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# ON THE HARARY INDEX OF $\Gamma(\mathbb{Z}_n)$

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Abstract. In this work, the Harary index of zero-divisor graphs of rings $\mathbb{Z}_n$ are calculated when $n$ is a member of the set $\{2p, p^2, p^\lambda, pq, p^2q, pqr\}$ where $p, q$ and $r$ are distinct prime numbers and $\lambda$ is an integer number . We give the formulas for computing the Harary index of $\Gamma(\mathbb{Z}_n)$. Moreover, the Harary index of graphs for products of rings were computed.

**Keywords**: Graph theory, Topological indeces, Harary index, Zero-divisor graph, Distance in graph.

**2010 Mathematics Subject Classification**: 05C09, 05C25, 05C12.

## 1. Introduction and Preliminaries

The numerical invariants of chemical graphs are used to characterize some properties of the graph of a molecule [35]. These invariants are named in the chemical literature as topological indices also known as molecular descriptors, which are a single number [21]. Topological indices have found application in various areas of chemistry, physics, mathematics, informatics, biology, etc. [1, 2, 20, 28, 29]. Topological indices have found some applications in theoretical chemistry, Chemical graph theory is a branch of mathematical chemistry that has a significant impact on the development of the chemical sciences. This study, due to its mathematical convergence, will attract many researchers.

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Many times, nearby atoms affect each other more than distant atoms. Ivanciuc et al. defined a new molecular graph matrix for researching this interaction, namely the Harary matrix [22]. It was also called initially the reciprocal distance matrix [24]. The Harary index has been introduced independently by Plavšić et al. [31]. The Harary index is derived from the Harary matrix and has a number of exciting properties. For this reason, many researchers have studied this notion for many years [3, 10, 11, 12, 13, 14, 16, 36, 37, 38].

Graphs are a powerful tool for exploring algebraic structures, and their use has become a prominent area of research. By mapping a graph to a ring or other algebraic structures, many academics have investigated the algebraic properties of these structures using the associated graphs [4, 6, 7, 15, 17, 19, 26, 27, 30].

Let $G = (V, E)$ be a connected graph with vertex set $V(G) = \{\nu_1, \nu_2, ..., \nu_n\}$ and edge set $E(G)$ such that $|V(G)| = n$ and $|E(G)| = m$. Let $d_{i,j}$ denote by the distance between the vertices $\nu_i$ and $\nu_j$ in $G$. The Harary matrix of $G$ denoted by $RD(G)$ is an $n \times n$ matrix $(RD_{i,j})$ such that [23, 31]

$$RD_{i,j} = \begin{cases} \frac{1}{d_{i,j}}, & i \neq j \\ 0, & i = j. \end{cases}$$

The Harary index of the graph $G$, denoted by $HI(G)$, is defined as

$$\begin{aligned} HI(G) &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} RD_{i,j} \\ &= \sum_{i<j} RD_{i,j}. \end{aligned}$$

Zero-divisor graph of a commutative ring was introduced by Beck [7]. In that study, Beck constitutes a connection between graph theory and commutative ring theory. Then, Anderson and Livingston modified the definition of the zero-divisor graph of a commutative ring [4]. They defined the zero-divisor graph of a commutative ring on nonzero zero-divisor elements of the ring as follows:

Let $\mathbb{Z}_n$ be the ring of integers modulo $n$. The zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is the simple undirected graph without loops which has its vertex set coincides with the nonzero zero-divisors of $\mathbb{Z}_n$ and two distinct vertices $\upsilon$ and $\nu$ in $\Gamma(\mathbb{Z}_n)$ are adjacent whenever $\upsilon\nu = 0$ in $\mathbb{Z}_n$. Zero-divisor graphs have been a topic of interest to many researchers for many years [8, 9, 32, 34].

Throughout this paper, we study Harary index of zero-divisor graphs of $\mathbb{Z}_n$ and find some formulas for computing the Harary index of $\Gamma(\mathbb{Z}_n)$ which are examined. In Section 2, we calculate Harary index of zero-divisor graphs of $\mathbb{Z}_n$ for $n \in \{2p, p^2, p^\lambda, pq, p^2q, pqr\}$ where $p, q$ and $r$ are distinct prime numbers and $\lambda > 2$ is an integer number. Moreover, we arrive at the Harary index of the Cartesian product of these graphs. Finally, we provide some examples to support these theorems.

## 2. Harary index of $\Gamma(\mathbb{Z}_n)$

Lately, the zero-divisor graph of the ring $\mathbb{Z}_n$ is popular research in spectral graph and chemical graph theory. Many researchers have examined some topological indices of zero-divisor graph of the $\mathbb{Z}_n$ [5, 17, 18, 25, 33].

**Theorem 2.1.** *Let $p > 2$ be a prime number, then*

$$HI(\Gamma(\mathbb{Z}_{2p})) = \frac{(p-1)(p+2)}{4}.$$

*Proof.* Since $\Gamma(\mathbb{Z}_{2p})$ is a star graph it is isomorphic to $K_{1,p-1}$. In this graph, the vertex set $V(\Gamma(\mathbb{Z}_{2p}))$ is divided into two distinct subsets as follow:

$$S_1 = \{p\},$$
$$S_2 = \{2x \mid x = 1, ..., p-1\},$$

where $|S_1| = \Phi(\frac{2p}{p}) = 1$ and $|S_2| = \Phi(\frac{2p}{2}) = p-1$. $d(\upsilon, \nu) = 1$ for $\forall \upsilon \in S_1, \forall \nu \in S_2$, and $d(\upsilon, \nu) = 2$ for $\forall \upsilon, \nu \in S_2$. Therefore,

$$\begin{aligned} HI(\Gamma(\mathbb{Z}_{2p})) &= \sum_{\upsilon,\nu \in V(\Gamma(\mathbb{Z}_{2p}))} \frac{1}{d(\upsilon,\nu)} \\ &= \sum_{\upsilon \in S_1, \nu \in S_2} \frac{1}{d(\upsilon,\nu)} + \sum_{\upsilon,\nu \in S_2} \frac{1}{d(\upsilon,\nu)} \\ &= |S_2| \frac{1}{d(\upsilon,\nu)} + \frac{|S_2|(|S_2|-1)}{2} \frac{1}{d(\upsilon,\nu)} \\ &= \frac{(p-1)(p+2)}{4}. \end{aligned}$$

□

**Theorem 2.2.** *Let $p > 2$ be a prime number, then*

$$HI(\Gamma(\mathbb{Z}_{p^2})) = \frac{(p-1)(p-2)}{2}.$$

*Proof.* Since $\Gamma(\mathbb{Z}_{p^2})$ is a complete graph having $p-1$ vertices, so $\Gamma(\mathbb{Z}_{p^2}) \cong K_{p-1}$. In a complete graph, $d(\upsilon,\nu)=1$ for $\forall \upsilon,\nu \in V(\Gamma(\mathbb{Z}_{p^2}))$. Therefore,

$$HI(\Gamma(\mathbb{Z}_{p^2})) = \sum_{\upsilon,\nu\in V(\Gamma(\mathbb{Z}_{p^2}))} \frac{1}{d(\upsilon,\nu)} = \frac{(p-1)(p-2)}{2}.$$

□

**Theorem 2.3.** *Let $p$ be a prime number and $\lambda > 2$ be an integer, then*

$$HI(\Gamma(\mathbb{Z}_{p^\lambda})) = \frac{(\lambda-1)}{4}p^\lambda - \frac{(\lambda+3)}{4}p^{\lambda-1} - \frac{p^{\lfloor \frac{\lambda}{2} \rfloor}}{4} + \frac{p^{2(\lambda-1)}}{4} + 1.$$

*Proof.* Firstly, we suppose that $\lambda$ is even.

**Case 1.** In the first case, there are two subpart to be considered. In the first subpart, it is considered the distance between a vertex from $S_i$ and a vertex from $S_j$ where $i = 2, ..., \frac{\lambda}{2}-1$ and $j = 1, 2, ..., i-1$ is 2 as $d(\upsilon,\nu) = 2$, $\upsilon \in S_i, \nu \in S_j$. So,

$$\sum_{i=2}^{\frac{\lambda}{2}-1}\sum_{j=1}^{i-1} |S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

The next subpart is related to the distance between a vertex from $S_i$ and a vertex from $S_j$ where $i = \dfrac{\lambda}{2}, ..., \lambda-2$ and $j = 1, ..., \lambda-i-1$

$$\sum_{i=\frac{\lambda}{2}}^{\lambda-2}\sum_{j=1}^{\lambda-i-1} |S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

**Case 2.** We consider vertex set $S_i$ and $S_j$ where $i = \frac{\lambda}{2}+1, ..., \lambda-1$ and $j = \lambda-1, .., i-1$. The distance between a vertex from $S_i$ and a vertex from $S_j$ is 1. From this,

$$\sum_{i=\frac{\lambda}{2}+1}^{\lambda-1}\sum_{j=\lambda-i}^{i-1} |S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

**Case 3.** In this case, we take into account vertices in $S_i$ where $i = 1, ..., \lambda-1$. When considering vertices $\upsilon,\nu \in S_i$ for $i \geq \frac{\lambda}{2}$, the distance is 1, otherwise 2. Hence, we get

$$\sum_{i=1}^{\frac{\lambda}{2}-1} \frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)} + \sum_{i=\frac{\lambda}{2}}^{\lambda-1} \frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

Using above three cases, when $\lambda$ is even, the Harary index of $\Gamma(\mathbb{Z}_{p^\lambda})$ is as follows:

$$HI(\Gamma(\mathbb{Z}_{p^\lambda})) = \sum_{i=2}^{\frac{\lambda}{2}-1}\sum_{j=1}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=\frac{\lambda}{2}}^{\lambda-2}\sum_{j=1}^{\lambda-i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=\frac{\lambda}{2}+1}^{\lambda-1}\sum_{j=\lambda-i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=1}^{\frac{\lambda}{2}-1}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)} + \sum_{i=\frac{\lambda}{2}}^{\lambda-1}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)}.$$

Now, we suppose that $\lambda$ is odd.

**Case 1.** In this case, we consider vertex sets $S_i$ and $S_j$ where $i = 2, ..., \frac{\lambda-1}{2}$ and $j = 1, ..., i-1$. The distance from $S_i$ to $S_j$ is 2 as $d(\upsilon, \nu) = 2$, where $\upsilon \in S_i$ and $\nu \in S_j$. Hence, we get

$$\sum_{i=2}^{\frac{\lambda-1}{2}}\sum_{j=i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

Also, in other part of this case, it is considered vertex sets $S_i$ and $S_j$ where $i = \frac{\lambda+1}{2}, ..., \lambda-2$ and $j = 1, ..., \lambda-i-1$. The distance between these vertices is also 2. So, we have

$$\sum_{i=\frac{\lambda+1}{2}}^{\lambda-2}\sum_{j=1}^{\lambda-i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

**Case 2.** In this case, we are interested in vertex sets $S_i$ and $S_j$ where $i = \frac{\lambda+1}{2}, ..., \lambda-1$ and $j = \lambda-i, ..., i-1$. The distance is $d(\upsilon, \nu) = 1$ where $\upsilon \in S_i$ and $\nu \in S_j$. Then, we have

$$\sum_{i=\frac{\lambda+1}{2}}^{\lambda-1}\sum_{j=\lambda-i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

**Case 3.** In this case, we are interested in vertex sets $S_i$ and $S_j$ where $i = \frac{\lambda+1}{2}, ..., \lambda-1$ and $j = \lambda-i, ..., i-1$. The distance is $d(\upsilon, \nu) = 1$ where $\upsilon \in S_i$ and $\nu \in S_j$. Then, we have

$$\sum_{i=\frac{\lambda+1}{2}}^{\lambda-1}\sum_{j=\lambda-i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j.$$

**Case 4.** In the last case, it is considered vertices in $S_i$ where $i = 1, ..., \lambda-1$. When considering vertices $\upsilon, \nu \in S_i$ for $i \geq \frac{\lambda+1}{2}$, the distance is 1, otherwise

2. So, we attain

$$\sum_{i=1}^{\frac{\lambda-1}{2}} \frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)} + \sum_{i=\frac{\lambda+1}{2}}^{\lambda-1} \frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)} \qquad \upsilon \in S_i, \nu \in S_j$$

When $\lambda$ is odd, using above three cases, the Harary index of $\Gamma(\mathbb{Z}_{p^\lambda})$ is as follows:

$$\begin{aligned} HI(\Gamma(\mathbb{Z}_{p^\lambda})) = & \sum_{i=2}^{\frac{\lambda-1}{2}}\sum_{j=i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=\frac{\lambda+1}{2}}^{\lambda-2}\sum_{j=1}^{\lambda-i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)}+ \\ & \sum_{i=\frac{\lambda+1}{2}}^{\lambda-1}\sum_{j=\lambda-i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=1}^{\frac{\lambda-1}{2}}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)}+ \\ & \sum_{i=\frac{\lambda+1}{2}}^{\lambda-1}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)}. \end{aligned}$$

Therefore, Harary index of $\Gamma(\mathbb{Z}_{p^\lambda})$ in a single form is as follows:

$$\begin{aligned} HI(\Gamma(\mathbb{Z}_{p^\lambda})) = & \sum_{i=2}^{\lfloor\frac{\lambda-1}{2}\rfloor}\sum_{j=i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=\lceil\frac{\lambda}{2}\rceil}^{\lambda-2}\sum_{j=1}^{\lambda-i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)}+ \\ & \sum_{i=\lceil\frac{\lambda+1}{2}\rceil}^{\lambda-1}\sum_{j=\lambda-i}^{i-1}|S_i||S_j|\frac{1}{d(\upsilon,\nu)} + \sum_{i=1}^{\lfloor\frac{\lambda-1}{2}\rfloor}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)}+ \\ & \sum_{i=\lceil\frac{\lambda}{2}\rceil}^{\lambda-1}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)}. \end{aligned}$$

Note that $|S_i| = \phi(\frac{\lambda}{i}) = p^{\lambda-i} - p^{\lambda-i-1}$.

$$\begin{aligned} HI(\Gamma(\mathbb{Z}_{p^\lambda})) = & \sum_{i=2}^{\lfloor\frac{\lambda-1}{2}\rfloor}\sum_{j=1}^{i}(p^{\lambda-i}-p^{\lambda-i-1})(p^{\lambda-j}-p^{\lambda-j-1})\frac{1}{2}+ \\ & \sum_{i=\lceil\frac{\lambda}{2}\rceil}^{\lambda-2}\sum_{j=1}^{\lambda-i-1}(p^{\lambda-i}-p^{\lambda-i-1})(p^{\lambda-j}-p^{\lambda-j-1})\frac{1}{2}+ \\ & \sum_{i=\lceil\frac{\lambda+1}{2}\rceil}^{\lambda-1}\sum_{j=\lambda-i}^{i-1}(p^{\lambda-i}-p^{\lambda-i-1})(p^{\lambda-j}-p^{\lambda-j-1})+ \\ & \sum_{i=1}^{\lfloor\frac{\lambda-1}{2}\rfloor}\frac{(p^{\lambda-i}-p^{\lambda-i-1})(p^{\lambda-i}-p^{\lambda-i-1}-1)}{2}\frac{1}{2}+ \\ & \sum_{i=\lceil\frac{\lambda}{2}\rceil}^{\lambda-1}\frac{(p^{\lambda-i}-p^{\lambda-i-1})(p^{\lambda-i}-p^{\lambda-i-1}-1)}{2}. \end{aligned} \tag{2.1}$$

