $\tilde{a}$ are demonstrated by $\tilde{a}^-_\alpha$ and $\tilde{a}^+_\alpha$, respectively. $\mathcal{F}_\mathbb{R}$ represents the set of all fuzzy numbers. For any $\lambda\in\mathbb{R}$ and $\tilde{a},\tilde{b}\in\mathcal{F}_\mathbb{R}$, the scalar multiplication $\lambda\odot\tilde{a}$ and the sum $\tilde{a}\oplus\tilde{b}$ are defined in that : $\left(\tilde{a}\oplus\tilde{b}\right)_\alpha=\tilde{a}_\alpha\oplus\tilde{b}_\alpha$ and $(\lambda\odot\tilde{a})_\alpha=\lambda\tilde{a}_\alpha$. Now, d is the Hausdorff metric and $d:\mathcal{F}_\mathbb{R}\times\mathcal{F}_\mathbb{R}\to\mathbb{R}$ is given by

$$d\left(\tilde{a},\tilde{b}\right)=\sup_{0<\alpha\leq 1}\max\left\{\left|\tilde{a}^-_\alpha-\tilde{b}^-_\alpha\right|,\left|\tilde{a}^+_\alpha-\tilde{b}^+_\alpha\right|\right\}=\sup_{0<\alpha\leq 1}d\left([\tilde{a}]_\alpha,\left[\tilde{b}\right]_\alpha\right).$$

For every $\tilde{a},\tilde{b},\tilde{c},\tilde{d}\in\mathcal{F}_\mathbb{R}$, we get

d1. the space $(\mathcal{F}_\mathbb{R},d)$ is a metric space that is complete [33],

d2. $d\left(p\tilde{a},p\tilde{b}\right)=|p|\,d\left(\tilde{a},\tilde{b}\right);p\in\mathbb{C}$ (the set of all complex scalars),

d3. $d\left(\tilde{a},\tilde{b}\right)=d\left(\tilde{a}\oplus\tilde{c},\tilde{b}\oplus\tilde{c}\right)$,

d4. $d\left(\tilde{a}\oplus\tilde{c},\tilde{b}\oplus\tilde{d}\right)\leq d\left(\tilde{a},\tilde{b}\right)+d\left(\tilde{c},\tilde{d}\right)$,

d5. $\left|d\left(\tilde{a},\tilde{0}\right)-d\left(\tilde{b},\tilde{0}\right)\right|\leq d\left(\tilde{a},\tilde{b}\right)\leq d\left(\tilde{a},\tilde{0}\right)+d\left(\tilde{b},\tilde{0}\right)$, where $\tilde{0}$ is the additive identity element of $\mathcal{F}_\mathbb{R}$.

Let $\tilde{a}=(\tilde{a}_n)$ be a sequence of fuzzy real numbers and if

$$\lim_{r\to\infty}1/r\left|\{n:n\leq r:d(\tilde{a}_n,\tilde{a}_0)\geq\epsilon\}\right|=0$$

for every $\epsilon>0$, then $(\tilde{a}_n)$ is statistically convergent to fuzzy number $\tilde{a}_0$.

Definition 2.1. *If there is a positive number M such that $d(\tilde{a}_{nkl},\tilde{0})<M$ for all n,k,l, then the triple sequence $\tilde{a}=(\tilde{a}_{nkl})$ of fuzzy numbers is said to be bounded. $\ell^{f,3}_\infty$ is the set that represents all bounded triple sequences of fuzzy numbers.*

Assume that $A=(a_{nkolpm})$ is a summability matrix with six-dimensions. If the series converges in the sense of Pringsheim for every $(n,o,p)\in\mathbb{N}^3$, the A-transform of a given triple sequence, $x=(x_{klm})$, is given by $Ax:=\left\{(Ax)_{nop}\right\}$. Recall that a six dimensional matrix

$A=(a_{nkolpm})$ is said to be Robinson-Hamilton (RH)-regular if it maps every bounded P-convergent sequence with the same P-limit. The RH-conditions state that a six dimensional matrix $A=(a_{nkolpm})$ is RH-regular iff

RH1. For each $(k,l,m)\in\mathbb{N}^3$, $P\text{-}\lim_{n,o,p} a_{nkolpm}=0$,

RH2. $P\text{-}\lim_{n,o,p}\sum_{k\in\mathbb{N}} a_{nkolpm}=0$ for every $l\in\mathbb{N}, m\in\mathbb{N}$,

RH3. $P\text{-}\lim_{n,o,p}\sum_{l\in\mathbb{N}} a_{nkolpm}=0$ for every $k\in\mathbb{N}, m\in\mathbb{N}$,

RH4. $P\text{-}\lim_{n,o,p}\sum_{m\in\mathbb{N}} a_{nkolpm}=0$ for every $k\in\mathbb{N}, l\in\mathbb{N}$,

RH5. $\sum_{(k,l,m)\in\mathbb{N}^3}|a_{nkolpm}|$ is P-convergent for all $(n,o,p)\in\mathbb{N}^3$,

RH6. There exist finite positive integers B and C such that $\sum_{k,l,m>C}|a_{nkolpm}|<B$ holds for all $(n,o,p)\in\mathbb{N}^3$,

RH7. $P\text{-}\lim_{n,o,p}\sum_{(k,l,m)\in\mathbb{N}^3} a_{nkolpm}=1$.

Now, assume that $K'\subset\mathbb{N}^3$ and $A=(a_{nkolpm})$ is non-negative RH-regular summability matrix. When the limit on the right-hand side exists in the sense of Pringsheim, the A-density of K' is then given by $\delta_3^A(K'):=P-\lim_{n,o,p}\sum_{(k,l,m)\in K'} a_{nkolpm}$, where $K':=\left\{(k,l,m)\in\mathbb{N}^3:|x_{klm}-\ell|\geq\varepsilon\right\}$. A real triple sequence $x=(x_{klm})$ is said to be A-statistically convergent to a number ℓ if $\delta_3^A(K')=0$ for every $\varepsilon>0$. $\left(A^3\text{-}stat\right)\text{-}\lim_{nop} x=\ell$ in this instance.

3. Main Results

This section introduces and investigates the concepts of strongly f-lacunary $A^{\mathcal{I}}$-summability of order γ and f-lacunary $A^{\mathcal{I}}$-statistical convergence of order γ for triple sequences of fuzzy numbers. Throughout this study, unless specified otherwise, we assume $0<\gamma\leq 1$ and that f is an unbounded modulus function.

Definition 3.1. *Let f be an unbounded modulus function, $\theta_3=\{(k_r,l_s,m_t)\}$ be a lacunary sequence and $\gamma\in(0,1]$. A sequence $\tilde{a}=(\tilde{a}_{klm})$ of fuzzy numbers is f-lacunary $A^{\mathcal{I}_3}$-statistical convergent of order γ ($\gamma\in(0,1]$) (or $A^{\mathcal{I}_3^f}-stat_{\gamma,\theta_3}$-convergent) to a fuzzy number $\tilde{a}_{000}$ if for every $\varepsilon>0$, $\zeta>0$,*

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{a})_{klm},\tilde{a}_{000}\right)\geq\varepsilon\right\}\right|\right)\geq\zeta\right\}$$

belongs to $\mathcal{I}_3$. In this case we write $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m\to\infty}\tilde{a}_{klm}=\tilde{a}_{000}$. $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$ represents the set of all f-lacunary $A^{\mathcal{I}_3^f}$-statistically convergent sequences of order γ.

Definition 3.2. *A triple sequence $\tilde{a} = (\tilde{a}_{klm})$ of fuzzy numbers is strongly f-lacunary $A^{\mathcal{I}_3^f}$-summable of order γ(or ${}^{A^{\mathcal{I}_3^f}}W_{\gamma,\theta_3}$-summable) if there exists a fuzzy number $\tilde{a}_{000}$ such that*

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{a})_{klm}, \tilde{a}_{000}\right)\right) \geq \varepsilon\right\} \in \mathcal{I}_3$$

for each $\varepsilon > 0$. $\left({}^{A^{\mathcal{I}_3^f}}W_{\gamma,\theta_3}\right)$ represents the set of all strongly f-lacunary $A^{\mathcal{I}_3^f}$-summable sequences of order γ.

Remark 3.1. $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$*-convergence is well defined for $\gamma \in (0,1]$. It is not necessary to define it for $\gamma > 1$. To illustrate this, consider $(\tilde{g}_{nop})$ to be a sequence of fuzzy numbers defined as*

$$\tilde{g}_{nop}(t) = \begin{cases} t-3, & \text{if } n,o,p \text{ are odd,} \\ 1-(t-3), & \text{otherwise} \end{cases},$$

for $t \in [3,4]$, and the matrix $A = (a_{nkolpm})$ defined as

$$a_{nkolpm} = \begin{cases} 1, & \begin{matrix} \text{if } n,o,p \text{ are a cube and} \\ k=n^3, l=o^3, m=p^3, \end{matrix} \\ 1, & \begin{matrix} \text{if } n,o,p \text{ are a non cube and} \\ k=n^3+1, l=o^3+1, m=p^3+1, \end{matrix} \\ 0, & \text{otherwise.} \end{cases}$$

One can easily verify that

$$\begin{aligned} (A\tilde{g}(t))_{nop} &= \sum_{k=1,l=1,m=1}^{\infty} a_{nkolpm}\tilde{g}_{klm} \\ &= \begin{cases} t-3=(\tilde{a}), & \begin{matrix} n,o,p \text{ is even non cube or} \\ n,o,p \text{ is odd cube,} \end{matrix} \\ 1-(t-3)=\left(\tilde{b}\right), & \begin{matrix} n,o,p \text{ is an even cube or} \\ n,o,p \text{ is odd non cube.} \end{matrix} \end{cases} \end{aligned}$$

Therefore, we have

$$d\left((A\tilde{g}(t))_{nop}, \tilde{a}\right) := \begin{cases} 0; & \begin{matrix} n,o,p \text{ are odd cubes or} \\ n,o,p \text{ are even non cubes,} \end{matrix} \\ 1; & \begin{matrix} n,o,p \text{ are even cubes or} \\ n,o,p \text{ are odd non cubes,} \end{matrix} \end{cases}$$

and

$$d\left((A\tilde{g}(t))_{nop},\tilde{b}\right):=\begin{cases}1; & \begin{matrix}n,o,p \text{ are odd cubes or}\\ n,o,p \text{ are even non cubes,}\end{matrix}\\ 0; & \begin{matrix}n,o,p \text{ are even cubes or}\\ n,o,p \text{ are odd non cubes.}\end{matrix}\end{cases}$$

Assume that $\gamma>1$, $f(x)=x$ *and* $\theta_3=\{(j_r,k_s,l_t)\}=r^2s^2t^2$. *For* $\varepsilon>0$, $\zeta>0$ *we have*

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{g})_{klm},\tilde{a}\right)\geq\varepsilon\}\right|\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{h_{rst}}{h_{rst}^{\gamma}}\geq\zeta\right\}\in\mathcal{I}_3$$

and

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{g})_{klm},\tilde{b}\right)\geq\varepsilon\right\}\right|\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{h_{rst}}{h_{rst}^{\gamma}}\geq\zeta\right\}\in\mathcal{I}_3.$$

Thus, $(\tilde{g}_{nop})$ *is* f*-lacunary* $A^{\mathcal{I}_3}$*-statistically convergent to both* $\tilde{a}$ *and* $\tilde{b}$*, which is impossible.*

Theorem 3.1. *Let* $y=(\tilde{y}_{klm})$ *and* $g=(\tilde{g}_{klm})$ *be two triple fuzzy sequences and* $\gamma\in(0,1]$. *Then, the subsequent statements are valid:*

(a) If $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}\tilde{y}_{klm}=\tilde{y}_{000}$ *and* $z\in\mathbb{C}$*, then* $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}z\tilde{y}_{klm}=z\tilde{y}_{000}$.

(b) If $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}\tilde{y}_{klm}=\tilde{y}_{000}$ *and* $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{k,l,m}\tilde{g}_{klm}=\tilde{g}_{000}$*, then* $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\lim_{k,l,m}(\tilde{y}_{klm}+\tilde{g}_{klm})=\tilde{y}_{000}+\tilde{g}_{000}$.

Proof. (a) For $z=0$, the result holds trivially. Let $z\neq0$, for given $\varepsilon>0$, we obtain

$$\{((k,l,m)\in I_{r,s,t}:d\left((Az\tilde{y})_{klm},z\tilde{y}_{000}\right)\geq\varepsilon)\}$$
$$=\{((k,l,m)\in I_{r,s,t}:|z|\,d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\geq\varepsilon)\}$$
$$\subseteq\left\{\left((k,l,m)\in I_{r,s,t}:d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\geq\frac{\varepsilon}{|z|}\right)\right\},$$

and, so we have

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\{(k,l,m)\in I_{r,s,t}:d\left((Az\tilde{y})_{klm},z\tilde{y}_{000}\right)\geq\varepsilon\}\right|\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f(h_{rst}^{\gamma})}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\geq\frac{\varepsilon}{|z|}\right\}\right|\right)\geq\zeta\right\}$$
$$\in\mathcal{I}_3,$$

for all $\zeta > 0$.

(b) It is derived from the fact that

$$\begin{aligned}
&\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}+\tilde{g}\right)\right)_{klm}, \tilde{y}_{000}+\tilde{g}_{000}\right) \geq \varepsilon\} \\
&\subseteq \{(k,l,m) \in I_{r,s,t} : d\left(\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) + \left(\left(A\left(\tilde{g}\right)\right)_{klm}, \tilde{g}_{000}\right)\right) \geq \varepsilon\} \\
&\subseteq \left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \frac{\varepsilon}{2}\right\} \\
&\cup \left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{g}\right)\right)_{klm}, \tilde{g}_{000}\right) \geq \frac{\varepsilon}{2}\right\}.
\end{aligned}$$

Additionally,

$$\begin{aligned}
&\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}+\tilde{g}\right)\right)_{klm}, \tilde{y}_{000}+\tilde{g}_{000}\right) \geq \varepsilon\}\right|\right) \geq \zeta\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \frac{\varepsilon}{2}\right\}\right|\right) \geq \zeta\right\} \\
&\cup \left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{g}\right)\right)_{klm}, \tilde{g}_{000}\right) \geq \frac{\varepsilon}{2}\right\}\right|\right) \geq \zeta\right\} \\
&\in \mathcal{I}_3,
\end{aligned}$$

is a consequence of this. Therefore, (b) follows. □

Theorem 3.2. *Let f be an unbounded modulus function such that $f(xy) \geq cf(x)f(y)$ for some positive constant $c, x, y \geq 0$ and $0 < \gamma \leq \delta \leq 1$. Then, we have* $\left(A^{\mathcal{I}_3^f} W_{\gamma,\theta_3}\right) \subseteq \left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$.

Proof. Let $\tilde{y} \in \left({}^{A^{\mathcal{I}_3}}W_{\gamma,\theta_3.f}\right)$. Then, for $\varepsilon > 0$ and $\zeta > 0$

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{(k,l,m) \in I_{r,s,t}} f\left(d\left(\left(A\tilde{y}\right)_{klm}, \tilde{y}_{000}\right)\right) \geq \varepsilon\right\} \in \mathcal{I}_3, \tag{3.1}$$

and, so we get

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$

$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}f\left(\sum_{(k,l,m)\in I_{r,s,t}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$

$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\delta}\right)}f\left(\sum_{\substack{(k,l,m)\in I_{r,s,t}\\ d\left((A(\tilde{y}))_{klm},\tilde{y}_0\right)\geq\varepsilon}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right.\right.$$

$$\left.\left.+\sum_{\substack{(k,l,m)\in I_{r,s,t}\\ d\left((A(\tilde{y}))_{klm},\tilde{y}_0\right)<\varepsilon}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$

$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\delta}\right)}f\left(\sum_{\substack{(k,l,m)\in I_{r,s,t}\\ d\left((A(\tilde{y}))_{klm},\tilde{y}_0\right)\geq\varepsilon}} d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}$$

$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\delta}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\left(\tilde{y}\right))_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|\varepsilon\right)\geq\zeta\right\}$$

$$\supseteq\left\{(r,s,t)\in\mathbb{N}^3:\left(\frac{c}{f\left(h_{rst}^{\delta}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\left(\tilde{y}\right))_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|\right)f\left(\varepsilon\right)\right)\geq\zeta\right\}$$

This implies

$$\left\{(r,s,t)\in\mathbb{N}^3:\frac{c}{f\left(h_{rst}^{\delta}\right)}f\left(\left|\left\{(k,l,m)\in I_{r,s,t}:d\left((A\left(\tilde{y}\right))_{klm},\tilde{y}_{000}\right)\geq\varepsilon\right\}\right|f\left(\varepsilon\right)\right)\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3:\frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm},\tilde{y}_{000}\right)\right)\geq\varepsilon\right\}.$$

Using (3.1), we obtain $\tilde{y}\in\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$. □

Using Lemma 2.1, we can give the following theorem.

Theorem 3.3. *Let f be an unbounded modulus function such that $\lim_{t\to\infty}\frac{f(t)}{t}>0$, $\lim_{r,s,t\to\infty}\frac{f(h_{rst})}{f\left(h_{rst}^{\gamma}\right)}=1$ and $0<\gamma\leq\delta\leq 1$. Then,*

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\cap\ell_\infty^3\left(A\right)\subseteq\left(A^{\mathcal{I}_3^f}W_{\delta,\theta_3}\right)\cap\ell_\infty^3\left(A\right).$$

Proof. Assume that $\lim_{t\to\infty}\frac{f(t)}{t}=L$. Then, by Lemma 2.1, $L\leq\frac{f(t)}{t}$, for all $t>0$. Let $\tilde{y}\in\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\cap\ell_\infty^3\left(A\right)$. Then, there exists positive real number M such that

$d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right) \leq M$, for any $k, l, m \in \mathbb{N}$, and

$$\left\{(r,s,t) \in \mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \varepsilon\right\}\right|\right) \geq \zeta\right\} \in \mathcal{I}_3,$$

supplies for every $\varepsilon > 0$ and $\zeta > 0$.

Now,

$$\begin{aligned}
&\left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\delta}\right)} \sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right) \geq \varepsilon\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} \sum_{(k,l,m)\in I_{r,s,t}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right) \geq \varepsilon\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} \left[\sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_0\right) \geq \varepsilon}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right)\right.\right. \\
&\left.\left. + \sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_0\right) < \varepsilon}} f\left(d\left((A\tilde{y})_{klm}, \tilde{y}_{000}\right)\right)\right] \geq \varepsilon\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} \left[\sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) \geq \varepsilon}} f(M) + \sum_{\substack{(k,l,m)\in I_{r,s,t} \\ d\left((A(\tilde{y}))_{klm}, \tilde{y}_{000}\right) < \varepsilon}} f(\varepsilon)\right]\right\} \\
&= \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{1}{f\left(h_{rst}^{\gamma}\right)} f(M) \left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \varepsilon\right\}\right| \geq \zeta\right\} \\
&+ \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{h_{rst}}{f\left(h_{rst}^{\gamma}\right)} f(\varepsilon) \geq \zeta\right\} \\
&\subseteq \left\{(r,s,t) \in \mathbb{N}^3 : \tfrac{L^{-1}}{f\left(h_{rst}^{\gamma}\right)} f(M)\, f\left(\left|\left\{(k,l,m) \in I_{r,s,t} : d\left(\left(A\left(\tilde{y}\right)\right)_{klm}, \tilde{y}_{000}\right) \geq \varepsilon\right\}\right|\right) \geq \zeta\right\} \\
&+ \left\{(k,l,m) \in I_{r,s,t} : \tfrac{L^{-1} f(h_{rst})}{f\left(h_{rst}^{\gamma}\right)} f(\varepsilon) \geq \zeta\right\}.
\end{aligned} \tag{3.2}$$

Using $\lim_{r,s,t\to\infty} \frac{f(h_{rst})}{f\left(h_{rst}^{\gamma}\right)} = 1$, we have $\tilde{y} \in \left(A^{\mathcal{I}_3^f} W_{\delta,\theta_3}\right) \cap \ell_\infty(A)$. □

3.1. **Fuzzy Korovkin-type theorems.** One notable theorem in mathematics is Korovkin's theorem, named after the mathematician Korovkin [21]. This theorem addresses how a sequence of positive linear operators can uniformly approximate continuous functions defined on compact metric spaces. Over time, the theorem's importance has grown across various mathematical disciplines. Researchers have explored its applications in numerous settings and have proposed several extensions in areas such as functional analysis, measure theory, probability theory, and summability theory (see [2], [16], [27], [32]). In this section, we apply lacunary triple sequences, modulus functions, and regular matrices to establish a fuzzy Korovkin-type theorem specifically for triple fuzzy number sequences.

A fuzzy valued function $\tilde{f}:[a,b]\times[a,b]\times[a,b]\to\mathcal{F}_{\mathbb{R}}$ is fuzzy continuous at $(u_{000},y_{000},z_{000})$ in $([a,b])^3=[a,b]\times[a,b]\times[a,b]$ if $(u_{klm},y_{klm},z_{klm})\to(u_{000},y_{000},z_{000})$, then

$$d^*\left(\tilde{f}\left(u_{klm},y_{klm},z_{klm}\right),\tilde{f}\left(u_{000},y_{000},z_{000}\right)\right)\to 0,\ \text{as } k,l,m\to\infty,$$

where

$$\begin{aligned}&d^*\left(\tilde{f}\left(u_{klm},y_{klm},z_{klm}\right),\tilde{f}\left(u_{000},y_{000},z_{000}\right)\right)\\&=\sup_{(u,y,z)\in([a,b])^3} d\left(\tilde{f}\left(u_{klm},y_{klm},z_{klm}\right),\tilde{f}\left(u_{000},y_{000},z_{000}\right)\right).\end{aligned}$$

If $\tilde{f}$ is fuzzy continuous at every point in $[a,b]$, then $\tilde{f}$ is fuzzy continuous on $([a,b])^3$. The set of all fuzzy continuous functions on the interval $([a,b])^3$ is denoted by $C_{\mathcal{F}}\left(([a,b])^3\right)$, and $C\left(([a,b])^3\right)$ represents the space of all continuous functions on $([a,b])^3$.

An operator $\tilde{\mathcal{T}}:C_{\mathcal{F}}\left(([a,b])^3\right)\to C_{\mathcal{F}}\left(([a,b])^3\right)$ is fuzzy linear, if

$$\tilde{\mathcal{T}}\left(\lambda_1\odot\tilde{f}_1\oplus\lambda_2\odot\tilde{f}_2;u,y,z\right)=\lambda_1\odot\tilde{\mathcal{T}}\left(\tilde{f}_1;u,y,z\right)\oplus\lambda_2\odot\tilde{\mathcal{T}}\left(\tilde{f}_2;u,y,z\right),$$

for every $\lambda_1,\lambda_2\in\mathbb{R}$ and $\tilde{f}_1,\tilde{f}_2\in C_{\mathcal{F}}\left(([a,b])^3\right)$. Furthermore, $\tilde{\mathcal{T}}$ is fuzzy positive linear operator, if it is fuzzy linear and

$$\tilde{\mathcal{T}}\left(\tilde{f}_1;u,y,z\right)\leq\tilde{\mathcal{T}}\left(\tilde{f}_2;u,y,z\right)$$

for all $\tilde{f}_1,\tilde{f}_2\in C_{\mathcal{F}}\left(([a,b])^3\right)$, and for any $(u,y,z)\in([a,b])^3$, and with $\tilde{f}_1(u,y,z)\leq\tilde{f}_2(u,y,z)$.

Theorem 3.4. *Assume that* $\left(\tilde{\mathcal{T}}_{klm}\right)$ *be a triple sequence of positive linear operators from* $C_{\mathcal{F}}\left(([a,b])^3\right)$ *to* $C_{\mathcal{F}}\left(([a,b])^3\right)$. *Suppose that there is a sequence* $(\mathcal{T}_{klm})$ *of positive linear operators from* $C\left(([a,b])^3\right)$ *into* $C\left(([a,b])^3\right)$ *such that*

$$\left\{\tilde{\mathcal{T}}_{klm}\left(\tilde{f};u,y,z\right)\right\}_{\alpha}^{\pm}=\mathcal{T}_{klm}\left(\tilde{f}_{\alpha}^{\pm};u,y,z\right),\ \ (k,l,m\in\mathbb{N}) \tag{3.3}$$

for each $\tilde{f}\in C_{\mathcal{F}}\left(([a,b])^3\right)$, $\alpha\in[0,1]$ *and* $(u,y,z)\in([a,b])^3$. *Then, if*

$$\left\{(k,l,m)\in I_{r,s,t}:\left\|\mathcal{T}_{klm}\left(g_i\right)-g_i\right\|\geq\varepsilon\right\}\in\mathcal{I}_3;\ \left(i=\overline{0,4}\right), \tag{3.4}$$

where $g_0=1,\ g_1=u,\ g_2=y,\ g_3=z,\ g_4=u^2+y^2+z^2$, *we have*

$$\left\{(k,l,m)\in I_{r,s,t}:d^*\left(\tilde{\mathcal{T}}_{klm}\left(\tilde{f}\right),\tilde{f}\right)\geq\varepsilon\right\}\in\mathcal{I}_3;\ \forall\tilde{f}\in C_{\mathcal{F}}\left[a,b\right], \tag{3.5}$$

for every $\varepsilon>0$.

Theorem 3.5. *Consider a fuzzy sequence* $\left(\tilde{\mathcal{T}}_{klm}\right)$ *of positive linear operators from* $C_{\mathcal{F}}\left(([a,b])^3\right)$ *into* $C_{\mathcal{F}}\left(([a,b])^3\right)$*. Suppose there exists a sequence* $(\mathcal{T}_{klm})$ *of positive linear operators from* $C\left(([a,b])^3\right)$ *into* $C\left(([a,b])^3\right)$ *such that equation (3.3) holds. If*

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}\left\|\mathcal{T}_{klm}\left(g_i\right)-g_i\right\|=0,\ \left(i=\overline{0,4}\right), \tag{3.6}$$

where $g_0=1, g_1=u, g_2=y, g_3=z$ *and* $g_4=u^2+y^2+z^2$*, then we have*

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}d^*\left(\tilde{\mathcal{T}}_{klm}\left(\tilde{f}\right),\tilde{f}\right)=0,\ \forall\tilde{f}\in C_{\mathcal{F}}\left[a,b\right]. \tag{3.7}$$

Proof. Let $\tilde{f}\in C_{\mathcal{F}}\left(([a,b])^3\right)$ and $(u,y,z)\in([a,b])^3$. Since $\tilde{f}_\alpha^{\pm}$ is continuous on $([a,b])^3$, for every $\varepsilon>0$ there exists $\delta>0$ such that $\left|\tilde{f}_\alpha^{\pm}(e,f,h)-\tilde{f}_\alpha^{\pm}(u,y,z)\right|<\varepsilon$, whenever $|e-u|<\delta, |f-y|<\delta, |h-z|<\delta$. Since $\tilde{f}$ is fuzzy bounded, we have $\left|\tilde{f}_\alpha^{\pm}(u,y,z)\right|\le\mathcal{K}_\alpha^{\pm}$ for all $(u,y,z)\in([a,b])^3$. Thus, we get

$$\left|\tilde{f}_\alpha^{\pm}(e,f,h)-\tilde{f}_\alpha^{\pm}(u,y,z)\right|\le\varepsilon+\frac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\left|(e-u)^2+(f-y)^2+(h-z)^2\right| \tag{3.8}$$

for every $(e,f,h),(u,y,z)\in([a,b])^3$.

Applying $(A\mathcal{T}(g_0;u,y,z))_{klm}$ on both the sides for a fixed (u,y,z) and by the monotonicity and linearity of $(A\mathcal{T}(g_0;u,y,z))_{klm}$, we have

$$\begin{aligned}
&\left|\left(A\mathcal{T}\left(\tilde{f}_\alpha^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\left(A\mathcal{T}\left(\tilde{f}_\alpha^{\pm}(u,y,z);u,y,z\right)\right)_{klm}\right|\\
&\le\left|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\tfrac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\left(A\mathcal{T}\left((e-u)^2+(f-y)^2+(h-z)^2;u,y,z\right)\right)_{klm}\right|\\
&=\left|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\tfrac{2\mathcal{K}_\alpha^{\pm}}{\delta^2}\left(\left(A\mathcal{T}\left(e^2+f^2+h^2;u,y,z\right)\right)_{klm}\right.\right.\\
&-2u\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-2y\left(A\mathcal{T}(f;u,y,z)\right)_{klm}|\\
&-2z\left(A\mathcal{T}(h;u,y,z)\right)_{klm}+\left(u^2+y^2+z^2\right)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}|.
\end{aligned} \tag{3.9}$$

Using (3.8) and (3.9), we have

$$
\begin{aligned}
&\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\right| \\
&=\Big|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(A\mathcal{T}(1;u,y,z)\right)_{klm} \\
&+\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\Big| \\
&=\Big|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(u,y,z);u,y,z\right)\right)_{klm} \\
&+\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\Big| \\
&\le\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(e,f,h);u,y,z\right)\right)_{klm}-\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}(u,y,z);u,y,z\right)\right)_{klm}\right| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\le\Big|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\Big[\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm} \\
&-2u\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-2y\left(A\mathcal{T}(f;u,y,z)\right)_{klm} \\
&-2z\left(A\mathcal{T}(h;u,y,z)\right)_{klm}+\left(u^{2}+y^{2}+z^{2}\right)\left(A\mathcal{T}(1;u,y,z)\right)_{klm}\Big]\Big| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\le\Big|\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}+\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\Big[\left(\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right) \\
&-2u\left(\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-u\right)-2y\left(\left(A\mathcal{T}(f;u,y,z)\right)_{klm}-y\right) \\
&-2z\left(\left(A\mathcal{T}(h;u,y,z)\right)_{klm}-z\right)+\left(u^{2}+y^{2}+z^{2}\right)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\Big]\Big| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\le\left|\varepsilon+\varepsilon\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-\varepsilon\right|+\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\Big[\left|\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right| \\
&+|2u|\left|\left(A\mathcal{T}(e;u,y,z)\right)_{klm}-u\right|+|2y|\left|\left(A\mathcal{T}(f;u,y,z)\right)_{klm}-y\right| \\
&+|2z|\left|\left(A\mathcal{T}(h;u,y,z)\right)_{klm}-z\right|+\left(u^{2}+y^{2}+z^{2}\right)\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right|\Big] \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\le\varepsilon+\varepsilon\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right|+\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left|\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right| \\
&+\tfrac{4\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}|u|\left|\left(A\mathcal{T}\left((e+f+h);u,y,z\right)\right)_{klm}-u\right|+\tfrac{4\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}|y|\left|\left(A\mathcal{T}\left((e+f+h);u,y,z\right)\right)_{klm}-y\right| \\
&+\tfrac{4\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}|z|\left|\left(A\mathcal{T}\left((e+f+h);u,y,z\right)\right)_{klm}-z\right|+\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left(u^{2}+y^{2}+z^{2}\right)\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right| \\
&+\left|\tilde{f}_{\alpha}^{\pm}(u,y,z)\right|\left|\left(\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right)\right| \\
&\le\varepsilon+\left(\varepsilon+\tfrac{2\mathcal{K}_{\alpha}^{\pm}\left(B^{2}+C^{2}+D^{2}\right)}{\delta^{2}}+\mathcal{K}_{\alpha}^{\pm}\right)\left|\left(A\mathcal{T}(1;u,y,z)\right)_{klm}-1\right| \\
&+\tfrac{4\mathcal{K}_{\alpha}^{\pm}B}{\delta^{2}}\left|\left(A\mathcal{T}\left((e+f+h);u,y,z\right)\right)_{klm}-u\right|+\tfrac{4\mathcal{K}_{\alpha}^{\pm}C}{\delta^{2}}\left|\left(A\mathcal{T}\left((e+f+h);u,y,z\right)\right)_{klm}-y\right| \\
&+\tfrac{4\mathcal{K}_{\alpha}^{\pm}D}{\delta^{2}}\left|\left(A\mathcal{T}\left((e+f+h);u,y,z\right)\right)_{klm}-z\right| \\
&+\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^{2}}\left|\left(A\mathcal{T}\left(\left(e^{2}+f^{2}+h^{2}\right);u,y,z\right)\right)_{klm}-\left(u^{2}+y^{2}+z^{2}\right)\right| \\
&\le\varepsilon+\mathcal{M}_{\alpha}^{\pm}\left(\left|\left(A\mathcal{T}(g_{0};u,y,z)\right)_{klm}-g_{0}\right|+\left|\left(A\mathcal{T}(g_{1};u,y,z)\right)_{klm}-g_{1}\right|\right. \\
&\left.+\left|\left(A\mathcal{T}(g_{2};u,y,z)\right)_{klm}-g_{2}\right|+\left|\left(A\mathcal{T}(g_{3};u,y,z)\right)_{klm}-g_{3}\right|+\left|\left(A\mathcal{T}(g_{4};u,y,z)\right)_{klm}-g_{4}\right|\right)
\end{aligned}
$$

where

$$\mathcal{M}_{\alpha}^{\pm}=\max\left\{\varepsilon+\tfrac{2\mathcal{K}_{\alpha}^{\pm}\left(B^2+C^2+D^2\right)}{\delta^2}+\mathcal{K}_{\alpha}^{\pm},\tfrac{4\mathcal{K}_{\alpha}^{\pm}(B+C+D)}{\delta^2},\tfrac{2\mathcal{K}_{\alpha}^{\pm}}{\delta^2}\right\},$$

$$B=\max\left\{|u|\right\},\ C=\max\left\{|y|\right\}\text{ and }D=\max\left\{|z|\right\}.$$

Then, taking supremum over $(u,y,z)\in([a,b])^3$, we obtain

$$\left\|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{\pm}\right)\right)_{klm}-\tilde{f}_{\alpha}^{\pm}(u,y,z)\right\|\leq\varepsilon+\mathcal{M}_{\alpha}^{\pm}\sum_{i=0}^{4}\left\|\left(A\mathcal{T}\left(g_i\right)\right)_{klm}-g_i\right\|. \tag{3.10}$$

Using the definition of $d^*(.,.)$ and the relation (3.3), we have

$$d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm},\tilde{f}\right)=\sup_{(u,y,z)\in([a,b])^3}d\left(\left(A\mathcal{T}\left(\tilde{f};u,y,z\right)\right)_{klm},\tilde{f}(u,y,z)\right) \tag{3.11}$$

$$=\sup_{(u,y,z)\in([a,b])^3}\sup_{\alpha\in[0,1]}\max\left\{\left|\left\{\left(A\tilde{\mathcal{T}}\left(\tilde{f};u,y,z\right)\right)_{klm}\right\}_{\alpha}^{-}-\left\{\tilde{f}(u,y,z)\right\}_{\alpha}^{-}\right|,\right.$$
$$\left.\left|\left\{\left(A\tilde{\mathcal{T}}\left(\tilde{f};u,y,z\right)\right)_{klm}\right\}_{\alpha}^{+}-\left\{\tilde{f}(u,y,z)\right\}_{\alpha}^{+}\right|\right\}$$
$$=\sup_{(u,y,z)\in([a,b])^3}\sup_{\alpha\in[0,1]}\max\left\{\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{-};u,y,z\right)\right)_{klm}-\tilde{f}_{\alpha}^{-}(u,y,z)\right|,\right.$$
$$\left.\left|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{+};u,y,z\right)\right)_{klm}-\tilde{f}_{\alpha}^{+}(u,y,z)\right|\right\}$$
$$=\sup_{\alpha\in[0,1]}\max\left\{\left\|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{-}\right)\right)_{klm}-\tilde{f}_{\alpha}^{-}\right\|,\left\|\left(A\mathcal{T}\left(\tilde{f}_{\alpha}^{+}\right)\right)_{klm}-\tilde{f}_{\alpha}^{+}\right\|\right\}.$$

From (3.10) and (3.11), we have

$$d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm},\tilde{f}\right)\leq\varepsilon+\mathcal{M}_{\alpha}\sum_{i=0}^{4}\left\|\left(A\mathcal{T}\left(g_i\right)\right)_{klm}-g_i\right\|,$$

where $\mathcal{M}_{\alpha}=\sup_{\alpha\in[0,1]}\max\left\{\mathcal{M}_{\alpha}^{-},\mathcal{M}_{\alpha}^{+}\right\}$.