After reducing and simplifying Equation 2.1, we get

$$HI(\Gamma(\mathbb{Z}_{p^\lambda})) = \frac{(\lambda-1)}{4}p^\lambda - \frac{(\lambda+3)}{4}p^{\lambda-1} - \frac{p^{\lfloor \frac{\lambda}{2} \rfloor}}{4} + \frac{p^{2(\lambda-1)}}{4} + 1.$$

□

**Example 2.1.** *Given* $\Gamma(\mathbb{Z}_{2^7})$ *where* $p = 2$ *and* $\lambda = 7$ *as in Figure 1. We consider Harary index of* $\Gamma(\mathbb{Z}_{2^7})$ *according to Theorem 2.3 while* $\lambda$ *is odd.*

$$\begin{aligned} HI(\Gamma(\mathbb{Z}_{p^\lambda})) =& \frac{(\lambda-1)}{4}p^\lambda - \frac{(\lambda+3)}{4}p^{\lambda-1} - \frac{p^{\lfloor \frac{\lambda}{2} \rfloor}}{4} + \frac{p^{2(\lambda-1)}}{4} + 1 \\ =& \frac{6}{4}2^7 - \frac{10}{4}2^6 - \frac{2^3}{4} + \frac{2^{12}}{4} + 1 \\ =& 1055. \end{aligned}$$

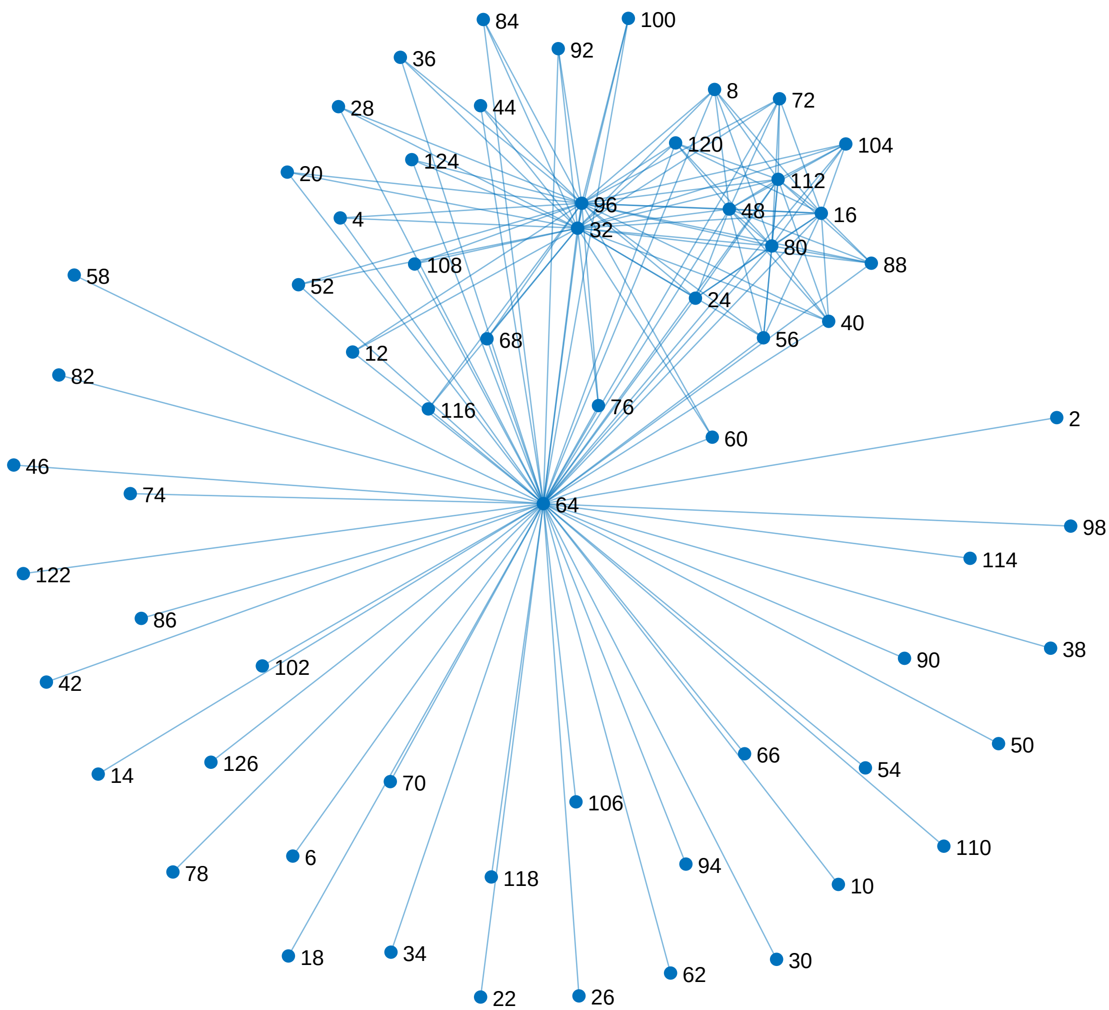


FIGURE 1. $\Gamma(\mathbb{Z}_{2^7})$

**Example 2.2.** *Given $\Gamma(\mathbb{Z}_{3^6})$ where $p = 3$ and $\lambda = 6$ as in Figure 2. In this example, we consider Harary index of $\Gamma(\mathbb{Z}_{3^6})$ according to Theorem 2.3 when $\lambda$ is even.*

$$
\begin{aligned}
HI(\Gamma(\mathbb{Z}_{p^\lambda})) =& \frac{(\lambda-1)}{4}p^\lambda - \frac{(\lambda+3)}{4}p^{\lambda-1} - \frac{p^{\lfloor \frac{\lambda}{2} \rfloor}}{4} + \frac{p^{2(\lambda-1)}}{4} + 1 \\
=& \frac{5}{4}3^6 - \frac{9}{4}3^5 - \frac{3^3}{4} + \frac{3^{10}}{4} + 1 \\
=& 15121.
\end{aligned}
$$

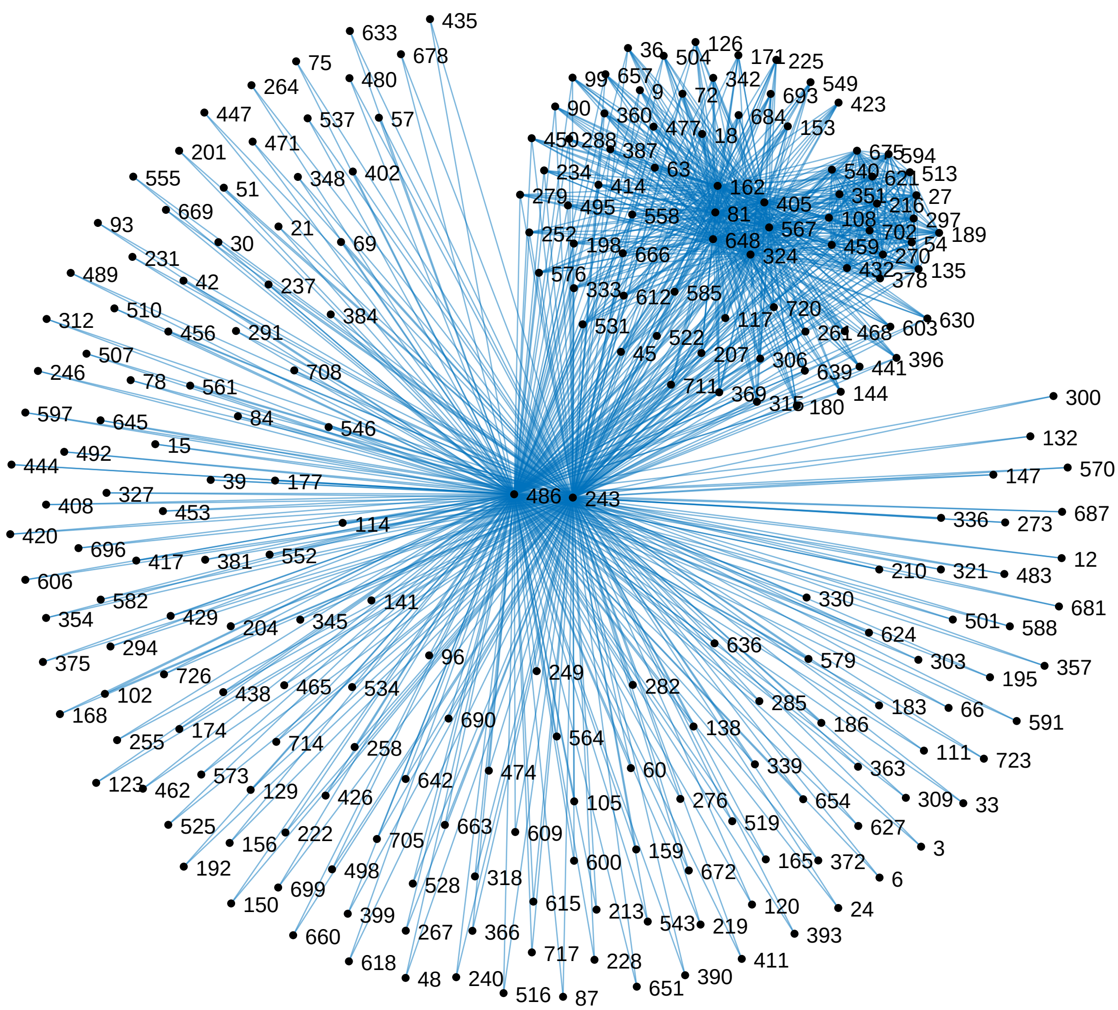


FIGURE 2. $\Gamma(\mathbb{Z}_{3^6})$

**Theorem 2.4.** *Let $\Gamma(\mathbb{Z}_{pq})$ be a zero divisor graph and $p$ and $q$ be distinct prime numbers, then*

$$
HI(\Gamma(\mathbb{Z}_{pq})) = (p-1)(q-1)\left[1 + \frac{(p-2)}{4(q-1)} + \frac{(q-2)}{4(p-1)}\right].
$$

*Proof.* Note that the graph $\Gamma(\mathbb{Z}_{pq})$ is isomorphic to $K_{p-1,q-1}$ which is a complete bipartite graph. The vertex set of this graph can be partitioned into two distinct subsets as

$$S_1 = \{px \mid x = 1, ..., q-1\},$$
$$S_2 = \{qx \mid x = 1, ..., p-1\},$$

where $|S_1| = \Phi(\frac{pq}{p}) = q-1$ and $|S_2| = \Phi(\frac{pq}{q}) = p-1$. It is clear that we have two cases to calculate the Harary Index.

**Case 1.** $d(\upsilon,\nu) = 1$ for $\forall\upsilon \in S_1$ and $\forall\nu \in S_2$ where $S_1 \cup S_2 = V(\Gamma(\mathbb{Z}_{pq}))$. Then

$$\begin{aligned}\sum_{\upsilon\in S_1,\nu\in S_2} \frac{1}{d(\upsilon,\nu)} &= |S_1| \cdot |S_2| \\ &= (p-1)(q-1).\end{aligned}$$

**Case 2.** $d(\upsilon,\nu) = 2$ for $\forall\upsilon,\nu \in S_i$ where $i = 1, 2$. Then

$$\begin{aligned}\sum_{i=1}^{2}\sum_{\upsilon\in S_1,\nu\in S_2} \frac{1}{d(\upsilon,\nu)} &= \binom{|S_1|}{2}\cdot\frac{1}{2} + \binom{|S_2|}{2}\cdot\frac{1}{2} \\ &= \frac{1}{4}\cdot\big[(p-1)(p-2) + (q-1)(q-2)\big].\end{aligned}$$

According to these cases, we attain that

$$HI(\Gamma(\mathbb{Z}_{pq})) = (p-1)(q-1)\left[1 + \frac{(p-2)}{4(q-1)} + \frac{(q-2)}{4(p-1)}\right].$$

□

**Theorem 2.5.** *Let $p$ and $q$ be two distinct prime numbers, then Harary index of zero divisor graph $\Gamma(\mathbb{Z}_{p^2q})$ is*

$$\begin{aligned}HI(\Gamma(\mathbb{Z}_{p^2q})) = (p-1)(q-1)\Bigg[&\frac{p(p+5)}{3} + \frac{(p+1)(q-1)-1}{4} \\ &+ \frac{p^3+p^2-p-4}{4(q-1)} + \frac{q-2}{4(p-1)}\Bigg].\end{aligned}$$

*Proof.* In zero divisor graph $\Gamma(\mathbb{Z}_{p^2q})$, the vertex set is split four subsets as

$$S_1 = \{px \mid x = 1, ..., pq-1, p \nmid x, q \nmid x\},$$
$$S_2 = \{qx \mid x = 1, ..., p^2-1, p \nmid x\},$$
$$S_3 = \{p^2x \mid x = 1, ..., q-1\},$$
$$S_4 = \{pqx \mid x = 1, ..., p-1\},$$

where $\bigcup_{i=1}^{4} S_i = V(\Gamma(\mathbb{Z}_{p^2q}))$ and $S_i \cap S_j = \emptyset$ for $i, j = 1, ..., 4$, $i \neq j$.

Using these four distinct subsets, there are three possible cases to evaluate Harary index of $\Gamma(\mathbb{Z}_{p^2q})$.

**Case 1.** In this case, we consider $(\upsilon, \nu)$ vertex couples where $\forall \upsilon \in S_i$ and $\forall \nu \in S_j$ for $i = 1, ..., 3$ and $j > i$. Then

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=i+1}^{4}\sum_{\upsilon\in S_i,\nu\in S_j}\frac{1}{d(\upsilon,\nu)} &= \sum_{i=1}^{3}\sum_{j=i+1}^{4}|S_i|\cdot|S_j|\cdot\frac{1}{d(\upsilon,\nu)}\Big|_{\upsilon\in S_i,\nu\in S_j}\\
&= |S_1||S_2|\frac{1}{3} + |S_1||S_3|\frac{1}{2} + |S_1||S_4|\\
&+ |S_2||S_3| + |S_2||S_4|\frac{1}{2} + |S_3||S_4|\\
&= p(p-1)(q-1)\left[2 + \frac{p-1}{3} + \frac{q-1}{2p} + \frac{p-1}{2(q-1)}\right].
\end{aligned}$$

**Case 2.** This case takes into account two distinct vertices $\upsilon$ and $\nu$ where $\forall \upsilon, \nu \in S_i$ for $i = 1, ..., 3$. Then

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{\upsilon,\nu\in S_i}\frac{1}{d(\upsilon,\nu)} &= \sum_{i=1}^{3}|S_i|\cdot|S_i|\cdot\frac{1}{d(\upsilon,\nu)}\Big|_{\upsilon,\nu\in S_i}\\
&= \sum_{i=1}^{3}\binom{|S_i|}{2}\cdot\frac{1}{2}\\
&= \left[\binom{|S_1|}{2} + \binom{|S_2|}{2} + \binom{|S_3|}{2}\right]\cdot\frac{1}{2}\\
&= \frac{1}{4}(p-1)(q-1)\Bigg[(p-1)(q-1) - 1 + \frac{p\,(p(p-1)-1)}{q-1}\\
&\quad + \frac{q-2}{p-1}\Bigg]
\end{aligned}$$

**Case 3.** In the last case, we consider the vertices from the $S_4$ which forms a complete subgraph in $\Gamma(\mathbb{Z}_{p^2q})$. Then

$$\sum_{\upsilon,\nu \in S_4} \frac{1}{d(\upsilon,\nu)} = \frac{|S_4|(|S_4|-1)}{2} = \frac{(p-1)(p-2)}{2}.$$