For a given $t>0$, choose $\epsilon>0$ such that $t>\epsilon$. Then, let

$$\mathcal{D}_{r,s,t}=\left\{(k,l,m)\in I_{r,s,t}:d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm},\tilde{f}\right)\geq\epsilon\right\}$$

and

$$\mathcal{D}_{r,s,t;i}=\left\{(k,l,m)\in I_{r,s,t}:\left\|\left(A\mathcal{T}\left(g_i\right)\right)_{klm}-g_i\right\|\geq\frac{t-\epsilon}{3\mathcal{M}_{\alpha}}\right\},$$

where $i=\overline{0,4}$ and $(r,s,t)\in\mathbb{N}^3$. Therefore, $\mathcal{D}_{r,s,t}\subseteq\cup_{i=0}^{4}\mathcal{D}_{r,s,t;i}$. This implies

$$\left|\left\{(k,l,m)\in I_{r,s,t}:d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right),\tilde{f}\right)\right)_{klm}\geq\epsilon\right\}\right|$$
$$\leq\sum_{i=0}^{4}\left|\left\{(k,l,m)\in I_{r,s,t}:\left\|\left(A\mathcal{T}\left(g_i\right)\right)_{klm}-g_i\right\|\geq\frac{t-\epsilon}{3\mathcal{M}_{\alpha}}\right\}\right|$$

and using (3.6) for $\zeta > 0$ we get

$$\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)} f\left(\left|\left\{(k,l,m)\in I_{r,s,t} : d^*\left(\left(A\tilde{\mathcal{T}}\left(\tilde{f}\right)\right)_{klm}, \tilde{f}\right)\geq\epsilon\right\}\right|\right)\geq\zeta\right\} \tag{3.12}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{f\left(h_{rst}^{\gamma}\right)}\sum_{i=0}^{4} f\left(\left|\left\{(k,l,m)\in I_{r,s,t} : \left\|\left(A\mathcal{T}\left(g_i\right)\right)_{klm}-g_i\right\|\geq\frac{t-\epsilon}{3\mathcal{M}_\alpha}\right\}\right|\right)\geq\zeta\right\}$$

belongs to $\mathcal{I}_3$. Therefore, we have

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)\text{-}d^*\left(\tilde{\mathcal{T}}_{klm}\left(\tilde{f}\right),\tilde{f}\right)=0,\ \forall\tilde{f}\in C_{\mathcal{F}}\left[a,b\right].$$

□

Example 3.1. *Let $(\tilde{y}_{nop})$ be a triple fuzzy sequence defined by*

$$\tilde{y}_{nop}\left(t\right)=\begin{cases}1, & \text{if } n,o,p \text{ are squares,}\\ 0, & \text{otherwise,}\end{cases}\quad\forall t\in\left[0,1\right].$$

Also, consider the matrix $A=(a_{nkolpm})$ defined by

$$a_{nkolpm}=\begin{cases}1, & \text{if } klm=\left(nop\right)^2,\\ 0, & \text{otherwise.}\end{cases}$$

Then

$$\left(A\tilde{y}\left(t\right)\right)_{nop}=\sum_{k=1,l=1,m=1}^{\infty}a_{nkolpm}\tilde{y}_{klm}=\begin{cases}1, & \text{if } n,o,p \text{ are squares,}\\ 0, & \text{otherwise, } \forall t\in\left[0,1\right].\end{cases}$$

Now, assume $f(x)=x$, we have for $\varepsilon>0,\zeta>0$

$$\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{h_{rst}^{\alpha}}\left|\left\{(k,l,m)\in I_{r,s,t} : d\left(\left(A\tilde{y}\right)_{klm},0\right)\geq\varepsilon\right\}\right|\geq\zeta\right\}$$
$$\subseteq\left\{(r,s,t)\in\mathbb{N}^3 : \frac{1}{h_{rst}^{\alpha}}\geq\zeta\right\}\in\mathcal{I}_3.$$

This implies $(\tilde{y}_{nop})$ is f-lacunary A-statistical convergent to 0 but it is not convergent to 0. Let $\tilde{y}\in C_{\mathcal{F}}\left([0,1]\right)^3$, $(a,b,c)\in\left([0,1]\right)^3$ and consider the fuzzy Bernstein operators

$$\tilde{\mathcal{B}}_{nop}\left(\tilde{y};a,b,c\right)$$
$$=\bigoplus_{i=0,j=0,k=0}^{n,o,p}\binom{n}{i}\binom{o}{j}\binom{p}{k}a^i\left(1-a\right)^{n-i}b^j\left(1-b\right)^{o-j}c^k\left(1-c\right)^{p-k}\odot\tilde{y}\left(\frac{i}{n},\frac{j}{o},\frac{k}{p}\right).$$

This implies

$$\begin{aligned}
&\left\{\tilde{\mathcal{B}}_{nop}\left(\tilde{y};a,b,c\right)\right\}_{\alpha}^{\pm} \\
&=\mathcal{B}_{nop}\left(\tilde{y}_{\alpha}^{\pm};a,b,c\right) \\
&=\sum_{i=0,j=0,k=0}^{n,o,p}\binom{n}{i}\binom{o}{j}\binom{p}{k}a^{i}\left(1-a\right)^{n-i}b^{j}\left(1-b\right)^{o-j}c^{k}\left(1-c\right)^{p-k}\tilde{y}_{\alpha}^{\pm}\left(\frac{i}{n},\frac{j}{o},\frac{k}{p}\right),
\end{aligned}$$

where $\tilde{y}_{\alpha}^{\pm}\in C\left([0,1]\right)^{3}$ *and* $\alpha\in[0,1]$. *We define the sequence of fuzzy positive linear operators on* $C_{\mathcal{F}}\left([0,1]\right)^{3}$ *as follows:*

$$A\tilde{\mathcal{T}}_{nop}\left(\tilde{y}\left(x\right);a,b,c\right)=\left(\left(A\tilde{y}\right)_{nop}+1\right)\odot\tilde{\mathcal{B}}_{nop}\left(\tilde{y};a,b,c\right),$$

using these polynomials. Currently,

$$\begin{aligned}
&A\mathcal{T}_{nop}\left(\tilde{y}_{\alpha}^{\pm};a,b,c\right) \\
&=\left(\left(A\tilde{y}\right)_{nop}+1\right)\sum_{i=0,j=0,k=0}^{n,o,p}\binom{n}{i}\binom{o}{j}\binom{p}{k}a^{i}\left(1-a\right)^{n-i}b^{j}\left(1-b\right)^{o-j}c^{k}\left(1-c\right)^{p-k}\tilde{y}_{\alpha}^{\pm}\left(\frac{i}{n},\frac{j}{o},\frac{k}{p}\right).
\end{aligned} \tag{3.13}$$

Then, we calculate

$$\begin{aligned}
\left(A\mathcal{T}\left(g_0;u,y,z\right)\right)_{nop}&=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_0\left(u,y,z\right),\\
\left(A\mathcal{T}\left(g_1;u,y,z\right)\right)_{nop}&=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_1\left(u,y,z\right),\\
\left(A\mathcal{T}\left(g_2;u,y,z\right)\right)_{nop}&=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_2\left(u,y,z\right),\\
\left(A\mathcal{T}\left(g_3;u,y,z\right)\right)_{nop}&=\left(\left(A\tilde{y}\right)_{nop}+1\right)g_3\left(u,y,z\right),\\
\left(A\mathcal{T}\left(g_4;u,y,z\right)\right)_{nop}&=\left(\left(A\tilde{y}\right)_{nop}+1\right)\left(g_4\left(u,y,z\right)+u^2+y^2+z^2+\frac{u-u^2}{n}+\frac{y-y^2}{o}+\frac{z-z^2}{p}\right).
\end{aligned}$$

Since $\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)$-$\lim_{nop}\tilde{y}_{nop}=0$, *we conclude thatwhere* $g_0=1,g_1=u,g_2=y,g_3=z,g_4=u^2+y^2+z^2$.

$$\begin{aligned}
&\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim\nolimits_{nop}\left(A\mathcal{T}\left(g_0;u,y,z\right)\right)_{nop}=1,\\
&\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim\nolimits_{nop}\left(A\mathcal{T}\left(g_1;u,y,z\right)\right)_{nop}=u,\\
&\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim\nolimits_{nop}\left(A\mathcal{T}\left(g_2;u,y,z\right)\right)_{nop}=y,\\
&\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim\nolimits_{nop}\left(A\mathcal{T}\left(g_3;u,y,z\right)\right)_{nop}=z,\\
&\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim\nolimits_{nop}\left(A\mathcal{T}\left(g_3;u,y,z\right)\right)_{nop}=u^2+y^2+z^2.
\end{aligned}$$

So, by using Theorem 3.5, we have

$$\left(A^{\mathcal{I}_3^f}\text{-}stat_{\gamma,\theta_3}\right)-\lim_{nop}d^{*}\left(\left(A\tilde{\mathcal{T}}\left(\tilde{y}\right)\right)_{nop},\tilde{y}\right)=0$$

However, since $(\tilde{y}_{nop})$ *is not convergent, Theorem 3.4 does not work for operator defined by (3.13). This demonstrates the superiority of our Theorem 3.5 over Theorem 3.4.*

4. Conclusion

In this study, we explore the concepts of strongly f-lacunary A-summability of order γ and f-lacunary $A^{\mathcal{I}_3}$-statistical convergence of order γ for sequences of fuzzy numbers. We also establish that for $0 < \gamma \leq 1$, the f-lacunary $A^{\mathcal{I}_3}$-statistical convergence of order γ is well-defined. Moreover, we investigate the relationships between newly defined spaces and show that, under certain conditions, these spaces are interconnected. As a significant application, we prove a fuzzy Korovkin-type theorem and provide an example that highlights the advantages of our result over the classical version. By utilizing f-lacunary $A^{\mathcal{I}_3}$-statistical convergence, this paper offers a new perspective on the fuzzy Korovkin-type approximation theorem. Further exploration is needed to fully understand these concepts and the results pertaining to double sequences of fuzzy numbers.

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LORENTZIAN β-KENMOTSU MANIFOLD ADMITTING GENERALIZED TANAKA-WEBSTER CONNECTION

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Abstract. In this manuscript, we investigate Lorentzian β-Kenmotsu manifold admitting generalized Tanaka-Webster connection (GTWC) $\widetilde{\widetilde{\nabla}}$. We study curvature tensor and its properties with respect to the above connection. Further, we study the connection on extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold. We also investigate the properties of projectively flat, ζ-projectively flat and η-parallel φ-tensor on Lorentzian β-Kenmotsu manifold admitting the connection $\widetilde{\widetilde{\nabla}}$. Moreover, we study Ricci soliton on the above manifold with respect to the connection (GTWC). Finally, we give an example of 3-dimensional Lorentzian β-Kenmotsu manifold verifying our results.

Keywords: Lorentzian β-Kenmotsu manifold, generalized Tanaka-Webster connection, generalized η-Einstein manifold, Ricci soliton, projectively flat.

2010 Mathematics Subject Classification: Primary 53C05, 53C20, 53C25, Secondary 53D15.

1. Introduction

The semi-Riemannian geometry [29] fascinates the researchers because of its abilities to determine the several problems of science, technology, medical and their related areas. A differentiable manifold $\mathfrak{M}$ of dimension $(2n+1)$ equipped with a semi-Riemannian metric g, whose signature is (p, q), $(p+q = 2n+1)$, referred to as $(2n+1)$-dimensional semi-Riemannian manifold. In particular, if we replace p by 1 and q by $2n$, then the semi-Riemannian manifold

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$\mathfrak{M}$ reduces into Lorentzian manifold. The basic characterization of the vectors in a Lorentzian manifold were the starting point to study the geometry of it. As a reason, Lorentzian manifold $\mathfrak{M}$ is the finest choice for the researchers to study the general theory of relativity and cosmological models. The material substance of the cosmos is referred to behave like a perfect fluid space-time in standard cosmological models. In describing the gravity of the space-time, the Riemannian curvature $\mathfrak{R}$, the Ricci tensor $\mathcal{S}$, and the scalar curvature $\mathfrak{r}$ play an essential role.

In the Gray-Hervella classification of almost Hermitian manifolds [7], there appears a class $\mathcal{W}_4$, of Hermitian manifolds which are closely related to locally conformal Kähler manifolds [5]. An almost contact metric structure $(\varphi, \zeta, \eta, g)$ on $\mathfrak{M}$ is referred to as trans-Sasakian structure [15] if $(\mathfrak{M} \times \mathbb{R}, \mathcal{J}, \mathcal{G})$ belongs to the class $\mathcal{W}_4$ [7], where $\mathcal{J}$ is the almost complex structure on $\mathfrak{M} \times \mathbb{R}$ defined by

$$\mathcal{J}\left(\mathfrak{U}_1, \frac{fd}{dt}\right) = \left(\varphi\mathfrak{U}_1 - f\zeta, \eta(\mathfrak{U}_1)\frac{fd}{dt}\right)$$

for all vector fields $\mathfrak{U}_1$ on $\mathfrak{M}$, smooth functions f on $\mathfrak{M} \times \mathbb{R}$ and $\mathcal{G}$ is the product metric on $\mathfrak{M} \times \mathbb{R}$. This can be defined by [4]

$$(\nabla_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2 = \alpha(g(\mathfrak{U}_1, \mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\mathfrak{U}_1) + \beta(g(\varphi\mathfrak{U}_1, \mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\varphi\mathfrak{U}_1) \tag{1.1}$$

for some smooth functions α, β on $\mathfrak{M}$ and we say that the trans-Sasakian structure is of type (α, β).

The concept of α-Sasakian and β-Kenmotsu manifolds was initiated by Janssens and Vanhecke in 1981, where α and β are non-zero real numbers. We know that [11] trans-Sasakian structure of type $(0,0)$, $(0,\beta)$, and $(\alpha, 0)$ are cosymplectic [3, 4], β-Kenmotsu, and α-Sasakian, respectively. Marrero [13] proved that a trans-Sasakian manifold of dimension $n \geq 5$ is either cosymplectic or α-Sasakian or β-Kenmotsu manifold.

Tanno [25] studied the generalized Tanaka-Webster connection (GTWC) for contact metric manifolds by using the canonical connection. This connection coincides with the Tanaka-Webster connection if the associated CR-structure is integrable. Using this connection, some characterizations of real hypersurfaces in complex space forms [23] have been studied by few geometers. Recently, many authors [6, 12, 16, 18, 20, 22] studied generalized Tanaka-Webster connection (GTWC) in Kenmotsu manifolds.

Hamilton [8] introduced the theory of Ricci flow to establish a canonical metric on a smooth manifold in 1982. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial t} g_{ij}(t) = -2\mathfrak{R}_{ij}.$$

A Ricci soliton $(g, \mathcal{V}, \Theta)$ on a Riemannian manifold $(\mathfrak{M}, g)$ is a generalization of an Einstein metric such that it satisfies the following condition [9, 10]:

$$\mathfrak{L}_{\mathcal{V}} g + 2\mathcal{S} + 2\Theta g = 0, \tag{1.2}$$

where $\mathcal{S}$ is the Ricci tensor, $\mathfrak{L}_{\mathcal{V}}$ is the Lie derivative operator along the vector field $\mathcal{V}$ on $(\mathfrak{M}, g)$ and Θ is a real number. The Ricci soliton $(g, \mathcal{V}, \Theta)$ is said to be shrinking, steady, and expanding according to $\Theta < 0, \Theta = 0$, and $\Theta > 0$, respectively.

In this paper, we have taken β as a real constant. Motivated by above studies, the present work is classified as follows: After the introduction, we give a brief account of Lorentzian β-Kenmotsu manifold in section 2. In section 3, we study the expressions for curvature tensor and some results on Lorentzian β-Kenmotsu manifold with respect to GTWC $\widetilde{\nabla}$. In section 4, we also study extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$. In section 5, we investigate the properties of projectively flat, ζ-projectively flat and η-parallel φ-tensor on Lorentzian β-Kenmotsu manifold with respect to the GTWC $\widetilde{\nabla}$. Moreover, in section 6, we study Ricci soliton on Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$. In the last section, we give an example of 3-dimensional Lorentzian β-Kenmotsu manifold with respect to the GTWC $\widetilde{\nabla}$ varifying our results.

2. Preliminaries

A differentiable manifold of dimension $(2n+1)$ is referred to as Lorentzian β-Kenmotsu manifold if it admits a $(1,1)$-tensor field φ, a contravariant vector field ζ, a covariant vector field η and Lorentzian metric g which satisfy

$$\eta(\zeta) = -1, \quad \varphi\zeta = 0, \quad \eta(\varphi\mathfrak{U}_1) = 0, \tag{2.3}$$

$$\varphi^2(\mathfrak{U}_1) = \mathfrak{U}_1 + \eta(\mathfrak{U}_1)\zeta, \quad g(\mathfrak{U}_1, \zeta) = \eta(\mathfrak{U}_1), \tag{2.4}$$

$$g(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = g(\mathfrak{U}_1, \mathfrak{U}_2) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2), \quad g(\varphi\mathfrak{U}_1, \mathfrak{U}_2) = g(\mathfrak{U}_1, \varphi\mathfrak{U}_2) \tag{2.5}$$

$\forall\, \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$, where $\mathfrak{X}(\mathfrak{M})$ is a set of all smooth vector fields on $\mathfrak{M}$. Then such a quartet $(\varphi, \zeta, \eta, g)$ is known as Lorentzian para-contact quartet and the manifold $\mathfrak{M}$ with a Lorentzian para-contact quartet is referred to as a Lorentzian para-contact manifold [14, 19, 21].

On a Lorentzian para-contact manifold, we also have

$$(\nabla_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2 = \beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\varphi\mathfrak{U}_1] \tag{2.6}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$, where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Therefore a Lorentzian para-contact manifold satisfying (2.6) is referred to as a Lorentzian β-Kenmotsu manifold [27].

On a Lorentzian β-Kenmotsu manifold $\mathfrak{M}$, the following relations hold [1, 2]:

$$\nabla_{\mathfrak{U}_1}\zeta = \beta[\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\zeta], \tag{2.7}$$

$$(\nabla_{\mathfrak{U}_1}\eta)\mathfrak{U}_2 = \beta[g(\mathfrak{U}_1, \mathfrak{U}_2) - \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)], \tag{2.8}$$

$$\eta(\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3) = \beta^2[g(\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2) - g(\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)], \tag{2.9}$$

$$\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\zeta = \beta^2[\eta(\mathfrak{U}_1)\mathfrak{U}_2 - \eta(\mathfrak{U}_2)\mathfrak{U}_1], \tag{2.10}$$

$$\mathfrak{R}(\zeta, \mathfrak{U}_1)\mathfrak{U}_2 = \beta^2[\eta(\mathfrak{U}_2)\mathfrak{U}_1 - g(\mathfrak{U}_1, \mathfrak{U}_2)\zeta], \tag{2.11}$$

$$\mathcal{S}(\mathfrak{U}_1, \zeta) = -2n\beta^2\eta(\mathfrak{U}_1), \tag{2.12}$$

$$\mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) = g(\mathfrak{Q}\mathfrak{U}_1, \mathfrak{U}_2), \tag{2.13}$$

$$\mathfrak{Q}\mathfrak{U}_1 = -2n\beta^2\mathfrak{U}_1, \tag{2.14}$$

$$\mathfrak{Q}\zeta = -2n\beta^2\zeta, \tag{2.15}$$

$$\mathcal{S}(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = g(\mathfrak{Q}\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2). \tag{2.16}$$

Using (2.5), (2.13), (2.14) and $\mathfrak{Q}\varphi = \varphi\mathfrak{Q}$, we have

$$\mathcal{S}(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) - 2n\beta^2\eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2), \tag{2.17}$$

$$\mathcal{S}(\zeta, \zeta) = 2n\beta^2 \tag{2.18}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$. Where $\mathfrak{R}$, $\mathcal{S}$, and $\mathfrak{Q}$ denote the curvature tensor of type $(1, 3)$, Ricci tensor of type $(0, 2)$, and Ricci operator, respectively with respect to the connection ∇.

Definition 2.1. *The projective curvature tensor $\mathcal{P}$ in $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold $\mathfrak{M}$ with respect to the connection ∇ is defined by*

$$\mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 = \mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 - \frac{1}{2n}[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{Q}\mathfrak{U}_1 - g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{Q}\mathfrak{U}_2] \tag{2.19}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$. *The manifold is said to be projectively flat if $\mathcal{P}$ vanishes identically on $\mathfrak{M}$.*

Definition 2.2. *A $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold is said to be ζ-projectively flat with respect to Levi-Civita connection ∇ if*

$$\mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = 0 \tag{2.20}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$.

Definition 2.3. *If the $(1,1)$ tensor φ is η-parallel in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$, then we have*

$$g((\nabla_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2,\mathfrak{U}_3) = 0 \tag{2.21}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$.

3. The generalized Tanaka-Webster connection (GTWC) $\widetilde{\widetilde{\nabla}}$

Tanno defined the generalized Tanaka-Webster connection (GTWC) $\widetilde{\widetilde{\nabla}}$ for contact metric manifolds. It is given by[24]

$$\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\mathfrak{U}_2 = \nabla_{\mathfrak{U}_1}\mathfrak{U}_2 + (\nabla_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2)\zeta - \eta(\mathfrak{U}_2)\nabla_{\mathfrak{U}_1}\zeta - \eta(\mathfrak{U}_1)\varphi\mathfrak{U}_2 \tag{3.22}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$.

By virtue of (2.7) and (2.8), equation (3.22) takes the form

$$\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\mathfrak{U}_2 = \nabla_{\mathfrak{U}_1}\mathfrak{U}_2 + \beta g(\mathfrak{U}_1,\mathfrak{U}_2)\zeta - \beta\eta(\mathfrak{U}_2)\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\varphi\mathfrak{U}_2. \tag{3.23}$$

Replacing $\mathfrak{U}_2$ by ζ in (3.23) and using (2.3), (2.4), (2.7), we have

$$\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\zeta = 2\beta\mathfrak{U}_1. \tag{3.24}$$

Now

$$(\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\varphi)(\mathfrak{U}_2) = \widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}(\varphi\mathfrak{U}_2) - \varphi(\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\mathfrak{U}_2). \tag{3.25}$$

Using (2.6) and (3.23) in (3.25), we have

$$(\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\varphi)(\mathfrak{U}_2) = \beta g(\varphi\mathfrak{U}_1,\mathfrak{U}_2)\zeta + \eta(\mathfrak{U}_1)\mathfrak{U}_2 + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)\zeta. \tag{3.26}$$

Now

$$(\widetilde{\nabla}_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2) = \widetilde{\nabla}_{\mathfrak{U}_1}\eta(\mathfrak{U}_2) - \eta(\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2). \tag{3.27}$$

Using (3.23) in (3.27), we have

$$(\widetilde{\nabla}_{\mathfrak{U}_1}\eta)(\mathfrak{U}_2) = 2\beta g(\mathfrak{U}_1, \mathfrak{U}_2). \tag{3.28}$$

Now

$$(\widetilde{\nabla}_{\mathfrak{U}_1}g)(\mathfrak{U}_2, \mathfrak{U}_3) = \widetilde{\nabla}_{\mathfrak{U}_1}g(\mathfrak{U}_2, \mathfrak{U}_3) - g(\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2, \mathfrak{U}_3) - g(\mathfrak{U}_2, \widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_3). \tag{3.29}$$

Using (3.23) in (3.29), we have

$$(\widetilde{\nabla}_{\mathfrak{U}_1}g)(\mathfrak{U}_2, \mathfrak{U}_3) = 2\eta(\mathfrak{U}_1)g(\varphi\mathfrak{U}_2, \mathfrak{U}_3) \neq 0. \tag{3.30}$$

Thus we can state the following :

Theorem 3.1. *The GTWC $\widetilde{\nabla}$ on a Lorentzian β-Kenmotsu manifold is a non-metric connection.*

Now the torsion tensor $\widetilde{\mathcal{T}}$ of the GTWC $\widetilde{\nabla}$ is given as:

$$\widetilde{\mathcal{T}}(\mathfrak{U}_1, \mathfrak{U}_2) = \widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_2 - \widetilde{\nabla}_{\mathfrak{U}_2}\mathfrak{U}_1 - [\mathfrak{U}_1, \mathfrak{U}_2]. \tag{3.31}$$

Using (3.23) in (3.31), we have

$$\widetilde{\mathcal{T}}(\mathfrak{U}_1, \mathfrak{U}_2) = \beta\eta(\mathfrak{U}_1)\mathfrak{U}_2 - \beta\eta(\mathfrak{U}_2)\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\varphi\mathfrak{U}_2 + \eta(\mathfrak{U}_2)\varphi\mathfrak{U}_1. \tag{3.32}$$

Now we have the following:

Theorem 3.2. *The GTWC $\widetilde{\nabla}$ on a Lorentzian β-Kenmotsu manifold associated to the connection ∇ of $\mathfrak{M}$ is just the only one affine connection, which is non-metric and its torsion has the form (3.32)*

Let $\mathfrak{R}$ and $\widetilde{\mathfrak{R}}$ denote the curvature tensors of the connections ∇ and $\widetilde{\nabla}$, respectively. Then

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 = \widetilde{\nabla}_{\mathfrak{U}_1}\widetilde{\nabla}_{\mathfrak{U}_2}\mathfrak{U}_3 - \widetilde{\nabla}_{\mathfrak{U}_2}\widetilde{\nabla}_{\mathfrak{U}_1}\mathfrak{U}_3 - \widetilde{\nabla}_{[\mathfrak{U}_1, \mathfrak{U}_2]}\mathfrak{U}_3. \tag{3.33}$$

Using (2.3), (2.4), (2.5), (2.6), (2.7) and (3.23) in (3.33), we have

$$\begin{aligned} \widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 &= \mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 + 3\beta^2[g(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - g(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2] \\ &\quad -2\beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta - g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta]. \end{aligned} \tag{3.34}$$

Contracting (3.34), we have

$$\widetilde{\mathcal{S}}(\mathfrak{U}_2,\mathfrak{U}_3)=\mathcal{S}(\mathfrak{U}_2,\mathfrak{U}_3)+6n\beta^2 g(\mathfrak{U}_2,\mathfrak{U}_3)-2\beta g(\varphi\mathfrak{U}_2,\mathfrak{U}_3). \tag{3.35}$$

Using (2.13) in (3.35), we have

$$\widetilde{\mathfrak{Q}}\mathfrak{U}_2=\mathfrak{Q}\mathfrak{U}_2+6n\beta^2\mathfrak{U}_2-2\beta(\varphi\mathfrak{U}_2). \tag{3.36}$$

Contracting (3.35), we have

$$\widetilde{\mathfrak{r}}=\mathfrak{r}+6n(2n+1)\beta^2-2\beta\Psi, \tag{3.37}$$

where $\Psi=trace(\varphi)$.

Replacing $\mathfrak{U}_3$ by ζ in (3.34) and using (2.3), (2.4), (2.10), we have

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta=-2\beta^2[\eta(\mathfrak{U}_1)\mathfrak{U}_2-\eta(\mathfrak{U}_2)\mathfrak{U}_1]=-2\mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta. \tag{3.38}$$

Replacing $\mathfrak{U}_1$ by ζ, $\mathfrak{U}_2$ by $\mathfrak{U}_1$ and $\mathfrak{U}_3$ by $\mathfrak{U}_2$ in (3.34) and using (2.3), (2.4), (2.11), we have

$$\widetilde{\mathfrak{R}}(\zeta,\mathfrak{U}_1)\mathfrak{U}_2=-2[\mathfrak{R}(\zeta,\mathfrak{U}_1)\mathfrak{U}_2+\beta g(\varphi\mathfrak{U}_1,\mathfrak{U}_2)\zeta]. \tag{3.39}$$

Replacing $\mathfrak{U}_3$ by ζ in (3.35) and using (2.3), (2.4), (2.12), we have

$$\widetilde{\mathcal{S}}(\mathfrak{U}_2,\zeta)=4n\beta^2\eta(\mathfrak{U}_2). \tag{3.40}$$

Replacing $\mathfrak{U}_2$ by ζ in (3.36) and using (2.3), (2.15), we have

$$\widetilde{\mathfrak{Q}}\zeta=4n\beta^2\zeta. \tag{3.41}$$

Taking the cyclic permutation of $\mathfrak{U}_1,\mathfrak{U}_2$ and $\mathfrak{U}_3$ in (3.34), we have

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 &= \mathfrak{R}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1+3\beta^2[g(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2-g(\mathfrak{U}_2,\mathfrak{U}_1)\mathfrak{U}_3]\\ &\quad -2\beta[g(\varphi\mathfrak{U}_2,\mathfrak{U}_1)\eta(\mathfrak{U}_3)\zeta-g(\varphi\mathfrak{U}_3,\mathfrak{U}_1)\eta(\mathfrak{U}_2)\zeta]\end{aligned} \tag{3.42}$$

and

$$\begin{aligned}\widetilde{\mathfrak{R}}(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2 &= \mathfrak{R}(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2+3\beta^2[g(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3-g(\mathfrak{U}_3,\mathfrak{U}_2)\mathfrak{U}_1]\\ &\quad -2\beta[g(\varphi\mathfrak{U}_3,\mathfrak{U}_2)\eta(\mathfrak{U}_1)\zeta-g(\varphi\mathfrak{U}_1,\mathfrak{U}_2)\eta(\mathfrak{U}_3)\zeta].\end{aligned} \tag{3.43}$$

Using Bianchi's first identity in the addition of (3.34), (3.42) and (3.43), we have

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3+\widetilde{\mathfrak{R}}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1+\widetilde{\mathfrak{R}}(\mathfrak{U}_3,\mathfrak{U}_1)\mathfrak{U}_2=0. \tag{3.44}$$

Hence we give the following:

Theorem 3.3. *The curvature tensor of a Lorentzian β-Kenmotsu manifold admitting GTWC $\widetilde{\nabla}$ satisfies the equation (3.44).*

4. Extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$

Definition 4.1. *A Lorentzian β-Kenmotsu manifold is said to be an extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold if its curvature tensor $\mathfrak{R}$ satisfies the relation*

$$\begin{aligned}\varphi^2((\nabla_{\mathcal{W}}\mathfrak{R})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3) &= \mathcal{A}(\mathcal{W})\varphi^2(\mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\\ &+\mathfrak{B}(\mathcal{W})\varphi^2[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1-g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2] \qquad (4.45)\end{aligned}$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2,\mathfrak{U}_3,\mathcal{W}\in\mathfrak{X}(\mathfrak{M})$. *Where $\mathcal{A},\mathfrak{B}$ are two non-vanishing 1-forms such that $g(\mathcal{W},\rho_1)=\mathcal{A}(\mathcal{W})$ and $g(\mathcal{W},\rho_2)=\mathfrak{B}(\mathcal{W})$ for all $\mathcal{W}\in\mathfrak{X}(\mathfrak{M})$ with ρ_1 and ρ_2 being the vector fields associated 1-forms $\mathcal{A}$ and $\mathfrak{B}$, respectively* [17].

Suppose an extended generalized ϕ-recurrent Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$. Then from definition (4.1), we have

$$\begin{aligned}\varphi^2((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3) &= \mathcal{A}(\mathcal{W})\varphi^2(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\\ &+\mathfrak{B}(\mathcal{W})\varphi^2[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1-g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2]. \qquad (4.46)\end{aligned}$$

Using (2.4) in (4.46), we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 &= -\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\zeta+\mathcal{A}(\mathcal{W})[\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3\\ &+\eta(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\zeta]+\mathfrak{B}(\mathcal{W})[g(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1\\ &-g(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2+g(\mathfrak{U}_2,\mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta\\ &-g(\mathfrak{U}_1,\mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta]. \qquad (4.47)\end{aligned}$$

Taking inner product in (4.47) with $\mathcal{V}$ and using (2.4), we have

$$\begin{aligned} g((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3,\mathcal{V}) &= -\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\eta(\mathcal{V}) \\ &+\mathcal{A}(\mathcal{W})[g(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3,\mathcal{V}) \\ &+\eta(\widetilde{\mathfrak{R}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3)\eta(\mathcal{V})] \\ &+\mathfrak{B}(\mathcal{W})[g(\mathfrak{U}_2,\mathfrak{U}_3)g(\mathfrak{U}_1,\mathcal{V}) \\ &-g(\mathfrak{U}_1,\mathfrak{U}_3)g(\mathfrak{U}_2,\mathcal{V}) \\ &+g(\mathfrak{U}_2,\mathfrak{U}_3)\eta(\mathfrak{U}_1)\eta(\mathcal{V}) \\ &-g(\mathfrak{U}_1,\mathfrak{U}_3)\eta(\mathfrak{U}_2)\eta(\mathcal{V})]. \qquad (4.48) \end{aligned}$$

Let $\{\varsigma_1,\varsigma_2,\varsigma_3,...,\varsigma_n\}$ be an orthonormal basis for the tangent space of $\mathfrak{M}^{2n+1}$ at a point $p\in\mathfrak{M}^{2n+1}$. Taking $\mathfrak{U}_1=\mathcal{V}=\varsigma_i$ and summation over $i\in[1,n]$ in (4.48), we have

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\mathfrak{U}_3) &= -\sum_{i=1}^{2n+1}\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\mathfrak{U}_3)\eta(\varsigma_i) \\ &+\mathcal{A}(\mathcal{W})[\widetilde{\mathcal{S}}(\mathfrak{U}_2,\mathfrak{U}_3)+\eta(\widetilde{\mathfrak{R}}(\zeta,\mathfrak{U}_2)\mathfrak{U}_3)] \\ &+\mathfrak{B}(\mathcal{W})[2ng(\mathfrak{U}_2,\mathfrak{U}_3)-g(\mathfrak{U}_2,\mathfrak{U}_3)-\eta(\mathfrak{U}_2)\eta(\mathfrak{U}_3)]. \qquad (4.49) \end{aligned}$$

Replacing $\mathfrak{U}_3$ by ζ in (4.49) and using (2.3), (2.4), (3.39), (3.40), we have

$$\begin{aligned} (\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) &= -\sum_{i=1}^{2n+1}\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta)\eta(\varsigma_i) \\ &+4n\beta^2\mathcal{A}(\mathcal{W})\eta(\mathfrak{U}_2)+2n\mathfrak{B}(\mathcal{W})\eta(\mathfrak{U}_2). \qquad (4.50) \end{aligned}$$

Taking second term of (4.50), we can calculate

$$\begin{aligned} \eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta) &= g(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)-g(\widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{W}}\varsigma_i,\mathfrak{U}_2)\zeta,\zeta) \\ &-g(\widetilde{\mathfrak{R}}(\varsigma_i,\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2)\zeta,\zeta)-g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\widetilde{\nabla}_{\mathcal{W}}\zeta,\zeta). \qquad (4.51) \end{aligned}$$

Let $p\in\mathfrak{M}^{2n+1}$, since ς_i is an orthonormal basis, therefore $\widetilde{\nabla}_{\mathcal{W}}\varsigma_i=0$ at p. Also

$$g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)=-g(\widetilde{\mathfrak{R}}(\zeta,\zeta)\mathfrak{U}_2,\varsigma_i)=0. \qquad (4.52)$$

Since $(\widetilde{\nabla}_{\mathcal{W}}g)=0$, we have

$$g(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)+g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\widetilde{\nabla}_{\mathcal{W}}\zeta)=0. \qquad (4.53)$$

Using (4.53) in (4.51), we have

$$g((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)$$
$$= -g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\zeta,\widetilde{\nabla}_{\mathcal{W}}\zeta) - g(\widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{W}}\varsigma_i,\mathfrak{U}_2)\zeta,\zeta)$$
$$-g(\widetilde{\mathfrak{R}}(\varsigma_i,\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2)\zeta,\zeta) - g(\widetilde{\mathfrak{R}}(\varsigma_i,\mathfrak{U}_2)\widetilde{\nabla}_{\mathcal{W}}\zeta,\zeta). \tag{4.54}$$

We also know that

$$g(\widetilde{\mathfrak{R}}(\varsigma_i,\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2)\zeta,\zeta) = 0 = g(\widetilde{\mathfrak{R}}(\widetilde{\nabla}_{\mathcal{W}}\varsigma_i,\mathfrak{U}_2)\zeta,\zeta). \tag{4.55}$$

Using (4.55) in (4.54) and using the fact that $\mathfrak{R}$ is skew-symmetric, we obtain

$$\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta) = 0. \tag{4.56}$$

Therefore second term of (4.50) is zero, i.e.

$$\sum_{i=1}^{2n+1}\eta((\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathfrak{R}})(\varsigma_i,\mathfrak{U}_2)\zeta)\eta(\varsigma_i) = 0. \tag{4.57}$$

Using (4.57) in (4.50), we have

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = 4n\beta^2\mathcal{A}(\mathcal{W})\eta(\mathfrak{U}_2) + 2n\mathfrak{B}(\mathcal{W})\eta(\mathfrak{U}_2). \tag{4.58}$$

Now we know that

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = \widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}}(\mathfrak{U}_2,\zeta) - \widetilde{\mathcal{S}}(\widetilde{\nabla}_{\mathcal{W}}\mathfrak{U}_2,\zeta) - \widetilde{\mathcal{S}}(\mathfrak{U}_2,\widetilde{\nabla}_{\mathcal{W}}\zeta). \tag{4.59}$$

Using (3.24), (3.27) and (3.40) in (4.59), we have

$$\begin{aligned}(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) &= 4n\beta^2(\widetilde{\nabla}_{\mathcal{W}}\eta)\mathfrak{U}_2 - 2\beta\mathcal{S}(\mathfrak{U}_2,\mathcal{W}) \\ &\quad -12n\beta^3 g(\mathfrak{U}_2,\mathcal{W}) + 4\beta^2 g(\varphi\mathfrak{U}_2,\mathcal{W}).\end{aligned} \tag{4.60}$$

Using (3.28) in (4.60), we have

$$(\widetilde{\nabla}_{\mathcal{W}}\widetilde{\mathcal{S}})(\mathfrak{U}_2,\zeta) = -2\beta\mathcal{S}(\mathfrak{U}_2,\mathcal{W}) - 4n\beta^3 g(\mathfrak{U}_2,\mathcal{W}) + 4\beta^2 g(\varphi\mathfrak{U}_2,\mathcal{W}). \tag{4.61}$$

By virtue of (4.58) and (4.61), we have

$$-\beta\mathcal{S}(\mathfrak{U}_2,\mathcal{W}) - 2n\beta^3 g(\mathfrak{U}_2,\mathcal{W}) + 2\beta^2 g(\varphi\mathfrak{U}_2,\mathcal{W})$$
$$= 2n\beta^2\mathcal{A}(\mathcal{W})\eta(\mathfrak{U}_2) + n\mathfrak{B}(\mathcal{W})\eta(\mathfrak{U}_2). \tag{4.62}$$

Replacing $\mathfrak{U}_2$ by ζ in (4.62) and using (2.3), (2.4), (2.12), we have

$$2n\beta^2\mathcal{A}(\mathcal{W}) + n\mathfrak{B}(\mathcal{W}) = 0. \tag{4.63}$$

By virtue of (4.62) and (4.63), we have

$$\mathcal{S}(\mathfrak{U}_2, \mathcal{W}) = -2n\beta^2 g(\mathfrak{U}_2, \mathcal{W}) + 2\beta g(\varphi\mathfrak{U}_2, \mathcal{W}). \tag{4.64}$$

Thus we can state the following:

Theorem 4.1. *An extended generalized φ-recurrent Lorentzian β-Kenmotsu manifold with respect to the GTWC $\widetilde{\nabla}$ is some class of generalized η-Einstein manifold and the 1-forms $\mathcal{A}$ and $\mathfrak{B}$ are related as $[2\beta^2\mathcal{A}(\mathcal{W}) + \mathfrak{B}(\mathcal{W})] = 0$.*

5. Certain conditions on Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$

The projective curvature tensor [28] $\widetilde{\mathcal{P}}$ on Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is defined by

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 = \widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 - \frac{1}{2n}[\widetilde{\mathcal{S}}(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - \widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2]. \tag{5.65}$$

If projective curvature tensor $\widetilde{\mathcal{P}}$ vanishes, then from (5.65), we have

$$\widetilde{\mathfrak{R}}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 = \frac{1}{2n}[\widetilde{\mathcal{S}}(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - \widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2]. \tag{5.66}$$

Using (3.34) and (3.35) in (5.66), we have

$$\begin{aligned}
&\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3 - 2\beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta - g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta] \\
&= \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1 - \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2 + 2\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\mathfrak{U}_2 \\
&-2\beta g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\mathfrak{U}_1].
\end{aligned} \tag{5.67}$$

Taking inner product in (5.67) with $\mathcal{V}$ and using (2.4), we have

$$\begin{aligned}
&g(\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3, \mathcal{V}) - 2\beta[g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)\eta(\mathcal{V}) - g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)\eta(\mathcal{V})] \\
&= \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2, \mathfrak{U}_3)g(\mathfrak{U}_1, \mathcal{V}) - \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_3)g(\mathfrak{U}_2, \mathcal{V}) + 2\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)g(\mathfrak{U}_2, \mathcal{V}) \\
&-2\beta g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)g(\mathfrak{U}_1, \mathcal{V})].
\end{aligned} \tag{5.68}$$