Evaluating above three cases, Harary index of $\Gamma(\mathbb{Z}_{p^2q})$ is

$$HI(\Gamma(\mathbb{Z}_{p^2q})) = (p-1)(q-1)\left[\frac{p(p+5)}{3} + \frac{(p+1)(q-1)-1}{4} + \frac{p^3+p^2-p-4}{4(q-1)} + \frac{q-2}{4(p-1)}\right].$$

□

**Theorem 2.6.** *Let $p$, $q$ and $r$ be three distinct prime numbers, then Harary index of zero divisor graph $\Gamma(\mathbb{Z}_{pqr})$ is*

$$\begin{aligned} HI(\Gamma(\mathbb{Z}_{pqr})) =& \alpha\beta\gamma\left(3+\frac{\alpha}{3}+\frac{\beta}{3}+\frac{\gamma}{3}\right) + \alpha\beta\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\alpha\beta}{4}+\frac{3}{4}\right) + \\ & \alpha\gamma\left(\frac{\alpha}{2}+\frac{\gamma}{2}+\frac{\alpha\gamma}{4}+\frac{3}{4}\right) + \beta\gamma\left(\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\beta\gamma}{4}+\frac{3}{4}\right) + \\ & \frac{1}{4}\left(\alpha^2+\beta^2+\gamma^2-\alpha-\beta-\gamma\right) \end{aligned}$$

*where $\alpha = p-1$, $\beta = q-1$, and $\gamma = r-1$.*

*Proof.* $V(\Gamma(\mathbb{Z}_{pqr}))$ is divided into six separate subsets such as

$$S_1 = \{px \mid x = 1, ..., qr-1, q \nmid x, r \nmid x\},$$

$$S_2 = \{qx \mid x = 1, ..., pr-1, p \nmid x, r \nmid x\},$$

$$S_3 = \{rx \mid x = 1, ..., pq-1, p \nmid x, q \nmid x\},$$

$$S_4 = \{pqx \mid x = 1, ..., r-1\},$$

$$S_5 = \{prx \mid x = 1, ..., q-1\},$$

$$S_6 = \{qrx \mid x = 1, ..., p-1\}.$$

where $\bigcup_{i=1}^{6} S_i = V(\Gamma(\mathbb{Z}_{pqr}))$ and $S_i \cap S_j = \emptyset$ for $i, j = 1, ..., 6$, $i \neq j$. Using these six distinct vertex subsets, there are three possible cases to evaluate Harary index of $\Gamma(\mathbb{Z}_{pqr})$.

**Case 1.** In this case, we consider $(\upsilon, \nu)$ vertex couples where $\forall \upsilon \in S_i$ and $\forall \nu \in S_j$ for $i = 1, ..., 2$ and $j = i+1, ..., 3$, and $d(\upsilon, \nu) = 3$ according to the graph. Then

$$\begin{aligned}
\sum_{i=1}^{2} \sum_{j=i+1}^{3} \sum_{\upsilon \in S_i, \nu \in S_j} \frac{1}{d(\upsilon,\nu)} &= \sum_{i=1}^{2} \sum_{j=i+1}^{3} |S_i| \cdot |S_j| \cdot \frac{1}{d(\upsilon,\nu)}\Big|_{\upsilon \in S_i, \nu \in S_j} \\
&= |S_1||S_2|\frac{1}{3} + |S_1||S_3|\frac{1}{3} + |S_2||S_3|\frac{1}{3} \\
&= \frac{1}{3}\Big[(p-1)(q-1)(r-1)^2 + (p-1)(q-1)^2(r-1) \\
&\quad + (p-1)^2(q-1)(r-1)\Big] \\
&= \frac{(p-1)(q-1)(r-1)}{3}(p+q+r-3).
\end{aligned}$$

**Case 2.** This case takes into account two distinct vertex set couples $S_i$ and $S_j$ $\upsilon$ where $d(\upsilon, \nu) = 2$, $\upsilon \in S_i$ and $\nu \in S_j$. Then

$$\begin{aligned}
&\sum_{j \in \{4,5\}} \sum_{\substack{\upsilon \in S_1, \\ \nu \in S_j}} \frac{1}{d(\upsilon,\nu)} + \sum_{j \in \{4,6\}} \sum_{\substack{\upsilon \in S_2, \\ \nu \in S_j}} \frac{1}{d(\upsilon,\nu)} + \sum_{j \in \{5,6\}} \sum_{\substack{\upsilon \in S_3, \\ \nu \in S_j}} \frac{1}{d(\upsilon,\nu)} \\
&= |S_1| \cdot |S_4| \cdot \frac{1}{2} + |S_1| \cdot |S_5| \cdot \frac{1}{2} + |S_2| \cdot |S_4| \cdot \frac{1}{2} + |S_2| \cdot |S_6| \cdot \frac{1}{2} \\
&\quad + |S_3| \cdot |S_5| \cdot \frac{1}{2} + |S_3| \cdot |S_6| \cdot \frac{1}{2} \\
&= \frac{1}{2}\big[(q-1)(r-1)^2 + (q-1)^2(r-1) + (p-1)(r-1)^2 + \\
&\quad (p-1)^2(r-1) + (p-1)(q-1)^2 + (p-1)^2(q-1)\big].
\end{aligned}$$

**Case 3.** In this case, it is considered the vertex set couples such as $S_i$ and $S_j$ where $d(\upsilon, \nu) = 1$, $\upsilon \in S_i$ and $\nu \in S_j$ in $\Gamma(\mathbb{Z}_{pqr})$. Then

$$\begin{aligned}
&\sum_{i=1}^{3} \sum_{\substack{\upsilon \in S_i, \\ \nu \in S_{7-i}}} \frac{1}{d(\upsilon,\nu)} + \sum_{i=4}^{5} \sum_{j=i+1}^{6} \sum_{\substack{\upsilon \in S_i, \\ \nu \in S_j}} \frac{1}{d(\upsilon,\nu)} \\
&= |S_1| \cdot |S_6| + |S_2| \cdot |S_5| + |S_3| \cdot |S_4| + |S_4| \cdot |S_5| + |S_4| \cdot |S_6| + |S_5| \cdot |S_6| \\
&= (q-1)(r-1)(p-1) + (p-1)(r-1)(q-1) + (p-1)(q-1)(r-1) + \\
&\quad (r-1)(q-1) + (r-1)(p-1) + (q-1)(p-1).
\end{aligned}$$

**Case 4.** In the last case, we consider the remaining vertex couples such as $d(\upsilon,\nu)=2$ where $\upsilon,\nu\in S_i$ and $i=1,...,6$ in $\Gamma(\mathbb{Z}_{pqr})$. Then

$$\begin{aligned}\sum_{i=1}^{6}\sum_{\upsilon,\nu\in S_i}\frac{1}{d(\upsilon,\nu)}=&\sum_{i=1}^{6}\frac{|S_i|(|S_i|-1)}{2}\frac{1}{d(\upsilon,\nu)}\Big|_{\upsilon,\nu\in S_i}\\ =&\frac{1}{2}\Bigg[\frac{(q-1)(r-1)\left[(q-1)(r-1)-1\right]}{2}\\ &+\frac{(p-1)(r-1)\left[(p-1)(r-1)-1\right]}{2}\\ &+\frac{(p-1)(q-1)\left[(p-1)(q-1)-1\right]}{2}\\ &+\frac{(r-1)(r-2)}{2}\\ &+\frac{(q-1)(q-2)}{2}+\frac{(p-1)(p-2)}{2}\Bigg].\end{aligned}$$

Evaluating above all four cases, Harary index of $\Gamma(\mathbb{Z}_{pqr})$ is

$$\begin{aligned}HI(\Gamma(\mathbb{Z}_{pqr}))=&\frac{(p-1)(q-1)(r-1)}{3}(p+q+r-3)\\ &+\frac{(p-1)(q-1)(r-1)}{2}\Bigg[\frac{r-1}{p-1}+\frac{q-1}{p-1}+\frac{r-1}{q-1}+\frac{p-1}{q-1}\\ &+\frac{q-1}{r-1}+\frac{p-1}{r-1}\Bigg]\\ &+(p-1)(q-1)(r-1)\left[3+\frac{1}{p-1}+\frac{1}{q-1}+\frac{1}{r-1}\right]\\ &+\frac{(p-1)(q-1)(r-1)}{4}\Bigg[\frac{(q-1)(r-1)-1}{p-1}+\frac{(p-1)(r-1)-1}{q-1}\\ &+\frac{(p-1)(q-1)-1}{r-1}+\frac{r-2}{(p-1)(q-1)}+\frac{q-2}{(p-1)(r-1)}\\ &+\frac{p-2}{(q-1)(r-1)}\Bigg]\\ =&\alpha\beta\gamma\left(3+\frac{\alpha}{3}+\frac{\beta}{3}+\frac{\gamma}{3}\right)+\alpha\beta\left(\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\alpha\beta}{4}+\frac{3}{4}\right)\\ &+\alpha\gamma\left(\frac{\alpha}{2}+\frac{\gamma}{2}+\frac{\alpha\gamma}{4}+\frac{3}{4}\right)+\beta\gamma\left(\frac{\beta}{2}+\frac{\gamma}{2}+\frac{\beta\gamma}{4}+\frac{3}{4}\right)\\ &+\frac{1}{4}\left(\alpha^2+\beta^2+\gamma^2-\alpha-\beta-\gamma\right).\end{aligned}$$

where $\alpha=p-1$, $\beta=q-1$, and $\gamma=r-1$. □

**Theorem 2.7.** *Let $\Gamma(\mathbb{Z}_p\times\mathbb{Z}_q)$, $\Gamma(\mathbb{Z}_p\times\mathbb{Z}_q\times\mathbb{Z}_r)$, $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{pqr})$ be zero-divisor graphs where $p$, $q$, and $r$ are distinct prime numbers. The followings hold:*

*i)* $HI(\Gamma(\mathbb{Z}_p\times\mathbb{Z}_q))=HI(\Gamma(\mathbb{Z}_{pq}))$

*ii)* $HI(\Gamma(\mathbb{Z}_p \times \mathbb{Z}_q \times \mathbb{Z}_r)) = HI(\Gamma(\mathbb{Z}_{pqr}))$

*Proof.* If $R_1 \cong R_2$, then $\Gamma(R_1) \cong \Gamma(R_2)$ [18]. Therefore proof of this theorem is clear. □

## 3. CONCLUSION

Topological indices is very important in chemical graph theory since they are one of these methods of studying graphs and obtaining new applications of them. We computed Harary index of the zero-divisor graphs of Zn this article. Some formulas was found for computing the Harary index of $\mathbb{Z}_n$ for $n \in \{2p, p^2, p^\lambda, pq, p^2q, pqr\}$ where $p, q$ and $r$ are distinct prime numbers and $\lambda > 2$ is an integer number. Finally, some examples were given support to the Theorems in this article.

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# SOME RESULTS ON GEODESICS AND $F$-GEODESICS IN TANGENT BUNDLE WITH $\varphi$-SASAKIAN METRIC OVER PARA-KÄHLER-NORDEN MANIFOLD

ABDERRAHIM ZAGANE *

Abstract. In this paper, we investigate some geodesics and $F$-geodesics problems on tangent bundle and $\varphi$-unit tangent bundle $T_1^{\varphi}M$ equipped with the $\varphi$-Sasaki metric over para-Kähler-Norden manifold $(M^{2m}, \varphi, g)$.

**Keywords**: Para-Kähler-Norden manifold, $\varphi$-Sasaki metric, geodesics, $F$-geodesics.

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## 1. Introduction

On the tangent bundle of a Riemannian manifold one can define natural Riemannian metrics. Their construction makes use of the Levi-Civita connection. Among them, the so called Sasaki metric [15] is of particular interest. That is why the geometry of tangent bundle equipped with this metric has been studied by many authors. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on tangent bundle. Among them, we mention [20, 22]. The geometry of tangent bundle remains a rich area of research in differential geometry to this day.

Geodesics on the tangent bundle has been studied by many authors. In particular the oblique geodesics, non-vertical geodesics and their projections onto the base manifold. Sasaki [16] and Sato [17] gave a complete description of the curves and vector fields along them which generated non-vertical geodesics on the tangent bundle and unit the tangent bundle

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respectively. They proved that the projected curves have constant geodesic curvatures (Frenet curvatures). Nagy [9] generalized these results to the case of locally symmetric base manifold. Yampolsky [18] also did the same studies on the tangent bundle and unit the tangent bundle with the Berger-type deformed Sasaki metric over Kählerian manifold, in the cases of locally symmetric base manifold and of the constant holomorphic curvature base manifold. Also, we refer to [2, 11, 13, 21].

The notion of $F$-planar curves generalizes the magnetic curves and implicitly the geodesics (see [5, 8]), but the notion of $F$-geodesic, which is slightly different from that of $F$-planar curve [1]. Recently, a number of articles on magnetic curves, $F$-planar curves and $F$-geodesics have been published in the mathematical literature (see [3, 4, 10]).

In previous works, [20, 22], we proposed the $\varphi$-Sasaki metric on the tangent bundle over para-Kähler-Norden manifold $(M^{2m}, \varphi, g)$, where we studied the para-Kähler-Norden properties on the tangent bundle and the geometry of $\varphi$-Sasaki metric on tangent bundle respectively. In this paper, after the introduction and generalities, in Section 3, we study the geodesics on $\varphi$-unit tangent bundle with respect to the $\varphi$-Sasaki metric, where we establish necessary and sufficient conditions under which a curve be a geodesic with respect to this metric (Theorem 3.1, Corollary3.1 and Corollary3.2), then we discuss the Frenet curvatures of the projected of the non-vertical geodesic (Theorem 3.2, Theorem 3.3, Corollary3.3 and Theorem 3.4). In section 4, we investigate the $F$-geodesics and $F$-planar curves on tangent bundle with respect to the $\varphi$-Sasaki metric (Theorem 4.1, Theorem 4.2 and Theorem 4.3). In the last section, we study the $F$-geodesics and $F$-planar curves on the $\varphi$-unit tangent bundle with respect to the $\varphi$-Sasaki metric (Theorem 5.1, Theorem 5.2 and Theorem 5.4).

## 2. Generalities on the $\varphi$-Sasaki metric

Let $TM$ be the tangent bundle over an $m$-dimensional Riemannian manifold $(M^m, g)$ and the natural (bundle) projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on $M$ induces a local chart $(\pi^{-1}(U), x^i, \xi^i)_{i=\overline{1,m}}$ on $TM$. Denote by $\Gamma^k_{ij}$ the Christoffel symbols of $g$ and by $\nabla$ the Levi-Civita connection of $g$.

The Levi Civita connection $\nabla$ defines a direct sum decomposition

$$T_{(x,\xi)}TM = V_{(x,\xi)}TM \oplus H_{(x,\xi)}TM.$$

of the tangent bundle to $TM$ at any $(x, \xi) \in TM$ into vertical subspace

$$V_{(x,\xi)}TM \quad = \quad Ker(d\pi_{(x,\xi)}) = \{a^i \frac{\partial}{\partial \xi^i}|_{(x,\xi)},\, a^i \in \mathbb{R}\},$$

and the horizontal subspace

$$H_{(x,\xi)}TM \quad = \quad \{a^i \frac{\partial}{\partial x^i}|_{(x,\xi)} - a^i \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}|_{(x,\xi)},\ a^i \in \mathbb{R}\}.$$

.

Let $Z = Z^i \frac{\partial}{\partial x^i}$ be a local vector field on $M$. The vertical and the horizontal lifts of $Z$ are defined by

$$\begin{aligned} {}^V Z &= Z^i \frac{\partial}{\partial \xi^i}, \\ {}^H Z &= Z^i (\frac{\partial}{\partial x^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}). \end{aligned}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial \xi^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on $TTM$.