Replacing $\mathcal{V}$ by ζ in (5.68) and using (2.3), (2.4), we have

$$\begin{aligned}
&\eta(\mathfrak{R}(\mathfrak{U}_1, \mathfrak{U}_2)\mathfrak{U}_3) - 2\beta[g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1) - g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2)] \\
&= \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1) - \mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2) + 2\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_3)\eta(\mathfrak{U}_2) \\
&-2\beta g(\varphi\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1)].
\end{aligned} \tag{5.69}$$

Replacing $\mathfrak{U}_1$ by ζ in (5.69) and using (2.3), (2.11), (2.12), we have

$$\mathcal{S}(\mathfrak{U}_2,\mathfrak{U}_3) = -2n\beta^2 g(\mathfrak{U}_2,\mathfrak{U}_3) - 6n\beta^2\eta(\mathfrak{U}_2)\eta(\mathfrak{U}_3) - 2\beta(2n-1)g(\varphi\mathfrak{U}_2,\mathfrak{U}_3). \tag{5.70}$$

Thus we have the following:

Theorem 5.1. *A projectively flat Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is a generalized η-Einstein manifold.*

Definition 5.1. *A Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with respect to the GTWC $\widetilde{\nabla}$ is said to be ζ-projectively flat* [26] *if*

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = 0$$

$\forall\ \mathfrak{U}_1,\mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$ *orthogonal to ζ, where $\widetilde{\mathcal{P}}$ is the projective curvature tensor of the GTWC $\widetilde{\nabla}$.*

Using (3.34) and (3.35) in (5.66), we have

$$\begin{aligned}\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 &= \mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 - \frac{\beta}{n}[g(\varphi\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2 - g(\varphi\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1] \\ &\quad -2\beta[g(\varphi\mathfrak{U}_1,\mathfrak{U}_3)\eta(\mathfrak{U}_2)\zeta - g(\varphi\mathfrak{U}_2,\mathfrak{U}_3)\eta(\mathfrak{U}_1)\zeta],\end{aligned} \tag{5.71}$$

where

$$\mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 = \mathfrak{R}(\mathfrak{U}_1,\mathfrak{U}_2)\mathfrak{U}_3 - \frac{1}{2n}[\mathcal{S}(\mathfrak{U}_2,\mathfrak{U}_3)\mathfrak{U}_1 - \mathcal{S}(\mathfrak{U}_1,\mathfrak{U}_3)\mathfrak{U}_2] \tag{5.72}$$

is a projective curvature tensor with respect to the connection ∇.
Putting $\mathfrak{U}_3 = \zeta$ in (5.71) and using (2.3), (2.4), we have

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = \mathcal{P}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta. \tag{5.73}$$

Now we give the following:

Theorem 5.2. *A $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ is ζ-projectively flat iff the manifold $\mathfrak{M}^{2n+1}$ is ζ-projectively flat with respect to the connection ∇.*

Now using (2.10), (2.12) and (5.72) in (5.73), we have

$$\widetilde{\mathcal{P}}(\mathfrak{U}_1,\mathfrak{U}_2)\zeta = 0. \tag{5.74}$$

Thus we can state the following:

Theorem 5.3. *A $(2n+1)$-dimensional Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\widetilde{\nabla}}$ is ζ-projectively flat.*

Next if the $(1,1)$-tensor φ is η-parallel with respect to the GTWC $\widetilde{\widetilde{\nabla}}$, then we have

$$g((\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\varphi)\mathfrak{U}_2, \mathfrak{U}_3) = 0 \tag{5.75}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2, \mathfrak{U}_3 \in \mathfrak{X}(\mathfrak{M})$.
By virtue of (3.26) and (5.75), we have

$$\beta g(\varphi\mathfrak{U}_1, \mathfrak{U}_2)\eta(\mathfrak{U}_3) + g(\mathfrak{U}_2, \mathfrak{U}_3)\eta(\mathfrak{U}_1) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2)\eta(\mathfrak{U}_3) = 0. \tag{5.76}$$

Taking $\mathfrak{U}_3 = \zeta$ in (5.76) and using (2.3), (2.4), we have

$$g(\varphi\mathfrak{U}_1, \mathfrak{U}_2) = 0. \tag{5.77}$$

Replacing $\mathfrak{U}_2$ by $\varphi\mathfrak{U}_2$ in (5.77) and using (2.5), we have

$$g(\mathfrak{U}_1, \mathfrak{U}_2) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2) = 0. \tag{5.78}$$

Replacing $\mathfrak{U}_1$ by $\mathfrak{Q}\mathfrak{U}_1$ in (5.78) and using (2.13), (2.14), we have

$$\mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) = 2n\beta^2\eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2). \tag{5.79}$$

Hence we have the following:

Theorem 5.4. *If the $(1,1)$-tensor φ is η-parallel on the Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\widetilde{\nabla}}$, then the manifold $\mathfrak{M}^{2n+1}$ is a special type of η-Einstein manifold.*

6. Ricci soliton on Lorentzian β-Kenmotsu manifold with GTWC $\widetilde{\widetilde{\nabla}}$

Let (g, ζ, Θ) be a Ricci soliton on Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with respect to the GTWC $\widetilde{\widetilde{\nabla}}$. Then we have

$$(\widetilde{\widetilde{\mathfrak{L}}}_\zeta g)(\mathfrak{U}_1, \mathfrak{U}_2) + 2\widetilde{\widetilde{\mathcal{S}}}(\mathfrak{U}_1, \mathfrak{U}_2) + 2\Theta g(\mathfrak{U}_1, \mathfrak{U}_2) = 0. \tag{6.80}$$

Now

$$(\widetilde{\widetilde{\mathfrak{L}}}_\zeta g)(\mathfrak{U}_1, \mathfrak{U}_2) = g(\widetilde{\widetilde{\nabla}}_{\mathfrak{U}_1}\zeta, \mathfrak{U}_2) + g(\mathfrak{U}_1, \widetilde{\widetilde{\nabla}}_{\mathfrak{U}_2}\zeta). \tag{6.81}$$

Using (3.24) in (6.81), we have

$$(\widetilde{\widetilde{\mathfrak{L}}}_\zeta g)(\mathfrak{U}_1, \mathfrak{U}_2) = 4\beta g(\mathfrak{U}_1, \mathfrak{U}_2). \tag{6.82}$$

Using (3.35) and (6.82) in (6.80), we have

$$\mathcal{S}(\mathfrak{U}_1, \mathfrak{U}_2) = -(\Theta + 2\beta + 6n\beta^2) g(\mathfrak{U}_1, \mathfrak{U}_2) + 2\beta g(\varphi \mathfrak{U}_1, \mathfrak{U}_2). \tag{6.83}$$

Now we give the following:

Theorem 6.1. *If (g, ζ, Θ) be a Ricci soliton on a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ with the GTWC $\widetilde{\nabla}$, then the manifold $\mathfrak{M}^{2n+1}$ is some class of generalized η-Einstein manifold.*

Using (6.82) in (6.80), we have

$$\widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_2) = -(2\beta + \Theta) g(\mathfrak{U}_1, \mathfrak{U}_2). \tag{6.84}$$

Contracting (6.84), we have

$$\widetilde{\mathfrak{r}} = -(2n+1)(2\beta + \Theta). \tag{6.85}$$

Replacing $\mathfrak{U}_2$ by ζ in (6.83) and using (2.3), (2.4), (2.12), we have

$$\Theta = -2\beta(1 + 2n\beta). \tag{6.86}$$

Thus we have the following:

Theorem 6.2. *A Ricci soliton (g, ζ, Θ) in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ is either steady or shrinking.*

Let $(g, \mathcal{V}, \Theta)$ be the Ricci soliton in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\nabla}$ such that $\mathcal{V}$ is pointwise collinear with ζ, i.e., $\mathcal{V} = \mathfrak{b}\zeta$, where $\mathfrak{b}$ is a function. Then (1.2) holds and follows that

$$\begin{aligned} &\mathfrak{b} g(\widetilde{\nabla}_{\mathfrak{U}_1} \zeta, \mathfrak{U}_2) + (\mathfrak{U}_1 \mathfrak{b}) \eta(\mathfrak{U}_2) + \mathfrak{b} g(\mathfrak{U}_1, \widetilde{\nabla}_{\mathfrak{U}_2} \zeta) \\ &+ (\mathfrak{U}_2 \mathfrak{b}) \eta(\mathfrak{U}_1) + 2\widetilde{\mathcal{S}}(\mathfrak{U}_1, \mathfrak{U}_2) + 2\Theta g(\mathfrak{U}_1, \mathfrak{U}_2) = 0. \end{aligned} \tag{6.87}$$

Replacing $\mathfrak{U}_2$ by ζ in (6.87) and using (2.3), (2.4), (3.24), (3.40), we have

$$(\mathfrak{U}_1 \mathfrak{b}) = (2\Theta + \zeta\mathfrak{b} + 4\mathfrak{b}\beta + 4\mathfrak{b}\beta + 8n\beta^2) \eta(\mathfrak{U}_1). \tag{6.88}$$

Replacing $\mathfrak{U}_1$ by ζ in (6.88) and using (2.3), we have

$$(\zeta \mathfrak{b}) = -(\Theta + 2\mathfrak{b}\beta + 4n\beta^2). \tag{6.89}$$

Equations (6.88) and (6.89), yield

$$(d\mathfrak{b}) = (\Theta + 2\mathfrak{b}\beta + 4n\beta^2)\eta. \tag{6.90}$$

Applying d on (6.90), we have

$$(\Theta + 2\mathfrak{b}\beta + 4n\beta^2)d\eta = 0. \tag{6.91}$$

Since $d\eta \neq 0$, from (6.91), we have

$$\Theta = -2\beta(\mathfrak{b} + 2n\beta). \tag{6.92}$$

Putting (6.92) in (6.90), we obtain $d\mathfrak{b} = 0$, i.e., $\mathfrak{b}$ is a constant. Hence we have the following:

Theorem 6.3. *If $(g, \mathcal{V}, \Theta)$ be the Ricci soliton in a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^{2n+1}$ admitting the GTWC $\widetilde{\widetilde{\nabla}}$ such that $\mathcal{V} = \mathfrak{b}\zeta$, then $\mathcal{V}$ is a constant multiple of ζ and the Ricci soliton is either steady or shrinking.*

7. Example of Lorentzian β-Kenmotsu manifold

Example 7.1. *Let $\mathfrak{M} = \{(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3) \in \mathbb{R}^3 : \mathfrak{t}_3 > 0\}$ be a 3-dimensional manifold, where $(\mathfrak{t}_1, \mathfrak{t}_2, \mathfrak{t}_3)$ are the standard coordinates of $\mathbb{R}^3$. The vector fields* [27]

$$\varsigma_1 = e^{\mathfrak{t}_3}\frac{\partial}{\partial \mathfrak{t}_2}, \quad \varsigma_2 = e^{\mathfrak{t}_3}\left(\frac{\partial}{\partial \mathfrak{t}_1} + \frac{\partial}{\partial \mathfrak{t}_2}\right), \quad \varsigma_3 = \beta\frac{\partial}{\partial \mathfrak{t}_3}$$

are linearly independent at each point of $\mathfrak{M}$, where β is a real constant. Let g be the Lorentzian metric defined by

$$\begin{aligned} &g(\varsigma_1, \varsigma_2) = g(\varsigma_1, \varsigma_3) = g(\varsigma_2, \varsigma_3) = 0, \\ &g(\varsigma_1, \varsigma_1) = g(\varsigma_2, \varsigma_2) = -g(\varsigma_3, \varsigma_3) = 1. \end{aligned} \tag{7.93}$$

Let η be the 1-form defined by $\eta(\mathfrak{U}_1) = g(\mathfrak{U}_1, \varsigma_3)$ for any $\mathfrak{U}_1 \in \mathfrak{X}(\mathfrak{M})$ and φ be the $(1,1)$-tensor field defined by

$$\varphi(\varsigma_1) = -\varsigma_2, \quad \varphi(\varsigma_2) = -\varsigma_1, \quad \varphi(\varsigma_3) = 0. \tag{7.94}$$

Now using the linearity of φ and g, we have

$$\eta(\varsigma_3) = -1, \quad \varphi^2(\mathfrak{U}_1) = \mathfrak{U}_1 + \eta(\mathfrak{U}_1)\varsigma_3 \tag{7.95}$$

and

$$g(\varphi\mathfrak{U}_1, \varphi\mathfrak{U}_2) = g(\mathfrak{U}_1, \mathfrak{U}_2) + \eta(\mathfrak{U}_1)\eta(\mathfrak{U}_2) \tag{7.96}$$

$\forall\ \mathfrak{U}_1, \mathfrak{U}_2 \in \mathfrak{X}(\mathfrak{M})$. *Therefore for* $\varsigma_3 = \zeta$, *the structure* $(\varphi, \zeta, \eta, g)$ *defines a Lorentzian para-contact structure on* $\mathfrak{M}$. *Let* ∇ *be the Levi-Civita connection with respect to the Lorentzian metric* g. *Then we have*

$$[\varsigma_1, \varsigma_2] = 0, \quad [\varsigma_1, \varsigma_3] = -\beta\varsigma_1, \quad [\varsigma_2, \varsigma_3] = -\beta\varsigma_2. \tag{7.97}$$

We recall Koszul's formula as

$$\begin{aligned} 2g(\nabla_{\mathfrak{U}_1}\mathfrak{U}_2, \mathfrak{U}_3) &= \mathfrak{U}_1 g(\mathfrak{U}_2, \mathfrak{U}_3) + \mathfrak{U}_2 g(\mathfrak{U}_3, \mathfrak{U}_1) - \mathfrak{U}_3 g(\mathfrak{U}_1, \mathfrak{U}_2) \\ &\quad - g(\mathfrak{U}_1, [\mathfrak{U}_2, \mathfrak{U}_3]) - g(\mathfrak{U}_2, [\mathfrak{U}_1, \mathfrak{U}_3]) \\ &\quad + g(\mathfrak{U}_3, [\mathfrak{U}_1, \mathfrak{U}_2]). \end{aligned} \tag{7.98}$$

By virtue of (7.98), we have

$$\begin{aligned} &\nabla_{\varsigma_1}\varsigma_1 = -\beta\varsigma_3, \quad \nabla_{\varsigma_1}\varsigma_2 = 0, \quad \nabla_{\varsigma_1}\varsigma_3 = -\beta\varsigma_1, \\ &\nabla_{\varsigma_2}\varsigma_1 = 0, \quad \nabla_{\varsigma_2}\varsigma_2 = -\beta\varsigma_3, \quad \nabla_{\varsigma_2}\varsigma_3 = -\beta\varsigma_2, \\ &\nabla_{\varsigma_3}\varsigma_1 = 0, \quad \nabla_{\varsigma_3}\varsigma_2 = 0, \quad \nabla_{\varsigma_3}\varsigma_3 = 0. \end{aligned} \tag{7.99}$$

Now for $\mathfrak{U}_1 = \mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3$ *and* $\zeta = \varsigma_3$, *we have*

$$\nabla_{\mathfrak{U}_1}\zeta = \nabla_{\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3}\varsigma_3 = -\beta(\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2) \tag{7.100}$$

and

$$\beta[\mathfrak{U}_1 - \eta(\mathfrak{U}_1)\zeta] = \beta[\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + 2\mathfrak{U}_1^3\varsigma_3], \tag{7.101}$$

where $\mathfrak{U}_1^1, \mathfrak{U}_1^2$ *and* $\mathfrak{U}_1^3$ *are scalars.*
Now using (7.100) and (7.101), we have

$$2\beta(\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3) = 0.$$

Since $(\mathfrak{U}_1^1\varsigma_1 + \mathfrak{U}_1^2\varsigma_2 + \mathfrak{U}_1^3\varsigma_3) \neq 0$, *therefore we have*

$$\beta = 0. \tag{7.102}$$

Hence it can be easily see that the structure $(\mathfrak{M}^3, \varphi, \zeta, \eta, g)$ *is a Lorentzian* β*-Kenmotsu manifold.*

By using (7.97) and (7.99), we can obtain the components of the curvature tensor $\mathfrak{R}$ with respect to the connection ∇ as follows:

$$\begin{aligned}
&\mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_3=0, \quad \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_2=0, \quad \mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_1=0,\\
&\mathfrak{R}(\varsigma_2,\varsigma_3)\varsigma_3=-\beta^2\varsigma_2, \quad \mathfrak{R}(\varsigma_1,\varsigma_3)\varsigma_3=-\beta^2\varsigma_1, \quad \mathfrak{R}(\varsigma_1,\varsigma_2)\varsigma_2=\beta^2\varsigma_1,\\
&\mathfrak{R}(\varsigma_3,\varsigma_1)\varsigma_1=\beta^2\varsigma_3, \quad \mathfrak{R}(\varsigma_2,\varsigma_1)\varsigma_1=\beta^2\varsigma_2, \quad \mathfrak{R}(\varsigma_3,\varsigma_2)\varsigma_2=\beta^2\varsigma_3.
\end{aligned} \tag{7.103}$$

Along with $\mathfrak{R}(\varsigma_i,\varsigma_i)\varsigma_i=0$, $\forall\ i=1,2,3$. By using (7.103), we can verify equations (2.9), (2.10) and (2.11).

Now using (3.23), (7.93), (7.94) and (7.99), we obtain

$$\begin{aligned}
&\widetilde{\nabla}_{\varsigma_1}\varsigma_1=0, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_2=0, \quad \widetilde{\nabla}_{\varsigma_1}\varsigma_3=0,\\
&\widetilde{\nabla}_{\varsigma_2}\varsigma_1=0, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_2=0, \quad \widetilde{\nabla}_{\varsigma_2}\varsigma_3=0,\\
&\widetilde{\nabla}_{\varsigma_3}\varsigma_1=-\varsigma_2, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_2=-\varsigma_1, \quad \widetilde{\nabla}_{\varsigma_3}\varsigma_3=0.
\end{aligned} \tag{7.104}$$

By using (3.30) and (3.32), we have

$$(\widetilde{\nabla}_{\varsigma_1}g)(\varsigma_2,\varsigma_3)=0, \quad (\widetilde{\nabla}_{\varsigma_2}g)(\varsigma_3,\varsigma_1)=0, \quad (\widetilde{\nabla}_{\varsigma_3}g)(\varsigma_1,\varsigma_2)=2\neq 0$$

and also, we have

$$\widetilde{\mathcal{T}}(\varsigma_1,\varsigma_2)=0, \quad \widetilde{\mathcal{T}}(\varsigma_1,\varsigma_3)=\beta\varsigma_1-\varsigma_2, \quad \widetilde{\mathcal{T}}(\varsigma_2,\varsigma_3)=\beta\varsigma_2+\varsigma_1.$$

Along with $\widetilde{\mathcal{T}}(\varsigma_i,\varsigma_i)=0$; $\forall\ i=1,2,3$. Hence $\mathfrak{M}^3$ is a 3-dimensional Lorentzian β-Kenmotsu manifold admitting the GTWC $\widetilde{\nabla}$ which is a non-metric connection.

Now using (3.33), (7.97) and (7.104), we can easily obtain the components of curvature tensor $\widetilde{\mathfrak{R}}$ with respect to the GTWC $\widetilde{\nabla}$ as follows:

$$\widetilde{\mathfrak{R}}(\varsigma_i,\varsigma_j)\varsigma_k=0 \tag{7.105}$$

$\forall\ i,j,k=1,2,3$. *In view of (7.105), we can verify equations (3.34), (3.38), (3.39), (3.42), (3.43) and (3.44). Therefore it is clear that the Theorem (3.3) is well satisfied.*

The Ricci tensor $\mathcal{S}(\varsigma_j,\varsigma_k)$; $j,k=1,2,3$ of the connection ∇ can be calculated as under:

$$\mathcal{S}(\varsigma_j,\varsigma_k)=\sum_{i=1}^{3} g(\mathfrak{R}(\varsigma_i,\varsigma_j)\varsigma_k,\varsigma_i).$$

It follows that

$$\mathcal{S}(\varsigma_1,\varsigma_1)=0, \quad \mathcal{S}(\varsigma_2,\varsigma_2)=0, \quad \mathcal{S}(\varsigma_3,\varsigma_3)=-2\beta^2. \tag{7.106}$$

Along with $\mathcal{S}(\varsigma_j, \varsigma_k) = 0$; $\forall$ $(j \neq k) = 1, 2, 3$. By virtue of (7.106), we can verify equations (4.64), (5.70) and (5.79).

The Ricci tensor $\widetilde{\mathcal{S}}(\varsigma_j, \varsigma_k)$; $j, k = 1, 2, 3$ of the connection $\widetilde{\nabla}$ can be calculated as under:

$$\widetilde{\mathcal{S}}(\varsigma_j, \varsigma_k) = \sum_{i=1}^{3} g(\widetilde{\mathfrak{R}}(\varsigma_i, \varsigma_j)\varsigma_k, \varsigma_i).$$

It follows that

$$\widetilde{\mathcal{S}}(\varsigma_j, \varsigma_k) = 0 \tag{7.107}$$

$\forall$ $j, k = 1, 2, 3$.

By virtue of (7.107), we can verify equations (3.35) and (3.40).

The scalar curvature $\mathfrak{r}$ is given by

$$\begin{aligned}\mathfrak{r} &= \sum_{i=1}^{3} g(\varsigma_i, \varsigma_i)\mathcal{S}(\varsigma_i, \varsigma_i)\\ &= g(\varsigma_1, \varsigma_1)\mathcal{S}(\varsigma_1, \varsigma_1) + g(\varsigma_2, \varsigma_2)\mathcal{S}(\varsigma_2, \varsigma_2) + g(\varsigma_3, \varsigma_3)\mathcal{S}(\varsigma_3, \varsigma_3)\\ &= 2\beta^2. \end{aligned}\tag{7.108}$$

Also, the scalar curvature $\widetilde{\mathfrak{r}}$ is given by

$$\begin{aligned}\widetilde{\mathfrak{r}} &= \sum_{i=1}^{3} g(\varsigma_i, \varsigma_i)\widetilde{\mathcal{S}}(\varsigma_i, \varsigma_i)\\ &= g(\varsigma_1, \varsigma_1)\widetilde{\mathcal{S}}(\varsigma_1, \varsigma_1) + g(\varsigma_2, \varsigma_2)\widetilde{\mathcal{S}}(\varsigma_2, \varsigma_2) + g(\varsigma_3, \varsigma_3)\widetilde{\mathcal{S}}(\varsigma_3, \varsigma_3)\\ &= 0. \end{aligned}\tag{7.109}$$

If (g, ζ, Θ) be the Ricci soliton on $\mathfrak{M}^3$ with respect to the GTWC $\widetilde{\nabla}$, then from (7.109) and (6.85), we have

$$-(2n+1)(2\beta + \Theta) = 0,$$

i.e.

$$\Theta = -2\beta. \tag{7.110}$$

Thus the Ricci soliton (g, ζ, Θ) on a Lorentzian β-Kenmotsu manifold $\mathfrak{M}^3$ admitting the GTWC $\widetilde{\nabla}$ is steady, expanding, and shrinking according to $\beta = 0$, $\beta < 0$, and $\beta > 0$, respectively. Hence Theorem (6.2) is verified.

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A CLASS OF INDEFINITE ALMOST PARACONTACT METRIC MANIFOLDS

S. SUNITHA DEVI * AND K. L. SAI PRASAD

Abstract. This research, we develop a new class of indefinite almost paracontact metric manifolds, termed (ϵ)-para Kenmotsu manifolds and we obtain some typical identities for the curvature tensor, scalar curvature and Ricci tensor. Furthermore, in particular, we investigate the curvature features of $three$-dimensional (ϵ)-para Kenmotsu manifolds. We establish an essential as well as sufficient condition for an (ϵ)-para Kenmotsu 3-manifold to have an indefinite space form. Furthermore, we classify and demonstrate that (ϵ)-para Kenmotsu 3-manifolds, which are either semi-symmetric, Ricci-semi-symmetric or semi-symmetric type, are η-Einstein. In conclusion, we create a 3-D (ϵ)-para Kenmotsu manifold example.

Keywords: Indefinite almost paracontact metric manifold, Ricci semi-symmetric manifold, (ϵ)-para Kenmotsu manifold, semi-symmetric and η-Einstein manifolds.

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1. Introduction

With an emphasis on Sasakian manifolds, Takahashi [16] introduced almost contact manifolds equipped with pseudo-Riemannian metrics in 1969. The terms (ϵ)-almost contact metric and (ϵ)-Sasakian have also been used to refer to indefinite almost contact metric manifolds and the indefinite Sasakian manifolds, respectively. The (ϵ)-Kenmotsu manifold which has

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been introduced by De and Sarkar [1] is based on a class of almost contact Riemannian manifolds called Kenmotsu manifolds [3]. They proved that the curvatures are influenced by the presence of a new structure with indefinite metrics.

On the other hand, in 1976, Sato [12] defined the notions of an almost paracontact structure, which is similar to the almost contact structure. By replacing the vector field ξ in almost paracontact manifold with $-\xi$, Matsumoto [4] first proposed the concept of Lorentzian almost paracontact in 1989. Lorentzian para-Sasakian (LP-Sasakian) manifolds connected to the Lorentzian metric are the outcome of this. While the structural vector field ξ is always time-like, the semi-Riemannian metric in a Lorentzian almost paracontact manifold has only an index of 1. Abdul Haseeb along with Rajendra Prasad [2] defined Lorentzian para-Kenmotsu (also called LP-Kenmotsu) manifolds in 2018. Afterward, numerous geometers, including [7, 8, 9, 10, 11, 14, 18], have extensively investigated these manifolds.

Inspired by these studies, Tripathi *et al.*, [17] presented the notion of an indefinite almost paracontact metric structure, also referred to as an (ϵ)-almost paracontact structure, by linking an almost paracontact structure with a semi-Riemannian metric, which need not be Lorentzian. In this instance, $\epsilon = 1$ or $\epsilon = -1$ indicates that the structure vector field ξ is either space-like or time-like. In addition, they introduced and examined the characteristics of (ϵ)-para Sasakian [17] and (ϵ)-para Sasakian 3-manifolds [6].

Inspired by the prior study, the current paper continues the discussion of indefinite almost paracontact metric manifolds, introducing the idea of (ϵ)-para Kenmotsu manifolds based on para-Kenmotsu manifolds, defined by Sinha and Sai Prasad in 1995 [13].

The format of the paper is as follows: We define an (ϵ)-para Kenmotsu manifold, investigate some of its fundamental characteristics and derive some typical identities for the Ricci tensor, scalar curvature, and curvature tensor in Section-2. Furthermore, we explore the curvature features of (ϵ)-para Kenmotsu three-dimensional manifolds. We attained an essential as well as sufficient condition for an (ϵ)-para Kenmotsu 3-dimensional manifold M_3 to have an indefinite space form. Furthermore, in Sections 3, 4, and 5, we classify and demonstrate that (ϵ)-para Kenmotsu 3-manifolds, which are either semi-symmetric, Ricci-semi-symmetric, or semi-symmetric type, are η-Einstein. In conclusion, we create a 3-D (ϵ)-para Kenmotsu manifold example.

2. (ϵ)-para Kenmotsu manifolds

A differentiable manifold (M_n, g) of n-dimension is regarded as an (ϵ)-almost paracontact metric manifold [17] with the structure tensors $(\phi,\ \xi,\ \eta,\ g,\ \epsilon)$, where the tensor field $(1,1)$ is represented by ϕ, the vector field by ξ, the 1-form by η, the semi-Riemannian metric by $g(X, Y)$, not necessarily Lorentzian, such that

$$\eta(\xi) = 1, \tag{2.1}$$

$$\overline{\overline{X}} \;=\; X - \eta(X)\xi,\ where\ \overline{X} = \phi X, \tag{2.2}$$

$$g\ (\xi,\ \xi) = \epsilon, \tag{2.3}$$

$$g\ (X,\ \xi) = \epsilon\ \eta(X), \tag{2.4}$$

$$g\ (\phi X,\ \phi Y) = g\ (X,\ Y) - \epsilon\ \eta(X)\ \eta(Y); \tag{2.5}$$

for every $X, Y \in \chi(M_n)$, and $\chi(M_n)$ is a collection of differentiable vector fields on M_n. Since the structure vector field ξ which has been vector field that is either space-like or time-like, and the rank of that tensor filed ϕ is $(n-1)$, in this case, (ϵ) is either 1 or -1.
If $g(X,\ Y)$ is positive definite, that is

$$d\eta(X,\ Y) = g(X,\ \phi Y), \tag{2.6}$$

then the manifold M_n is referred as an almost paracontact metric manifold [12]. Evidently, on M_n, we have

$$\phi\ \xi = 0,\ \ \eta(\phi X) = 0. \tag{2.7}$$

Definition 2.1. *An (ϵ)-paracontact metric structure is referred to as an (ϵ)-para Kenmotsu structure if*

$$(\nabla_X \Phi)Y = g\ (X, \phi Y)\xi - \epsilon\ \eta(Y)\phi X, \tag{2.8}$$

where, for all vector fields X and Y, the Levi-Civita connection is given by ∇ with regard to the indefinite metric $g(X,\ Y)$. An (ϵ)-para Kenmotsu manifold is a manifold M_n with the (ϵ)-para Kenmotsu structure.

For $\epsilon = 1$ and the Riemannian metric $g(X,\ Y)$, the manifold M_n is the standard para-Kenmotsu manifold.

An (ϵ)-almost paracontact metric manifold is an (ϵ)-para Kenmotsu manifold if and only if

$$\nabla_X \xi = \epsilon\ \phi^2(X) = \epsilon\ (X - \eta(X)\xi). \tag{2.9}$$

Furthermore, from (2.4), we get

$$(\nabla_X \eta)\ Y = \epsilon\ g(\nabla_X \xi, Y).$$

Then by using the above expression and (2.9), we have

$$(\nabla_X \eta)\ Y = \epsilon\ g(X,\ Y) - \eta(X)\ \eta(Y). \tag{2.10}$$

Lemma 2.1. *Let M_n be an (ϵ)-para Kenmotsu manifold. Then, the type $(1, 3)$ Riemannian Christoffel curvature tensor $R(X, Y)$ satisfies*

$$R\ (X,\ Y)\xi = \ \eta(X)\ Y - \eta(Y)\ X. \tag{2.11}$$

Consequently*,*

$$R\ (\xi,\ X)\ Y = \epsilon\ \eta(Y)X\ -\ g(X,\ Y)\ \xi, \tag{2.12}$$

$$R\ (\xi,\ X)\xi = \ \epsilon X - \epsilon\ \eta(X)\ \xi, \tag{2.13}$$

$$\eta(R(X, Y)\ Z) = \ \epsilon\ g(X,\ Z)\ \eta(Y) - \epsilon\ \eta(X)\ g(Y,\ Z), \tag{2.14}$$

$$S\ (Y,\ \xi) = \ -(n-1)\ \eta(Y), \tag{2.15}$$

for all vector fields X, Y and Z, where $S(X, Y)$ denotes the Ricci tensor and Q is known to be the Ricci operator with regard to ∇.

Proof. By using the equations (2.9), (2.1), and (2.10) in

$$R(X,\ Y)\ \xi = \ \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,\ Y]}\xi,$$

we obtain (2.11). Moreover, we have

$$R(X, Y, Z, W) = \ g(X,\ Z)\ g(Y,\ W) - g(Y,\ Z)\ g(X,\ W).$$

Then, by using (2.4) and from the above expression, we obtain the results (2.12), (2.13), and (2.14). Further, on the contraction of the above expression with respect to X and W, we get (2.15), and hence it completes the proof. □

Furthermore, it is recognized that we have in a semi-Riemannian 3-manifold

$$\begin{aligned} R\ (X,\ Y)Z = g(X,\ Z)\ QY - g(Y,\ Z)\ QX\ +\ S(X,\ Z)Y \\ - S(Y,\ Z)X - \frac{r}{2}\ [g(X,\ Z)Y\ -\ g(Y,\ Z)X], \end{aligned} \tag{2.16}$$

where r is the manifold's scalar curvature.

By substituting ξ for Z in (2.16) as well as utilizing the equation (2.11) for $n = 3$, we have

$$\epsilon\ [\eta(Y)\ QX - \eta(X)\ QY] = \left[3 + \frac{r\epsilon}{2}\right]\ [\eta(Y)\ X - \eta(X)\ Y]. \tag{2.17}$$

Then for $Y=\xi$ in (2.17) and utilizing (2.2) & (2.15), we get

$$QX=\frac{1}{2}(r+6\epsilon)\ X-\frac{1}{2}(r+10\epsilon)\ \eta(X)\ \xi,$$

and hence

$$S(X,\ Y)=g(QX,\ Y)\ =\ \frac{1}{2}\big[(r+6\epsilon)\ g(X,\ Y)-\ \epsilon\ (r+10\epsilon)\ \eta(X)\ \eta(Y)\big]. \tag{2.18}$$

Therefore from (2.16) and (2.18)

$$R(X,Y)\ Z=\big[g(X,\ Z)\ Y\ -\ g(Y,\ Z)\ X\big]\Big[\frac{r}{2}\ +\ 6\epsilon\Big]$$
$$+\ \big[g(Y,\ Z)\ \eta(X)\ \xi\ -\ g(X,\ Z)\ \eta(Y)\ \xi\ +\ \epsilon\eta(Y)\ \eta(Z)X-\epsilon\eta(X)\ \eta(Z)Y\big]\Big[\frac{r}{2}+5\epsilon\Big]. \tag{2.19}$$

It demonstrates that an (ϵ)-para Kenmotsu manifold with constant scalar curvature is an indefinite space form.

Lemma 2.2. *If the scalar curvature of an (ϵ)-para Kenmotsu manifold of dimension 3 is -6ϵ, then the manifold has an indefinite space form. Also, the converse.*

Proof. Consider a 3-D (ϵ)-para Kenmotsu manifold M_3 which has an indefinite space form. Then

$$R(X,\ Y)\ Z=c\ [g(X,\ Z)\ Y-g(Y,\ Z)\ X], \tag{2.20}$$

where c represents the manifold's constant curvature. Using the definition of Ricci curvature as well as equation (2.20), we get

$$S(X,\ Y)\ =\ 2c\ g(X,\ Y). \tag{2.21}$$

Utilizing (2.21) in the scalar curvature definition yields

$$r=\ 6c. \tag{2.22}$$

Next, it is evident from (2.21) and (2.22) that

$$S(X,\ Y)=\frac{r}{3}\ g(X,\ Y). \tag{2.23}$$

Using (2.23) and entering $X=\ Y=\ \xi$ in (2.18), we get

$$r=\ -6\epsilon. \tag{2.24}$$

On the other hand, the proof is completed if $r=-6\epsilon$, in which case the manifold is clearly an indefinite space form as shown by equation (2.19). □

Theorem 2.1. *Each (ϵ)-para Kenmotsu manifold of dimension 3 is η-Einstein.*

Proof. The theorem's proof is derived from (2.18) and (2.11). □

3. Semi-symmetric (ϵ)-para Kenmotsu 3-manifolds

Definition 3.1. *An (ϵ)-para Kenmotsu manifold of dimension* 3 *is semi-symmetric* [15] *if*

$$R(X,\ Y)\cdot R\ =\ 0, \tag{3.25}$$

holds for all vector fields X and Y.

Theorem 3.1. *M_3 is an η-Einstein manifold, if it is a semi-symmetric (ϵ)-para Kenmotsu 3-manifold.*

Proof. Consider

$$(R(X,Y)\cdot R)\ (Z,W,U) = \ 0, \tag{3.26}$$

for all vector fields $X,\ Y,\ Z$, and U.

The above equation implies that

$$\begin{aligned}(R(X,\ Y,R\ (Z,\ W,\ U))\ -\ R(R(X,\ Y,\ Z),\ W,\ U)\\ -R(Z,\ R(X,\ Y,\ W),U)\ -\ R(Z,\ W,R(X,\ Y)U)=0.\end{aligned} \tag{3.27}$$

Afterward, specifically for $X=\xi$, we have

$$\begin{aligned}(R(\xi,\ Y,R\ (Z,\ W,\ U))\ -\ R(R(\xi,\ Y,\ Z),\ W,\ U)\\ -\ R(Z,\ R(\xi,\ Y,\ W),U)\ -\ R(Z,\ W,R(\xi,\ Y)U)=0.\end{aligned} \tag{3.28}$$

Using the aforementioned equation along with (2.12) and (2.14), we now obtain

$$\begin{aligned}'R\ (Z,\ W,U,\ Y)\xi =&\ \epsilon\, g(Z,\ U)\ \eta(W)Y - \ \epsilon\, g(W,\ U)\ \eta(Z)Y - \ \epsilon\, \eta(Z)\ R(Y,\ W,\ U)\\ &+\ g(Y,Z)\ R(\xi,\ W,\ U) - \ \epsilon\, \eta(W)\ R(Z,\ Y,\ U) + \ g(Y,\ W)\ R(Z,\ \xi,\ U)\\ &-\epsilon\, \eta(U)\ R(Z,\ W,\ Y)\ +\ g(Y,\ U)\ R(Z,\ W,\ \xi).\end{aligned} \tag{3.29}$$

Then by using equations (2.11), (2.12), (2.14), and the inner product with ξ, the above equation is reduced to

$$'R(Z,\ W,\ U,\ Y)\ =\ g(Y,\ W)\ g(Z,\ U)\ - g(Y,\ Z)\ g(W,\ U), \tag{3.30}$$

which, when contracted with regard to U and W, results in

$$S(Y,\ Z) = \ \eta(Y)\ \eta(Z) - n\ \epsilon\ g(Y,Z). \tag{3.31}$$

For $Z = \xi$ in (3.31), we obtain

$$S(Y, \xi) = -(n-1)\, \eta(Y). \tag{3.32}$$

This concludes the proof of the theorem. □

4. Ricci semi-symmetric (ϵ)-para Kenmotsu 3-manifolds

If a semi-Riemannian manifold, M_n, satisfies the following condition, its Ricci tensor, S, is deemed Ricci-semi-symmetric [5].

$$R\,(X, Y) \cdot S = 0, \; for\ all\ X, Y \in \chi(M_n), \tag{4.33}$$

where $R(X, Y)$ serves as a derivation on S.

Let us suppose that M_3 be a Ricci-semi-symmetric (ϵ)-para Kenmotsu *three*-dimensional manifold. That is

$$(R\,(X, Y) \cdot S)\,(Z, U) = 0. \tag{4.34}$$

The above equation further implies that

$$S(R(X, Y)Z, U) + S(U, R(X, Y)\,Z) = 0. \tag{4.35}$$

For $X = \xi$ in (4.35), we have

$$S\,(R(\xi, Y)Z, U) + S(U, R(\xi, Y)\,Z) = 0. \tag{4.36}$$

Now by using (2.12) and (2.15), we have, from the above equation

$$\epsilon\,\eta(Z)\,S(Y, U) + (n-1)\,g(Y, Z)\,\eta(U) + \epsilon\,\eta(U)\,S(Y, Z) + (n-1)\,g(Y, U)\,\eta(Z) = 0. \tag{4.37}$$

Using equations (2.2) and (2.4) and substituting $U = Z = \xi$ in (4.37), we obtain

$$S(Y, \xi) = -(n-1)\,\eta(Y). \tag{4.38}$$

Based on this, we could say the following:

Theorem 4.1. *M_3 is an η-Einstein manifold, if it is a Ricci-semi-symmetric (ϵ)-para Kenmotsu 3-manifold.*

5. Semi-symmetric type (ϵ)-para Kenmotsu 3-manifolds

A semi-Riemannian manifold M_n is considered semi-symmetric type if

$$S(X,\ Y)\cdot R = 0, \tag{5.39}$$

holds for all vector fields X and Y.

Theorem 5.1. *The semi-symmetric type (ϵ)-para Kenmotsu 3-manifold is η-Einstein.*

Proof. Let M_3 be a semi-symmetric type (ϵ)-para Kenmotsu 3-manifold. Then

$$(S(X,\ Y)\cdot R)(Z,\ U,\ V) = 0, \tag{5.40}$$

for all vector fields $X,\ Y,\ Z,\ U$, and V.

The above equation implies that

$$\begin{aligned} &S\ (Y,\ R(Z,\ U,\ V))X - S(X,\ R(Z,\ U,\ V))Y + S(Y,\ Z)\ R(X,\ U,\ V)\\ &- S(Z,\ X)\ R(Y,\ U,\ V) + S(Y,\ U)\ R(Z,\ X,\ V) - S(U,\ X)\ R(Z,\ Y,\ V)\\ &+ S(V,\ Y)\ R(Z,\ U,\ X) - S(V,\ X)\ R(Z,\ U,\ Y) = 0. \end{aligned} \tag{5.41}$$

For $X = \xi$ in (5.41), we have

$$\begin{aligned} &S(Y,\ R(Z,\ U,\ V))\xi - S(\xi,\ R(Z,\ U,\ V))Y + S(Y,\ Z)\ R(\xi,\ U,\ V)\\ &- S(Z,\ \xi)\ R(Y,\ U,\ V) + S(Y,\ U)\ R(Z,\ \xi,\ V) - S(U,\ \xi)\ R(Z,\ Y,\ V)\\ &+ S(V,\ Y)\ R(Z,\ U,\ \xi) - S(V,\ \xi)\ R(Z,\ U,\ Y) = 0. \end{aligned} \tag{5.42}$$

Taking the inner product with ξ and using equations (2.22), (2.12), (2.14), (2.15) in (5.42), we get

$$\begin{aligned} &S(Y,R(Z,U,V)) + 2(n-1)g(Z,V)\eta(U)\eta(Y) - 2(n-1)g(U,V)\eta(Y)\eta(Z)\\ &+\epsilon\eta(V)\eta(U)S(Y,Z) - g(U,V)S(Y,Z) + g(Z,V)S(Y,U)\\ &-\epsilon S(Y,U)\eta(V)\eta(Z) + (n-1)g(Z,Y)\eta(U)\eta(V) - (n-1)g(U,Y)\eta(V)\eta(Z) = 0. \end{aligned} \tag{5.43}$$

If we put ξ in place of V in (5.43) and on using (2.11), we get

$$\epsilon\ \eta(Z)\ S(Y,\ U) - \epsilon\ \eta(U)\ S(Y,\ Z) + (n-1)\ g(Y,\ Z)\eta(U) - (n-1)\ g(U,\ Y)\ \eta(Z) = 0. \tag{5.44}$$

Put $U = Y = \xi$ in (5.44). Then by using (2.2), (2.4), we get

$$S\ (Z,\ \xi) = -(n-1)\ \eta(Z), \tag{5.45}$$

which proves the theorem. ∎

6. Example of a 3-dimensional (ϵ)-para Kenmotsu manifold

In this section, we create a 3-D (ϵ)-para Kenmotsu manifold example.

Example 6.1. *Let $M_3 = \{(x, y, z) \in R^3\}$, where (x, y, z)-represent the standard coordinates in R^3, be a 3-D manifold. Let e_1, e_2, and e_3 be the vector fields on M_3, given by*

$$e_1 = -x\frac{\partial}{\partial x},\ e_2 = x\frac{\partial}{\partial y},\ e_3 = x\frac{\partial}{\partial z}. \tag{6.46}$$

Clearly, at any point in M_3, $\{e_1,\ e_2,\ e_3\}$ represent a set of linearly independent vectors. The Riemannian metric $g(X,\ Y)$ is explained by

$$g\ (e_i, e_j) = \begin{cases} \epsilon, & if\ i = j \\ 0, & if\ i \neq j;\ i,\ j = 1, 2, 3. \end{cases}$$

Let η be the 1-form defined by:

$$g(X, e_1) = \epsilon\ \eta(X).$$

Let ϕ be a $(1,\ 1)$-tensor field on M_3 explained by:

$$\phi(e_1) = 0,\ \phi(e_2) = -\epsilon\ e_2,\ \phi(e_3) = -\epsilon\ e_3.$$

Then the linearity of ϕ & $g(X,\ Y)$ yields that

$$\eta(e_1) = 1,\ \phi^2(X) = X - \eta(X)e_1\ ;$$

$$\text{and } g(\phi X,\ \phi Y) = g(X,\ Y) - \epsilon\ \eta(X)\ \eta(Y),$$

for all $X,\ Y,\ Z \in M_3$.
The structure $(\phi,\ \xi,\ \eta,\ g,\ \epsilon)$ therefore establishes an (ϵ)-almost paracontact structure on M_3 for $e_1 = \xi$.
Now from (6.46), we also have

$$[e_1,\ e_2] = -\epsilon\ e_2,\ [e_1,\ e_3] = -\epsilon\ e_3,\ [e_2,\ e_3] = 0.$$

Koszul's formula provides the Levi-Civita connection ∇ of the metric tensor $g(X,\ Y)$ as follows:

$$\begin{aligned} 2g\ (\nabla_X Y, Z) = {} & Xg\ (Y, Z) + Yg\ (Z, X) - Zg\ (X, Y) \\ & - g\ (X, [Y, Z]) - g\ (Y, [X, Z]) + g\ (Z, [X, Y]). \end{aligned} \tag{6.47}$$

Utilizing the above formula and $e_1 = \xi$ yields the following result:

$$\begin{aligned} &\nabla_{e_1}\ e_1 = 0,\ \nabla_{e_1}\ e_2 = 0,\ \nabla_{e_1}\ e_3 = 0; \\ &\nabla_{e_2}\ e_1 = \epsilon\ e_2,\ \nabla_{e_2}\ e_2 = -\epsilon\ e_1,\ \nabla_{e_2}\ e_3 = 0; \\ &\nabla_{e_3}\ e_1 = \epsilon\ e_2,\ \nabla_{e_3}\ e_2 = 0,\ \nabla_{e_3}\ e_3 = -\epsilon\ e_1. \end{aligned} \tag{6.48}$$

The preceding computations show that the manifold M_3 under consideration meets the conditions $\nabla_X\ \xi = \epsilon\ (X -\ \eta(X)\ \xi)$, for all $e_1 =\ \xi$.
It can be seen from this that the manifold M_3, that is being studied is a dimension three (ϵ)-para Kenmotsu manifold having the structure $(\phi,\ \xi,\ \eta,\ g,\ \epsilon)$.

7. Conclusion

This paper defines a new class of indefinite almost paracontact metric manifolds, termed (ϵ)-para Kenmotsu manifolds, using a semi-Riemannian metric. When these manifolds are semi-symmetric or Ricci-semi-symmetric, the metric described by them is both geometrical and physical in nature. The geometrical features of these manifolds are widely applied in a variety of physical and geometrical fields, including the construction of super resolution sensors in electronic and communication systems, in electrical engineering, and in the general theory of relativity.

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SOME INEQUALITIES ON SUBMANIFOLDS OF A COMPLEX SPACE FORM EQUIPPED WITH COMPLEX SEMI-SYMMETRIC METRIC CONNECTION

BURÇIN DOĞAN *, NERGIZ (ÖNEN) POYRAZ , AND EROL YAŞAR

Abstract. The aim of this study is to introduce geometric inequalities on a complex space form equipped with complex semi-symmetric metric connection (complex s-s.m.c) and to get a formula between intrinsic and extrinsic invariants with the help of these.

Keywords: Submanifold, Chen inequalities, Complex Semi-Symmetric Metric Connection.

2010 Mathematics Subject Classification: 53B15, 53B30, 53C05, 53C50

1. Introduction

According to the famous embedding theory of J. F. Nash, any Riemannian manifold can be isometrically immersed in a suitable Euclidean space. Thus, one of the most fundamental problems of Riemannian submanifold theory is to establish relationships between intrinsic and extrinsic invariants. The Riemannian invariants characterizing a Riemannian manifold have been studied by several geometers for a long time. We note that the sectional curvature and the scalar curvature are called *the main intrinsic curvatures* and the squared mean curvature is called *the main extrinsic curvature* of a Riemann manifold. B.Y. Chen introduced some specific submanifolds which have important intrinsic invariants in [4, 6, 7].

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Let N be a Riemann manifold and $\tau(p)$ is scalar curvature of N. Then $\inf(K)(p)$ is defined as follows

$$\inf(K)(p) = \inf\{K(\Pi)\}$$

where $K(\Pi)$ is a plane section of T_pN. Thus, a new Riemannian invariant δ_N for N was introduced by Chen in [4] as

$$\delta_N = \tau(p) - \inf(K)(p). \tag{1.1}$$

In [4] and [3], Chen established the general optimal inequality and a sharp inequality which named *Chen inequality* for a submanifold N^n of a real space form $R(\widetilde{\overline{c}})$, respectively,

$$\delta_N \leq \frac{n^2(n-2)}{2(n-2)} \|H\|^2 + \frac{1}{2}(n+1)(n-2)\overline{c} \tag{1.2}$$

and

$$\|H\|^2(p) \geq \frac{4}{n^2}\{Ric(U_1) - (n-1)\widetilde{\overline{c}}\}, \quad \forall U_1 \in T_p^1 N^n, \tag{1.3}$$

where $\|H\|^2$ is the squared mean curvature and $Ric(U_1)$ is Ricci curvature of N^n at U_1. Using the above last inequality, many authors established similar inequalities for different kind of submanifolds in ambient manifolds which have different kind of structures [3, 12, 13, 16, 17, 22, 23] and so on *Chen-Ricci inequality* was introduced by Hong and Tripathi in [11]. Later, Chen inequalities for submanifolds of real space forms admitting a semi-symmetric metric conection (s-s.m.c.) was studied by Mihai and Özgür in [14]. On the other hand, Yücesan studied totally real submanifolds of an indefinite Kaehler manifold with a complex s-s.m.c. in [21].

The study is organized as follows:

In section 2, we present preliminaries which will be used throughout this paper. We give some basic information about s-s.m.c. and complex s-s.m.c., respectively. In the last section, we study geometric inequalities for submanifold of complex space forms equipped with a complex s-s.m.c. and present important characterization theorems.

2. Preliminaries

Let $(\widetilde{N}, \widetilde{g})$ be a real $2m-$dimensional semi-Riemannian manifold and J be an almost complex structure such that, for any $U_1, U_2 \in T_p\widetilde{N}$,

$$\widetilde{g}(JU_1, JU_2) = \widetilde{g}(U_1, U_2), \qquad J^2 = -I \tag{2.4}$$

where $T_p\widetilde{N}$ is the tangent space of $\widetilde{N}$ at p.

If complex structure J is parallel according to Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ of $\widetilde{g}$, that is, the following equation is satisfied, then $(\widetilde{N}, \widetilde{g}, J)$ will be called an *indefinite Kaehler manifold*

$$(\overset{\circ}{\widetilde{\nabla}}_{U_1} J)U_2 = 0. \tag{2.5}$$

For a Kaehler manifold, J is integrable and the index of $\widetilde{g}$ is even, say $2v,\ 0 \leq v \leq m$. Note that if $v = 0$, then $\widetilde{N}$ is a positive definite Kaehler manifold (i.e., a classical Kaehler manifold). Moreover, the opposite $-\widetilde{g}$ of an indefinite Kaehler metric $\widetilde{g}$ is also Kaehler with index $2(m - v)$, where $2v$ is the index of $\widetilde{g}$. The indefinite Kaehler metric with index $v = 2$ is a complex version of the Lorentzian metric [1].

2.1. **Semi-symmetric metric connections.** Let $\widetilde{N}$ be a real $n-$dimensional semi-Riemannian manifold with a metric tensor $\widetilde{g}$ of index $v,\ 0 \leq v \leq n$, and its Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$. A linear connection $\overset{\circ}{\nabla}$ on $\widetilde{N}$ is said to be semi-symmetric if the torsion tensor of the connection $\overset{\circ}{\nabla}$ satisfies

$$\overset{\circ}{T}(U_1, U_2) = \pi(U_2)U_1 - \pi(U_1)U_2, \quad \forall U_1, U_2 \in T_pN, \tag{2.6}$$

where π is a $1-$form. A semi-symmetric connection $\overset{\circ}{\nabla}$ is called a semi-symmetric metric connection. [10] if it further satisfies the equation

$$\overset{\circ}{\nabla}\widetilde{g} = 0. \tag{2.7}$$

A relation between a s-s.m.c. $\overset{\circ}{\nabla}$ and the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ of $\widetilde{N}$, which has been obtained by Yano [19], is given by

$$\overset{\circ}{\nabla}_{U_1} U_2 = \overset{\circ}{\widetilde{\nabla}}_{U_1} U_2 + \widetilde{\pi}(U_2)U_1 - \widetilde{g}(U_1, U_2)P, \tag{2.8}$$

where P is the tangent vector on $\bar{N}$ associated with the $1-$form $\widetilde{\pi}$ by

$$\widetilde{\pi}(U_1) = \widetilde{g}(U_1, P), \tag{2.9}$$

for any tangent vector U_1.

2.2. **Complex semi-symmetric metric connections.** [21] Let $\widetilde{N}$ be a real $2m-$dimensional indefinite Kaehler manifold. Now, we consider a linear connection $\widetilde{\nabla}$ on $\widetilde{N}$. When

$$\widetilde{\nabla}\widetilde{g} = 0, \quad \widetilde{\nabla}J = 0 \tag{2.10}$$

and the torsion tensor $\widetilde{T}$ is of the form

$$\widetilde{T}(U_1, U_2) = \widetilde{\pi}(U_2)U_1 - \widetilde{\pi}(U_1)U_2 - 2\widetilde{g}(JU_1, U_2)J\widetilde{P}, \tag{2.11}$$

the connection $\widetilde{\nabla}$ is called a complex semi-symmetric metric connection (complex s-s.m.c.), where $\widetilde{\pi}$ is a 1−form and $\widetilde{P}$ is the tangent vector defined by

$$\widetilde{\pi}(U_1) = \widetilde{g}(\widetilde{P}, U_1). \tag{2.12}$$

Let $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{\nabla}}$ be a complex s-s.m.c. and the Levi-Civita connection defined on $\widetilde{N}$, respectively. Then

$$\begin{aligned} \widetilde{\nabla}_{U_1}U_2 &= \overset{\circ}{\widetilde{\nabla}}_{U_1}U_2 + \widetilde{\pi}(U_2)U_1 - \widetilde{g}(U_1,U_2)\widetilde{P} + \widetilde{\Gamma}(U_2)JU_1 \\ &+ \widetilde{\Gamma}(U_1)JU_2 - \widetilde{g}(JU_1,U_2)J\widetilde{P}, \end{aligned} \tag{2.13}$$

where $\widetilde{\pi}$ and $\widetilde{\Gamma}$ are 1−forms with (2.12) and

$$\widetilde{\Gamma}(U_1) = \widetilde{g}(J\widetilde{P}, U_1), \tag{2.14}$$

for any tangent vector U_1.

Let N be a $n-$dimensional submanifold of a Riemannian manifold $\widetilde{N}$ and $\overset{\circ}{\nabla}$ and ∇ be the Levi-Civita connection and the complex s-s.m.c. on N induced by the Levi-Civita connection $\overset{\circ}{\widetilde{\nabla}}$ and the complex s-s.m.c. $\widetilde{\nabla}$ of $\widetilde{N}$, respectively. Then the *Gauss formulas* with $\overset{\circ}{\nabla}$ and ∇, respectively, are as followings:

$$\overset{\circ}{\widetilde{\nabla}}_{U_1}U_2 = \overset{\circ}{\nabla}_{U_1}U_2 + \overset{\circ}{h}(U_1,U_2) \tag{2.15}$$

and

$$\widetilde{\nabla}_{U_1}U_2 = \nabla_{U_1}U_2 + h(U_1,U_2) \tag{2.16}$$

where $\overset{\circ}{h}$ is the second fundamental form of N in $\widetilde{N}$ and h is a $(0,2)$-tensor on N.

We denote by $\widetilde{R}$ and $\overset{\circ}{\widetilde{R}}$ the Riemannian curvature tensors of an indefinite Kaehler manifold $\widetilde{N}$ with respect to $\widetilde{\nabla}$ and $\overset{\circ}{\widetilde{\nabla}}$, respectively. Also, let R and $\overset{\circ}{R}$ be the Riemannian curvature tensors of a submanifold N of $\widetilde{N}$ with respect to ∇ and $\overset{\circ}{\nabla}$. Then the Gauss equations are with respect to the Levi-Civita connection and the complex s-s.m.c. can be written as

$$\overset{\circ}{\widetilde{R}}(U_1,U_2,V_1,V_2) = \overset{\circ}{R}(U_1,U_2,V_1,V_2) + \widetilde{g}(\overset{\circ}{h}(U_1,V_1),\overset{\circ}{h}(U_2,V_2)) - \widetilde{g}(\overset{\circ}{h}(U_2,V_1),\overset{\circ}{h}(U_1,V_2)) \tag{2.17}$$

and

$$\widetilde{R}(U_1,U_2,V_1,V_2) = R(U_1,U_2,V_1,V_2) + \widetilde{g}(h(U_1,V_1),h(U_2,V_2)) - \widetilde{g}(h(U_2,V_1),h(U_1,V_2)) \tag{2.18}$$

respectively, [21].

Then, by a straightforward computation, we find

$$\begin{aligned} \widetilde{R}(U_1,U_2)V_1 &= \overset{\circ}{\widetilde{R}}(U_1,U_2)V_1 - \widetilde{\alpha}(U_2,V_1)U_1 + \widetilde{\alpha}(U_1,V_1)U_2 \\ &- \widetilde{F}(U_1)\widetilde{g}(U_2,V_1) + \widetilde{F}(U_2)\widetilde{g}(U_1,V_1) - \widetilde{\beta}(U_2,V_1)JU_1 \\ &+ \widetilde{\beta}(U_1,V_1)JU_2 - \widetilde{G}(U_1)\widetilde{g}(JU_2,V_1) + \widetilde{G}(U_2)\widetilde{g}(JU_1,V_1) \\ &+ \widetilde{\gamma}(U_1,U_2)JV_1 - \widetilde{E}(V_1)\widetilde{g}(JU_1,U_2) \end{aligned} \tag{2.19}$$

where

$$\begin{aligned} \widetilde{\alpha}(U_2,V_1) &= (\overset{\circ}{\widetilde{\nabla}}_{U_2}\widetilde{\pi})V_1 - \widetilde{\pi}(U_2)\widetilde{\pi}(V_1) \\ &+ \widetilde{\Gamma}(U_2)\widetilde{\Gamma}(V_1) + \tfrac{1}{2}\widetilde{g}(U_2,V_1)\widetilde{\pi}(\widetilde{P}), \end{aligned} \tag{2.20}$$

$$\begin{aligned} \widetilde{\beta}(U_2,V_1) &= (\overset{\circ}{\widetilde{\nabla}}_{U_2}\widetilde{\Gamma})V_1 - \widetilde{\pi}(U_2)\widetilde{\Gamma}(V_1) \\ &- \widetilde{\Gamma}(U_2)\widetilde{\pi}(V_1) + \tfrac{1}{2}\widetilde{g}(JU_2,V_1)\widetilde{\pi}(\widetilde{P}), \end{aligned} \tag{2.21}$$

$$\overline{\gamma}(U_1,U_2) = (\overset{\circ}{\widetilde{\nabla}}_{U_1}\widetilde{\Gamma})U_2 - (\overset{\circ}{\widetilde{\nabla}}_{U_2}\widetilde{\Gamma})U_1, \tag{2.22}$$

$$\overline{E}(V_1) = 2(\widetilde{\pi}(V_1)J\widetilde{P} - \widetilde{\Gamma}(V_1)\widetilde{P}) \tag{2.23}$$

and

$$\widetilde{g}(\widetilde{F}(U_1),U_2) = \widetilde{\alpha}(U_1,U_2), \quad \widetilde{g}(\widetilde{G}(U_1),U_2) = \widetilde{\beta}(U_1,U_2). \tag{2.24}$$

On the other hand, we have

$$\widetilde{\beta}(U_2,V_1) = -\widetilde{\alpha}(U_2,JV_1), \quad \widetilde{\alpha}(U_2,V_1) = \widetilde{\beta}(U_2,JV_1), \tag{2.25}$$

$$\widetilde{\gamma}(U_1,U_2) = \widetilde{\beta}(U_1,U_2) - \widetilde{\beta}(U_2,U_1) - \widetilde{\pi}(\widetilde{P})\widetilde{g}(JU_1,U_2), \tag{2.26}$$

$$\widetilde{\gamma}(U_1,U_2) = -\widetilde{\gamma}(U_2,U_1), \quad \widetilde{g}(\widetilde{E}(V_1),V_2) = -\,\widetilde{g}(\widetilde{E}(V_2),V_1). \tag{2.27}$$

From now on, in this paper, we assume that $v = 0$, that is, $\widetilde{N}$ is a classical Kaehler manifold.

Let $\{e_1, ..., e_n\}$ be an orthonormal basis of the T_pN^n. Then, following equation can be written for the mean curvature vector

$$H(p) = \frac{1}{n}\sum_{l=1}^{n} h(e_l,e_l). \tag{2.28}$$

We note that, N is totally geodesic if $h = 0$; minimal if $H = 0$ and totally umbilical if $h(U_1,U_2) = g(U_1,U_2)H,\ \forall\ U_1,U_2 \in TN$.

If we consider a 2−dimensional non-degenerate plane $\Pi = Span\{e_l, e_s\}$, then we can calculate the sectional curvature of the section Π at $p \in N$ by

$$K_{ls} = \frac{g(R(e_s, e_l)e_l, e_s)}{g(e_l, e_l)g(e_s, e_s) - g(e_l, e_s)^2}. \tag{2.29}$$

We denote by $K(\pi)$ the sectional curvature of N^n. For $\{e_1, ..., e_n\}$ orthonormal basis and a k-plane section L of TpN^n, the scalar curvature τ at p and the Ricci curvature (or k-Ricci curvature) of L at U_1 is respectively defined by

$$\tau(p) = \sum_{1 \leq l < s \leq n} K_{ls}, \tag{2.30}$$

$$Ric_L(U_1) = K_{12} + K_{13} + ... + K_{1k} \tag{2.31}$$

where $\pi \subset TpN^n$ is a plane section and U_1 be a unit vector in L. We note that for $\{e_1, ..., e_k\}$ is an orthonormal basis of L such that $e_1 = U_1$, K_{ls} is spanned by e_l, e_s [3].

The Riemannian invariant θ_k is defined as:

$$\theta_k(p) = \frac{1}{k-1} \inf_{L, U_1} Ric_L(U_1), \quad p \in N \tag{2.32}$$

where k is a integer such that $2 \leq k \leq n$, L runs over all s-plane sections in TpN^n and U_1 runs over all unit vectors in L.

Let $\widetilde{N}$ be a real $2n-$dimensional Kaehler manifold and J almost complex structure. The sectional curvature of $\widetilde{N}$ in the direction of an invariant 2-plane section by J is called the holomorphic sectional curvature. If the holomorphic sectional curvature is constant $4\overline{c}$ for all plane sections π of $T_p\widetilde{N}$ invariant by J for any $p \in \widetilde{N}$, then $\widetilde{N}$ is called a complex space form and is denoted by $\widetilde{N}(4\overline{c})$. The curvature tensor $\overset{\circ}{\widetilde{R}}$ with respect to $\overset{\circ}{\widetilde{\nabla}}$ on $\widetilde{N}(4\overline{c})$ is calculated by

$$\begin{aligned} \overset{\circ}{\widetilde{R}}(U_1, U_2, V_1, V_2) &= \overline{c}\{ \widetilde{g}(U_1, V_2)\, \widetilde{g}(U_2, V_1) - \widetilde{g}(U_2, V_2)\, \widetilde{g}(U_1, V_1) \\ &+ \widetilde{g}(JU_1, V_2)\, \widetilde{g}(JU_2, V_1) - \widetilde{g}(JU_1, V_1)\, \widetilde{g}(JU_2, V_2) \\ &- 2\, \widetilde{g}(U_1, JU_2)\, \widetilde{g}(V_1, JV_2)\}. \end{aligned} \tag{2.33}$$

3. k-Ricci Curvature and k-Scalar Curvature

Let $\widetilde{N}$ be a Kaehler manifold endowed with a complex s-s.m.c.. Then from (2.19) and (2.33) we get

$$\begin{aligned}\widetilde{R}(e_l,e_s,e_s,e_l) &= \overline{c}(1+\widetilde{g}^2(Je_l,e_s))-\alpha(e_l,e_l)-\alpha(e_s,e_s)\\ &- (2\widetilde{\gamma}(e_l,e_s)+\ \widetilde{g}(\widetilde{E}(e_s),e_l))\widetilde{g}(Je_l,e_s)-\pi(P)\widetilde{g}^2(Je_l,e_s).\end{aligned}\tag{3.34}$$

From (3.34) we derive

$$\begin{aligned}\widetilde{R}(e_l,e_s,e_s,e_l) &= \overline{c}(1+g^2(Je_l,e_s))-\alpha(e_l,e_l)-\alpha(e_s,e_s)\\ &\quad -\widetilde{\pi}(p)\widetilde{g}^2(Je_l,e_s)-N_{ls}.\end{aligned}\tag{3.35}$$

Thus, taking $U_1=V_2=e_l$ and $U_2=V_1=e_s$ and using (3.35), we have

$$\begin{aligned}R(e_l,e_s,e_s,e_l) &= \overline{c}(1+g^2(Je_l,e_s))-\alpha(e_l,e_l)-\alpha(e_s,e_s)-\pi(p)\widetilde{g}^2(Je_l,e_s)\\ &- N_{ls}+\sum_{r=n+1}^{2n+2}h^r(e_l,e_l)h^r(e_s,e_s)-h^r(e_l,e_s)h^r(e_s,e_l).\end{aligned}\tag{3.36}$$

Then, we find

$$\begin{aligned}2\tau(p) &= n(n-1)\overline{c}+\overline{c}\,\|T\|^2-2(n-1)\lambda-\pi(p)\,\|T\|^2\\ &- \sum_{l,s=1}^{m}m_{ij}+\sum_{l,s=1}^{n}\sum_{r=m+1}^{2m}h^r_{ll}h^r_{ss}-(h^r_{ls})^2.\end{aligned}\tag{3.37}$$

If we write JU_1 with its components as $JU_1=TU_1+FU_1$, then we get

$$\|T\|^2=\sum_{l,s=1}^{n}\widetilde{g}^2(Je_l,e_s).$$

Thus, (3.37) can be written by

$$\begin{aligned}2\tau(p) &= n(n-1)\overline{c}-2(n-1)\lambda-\pi(p)\,\|T\|^2+n^2\,\|H\|^2\\ &- \overline{c}\,\|T\|^2-\sum_{l,s=1}^{m}m_{ij}-\sum_{l,s=1}^{n}\sum_{r=m+1}^{2m}(h^r_{ls})^2.\end{aligned}\tag{3.38}$$

Theorem 3.1. *Let N^n be a real $n-$dimensional submanifold of a real $2m-$ dimensional Kaehler manifold of constant holomorphic sectional curvature is constant $4\overline{c}$ endowed with complex s-s.m.c.. Then, the followings are true.*

(i) For each unit vector $U_1\in T^1_p(N)$, we have

$$\begin{aligned}Ric(U_1) &\leq (n-1)\overline{c}+\overline{c}\sum_{s=2}^{n}g^2(JU_1,e_s)-\lambda-\tfrac{1}{2}\sum_{l=1}^{n}N_{ll}-\sum_{s=2}^{n}N_{1s}\\ &- \tfrac{1}{2}\pi(p)\sum_{s=2}^{n}g^2(JU_1,e_s)+\tfrac{1}{4}n^2\,\|H\|^2-(n-2)\alpha(U_1,U_1).\end{aligned}\tag{3.39}$$

(ii) The equality case of (3.39) is satisfied by unit $U_1\in T^1_p(N)$, if and only if (iff)

$$\begin{aligned}&h(U_1,U_2)=0,\ \ \text{for all } U_2\in T^1_p(N) \text{ orthogonal to } U_1,\\ &h(U_1,U_1)=\tfrac{n}{2}H(p).\end{aligned}\tag{3.40}$$

(iii) For $\forall U_1 \in T_p^1(N)$, equality of (3.39) is satisfied iff either p is a totally geodesic point or p is a totally umbilical point and $n = 2$.

Proof. Let $\{e_1, e_2, ..., e_n\}$and $\{e_{n+1}, ..., e_{2m}\}$ be orthonormal basis of $T_{U_1}N$ and $T_{U_1}^{\perp}N$ at $U_1 \in N$, respectively, where $e_n + 1$ is parallel to the mean curvature vector H. Then, from (3.38) we have

$$\begin{aligned}\sum_{l,s=1}^{n}\sum_{r=n+1}^{2m}(h_{ls}^r)^2 &= n(n-1)\overline{c} + \overline{c}\,\|T\|^2 - 2(n-1)\lambda \\ &- \pi(p)\,\|T\|^2 - \sum_{l,s=1}^{n} N_{ls} + n^2\,\|H\|^2 - 2\tau(p). \end{aligned} \tag{3.41}$$

From (3.41) we get

$$\begin{aligned}\frac{1}{4}n^2\,\|H\|^2 &= \tau(p) - \frac{1}{2}(n(n-1) + \|T\|^2)\overline{c} + (n-1)\lambda + \frac{1}{2}\sum_{l,s=1}^{n} N_{ls} \\ &+ \frac{1}{2}\pi(p)\,\|T\|^2 + \frac{1}{4}\sum_{r=n+1}^{2m}(h_{11}^r - h_{22}^r - ... - h_{nn}^r)^2 \\ &+ \sum_{r=n+1}^{2m}\sum_{s=2}^{n}(h_{1s}^r)^2 - \sum_{r=n+1}^{2m}\sum_{2\leq l<s\leq n}(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2). \end{aligned} \tag{3.42}$$

Then using (3.36), we have

$$\begin{aligned}\sum_{r=n+1}^{2m}(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2) &= \sum_{2\leq l<s\leq n} K_{ls} - \frac{(n-1)(n-2)}{2}\overline{c} \\ &- \overline{c}\sum_{2\leq l<s\leq n} g^2(Je_l, e_s) + (n-2)(\lambda - \alpha(e_1, e_1)) \\ &+ \sum_{2\leq l<s\leq n} m_{ls} + \pi(p)\sum_{2\leq l<s\leq n} g^2(Je_l, e_s). \end{aligned} \tag{3.43}$$

From (3.42) and (3.43) we derive

$$\begin{aligned}Ric(e_1) &= (n-1)\overline{c} + \overline{c}\sum_{s=2}^{n} g^2(Je_1, e_s) - \lambda - \frac{1}{2}\sum_{l}^{n} m_{ll} - \sum_{s=2}^{n} N_{1s} \\ &- \frac{1}{2}\pi(p)\sum_{s=2}^{n} g^2(Je_1, e_s) + \frac{1}{4}n^2\,\|H\|^2 \\ &-\frac{1}{4}\sum_{r=n+1}^{2m}(h_{11}^r - h_{22}^r - ... - h_{nn}^r)^2 \\ &- \sum_{r=n+1}^{2n+2}\sum_{s=2}^{n}(h_{1s}^r)^2 - (n-2)\alpha(e_1, e_1). \end{aligned} \tag{3.44}$$

By choosing $e_1 = U_1$ in equation (3.44), (3.39) is obtained.

When the equality case of (3.39), the followings are satisfied:

$$h_{12}^r = h_{13}^r = ... = h_{1n}^r = 0 \text{ and } h_{11}^r = h_{22}^r + ... + h_{nn}^r \tag{3.45}$$

where $r \in \{n+1, ..., 2n+2\}$. Thus, (3.40) is holded.

Let inequality (3.39) satisfy case of equality for $\forall U_1 \in T_pN^n$. Then, from (3.45), $\forall r \in \{n+1, ..., 2n+2\}$, we get $i \in \{1, ..., n\}$,

$$h_{ls}^r = 0, \quad l \neq s, \tag{3.46}$$

$$2h_{ll}^r = h_{11}^r + h_{22}^r + ... + h_{nn}^r. \tag{3.47}$$

Using (3.47), we derive

$$(n-2)(h_{11}^r + h_{22}^r + ... + h_{nn}^r) = 0.$$

It is clear that, there are two situations for the last equality. For $h_{11}^r + h_{22}^r + ... + h_{nn}^r = 0$, if we consider (3.47) and (3.46) together, then, we can write $h_{ls}^r = 0$ for all $l, s \in \{1, ..., n\}$ and $r \in \{n+1, ..., 2n+2\}$ which gives that p is a totally geodesic point. On the other hand, if $n = 2$, then from 3.47, $2h_{11}^r = 2h_{22}^r = h_{11}^r + h_{22}^r$, which completes the proof. The converse is clear. □

Theorem 3.2. *Let N^n be a real $n-$dimensional submanifold of a real $2m-$ dimensional Kaehler manifold of constant holomorphic sectional curvature is constant $4\overline{c}$ endowed with complex s-s.m.c.Then, we get*

$$\begin{aligned} \tau(p) &\leq \frac{1}{2}(n(n-1) + \|T\|^2)\overline{c} - (n-1)\lambda \\ &- \frac{1}{2}\pi(p)\|T\|^2 - \frac{1}{2}\sum_{l,s=1}^{n} m_{ij} + \frac{1}{2}n^2\|H\|^2. \end{aligned} \tag{3.48}$$

Equality case of 3.48 *holds iff* N *is totally geodesic.*

Theorem 3.3. *Let $\widetilde{N}(\overline{c})$ be an $m-$dimensional real space form of constant holomorphic sectional curvature $4\overline{c}$ equipped with complex s-s.m.c. $\widetilde{\overline{\nabla}}$ and N^n be $n-$dimensional Einstein submanifold of $\widetilde{N}(\overline{c})$. Then,*

$$\begin{aligned} \tau(p) &\leq \frac{n(n-1)}{2}(\overline{c} + \|H\|^2) + \frac{\overline{c}}{2}\|T\|^2 - (n-1)\lambda \\ &- (n-1)\lambda - \frac{1}{2}\pi(p)\|T\|^2 - \frac{1}{2}\sum_{l,s=1}^{n} m_{ij} \end{aligned} \tag{3.49}$$

is satisfied and the equality case of (3.49) holds at $p \in N^n$ iff p is a totally umbilical point.