An almost product structure $\varphi$ on a manifold $M$ is a $(1,1)$-tensor field on $M$ such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ ($id_M$ is the identity tensor field of type $(1,1)$ on $M$). The pair $(M, \varphi)$ is called an almost product manifold.

An almost para-complex manifold is an almost product manifold $(M, \varphi)$, such that the two eigenbundles $TM^+$ and $TM^-$ associated to the two eigenvalues $+1$ and $-1$ of $\varphi$, respectively, have the same rank. Note that the dimension of an almost para-complex manifold is necessarily even.

An almost para-complex structure $\varphi$ is integrable if the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [X,Y]$$

vanishes identically on $M$. On the other hand, in order that an almost para-complex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection $\nabla$ such that $\nabla \varphi = 0$ [14].

An almost para-complex Norden manifold $(M^{2m}, \varphi, g)$ is a $2m$-dimensional differentiable manifold $M$ with an almost para-complex structure $\varphi$ and a Riemannian metric $g$ such that:

$$g(\varphi X, Y) = g(X, \varphi Y) \quad \Leftrightarrow \quad g(\varphi X, \varphi Y) = g(X,Y),$$

for any vector fields $X$ and $Y$ on $M$, in this case $g$ is called a pure metric with respect to $\varphi$ or para-Norden metric (B-metric)[14].

Also note that

$$G(X,Y) = g(\varphi X, Y), \tag{2.1}$$

is a bilinear, symmetric tensor field of type $(0,2)$ on $(M,\varphi)$ and pure with respect to the paracomplex structure $\varphi$, which is called the twin (or dual) metric of $g$, and it plays a role similar to the Kähler form in Hermitian Geometry. Some properties of twin Norden metric are investigated in [6, 14].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold $(M^{2m},\varphi,g)$ such that $\varphi$ is integrable i.e. $\nabla\varphi = 0$, where $\nabla$ is the Levi-Civita connection of $g$ [12, 14].

It is well known that if $(M^{2m},\varphi,g)$ is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [14].

**Definition 2.1.** [20] *Let $(M^{2m},\varphi,g)$ be a para-Kähler-Norden manifold. On the tangent bundle $TM$, we define a $\varphi$-Sasaki metric noted $g^\varphi$ by*

$$
\begin{aligned}
(1)\quad g^{\varphi}({}^{H}X,{}^{H}Y)_{(x,\xi)} &= g_x(X,Y),\\
(2)\quad g^{\varphi}({}^{H}X,{}^{V}Y)_{(x,\xi)} &= 0,\\
(3)\quad g^{\varphi}({}^{V}X,{}^{V}Y)_{(x,\xi)} &= g_x(X,\varphi Y) = G_x(X,Y),
\end{aligned}
$$

*for any vector fields $X$ and $Y$ on $M$ and $(x,\xi)\in TM$, where $G$ is the twin Norden metric of $g$ defined by* (2.1).

**Theorem 2.1.** [20] *Let $(M^{2m},\varphi,g)$ be a para-Kähler-Norden manifold and $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric $g^\varphi$. If $\nabla$ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of $(M^{2m},\varphi,g)$ (resp $(TM,g^\varphi)$ ), then we have*

$$
\begin{aligned}
(1)\quad (\widetilde{\nabla}_{{}^{H}X}{}^{H}Y)_{(x,\xi)} &= {}^{H}(\nabla_X Y)_{(x,\xi)} - \frac{1}{2}{}^{V}(R_x(X,Y)\xi),\\
(2)\quad (\widetilde{\nabla}_{{}^{H}X}{}^{V}Y)_{(x,\xi)} &= {}^{V}(\nabla_X Y)_{(x,\xi)} + \frac{1}{2}{}^{H}(R_x(\varphi\xi,Y)X),\\
(3)\quad (\widetilde{\nabla}_{{}^{V}X}{}^{H}Y)_{(x,\xi)} &= \frac{1}{2}{}^{H}(R_x(\varphi\xi,X)Y),\\
(4)\quad (\widetilde{\nabla}_{{}^{V}X}{}^{V}Y)_{(x,\xi)} &= 0,
\end{aligned}
$$

*for all vector fields $X$ and $Y$ on $M$ and $(x,\xi)\in TM$, where $R$ denote the curvature tensor of $(M^{2m},\varphi,g)$.*

The $\varphi$-unit tangent sphere bundle over a para-Kähler-Norden manifold $(M^{2m},\varphi,g)$, is the hypersurface

$$T_1^{\varphi}M = \big\{(x,\xi)\in TM,\, g(\xi,\varphi\xi) = 1\big\}.$$

The unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N} = {}^V\xi.$$

The tangential lift ${}^TX$ with respect to $g^{\varphi}$ of a vector $X \in T_xM$ to $(x,\xi) \in T_1^{\varphi}M$ as the tangential projection of the vertical lift of $X$ to $(x,\xi)$ with respect to $\mathcal{N}$, that is

$${}^TX = {}^VX - g^{\varphi}_{(x,\xi)}({}^VX, \mathcal{N}_{(x,\xi)})\mathcal{N}_{(x,\xi)} = {}^VX - g_x(X, \varphi\xi)^V\xi.$$

The tangent space $T_{(x,\xi)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at $(x,\xi)$ is given by

$$T_{(x,\xi)}T_1^{\varphi}M = \{{}^HX + {}^TY \,/\, X \in T_xM, Y \in \xi^{\perp} \subset T_xM\}.$$

where $\xi^{\perp} = \big\{Y \in T_xM,\ g(Y, \varphi\xi) = 0\big\}$, see [22].

**Theorem 2.2.** [22] *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. If $\widehat{\nabla}$ denote the Levi-Civita connection of $\varphi$-Sasaki metric on $T_1^{\varphi}M$, then we have the following formulas.*

$$\begin{aligned}
1.\ \widehat{\nabla}_{{}^HX}{}^HY &= {}^H(\nabla_XY) - \frac{1}{2}{}^T(R(X,Y)\xi),\\
2.\ \widehat{\nabla}_{{}^HX}{}^TY &= {}^T(\nabla_XY) + \frac{1}{2}{}^H(R(\varphi\xi, Y)X),\\
3.\ \widehat{\nabla}_{{}^TX}{}^HY &= \frac{1}{2}{}^H(R(\varphi\xi, X)Y),\\
4.\ \widehat{\nabla}_{{}^TX}{}^TY &= -g(Y, \varphi\xi)^TX,
\end{aligned}$$

*for all vector fields $X$ and $Y$ on $M$, where $\nabla$ is the Levi-Civita connection and $R$ is its curvature tensor of $(M^{2m}, \varphi, g)$.*

## 3. Geodesics on $\varphi$-unit tangent bundle with the $\varphi$-Sasaki metric

Let $\Gamma = (\gamma(t), \xi(t))$ be a naturally parameterized curve on the tangent bundle $TM$ (i.e. $t$ is an arc length parameter on $\Gamma$), where $\gamma$ is a curve on $M$ and $\xi$ is a vector field along this curve. Denote $\gamma'_t = \frac{d\gamma}{dt}$, $\gamma''_t = \nabla_{\gamma'_t}\gamma'_t$, $\xi'_t = \nabla_{\gamma'_t}\xi$, $\xi''_t = \nabla_{\gamma'_t}\xi'_t$ and $\Gamma'_t = \frac{d\Gamma}{dt}$. Then

$$\Gamma'_t = {}^H\gamma'_t + {}^V\xi'_t. \tag{3.2}$$

**Lemma 3.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold, $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^{\varphi}M$. Then we have*

$$\Gamma'_t = {}^H\gamma'_t + {}^T\xi'_t, \tag{3.3}$$

*Proof.* Using (3.2), we have

$$\Gamma'_t = {}^H\gamma'_t + {}^V\xi'_t = {}^H\gamma'_t + {}^T\xi'_t + g(\xi'_t, \varphi\xi)^V\xi.$$

Since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$ then $g(\xi, \varphi\xi) = 1$, on the other hand

$$0 = \gamma'_t g(\xi, \varphi\xi) = 2g(\xi'_t, \varphi\xi),$$

i.e.

$$g(\xi'_t, \varphi\xi) = 0. \tag{3.4}$$

Hence, the proof of the lemma is completed. □

From (3.3), we have

$$1 = |\gamma'_t|^2 + g(\xi'_t, \varphi\xi'_t), \tag{3.5}$$

where $|\,.\,|$ mean the norm of vectors with respect to the $(M^{2m}, \varphi, g)$.

**Theorem 3.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold, $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^{\varphi}M$. Then $\Gamma$ is a geodesic on $T_1^{\varphi}M$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = 0 \end{cases} \tag{3.6}$$

*Moreover,*

$$\begin{cases} |\gamma'_t| = 1 \\ |\xi'_t| = \kappa = const \end{cases} \tag{3.7}$$

*i.e. $t$ is an arc length parameter on $\gamma$.*

*Proof.* Using formula (3.3)and Theorem 2.2, we compute the derivative $\widehat{\nabla}_{\Gamma'_t}\Gamma'_t$.

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \widehat{\nabla}_{({}^H\gamma'_t + {}^T\xi'_t)}({}^H\gamma'_t + {}^T\xi'_t) \\ &= \widehat{\nabla}_{{}^H\gamma'_t}{}^H\gamma'_t + \widehat{\nabla}_{{}^H\gamma'_t}{}^T\xi'_t + \widehat{\nabla}_{{}^T\xi'_t}{}^H\gamma'_t + \widehat{\nabla}_{{}^T\xi'_t}{}^T\xi'_t \\ &= {}^H\gamma''_t + {}^H(R(\varphi\xi, \xi'_t)\gamma'_t) + {}^T\xi''_t. \end{aligned}$$

If we put $\widehat{\nabla}_{\Gamma'_t}\Gamma'_t$ equal to zero, we find (3.6).

From (3.4) we have, $0 = \gamma'_t g(\xi'_t, \varphi\xi) = g(\xi''_t, \varphi\xi) + g(\xi'_t, \varphi\xi'_t)$ then,

$$g(\xi'_t, \varphi\xi'_t) = -g(\xi''_t, \varphi\xi),$$

using the second equation of the formula (3.6), we get $g(\xi'_t, \varphi\xi'_t) = 0$, then, by (3.5), we find,

$$|\gamma'_t| = 1.$$

On the other hand, as well

$$\gamma'_t|\xi'_t|^2 = \gamma'_t g(\xi'_t, \xi'_t) = 2g(\xi''_t, \xi'_t) = 0,$$

then $|\xi'_t| = \kappa = const$, □

A curve $\Gamma = (\gamma(t), \xi(t))$ on $TM$ is said to be a horizontal lift of the curve $\gamma$ on $M$ if and only if $\xi'_t = 0$ [19]. In general, the horizontal lift $\Gamma = (\gamma(t), \xi(t))$ of the curve $\gamma$ on $M$ does not belong to $T_1^{\varphi}M$, we have $\xi'_t = 0$, then $0 = 2g(\xi'_t, \varphi\xi) = \gamma'_t g(\xi, \varphi\xi)$, hence $g(\xi, \varphi\xi) = const \neq 1$ (in general). Then, we have the following corollary.

**Corollary 3.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of $\gamma$ and $\Gamma \in T_1^{\varphi}M$, then $\Gamma$ is a geodesic on $T_1^{\varphi}M$ if and only if $\gamma$ is a geodesic on $M$.*

The curve $\Gamma = (\gamma(t), \gamma'_t(t))$ is called a natural lift of the curve $\gamma$ on $TM$ [19]. Likewise as for the natural lift, in the general case it does not belong to $T_1^{\varphi}M$. If $\gamma$ is a geodesic on $M$. we have $\gamma''_t = 0$, then $0 = 2g(\gamma''_t, \varphi\gamma'_t) = \gamma'_t g(\gamma'_t, \varphi\gamma'_t)$, hence $g(\gamma'_t, \varphi\gamma'_t) = const \neq 1$ (in general). Then, we get the following corollary.

**Corollary 3.2.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. If $\Gamma = (\gamma(t), \gamma'_t(t))$ is a natural lift of $\gamma$ and $\Gamma \in T_1^{\varphi}M$, then $\Gamma$ is a geodesic on $T_1^{\varphi}M$ if and only if $\gamma$ is a geodesic on $M$.*

**Remark 3.1.** *As a reminder, note that locally we have:*

$$\gamma''_t = \sum_{l=1}^{2m}\Big(\frac{d^2\gamma^l}{dt^2} + \sum_{i,j=1}^{2m}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^l\Big)\frac{\partial}{\partial x^l}, \tag{3.8}$$

*and*

$$\xi'_t = \sum_{l=1}^{2m}\Big(\frac{d\xi^l}{dt} + \sum_{i,j=1}^{2m}\frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^l\Big)\frac{\partial}{\partial x^l}. \tag{3.9}$$

**Example 3.1.** *Let $(\mathbb{R}^2, \varphi, g)$ be a para-Kähler-Norden manifold such that*

$$g = e^{2x}dx^2 + e^{2y}dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

*The non-null Christoffel symbols of the Riemannian connection are:*

$$\Gamma_{11}^{1} = \Gamma_{22}^{2} = 1.$$

1) *Let $\gamma$ be a curve on $\mathbb{R}^2$, such that $\gamma(t) = (x(t), y(t))$, from* (3.8)*, $\gamma$ is a geodesic if and only if $\gamma_t'' = 0$ or equivalently $\gamma$ satisfies the system of differential equations,*

$$\frac{d^2\gamma^l}{dt^2} + \sum_{i,j=1}^{2} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^{l} = 0 \Leftrightarrow \begin{cases} \dfrac{d^2x}{dt^2} + (\dfrac{dx}{dt})^2 = 0 \\ \\ \dfrac{d^2y}{dt^2} + (\dfrac{dy}{dt})^2 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x(t) = \ln(c_1 t + c_2) \\ \\ y(t) = \ln(c_3 t + c_4) \end{cases}$$

*where $c_1, c_2, c_3$ and $c_4$ are real constants, hence*

$$\gamma(t) = (\ln(c_1 t + c_2), \ln(c_3 t + c_4)),\ \gamma_t'(t) = \frac{c_1}{c_1 t + c_2}\frac{\partial}{\partial x} + \frac{c_3}{c_3 t + c_4}\frac{\partial}{\partial y}.$$

*On the other hand we have*

$$g(\gamma_t', \varphi\gamma_t') = 1 \Leftrightarrow c_1 = \pm\sqrt{1 + c_3^2},$$

*become $\Gamma_1 = (\gamma(t), \gamma_t'(t)) \in T_1^{\varphi}\mathbb{R}^2$. Hence from Corollary* 3.2*, the curve $\Gamma_1$ is a geodesic on $T_1^{\varphi}\mathbb{R}^2$.*

2) *If $\Gamma_2 = (\gamma(t), \xi(t))$ is horizontal lift of $\gamma$, such that $\xi(t) = (u(t), v(t))$, from* (3.9)*, we have,*

$$\frac{d\xi^l}{dt} + \sum_{i,j=1}^{2} \frac{dx^j}{dt}\xi^i\Gamma_{ij}^{l} = 0 \Leftrightarrow \begin{cases} \dfrac{du}{dt} + \dfrac{dx}{dt}u = 0 \\ \dfrac{dv}{dt} + \dfrac{dy}{dt}v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = \dfrac{c_5}{c_1 t + c_2} \\ v(t) = \dfrac{c_6}{c_3 t + c_4} \end{cases}$$

*where $c_5$ and $c_6$ are real constants, hence*

$$\xi(t) = \frac{c_5}{c_1 t + c_2}\frac{\partial}{\partial x} + \frac{c_6}{c_3 t + c_4}\frac{\partial}{\partial y},$$

*but when*

$$g(\xi, \varphi\xi) = 1 \Leftrightarrow c_5 = \pm\sqrt{1 + c_6^2},$$

*become $\Gamma_2 = (\gamma(t), \xi(t)) \in T_1^{\varphi}\mathbb{R}^2$. Hence from Corollary* 3.1*, the curve $\Gamma_2$ is a geodesic on $T_1^{\varphi}\mathbb{R}^2$.*

Let $\Gamma$ be a curve on $T_1^{\varphi}M$, the cure $\pi \circ \Gamma$ is called the projection (projected curve) of the curve $\Gamma$ on $M$, where $\pi : T_1^{\varphi}M \to M$ is a bundle projection.

**Theorem 3.2.** *Let $(M^{2m}, \varphi, g)$ be a locally symmetric para-Kähler-Norden manifold, $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a geodesic on $T_1^{\varphi}M$, then all Frenet curvatures of the projected curve $\pi \circ \Gamma$ are constants.*

*Proof.* Using the first equation of (3.7), we have $|\gamma_t'| = 1 = const$.

On the other hand we have

$$\begin{aligned}\gamma_t''' &= (\nabla_{\gamma_t'} R)(\xi_t', \varphi\xi)\gamma_t' + R(\xi_t'', \varphi\xi)\gamma_t' + R(\xi_t', \varphi\xi_t')\gamma_t' + R(\xi_t', \varphi\xi)\gamma_t'' \\ &= R(\xi_t', \varphi\xi)\gamma_t''.\end{aligned}$$

Since

$$\gamma_t' g(\gamma_t'', \gamma_t'') = 2g(\gamma_t''', \gamma_t'') = 2g(R(\xi_t', \varphi\xi)\gamma_t'', \gamma_t'') = 0,$$

hence, $|\gamma_t''| = const$.

Continuing the process, by recurrence we obtain the following

$$\gamma_t^{(p+1)} = R(\xi_t', \varphi\xi)\gamma_t^{(p)}, \quad p \geq 1 \tag{3.10}$$

and

$$\gamma_t' g(\gamma_t^{(p)}, \gamma_t^{(p)}) = 2g(\gamma_t^{(p+1)}, \gamma_t^{(p)}) = 2g(R(\xi_t', \varphi\xi)\gamma_t^{(p)}, \gamma_t^{(p)}) = 0.$$

Thus, we get

$$|\gamma_t^{(p)}| = const, \quad p \geq 1. \tag{3.11}$$

Let $\nu_1 = \gamma_t'$ be the first vector in the Frenet frame $\nu_1, \ldots, \nu_{2m-1}$ along $\gamma$ and let $k_1, \ldots, k_{2m-1}$ the Frenet curvatures of $\gamma$. Then from the Frenet formulas

$$\begin{cases} (\nu_1)_t' &= k_1\nu_2 \\ (\nu_i)_t' &= -k_{i-1}\nu_{i-1} + k_i\nu_{i+1}, \quad 2 \leq i \leq 2m-2 \\ (\nu_{2m-1})_t' &= -k_{2m-2}\nu_{2m-2} \end{cases}$$

we obtain

$$\gamma_t'' = (\nu_1)_t' = k_1\nu_2. \tag{3.12}$$

Now (3.11) implies $k_1 =$ const. Next, in a similar way, we have

$$\gamma_t''' = k_1(\nu_2)_t' = -k_1^2\nu_1 + k_1k_2\nu_3. \tag{3.13}$$

and again (3.11) implies $k_2 =$ const.

By continuing the process, we finish the proof. □

**Proposition 3.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. If $\Gamma = (\gamma(t), \xi(t))$ is a curve on $T_1^{\varphi}M$, then we have*

(1) *$\Phi = (\gamma(t), \varphi\xi(t))$ is a curve on $T_1^{\varphi}M$.*

(2) *$\Phi$ is a geodesic on $T_1^{\varphi}M$ if and only if $\Gamma$ is a geodesic on $T_1^{\varphi}M$.*

*Proof.* (1) We put $\mu(t) = \varphi\xi(t)$, since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$, then $g(\xi, \varphi\xi) = 1$. On the other hand, $g(\mu, \varphi\mu) = g(\varphi\xi, \varphi(\varphi\xi)) = g(\varphi\xi, \xi) = 1$ i.e.

$$\Phi(t) = (\gamma(t), \mu(t)) \in T_1^{\varphi}M.$$

(2) In a similar way proof of (3.6), and using $\mu_t' = \varphi\xi_t'$ and $\mu_t'' = \varphi\xi_t''$, we have

$$\begin{aligned} \widehat{\nabla}_{\Phi_t'}\Phi_t' &= {}^{H}(\gamma_t'' + R(\varphi\mu, \mu_t')\gamma_t') + {}^{T}\mu_t'' \\ &= {}^{H}\big(\gamma_t'' + R(\xi, \varphi\xi_t')\gamma_t'\big) + {}^{T}(\varphi\xi_t''). \end{aligned}$$

Since the Riemannian curvature tensor is pure, we get

$$\widehat{\nabla}_{\Phi_t'}\Phi_t' = {}^{H}\big(\gamma_t'' + R(\varphi\xi, \xi_t')\gamma_t'\big) + {}^{T}(\varphi\xi_t''),$$

hence,

$$\begin{aligned} \widehat{\nabla}_{\Phi_t'}\Phi_t' = 0 \quad &\Leftrightarrow \quad \begin{cases} \gamma_t'' = -R(\varphi\xi, \xi_t')\gamma_t' \\ \varphi\xi_t'' = 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} \gamma_t'' = R(\xi_t', \varphi\xi)\gamma_t' \\ \xi_t'' = 0 \end{cases} \\ &\Leftrightarrow \quad \widehat{\nabla}_{\Gamma_t'}\Gamma_t' = 0. \end{aligned}$$

□

From Theorem 3.2 and Proposition 3.1, we have the following theorem

**Theorem 3.3.** *Let $(M^{2m}, \varphi, g)$ be a locally symmetric para-Kähler-Norden manifold, $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a geodesic on $T_1^{\varphi}M$ then, all Frenet curvatures of the projected curve $\pi \circ \Phi$ are constants, where $\Phi = (\gamma(t), \varphi\xi(t))$.*

Now we study the geodesics on $\varphi$-unit tangent bundle with the $\varphi$-Sasaki metric over para-Kähler-Norden manifold of constant sectional curvature. From Theorem 3.1, we have the following

**Corollary 3.3.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold of constant curvature $c \neq 0$, $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^{\varphi}M$. Then $\Gamma$ is a geodesic on $T_1^{\varphi}M$ if and only if*

$$\left\{ \begin{array}{lcl} \gamma_t'' & = & cg(\varphi\xi, \gamma_t')\xi_t' - cg(\xi_t', \gamma_t')\varphi\xi \\ \xi_t'' & = & 0 \end{array} \right. \tag{3.14}$$

**Theorem 3.4.** *Let $(\mathbb{R}^{2m}, \varphi, <,>)$ be a para-Kähler-Norden real euclidean space , $T_1^{\varphi}\mathbb{R}^{2m}$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. Any geodesics $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}\mathbb{R}^{2m}$ is the following form*

$$\left\{ \begin{array}{lcl} \gamma^i(t) & = & a^i t + b^i \\ \xi^i(t) & = & c^i t + d^i \end{array} \right. \tag{3.15}$$

*where $\gamma(t) = (\gamma^1(t), \ldots, \gamma^{2m}(t)), \xi(t) = (\xi^1(t), \ldots, \xi^{2m-1}(t))$ and $a^i, b^i, c^i, d^i$ are real constants.*

## 4. $F$-geodesics on tangent bundle with the $\varphi$-Sasaki metric

Let $(M^m, g)$ be an Riemannian manifold and $F$ be a $(1,1)$-tensor field on $(M^m, g)$. A curve $\gamma$ on $M$ is called $F$-planar if its speed remains, under parallel translation along the curve $\gamma$, in the distribution generated by the vector $\gamma_t'$ and $F\gamma_t'$ along $\gamma$. This is equivalent to the fact that the tangent vector $\gamma_t'$ satisfies

$$\gamma_t'' = \varrho_1(t)\gamma_t' + \varrho_2 F\gamma_t', \tag{4.16}$$

where $\varrho_1$ and $\varrho_2$ are some functions of the parameter $t$, see [5, 7, 8].

We say that a curve $\gamma$ on $M$ is an $F$-geodesic if $\gamma$ satisfies:

$$\gamma_t'' = F\gamma_t', \tag{4.17}$$

One can see that an $F$-geodesic is an $F$-planar curve, but in general an $F$-planar curve is not always an $F$-geodesic, the $F$-geodesic generalize the geodesics, see [1, 3].

Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\varphi$-Sasaki metric on tangent bundle $TM$, given in the Theorem 2.1.

**Theorem 4.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold, $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric and $F$ be a $(1,1)$-tensor field on $M$. A curve $\Gamma = (\gamma(t), \xi(t))$ on $TM$ is an ${}^{H}F$-planar with respect to $\widetilde{\nabla}$ if and only if the*

$$\left\{ \begin{array}{l} \gamma_t'' = \varrho_1\gamma_t' + \varrho_2 F\gamma_t' + R(\xi_t', \varphi\xi)\gamma_t' \\ \xi_t'' = \varrho_1\xi_t' + \varrho_2 F\xi_t' \end{array} \right.$$

*Proof.* Using Theorem 2.1 and (3.2), we find

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= \widetilde{\nabla}_{({}^H\gamma'_t + {}^V\xi'_t)}({}^H\gamma'_t + {}^V\xi'_t) \\ &= \widetilde{\nabla}_{{}^H\gamma'_t}{}^H\gamma'_t + \widetilde{\nabla}_{{}^H\gamma'_t}{}^V\xi'_t + \widetilde{\nabla}_{{}^V\xi'_t}{}^H\gamma'_t + \widetilde{\nabla}_{{}^V\xi'_t}{}^V\xi'_t \\ &= {}^H(\gamma''_t + R(\varphi\xi, \xi'_t)\gamma'_t) + {}^V\xi''_t \qquad (4.18) \end{aligned}$$

On the other hand,

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1\Gamma'_t + \varrho_2{}^HF\Gamma'_t \\ &= \varrho_1({}^H\gamma'_t + {}^V\xi'_t) + \varrho_2{}^HF({}^H\gamma'_t + {}^V\xi'_t) \\ &= \varrho_1{}^H\gamma'_t + \varrho_2{}^HF{}^H\gamma'_t + \varrho_1{}^V\xi'_t + \varrho_2{}^HF{}^V\xi'_t \\ &= {}^H(\varrho_1\gamma'_t + \varrho_2F\gamma'_t) + {}^V(\varrho_1\xi'_t + \varrho_2F\xi'_t). \qquad (4.19) \end{aligned}$$

From (4.18) and (4.19), the result immediately follows. □

**Corollary 4.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $TM$ is an ${}^H\varphi$-planar with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t \end{cases}$$

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 4.1, we obtain the following result.

**Theorem 4.2.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold, $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric and $F$ be a $(1,1)$-tensor field on $M$. A curve $\Gamma = (\gamma(t), \xi(t))$ on $TM$ is an ${}^HF$-geodesic with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = F\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = F\xi'_t \end{cases}$$

**Corollary 4.2.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $TM$ is an ${}^H\varphi$-geodesic with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varphi\xi'_t \end{cases}$$

**Theorem 4.3.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold, $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric and $F$ be a $(1,1)$-tensor field on $M$. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of a curve $\gamma$, then $\Gamma$ is an ${}^{H}F$-planar curve (resp. ${}^{H}F$-geodesic) if and only if $\gamma$ is an $F$-planar curve (resp. $F$-geodesic).*

*Proof.* Let $\gamma$ be an $F$-planar with respect to $\nabla$ on $M$, i.e. $\gamma$ satisfies

$$\gamma''_t = \varrho_1 \gamma'_t + \varrho_2 F \gamma'_t,$$

where $\varrho_1$ and $\varrho_2$ are some functions of the parameter $t$. Since $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of a curve $\gamma$ then $\xi'_t = 0$ and from (3.2), we have $\Gamma'_t = {}^{H}\gamma'_t$. Using (4.18), we get,

$$\begin{aligned} \widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^{H}\gamma''_t \\ &= {}^{H}(\varrho_1 \gamma'_t + \varrho_2 F \gamma'_t) \\ &= \varrho_1 {}^{H}\gamma'_t + \varrho_2 {}^{H}F {}^{H}\gamma'_t \\ &= \varrho_1 \Gamma'_t + \varrho_2 {}^{H}F \Gamma'_t. \end{aligned}$$

i.e. $\Gamma$ be an ${}^{H}F$-planar with respect to $\widetilde{\nabla}$. In the case of $\varrho_1 = 0$ and $\varrho_2 = 1$, we get that $\Gamma$ is an ${}^{H}F$-geodesic if and only $\gamma$ is an $F$-geodesic. □

**Corollary 4.3.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold, $TM$ its tangent bundle equipped with the $\varphi$-Sasaki metric. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of a curve $\gamma$, then $\Gamma$ is an ${}^{H}\varphi$-planar curve (resp., ${}^{H}\varphi$-geodesic) if and only if $\gamma$ is an $\varphi$-planar curve (resp., $\varphi$-geodesic).*

**Example 4.1.** *Let $(\mathbb{R}^2, \varphi, g)$ be a para-Kähler-Norden manifold such that*

$$g = dx^2 + dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

*Let $\Gamma = (\gamma(t), \xi(t))$ such that $\gamma(t) = (x(t), y(t))$ and $\xi(t) = (u(t), v(t))$*

1) *From the Corollary 4.2, $\Gamma$ is an ${}^{H}\varphi$-geodesic if and only if the*

$$\begin{cases} \gamma''_t = \varphi \gamma'_t \\ \xi''_t = \varphi \xi'_t \end{cases} \Leftrightarrow \begin{cases} x'' = x' \\ y'' = -y' \\ u'' = u' \\ v'' = -v' \end{cases} \Leftrightarrow \begin{cases} x(t) = a_1 e^t + a_2 \\ y(t) = a_3 e^{-t} + a_4 \\ u(t) = b_1 e^t + b_2 \\ v(t) = b_3 e^{-t} + b_4 \end{cases}$$

*where* $a_i$ *and* $b_i$, $i=\overline{1,4}$ *are real constants.*

2) *From the Corollary 4.1,* $\Gamma$ *is an* ${}^H\varphi$*-planar if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t \end{cases} \Leftrightarrow \begin{cases} x'' = (\varrho_1+\varrho_2)x' \\ y'' = (\varrho_1-\varrho_2)y' \\ u'' = (\varrho_1+\varrho_2)u' \\ v'' = (\varrho_1-\varrho_2)v' \end{cases}$$

$$\Leftrightarrow \begin{cases} x(t) = \pm\int(e^{\int(\varrho_1+\varrho_2)dt})dt \\ y(t) = \pm\int(e^{\int(\varrho_1-\varrho_2)dt})dt \\ u(t) = \pm\int(e^{\int(\varrho_1+\varrho_2)dt})dt \\ v(t) = \pm\int(e^{\int(\varrho_1-\varrho_2)dt})dt \end{cases}$$