Proof. The relation (3.38) at $p \in N^n$ is equivalent with

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau(p) - n(n-1)\bar{c} - \bar{c}\|T\|^2) + 2(n-1)\lambda \\ &+ \pi(p)\|T\|^2 + \sum_{l,s=1}^{n} m_{ij} + \sum_{r=n+2}^{2m} \sum_{l,s=1}^{n} (h_{ls})^2 \\ &+ \sum_{l=1}^{n} (h_{ll}^{n+1})^2 + \sum_{l\neq s} (h_{ls}^{n+1})^2. \end{aligned} \tag{3.50}$$

For a choosen orthonormal basis, let $\{e_1, e_2, ..., e_n\}$ diagonalize the shape operator $A_{e_{n+1}}$. Then the shape operators take the forms

$$A_{e_{n+1}} = \begin{bmatrix} a_1 & 0 & . & . & . & 0 \\ 0 & a_2 & . & . & . & 0 \\ . & . & . & & & . \\ . & . & & . & & . \\ . & . & & & . & . \\ 0 & 0 & . & . & . & a_n \end{bmatrix}, \tag{3.51}$$

$$A_{e_r} = (h_{ls}^r), \quad l, s = 1, ..., n; \quad r = n+2, ..., n+p, \quad trace A_{e_r} = 0. \tag{3.52}$$

From (3.50), we get

$$\begin{aligned} n^2 \|H\|^2 &= 2\tau(p) - n(n-1)\bar{c} - \bar{c}\|T\|^2) + 2(n-1)\lambda \\ &+ \pi(p)\|T\|^2 + \sum_{l,s=1}^{n} m_{ij} + \sum_{l=1}^{n} (a_l^2) + \sum_{r=n+2}^{2m} (h_{ls}^r)^2. \end{aligned} \tag{3.53}$$

On the other hand, since

$$0 \leq \sum_{l<s} (a_l - a_s)^2 = (n-1)\sum_{l} a_l^2 - 2\sum_{l<s} a_l a_s \tag{3.54}$$

we obtain

$$n^2 \|H\|^2 = \left(\sum_{l=1}^{n} a_l\right)^2 = \sum_{l=1}^{n} a_l^2 + 2\sum_{l<s} a_l a_s \leq n \sum_{l=1}^{n} a_l^2 \tag{3.55}$$

which implies

$$\sum_{l=1}^{n} a_l^2 \geq n\|H\|^2. \tag{3.56}$$

So from (3.53) and (3.56), we have

$$\begin{aligned} n^2 \|H\|^2 &\geq 2\tau(p) - n(n-1)\bar{c} - \bar{c}\|T\|^2 + 2(n-1)\lambda + \pi(p)\|T\|^2 \\ &+ \sum_{l,s=1}^{n} m_{ls} + n\|H\|^2 + \sum_{r=n+2}^{2n+2} \sum_{l,s=1}^{n} (h_{ls}^r)^2. \end{aligned} \tag{3.57}$$

If (3.49) is case of equality, using (3.54) and (3.57) we obtain

$$a_1 = a_2 = ... = a_n \text{ and } A_{e_r} = 0, \quad r = n+2, ..., m. \tag{3.58}$$

which gives p is a totally umbilical point. The converse is obvious. □

Theorem 3.4. *Let $\widetilde{N}(\bar{c})$ be $2m-$dimensional real space form of constant holomorphic sectional curvature $4\bar{c}$ equipped with complex s-s.m.c. $\widetilde{\bar{\nabla}}$ and N^n be $n-$dimensional submanifold of $\widetilde{N}(\bar{c})$. Then we have*

$$\begin{aligned} \theta_k(p) &\leq \bar{c} + \|H\|^2 + \frac{\bar{c}}{n(n-1)}\|T\|^2 - \frac{2}{n}\lambda \\ &- \frac{\lambda}{n(n-1)}\pi(p)\|T\|^2 - \frac{1}{n(n-1)}\sum_{l,s=1}^{n} m_{ls}. \end{aligned} \tag{3.59}$$

Lemma 3.1. *If $n > k \geq 2$ and $a_1, ..., a_n, a$ are real numbers such that*

$$\left(\sum_{l=1}^{n} a_l\right)^2 = (n-k+1)\left(\sum_{l=1}^{n} a_l^2 + a\right) \tag{3.60}$$

then

$$2\sum_{1\leq l<s\leq k} a_l a_s \geq a \tag{3.61}$$

with equality holding iff

$$a_1 + a_2 + ... + a_k = a_{k+1} = ... = a_n. \tag{3.62}$$

Theorem 3.5. *N^n be $n-$dimensional submanifold of an $2m-$dimensional real space form $\widetilde{N}(\bar{c})$ of constant holomorphic sectional curvature $4\bar{c}$ endowed with complex s-s.m.c. $\widetilde{\bar{\nabla}}$. Then, for each k-plane section ($n > k \geq 2$) and $p \in N^n$, we obtain*

$$\begin{aligned} \tau(p) - \tau(\pi_k) &\leq \frac{1}{2}(n+k-1)(n-k)\bar{c} + \frac{1}{2}\bar{c}\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) \\ &- (n-k)\lambda - (k-1)trace(N_{|_{\pi_k^\perp}}) \\ &- \frac{1}{2}\left(\pi(p)\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) + \sum_{l,s=k+1}^{n} m_{ls}\right) \\ &+ \frac{n^2(n-k)}{2(n-k+1)}\|H\|^2. \end{aligned} \tag{3.63}$$

(3.63) is the equation of equality at $p \in N^n$ iff there exist $\{e_1, ..., e_n\}$ and $\{e_{n+1}, ..., e_{2m}\}$ orthonormal basis of TpN^n and $T_p^\perp N^n$, respectively, such that (a) Π_k=Span$\{e_1, ..., e_k\}$ and

(b) the shape operators A_{e_r}, take the forms

$$A_{e_{n+1}} = \begin{bmatrix} h_{11}^{n+1} & 0 & . & . & . & 0 & & \\ 0 & h_{22}^{n+1} & . & . & . & 0 & & \\ . & . & . & & & & 0 & \\ . & . & & . & & & & \\ . & . & & & . & & & \\ 0 & 0 & . & . & . & h_{kk}^{n+1} & & \\ & & 0 & & & & \left(\sum_{l=1}^{k} h_{ll}^{n+1}\right) I_{n-k} & \end{bmatrix}, \tag{3.64}$$

$$A_{e_r} = \left(h_{ls}^r\right), \quad l,s = 1,...,n; \quad r = n+2,...,2n+2, \quad traceA_{e_r} = 0. \tag{3.65}$$

Proof. Let Π_k is a $k-$plane section and we choose orthonormal basis $\{e_1,...,e_n\}$ and $\{e_{n+1},...,e_{2m}\}$ of TpN^n and $T_p^{\perp}N^n$ at p, respectively, such that $\Pi_k = Span\{e_1,...,e_k\}$. If we consider that the mean curvature vector H is in the direction of the normal vector to e_{n+1} and $e_1,...,e_n$ diagonalize the shape operator $A_{e_{n+1}}$, then the shape operators take the forms (3.51) and (3.52). So, we can rewrite (3.38) as

$$\left(\sum_{l=1}^{n} h_{ll}^{n+1}\right)^2 = (n-k+1)\left(\sum_{l=1}^{n}\left(h_{ll}^{n+1}\right)^2 + \sum_{l\neq s}\left(h_{ls}^{n+1}\right) + \sum_{r=n+2}^{2n+2}\sum_{l,s=1}^{n}(h_{ls}^r)^2 + \epsilon\right) \tag{3.66}$$

where

$$\begin{aligned} \epsilon &= 2\tau(p) - n(n-1)\overline{c} - \overline{c}\left\|T\right\|^2 + 2(n-1)\lambda \\ &+ \pi(p)\left\|T\right\|^2 + \sum_{l,s=1}^{n} m_{ls} - \frac{n^2(n-k)}{(n-k+1)}\left\|H\right\|^2. \end{aligned} \tag{3.67}$$

Applying Lemma 1 in (3.66), we get

$$2\sum_{1\leq l<s\leq k} h_{ll}^{n+1}h_{ss}^{n+1} \geq \sum_{l\neq s}\left(h_{ls}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{l,s=1}^{n}(h_{ls}^r)^2 + \epsilon. \tag{3.68}$$

From equation (3.36) it also follows that

$$\begin{aligned} 2\tau(\pi_k) &= k(k-1)\overline{c} + \overline{c}\sum_{l,s=1}^{k} g^2(Je_l,e_s) - 2(k-1)\sum_{l=1}^{k}\alpha(e_l,e_l) \\ &- \pi(p)\sum_{l,s=1}^{k} g^2(Je_l,e_s) - \sum_{l,s=1}^{k} m_{ls} + \sum_{l,s=1}^{k}\sum_{r=n+2}^{2n+2}\left(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2\right) \\ &+ \sum_{l=1}^{k}(h_{ll}^{n+1})^2 + 2\sum_{1\leq l<s\leq k} h_{ll}^{n+1}h_{ss}^{n+1} - \sum_{l,s=1}^{k}(h_{ls}^{n+1})^2. \end{aligned} \tag{3.69}$$

Using (3.68) and (3.69)

$$\begin{aligned} 2\tau(\pi_k) \geq\ & k(k-1)\overline{c} + \overline{c}\sum_{l,s=1}^{k} g^2(Je_l, e_s) - \sum_{l,s=1}^{k} m_{ls} \\ & -2(k-1)\sum_{l=1}^{k}\alpha(e_l, e_l) - \pi(p)\sum_{l,s=1}^{k} g^2(Je_l, e_s) \\ & + \sum_{l,s=1}^{k}\sum_{r=n+2}^{2m}\left(h_{ll}^r h_{ss}^r - (h_{ls}^r)^2\right) + \sum_{l=1}^{k}(h_{ll}^{n+1})^2 \\ & + \sum_{l\neq s}^{n}(h_{ls}^{n+1})^2 + \sum_{r=n+2}^{2m}\sum_{l,s=1}^{n}(h_{ls}^r)^2 + \epsilon - \sum_{l,s=1}^{k}(h_{ls}^{n+1})^2 \end{aligned} \tag{3.70}$$

is obtained. From this, we can write that

$$\begin{aligned} 2\tau(\pi_k) \geq\ & k(k-1)\overline{c} + \overline{c}\sum_{l,s=1}^{k} g^2(Je_l, e_s) - 2(k-1)\sum_{l=1}^{k}\alpha(e_l, e_l) \\ - \ & \pi(p)\sum_{l,s=1}^{k} g^2(Je_l, e_s) - \sum_{l,s=1}^{k} m_{ls} \\ & + \sum_{r=n+2}^{2m}(h_{11}^r + h_{22}^r + ... + ...h_{kk}^r)^2 + \sum_{r=n+2}^{2m}\sum_{l,s>k}(h_{ls}^r)^2 \\ & + \sum_{r=n+2}^{2m}\sum_{s>k}\left((h_{1s}^r)^2 + (h_{2s}^r)^2 + ... + (h_{ks}^r)^2\right) + \epsilon, \end{aligned} \tag{3.71}$$

or

$$\begin{aligned} \tau(\pi_k) \geq\ & \frac{k(k-1)}{2}\overline{c} + \frac{\overline{c}}{2}\sum_{l,s=1}^{k} g^2(Je_l, e_s) \\ & -(k-1)\sum_{l=1}^{k}\alpha(e_l, e_l) \\ - \ & \frac{\pi(p)}{2}\sum_{l,s=1}^{k} g^2(Je_l, e_s) - \frac{1}{2}\sum_{l,s=1}^{k} m_{ls} + \frac{1}{2}\epsilon. \end{aligned} \tag{3.72}$$

We remark that

$$\alpha(e_1, e_1) + \alpha(e_2, e_2) + ... + \alpha(e_k, e_k) = \lambda - trace(\lambda_{|_{\pi_k^\perp}}). \tag{3.73}$$

From (3.67), (3.71) and (3.72), we get

$$\begin{aligned} 2\tau(\pi_k) &\geq -(n+k-1)(n+k)\bar{c} - \bar{c}\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) + 2(n-k)\lambda \\ &+ (k-1)trace(N_{|_{\pi_k^{\perp}}}) - \pi(p)\sum_{l,s=k+1}^{n} g^2(Je_l, e_s) - \sum_{l,s=1}^{k} m_{ls} \\ &+ 2\tau(p) + \sum_{l,s=1}^{n} m_{ls} - \frac{n^2(n-k)}{(n-k+1)}\|H\|^2 \end{aligned}$$

which completes the proof. □

By Theorem 5 we obtain the following corollary.

Corollary 3.1. *Let N^n be $n-$dimensional submanifold of an $2m-$dimensional real space form $\widetilde{N}(\bar{c})$ of constant holomorphic sectional curvature $4\bar{c}$ endowed with complex s-s.m.c. $\tilde{\overline{\nabla}}$. Then, for each k-plane section and $p \in N^n$, we get*

$$\begin{aligned} \delta_N &\leq \frac{1}{2}(n+1)(n-2)\bar{c} + \frac{1}{2}\bar{c}\sum_{l,s=3}^{n} g^2(Je_l, e_s) \\ &- (n-2)\lambda - trace(N_{|_{\pi_k^{\perp}}}) \\ &- \frac{1}{2}\left(\pi(p)\sum_{l,s=3}^{n} g^2(Je_l, e_s) + \sum_{l,s=3}^{n} m_{ls}\right) \\ &+ \frac{n^2(n-2)}{2(n+1)}\|H\|^2. \end{aligned}$$

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SHARP INEQUALITIES FOR QUASI HEMI-SLANT RIEMANNIAN SUBMERSIONS ($\mathcal{QHSS}$)

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Abstract. The purpose of this article, we obtain sharp inequalities involving the Ricci curvature and the scalar curvature on the horizontal and the vertical distributions for quasi-hemi-slant Riemannian submersions (briefly, $\mathcal{QHSS}$)) from complex space forms onto Riemannian manifolds and debate the equivalence posture the acquired inequality. Lastly, we adduce some examples for $\mathcal{QHSS}$.

Keywords: Riemannian submersion, quasi hemi-slant Riemannian submersion, Chen inequality, complex space form, vertical distribution

2010 Mathematics Subject Classification: Primary 53C15, 53B20.

1. Introduction

In 1990, the notion of slant submanifolds of almost Hermitian manifolds was introduced by [8]. It was a natural generalization of both holomorphic and real submanifolds. Inspired by this notion, several geometers have worked on several types of slant submanifolds (see: [1], [5], [6], [7], [27], [32], [33], [34], [35], [36], [38], [44], [45]).

In the 1960s, O'Neills [53] and Gray [20] studied separately Riemannian submersions. In 1976, Watson studied almost complex types of Riemannian submersions [54] and this invention revealed Hermitian submersions between almost Hermitian manifolds. After these

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studies, Şahin [47] introduced the semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds that it was a generalization of holomorphic submersions and anti-invariant submersions [46] and slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds in 2013 [48]. Subsequently, different kinds of structures have been studied in several types of Riemannian submersions(see: [3], [16], [18], [22], [23], [25], [49]). Prasad, Shukla, and Kumar, as a natural generalization of hemi-slant submersions, semi-slant submersions, and bi-slant submersions, identified the notion of quasi bi-slant submersions from Kaehler manifold onto a Riemannian manifold [37]. Longwap, Massamba, and Homti [28]. introduced $\mathcal{QHSS}$ as a generalization of slant, semi-slant, and hemi-slant Riemannian submersions in 2019. On the contrary, Chen established Chen inequalities [9], [10], [11], [14] which as a solution "one of the basic problems in submanifold theory finds simple relationships between the extrinsic and intrinsic invariants of a submanifold". According to Chen [13], a generalization of this inequality was proved arbitrary submanifolds of an arbitrary Riemannian manifold in 2005. Subsequently, several authors investigated Chen-Ricci inequality of submersions and submanifolds (see: [2], [4], [15], [17], [19], [21], [24], [29], [30], [31], [39], [40], [41], [42], [43], [50], [51], [52], [55]). The main purpose of this article acquire some inequalities bearing Ricci curvatures and running Chen-Ricci inequality on the horizontal and the vertical distributions for $\mathcal{QHSS}$ from complex space forms onto Riemannian manifolds.

This article is organized as follows; in Section 2, we recall respectively some basic geometric properties of Riemannian submersions, O'Neill tensors, curvature relations, complex space form, and $\mathcal{QHSS}$. In Section 3, we attain Chen-Ricci inequalities on the horizontal the vertical distributions for $\mathcal{QHSS}$ from complex space forms onto Riemannian manifolds and dispute the equivalence case of the acquired inequality. Eventually, we ensure some examples for $\mathcal{QHSS}$.

2. Quasi hemi-slant Riemannian submersions($\mathcal{QHSS}$)

In this working, unless stated otherwise, all concepts such as manifolds, maps and so on, expressed will be considered differentiable. First let's give the following description.

Definition 2.1. *Let (M_1, g_1) and (M_2, g_2) be Riemannian manifolds, where $dim(M_1)$ is greater than $dim(M_2)$. A surjective mapping $\varphi : (M_1, g_1) \to (M_2, g_2)$ is called a Riemannian submersion if* *(i)* φ *has maximal rank, and*

(ii) φ_**, restricted to* $ker\varphi_*^{\perp}$ *is a linear isometry* [53].

Describe the O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by [53]:

$$\mathcal{T}_\xi\eta = \mathcal{V}\nabla_{\mathcal{V}\xi}\mathcal{H}\eta + \mathcal{H}\nabla_{\mathcal{V}\xi}\mathcal{V}\eta, \tag{2.1}$$

$$\mathcal{A}_\xi\eta = \mathcal{V}\nabla_{\mathcal{H}\xi}\mathcal{H}\eta + \mathcal{H}\nabla_{\mathcal{H}\xi}\mathcal{V}\eta \tag{2.2}$$

for any vector fields $\xi, \eta \in \Gamma(M_1)$, where ∇ is the Levi-Civita connection of g_1. Moreover, from (2.1) and (2.2), we have

$$\nabla_{V_1}V_2 = \mathcal{T}_{V_1}V_2 + \hat{\nabla}_{V_1}V_2, \tag{2.3}$$

$$\nabla_{V_1}X_1 = \mathcal{T}_{V_1}X_1 + \mathcal{H}\nabla_{V_1}X_1, \tag{2.4}$$

$$\nabla_{X_1}V_1 = \mathcal{A}_{X_1}V_1 + \mathcal{V}\nabla_{X_1}V_1, \tag{2.5}$$

$$\nabla_{X_1}X_2 = \mathcal{H}\nabla_{X_1}X_2 + \mathcal{A}_{X_1}X_2, \tag{2.6}$$

for $V_1, V_2 \in \Gamma(ker\varphi_*)$ and $X_1, X_2 \in \Gamma((ker\varphi_*)^{\perp})$ where $\hat{\nabla}_{V_1}V_2 = \mathcal{V}\nabla_{V_1}V_2$. It is not difficult to observe that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution [53].

Specify by R_1, R_2, R_3 and R_4 the Riemannian curvature tensor of Riemannian manifolds M_1, M_2, the vertical distribution $ker\varphi_*$ and the horizontal distribution $(ker\varphi_*)^{\perp}$, seriatim. Then the Gauss-Codazzi type equivalences are dedicated by

$$R_1(U_1, U_2, V_1, V_2) = R_3(U_1, U_2, V_1, V_2) + g_1(\mathcal{T}_{U_1}V_2, \mathcal{T}_{U_2}V_1) - g_1(\mathcal{T}_{U_2}V_2, \mathcal{T}_{U_1}V_1) \tag{2.7}$$

$$\begin{aligned} R_1(X_1, X_2, Y_1, Y_2) = {} & R_4(X_1, X_2, Y_1, Y_2) - 2g_1(\mathcal{A}_{X_1}X_2, \mathcal{A}_{Y_1}Y_2), \\ & + g_1(\mathcal{A}_{X_2}Y_1, \mathcal{A}_{X_1}Y_2) - g_1(\mathcal{A}_{X_1}Y_1, \mathcal{A}_{X_2}Y_2), \end{aligned} \tag{2.8}$$

$$\begin{aligned} R_1(X_1, V_1, Y_1, U_1) = {} & g_1((\nabla_{X_1}\mathcal{T})(V_1, U_1), Y_1) + g_1((\nabla_{V_1}\mathcal{A})(X_1, Y_1), U_1), \\ & - g_1(\mathcal{T}_{V_1}X_1, \mathcal{T}_{U_1}Y_1) + g_1(\mathcal{A}_{Y_1}U_1, \mathcal{A}_{X_1}V_1), \end{aligned} \tag{2.9}$$

where

$$\varphi_*(R_4(X_1, X_2)Y_1)) = R_2(\varphi_*X_1, \varphi_*X_2)\varphi_*Y_1 \tag{2.10}$$

for all $U_1, U_2, V_1, V_2 \in \Gamma(ker\varphi_*)$ and $X_1, X_2, Y_1, Y_2 \in \Gamma(ker\varphi_*)^{\perp}$ [53].

Conversely, the mean curvature vector field $\acute{H}$ of any fibre of Riemannian submersion φ is dedicated by

$$\grave{N} = t\acute{H}, \grave{N} = \sum_{j=1}^{t} \mathcal{T}_{V_j} V_j \tag{2.11}$$

where $\{V_1, ..., V_t\}$ is an orthonormal basis of the vertical distribution $\mathcal{V}$. Additionally, φ has totally geodesic fibers if $\mathcal{T}$ vanishes on $ker\varphi_*$ and $(ker\varphi_*)^\perp$ [53].

Let M_1 be an almost Hermitian manifold with an almost complex structure J_1 and a Hermitian metric g_1. If J_1 is parallel as far as concerns the Levi-Civita connection ∇ on M_1, that mean

$$(\nabla_{X_1} J_1) X_2 = 0$$

for all $X_1,\ X_2 \in \Gamma(TM_1)$, then (M_1, J_1, g_1, ∇) is yclepted a Kaehler manifold. A Kaehler manifold M_1 is named a complex space form if it has fixed holomorphic sectional curvature represented by $M_1(c_1)$. The curvature tensor of the complex space form $M_1(c)$ is dedicated by

$$\begin{aligned} R_5(X_1, X_2)Y_1 = \frac{c_1}{4}\{&g_1(X_2, Y_1)X_1 - g_1(X_1, Y_1)X_2 + g_1(J_1X_2, Y_1)J_1X_1 \\ &- g_1(J_1X_1, Y_1)J_1X_2 + 2g_1(X_1, J_1X_2)J_1Y_1\} \end{aligned} \tag{2.12}$$

for any $X_1, X_2, Y_1 \in \Gamma(TM_1)$.

Definition 2.2. *Let (M_1, g_1, J_1) be an almost Hermitian manifold and (M_2, g_2) be a Riemannian manifold. A Riemannian submersion $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ is called a $\mathcal{QHSS}$ if there exist three mutually orthogonal distribution $\mathcal{D}$, $\mathcal{D}^\perp$ and $\mathcal{D}^\theta$ such that*
(i)$ker\varphi_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$,
(ii)$J_1(\mathcal{D}) = \mathcal{D},\ J_1\mathcal{D}^\perp \subseteq (ker\varphi_*)^\perp$
(iii) for any non-zero vector field $Z_1 \in \Gamma(\mathcal{D}^\theta_p)$, $p \in M_1$ the angle θ between $J_1(Z_1)$ and $\mathcal{D}^\theta_p$ is constant and independent of the choice of point p and Z_1 in $\mathcal{D}^\theta_p$ [28].

We name the angle θ a quasi hemi-slant angle. In this article, we will presume all horizontal vector fields as basic vector fields.

Let $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ be a $\mathcal{QHSS}$, at present. Then [28], we have for all $V \in \Gamma(ker\varphi_*)$, we get

$$J_1V_1 = \psi V_1 + \omega V_1 \tag{2.13}$$

where $\psi V \in \Gamma(ker\varphi_*)$ and $\omega V_1 \in \Gamma(\omega\mathcal{D}^\theta \oplus \omega\mathcal{D}^\perp)$. For any $X_1 \in \Gamma((ker\varphi_*)^\perp)$, we get

$$J_1X_1 = \mathcal{B}_1X_1 + \mathcal{B}_2X_1 \tag{2.14}$$

where $\mathcal{B}_1X_1 \in \Gamma(ker\varphi_*)$ and $\mathcal{B}_2X_1 \in \Gamma(\mathcal{V})$.

Theorem 2.1. [28] *Let M_1 be a $2m$-dimensional almost Hermitian manifold with g_1 a Riemannian metric on M_1 and almost complex structure J_1, and M_2 be a Riemannian manifold with Riemannian metric g_2. Then there is a Riemannian submersion $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ such that its vertical distribution $\ker\varphi_*$ admits three orthogonal distributions $\mathcal{D}, \mathcal{D}^\theta$ and $\mathcal{D}^\perp$ which are invariant, slant and anti-invariant respectively, i.e.*

$$\ker\varphi_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta,$$

with $J_1\mathcal{D} = \mathcal{D}$, the angle θ between $J_1\mathcal{D}^\theta$ and $\mathcal{D}^\theta$ being constant and $J_1\mathcal{D}^\perp \subseteq (\ker\varphi_*)^\perp$. If we denote the dimension of $\mathcal{D}, \mathcal{D}^\theta$ and $\mathcal{D}^\perp$ by m_1, m_2 and m_3, respectively, then we easily see the following particular cases:

(1) *If $m_1 = 0$, then M_1 is a hemi-slant submersion.*
(2) *If $m_2 = 0$, then M_1 is a semi-invariant submersion.*
(3) *If $m_3 = 0$, then M_1 is a semi-slant submersion.*

The submersion in Theorem 2.1 will be called $\mathcal{QHSS}$ and the angle θ is called the quasi hemi-slant angle of the submersion. This means that a $\mathcal{QHSS}$ is a generalization of hemi-slant, semi-invariant and semi-slant submersions.

We say that the $\mathcal{QHSS}$ $\varphi : (M_1, g_1, J_1) \to (M_2, g_2)$ is proper if $\mathcal{D} \neq \{0\}$, $\mathcal{D}^\perp \neq \{0\}$ and $\theta \neq 0, \frac{\pi}{2}$. From the above items, hemi-slant submersions, semi-invariant submersions, and semi-slant submersions are all examples of $\mathcal{QHSS}$. The undermentioned theorem is a characterization for $\mathcal{QHSS}$ of a complex space form. The proof of it completely identical with slant immersions see:[28]. Hence we omit its substantiation.

Theorem 2.2. [28] *Let φ be a Riemannian submersion from a complex manifold (M_1, g_1, J_1) onto a Riemannian manifold (M_2, g_2). Then, φ is a $\mathcal{QHSS}$ if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$\phi^2 U_1 = -\lambda U_1.$$

where $U_1 \in \Gamma(D^\theta)$. Furthermore, in such a case, if θ is the slant angle of φ, it satisfies that $\lambda = \cos^2\theta$.

Lemma 2.1. *Let* $(M_1(c_1), g_1)$, (M_2, g_2) *be a complex space form and a Riemannian manifold, seriatim and* $\varphi : M_1(c_1) \to M_2$ *a* $\mathcal{QHSS}$. *Then the undermentioned relations are current,*

$$g_1(\phi U_1, \phi V_1) = \cos^2\theta g_1(U_1, V_1),$$
$$g_1(\omega U_1, \omega V_1) = \sin^2\theta g_1(U_1, V_1),$$

for any $U_1, V_1 \in \Gamma(ker\varphi_*)$ [28].

Lemma 2.2. *If* φ *is a* $\mathcal{QHSS}$ *then we have*
i) $\phi^2 V = -(\cos^2\theta)V$,
ii) $g_1(\phi V_1, \phi V_2) = \cos^2\theta g_1(V_1, V_2)$,
iii) $g_1(\omega V_1, \omega V_2) = \sin^2\theta g_1(V_1, V_2)$
for all $V_1, V_2 \in \Gamma(D^\theta)$ [28].

3. Chen-Ricci inequality and Chen inequalities

Let $(M_1(c_1), g_1)$, (M_2, g_2) be a complex space form and a Riemannian manifold, seriatim and $\varphi : M_1(c_1) \to M_2$ a $\mathcal{QHSS}$. Additionally, let $\{V_1, ..., V_t, Y_1, ..., Y_n\}$ be an orthonormal basis of $T_pM_1(c_1)$ such that $\mathcal{V} = Sp\{V_1, ..., V_t\}$, $\mathcal{H} = Sp\{Y_1, ..., Y_n\}$ and $t = 2t_1 + 2t_2 + t_3$, where $dim\mathcal{D} = 2t_1$, $dim\mathcal{D}^\theta = 2t_2$ and $dim\mathcal{D}^\perp = t_3$. Then we may consider an adapted quasi hemi-slant orthonormal frames as follows:

$$V_1, V_2 = J_1V_1, ..., V_{2t_1-1}, V_{2t_1} = J_1V_{2t_1-1}, V_{2t_1+1},$$

$$V_{2t_1+2} = \sec\theta\psi V_{2t_1+1}, , ..., V_{2t_1+2t_2-1}, V_{2t_1+2r_2} = \sec\theta\psi V_{2t_1+2t_2-1},$$

$$V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+t_3}.$$

Obviously, we obtain

$$g_1^2(J_1V_i, V_{i+1}) = \begin{cases} 1, & \text{for } i \in \{1, ..., 2t_1 - 1\}, \\ \cos^2\theta, & \text{for } i \in \{1, ..., 2t_1 + 2t_2 - 1\}, \\ 0, & \text{for } i \in \{2t_1 + 2t_2 + 1, ..., 2t_1 + 2t_2 + t_3 - 1\}, \end{cases}$$

then

$$\sum_{i,j=1}^{t} g_1^2(J_1V_i, V_j) = 2(t_1 + t_2\cos^2\theta).$$

Furthermore, let $\{V_1, ..., V_t, Y_1, ..., Y_n\}$ be an orthonormal basis of $T_pM_1(c_1)$ such that $\mathcal{V} = Sp\{V_1, ..., V_t\}$, $\mathcal{H} = Sp\{Y_1, ..., Y_n\}$. Then Ric_1 and Ric_2 are dedicated by

$$Ric_1(V_1) = \sum_{i=1}^{t} R_3(V_1, V_i, V_i, V_1), \tag{3.15}$$

$$Ric_2(Y_1) = \sum_{s=1}^{n} R_2(Y_1, Y_j, Y_j, Y_1). \tag{3.16}$$

Furthermore, scalar curvature τ_1 and τ_2 are defined

$$\tau_1 = \sum_{1\le i<j\le t} R_3(V_i, V_j, V_j, V_i), \tag{3.17}$$

$$\tau_2 = \sum_{1\le i<j\le n} R_4(Y_i, Y_j, Y_j, Y_i). \tag{3.18}$$

Moreover, utilizing (2.7), (2.8) and (2.12), we get

$$\begin{aligned} R_3(V_1, V_2, V_3, V_4) = \frac{c_1}{4}\{&g_1(V_2, V_3)g_1(V_1, V_4) - g_1(V_1, V_3)g_1(V_2, V_4) + g_1(J_1V_2, V_3)g_1(J_1V_1, V_4) \\ &- g_1(J_1V_1, V_3)g_1(J_1V_2, V_4) + 2g_1(V_1, J_1V_2)g_1(J_1V_3, V_4)\} \\ &- g_1(\mathcal{T}_{V_1}V_4, \mathcal{T}_{V_2}V_3) + g_1(\mathcal{T}_{V_2}V_4, \mathcal{T}_{V_1}V_3), \end{aligned} \tag{3.19}$$

$$\begin{aligned} R_4(Y_1, Y_2, Y_3, Y_4) = \frac{c_1}{4}\{&g_1(Y_2, Y_3)g_1(Y_1, Y_4) - g_1(Y_1, Y_3)g_1(Y_2, Y_4) + g_1(J_1Y_2, Y_3)g_1(J_1Y_1, Y_4) \\ &- g_1(J_1Y_1, Y_3)g_1(J_1Y_2, Y_4) + 2g_1(Y_1, J_1Y_2)g_1(J_1Y_3, Y_4)\} \\ &+ 2g_1(\mathcal{A}_{Y_1}Y_2, \mathcal{A}_{Y_3}Y_4) - g_1(\mathcal{A}_{Y_2}Y_3, \mathcal{A}_{Y_1}Y_4) + g_1(\mathcal{A}_{Y_1}Y_3, \mathcal{A}_{Y_2}Y_4). \end{aligned} \tag{3.20}$$

Theorem 3.1. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case, the undermentioned expressions are actual.*

i) If $V_1 \in \Gamma(\mathcal{D})$, in that case

$$Ric_1(V_1) \ge \frac{c_1}{4}(t+2) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}), \tag{3.21}$$

In case of (3.21)' equality holds for a unit vertical vector $V_1 \in \Gamma(\mathcal{D})$ if and only if each fiber is totally geodesic.

ii) If $V_1 \in \Gamma(\mathcal{D}^\theta)$, in that case

$$Ric_1(V_1) \ge \frac{c_1}{4}(t-1+3\cos^2\theta) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}), \tag{3.22}$$

In case of (3.22)'s equality holds for a unit vertical vector $V_1 \in \Gamma(\mathcal{D}^\theta)$ if and only if each fiber is totally geodesic.

iii) If $V_1 \in \Gamma(\mathcal{D}^\perp)$, in that case

$$Ric_1(V_1) \ge \frac{c_1}{4}(t-1) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}), \tag{3.23}$$

In case of (3.23)'s equality holds for a unit vertical vector $V_1 \in \Gamma(\mathcal{D}^\perp)$ if and only if each fiber is totally geodesic.

Proof. Using (3.15) and (3.19) we have,

$$Ric_1(V_1) = \frac{c_1}{4}(t - 1 + 3\sum_{i=1}^{t} g_1^2(J_1V_1, V_i)) - tg_1(\mathcal{T}_{V_1}V_1, \acute{H}) + \|\mathcal{T}_{V_1}V_i\|^2 . \tag{3.24}$$

In that case we have

$$\sum_{i=1}^{t} g_1^2(J_1V_1, V_i) = \begin{cases} 1, & \text{if } V_1 \in \Gamma(\mathcal{D}) \\ \cos^2\theta, & \text{if } V_1 \in \Gamma(\mathcal{D}^\theta) \\ 0, & \text{if } V_1 \in \Gamma(\mathcal{D}^\perp). \end{cases}$$

Using last equivalence in (3.24), we get (3.21), (3.22) and (3.23). □

Theorem 3.2. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case*

$$2\tau_1 \geq \frac{c_1}{4}\{t(t-1) + 6(t_1 + t_2\cos^2\theta)\} - t^2 \left\|\acute{H}\right\|^2 . \tag{3.25}$$

The equivalence case of (3.25) holds if and only if each fiber is totally geodesic.

Proof. From (3.17) and (3.19) we have:

$$2\tau_1 = \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - t^2 \left\|\acute{H}\right\|^2 + \sum_{i,j=1}^{t} g_1(\mathcal{T}_{V_i}V_j, \mathcal{T}_{V_i}V_j). \tag{3.26}$$

Here we have use $\mathcal{T}$ is a symmetric operator. Hence from (3.26) the proof is completed. □

Since φ is $\mathcal{QHSS}$ and $\mathcal{A}$ is an anti-symmetric operator, from (3.18) and (3.20) we have

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\sum_{i,j=1}^{n} g_1(\mathcal{B}_2Y_i, Y_j)g_1(\mathcal{B}_2Y_i, Y_j)) - 3\sum_{i,j=1}^{n} g_1(\mathcal{A}_{Y_i}Y_j, \mathcal{A}_{Y_i}Y_j). \tag{3.27}$$

If we portray

$$\|\mathcal{B}_2\|^2 = \sum_{i=1}^{n} g_1^2(\mathcal{B}_2Y_i, Y_j), \tag{3.28}$$

In that case from (3.27) and (3.28) we get

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\|\mathcal{B}_2\|^2) - 3\sum_{i,j=1}^{n} g_1(\mathcal{A}_{Y_i}Y_j, \mathcal{A}_{Y_i}Y_j). \tag{3.29}$$

From (3.29) we have:

Theorem 3.3. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case*

$$2\tau_2 \leq \frac{c_1}{4}(n(n-1) + 3\|\mathcal{B}_2\|^2). \tag{3.30}$$

In case of (3.30)'s equality holds if and only if $\mathcal{H}(M_1)$ is integrable.

Let $(M_1(c_1), g_1)$ be a complex space form and (M_2, g_2) a Riemannian manifold. Assume that $\varphi : M_1(c_1) \to M_2$ is a $\mathcal{QHSS}$ and $\{V_1, ..., V_t, Y_1, ..., Y_n\}$ is an orthonormal basis of $TpM_1(c_1)$ such that $\mathcal{V}p(M_1) = Sp\{V_1, ..., V_t\}$, $\mathcal{H}p(M_1) = Sp\{Y_1, ..., Y_n\}$. Now, if we denote $\mathcal{T}^s_{ij}$ by

$$\mathcal{T}^s_{ij} = g_1(\mathcal{T}_{V_i} V_j, Y_s), \tag{3.31}$$

where $1 \leq i, j \leq t$ and $1 \leq s \leq n$ (see [17]). The same, if we denote $\mathcal{A}^\alpha_{ij}$ by

$$\mathcal{A}^\alpha_{ij} = g_1(\mathcal{A}_{Y_i} Y_j, V_\alpha), \tag{3.32}$$

where $1 \leq i, j \leq n$ and $1 \leq \alpha \leq t$. From [17], we use

$$\delta(\grave{N}) = \sum_{i=1}^{n} \sum_{k=1}^{t} g_1((\nabla_{Y_i} \mathcal{T})_{V_k} V_k, Y_i). \tag{3.33}$$

$$\sum_{s=1}^{n} \sum_{i,j=1}^{t} (\mathcal{T}^s_{ij})^2 = \frac{1}{2} t^2 \left\| \acute{H} \right\|^2 + \frac{1}{2} (\mathcal{T}^s_{11} - \mathcal{T}^s_{22} - ... - \mathcal{T}^s_{tt})^2$$
$$+ 2 \sum_{s=1}^{n} \sum_{j=2}^{t} (\mathcal{T}^s_{1j})^2 - 2 \sum_{s=1}^{n} \sum_{2 \leq i < j \leq t} (\mathcal{T}^s_{ii} \mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{3.34}$$

The above equations, the Binomial theorem we have like equivalence between the tensor fields $\mathcal{T}$:

Theorem 3.4. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case, the undermentioned statements are actual.*

i) If $V \in \Gamma(\mathcal{D})$, in that case

$$Ric_1(V) \geq \frac{c_1}{4}(t + 2) - \frac{1}{4} t^2 \left\| \acute{H} \right\|^2. \tag{3.35}$$

ii) If $V \in \Gamma(\mathcal{D}^\theta)$, in that case

$$Ric_1(V) \geq \frac{c_1}{4}(t - 1 + 3\cos^2\theta) - \frac{1}{4} t^2 \left\| \acute{H} \right\|^2. \tag{3.36}$$

iii) If $V \in \Gamma(\mathcal{D}^{\perp})$, in that case

$$Ric_1(V) \geq \frac{c_1}{4}(t-1) - \frac{1}{4}t^2 \left\| \acute{H} \right\|^2. \tag{3.37}$$

In case of (3.35)'s, (3.36)'s and (3.37)'s equalities hold if and only if

$$\mathcal{T}^s_{11} = \mathcal{T}^s_{22} + ... + \mathcal{T}^s_{tt},$$

$$\mathcal{T}^s_{1j} = 0, \quad j = 2, ..., r.$$

Proof. Let $\{V_1, ..., V_{2t_1}, V_{2t_1+1}, V_{2t_1+2}, ..., V_{2t_1+2t_2-1}, V_{2t_1+2t_2}, V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+2t_3-1}, V_{2t_1+2t_2+2t_3}\}$ be an adapted quasi hemi-slant basis of $\mathcal{V}p(M_1)$.

i) Because in this case one can comprehend the concerted quasi hemi-slant basis such that $V_1 = V$, it suffices to prove (3.35) for $V = V_1$. Using (3.31) in (3.26) and the symmetry of $\mathcal{T}$, we get

$$2\tau_1 = \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - t^2 \left\| \acute{H} \right\|^2 + \sum_{s=1}^{n}\sum_{i,j=1}^{t}(\mathcal{T}^s_{ij})^2. \tag{3.38}$$

Thus using (3.34) in (3.38) we have

$$\begin{aligned} 2\tau_1 = {} & \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\| \acute{H} \right\|^2 + \frac{1}{2}(\mathcal{T}^s_{11} - \mathcal{T}^s_{22} - ... - \mathcal{T}^s_{tt})^2 \\ & + 2\sum_{s=1}^{n}\sum_{j=2}^{t}(\mathcal{T}^s_{1j})^2 - 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq t}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \end{aligned} \tag{3.39}$$

In that case from (3.39) we get

$$2\tau_1 \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta))) - \frac{1}{2}t^2 \left\| \acute{H} \right\|^2 - 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq t}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{3.40}$$

In addition to, letting $V_1 = V_2 = V_i$, $V_3 = V_4 = V_j$ in (3.19) and using (3.31), we have

$$2\sum_{2\leq i<j\leq t} R(V_i, V_j, V_j, V_i) = 2\sum_{2\leq i<j\leq t} R_3(V_i, V_j, V_j, V_i) + 2\sum_{s=1}^{n}\sum_{2\leq i<j\leq t}(\mathcal{T}^s_{ii}\mathcal{T}^s_{jj} - (\mathcal{T}^s_{ij})^2). \tag{3.41}$$

From (3.41) in (3.40), we have

$$\begin{aligned} 2\tau_1 \geq {} & \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\| \acute{H} \right\|^2 \\ & + 2\sum_{2\leq i<j\leq t} R_3(V_i, V_j, V_j, V_i) - 2\sum_{2\leq i<j\leq t} R(V_i, V_j, V_j, V_i). \end{aligned} \tag{3.42}$$

In addition to, we know

$$2\tau_1 = 2\sum_{2\leq i<j\leq t} R_3(V_i, V_j, V_j, V_i) + 2\sum_{j=1}^{t} R_3(V_1, V_j, V_j, V_1). \tag{3.43}$$

Considering (3.43) in (3.42), we obtain

$$2Ric_1(V_1) \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\|\acute{H}\right\|^2 - 2\sum_{2\leq i<j\leq t} R(V_i, V_j, V_j, V_i). \tag{3.44}$$

Since M_1 is a complex space form, its curvature tensor R satisfies the equality (2.12), we have

$$\sum_{2\leq i<j\leq t} R_1(V_i, V_j, V_j, V_i) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + 3\sum_{2\leq i<j\leq t} g_1^2(J_1V_i, V_j)). \tag{3.45}$$

Taking $V_1 \in \Gamma(\mathcal{D})$ in (3.45), we get

$$\sum_{2\leq i<j\leq t} R_1(V_i, V_j, V_j, V_i) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + 3(t_1 - 1 + t_2\cos^2\theta)). \tag{3.46}$$

Using last equation in (3.44) we have (3.35).

ii) Because in this case one can comprehend the concerted semi-slant basis $\{V_1, ..., V_{2t_1}, V_{2t_1+1}, V_{2t_1+2}, ..., V_{2t_1+2t_2-1}, V_{2t_1+2t_2}, V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+2t_3-1}, V_{2t_1+2t_2+2t_3}\}$ such that $V_{2t_1+1} = V$, it suffices to prove (3.37) for $V = V_{2t_1+1}$.