*For example:* $\varrho_1(t) = \dfrac{1}{t+1}$ *and* $\varrho_2(t) = \dfrac{1}{t-1}$, *we find*

$$\begin{cases} x(t) = c_1t^3 - 3c_1t + c_2 \\ y(t) = c_3\ln(t-1)^2 + c_3t + c_4 \\ u(t) = d_1t^3 - 3d_1t + d_2 \\ v(t) = d_3\ln(t-1)^2 + d_3t + d_4 \end{cases}$$

*where* $c_i$ *and* $d_i$, $i=\overline{1,4}$ *are real constants.*

**Example 4.2.** *Let* $(\mathbb{R}^2,\varphi,g)$ *be a para-Kähler-Norden manifold such that*

$$g = e^{2x}dx^2 + e^{2y}dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \ a,b\in\mathbb{R}^*.$$

*The non-null Christoffel symbols of the Riemannian connection are:*

$$\Gamma^1_{11} = \Gamma^2_{22} = 1.$$

*Let* $\Gamma = (\gamma(t),\xi(t))$ *be a horizontal lift of a curve* $\gamma$, *such that* $\gamma(t) = (x(t),y(t))$ *and* $\xi(t) = (u(t),v(t))$ *then* $\xi'_t = 0$, *from* (3.9) *we have,*

$$\frac{d\xi^h}{dt} + \sum_{i,j=1}^{2}\frac{d\gamma^j}{dt}\xi^i\Gamma^h_{ij} = 0 \Leftrightarrow \begin{cases} u' + x'u = 0 \\ v' + y'v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = \dfrac{\lambda_1}{e^{x(t)}} \\ v(t) = \dfrac{\lambda_2}{e^{y(t)}} \end{cases}$$

*where* $\lambda_1, \lambda_2$ *are real constants.*

$\gamma$ *is an* $F$*-geodesic if and only if* $\gamma''_t = F\gamma'_t$, *from* (3.8) *we have*

$$\begin{cases} x'' + (x')^2 = ax' \\ y'' + (y')^2 = by' \end{cases} \Leftrightarrow \begin{cases} x(t) = \ln(\frac{\mu_1}{a}e^{at} + \mu_2) \\ y(t) = \ln(\frac{\mu_3}{b}e^{bt} + \mu_4) \end{cases}$$

*where* $\mu_1, \mu_2, \mu_3, \mu_4$ *are real constants.*

*The horizontal lift* $\Gamma = (\ln(\frac{\mu_1}{a}e^{at} + \mu_2), \ln(\frac{\mu_3}{b}e^{bt} + \mu_4), \frac{\lambda_1}{\frac{\mu_1}{a}e^{at} + \mu_2}, \frac{\lambda_2}{\frac{\mu_3}{b}e^{bt} + \mu_4})$ *is an* ${}^{H}F$*-geodesic on* $T\mathbb{R}^2$.

$\gamma$ *is an* $F$*-planar if and only if* $\gamma_t'' = \varrho_1\gamma_t' + \varrho_2 F\gamma_t'$*, where* $\varrho_1$ *and* $\varrho_2$ *are some functions of the parameter* $t$*, from* (3.8) *we have*

$$\begin{cases} x'' + (x')^2 = (\varrho_1 + a\varrho_2)x' \\ y'' + (y')^2 = (\varrho_2 + b\varrho_2)y' \end{cases} \Leftrightarrow \begin{cases} x(t) = \ln(\pm\int(e^{\int(\varrho_1 + a\varrho_2)dt})dt) \\ y(t) = \ln(\pm\int(e^{\int(\varrho_1 + b\varrho_2)dt})dt) \end{cases}$$

*For example: If* $\varrho_1(t) = \dfrac{-1}{t}$ *and* $\varrho_2(t) = \dfrac{1}{t}$*, we find*

$$\begin{cases} x(t) = \ln(\frac{\alpha_1}{a}t^a + \alpha_2) \\ y(t) = \ln(\frac{\alpha_3}{b}t^b + \alpha_4) \\ u(t) = \dfrac{\lambda_1}{\frac{\alpha_1}{a}t^a + \alpha_2} \\ v(t) = \dfrac{\lambda_2}{\frac{\alpha_3}{b}t^b + \alpha_4} \end{cases}$$

*where* $\alpha_i$*,* $i = \overline{1,4}$ *are real constants, then* $\Gamma = (x(t), y(t), u(t), v(t))$ *is an* ${}^{H}\varphi$*-planar on* $T\mathbb{R}^2$.

## 5. $F$-geodesics on $\varphi$-unit tangent bundle with the $\varphi$-Sasaki metric

Let $\widehat{\nabla}$ be the Levi-Civita connection of $\varphi$-Sasaki metric on $\varphi$-unit tangent bundle $T_1^{\varphi}M$, given in the Theorem 2.2.

**Theorem 5.1.** *Let* $(M^{2m}, \varphi, g)$ *be a para-Kähler-Norden manifold and* $T_1^{\varphi}M$ *its* $\varphi$*-unit tangent bundle equipped with the* $\varphi$*-Sasaki metric and* $F$ *be a* $(1,1)$*-tensor field on* $M$*. A curve* $\Gamma = (\gamma(t), \xi(t))$ *on* $T_1^{\varphi}M$ *is an* ${}^{H}F$*-planar with respect to* $\widehat{\nabla}$ *if and only if the*

$$\begin{cases} \gamma_t'' = \varrho_1\gamma_t' + \varrho_2 F\gamma_t' + R(\xi_t', \varphi\xi)\gamma_t' \\ \xi_t'' = \varrho_1\xi_t' + \varrho_2 F\xi_t' \end{cases}$$

*Proof.* From the proof of Theorem 3.1, we find

$$\widehat{\nabla}_{\Gamma_t'}\Gamma_t' = {}^{H}(\gamma_t'' + R(\varphi\xi, \xi_t')\gamma_t') + {}^{T}\xi_t''. \tag{5.20}$$

On the other hand,

$$\begin{aligned} \widehat{\nabla}_{\Gamma_t'}\Gamma_t' &= \varrho_1\Gamma_t' + \varrho_2{}^{H}F\Gamma_t' \\ &= \varrho_1({}^{H}\gamma_t' + {}^{T}\xi_t') + \varrho_2{}^{H}F({}^{H}\gamma_t' + {}^{T}\xi_t'). \end{aligned}$$

From (3.4), we have ${}^{T}\xi'_t = {}^{V}\xi'_t$, then

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1{}^{H}\gamma'_t + \varrho_2{}^{H}F^{H}\gamma'_t + \varrho_1{}^{V}\xi'_t + \varrho_2{}^{H}F^{V}\xi'_t \\ &= {}^{H}(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^{V}(\varrho_1\xi'_t + \varrho_2 F\xi'_t) \\ &= {}^{H}(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^{T}(\varrho_1\xi'_t + \varrho_2 F\xi'_t) \end{aligned} \tag{5.21}$$

From (5.20) and (5.21), the result immediately follows. □

**Corollary 5.1.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-planar with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t \end{cases}$$

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 5.1, we obtain the following result.

**Theorem 5.2.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $F$ be a $(1,1)$-tensor field on $M$. A curve $\Gamma = (\gamma(t), \xi(t))$ on $TM$ is an ${}^{H}F$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = F\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = F\xi'_t \end{cases}$$

**Corollary 5.2.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varphi\xi'_t \end{cases}$$

**Theorem 5.3.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}(R(\xi'_t, \varphi\xi))$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = 2R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = R(\xi'_t, \varphi\xi)\xi'_t \end{cases}$$

**Corollary 5.3.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold of constant sectional curvature $c \neq 0$ and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^H(R(\xi'_t, \varphi\xi))$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = 2c(g(\varphi\xi, \gamma'_t)\xi'_t - g(\xi'_t, \gamma'_t)\varphi\xi) \\ \xi''_t = -cg(\xi'_t, \xi'_t)\varphi\xi \end{cases}$$

**Theorem 5.4.** *Let $(M^{2m}, \varphi, g)$ be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its $\varphi$-unit tangent bundle equipped with the $\varphi$-Sasaki metric and $F$ be a $(1,1)$-tensor field on $M$. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of $\gamma$ and $\Gamma \in T_1^{\varphi}M$, then $\Gamma$ is an ${}^HF$-planar curve (resp., ${}^HF$-geodesic) if and only if $\gamma$ is an $F$-planar curve (resp., $F$-geodesic).*

*Proof.* Let $\gamma$ be an $F$-planar with respect to $\nabla$ on $M$, i.e. $\gamma$ satisfies

$$\gamma''_t = \varrho_1\gamma'_t + \varrho_2 F\gamma'_t,$$

where $\varrho_1$ and $\varrho_2$ are some functions of the parameter $t$. Since $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of a curve $\gamma$ then $\xi'_t = 0$ and from (5.20), we have,

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^H\gamma''_t \\ &= {}^H(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) \\ &= \varrho_1 {}^H\gamma'_t + \varrho_2 {}^HF {}^H\gamma'_t \\ &= \varrho_1\Gamma'_t + \varrho_2 {}^HF\Gamma'_t. \end{aligned}$$

i.e. $\Gamma$ be an ${}^HF$-planar with respect to $\widehat{\nabla}$. In the case of $\varrho_1 = 0$ and $\varrho_2 = 1$, we get that $\Gamma$ is an ${}^HF$-geodesic if and only $\gamma$ is an $F$-geodesic. □

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# A NOTE ON POINTWISE SEMI-SLANT CONFORMAL SUBMERSIONS

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Abstract. As a generalization of pointwise slant submersions, we investigate pointwise semi-slant conformal submersions from almost Hermitian manifolds onto Riemannian manifolds in the present work. With the investigation of the distributions' leaves geometry, we explore integrability conditions for distributions. In this study, we additionally explore the notion of pluriharminicty.

**Keywords**: Almost Hermitian manifolds, Riemannian submersion, Pointwise semi-slant conformal submersions, Conformal submersions.

**2020 Mathematics Subject Classification**: 53C55, 53C22, 32Q15.

## 1. Introduction

The theory of submersions and immersions had originally been developed and proposed by B. O'Neill [27] and A. Gray [14]. They studied the geometrical properties of Riemannian manifolds and discovered certain Riemannian equations for them. When discussing the characteristics between differentiable structures in differential geometry, submersions theory becomes an intriguing subject. Mathematics and physics identically study Riemannian submersions because of their many applications, most prominent among them being Yang-Mills and Kaluza-Klein theories.(see [9], [42], [25], [21]). In 1976, B. Watson [41] glanced into Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. Afterwards, B. Sahin [34] investigated the geometry of Riemannian submersions and geometric properties. He defined anti-invariant Riemannian submersions onto Riemannian manifolds

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by using an almost Hermitian manifold. He establishes that their vertical distribution is anti-invariant under the almost complex structure of the total manifold. Numerous writers examined and developed this work by examining anti-invariant submersions [4], [34], semi-invariant submersions [35], slant submersions [12], [36], and semi-slant submersions [18], [28], among other topics. Tastan, Sahin, and Yanan [40] defined and examined hemi-slant submersions from almost Hermitian manifolds as a generalization case of semi-invariant and semi-slant submersions.

In this contribution, T. W. Lee and B. Sahin [24] extended their concept of slant submersion a step further by expanding it to include pointwise slant submersions from almost Hermitian manifolds onto Riemannian manifolds. In doing so, they discovered a technique for illustrating examples for this kind of submersions. Additionally, they established characterizations for pointwise slant submersions. B. Fuglede [15] and T. Ishihara [22] introduced the concept of conformal submersion as a generalisation of Riemannian submersions and talked about some of their geometric characteristics. It is clear that conformal submersion with dilation $\lambda = 1$ is a Riemannian submersion. Gudmundsson and Wood [17] investigated conformal holomorphic submersion as a generalisation of holomorphic submersion. The necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions have been established. Later on, conformal anti-invariant submersions , [37], [31], conformal semi-invariant submersions [5], conformal slant submersions [3], and conformal semi-slant submersions [2] have been studied and defined by Akyol and Sahin. Conformal hemi-slant submersions [38], [39], conformal bi-slant submersions [6], and quasi bi-slant conformal submersions [7] have all been studied geometrically recently, and several decomposition theorems have been covered. They also extended the notion of pluriharmonicity to almost contact metric manifolds, from almost Hermitian manifolds.

In this paper, we investigate pointwise semi-slant conformal submersions from Almost Hermtian manifold onto a Riemannian manifold. The structure of the paper is as follows. Section 2 introduces almost contact manifolds, precisely Kaehler manifold with the properties required for this study. In the third section of our paper, we define pointwise semi-slant conformal submersion and report a few intriguing findings. The prerequisites for distribution integrability and the totally geodesicness of its leaves were covered in detail in Section 4. Lastly, the notion of $J$-pluriharmonicity is discussed at the end of the study.

**Note:** In this paper, we will use abbreviation as follows:

Pointwise semi-slant conformal submersion- $\mathcal{PWSSCS}$

Almost Hermitian manifold- AHM

Kaehler manifold- KM

Riemannian manifold- RM

Horizontal conformal submersion -HCS

## 2. Preliminaries

We shall provide a few fundamental ideas and consequences that are highly productive for our paper.

**Definition 2.1.** [8] *Let* $\Pi$ *be a Riemannian submersions between two Riemannian manifolds. Then* $\Pi$ *is called a horizontally conformal submersion (HCS), if there is a positive function* $\lambda$ *such that*

$$g_1(\hat{U}_1, \hat{V}_1) = \frac{1}{\lambda^2} g_2(\Pi_* \hat{U}_1, \Pi_* \hat{V}_1) \tag{2.1}$$

*for any* $\hat{U}_1, \hat{V}_1 \in \Gamma(\ker \Pi_*)^{\perp}$. *If the dilation function* $\lambda = 1$ *then, HCS become RS.*

Let $\Pi : (\Theta_1, g_1, J) \to (\Theta_2, g_2)$ be a Riemannian submersion. A vector field $\hat{X}$ on $\Theta_1$ is called a basic vector field if $\hat{X} \in \Gamma(\ker \Pi_*)^{\perp}$ and $\Pi$-related with a vector field $\hat{X}$ on $\Theta_2$ i.e $\Pi_*(\hat{X}(q)) = \hat{X}\Pi(q)$ for $q \in \Theta_1$.

The formulae given by B . O'Neill of two $(1, 2)$ tensor fields $\mathscr{T}$ and $\mathfrak{A}$ are

$$\mathfrak{A}_{E_1} F_1 = \mathfrak{H} \nabla_{\mathfrak{H} E_1} \mathscr{V} F_1 + \mathscr{V} \nabla_{\mathfrak{H} E_1} \mathfrak{H} F_1, \tag{2.2}$$

$$\mathscr{T}_{E_1} F_1 = \mathfrak{H} \nabla_{\mathscr{V} E_1} \mathscr{V} F_1 + \mathscr{V} \nabla_{\mathscr{V} E_1} \mathfrak{H} F_1, \tag{2.3}$$

for any $E_1, F_1 \in \Gamma(T\Theta_1)$ and $\nabla$ is Levi-Civita connection of $g_1$. From equations (2.2) and (2.3), we can deduce

$$\nabla_{\hat{U}_1} \hat{V}_1 = \mathscr{T}_{\hat{U}_1} \hat{V}_1 + \mathscr{V} \nabla_{\hat{U}_1} \hat{V}_1 \tag{2.4}$$

$$\nabla_{\hat{U}_1} \hat{X}_1 = \mathscr{T}_{\hat{U}_1} \hat{X}_1 + \mathfrak{H} \nabla_{\hat{U}_1} \hat{X}_1 \tag{2.5}$$

$$\nabla_{\hat{X}_1} \hat{U}_1 = \mathfrak{A}_{\hat{X}_1} \hat{U}_1 + \mathscr{V}_1 \nabla_{\hat{X}_1} \hat{U}_1 \tag{2.6}$$

$$\nabla_{\hat{X}_1} \hat{Y}_1 = \mathfrak{H} \nabla_{\hat{X}_1} \hat{Y}_1 + \mathfrak{A}_{\hat{X}_1} \hat{Y}_1 \tag{2.7}$$

for any vector fields $\hat{U}_1, \hat{V}_1 \in \Gamma(\ker \Pi_*)$ and $\hat{X}_1, \hat{Y}_1 \in \Gamma(\ker \Pi_*)^{\perp}$ [13].

It is obvious that $\mathscr{T}$ and $\mathfrak{A}$ are skew-symmetric, that is

$$g(\mathfrak{A}_{\hat{X}} E_1, F_1) = -g(E_1, \mathfrak{A}_{\hat{X}} F_1), \ g(\mathscr{T}_{\hat{V}} E_1, F_1) = -g(E_1, \mathscr{T}_{\hat{V}} F_1), \tag{2.8}$$

for any vector fields $E_1, F_1 \in \Gamma(T_p\Theta_1)$. Since $\mathscr{T}_{\hat{V}}$ is skew-symmetric, we say that $\Pi$ has totally geodesic fibres if and only if $\mathscr{T} = 0$. For the special case when $\Pi$ is HCS, we have

**Proposition 2.1.** *Let* $\Pi : (\Theta_1, g_M) \to (\Theta_2, g_2)$ *be a HCS with dilation* $\lambda$ *and* $\hat{X}, Y$ *be the horizontal vectors, then*

$$A_{\hat{X}}\hat{Y} = \frac{1}{2}\{\mathscr{V}[\hat{X}, \hat{Y}] - \lambda^2 g(\hat{X}, \hat{Y}) grad_{\mathscr{V}}(\frac{1}{\lambda^2})\} \tag{2.9}$$

*measures the obstruction integrability of the horizontal distribution*

The second fundamental form of smooth map $\Pi$ is provided by the formula

$$(\nabla\Pi_*)(\hat{U}_1, \hat{V}_1) = \nabla^{\Pi}_{\hat{U}_1}\Pi_*\hat{V}_1 - \Pi_*\nabla_{\hat{U}_1}\hat{V}_1, \tag{2.10}$$

if $(\nabla\Pi_*)(\hat{U}_1, \hat{V}_1) = 0$ for all $\hat{U}_1, \hat{V}_1 \in \Gamma(T_p\Theta_1)$, then $\Pi$ is said to be a totally geodesic map where $\nabla$ and $\nabla^{\Pi_*}$ are Levi-Civita and pullback connections.

**Lemma 2.1.** *Let* $\Pi : \Theta_1 \to \Theta_2$ *be a HCS. Then, we have*

(i) $(\nabla\Pi_*)(\hat{X}_1, \hat{Y}_1) = \hat{X}_1(ln\lambda)\Pi_*(\hat{Y}_1) + \hat{Y}_1(ln\lambda)\Pi_*(\hat{X}_1) - g_1(\hat{X}_1, \hat{Y}_1)\Pi_*(grad\ ln\lambda,$

(ii) $(\nabla\Pi_*)(\hat{U}_1, \hat{V}_1) = -\Pi_*(\mathscr{T}_{\hat{U}_1}\hat{V}_1)$

(iii) $(\nabla\Pi_*)(\hat{X}_1, \hat{U}_1) = -\Pi_*(\nabla_{\hat{X}_1}\hat{U}_1) = -\Pi_*(\mathfrak{A}_{\hat{X}_1}\hat{U}_1)$

*for any horizontal vector fields* $\hat{X}_1, \hat{Y}_1$ *and vertical vector fields* $\hat{U}_1, \hat{V}_1$ [8].

Let $(\Theta, g)$ be an AHM. This means that $\Theta$ admits a tensor field $J$ of type $(1,1)$ on $\Theta$ such that

$$J^2 = -I,\ g(J\hat{X}, J\hat{Y}) = g(\hat{X}, \hat{Y}) \text{ for all } \hat{X}, \hat{Y} \in \Gamma(T\Theta). \tag{2.11}$$

An AHM $\Theta$ is called KM if

$$(\nabla_{\hat{X}}J)\hat{Y} = 0, \text{ for all } \hat{X}, \hat{Y} \in \Gamma(T\Theta) \tag{2.12}$$

where $\nabla$ is the Levi-Civita connection on $\Theta$. The covariant derivative of $J$ is defined by

$$(\nabla_{\hat{X}}J)\hat{Y} = \nabla_{\hat{X}}J\hat{Y} - J\nabla_{\hat{X}}\hat{Y} \tag{2.13}$$

for all vector fields $\hat{X}, \hat{Y}$ in $\Theta$.

Here, we recall the definitions which will be helpful for our text.

**Definition 2.2.** *Let* $\Pi$ *be a Riemannian submersion from AHM* $(\bar{\Theta}_1, g_1, J)$ *onto RM* $(\bar{\Theta}_2, g_2)$. *If for any non-zero vector* $\hat{X} \in \Gamma(\ker\Pi_*)$, *the angle* $\theta(\hat{X})$ *between* $J\hat{X}$ *and the space* $\ker\Pi_*$ *is constant,i.e., it is independent of the choice of point* $p \in \bar{\Theta}_1$, *and choice of tangent vector*

*$\hat{X}$ in $\ker \Pi_*$, then we said $\Pi$ is slant submersion. In this case, the angle $\theta$ is called the slant angle of submersion.*

Now, we recall the definition of pointwise slant submersion defined by T.W. Lee and B. Sahin [24]

**Definition 2.3.** *Let $\Pi$ be a Riemannian submersion from AHM $(\bar{\Theta_1}, g_1, J)$ onto RM $(\bar{\Theta_2}, g_2)$. If at each given point $q \in \bar{\Theta_2}$, the wirtinger angle $\theta(\hat{X})$ between $J\hat{X}$ and the space $\ker \Pi_*$ is independent of choice of the non-zero vector $\hat{X} \in \Gamma(\ker \Pi_*)$, then we say that $\Pi$ is a pointwise slant submersion. In this case, the angle $\theta$ can be regarded as a function on $\bar{\Theta_1}$, which is called slant function of the pointwise slant submersion.*

## 3. Pointwise semi-slant conformal submersions ($\mathcal{PWSSCS}$)

In this section, we will review the definition that will aid us in discussing and investigating the concept of pointwise semi-slant conformal submersions $\mathcal{PWSSCS}$ from almost Hermitian manifolds.

**Definition 3.1.** *Let $\Pi : (\bar{\Theta_1}, g_1, J) \to (\bar{\Theta_2}, g_2)$ be a HCS where $(\bar{\Theta_1}, g_1, J)$ is a AHM and $(\bar{\Theta_2}, g_2)$ is a RM. A HCS $\Pi$ is called a $\mathcal{PWSSCS}$ if there exists a distribution $\mathfrak{D}$ such that $\ker \Pi_* = \mathfrak{D} \oplus \mathfrak{D}^\theta$, $J(\mathfrak{D}) = \mathfrak{D}$ and for any given point $q \in \bar{\Theta_1}$ and $\hat{X} \in (\mathfrak{D}^\theta)_q$, the angle $\theta = \theta(\hat{X})$ between $J\hat{X}$ and space $(\mathfrak{D}^\theta)_q$ is independent of choice of non-zero vector $\hat{X} \in (\mathfrak{D}^\theta)_q$, where $\mathfrak{D}^\theta$ is the orthogonal complement of $\mathfrak{D}$ in $\ker \Pi_*$. In this case, the angle $\theta$ can be regarded as a slant function and called pointwise semi-slant function of submersion.*

If we suppose $m_1$ and $m_2$ are the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, then we have the following:

(i) If $m_1 = 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then $\Pi$ is a pointwise slant submersion.

(i) If $m_1 \neq 0$ and $m_2 = 0$, then $\Pi$ is a invariant submersion

(ii) If $m_1 \neq 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$ , then $\Pi$ is a pointwise semi-slant submersion.

We are providing the example of $\mathcal{PWSSCS}$ for support of our study.

Let $\Pi$ be a $\mathcal{PWSSCS}$ from an AHM $(\bar{\Theta_1}, g_1, J)$ onto a RM $(\bar{\Theta_2}, g_2)$. Then, for any $\hat{W} \in (\ker \Pi_*)$, we have

$$\hat{W} = \mathbb{P}\hat{W} + \mathbb{Q}\hat{W} \tag{3.14}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are the projections morphism onto $\mathfrak{D}$ and $\mathfrak{D}^\theta$. Now, for any $\hat{W} \in (\ker \Pi_*)$, we have

$$J\hat{W} = \psi\hat{W} + \zeta\hat{W} \tag{3.15}$$

where $\psi\hat{W} \in \Gamma(\ker \Pi_*)$ and $\zeta\hat{W} \in \Gamma(\ker \Pi_*)^\perp$. From equations (3.14) and (3.15), we have

$$\begin{aligned} J\hat{U} =& J(\mathbb{P}\hat{W}) + J(\mathbb{Q}\hat{W}) \\ =& \psi(\mathbb{P}\hat{W}) + \zeta(\mathbb{P}\hat{W}) + \psi(\mathbb{Q}\hat{W}) + \zeta(\mathbb{Q}\hat{W}). \end{aligned}$$

Since $J\mathfrak{D} = \mathfrak{D}$ and $\zeta(\mathbb{P}\hat{W}) = 0$, we have

$$J\hat{U} = \psi(\mathbb{P}\hat{W}) + \psi(\mathbb{Q}\hat{W}) + \zeta(\mathbb{Q}\hat{W}).$$

Now, we have the following decomposition

$$(\ker \Pi_*)^\perp = \zeta\mathfrak{D}^\theta \oplus \mu, \tag{3.16}$$

where $\mu$ is the orthogonal complement to $\zeta\mathfrak{D}^\theta$ in $(\ker \Pi_*)^\perp$ such that $\mu$ is invariant with respect to $J$. Now, for any $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$, we have

$$J\hat{X} = \mathfrak{B}\hat{X} + \mathfrak{C}\hat{X} \tag{3.17}$$

where $\mathfrak{B}\hat{X} \in \Gamma(\ker \Pi_*)$ and $\mathfrak{C}\hat{X} \in \Gamma(\ker \Pi_*)^\perp$.

**Lemma 3.1.** *Let $(\bar{\Theta}_1, g_1, J)$ be an KM and $(\bar{\Theta}_2, g_2)$ be a RM. If $\Pi : \bar{\Theta}_1 \to \bar{\Theta}_2$ is a $\mathcal{PWSSCS}$, then we have*

$$-\hat{W} = \psi^2\hat{W} + \mathbb{P}\zeta\hat{W},\ \zeta\psi\hat{W} + \mathfrak{C}\zeta\hat{W} = 0,\ -\hat{Y} = \zeta\mathfrak{B}\hat{Y} + \mathfrak{C}^2\hat{Y},\ \psi\mathfrak{B}\hat{Y} + \mathfrak{B}\mathfrak{C}\hat{Y},$$

*for any vector field $\hat{W} \in \Gamma(\ker \Pi_*)$ and $\hat{Y} \in \Gamma(\ker \Pi_*)^\perp$.*

*Proof.* On considering the equations (2.11), (3.15) and (3.17), the proof of Lemma exists. □

Since $\Pi : \bar{\Theta}_1 \to \bar{\Theta}_2$ is a $\mathcal{PWSSCS}$, let us present some helpful investigations that will be applied in this paper.

**Lemma 3.2.** *Let $\Pi$ be a $\mathcal{PWSSCS}$ from an AHM $(\bar{\Theta}_1, g_1, J)$ onto a RM $(\bar{\Theta}_2, g_2)$, then we have*

$$\psi^2\hat{W} = (-\cos^2\theta)\hat{W}, \tag{3.18}$$

*for any vector fields $\hat{W} \in \Gamma(\mathfrak{D}^\theta)$.*

**Lemma 3.3.** *Let $\Pi$ be a $\mathcal{PWSSCS}$ from an AHM $(\bar{\Theta}_1, g_1, J)$ onto a RM $(\bar{\Theta}_2, g_2)$, then we have*

(i) $g_1(\psi\hat{Z}, \psi\hat{W}) = \cos^2\theta\, g_1(\hat{Z}, \hat{W})$,

(ii) $g_1(\zeta\hat{Z}, \zeta\hat{W}) = \text{operatornamesin}^2\theta\, g_1(\hat{Z}, \hat{W})$,

*for any vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta)$.*

*Proof.* The proof of the preceding Lemmas is identical to the proof of Theorem (2.2) of [11]. As a result, we omit the proofs. □

Let us suppose that $(\bar{\Theta_2}, g_2)$ be a RM and $(\Theta_1, g_1, J)$ be an AHM. We now analyse how the Hermitian structure on $\Theta_1$ influences the tensor fields $\mathscr{T}$ and $\mathfrak{A}$ of $\mathcal{PWSSCS}$ $\Pi : (\Theta_1, g_1, J) \to (\Theta_2, g_2)$.

**Lemma 3.4.** *Let* $\Pi : \bar{\Theta_1} \to \bar{\Theta_2}$ *be* $\mathcal{PWSSCS}$ *with semi-slant function* $\theta$ *where,* $(\bar{\Theta_1}, g_1, J)$ *KM and* $(\bar{\Theta_2}, g_2)$ *be a RM, then we have*

$$\mathfrak{A}_{\hat{X}}\mathfrak{C}\hat{Y} + \mathscr{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y} = \mathfrak{B}\mathfrak{H}\nabla_{\hat{X}}\hat{Y} + \psi\mathfrak{A}_{\hat{X}}\hat{Y} \tag{3.19}$$

$$\mathfrak{H}\nabla_{\hat{X}}\mathfrak{C}\hat{Y} + \mathfrak{A}_{\hat{X}}\mathfrak{B}\hat{Y} = \mathfrak{C}\mathfrak{H}\nabla_{\hat{X}}\hat{Y} + \zeta\mathfrak{A}_{\hat{X}}\hat{Y} \tag{3.20}$$

$$\mathscr{V}\nabla_{\hat{X}}\psi\hat{V} + \mathfrak{A}_{\hat{X}}\zeta\hat{V} = \mathfrak{B}\mathfrak{A}_{\hat{X}}\hat{V} + \psi\mathscr{V}\nabla_{\hat{X}}\hat{V} \tag{3.21}$$

$$\mathfrak{A}_{\hat{X}}\psi\hat{V} + \mathfrak{H}\nabla_{\hat{X}}\zeta\hat{V} = \mathfrak{C}\mathfrak{A}_{\hat{X}}\hat{V} + \zeta\mathscr{V}\nabla_{\hat{X}}\hat{V} \tag{3.22}$$

$$\mathscr{V}\nabla_{\hat{V}}\mathfrak{B}\hat{X} + \mathscr{T}_{\hat{V}}\mathfrak{C}\hat{X} = \psi\mathscr{T}_{\hat{V}}\hat{X} + \mathfrak{B}\mathfrak{H}\nabla_{\hat{V}}\hat{X} \tag{3.23}$$

$$\mathscr{T}_{\hat{V}}\mathfrak{B}\hat{X} + \mathfrak{H}\nabla_{\hat{V}}\mathfrak{C}\hat{X} = \zeta\mathscr{T}_{\hat{V}}\hat{X} + \mathfrak{C}\mathfrak{H}\nabla_{\hat{V}}\hat{X} \tag{3.24}$$

$$\mathscr{V}\nabla_{\hat{U}}\psi\hat{V} + \mathscr{T}_{\hat{U}}\zeta\hat{V} - \psi\mathscr{V}\nabla_{\hat{U}}\hat{V} = \mathfrak{B}\mathscr{T}_{\hat{U}}\hat{V} \tag{3.25}$$

$$\mathscr{T}_{\hat{U}}\psi\hat{V} + \mathfrak{H}\nabla_{\hat{U}}\zeta\hat{V} = \mathfrak{C}\mathscr{T}_{\hat{U}}\hat{V} + \zeta\mathscr{V}\nabla_{\hat{U}}\hat{V}, \tag{3.26}$$

*for any vector fields* $\hat{U}, \hat{V} \in \Gamma(\ker \Pi_*)$ *and* $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^{\perp}$.