With like arguments as in case *i*), we obtain

$$2Ric_1(V_{2t_1+1}) \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\|\acute{H}\right\|^2 -2\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} R(V_k, V_s, V_s, V_k). \tag{3.47}$$

and

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} R_1(V_k, V_s, V_s, V_k) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + \sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} g_1^2(J_1V_k, V_s)). \tag{3.48}$$

As $V_{2t_1+1} \in \Gamma(\mathcal{D}^\theta)$, we acquire immediately

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} g_1^2(J_1V_k, V_s) = t_1 + (t_2 - 1)\cos^2\theta_2$$

and therefore (3.48) can be written as

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+1} R_1(V_k, V_s, V_s, V_k) = \frac{c_1}{4}\left[\frac{(t-2)(t-1)}{2} + 3(t_1 + (t_2-1)\cos^2\theta)\right]. \tag{3.49}$$

Considering now the last equation in (3.47), we have

$$Ric_1(V_{2t_1+1}) \geq \frac{c_1}{4}(t - 1 + 3\cos^2\theta) - \frac{1}{4}t^2 \left\|\acute{H}\right\|^2$$

which implies (3.36).

iii) Because in this case one can comprehend the concerted semi-slant basis $\{V_1, ..., V_{2t_1}, V_{2t_1+1}, V_{2t_1+2}, ..., V_{2t_1+2t_2-1}, V_{2t_1+2t_2}, V_{2t_1+2t_2+1}, ..., V_{2t_1+2t_2+2t_3-1}, V_{2t_1+2t_2+2t_3}\}$ such that $V_{2t_1+2t_2+1} = V$, it suffices to prove (3.37) for $V = V_{2t_1+2t_2+1}$.

With similar arguments as in case i), we obtain

$$2Ric_1(V_{2t_1+2t_2+1}) \geq \frac{c_1}{4}(t(t-1) + 6(t_1 + t_2\cos^2\theta)) - \frac{1}{2}t^2 \left\|\acute{H}\right\|^2 -2\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} R_1(V_k, V_s, V_s, V_k) \tag{3.50}$$

and

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} R_1(V_k, V_s, V_s, V_k) = \frac{c_1}{4}(\frac{(t-2)(t-1)}{2} + \sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} g_1^2(J_1V_k, V_s)). \tag{3.51}$$

As $V_{2t_1+2t_2+1} \in \Gamma(\mathcal{D}^\perp)$, we obtain immediately

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} g_1^2(J_1V_k, V_s) = t_1 + t_2\cos^2\theta$$

and therefore (3.51) can be written as

$$\sum_{1\leq k<s\leq t;\ k,s\neq 2t_1+2t_2+1} R_1(V_k, V_s, V_s, V_k) = \frac{c_1}{4}\left[\frac{(t-2)(t-1)}{2} + 3(t_1 + t_2\cos^2\theta)\right]. \tag{3.52}$$

Thinking now the last equation in (3.50), we get

$$Ric_1(V_{2t_1+2t_2+1}) \geq \frac{c_1}{4}(t-1) - \frac{1}{4}t^2 \left\|\acute{H}\right\|^2$$

which implies (3.37). □

Theorem 3.5. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$Ric_2(Y_1) \leq \frac{c_1}{4}(n - 1 + 3\left\|\mathcal{B}_2Y_1\right\|^2). \tag{3.53}$$

In case of (3.53)'s equality holds if and only if

$$\mathcal{A}_{1j}^\alpha = 0, \qquad j = 2, ..., n.$$

Proof. Considering (3.29) and (3.32), we get

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\left\|\mathcal{B}_2\right\|^2) - 3\sum_{\alpha=1}^{t}\sum_{i,j=1}^{n}(\mathcal{A}_{ij}^\alpha)^2. \tag{3.54}$$

In that case (3.54) can be written as

$$2\tau_2 = \frac{c_1}{4}(n(n-1) + 3\left\|\mathcal{B}_2\right\|^2) - 6\sum_{\alpha=1}^{t}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 - 6\sum_{\alpha=1}^{t}\sum_{2\leq i<j\leq n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{3.55}$$

Besides, letting $X_1 = Y_2 = Y_i,\ X_2 = Y_1 = Y_j$ in (3.20) and considering (3.32), we derive

$$2\sum_{2\leq i<j\leq n} R(Y_i, Y_j, Y_j, Y_i) = 2\sum_{2\leq i<j\leq n} R_4(Y_i, Y_j, Y_j, Y_i) + 6\sum_{\alpha=1}^{t}\sum_{2\leq i<j\leq n}(\mathcal{A}_{ij}^{\alpha})^2. \tag{3.56}$$

Using (3.56) in (3.55), we get

$$\begin{aligned} 2\tau_2 = {} & \frac{c_1}{4}(n(n-1) + 3\left\|\mathcal{B}_2\right\|^2) - 6\sum_{\alpha=1}^{t}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2 \\ & + 2\sum_{2\leq i<j\leq n} R_4(Y_i, Y_j, Y_j, Y_i) - 2\sum_{2\leq i<j\leq n} R_1(Y_i, Y_j, Y_j, Y_i). \end{aligned} \tag{3.57}$$

Moreover, using (3.20) we have

$$\sum_{2\leq i<j\leq n} R_1(Y_i, Y_j, Y_j, Y_i) = \frac{c_1}{4}(\frac{(n-2)(n-1)}{2} + 3\sum_{2\leq i<j\leq n} g_1^2(\mathcal{B}_2 Y_i, Y_j)). \tag{3.58}$$

Taking into account that

$$\left\|\mathcal{B}_2\right\|^2 - 2\sum_{2\leq i<j\leq n} g_1^2(\mathcal{B}_2 Y_i, Y_j) = 2\left\|\mathcal{B}_2 Y_1\right\|^2. \tag{3.59}$$

and using (3.57), (3.58) and (3.59), we get

$$2Ric_2(Y_1) = \frac{c_1}{2}(n - 1 + 3\left\|\mathcal{B}_2 Y_1\right\|^2) - 6\sum_{\alpha=1}^{t}\sum_{j=2}^{n}(\mathcal{A}_{1j}^{\alpha})^2. \tag{3.60}$$

Hence the assertion follows. □

Now, we calculate the Chen-Ricci inequality between horizontal and the vertical distributions. For the scalar curvature τ of $M_1(c_1)$, we provide

$$2\tau = \sum_{s=1}^{n} Ric(Y_s, Y_s) + \sum_{k=1}^{t} Ric(V_k, V_k). \tag{3.61}$$

Additionally, we can write

$$\begin{aligned} 2\tau = {} & \sum_{j,k=1}^{t} R_1(V_j, V_k, V_k, V_j) + \sum_{i=1}^{n}\sum_{k=1}^{t} R_1(Y_i, V_k, V_k, Y_i) \\ & + \sum_{i,s=1}^{n} R_1(Y_i, Y_s, Y_s, Y_i) + \sum_{s=1}^{n}\sum_{j=1}^{t} R_1(V_j, Y_s, Y_s, V_j). \end{aligned} \tag{3.62}$$

Next, let us denote as usual (see [17]):

$$\left\|\mathcal{T}^{\mathcal{V}}\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{t} g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i), \tag{3.63}$$

$$\left\|\mathcal{T}^{\mathcal{H}}\right\|^2 = \sum_{k,j=1}^{t} g_1(\mathcal{T}_{V_k}V_j, \mathcal{T}_{V_k}V_j), \tag{3.64}$$

$$\left\|\mathcal{A}^{\mathcal{V}}\right\|^2 = \sum_{i,j=1}^{n} g_1(\mathcal{A}_{Y_i}Y_j, \mathcal{A}_{Y_i}Y_j), \tag{3.65}$$

$$\left\|\mathcal{A}^{\mathcal{H}}\right\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{t} g_1(\mathcal{A}_{Y_i}V_k, \mathcal{A}_{Y_i}V_k). \tag{3.66}$$

Theorem 3.6. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2).*

i) If $V_1 \in \Gamma(\mathcal{D})$, in that case

$$\frac{c_1}{4}(nt + n + t + 3(1 + \|\mathcal{B}_1\|^2 + \|\mathcal{B}_2 Y_1\|^2)) \leq Ric_1(V_1) + Ric_2(Y_1) + \frac{1}{4}t^2 \left\|\acute{H}\right\|^2 + 3\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2 - \delta(\grave{N}) + \left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - \left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.67}$$

ii) If $V_1 \in \Gamma(\mathcal{D}^{\theta})$, in that case

$$\frac{c_1}{4}(nt + n + t + 3(\cos^2\theta + \|\mathcal{B}_1\|^2 + \|\mathcal{B}_2 Y_1\|^2)) \leq Ric_1(V_1) + Ric_2(Y_1) + \frac{1}{4}t^2 \left\|\acute{H}\right\|^2 + 3\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2 - \delta(\grave{N}) + \left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - \left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.68}$$

iii) If $V_1 \in \Gamma(\mathcal{D}^{\perp})$, in that case

$$\frac{c_1}{4}(nt + n + t + 3(t_1 + t_2\cos^2\theta + \|\mathcal{B}_1\|^2 + \|\mathcal{B}_2 Y_1\|^2)) \leq Ric_1(V_1) + Ric_2(Y_1) + \frac{1}{4}t^2 \left\|\acute{H}\right\|^2 + 3\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^{\alpha})^2 - \delta(\grave{N}) + \left\|\mathcal{T}^{\mathcal{V}}\right\|^2 - \left\|\mathcal{A}^{\mathcal{H}}\right\|^2. \tag{3.69}$$

In case of (3.67)'s, (3.68)'s and (3.69)'s equalties hold if and only if

$$\mathcal{T}_{11}^{s} = \mathcal{T}_{22}^{s} + ... + \mathcal{T}_{tt}^{s},$$

$$\mathcal{T}_{1j}^{s} = 0, \qquad j = 2, ..., t.$$

Proof. Since $M_1(c_1)$ is a complex space form, from (3.62) we have

$$2\tau = \frac{c_1}{4}[(n+t)(n+t-1) + 6(t_1 + t_2\cos^2\theta)) + 3(\|\mathcal{B}_2\|^2 + 2\sum_{i=1}^{n}\sum_{k=1}^{t} g_1^2(\mathcal{B}_1 Y_i, V_k))]. \tag{3.70}$$

Now, we define

$$\|\mathcal{B}_1\|^2 = \sum_{i=1}^{n}\sum_{k=1}^{t} g^2(\mathcal{B}_1 Y_i, V_k). \tag{3.71}$$

Moreover, handling the Gauss-Codazzi type equations (2.7)-(2.9), we have

$$\begin{aligned}
2\tau &= 2\tau_1 + 2\tau_2 + t^2 \left\|\acute{H}\right\|^2 - \sum_{k,j=1}^{t} g_1(\mathcal{T}_{V_k}V_j, \mathcal{T}_{V_k}V_j) + 3\sum_{i,s=1}^{n} g_1(\mathcal{A}_{Y_i}X_s, \mathcal{A}_{Y_i}X_s)\\
&\quad - \sum_{i=1}^{n}\sum_{k=1}^{t} g_1((\nabla_{Y_i}\mathcal{T})_{V_k}V_k, Y_i) + \sum_{i=1}^{n}\sum_{k=1}^{t}(g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i) - g_1(\mathcal{A}_{Y_i}V_k, \mathcal{A}_{Y_i}V_k))\\
&\quad - \sum_{s=1}^{n}\sum_{j=1}^{r} g_1((\nabla_{Y_s}\mathcal{T})_{V_j}V_j, Y_s) + \sum_{s=1}^{n}\sum_{j=1}^{t}(g_1(\mathcal{T}_{V_j}Y_s, \mathcal{T}_{V_j}Y_s) - g_1(\mathcal{A}_{Y_s}V_j, \mathcal{A}_{Y_s}V_j)).
\end{aligned} \tag{3.72}$$

Thus considering (3.34) and (3.72), we get

$$\begin{aligned}
2\tau &= 2\tau_1 + 2\tau_2 + \frac{1}{2}t^2 \left\|\acute{H}\right\|^2 - \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - \ldots - \mathcal{T}_{tt}^s)^2 - 2\sum_{s=1}^{n}\sum_{j=2}^{t}(\mathcal{T}_{1j}^s)^2\\
&\quad + 2\sum_{s=1}^{n}\sum_{2\le i<j\le t}(\mathcal{T}_{ii}^s\mathcal{T}_{jj}^s - (\mathcal{T}_{ij}^s)^2) + 6\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2 + 6\sum_{\alpha=1}^{t}\sum_{2\le i<s\le n}(\mathcal{A}_{is}^\alpha)^2\\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{t}(g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i) - g_1(\mathcal{A}_{Y_i}V_k, \mathcal{A}_{Y_i}V_k)) - 2\delta(\grave{N})\\
&\quad + \sum_{s=1}^{n}\sum_{j=1}^{t}(g_1(\mathcal{T}_{V_j}Y_s, \mathcal{T}_{V_j}Y_s) - g_1(\mathcal{A}_{Y_s}V_j, \mathcal{A}_{Y_s}V_j)).
\end{aligned} \tag{3.73}$$

Considering (3.41), (3.56), (3.70) and (3.71) in (3.73), we get

$$\begin{aligned}
&\frac{c_1}{4}[(n+t)(n+t-1) + 3(2(t_1 + t_2\cos^2\theta) + 2\|\mathcal{B}_1\|^2 + \|\mathcal{B}_2\|^2)] = 2Ric_1(V_1)\\
&+ 2Ric_2(Y_1) + \frac{1}{2}t^2 \left\|\acute{H}\right\|^2 - \frac{1}{2}(\mathcal{T}_{11}^s - \mathcal{T}_{22}^s - \ldots - \mathcal{T}_{tt}^s)^2 - 2\sum_{s=1}^{n}\sum_{j=2}^{t}(\mathcal{T}_{1j}^s)^2\\
&+ 6\sum_{\alpha=1}^{t}\sum_{s=2}^{n}(\mathcal{A}_{1s}^\alpha)^2 + \sum_{i=1}^{n}\sum_{k=1}^{t}(g_1(\mathcal{T}_{V_k}Y_i, \mathcal{T}_{V_k}Y_i) - g_1(\mathcal{A}_{Y_i}V_k, A_{Y_i}V_k))\\
&- 2\delta(\grave{N}) + \sum_{s=1}^{n}\sum_{j=1}^{r}(g_1(\mathcal{T}_{V_j}Y_s, \mathcal{T}_{V_j}Y_s) - g_1(\mathcal{A}_{Y_s}V_j, \mathcal{A}_{Y_s}V_j))\\
&+ \sum_{2\le i<j\le t} R_1(V_i, V_j, V_j, V_i) + \sum_{2\le i<j\le n} R_1(Y_i, Y_j, Y_j, Y_i).
\end{aligned} \tag{3.74}$$

If we take $V_1 \in \Gamma(\mathcal{D})$, considering (3.46), (3.58), (3.59), (3.63) and (3.66) in (3.74) we obtain (3.67). If we take $V_1 \in \Gamma(\mathcal{D}^\theta)$, considering (3.49), (3.58), (3.59), (3.63) and (3.66) in (3.74) we obtain (3.68). Similarly, if we take $V_1 \in \Gamma(\mathcal{D}^\perp)$, considering (3.52), (3.58), (3.59), (3.63) and (3.66) in (3.74) we obtain (3.69). This completes the proof. □

Considering (3.70), (3.71) and (3.72) we have

$$\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)] = 2\tau_1+2\tau_2$$
$$+t^2\left\|\acute{H}\right\|^2-\|\mathcal{T}^{\mathcal{H}}\|^2+3\|\mathcal{A}^{\mathcal{V}}\|^2-2\delta(\grave{N})+2\|\mathcal{T}^{\mathcal{V}}\|^2-2\|\mathcal{A}^{\mathcal{H}}\|^2. \quad (3.75)$$

Considering (3.75) we get the following theorem.

Theorem 3.7. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$2\tau_1+2\tau_2 \leq \frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+\|\mathcal{T}^{\mathcal{H}}\|^2+2\delta(\grave{N})-2\|\mathcal{T}^{\mathcal{V}}\|^2+2\|\mathcal{A}^{\mathcal{H}}\|^2, \quad (3.76)$$

$$2\tau_1+2\tau_2 \geq \frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+\|\mathcal{T}^{\mathcal{H}}\|^2-3\|\mathcal{A}^{\mathcal{V}}\|^2+2\delta(\grave{N})-2\|\mathcal{T}^{\mathcal{V}}\|^2. \quad (3.77)$$

In case of (3.76)'s and (3.77)'s equalities hold for all $p \in M_1$ if and only if horizontal distribution $\mathcal{H}$ is integrable.

Considering (3.75) we have the following theorem.

Theorem 3.8. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$2\tau_1+2\tau_2 \geq \frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+2\delta(\grave{N})-2\|\mathcal{T}^{\mathcal{V}}\|^2+2\|\mathcal{A}^{\mathcal{H}}\|^2-3\|\mathcal{A}^{\mathcal{V}}\|^2, \quad (3.78)$$

$$2\tau_1+2\tau_2 \leq \frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]$$
$$-t^2\left\|\acute{H}\right\|^2+\|\mathcal{T}^{\mathcal{H}}\|^2+2\delta(\grave{N})+2\|\mathcal{A}^{\mathcal{H}}\|^2-3\|\mathcal{A}^{\mathcal{V}}\|^2. \quad (3.79)$$

In case of (3.78)'s and (3.79)'s equalities hold for all $p \in M_1$ if and only if the fiber through p of φ is a totally geodesic submanifold of M_1.

Lemma 3.1. *Let k ve l be non-negative real number, In that case*

$$\frac{k+l}{2} \geq \sqrt{kl}$$

with equality iff $k = l$.

Applying Lemma 3.1 in (3.75), we have.

Theorem 3.9. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we have*

$$\frac{c_1}{4}[(n+t)(n+t-1) + 3(2(t_1 + t_2\cos^2\theta) + 2\|\mathcal{B}_1\|^2 + \|\mathcal{B}_2\|^2)] \leq 2\tau_1 + 2\tau_2$$
$$+ t^2 \left\|\acute{H}\right\|^2 + 2\|\mathcal{T}^{\mathcal{V}}\|^2 + 3\|\mathcal{A}^{\mathcal{V}}\|^2 - 2\delta(\grave{N}) - 2\sqrt{2}\|\mathcal{A}^{\mathcal{H}}\|\|\mathcal{T}^{\mathcal{H}}\|. \tag{3.80}$$

In case of (3.80)'s equality hold for all $p \in M_1$ if and only if $\|\mathcal{A}^{\mathcal{H}}\| = \|\mathcal{T}^{\mathcal{H}}\|$.

Theorem 3.10. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we get*

$$\frac{c_1}{4}[(n+t)(n+t-1) + 3(2(t_1 + t_2\cos^2\theta) + 2\|\mathcal{B}_1\|^2 + \|\mathcal{B}_2\|^2)] \geq 2\tau_1 + 2\tau_2$$
$$+ t^2 \left\|\acute{H}\right\|^2 - \|\mathcal{T}^{\mathcal{H}}\|^2 - 2\delta(\grave{N}) - 2\|\mathcal{A}^{\mathcal{H}}\|^2 + 2\sqrt{6}\|\mathcal{A}^{\mathcal{V}}\|\|\mathcal{T}^{\mathcal{V}}\|. \tag{3.81}$$

In case of (3.81)'s equality hold for all $p \in M_1$ if and only if $\|\mathcal{A}^{\mathcal{V}}\| = \|\mathcal{T}^{\mathcal{V}}\|$.

Lemma 3.2. [50] *Let $k_1, k_2, \ldots, k_n$, be $n-$real number $(n > 1)$, In that case*

$$\frac{1}{n}(\sum_{i=1}^{n} k_i)^2 \leq \sum_{i=1}^{n} k_i^2$$

with equality iff $k_1 = k_2 = \ldots = k_n$.

Theorem 3.11. *Let $\varphi : M_1(c_1) \to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1), g_1)$ onto a Riemannian manifold (M_2, g_2). In that case we get*

$$\frac{c_1}{4}[(n+t)(n+t-1) + 3(2(t_1 + t_2\cos^2\theta) + 2\|\mathcal{B}_1\|^2 + \|\mathcal{B}_2\|^2)] \leq 2\tau_1 + 2\tau_2$$
$$+ t(t-1)\left\|\acute{H}\right\|^2 + 3\|\mathcal{A}^{\mathcal{V}}\|^2 - 2\delta(\grave{N}) + 2\|\mathcal{T}^{\mathcal{V}}\|^2 - 2\|\mathcal{A}^{\mathcal{H}}\|^2. \tag{3.82}$$

In case of (3.82)'s equality holds for all $p \in M_1$ if and only if we get statements:

i) φ is a Riemannian submersion that has a totally umbilical fiber.

ii) $\mathcal{T}_{ij} = 0$, for $i \neq j \in \{1, 2, \ldots, r\}$.

Proof. Using (3.75) we get

$$\frac{c_1}{4}[(n+t)(n+t-1) + 3(2(t_1 + t_2\cos^2\theta) + 2\|\mathcal{B}_1\|^2 + \|\mathcal{B}_2\|^2)]$$
$$= 2\tau_1 + 2\tau_2 + t^2\left\|\acute{H}\right\|^2 - \sum_{j=1}^{t}(\mathcal{T}_{jj}^s)^2 - \sum_{i=1}^{n}\sum_{j\neq k}^{t}(\mathcal{T}_{jk}^s)^2$$
$$+ 3\|\mathcal{A}^{\mathcal{V}}\|^2 - 2\delta(\grave{N}) + 2\|\mathcal{T}^{\mathcal{V}}\|^2 - 2\|\mathcal{A}^{\mathcal{H}}\|^2. \tag{3.83}$$

Applying Lemma 3.2 in (3.83), we have

$$\begin{aligned}&\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]\\&\leq 2\tau_1+2\tau_2+t^2\left\|\acute{H}\right\|^2-\frac{1}{t}(\sum_{j=1}^{t}\mathcal{T}_{jj}^s)^2-\sum_{s=1}^{n}\sum_{j\neq k}^{t}(\mathcal{T}_{jk}^s)^2\\&+3\left\|\mathcal{A}^{\mathcal{V}}\right\|^2-2\delta(\grave{N})+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2.\end{aligned}\tag{3.84}$$

From this we have (3.82). In case of (3.82)'s equality holds for all $p\in M_1$ if and only if

$$\mathcal{T}_{11}=\mathcal{T}_{22}=...=\mathcal{T}_{tt} \text{ and } \sum_{s=1}^{n}\sum_{j\neq k}^{t}(\mathcal{T}_{jk}^s)^2=0.$$

This completes proof of the theorem. □

By compairing the proof of Theorem 3.11, we have

Theorem 3.12. *Let $\varphi: M_1(c_1)\to M_2$ be a $\mathcal{QHSS}$ from a complex space form $(M_1(c_1),g_1)$ onto a Riemannian manifold (M_2,g_2). In that case we have*

$$\begin{aligned}&\frac{c_1}{4}[(n+t)(n+t-1)+3(2(t_1+t_2\cos^2\theta)+2\|\mathcal{B}_1\|^2+\|\mathcal{B}_2\|^2)]\geq 2\tau_1+2\tau_2\\&+t^2\left\|\acute{H}\right\|^2-\left\|\mathcal{T}^{\mathcal{H}}\right\|^2+\frac{3}{n}(\mathcal{A}^{\mathcal{V}})^2-2\delta(\grave{N})+2\left\|\mathcal{T}^{\mathcal{V}}\right\|^2-2\left\|\mathcal{A}^{\mathcal{H}}\right\|^2.\end{aligned}\tag{3.85}$$

Equality case of (3.85) holds for all $p\in M_1$ if and only if $\mathcal{A}_{11}=\mathcal{A}_{22}=...=\mathcal{A}_{nn}$ and $\mathcal{A}_{ij}=0$, for $i\neq j\in\{1,2,...,n\}$.

4. Examples

In this section, we provide examples of $\mathcal{QHSS}$, illustrating the main results stated above and the examples of $\mathcal{QHSS}$ satisfying the equality case of all inequalities established in the above section.

Example 4.1. *Let $(\mathbb{R}^8,g_{\mathbb{R}^8},J_1)$ be an almost Hermitian manifold which*
$J_1(x_1,...,x_8)=(-x_6,x_5,-x_7,x_8,-x_2,x_1,x_3,-x_4)$ be a complex structure and $(\mathbb{R}^3,g_{\mathbb{R}^3})$ be a Riemanniann manifold.
$\varphi:(\mathbb{R}^8,g_{\mathbb{R}^8},J_1)\to(\mathbb{R}^3,g_{\mathbb{R}^3})$ be a map defined by

$$\varphi(x_1,x_2,...,x_8)=(\frac{1}{\sqrt{2}}x_1-\frac{1}{\sqrt{2}}x_5,x_2,x_3cos\alpha-x_7sin\alpha)$$

where $\theta\in(0,\frac{\pi}{2})$. In that case φ is a $\mathcal{QHSS}$ (where rank $\varphi_*=3$) such that

$$V_1=\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1}+\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5},V_2=sin\alpha\frac{\partial}{\partial x_3}+cos\alpha\frac{\partial}{\partial x_7},V_3=\frac{\partial}{\partial x_4},V_4=\frac{\partial}{\partial x_6},V_5=\frac{\partial}{\partial x_8},$$

$ker\varphi_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$, *where*

$$\mathcal{D} = \left\langle V_3 = \frac{\partial}{\partial x_4}, V_5 = \frac{\partial}{\partial x_8} \right\rangle$$

$$\mathcal{D}^\perp = \left\langle V_2 = sin\alpha\frac{\partial}{\partial x_3} + cos\alpha\frac{\partial}{\partial x_7} \right\rangle$$

$$\mathcal{D}^\theta = \left\langle V_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5}, V_4 = \frac{\partial}{\partial x_6} \right\rangle$$

and

$$(\ker\varphi_*)^\perp = \left\langle H_1 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_1} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_5}, H_2 = \frac{\partial}{\partial x_2} H_3 = cos\alpha\frac{\partial}{\partial x_3} - sin\alpha\frac{\partial}{\partial x_3} \right\rangle$$

which $\mathcal{D} = \langle V_3, V_5\rangle$ *is invariant,* $\mathcal{D}^\perp = \langle V_2\rangle$ *is anti-invariant and* $\mathcal{D}^\theta = \langle V_1, V_7\rangle$ *slant with slant angle* $\theta = \frac{\pi}{4}$.

Example 4.2. *Let* $(\mathbb{R}^{10}, g_{\mathbb{R}^{10}}, J_1)$ *be an almost Hermitian manifold which* $J_1(x_1, ..., x_{10}) = (-x_7, -x_9, -x_6, x_{10}, x_8, x_3, x_1, -x_5, x_2, -x_4)$ *be a complex structure and* $(\mathbb{R}^5, g_{\mathbb{R}^3})$ *be a Riemanniann manifold.*

$F : (\mathbb{R}^{10}, g_{\mathbb{R}^{10}}, J_1) \to (\mathbb{R}^5, g_{\mathbb{R}^5})$ *be a map defined by*

$$F(x_1, x_2, ..., x_{10}) = (cos\alpha x_1 - sin\alpha x_{10}, \frac{1}{\sqrt{2}}x_2 - \frac{1}{\sqrt{2}}x_6, \frac{1}{\sqrt{2}}x_3 + \frac{1}{\sqrt{2}}x_9, x_4, \frac{\sqrt{3}}{\sqrt{2}}x_5 - \frac{1}{2}x_8)$$

where $\theta \in (0, \frac{\pi}{2})$. *In that case* F *is a* $\mathcal{QHSS}$ *(where rank* $F_* = 5$ *) such that*

$$V_1 = -sin\alpha\frac{\partial}{\partial x_1} + cos\alpha\frac{\partial}{\partial x_{10}}, V_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_6}, V_3 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_9},$$

$$V_4 = \frac{1}{2}\frac{\partial}{\partial x_5} - \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_8}, V_5 = \frac{\partial}{\partial x_7},$$

$kerF_* = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \mathcal{D}^\theta$, *where*

$$\mathcal{D} = \left\langle V_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_6}, V_3 = -\frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_9} \right\rangle$$

$$\mathcal{D}^\perp = \left\langle V_4 = \frac{1}{2}\frac{\partial}{\partial x_5} + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_8} \right\rangle$$

$$\mathcal{D}^\theta = \left\langle V_1 = -sin\alpha\frac{\partial}{\partial x_1} - cos\alpha\frac{\partial}{\partial x_{10}}, V_5 = \frac{\partial}{\partial x_7} \right\rangle$$

and

$$(kerF_*)^\perp = \langle H_1 = -cos\alpha\frac{\partial}{\partial x_1} + sin\alpha\frac{\partial}{\partial x_{10}}, H_2 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_2} - \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_6}, H_3 = \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_3} + \frac{1}{\sqrt{2}}\frac{\partial}{\partial x_9},$$

$$H_4 = \frac{\sqrt{3}}{2}\frac{\partial}{\partial x_5} - \frac{1}{2}\frac{\partial}{\partial x_8}, H_5 = \frac{\partial}{\partial x_4}\rangle$$

which $\mathcal{D} = \langle V_2, V_3\rangle$ *is invariant,* $\mathcal{D}^\perp = \langle V_4\rangle$ *is anti-invariant and* $\mathcal{D}^\theta = \langle V_1, V_5\rangle$ *slant with slant angle* $\theta = arccos(sin\alpha)$.

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MORPHISMS AND ALGEBRAIC POINTS ON THE QUOTIENTS OF FERMAT CURVES AND HURWITZ CURVES

MOUSSA FALL * AND MOUSTAPHA CAMARA

ABSTRACT. In this paper we determine rational morphisms between the Hurwitz curves of affine equation : $u^n v^l + v^n + u^l = 0$ and the quotients of Fermat curves of affine equation $v^m = u^\lambda (u-1)$ where the integers $n > l \geq 1$ are coprime and $m = n^2 - ln + l^2$ and $\lambda \geq 1$. We also give a parametrization of the algebraic points of low degree on the quotient of Fermat curve : $v^7 = u(u-1)^2$. Using these morphisms, we explicitly determine the algebraic points of degree at most 3 on the Hurwitz curve $u^3v^2 + v^3 + u^2 = 0$ birationally isomorphic to the quotient of Fermat curve $v^7 = u^2(u-1)$.

Keywords: Hurwitz curve, Quotient of Fermat curve, Morphism, Degree of algebraic point

2010 Mathematics Subject Classification:14H40, 14L40.

1. Introduction

Let $\mathcal{C}$ be an algebraic curve defined over the rational number field $\mathbb{Q}$ and K an extension field of $\mathbb{Q}$. We denote by $\mathcal{C}(K)$ the set of rational points of $\mathcal{C}$ witch coordinates ares in K. A point $P \in \mathcal{C}(\overline{\mathbb{Q}})$ is said to be of degree d over $\mathbb{Q}$ if its field of definition L is an extension of $\mathbb{Q}$ of degree d. We denote by $\mathcal{C}^{(d)}(\mathbb{Q})$ the set of algebraic points of degree at most d on the curve $\mathcal{C}$ over $\mathbb{Q}$. A famous theorem of Faltings states that the number of rational points on an algebraic curve defined over a number field K is finite if the genus g of the curve is greater than 1. Currently, for a curve $\mathcal{C}$ of genus $g \geq 2$ defined over a number field K, there is no general method for computing the set $\mathcal{C}(K)$ or showing that $\mathcal{C}(K)$ is empty. But there are

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several methods for finding $\mathcal{C}(K)$ in special cases. These methods include the local method, the Chabauty elliptic method [4], the descent method [10], the Mordell-Weil Sieves method [3], the Sall-Fall method [9]. These methods are only applicable if the Mordell-Weil group $J_{\mathcal{C}}(\mathbb{Q})$ is known and is of a finite type. If $J_{\mathcal{C}}(\mathbb{Q})$ is finite, it is possible to determine $\mathcal{C}(\mathbb{Q})$ and to generalize to all number fields K and thus deduce $\mathcal{C}^{(d)}(\mathbb{Q})$ [7]. If we don't know the structure of the Mordell-Weil group, then we need to find a way to working around of it.

The purpose of this paper is to describe explicitly the morphisms between Hurwitz curves of affine equation $u^n v^l + v^n + u^l = 0$ and the Fermat quotient curves of affine equation $v^m = u^\lambda(u-1)$ where $n > l \geq 1$, $\gcd(n,l) = 1$, $\lambda \geq 1$ and $m = n^2 - ln + l^2$. Using these morphisms, we explicitly determine the algebraic points of degree at most 3 on the Hurwitz curves of affine equation $u^3v^2 + v^3 + u^2 = 0$, birationally isomorphic to the Fermat quotient curve of affine equation $v^7 = u^2(u-1)$. Our main results are Theorem 3.2 and Theorem 4.1.

2. Morphisms on Fermat curves and Hurwitz curves

2.1. Fermat curves and quotients of Fermat curves.

Let p be positive integer and K be an number field.

Definition 2.1. *The Fermat curve of degree p over a number field K is given by the projective equation*

$$F_p : U^p + V^p + W^p = 0.$$

The affine equation of F_p is

$$F_p : u^p + v^p + 1 = 0.$$

The Fermat curve F_p is smooth when the characteristic $car(K)$ of K does not divide p and has genus

$$g = \frac{(p-1)(p-2)}{2}.$$

For a pair (r,s) of positive integers such that $1 \leq r, s, r+s < p$ and $\gcd(r,s,p) = 1$, we denote by $C_{r,s}(p)$ the quotient of F_p defined by the equation

$$v^p = u^r(u-1)^s$$

where the projection $F_p \longrightarrow C_{r,s}(p)$ is defined by

$$\begin{array}{rccc}\phi : & F_p & \longrightarrow & C_{r,s}(p) \\ & (u,v) & \longmapsto & (-u^p,\, (-u)^r v^s).\end{array}$$

Lemma 2.1. *Let $C_{r,s}(p)$ and $C_{r',s'}(p)$ be two quotients of Fermat curve F_P. If it exists three integers k, i and j such that :*

$$(r,s) = k(r',s') + p(i,j) \quad and \quad gcd(k,p) = 1,$$

then we have the birational equivalences

(1) $C_{r,s}(p) \cong C_{r',s'}(p)$,
(2) $C_{r,s}(p) \cong C_{s,r}(p)$,
(3) $C_{r,s}(p) \cong C_{p-s-r,s}(p)$.

Proof.

(1) Consider the following covering map:

$$\begin{array}{rccc} f_{rs}: & C_{r',s'}(p) & \longrightarrow & C_{r,s}(p) \\ & (u,v) & \longmapsto & \left(u,\, v^k u^i (u-1)^j\right). \end{array}$$

We have the following successive equivalences :

$$\begin{aligned} \left(u,\, v^k u^i (u-1)^j\right) \in C_{r,s}(p) \ &\Leftrightarrow \left(v^k u^i (u-1)^j\right)^p - u^r (u-1)^s = 0 \\ &\Leftrightarrow v^{pk} u^{pi} (u-1)^{pj} - u^r (u-1)^s = 0 \\ &\Leftrightarrow u^{pi} (u-1)^{pj} \left(v^{pk} - u^{r-pi} (u-1)^{s-pj}\right) = 0 \\ &\Leftrightarrow u^{pi} (u-1)^{pj} \left(v^{pk} - u^{kr'} (u-1)^{ks'}\right) = 0 \\ &\Leftrightarrow u^{pi} (u-1)^{pj} \left(v^{p} - u^{r'} (u-1)^{s'}\right) \left(v^{p(k-1)} + \cdots\right) = 0. \end{aligned}$$

So

$$(u,v) \in C_{r',s'}(p) : v^p - u^{r'} (u-1)^{s'} = 0,$$

then $C_{r,s}(p)$ is isomorphic to $C_{r',s'}(p)$.

(2) Consider the following covering map:

$$\begin{array}{rccc} f_{rs}: & C_{r,s}(p) & \longrightarrow & C_{s,r}(p) \\ & (u,v) & \longmapsto & (1-u, (-1)^{s+r} v). \end{array}$$

We have

$$\begin{aligned} (1-u, (-1)^{s+r} v) \in C_{s,r}(p) \ &\Longleftrightarrow ((-1)^{s+r} v)^p = (1-u)^s ((1-u)-1)^r \\ &\Longleftrightarrow (-1)^{r+s} v^p = (-1)^{r+s} u^r (u-1)^s \\ &\Longleftrightarrow (u,v) \in C_{r,s}(p). \end{aligned}$$

(3) Consider the following covering map:

$$\begin{array}{rccc} f_{rs}: & C_{r,s}(p) & \longrightarrow & C_{p-r-s,s}(p) \\ & (u,v) & \longmapsto & \left(\dfrac{1}{u}, \dfrac{(-1)^s v}{u}\right). \end{array}$$

We have

$$\begin{aligned}\left(\frac{1}{u},\frac{(-1)^s v}{u}\right)\in C_{p-r-s,s}(p) &\Longleftrightarrow \left(\frac{(-1)^s v}{u}\right)^p = \left(\frac{1}{u}\right)^{p-r-s}\left(\frac{1}{u}-1\right)^s\\ &\Longleftrightarrow (-1)^{sp}v^p = (-1)^s u^r(u-1)^s\\ &\Longleftrightarrow (u,v)\in C_{r,s}(p).\end{aligned}$$

□

The following corollary is the consequence of the lemma 2.1.

Corollary 2.1. *Let $C_{r,s}(p)$ be the quotient curve of Fermat. We have*

(i) $C_{1,s}(p) \cong C_{s,1}(p) \cong C_{p-s-1,1}(p) \cong C_{1,p-s-1}(p)$.

(ii) *For $2s \leq p-1$, the curves $C_{1,1}(p),\ C_{1,2}(p),\ C_{1,3}(p,)\ldots, C_{1,\frac{p-1}{2}}(p)$ form a complete list (with repetition).*

(iii) *Any curve $C_{r,s}(7)$ is birationally isomorphic either to the hyperelliptic curve $C_{1,1}(7)$, or to the non-hyperelliptic curve $C_{1,2}(7)$ which is itself isomorphic to the Klein curve.*

2.2. **Hurwitz curves.**

Let n and l be positive integers $n > l \geq 1$ and K be an number field.

Definition 2.2. *The Hurwitz curve $H_{n,l}$ over K is given by the projective equation*

$$H_{n,l} : U^nV^l + V^nW^l + U^lW^n = 0.$$

The affine equation of $H_{n,l}$ is

$$H_{n,l} : u^nv^l + v^n + u^l = 0.$$

Let $m = n^2 - nl + l^2$. The Hurwitz curve $H_{n,l}$ has the following genus

$$g = \frac{m + 2 - 3\gcd(n,l)}{2}.$$

The curve $H_{n,l}$ is smooth when the characteristic car(K) of K is relatively prime to m.

Lemma 2.2. *Let n, l be two positive integers such that $gcd(n,l) = 1$. An integer $m > 3$ of the form $m = n^2 - nl + l^2$ is prime if and only if $m \equiv 1 \pmod 6$.*

Proof. See Bennama and Carbonne [2]. □

Lemma 2.3. *Let n and l be integers satisfying $1 \leq l < n$. The Hurwitz curve $H_{n,l}$ is covered by the Fermat curve F_m of degree m where $m = n^2 - nl + l^2$.*

Proof. Consider the following covering map is provided by [1]

$$\begin{array}{rccc} \phi_{n,l}: & F_m & \longrightarrow & H_{n,l} \\ & (u,v) & \longmapsto & \left(u^n v^{-l},\, u^l v^{n-l}\right). \end{array}$$

The image of the Fermat curve of affine equation $F_{n^2-ln+l^2}$ by the morphism $\phi_{n,l}$ include in $H_{n,l}$.

$$\begin{aligned} \left(u^n v^{-l},\, u^l v^{n-l}\right) \in H_{n,l} &\implies (u^n v^{-l})^n (u^l v^{n-l})^l + (u^l v^{n-l})^n + (u^n v^{-l})^l = 0 \\ &\implies (u^l v^{n-l})^n (u^{n^2-ln+l^2} + v^{n^2-ln+l^2} + 1) = 0 \\ &\implies (u,v) \in F_{n^2-ln+l^2}. \end{aligned}$$

Therefore the Hurwitz curve $H_{n,l}$ is covered by the Fermat curve $F_{n^2-ln+l^2}$. □

In the Table 2.1 we have the following correspondence with Hurwitz curve $H_{n,l}$ and Fermat curve F_m where $m = n^2 - nl + l^2$.

TABLE 2.1. Covering map $\phi_{n,l} : F_m \to H_{n,l}$

n	l	m	Hurwitz curve $H_{n,l}$	Fermat curve F_m	Covering map
3	1	7	$H_{3,1}$	F_7	$\left(u^3v^{-1},\, uv^2\right)$
3	2	7	$H_{3,2}$	F_7	$\left(u^3v^{-2},\, u^2v\right)$
4	1	13	$H_{4,1}$	F_{13}	$\left(u^4v^{-1},\, uv^3\right)$
4	3	13	$H_{4,3}$	F_{13}	$\left(u^4v^{-3},\, u^3v\right)$
5	2	19	$H_{5,2}$	F_{19}	$\left(u^5v^{-2},\, u^2v^3\right)$
5	3	19	$H_{5,3}$	F_{19}	$\left(u^5v^{-3},\, u^3v^2\right)$
6	1	31	$H_{6,1}$	F_{31}	$\left(u^6v^{-1},\, uv^5\right)$
6	5	31	$H_{6,5}$	F_{31}	$\left(u^6v^{-5},\, u^5v\right)$

2.3. **Birational maps.**

Suppose that $1 \leq l < n$ and $\gcd(n,l) = 1$. Then there exist integers δ and σ verifying

$$1 \leq \delta \leq l, \quad 1 \leq \sigma \leq n-1 \quad \text{and} \quad n\delta - \sigma l = 1.$$

Put $\lambda = \sigma n - \delta(n-l) = \sigma(n-l) + \delta l - 1$. We have $1 \leq \lambda \leq m-2$.

In [2], Bennama and Carbonne show the following proposition :

Proposition 2.1. *The Hurwitz curve* $H_{n,l} : x^n y^l + y^n + x^l = 0$ *is isomorphic to Fermat quotient curve* $C_{\lambda,1}(m) : v^m = u^{\lambda}\,(u-1)$.

Proof. The birational transformation is as follows

$$\begin{array}{rcll} f_{n,l}: & C_{\lambda,1}(m) & \longrightarrow & H_{n,l} \\ & (u,v) & \longmapsto & \left(\dfrac{\left((-1)^{\lambda}v\right)^{n}}{(-u)^{\sigma}}, \dfrac{\left((-1)^{\lambda}v\right)^{l}}{(-u)^{\delta}} \right) \end{array}$$

and

$$\begin{array}{rcll} g_{n,l}: & H_{n,l} & \longrightarrow & C_{\lambda,1}(m) \\ & (x,y) & \longmapsto & \left(\dfrac{-x^{l}}{y^{n}}, \dfrac{(-1)^{\lambda}x^{\delta}}{y^{\sigma}} \right). \end{array}$$

The composition of applications gives $(g_{n,l} \circ f_{n,l})(u,v) = (u,v)$ and $(f_{n,l} \circ g_{n,l})(x,y) = (x,y)$.

□

The following Table 2.2 shows the correspondence between Hurwitz curve $H_{n,l}$ and Fermat quotient curve $C_{\lambda,1}(m)$ where $m = n^2 - nl + l^2$ and $\lambda = \sigma n - \delta(n-l)$.

TABLE 2.2. Birational map $f_{n,l} : \mathcal{C}_{\lambda,1}(m) \longrightarrow H_{n,l}$

n	l	m	$H_{n,l}$	σ	δ	λ	$C_{\lambda,1}(m)$	$f_{n,l}(u,v)$
3	1	7	$H_{3,1}$	2	1	4	$C_{4,1}(7)$	$\left(\frac{v^3}{u^2}, -\frac{v}{u}\right)$
3	2	7	$H_{3,2}$	1	1	2	$C_{2,1}(7)$	$\left(-\frac{v^3}{u}, -\frac{v^2}{u}\right)$
4	1	13	$H_{4,1}$	3	1	9	$C_{9,1}(13)$	$\left(-\frac{v^4}{u^3}, \frac{v}{u}\right)$
4	3	13	$H_{4,3}$	1	1	3	$C_{3,1}(13)$	$\left(-\frac{v^4}{u}, \frac{v^3}{u}\right)$
5	2	19	$H_{5,2}$	2	1	7	$C_{7,1}(19)$	$\left(-\frac{v^5}{u^2}, -\frac{v^2}{u}\right)$
5	3	19	$H_{5,3}$	2	3	9	$C_{4,1}(19)$	$\left(-\frac{v^5}{u^2}, \frac{v^3}{u^3}\right)$

Remark 2.1. *By combining the Lemma 2.1 and Proposition 2.1, we have*

$$C_{2,1}(7) \cong C_{4,1}(7) \Longrightarrow H_{3,2} \cong H_{3,1}.$$

3. ALGEBRAIC POINTS ON THE CURVES $C_{1,2}(7)$

3.1. **Auxiliary results.**

For a divisor D on $\mathcal{C}_{1,2}(7)$, let $\mathcal{L}(D)$ denote the $\overline{\mathbb{Q}}-$vector space of all rational functions f on $\mathcal{C}_{1,2}(7)$ such that $f = 0$ or $\mathrm{div}(f) \geq -D$. Let $l(D)$ be the $\overline{\mathbb{Q}}-$dimension of $\mathcal{L}(D)$, u and v denote the rational functions on $\mathcal{C}_{1,2}(7)$ given by

$$u(U,V,W) = \frac{U}{W} \quad \text{and} \quad v(U,V,W) = \frac{V}{W}.$$

The projective equation of the curve $\mathcal{C}_{1,2}(7)$ is

$$\mathcal{C}_{1,2}(7) : V^7 = W^4U(U-W)^2.$$

Let $Q_0 = (0, 0, 1)$, $Q_1 = (1, 0, 1)$, $Q_\eta = (\eta, \overline{\eta}, 1)$, $\overline{Q_\eta} = (\overline{\eta}, \eta, 1)$, $Q_\infty = (1, 0, 0)$ and $R_0 = -Q_\eta - \overline{Q_\eta} + 2Q_\infty$ where η is a primitive 6−th root of unity in $\overline{\mathbb{Q}}$ and $\overline{\eta}$ is the complex conjugate of η. The Abel-Jacobi map associated to Q_∞ is the embedding

$$\begin{array}{rccc} j: & \mathcal{C}_{1,2}(7) & \longrightarrow & J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q}) \\ & P & \longmapsto & [P - Q_\infty] \end{array}$$

where $[P - Q_\infty]$ denotes the class of the divisor $P - Q_\infty$. The map j extends by linearity to the divisors of degree 0 : $Div^0(\mathcal{C}_{1,2}(7))$ to $J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q})$ where

$$Div^0(\mathcal{C}_{1,2}(7)) = \left\{ \sum_{i=1}^{n} n_i P_i \;\middle|\; \sum_{i=1}^{n} n_i = 0,\ n \in \mathbb{N}^*,\ n_i \in \mathbb{Z},\ P_i \in \mathcal{C}_{1,2}(7) \right\}.$$

The Abel Jacobi theorem is an important result. A simple version is the following.

Theorem 3.1. *(Abel-Jacobi) The application j is surjective and its kernel is formed by the divisors of functions on $\mathcal{C}$. In other words, for a divisor $D \in Div^0(\mathcal{C})$, there exists $f \in K^*(\mathcal{C})$ such that $div(f) = D$.*

Proof. See Griffiths [6] □

Lemma 3.1. *Let $\mathcal{C}_{1,2}(7)$ be the curve of affine equation $v^7 = u(u-1)^2$. We have*

(1) $J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q}) \cong \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$.
(2) $J_{\mathcal{C}_{1,2}(7)}(\mathbb{Q}) = \left\{ mj(Q_0) + pR_0 \;\middle|\; 0 \leq m \leq 6 \text{ and } 0 \leq p \leq 1 \right\}$.
(3) $div(u) = 7Q_0 - 7Q_\infty$, $div(u-1) = 7Q_1 - 7Q_\infty$ *and* $div(v) = Q_0 + 2Q_1 - 3Q_\infty$.
(4) $7j(Q_0) = 7j(Q_1) = 0, \quad j(Q_0) + 2j(Q_1) = 0, \quad 2j(R_0) = 0.$

Proof. See Sall [7] □

Lemma 3.2. *The $\overline{\mathbb{Q}}$−basis of the $\mathcal{L}(mQ_\infty)$ on the curve $\mathcal{C}_{1,2}(7)$ for $1 \leq m \leq 11$ are*

- $\mathcal{L}(Q_\infty) = \mathcal{L}(2Q_\infty) =< 1 >$,
- $\mathcal{L}(3Q_\infty) = \mathcal{L}(4Q_\infty) =< 1, v >$,
- $\mathcal{L}(5Q_\infty) =< 1, v, \frac{v^4}{u-1} >$,
- $\mathcal{L}(6Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2 >$,
- $\mathcal{L}(7Q_\infty) =< 1, v, \frac{v^4}{u-1} v^2, u >$,
- $\mathcal{L}(8Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1} >$,
- $\mathcal{L}(9Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1}, v^3 >$,
- $\mathcal{L}(10Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1}, v^3, uv >$,
- $\mathcal{L}(11Q_\infty) =< 1, v, \frac{v^4}{u-1}, v^2, u, \frac{v^5}{u-1}, v^3, uv, \frac{v^6}{u-1} >$.

Proof. See Sall [7] □

3.2. **The Main result on $\mathcal{C}_{1,2}(7)$.**

The main result in the curve $\mathcal{C}_{1,2}(7)$ is the following theorem.

Theorem 3.2. *The set of algebraic points of degree at most* 3 *on the quotient of Fermat curve* $\mathcal{C}_{1,2}(7)$ *over* $\mathbb{Q}$ *is* $\mathcal{C}_{1,2}(7)^{(3)}(\mathbb{Q}) = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ *where*

(1) *The set of rational points is* $\mathcal{M}_1 = \{Q_0,\ Q_1,\ Q_\infty\}$.

(2) *The set of quadratic points is* $\mathcal{M}_2 = \{Q_\eta,\ \overline{Q_\eta}\}$.

(3) *The set of cubic points is* $\mathcal{M}_3 = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3 \cup \mathcal{P}_4$ *with*

$$\mathcal{P}_1 = \{(u,\theta) \mid u^3 - 2u^2 + u - \theta^7 = 0,\ \theta \in \mathbb{Q}^*\},$$
$$\mathcal{P}_2 = \{(1+\theta v^{2+\alpha}, v) \mid v^{3-2\alpha} - \theta^3 v^{2+\alpha} - \theta^2 = 0,\ \theta \in \mathbb{Q}^*,\ \alpha \in \{0,1\}\},$$
$$\mathcal{P}_3 = \{(1+\alpha v^4 + (\alpha-1)v, v) \mid v^3 + v^2 - 1 = 0,\ \alpha \in \{0,1\}\},$$
$$\mathcal{P}_4 = \{(\alpha v^3 - v^2 + \alpha, v) \mid v^3 + (-1)^\alpha(2v^2+v) + 1 = 0,\ \alpha \in \{0,1\}\}.$$

Proof. Let P be an algebraic point on $\mathcal{C}_{1,2}(7)$ of degree $d \leq 3$ over $\mathbb{Q}$; if $d \leq 2$ these points are described by Faddeev ([5]) and Sall ([7]), so we can assume that $d = 3$. Let P_1, P_2, P_3 be the Galois conjugates of P. Then none of the points P_i is equal to the algebraic points on $\mathcal{C}_{1,2}(7)$ of degree ≤ 2 over $\mathbb{Q}$. We have

$$[P_1 + P_2 + P_3 - 3Q_\infty] \in J(\mathcal{C}_{1,2}(7))(\mathbb{Q})$$

and Lemma 3.1 gives

$$[P_1 + P_2 + P_3 - 3Q_\infty] = mj(Q_0) + pj(R_0) \ \text{ with } \ 0 \leq m \leq 6 \text{ and } 0 \leq p \leq 1. \tag{3.1}$$

The possible combinations for m and p are given in the Table 3.3

TABLE 3.3. combinations for m and p

m	0	1	2	3	4	5	6	0	1	2	3	4	5	6
p	0	0	0	0	0	0	0	1	1	1	1	1	1	1

We distinguish 14 cases to study.

Case 1 : $m = 0$ and $p = 0$.

The formula (3.1) becomes $[P_1 + P_2 + P_3 - 3Q_\infty] = 0$. The Abel-Jacobi Theorem 3.1 implies the existence of a rational function f defined over $\mathbb{Q}$ such that

$$\operatorname{div}(f) = P_1 + P_2 + P_3 - 3Q_\infty.$$

So $f \in \mathcal{L}(3P_\infty)$, hence $f = a_0 + a_1 v$ with $a_i \neq 0$. At points P_i, we have $a_0 + a_1 v = 0$ so $v = -\frac{a_0}{a_1} \in \mathbb{Q}^*$. By putting $v = \theta$ in the equation $v^7 = u(u-1)^2$ we have

$$u^3 - 2u^2 + u - \theta^7 = 0.$$

So we have the set of family cubic points

$$\mathcal{P}_1 = \left\{(u, \theta) \mid u^3 - 2u^2 + u - \theta^7 = 0,\, \theta \in \mathbb{Q}^*\right\}.$$

Case 2 : $m = 1$ and $p = 0$:

The relation (3.1) becomes $[P_1 + P_2 + P_3 - 3Q_\infty] = [Q_0 - Q_\infty] = -6[Q_0 - Q_\infty]$. This means

$$[P_1 + P_2 + P_3 + 6Q_0 - 9Q_\infty] = 0.$$

There exists a function f such that

$$\operatorname{div}(f) = P_1 + P_2 + P_3 + 6Q_0 - 9Q_\infty.$$

Therefore $f \in \mathcal{L}(9Q_\infty)$, hence

$$f = a_0 + a_1 v + a_2 \frac{v^4}{u-1} + a_3 v^2 + a_4 u + a_5 \frac{v^5}{u-1} + a_6 v^3.$$

The function f is of order 6 at the point Q_0, so $a_0 = a_1 = a_2 = a_3 = a_5 = a_6 = 0$, thus $f = a_4 u$. At points P_i, $a_4 u = 0$, hence $a_4 = 0$ or $u = 0$ which is absurd.

Cases 1 to 14 : By similar reasoning to the two previous cases, the results obtained can be summarized in the Table 3.4.

TABLE 3.4. Summary of solutions for all cases

m	p	Set of cubic points	m	p	Set of cubic points
0	0	$\mathcal{P}_1$	0	1	$\mathcal{P}_3$ with $\alpha = 0$
1	0	Absurd	1	1	$\mathcal{P}_3$ with $\alpha = 1$
2	0	$\mathcal{P}_2$ with $\alpha = 0$	2	1	$\mathcal{P}_4$ with $\alpha = 0$
3	0	$\mathcal{P}_2$ with $\alpha = 1$	3	1	Absurd
4	0	Absurd	4	1	Absurd
5	0	absurd	5	1	$\mathcal{P}_4$ with $\alpha = 1$
6	0	Absurd	6	1	Absurd

$\square$

4. Algebraic points of low degree on Hurwitz curve

In this section we use birational maps to give algebraic points of low degree on $H_{3,2}$.

4.1. Preliminary results.

Lemma 4.1. *If two curves $\mathcal{X}$ and $\mathcal{Y}$ defined over a number field K are birationally equivalent then $\mathcal{X}$ is isomorphic to $\mathcal{Y}$ and $\mathcal{X}(K) \cong \mathcal{Y}(K)$.*

Proof. See Perrin [8]. □

4.2. Main result on the Hurwitz curve $H_{3,2}$.

Let $P_0 = (0,0)$, $\infty_- = (1,0)$ and $\infty_+ = (0,1)$. The main result is the following theorem :

Theorem 4.1. *Let $H^{(3)}_{3,2}(\mathbb{Q})$ be the set of algebraic points of degree at most 3 on the Hurwitz curves $H_{3,2}$ over $\mathbb{Q}$, then $H^{(3)}_{3,2}(\mathbb{Q}) = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \mathcal{L}_3$ where*

(1) *The set of rational points is $\mathcal{L}_1 = \{P_0, \infty_-, \infty_+\}$*
(2) *The set of quadratic points is $\mathcal{L}_2 = \{(-\eta, -\overline{\eta}), (-\overline{\eta}, -\eta)\}$*
(3) *The set of cubic points is $\mathcal{L}_3 = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{G}_4$ with*

$$\mathcal{G}_1 = \left\{\left(\tfrac{\theta^3}{1-u}, \tfrac{\theta^2}{u-1}\right) \mid u^3 - 2u^2 + u - \theta^7 = 0,\ \theta \in \mathbb{Q}^*\right\},$$
$$\mathcal{G}_2 = \left\{\left(-\tfrac{v^{1-\alpha}}{\theta}, \tfrac{v^{-\alpha}}{\theta}\right) \mid v^{3-2\alpha} - \theta^3 v^{2+\alpha} - \theta^2 = 0,\ \theta \in \mathbb{Q}^*,\ \alpha \in \{0,1\}\right\},$$
$$\mathcal{G}_3 = \left\{\left(\tfrac{v^2}{1-\alpha-\alpha v^3}, -\tfrac{v}{1-\alpha-\alpha v^3}\right) \mid v^3 + v^2 - 1 = 0,\ \alpha \in \{0,1\}\right\},$$
$$\mathcal{G}_4 = \left\{\left(\tfrac{v^3}{1-\alpha+v^2-\alpha v^3}, -\tfrac{v^2}{1-\alpha+v^2-\alpha v^3}\right) \mid v^3 + (-1)^\alpha(2v^2 + v) + 1 = 0,\ \alpha \in \{0,1\}\right\}.$$

Proof.

- The Remark 2.1 gives $H_{3,2} \cong C_{2,1}(7)$ and by using theorem 3.2, we have $\#H_{3,2}(\mathbb{Q}) = 3$. An elementary search give us the set

$$\mathcal{L}_1 = \{P_0, \infty_-, \infty_+\}.$$

- We use birational maps to determine the quadratic and cubic points on the curve $H_{3,2}$. Let

$$\begin{array}{rccc} \varphi: & C_{1,2}(7) & \longrightarrow & C_{2,1}(7) \\ & (u,v) & \longmapsto & (1-u, -v) \end{array}$$

and

$$\begin{array}{rccc} \psi: & C_{2,1}(7) & \longrightarrow & H_{3,2} \\ & (u,v) & \longmapsto & (-\frac{v^3}{u}, -\frac{v^2}{u}). \end{array}$$

Thus, we have

$$\begin{array}{rccc} \psi\circ\varphi : & C_{1,2}(7) & \longrightarrow & H_{3,2} \\ & (u,v) & \longmapsto & \left(\frac{v^3}{1-u}, -\frac{v^2}{1-u}\right). \end{array}$$

(a) The set of quadratic points on $H_{3,2}$ are given by

$$\mathcal{L}_2 = (\psi\circ\varphi)(\mathcal{M}_2).$$

We obtain

$$\mathcal{L}_2 = \Big\{(\psi\circ\varphi)(\eta,\overline{\eta}),\ (\psi\circ\varphi)(\overline{\eta},\eta)\Big\} = \Big\{(-\eta,-\overline{\eta}),\ (-\overline{\eta},-\eta)\Big\}.$$

(b) The set of cubic points on $H_{3,2}$ are given by $\mathcal{G}_1\cup\mathcal{G}_2\cup\mathcal{G}_3\cup\mathcal{G}_4$ with

$$\mathcal{G}_i = (\psi\circ\varphi)(\mathcal{P}_i), \qquad \text{for } i\in\{1,\,2,\,3,\,4\}.$$

We obtain

$$\mathcal{G}_1 = \left\{\left(\frac{\theta^3}{1-u}, \frac{\theta^2}{u-1}\right) \mid u^3-2u^2+u-\theta^7=0,\ \theta\in\mathbb{Q}^*\right\};$$

$$\mathcal{G}_2 = \left\{\left(-\frac{v^{1-\alpha}}{\theta}, \frac{v^{-\alpha}}{\theta}\right) \mid v^{3-2\alpha}-\theta^3v^{2+\alpha}-\theta^2=0,\ \theta\in\mathbb{Q}^*,\ \alpha\in\{0,1\}\right\};$$

$$\mathcal{G}_3 = \left\{\left(\frac{v^2}{1-\alpha-\alpha v^3}, -\frac{v}{1-\alpha-\alpha v^3}\right) \mid v^3+v^2-1=0,\ \alpha\in\{0,1\}\right\};$$

$$\mathcal{G}_4 = \left\{\left(\frac{v^3}{1-\alpha+v^2-\alpha v^3}, -\frac{v^2}{1-\alpha+v^2-\alpha v^3}\right) \mid v^3+(-1)^\alpha(2v^2+v)+1=0,\ \alpha\in\{0,1\}\right\}.$$

□

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RIEMANNIAN CR MANIFOLDS AND ρ-EINSTEIN SOLITONS: A GEOMETRIC ANALYSIS AND APPLICATIONS

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ABSTRACT. In this article, we investigate ρ-Einstein solitons on Riemannian CR manifolds. Specifically, we explore the properties of ρ-Einstein solitons in the presence of cyclic η-recurrent Ricci tensors on Riemannian CR manifolds. We also examine these solitons with respect to Torse-forming vector fields. Additionally, we study ρ-Einstein solitons satisfying Ricci semi-symmetric condition on Riemannian CR manifolds. Furthermore, we examine the properties of conharmonic and conformal curvature tensors on Riemannian CR manifolds admitting ρ-Einstein solitons. Finally, we discuss the applications of ρ-Einstein solitons and their potential uses in various fields.

Keywords: ρ-Einstein soliton, Einstein manifolds, Riemannian CR manifolds.

2010 Mathematics Subject Classification: 53C25, 32V20, 53C21, 53D10, 58D17.

1. Introduction

Over the last twenty years, the study of geometric flow has become the main focus of many mathematicians as it helps in understanding the geometric structures of manifolds in Riemannian geometry. In order to better comprehend these structures on Riemannian manifolds (M, g), Hamilton [15] developed the 'Ricci flow' in 1982, which is described by

$$g_t = -2S,$$

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where S denotes the Ricci tensor and g is the Riemannian metric on M that satisfies the Ricci soliton equation

$$\pounds_V g + 2S + 2\lambda g = 0,$$

where $\pounds_V$ denotes the Lie derivative along the vector field V on M, and λ is a constant. The manifolds admitting such type of structures are called Ricci solitons. The nature of these solitons depends on λ, i.e. $\lambda = 0$ is steady, $\lambda > 0$ is shrinking and $\lambda < 0$ is expanding. Bourguignon [3] gave the generalization of the Ricci flow in 1980s by introducing the notion of Ricci-Bourguignon flow, which is described by

$$g_t = -2S + 2\rho r g, \;\; g(0) = g_0, \tag{1.1}$$

where r represents the scalar curvature and $\rho(\neq 0)$ is a real constant. For specific values of ρ, we get certain tensors associated to equation(1.1):

1. $\rho = \frac{1}{2}$, Einstein tensor $S - \frac{r}{2}g$.
2. $\rho = \frac{1}{n}$, traceless Ricci tensor $S - \frac{r}{n}g$.
3. $\rho = \frac{1}{2(n-1)}$, Schouten tensor $S - \frac{r}{2(n-1)}g$.
4. $\rho = 0$, Ricci tensor S.

The self-similar solution to the Ricci-Bourguignon flow is the Ricci-Bourguignon soliton (also called as 'ρ-Einstein soliton') whose equation is

$$\pounds_V g + 2S + 2(\lambda - \rho r)g = 0. \tag{1.2}$$

The ρ-Einstein soliton is steady if $\lambda = 0$, shrinking if $\lambda < 0$ and expanding if $\lambda > 0$.

Recent mathematical research focuses on classifying Ricci solitons in Riemannian manifolds under particular geometric conditions. Chen and Deshmukh [5] investigated potential fields as concurrent fields and provided a classification of Ricci solitons. Sharma [29] studied gradient Ricci solitons with scalar curvature which is constant and non-homothetic conformal vector fields on Riemannian manifolds, yielding significant findings. Naik [22] characterized Ricci solitons and gradient Ricci almost solitons on Riemannian manifolds, specifically those admitting concurrent recurrent vector fields, known as Riemannian CR manifolds. For further studies on Ricci solitons across various classes of Riemannian manifolds, we suggest [7, 24, 36].

Now, let's recall some concepts on vector fields in Riemannian manifolds. A smooth vector field ξ, on M is said to be conformal if $\exists$ a smooth function ψ (referred to as 'conformal coefficient') on M such as

$$\pounds_{\xi} g = 2\psi g. \tag{1.3}$$

In specific, ξ is said to be homothetic (Killing) vector field if ψ is constant, i.e. $d\psi = 0$. If the dual 1-form η is closed, i.e. ξ is closed, then (1.3) becomes

$$\nabla_{Z_2}\xi = \psi Z_2 \tag{1.4}$$

for all Z_2, a vector field and ∇, the Levi-Civita connection on M. From (1.4), we observe that ξ is a closed homothetic vector field (parallel) if ψ is constant, i.e. $d\psi = 0$ satisfies. In particular, ξ is said to be concurrent when $\psi = 1$ in (1.4). For additional information, we suggest [4, 8, 35]. In contrast, ξ is called a recurrent vector field when

$$\nabla\xi = \eta \otimes \xi.$$

We recommend [6, 13] for results on Riemannian manifolds admitting recurrent vector field. A vector field Z_2 that satisfies the expression

$$\nabla_{Z_2}\xi = \alpha Z_2 + \beta\eta(Z_2)\xi, \quad \alpha,\ \beta\ \in \mathbb{R},$$

generalizes both closed homothetic (concurrent), as well as recurrent vector fields. Thus, we observe that $\pounds_{\xi} g = 2\alpha id + 2\beta\eta \otimes \eta$. If ξ is a unit vector field, then $(\pounds_{\xi} g)(\xi, \xi) = 0 = 2(\alpha + \beta)$, which implies

$$\nabla_{Z_2}\xi = \alpha[Z_2 - \eta(Z_2)\xi] \tag{1.5}$$

for any $Z_2 \in \chi(M)$ and $\alpha \in \mathbb{R}$ is a constant. Here, we regard a unit vector field ξ, which is non-parallel and satisfies (1.5), as a concurrent-recurrent vector field.

The existence of certain vector fields within a Riemannian manifold is a key element of differential geometry. These vector fields are classified into two types: Killing and conformal vector fields [9, 10]. This article highlights Killing vector fields because of their significant prospects for future applications. Drawing on the works of Ahmad et al [1] on ρ-Einstein solitons in Lorentzian para-Kenmotsu manifolds, and inspired by the works of [8, 12, 17, 20, 22, 27, 37], it is essential to explore the geometry of ρ-Einstein solitons in Riemannian CR manifolds.

The current paper investigates ρ-Einstein solitons on Riemannian CR manifolds. The paper is structured as follows: Section 2 describes the preliminary concepts. Section 3 centers on the analysis of Riemannian CR manifolds admitting ρ-Einstein solitons. Section 4 examines ρ-Einstein solitons on Riemannian CR manifolds with cyclic η-recurrent Ricci tensor.

Section 5 delves into ρ-Einstein solitons on Riemannian CR manifolds with a torse-forming vector field. Section 6 addresses Riemannian CR manifolds admitting ρ-Einstein solitons that satisfy $R(\xi, Z_1) \cdot S = 0$. Section 7 highlights Riemannian CR manifolds admitting ρ-Einstein solitons concerning the conharmonic curvature tensor. Section 8 focuses on Riemannian CR manifolds admitting ρ-Einstein solitons with respect to the conformal curvature tensor. Section 9 gives some applications of ρ-Einstein solitons on various fields. The final section, section 10, draws the concluding remarks based on the obtained results.

2. Preliminaries

Let M be a Riemannian manifold of dimension n, admitting a concurrent recurrent vector field ξ, a 1-form η and a Riemannian metric g that satisfies the following relations:

$$\eta(\xi) = 1, \tag{2.6}$$

$$g(Z_1, \xi) = \eta(Z_1), \; g(Z_2, \xi) = \eta(Z_2), \tag{2.7}$$

$$\nabla_{Z_1}\xi = \alpha[Z_1 - \eta(Z_1)\xi], \tag{2.8}$$

$$(\nabla_{Z_1}\eta)(Z_2) = \alpha[g(Z_2, Z_1) - \eta(Z_1)\eta(Z_2)] \tag{2.9}$$

for all $Z_1, Z_2 \in \chi(M)$, ∇ denotes the Levi-Civita connection on M, and $\alpha \in \mathbb{R}$ is a constant.

A Riemannian CR manifold satisfies the following relations [22]:

$$g(R(Z_1, Z_2)Z_3, \xi) = \eta(R(Z_1, Z_2)Z_3) = -\alpha^2[g(Z_2, Z_3)\eta(Z_1) - g(Z_1, Z_3)\eta(Z_2)], \tag{2.10}$$

$$R(\xi, Z_1)Z_2 = -\alpha^2[g(Z_1, Z_2)\xi - \eta(Z_2)Z_1], \tag{2.11}$$

$$R(Z_1, Z_2)\xi = -\alpha^2[\eta(Z_2)Z_1 - \eta(Z_1)Z_2], \tag{2.12}$$

$$R(\xi, Z_1)\xi = -\alpha^2[\eta(Z_1)\xi - Z_1], \tag{2.13}$$

$$S(Z_1, \xi) = -(n-1)\alpha^2\eta(Z_1), \; S(\xi, \xi) = -(n-1)\alpha^2, \tag{2.14}$$

$$Q\xi = -(n-1)\alpha^2\xi, \tag{2.15}$$

where R denotes the Riemannian curvature, S represents the Ricci tensor and Q indicates the Ricci operator which is given as $S(Z_1, Z_2) = g(QZ_1, Z_2), \;\; \forall \; Z_1, Z_2 \in \chi(M)$.

Definition 2.1. [33] *A Riemannian CR manifold is said to be an η-Einstein manifold when its Ricci tensor S satisfies the following relation*

$$S(Z_1, Z_2) = ag(Z_1, Z_2) + b\eta(Z_1)\eta(Z_2),$$

for smooth functions a and b. If $b = 0$, then the Riemannian CR manifold reduces to an Einstein manifold.

Remark 2.1. [23] *In a Riemannian CR manifold, we have*

$$\xi(r) = -2\alpha(r + n(n-1)\alpha^2). \tag{2.16}$$

Remark 2.2. *From (2.16), if an n-dimensional Riemannian CR manifold is of constant curvature, then*

$$r = -\alpha^2 n(n-1). \tag{2.17}$$

3. Riemannian CR manifolds admitting ρ-Einstein solitons

In this section, we examine ρ-Einstein solitons on a Riemannian CR manifold and we explore the nature of solitons for various values of ρ.

Let an n-dimensional Riemannian CR manifold M admit ρ-Einstein soliton. Then (1.2) holds. So we have

$$(\pounds_\xi g)(Z_1, Z_2) + 2S(Z_1, Z_2) + 2(\lambda - \rho r)g(Z_1, Z_2) = 0 \tag{3.18}$$

for all $Z_1, Z_2 \in \chi(M)$.
We know that

$$(\pounds_\xi g)(Z_1, Z_2) = g(\nabla_{Z_1}\xi, Z_2) + g(Z_1, \nabla_{Z_2}\xi) = 2\alpha[g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2)]. \tag{3.19}$$

Hence, (3.18) leads to

$$S(Z_1, Z_2) = -(\lambda - \rho r + \alpha)g(Z_1, Z_2) + \alpha\eta(Z_1)\eta(Z_2). \tag{3.20}$$

Substituting $Z_2 = \xi$ in (3.20) and making use of (2.6) and (2.7), we get

$$S(Z_1, \xi) = -(\lambda - \rho r)\eta(Z_1), \tag{3.21}$$

which implies

$$Q\xi = -(\lambda - \rho r)\xi. \tag{3.22}$$

From (2.14) and (3.21), we obtain

$$\lambda = \alpha^2(n-1) + \rho r. \tag{3.23}$$

Now, if r is constant, then by Remark (2.2), (3.23) becomes

$$\lambda = \alpha^2(n-1)(1 - \rho n). \tag{3.24}$$

Therefore, we state:

Theorem 3.1. *If a Riemannian CR manifold of dimension* n *admits* ρ*-Einstein soliton then it is an* η*-Einstein manifold with the soliton constant* $\lambda = \alpha^2(n-1)(1-\rho n)$.

Now from the above theorem we obtain the following corollary:

Corollary 3.1. *Let an* n*-dimensional Riemannian CR manifold admit a* ρ*-Einstein soliton, then we have:*

Values of ρ	Soliton type	Soliton constant	Nature of soliton
$\rho = \frac{1}{2}$	Einstein soliton	$\lambda = -\frac{\alpha^2(n-1)(n-2)}{2}$	Shrinking
$\rho = \frac{1}{n}$	Traceless Ricci soliton	$\lambda = 0$	Steady
$\rho = \frac{1}{2(n-1)}$	Schouten soliton	$\lambda = \frac{\alpha^2(n-2)}{2}$	Expanding
$\rho = 0$	Ricci soliton	$\lambda = \alpha^2(n-1)$	Expanding

Lemma 3.1. *Let an* n*-dimensional Riemannian CR manifold admit a* ρ*-Einstein soliton* $\ni V = b\xi$*, where* b *is a function. Then,* V *is a constant multiple of* ξ *and an* n*-dimensional Riemannian CR manifold is an* η*-Einstein manifold of the type*

$$S(Z_1, Z_2) = -(b\alpha + (\lambda - \rho r))g(Z_1, Z_2) + b\alpha\eta(Z_1)\eta(Z_2).$$

Proof. Let (g, V, λ, ρ) be a ρ-Einstein soliton on an n-dimensional Riemannian CR manifold, $\ni V$ is pointwise collinear with ξ i.e. $V = b\xi$. Then (1.2) holds. Hence, we have

$$bg(\nabla_{Z_1}\xi, Z_2) + Z_1(b)\eta(Z_2) + bg(Z_1, \nabla_{Z_2}\xi)$$
$$+Z_2(b)\eta(Z_1) + 2S(Z_1, Z_2) + 2(\lambda - \rho r)g(Z_1, Z_2) = 0,$$

which from (2.8) implies

$$2b\alpha g(Z_1, Z_2) - 2b\alpha\eta(Z_1)\eta(Z_2) + Z_1(b)\eta(Z_2)$$
$$+\eta(Z_1)Z_2(b) + 2S(Z_1, Z_2) + 2(\lambda - \rho r)g(Z_1, Z_2) = 0. \tag{3.25}$$

Substituting $Z_2 = \xi$ in (3.25) and making use of (2.6), (2.7) and (2.14), we have

$$Z_1(b) + \xi(b)\eta(Z_1) - 2[\alpha^2\eta(Z_1)(n-1) - (\lambda - \rho r)\eta(Z_1)] = 0. \tag{3.26}$$

Now putting $Z_1 = \xi$ in (3.26) and making use of (2.6), we get

$$\xi(b) - \alpha^2(n-1) + (\lambda - \rho r) = 0. \tag{3.27}$$

Using (3.27) in (3.26), we have

$$db = [\alpha^2(n-1) - (\lambda - \rho r)]\eta. \tag{3.28}$$

Applying d on both sides of (3.28), we obtain

$$[\alpha^2(n-1) - (\lambda - \rho r)]d\eta = 0 \implies \lambda = \alpha^2(n-1) + \rho r,\ d\eta \neq 0. \tag{3.29}$$

Therefore, by (3.28) and (3.29) $db = 0$, i.e. b is constant. Thus, (3.25) becomes

$$S(Z_1, Z_2) = -[b\alpha + (\lambda - \rho r)]g(Z_1, Z_2) + b\alpha\eta(Z_1)\eta(Z_2).$$

□

4. ρ-Einstein solitons on n-dimensional Riemannian CR manifolds with respect to cyclic η-recurrent Ricci tensor

In this section, we discuss the characteristic of ρ-Einstein soliton on a Riemannian CR manifold with respect to cyclic η-recurrent Ricci tensor and explore the nature of solitons for various values of ρ.

Definition 4.1. [34] *A Riemannian CR manifold M of dimension n is said to have cyclic η-recurrent Ricci tensor if*

$$\begin{aligned}(\nabla_{Z_1}S)(Z_2, Z_3) + (\nabla_{Z_2}S)(Z_3, Z_1) + (\nabla_{Z_3}S)(Z_1, Z_2)\\ = \eta(Z_1)S(Z_2, Z_3) + \eta(Z_2)S(Z_3, Z_1) + \eta(Z_3)S(Z_1, Z_2),\end{aligned} \tag{4.30}$$

for any $Z_1, Z_2, Z_3 \in \chi(M)$.

Now, considering an n-dimensional Riemannian CR manifold M admitting a ρ-Einstein soliton with cyclic η-recurrent Ricci tensor, then equation (4.30) holds.
Applying covariant derivative on (3.20) with respect to Z_1 yields

$$\begin{aligned}(\nabla_{Z_1}S)(Z_2, Z_3) &= \rho Z_1(r)g(Z_2, Z_3) + \alpha^2 g(Z_1, Z_2)\eta(Z_3)\\ &\quad + \alpha^2 g(Z_1, Z_3)\eta(Z_2) - 2\alpha^2\eta(Z_1)\eta(Z_2)\eta(Z_3).\end{aligned} \tag{4.31}$$

Similarly, we get

$$\begin{aligned}(\nabla_{Z_2}S)(Z_3, Z_1) &= \rho Z_2(r)g(Z_3, Z_1) + \alpha^2 g(Z_2, Z_3)\eta(Z_1)\\ &\quad + \alpha^2 g(Z_1, Z_2)\eta(Z_3) - 2\alpha^2\eta(Z_1)\eta(Z_2)\eta(Z_3),\end{aligned} \tag{4.32}$$

and

$$(\nabla_{Z_3}S)(Z_1,Z_2) = \rho Z_3(r)g(Z_1,Z_2)+\alpha^2 g(Z_3,Z_1)\eta(Z_2) +\alpha^2 g(Z_3,Z_2)\eta(Z_1)-2\alpha^2\eta(Z_1)\eta(Z_2)\eta(Z_3). \tag{4.33}$$

Making use of (4.31)-(4.33) in (4.30) we get

$$\rho[Z_1(r)g(Z_2,Z_3)+Z_2(r)g(Z_3,Z_1)+Z_3(r)g(Z_1,Z_2)] = 3\alpha(1+2\alpha)\eta(Z_1)\eta(Z_2)\eta(Z_3) -(\lambda-\rho r+\alpha(1+2\alpha))[g(Z_2,Z_3)\eta(Z_1)+g(Z_1,Z_3)\eta(Z_2)+g(Z_1,Z_2)\eta(Z_3)],$$

which on substitution of $Z_2 = Z_3 = \xi$ and utilization of (2.6) and (2.7) gives

$$\rho[Z_1(r)+\xi(r)\eta(Z_1)+\xi(r)\eta(Z_1)] = 3\alpha(1+2\alpha)\eta(Z_1)-3(\lambda-\rho r+\alpha(1+2\alpha))\eta(Z_1). \tag{4.34}$$

Now taking $Z_1 = \xi$ and making use of (2.6) gives

$$3\rho\xi(r) = 3\alpha(1+2\alpha)-3(\lambda-\rho r+\alpha(1+2\alpha)). \tag{4.35}$$

Let r be a constant. Then $\xi(r) = 0$. Hence by (2.16) and (2.17), equation (4.35) implies

$$\lambda = -\alpha^2\rho n(n-1).$$

Thus, we state the following result:

Theorem 4.1. *If an n-dimensional Riemannian CR manifold with constant scalar curvature admitting ρ-Einstein solitons has cyclic η-recurrent Ricci tensor, then the soliton constant is given by $\lambda = -\alpha^2\rho n(n-1)$.*

Now we have the following corollary:

Corollary 4.1. *Let the metric of n-dimensional Riemannian CR manifold with constant scalar curvature be a ρ-Einstein soliton. Then we have:*

Values of ρ	Soliton type	Soliton constant	Nature of soliton
$\rho = \frac{1}{2}$	Einstein soliton	$\lambda = -\frac{\alpha^2 n(n-1)}{2}$	Shrinking
$\rho = \frac{1}{n}$	Traceless Ricci soliton	$\lambda = -\alpha^2(n-1)$	Shrinking
$\rho = \frac{1}{2(n-1)}$	Schouten soliton	$\lambda = -\frac{\alpha^2 n}{2}$	Shrinking
$\rho = 0$	Ricci soliton	$\lambda = 0$	Steady

5. ρ-Einstein solitons on n-dimensional Riemannian CR manifolds with respect to Torse-forming vector field

In this section, we examine the nature of ρ-Einstein soliton on a Riemannian CR manifold with a torse-forming vector field and discuss the nature of solitons for various values of ρ.

Definition 5.1. [32] *On a Riemannian manifold (M, g), a vector field V is said to be torse-forming if*

$$\nabla_{Z_1} V = fZ_1 + \omega(Z_1)V, \tag{5.36}$$

where f is a smooth function, ω is a 1-form and ∇ is the Levi-Civita connection of g.

Considering an n-dimensional Riemannian CR manifold admitting a ρ-Einstien soliton and ξ, the Reeb vector field to be a torse-forming vector field, then (5.36) implies

$$\nabla_{Z_1} \xi = fZ_1 + \omega(Z_1)\xi \tag{5.37}$$

for any $Z_1 \in \chi(M)$. Taking the inner product on (5.37) with ξ gives

$$\eta(\nabla_{Z_1} \xi) = f\eta(Z_1) + \omega(Z_1). \tag{5.38}$$

Using (2.8) in (5.38), we get

$$\omega(Z_1) = -f\eta(Z_1). \tag{5.39}$$

It follows from (5.39), equation (5.37) becomes

$$\nabla_{Z_1} \xi = f(Z_1 - \eta(Z_1)\xi). \tag{5.40}$$

By virtue of (5.40), we have

$$(\pounds_\xi g)(Z_1, Z_2) = 2f[g(Z_1, Z_2) - \eta(Z_1)\eta(Z_2)]. \tag{5.41}$$

In view of (5.41), equation (3.18) leads to

$$S(Z_1, Z_2) = -(f + \lambda - \rho r)g(Z_1, Z_2) + f\eta(Z_1)\eta(Z_2). \tag{5.42}$$

Substituting $Z_1 = Z_2 = \xi$ in (5.42) then utilizing (2.6), (2.14) and (2.17) we obtain

$$\lambda = \alpha^2(n-1)(1-\rho n).$$

Thus, we have:

Theorem 5.1. *If an n-dimensional Riemannian CR manifold of constant scalar curvature admit a ρ-Einstien soliton with torse-forming vector field ξ, then it is an η-Einstein manifold with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$. Further, for particular values of ρ, the nature of solitons can be discussed which is same as Corollary 3.1.*

6. ρ-Einstein solitons on n-dimensional Riemannian CR manifolds satisfying $R(\xi, Z_1) \cdot S = 0$

In this section, we investigate the nature of ρ-Einstein solitons on Riemannian CR manifolds that satisfies Ricci semi-symmetric condition.

Let an n-dimensional Riemannian CR manifold admit a ρ-Einstein soliton that satisfy the condition $R(\xi, Z_1) \cdot S = 0$. Then, we have

$$S(R(\xi, Z_1)Z_2, Z_3) + S(Z_2, R(\xi, Z_1)Z_3) = 0,$$

which by using (2.11) yields

$$-\alpha^2 g(Z_2, Z_1)S(\xi, Z_3) + \alpha^2\eta(Z_2)S(Z_1, Z_3) - \alpha^2 g(Z_3, Z_1)S(Z_2, \xi) + \alpha^2\eta(Z_3)S(Z_2, Z_1) = 0.$$

Substituting $Z_3 = \xi$ in the above equation and utilizing (2.6) and (3.21), we obtain

$$S(Z_1, Z_2) = -(\lambda - \rho r)g(Z_1, Z_2). \tag{6.43}$$

Now substituting $Z_2 = \xi$ in (6.43) and utilizing (2.14) and (2.17) we get

$$\lambda = \alpha^2(n-1)(1-\rho n).$$

Now we state:

Theorem 6.1. *If an n-dimensional Riemannian CR manifold of constant scalar curvature tensor admit a ρ-Einstein soliton and satisfies the condition $R(\xi, Z_1) \cdot S = 0$, then it is an Einstein manifold with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$. Further, for particular values of ρ, the nature of solitons can be discussed which is same as Corollary 3.1.*

7. Conharmonic curvature tensor on n-dimensional Riemannian CR manifolds admitting ρ-Einstein solitons

In this section, we inspect the nature of ρ-Einstein soliton on Riemannian CR manifolds with respect to conharmonic curvature tensor.

On a Riemannian manifold M of dimension n, the conharmonic curvature tensor K is defined by [18]

$$\begin{aligned} K(Z_1, Z_2)Z_3 &= R(Z_1, Z_2)Z_3 + \frac{1}{n-2}\Big[S(Z_1, Z_3)Z_2 - S(Z_2, Z_3)Z_1 \\ &\quad + g(Z_1, Z_3)QZ_2 - g(Z_2, Z_3)QZ_1\Big], \end{aligned} \tag{7.44}$$

for all $Z_1, Z_2, Z_3 \in \chi(M)$.

Now, considering a conharmonically flat Riemannian CR manifold of dimension n admitting a ρ-Einstein soliton i.e. $K(Z_1, Z_2)Z_3 = 0$. Then from (7.44), we have

$$R(Z_1, Z_2)Z_3 = -\frac{1}{n-2}\Big[S(Z_1, Z_3)Z_2 - S(Z_2, Z_3)Z_1 + g(Z_1, Z_3)QZ_2 - g(Z_2, Z_3)QZ_1\Big].$$

Putting $Z_3 = \xi$ and making use of (2.12), (3.21) and (3.22), the above equation yields

$$\begin{aligned} -\alpha^2[\eta(Z_2)Z_1 - \eta(Z_1)Z_2] &= -\frac{1}{n-2}\Big[(\lambda - \rho r)(\eta(Z_2)Z_1 - \eta(Z_1)Z_2) \\ &\quad + \eta(Z_1)QZ_2 - \eta(Z_2)QZ_1\Big]. \end{aligned} \tag{7.45}$$

Now substituting $Z_2 = \xi$ in (7.45), equation (7.45) yields

$$QZ_1 = [\lambda - \rho r - \alpha^2(n-2)]Z_1 - [2\lambda - 2\rho r - \alpha^2(n-2)]\eta(Z_1)\xi. \tag{7.46}$$

Applying the inner product on (7.46) with Z_2 leads to

$$S(Z_1, Z_2) = [\lambda - \rho r - \alpha^2(n-2)]g(Z_1, Z_2) - [2\lambda - 2\rho r - \alpha^2(n-2)]\eta(Z_1)\eta(Z_2). \tag{7.47}$$

Now substituting $Z_2 = \xi$ in (7.47) and utilizing (2.6), (2.7) and (2.14), we get

$$\lambda = \alpha^2(n-1) + \rho r. \tag{7.48}$$

Let r be a constant. Then by (2.17), equation (7.48) turns to

$$\lambda = \alpha^2(n-1)(1 - \rho n).$$

Thus, we state:

Theorem 7.1. *If the metric of a conharmonically flat Riemannian CR manifold of dimension n with constant scalar curvature r is a ρ-Einstein soliton, then it is η-Einstein with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$.*

8. Conformal curvature tensor on n-dimensional Riemannian CR manifolds admitting ρ-Einstein solitons

In this section, we examine the nature of ρ-Einstein soliton on Riemannian CR manifolds with respect to conformal curvature tensor.

On a Riemannian manifold M of dimension n, the conformal curvature tensor C is defined by [21]

$$\begin{aligned} C(Z_1,Z_2)Z_3 &= R(Z_1,Z_2)Z_3 + \frac{1}{n-2}\Big[S(Z_1,Z_3)Z_2 - S(Z_2,Z_3)Z_1 + g(Z_1,Z_3)QZ_2 \\ &\quad -g(Z_2,Z_3)QZ_1\Big] + \frac{r}{(n-1)(n-2)}\Big[g(Z_2,Z_3)Z_1 - g(Z_1,Z_3)Z_2\Big] \quad (8.49)\end{aligned}$$

for all $Z_1, Z_2, Z_3 \in \chi(M)$.

Now, considering a conformally flat Riemannian CR manifold of dimension n admitting a ρ-Einstein soliton i.e. $C(Z_1,Z_2)Z_3 = 0$. Then from (8.49), we have

$$\begin{aligned} R(Z_1,Z_2)Z_3 &= -\frac{1}{n-2}\Big[S(Z_1,Z_3)Z_2 - S(Z_2,Z_3)Z_1 + g(Z_1,Z_3)QZ_2 - g(Z_2,Z_3)QZ_1\Big] \\ &\quad -\frac{r}{(n-1)(n-2)}\Big[g(Z_2,Z_3)Z_1 - g(Z_1,Z_3)Z_2\Big].\end{aligned}$$

Substituting $Z_3 = \xi$ in the above equation and making use of (2.12), (3.21) and (3.22), the above equation yields

$$\begin{aligned} -\alpha^2[\eta(Z_2)Z_1 - \eta(Z_1)Z_2] &= -\frac{1}{n-2}\Big[(\lambda-\rho r)(\eta(Z_2)Z_1 - \eta(Z_1)Z_2) \\ &\quad +\eta(Z_1)QZ_2 - \eta(Z_2)QZ_1\Big] \\ &\quad -\frac{r}{(n-1)(n-2)}\Big[\eta(Z_2)Z_1 - \eta(Z_1)Z_2\Big]. \quad (8.50)\end{aligned}$$

Now putting $Z_2 = \xi$ in (8.50), equation (8.50) yields

$$QZ_1 = [\lambda - \rho r - 2\alpha^2(n-1)]Z_1 - [2\lambda - 2\rho r - 2\alpha^2(n-1)]\eta(Z_1)\xi. \quad (8.51)$$

Applying the inner product on (8.51) with Z_2 leads to

$$S(Z_1,Z_2) = [\lambda - \rho r - 2\alpha^2(n-1)]g(Z_1,Z_2) - [2\lambda - 2\rho r - 2\alpha^2(n-1)]\eta(Z_1)\eta(Z_2). \quad (8.52)$$

Now substituting $Z_2 = \xi$ in (8.52) and utilizing (2.6), (2.7) and (2.14), we get

$$\lambda = \alpha^2(n-1) + \rho r. \quad (8.53)$$

Let r be a constant. Then, by (2.17), equation (8.53) turns to

$$\lambda = \alpha^2(n-1)(1-\rho n).$$

Thus, we state:

Theorem 8.1. *If the metric of a conformally flat Riemannian CR manifold of dimension n with constant scalar curvature r is a ρ-Einstein soliton, then it is η-Einstein with the soliton constant $\lambda = \alpha^2(n-1)(1-\rho n)$.*

9. Applications

As generalized fixed points of Hamilton's Ricci flow $g_t = -2S$ [16], Ricci solitons are a natural generalization of Einstein metrics on a Riemannian manifold. This evolution equation permits a metric to smooth out irregularities based on the Ricci curvature of the manifold, i.e. expands for negative Ricci curvature and shrinks in positive case. It is a nonlinear diffusion equation comparable to the heat equation for metrics. For ρ-Einstein solitons, we have obtained the steady, expanding, and shrinking conditions.

Ricci solitons, known as quasi-Einstein metrics in physics literature, are of great interest to physicists as they have wide applications in the fields of physics [14], biology, chemistry, [19] and economics [28]. As Ricci solitons are self-similar solutions to the Ricci flow, they were instrumental in resolving the century-old Poincaré conjecture [25, 26]. Additionally, Ricci flow and Ricci solitons play a major role in medical imaging for brain surfaces [31], illustrating their wide-ranging impact.

The study of ρ-Einstein solitons aids in comprehending the geometry and topology of Riemannian manifolds. In general relativity, ρ-Einstein solitons provide models for space-time metrics with specific properties, helping to understand solutions to the Einstein field equations and contributing to cosmological models and the study of gravitational waves [30]. With respect to Ricci flow, ρ-Einstein solitons model the formation of singularities and provides insights on the nature of singularities that form during the flow, leading to a better understanding of the long-term behaviors in cosmology and the potential fate of the universe.

Further, Dey and Roy [11] discussed some applications of η-Ricci-Bourguignon solitons to general relativity. According to them, symmetry is a fascinating feature of our universe, governed by the laws of nature and applicable to various physical phenomena, including general relativity. Early in the 1800s, Albert Einstein made the discovery of the "Theory of General Relativity." In this theory, the gravitational field is defined by the curvature of space-time, and its source is the energy-momentum tensor. The most effective tools for comprehending general relativity in mathematics are differential geometry and relativistic models. A connected 4-dimensional Lorentzian manifold, a particular subclass of pseudo-Riemannian manifolds with Lorentzian metric g with signature (-,+,+,+), can be used to model the spacetime of general relativity and cosmology. The matter content of spacetime is represented by the energy-momentum tensor and is believed to behave as a fluid with properties such as pressure, density, dynamical and kinematic quantities like acceleration, velocity, vorticity, shear, and expansion [2]. These properties can be better understood by studying the nature of solitons. In this paper, we have examined the soliton constants in various cases and also have discussed on soliton constants for different values of ρ.

Moreover, ρ-Einstein solitons are also useful in classifying compact Riemannian manifolds with prescribed curvature conditions. The conformal curvature tensor (Weyl tensor in four dimensions) quantifies the deviation of a manifold from being conformally flat and remains invariant under conformal transformations of the metric. Studying these solitons in the context of conformal curvature involves examining how they affect the conformal geometry of the manifold. In this article, we have obtained the soliton constants of ρ-Einstein solitons with respect to conformal and conharmonic curvature tensors on Riemannian CR manifold.

In summary, ρ-Einstein solitons are fundamental in understanding manifold geometry, contributing to the study of geometric flows, singularity formation, manifold classification, and various interdisciplinary applications. Their study bridges theoretical mathematics and practical applications, highlighting their significance in both realms.

10. **Conclusion**

In this paper, we have systematically explored the behavior of ρ-Einstein solitons on Riemannian CR manifolds under various geometric conditions. Our analysis demonstrates that such solitons consistently reveal deep connections to η-Einstein and Einstein manifolds structures. Key findings from the Theorems 3.1, 4.1, 5.1, 6.1, 7.1, and 8.1 highlights that, regardless of the specific geometric conditions-whether involving torse-forming vector fields, or conditions like conharmonic or conformal flatness-these solitons share a common characteristic. They are invariably associated with η-Einstein manifolds, and their soliton constant takes the form $\lambda = \alpha^2(n-1)(1-\rho n)$, emphasizing the uniformity of their geometric structure.

Moreover, when additional curvature conditions are imposed, such as the Ricci semi-symmetric condition $R(\xi, Z_1) \cdot S = 0$, the manifold transforms into an Einstein manifold, reinforcing the broader geometric significance of ρ-Einstein solitons.

This study highlights the rich interplay between ρ-Einstein solitons and the curvature properties of Riemannian CR manifolds, contributing to a deeper understanding of their geometry. Through these results, we gain valuable insights into the solitons behavior across various curvature contexts, providing a foundation for future investigations in both theoretical mathematics and practical applications in physics and cosmology.

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UNIT TANGENT SPHERE BUNDLES WITH THE KALUZA-KLEIN METRIC SATISFYING SOME COMMUTATIVE CONDITIONS

MURAT ALTUNBAŞ *

ABSTRACT. Let (M, g) be an $n-$dimensional Riemannian manifold and T_1M its tangent sphere bundle with the contact metric structure $(\tilde{G}, \eta, \phi, \xi)$, where $\tilde{G}$ is the Kaluza-Klein metric. Let $\tilde{S}$ be the Ricci operator and h be the structural operator on T_1M. In this paper, we find some conditions for the relations $\tilde{S}h = h\tilde{S}$ and $\tilde{S}\phi h = \phi h\tilde{S}$ to be satisfied.

Keywords: Tangent sphere bundle, Kaluza-Klein metric, Ricci operator, contact metric structure.

2010 Mathematics Subject Classification: 53C25, 53D10.

1. Introduction

Let (M, g) be a Riemannian manifold and T_1M its tangent sphere bundle. There can be defined a lot of metrics on T_1M such as the Sasaki metric [9], the Kaluza-Klein metric [10], the Kaluza-Klein type metric [1] and a general natural metric ([5],[12]). All of these metrics are sub-classes of the $g-$natural metric introduced in [2]. These metrics are restrictions of the metrics on the tangent bundle TM to T_1M. Remark that the restrictions of the two well-known metrics on TM, say the Cheeger-Gromoll metric [15] and a metric with two parameter ([6], [13]), yield nothing new thank to isometries to the Sasaki metric (see [8] and [14]).

In [11], Cho and Chun studied the commutativity of the Ricci operator with the structural operator in (T_1M, g^s) by considering the base manifold M is conformally flat. Here, g^s denotes the Sasaki metric.

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In this paper, we endow the unit tangent sphere bundle T_1M with the Kaluza-Klein metric $\tilde{G}$ and give a contact metric structure (η, ϕ, ξ) associated to the metric $\tilde{G}$. We investigate the commutative properties of the Ricci operator $\tilde{S}$ with h and ϕh, where h is the structural operator on T_1M.

2. Preliminaries

The aim of this section is to report some fundamental facts about contact metric manifolds. Every manifolds are supposed to be smooth and connected. We can refer to [7] for a survey about contact metric geometry.

Given a $(2n-1)$ dimensional differentiable manifold $\bar{M}$. If $\bar{M}$ acknowledges a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$, then it is called a contact manifold. When η is given, there is a vector field ξ (so called characteristic vector field) such that $\eta(\xi) = 1$ and $d\eta(\xi, \bar{A}) = 0$ for all vector fields $\bar{A}$ on $\bar{M}$. Moreover, a Riemannian metric $\bar{g}$ is called an associated metric if there exists a $(1,1)-$tensor such that

$$\eta(\bar{A}) = g(\bar{A}, \xi),\ d\eta(\bar{A}, \bar{B}) = \bar{g}(\bar{X}, \phi\overline{B}),\ \phi^2\bar{A} = -\bar{A} + \eta(\bar{A})\xi, \tag{2.1}$$

for all vector fields $\bar{A}, \bar{B}$ on $\bar{M}$. It follows that

$$\phi\xi = 0,\ \eta \circ \phi = 0,\ \bar{g}(\phi\bar{A}, \phi\overline{B}) = \bar{g}(\bar{A}, \bar{B}) - \eta(\bar{A})\eta(\bar{B}).$$

The quartet $(\eta, \bar{g}, \phi, \xi)$ satisfying (2.1) is called a contact metric structure and the quintet $(\bar{M}, \eta, \bar{g}, \phi, \xi)$ a contact metric manifold.

Let $\bar{M}$ be a contact metric manifold. The structural operator h is defined by $h = \frac{1}{2}L_\xi\phi$, where L is the Lie derivative operator. The operator h is self-adjoint and it satisfies the following relations:

$$h\xi = 0,\ h\phi = -\phi h,\ \bar{\nabla}_{\bar{A}}\xi = -\phi\overline{A} - \phi h\overline{A},\ (\bar{\nabla}_\xi h)\phi = -\phi(\bar{\nabla}_\xi h),$$

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$.

Let (M, g) be an $n-$Riemannian manifold with the Levi-Civita connection ∇. The tangent bundle TM of M is a $2n-$dimensional manifold with the projection map $\pi : TM \to M,\ \pi(p, u) = u$. The $g-$natural metric G on TM is defined by, [4]

$$\begin{cases} G(A^h, B^h) = (\alpha_1 + \alpha_3)(r^2)g(A, B) + (\beta_1 + \beta_3)(r^2)g(A, u)g(B, u), \\ G(A^h, B^v) = \alpha_2(r^2)g(A, B) + \beta_2(r^2)g(A, u)g(B, u), \\ G(A^v, B^v) = \alpha_1(r^2)g(A, B) + \beta_1(r^2)g(A, u)g(B, u), \end{cases}$$

for all vector fields A, B on M, where $\alpha_i, \beta_i : \mathbb{R}^+ \to \mathbb{R}$, $i = 1, 2, 3$ are smooth functions, $r^2 = g(u, u)$ and A^h, A^v denote the horizontal lift and the vertical lift of A, respectively.

The unit tangent sphere bundle T_1M is a hypersurface of TM defined by $T_1M = \{(x, u) \in TM : g(u, u) = 1\}$. By definition, the $g-$natural metric on T_1M is the restriction of the $g-$natural metric of TM to its hypersurface T_1M. The $g-$natural metric on T_1M is defined by

$$\begin{cases} G(A^h, B^h) = (a + c)g(A, B) + \beta(r^2)g(A, u)g(B, u), \\ G(A^h, B^v) = bg(A, B), \\ G(A^v, B^v) = ag(A, B), \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $\beta : [0, \infty) \to \mathbb{R}$. The vector field $N = \frac{1}{\sqrt{(a+c+d)\varphi}}[-bu^h + (a + c + d)u^v]$ is the unit normal vector field, where $d = \beta(1)$ and $\varphi = a(a + c + d) - b^2$. The tangential lift A^t is given by $A^t = A^v - \sqrt{\frac{\varphi}{a+c+d}} g(A, u)N$. Inasmuch as the tangent space $(T_1M)_{(x,u)}$ of T_1M at (x, u) is produced by vectors of the form A^h and B^t, the Riemannian metric g on T_1M, induced from G, is established by

$$\begin{cases} G(A^h, B^h) = (a + c)g(A, B) + dg(A, u)g(B, u), \\ G(A^h, B^t) = bg(A, B), \\ G(A^t, B^t) = ag(A, B) - \frac{\varphi}{a+c+d} g(A, u)g(B, u), \end{cases} \tag{2.2}$$

for all vector fields A, B on M. The particular cases of the metric G are listed below:

i) The Sasaki metric if $a = 1,\ b = c = d = 0$,

ii) The Cheeger-Gromoll type metric if $b = d = 0,\ a = 1/2^m,\ c = 1 - a$,

iii) The Kaluza-Klein type metric if $b = 0$ (and $a,\ a + c > 0,\ a + c + d > 0$),

iv) The Kaluza-Klein metric if $b = d = 0$ (and $a,\ a + c > 0$).

In this paper, we deal with the Kaluza-Klein metric. From (2.2), the Kaluza-Klein metric $\tilde{G}$ is defined by

$$\begin{cases} \tilde{G}(A^h, B^h) = (a + c)g(A, B), \\ \tilde{G}(A^h, B^t) = 0, \\ \tilde{G}(A^t, B^t) = a(g(A, B) - g(A, u)g(B, u)). \end{cases} \tag{2.3}$$

The Levi-Civita connection $\tilde{\nabla}$ of $\tilde{G}$ is given by, [10]

$$\begin{aligned} \tilde{\nabla}_{A^h} B^h &= (\nabla_A B)^h - \frac{1}{2}(R(A, B)u)^t, \\ \tilde{\nabla}_{A^h} B^t &= (\nabla_A B)^t + \frac{a}{2(a + c)}(R(u, A)B)^h, \\ \tilde{\nabla}_{A^t} B^h &= \frac{a}{2(a + c)}(R(u, A)B)^h, \\ \tilde{\nabla}_{A^t} B^t &= -g(B, u)A^t. \end{aligned} \tag{2.4}$$

For an orthonormal basis $e_1, e_2, ..., e_n = u$, the Ricci tensor $\widetilde{Ric}$ of $\tilde{G}$ is computed as, [3]

$$\begin{aligned}
\widetilde{Ric}(A^h, B^h) &= Ric(A,B) - \frac{a}{2(a+c)}\sum_{i=1}^{n} g(R(u,e_i)A, R(u,e_i)B), \qquad (2.5)\\
\widetilde{Ric}(A^t, B^h) &= \frac{a}{2(a+c)}((\nabla_u Ric)(A,B) - (\nabla_A Ric)(u,B)),\\
\widetilde{Ric}(A^t, B^t) &= (n-2)(g(A,B) - g(A,u)g(B,u))\\
&\quad + \frac{a^2}{4(a+c)^2}\sum_{i=1}^{n} g(R(u,A)e_i, R(u,B)e_i),
\end{aligned}$$

where Ric denotes the Ricci tensor of g.

From [2], we have a contact metric structure $(\tilde{G}, \eta, \phi, \xi)$ on T_1M satisfying

$$\begin{aligned}
&\xi = 2u^h,\ \phi A^t = \frac{1}{4(a+c)}(-A^h + \frac{1}{2}g(A,u)\xi),\ \phi A^h = \frac{1}{4a}A^t, \qquad (2.6)\\
&\eta(A^h) = \frac{1}{2}g(A,u),\ \eta(A^t) = 0.
\end{aligned}$$

We also have

$$\begin{aligned}
hA^t &= \frac{1}{4a}A^t - \frac{1}{4(a+c)}(R_uA)^t, \qquad (2.7)\\
hA^h &= -\frac{1}{4a}A^h + \frac{1}{8a}g(A,u)\xi + \frac{1}{4(a+c)}(R_uA)^h,
\end{aligned}$$

where $R_u = R(\cdot, u)u$ is the Jacobi operator related with the unit vector u.

3. T_1M satisfying some commutative conditions

Let (M, g) be a Riemannian manifold and T_1M be its unit tangent sphere bundle with the Kaluza-Klein metric $\tilde{G}$ (2.2). Denote the Ricci operator of $\tilde{G}$ by $\tilde{S}$. First, we suppose that $\tilde{S}h = h\tilde{S}$. Then, using (2.5), (2.7) and self-adjoint property of the Jacobi operator R_u, we occur

$$\begin{aligned}
0 &= \tilde{G}(\tilde{S}hA^t - h\tilde{S}A^t, B^t)\\
&= \widetilde{Ric}(hA^t, B^t) - \widetilde{Ric}(A^t, hB^t) \qquad (3.8)\\
&= \frac{a^2}{16(a+c)^3}\sum_{i=1}^{n}[g(R(u,A)e_i, R(u,R_uB)e_i) - g(R(u,B)e_i, R(u,R_uA)e_i)],
\end{aligned}$$

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}hA^t - h\tilde{S}A^t, B^h) \\ &= \widetilde{Ric}(hA^t, B^h) - \widetilde{Ric}(A^t, hB^h) \\ &= \frac{1}{4(a+c)}[(\nabla_u Ric)(A,B) - (\nabla_A Ric)(u,B)] \\ &\quad -\frac{a}{8(a+c)^2}[(\nabla_u Ric)(R_u A, B) - (\nabla_{R_u A} Ric)(u,B)] \\ &\quad -\frac{a}{8(a+c)^2}[(\nabla_u Ric)(A, R_u B) - (\nabla_A Ric)(u, R_u B)] \\ &\quad -\frac{1}{8(a+c)} g(B,u)[(\nabla_u Ric)(A,u) - (\nabla_u Ric)(u,u)], \end{aligned} \tag{3.9}$$

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}hA^h - h\tilde{S}A^h, B^h) \\ &= \widetilde{Ric}(hA^h, B^h) - \widetilde{Ric}(A^h, hB^h) \\ &= \frac{1}{4a} g(A,u)[Ric(B,u) - \frac{a}{2(a+c)}\sum_{i=1}^{n} g(R(u,e_i)u, R(u,e_i)B)] \\ &\quad -\frac{1}{4a} g(B,u)[Ric(A,u) - \frac{a}{2(a+c)}\sum_{i=1}^{n} g(R(u,e_i)A, R(u,e_i)u)] \\ &\quad +\frac{1}{4(a+c)}[Ric(R_u A, B) - \frac{a}{2(a+c)}\sum_{i=1}^{n} g(R(u,e_i)R_u A, R(u,e_i)B)] \\ &\quad -\frac{1}{4(a+c)}[Ric(A, R_u B) - \sum_{i=1}^{n} g(R(u,e_i)A, R(u,e_i)R_u B)]. \end{aligned} \tag{3.10}$$

Thus, T_1M fulfills $\tilde{S}h = h\tilde{S}$ if and only if M fulfills (3.8)-(3.10).

Let M be a space with constant curvature k. In this case, we observe that it fulfills (3.8)-(3.10). Therefore, we obtain the following proposition.

Proposition 3.1. *Let M be a space with constant curvature k. Then T_1M fulfills the relation $\tilde{S}h = h\tilde{S}$.*

If we consider M as a surface (i.e., 2-dimensional manifold), we obtain the following theorem.

Theorem 3.1. *Let M be a 2-dimensional Riemannian manifold. Then T_1M fulfills the relation $\tilde{S}h = h\tilde{S}$ if and only if M is a space with constant Gaussian curvature.*

Proof. The curvature tensor of a 2-dimensional Riemannian manifold is expressed by the relation $R(A,B)C = \kappa(g(B,C)A - g(A,C)B)$, where κ is a smooth function on M. This relation gives us $R_uA = R(A,u)u = \kappa(A - g(A,u)u)$ and $Ric(A,B) = (n-1)\kappa g(A,B)$. Using these formulas, we see that (3.8) and (3.10) are valid. From (3.9), we deduce

$$\frac{a(1-\kappa)+c}{8(a+c)^2}\{(u\kappa)[2g(A,B)-g(A,u)g(B,u)]-(A\kappa)g(B,u)\}=0. \tag{3.11}$$

Taking $A = B \perp u$ and $\|B\| = 1$ in (3.11), we get $\frac{a(1-\kappa)+c}{4(a+c)^2}(u\kappa) = 0$. This shows that the Gaussian curvature κ is constant. The converse is true from Proposition 3.1. □

Now, we suppose that the relation $\tilde{S}\phi h = \phi h\tilde{S}$ holds for $(T_1M, \tilde{g})$. From (2.5)-(2.7), we obtain

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}\phi hA^t - \phi h\tilde{S}A^t, B^t) \qquad (3.12)\\ &= \widetilde{Ric}(\phi hA^t, B^t) + \widetilde{Ric}(A^t, h\phi B^t)\\ &= \frac{a}{2(a+c)}[(\nabla_B Ric)(u,A) - (\nabla_X Ric)(u,B) - (\nabla_u Ric)(A, R_uB)\\ &\quad +(\nabla_u Ric)(R_uA, B) + (\nabla_A Ric)(u, R_uB) - (\nabla_B Ric)(u, R_uA)\\ &\quad +g(A,u)((\nabla_u Ric)(u,B) - (\nabla_B Ric)(u,u)\\ &\quad -g(B,u)((\nabla_u Ric)(u,A) - (\nabla_A Ric)(u,u))\}, \end{aligned}$$

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}\phi hA^t - \phi h\tilde{S}A^t, B^h)\\ &= \widetilde{Ric}(\phi hA^t, B^h) + \widetilde{Ric}(A^t, h\phi B^h) \qquad (3.13)\\ &= (n-2)(g(A,B) - g(A,u)g(B,u)) + \frac{a^2}{4(a+c)^2}\sum_{i=1}^n g(R(u,A)e_i, R(u,B)e_i)\\ &\quad -(n-2)g(A, R_uB) - \frac{a^2}{4(a+c)^2}\sum_{i=1}^n g(R(u,A)e_i, R(u,R_uB)e_i)\\ &\quad -Ric(A,B) + \frac{a}{2(a+c)}\sum_{i=1}^n g(R(u,e_i)A, R(u,e_i)B)\\ &\quad +g(A,u)[Ric(B,u) - \frac{a}{2(a+c)}\sum_{i=1}^n g(R(u,e_i)u, R(u,e_i)B)]\\ &\quad +Ric(R_uA, B) - \frac{a}{2(a+c)}\sum_{i=1}^n g(R(u,e_i)R_uA, R(u,e_i)B), \end{aligned}$$

$$\begin{aligned} 0 &= \tilde{G}(\tilde{S}\phi hA^h - \phi h\tilde{S}A^h, B^h) \qquad (3.14)\\ &= \widetilde{Ric}(\phi hA^h, B^h) + \widetilde{Ric}(A^h, h\phi B^h)\\ &= \frac{a}{2(a+c)}[\nabla_A Ric(u,B) - (\nabla_B Ric)(u,A) + (\nabla_u Ric)(R_uA, B)\\ &\quad -(\nabla_u Ric)(A, R_uB) - (\nabla_{R_uA} Ric)(u,B) + (\nabla_{R_uB} Ric)(u,A)\}. \end{aligned}$$

Let's assume that M is a space with constant curvature k. Then, using the relations $R(A,B)C = k(g(B,C)A - g(A,C)B)$, $R_u A = R(A,u)u = k(A - g(A,u)u)$ and $Ric(A,B) = (n-1)kg(A,B)$ in (3.13), we get

$$\frac{2a^2+ac}{2(a+c)^2}k^3 + \frac{2c^2+3ac-2n(a+c)^2}{2(a+c)^2}k^2 + (2n-3)k - n + 2 = 0.$$

So, we have the following theorem.

Theorem 3.2. *Let (M,g) be an $n-$dimensional space with constant curvature k. Then T_1M fulfills the relation $\tilde{S}\phi h = \phi h\tilde{S}$ if and only if the following relation holds:*

$$\frac{2a^2+ac}{2(a+c)^2}k^3 + \frac{2c^2+3ac-2n(a+c)^2}{2(a+c)^2}k^2 + (2n-3)k - n + 2 = 0.$$

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