*Proof.* By using (2.12), (2.13) and (2.7) (3.17), we get first two relations (3.19) and (3.20). Similarly, by considering equations (2.12), (2.13) (2.7), (2.4)-(2.7) and (3.15) (3.17), the desired results holds good. □

We will now go through some key conclusions that can be utilised to examine the geometry of $\mathcal{PWSSCS}$ $\Pi : \Theta_1 \to \Theta_2$. From the direct calculations, we can conclude the following:

$$(\nabla_{\hat{U}}\psi)\hat{V} = \mathscr{V}\nabla_{\hat{U}}\psi\hat{V} - \psi\mathscr{V}\nabla_{\hat{U}}\hat{V} \tag{3.27}$$

$$(\nabla_{\hat{U}}\zeta)\hat{V} = \mathfrak{H}\nabla_{\hat{U}}\zeta\hat{V} - \zeta\mathscr{V}\nabla_{\hat{U}}\hat{V} \tag{3.28}$$

$$(\nabla_{\hat{X}}\mathfrak{B})\hat{Y} = \mathscr{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y} - \mathfrak{B}\mathfrak{H}\nabla_{\hat{X}}\hat{Y} \tag{3.29}$$

$$(\nabla_{\hat{X}}\mathfrak{C})\hat{Y} = \mathfrak{H}\nabla_{\hat{X}}\mathfrak{C}\hat{Y} - \mathfrak{H}\nabla_{\hat{X}}\hat{Y}, \tag{3.30}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(\ker \Pi_*)$ and $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^{\perp}$.

**Lemma 3.5.** *Let $(\Theta_1, g_1, J)$ be a KM and $(\Theta_2, g_2)$ be a RM. If $\Pi : \Theta_1 \to \Theta_2$ is a $\mathcal{PWSSCS}$ with semi-slant function $\theta$, then we have*

$$(\nabla_{\hat{U}}\psi)\hat{V} = \mathfrak{B}\mathscr{T}_{\hat{U}}\hat{V} - \mathscr{T}_{\hat{U}}\zeta\hat{V}$$

$$(\nabla_{\hat{U}}\zeta)\hat{V} = \mathfrak{C}\mathscr{T}_{\hat{U}}\hat{V} - \mathscr{T}_{\hat{U}}\psi\hat{V}$$

$$(\nabla_{\hat{X}}\mathfrak{B})\hat{Y} = \psi\mathfrak{A}_{\hat{X}}\hat{Y} - \mathfrak{A}_{\hat{X}}\mathfrak{C}\hat{Y}$$

$$(\nabla_{\hat{X}}\mathfrak{C})\hat{Y} = \zeta\mathfrak{A}_{\hat{X}}\hat{Y} - \mathfrak{A}_{\hat{X}}\mathfrak{B}\hat{Y},$$

*for all vector fields $\hat{U}, \hat{V} \in \Gamma(\ker \Pi_*)$ and $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^{\perp}$.*

*Proof.* By using equations (2.13), (2.4)- (2.7) and equations (3.27)-(3.30), we can obtain the results. □

The tensor fields $\psi$ and $\zeta$, if they are parallel with regard to the Levi- Civita connection $\nabla$ of $\Theta_1$, then we obtain

$$\mathfrak{B}\mathscr{T}_{\hat{U}}\hat{V} = \mathscr{T}_{\hat{U}}\zeta\hat{V},\ \mathfrak{C}\mathscr{T}_{\hat{U}}\hat{V} = \mathscr{T}_{\hat{U}}\psi\hat{V}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(T\Theta_1)$.

## 4. CONDITIONS FOR INTEGRABILITY AND TOTALLY GEODESICNESS

In this section, we discuss the geometry of $\mathcal{PWSSCS}$ $\Pi : (\Theta_1, g_1, J) \to (\Theta_2, g_2)$ from KM onto RM in terms of integrability of invariant and slant distribution. Apart from this, we also examine the necessary and sufficient conditions for the leaves of distribution to be define totally geodesic foliation on $\Theta_1$. We start the condition for integrability for invariant distribution as follows :

**Theorem 4.1.** *Let $\Pi : \bar{\Theta_1} \to \bar{\Theta_2}$ be $\mathcal{PWSSCS}$ with semi-slant function $\theta$ where, $(\bar{\Theta_1}, g_1, J)$ is a KM and $(\bar{\Theta_2}, g_2)$ be a RM. Then the invariant distribution $\mathfrak{D}$ is integrable if and only if*

$$\mathscr{V}\nabla_{\hat{U}}\psi\hat{Z} + \mathscr{T}_{\hat{U}}\zeta\hat{Z} \in \Gamma(\mathfrak{D}^{\theta}) \text{ and } \mathscr{V}\nabla_{\hat{V}}\psi\hat{Z} + \mathscr{T}_{\hat{V}}\zeta\hat{Z} \in \Gamma(\mathfrak{D}^{\theta}), \tag{4.31}$$

*for any vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{Z} \in \Gamma(\mathfrak{D}^{\theta})$.*

*Proof.* For all vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{Z} \in \Gamma(\mathfrak{D}^{\theta})$ and by using equations (2.11), (2.12) and (2.13), we have

$$\begin{aligned} g_1([\hat{U}, \hat{V}], \hat{Z}) &= g_1(\nabla_{\hat{U}}J\hat{V}, J\hat{Z}) - g_1(\nabla_{\hat{V}}J\hat{U}, J\hat{Z}) \\ &= -g_1(\nabla_{\hat{U}}J\hat{Z}, J\hat{V}) + g_1(\nabla_{\hat{V}}J\hat{Z}, J\hat{U}). \end{aligned}$$

Taking account the fact from equations (2.4) and (2.5) in both part of the above equation in right hand side, takes the form

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\nabla_{\hat{U}}\psi\hat{Z},\psi\hat{V}) - g_1(\nabla_{\hat{U}}\zeta\hat{Z},\psi\hat{V}).$$

By using equation (3.15) in above relation, we have

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{Z},\psi\hat{V}) - g_1(\mathscr{T}_{\hat{U}}\zeta\hat{Z},\psi\hat{V}).$$

In above equation, change the role of $\hat{U}$ and $\hat{V}$, we may yield

$$g_1([\hat{U},\hat{V}],\hat{Z}) = -g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{Z} + \mathscr{T}_{\hat{U}}\zeta\hat{Z},\psi\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{V}}\psi\hat{Z} + \mathscr{T}_{\hat{V}}\zeta\hat{Z},\psi\hat{U}).$$

□

**Theorem 4.2.** *Let* $\Pi$ *be* $\mathcal{PWSSCS}$ *with semi-slant function* $\theta$ *from KM* $(\Theta_1, g_1, J)$ *onto a RM* $(\Theta_2, g_2)$*. Then* $\mathfrak{D}^\theta$ *is integrable if and only if*

$$\psi(\mathscr{T}_{\hat{Z}}\zeta\hat{W} - \mathscr{T}_{\hat{W}}\zeta\hat{Z}) = (\mathscr{T}_{\hat{W}}\zeta\psi\hat{Z} - \mathscr{T}_{\hat{Z}}\zeta\psi\hat{W}), \tag{4.32}$$

*for any vector fields* $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta)$ *and* $\hat{U} \in \Gamma(\mathfrak{D})$*.*

*Proof.* By using equation (2.11), (2.12) and (2.13), we may yield

$$g_1([\hat{Z},\hat{W}],\hat{U}) = g_1(\nabla_{\hat{Z}}J\hat{W},J\hat{U}) - g_1(\nabla_{\hat{W}}J\hat{Z},J\hat{U}),$$

for every vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta)$ and $\hat{U} \in \Gamma(\mathfrak{D})$. By using equation (3.15), we can write

$$g_1([\hat{Z},\hat{W}],\hat{U}) = -g_1(\nabla_{\hat{Z}}\psi\hat{W},J\hat{U}) - g_1(\nabla_{\hat{W}}\psi\hat{Z},J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\hat{W},J\hat{U}) - g_1(\nabla_{\hat{W}}\zeta\hat{Z},J\hat{U}).$$

Now, considering the equation (2.11) and equation (2.5) in third and fourth terms, above equation takes the form

$$g_1([\hat{Z},\hat{W}],\hat{U}) = g_1(\nabla_{\hat{Z}}J\psi\hat{W},\hat{U}) + g_1(\nabla_{\hat{W}}J\psi\hat{Z},\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\hat{W},J\hat{U}) - g_1(\mathscr{T}_{\hat{W}}\zeta\hat{Z},J\hat{U}). \tag{4.33}$$

Taking account the fact from (3.15) in first term, we get $g_1(\nabla_{\hat{Z}}J\psi\hat{W},\hat{U}) = -g_1(\nabla_{\hat{Z}}\psi^2\hat{W},\hat{U}) - g_1(\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{U})$. By using Lemma 3.2, $g_1(\nabla_{\hat{Z}}J\psi\hat{W},\hat{U}) = \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{W},\hat{U}) - g_1(\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{U})$. The same calculation in second term, we get $-g_1(\nabla_{\hat{W}}J\psi\hat{Z},\hat{U}) = -\cos^2\theta g_1(\nabla_{\hat{W}}\hat{Z},\hat{U}) + g_1(\nabla_{\hat{W}}\zeta\psi\hat{Z},\hat{U})$. On combining these calculations, finally equation (4.33) takes the form

$$g_1([\hat{Z},\hat{W}],\hat{U}) = \cos^2\theta g_1([\hat{Z},\hat{W}],\hat{U}) - g_1(\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{U}) + g_1(\nabla_{\hat{W}}\zeta\psi\hat{Z},\hat{U})$$
$$+ g_1(\mathscr{T}_{\hat{Z}}\zeta\hat{W},J\hat{U}) - g_1(\mathscr{T}_{\hat{W}}\zeta\hat{Z},J\hat{U}).$$

Finally, by using equation (2.5), we can write

$$\sin^2\theta g_1([\hat{Z},\hat{W}],\hat{U}) = g_1(\mathscr{T}_{\hat{W}}\zeta\psi\hat{Z},\hat{U}) - g_1(\mathscr{T}_{\hat{Z}}\zeta\psi\hat{W},\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\hat{W} - \mathscr{T}_{\hat{W}}\zeta\hat{Z},J\hat{U}).$$

From which, we can conclude the result. □

Since $\Pi : (\Theta_1, g_1, J) \rightarrow (\Theta_2, g_2)$ be a $\mathcal{PWSSCS}$ which ensure the availability of slant distributions. After discussing the integrability conditions of distributions, we are going to examine the necessary and sufficient condition for which the leaves of distributions defined totally geodesic foliation on $\Theta_1$.

**Theorem 4.3.** *Let $\Pi$ be $\mathcal{PWSSCS}$ with semi-slant function $\theta$ from KM $(\bar{\Theta}_1, g_1, J)$ onto a RM $(\bar{\Theta}_2, g_2)$. Then $\mathfrak{D}$ is defines totally geodesic foliation on $\bar{\Theta}_1$ if and only if*

$$\mathscr{T}_{\hat{U}}\zeta\psi\hat{Z} = -\psi(\mathscr{T}_{\hat{U}}\zeta\hat{Z}) \text{ and } g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{V},\mathfrak{B}\hat{X}) + g_1(\mathscr{T}_{\hat{U}}\psi\hat{V},\mathfrak{C}\hat{X}) = 0, \tag{4.34}$$

*for any vector fields $\hat{U},\hat{V} \in \Gamma(\mathfrak{D}), \hat{Z} \in \Gamma(\mathfrak{D}^\theta)$ and $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$.*

*Proof.* By considering $g_1(\nabla_{\hat{U}}\hat{V},\hat{Z})$, for any vector fields $\hat{U},\hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{Z} \in \Gamma(\mathfrak{D}^\theta)$. Since, $\hat{V}$ and $\hat{Z}$ are orthogonal to each other, this can be write as $g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\nabla_{\hat{U}}\hat{Z},\hat{V})$. Operating almost complex structure $J$ on both side and using equations (2.11), (2.12), (2.13) and (3.15), we have

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\nabla_{\hat{U}}\psi\hat{Z},JV) - g_1(\nabla_{\hat{U}}\zeta\hat{Z},JV).$$

Further, in the light of equations (3.15) and (2.5), we get

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\nabla_{\hat{U}}\psi^2\hat{Z},\hat{V}) + g_1(\nabla_{\hat{U}}\zeta\psi\hat{Z},\hat{V}) - g_1(\mathscr{T}_{\hat{U}}\zeta\hat{Z},JV).$$

Since, $\Pi$ is a $\mathcal{PWSSCS}$ with semi-slant function $\theta$, then by using Lemma 3.2 in first term of above equation, finally this will takes the form

$$\sin^2\theta g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = g_1(\nabla_{\hat{U}}\zeta\psi\hat{Z},\hat{V}) - g_1(\mathscr{T}_{\hat{U}}\zeta\hat{Z},JV).$$

From this we can get the first part of theorem. For next one, we consider $g_1(\nabla_{\hat{U}}\hat{V},\hat{X})$ for any vector fields $\hat{U},\hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$. By using equation (2.11), (2.12), (2.13) and (3.17), (3.15), this term will takes the form as $g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = g_1(\nabla_{\hat{U}}\psi\hat{V},\mathfrak{B}\hat{X} + \mathfrak{C}\hat{X})$. Finally, considering equation (2.4), we can write

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{V},\mathfrak{B}\hat{X}) + g_1(\mathscr{T}_{\hat{U}}\psi\hat{V},\mathfrak{C}\hat{X}).$$

From which the second part of theorem holds good. □

Since, $\Pi$ is $\mathcal{PWSSCS}$ with semi-slant function $\theta$ from $(\Theta_1, g_1, J)$ onto $(\Theta_2, g_2)$. The slant distribution is mutually othogonal to invariant distribution. After discussion geometry of leaves of invariant distribution, it is quite interesting to study the leaves of slant distribution geometrical point of view in following manner.

**Theorem 4.4.** *Let $\Pi : \Theta_1 \to \Theta_2$ be $\mathcal{PWSSCS}$ with semi-slant function $\theta$ where, $(\Theta_1, g_1, J)$ a KM and $(\Theta_2, g_2)$ a RM. Then $\mathfrak{D}^\theta$ is defines totally geodesic foliation on $\Theta_1$ if and only if*

$$\psi(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W} + \mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W} + \mathscr{T}_{\hat{Z}}\zeta\hat{W}\mathbb{Q}) \in \Gamma(\mathfrak{D}^\theta) \tag{4.35}$$

*and*

$$\begin{aligned} &g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z}, \psi\hat{W}) - g_1(\mathscr{T}_{\hat{Z}}\psi\mathbb{P}\hat{Z}, \zeta\hat{W}) + g_1(\nabla_{\hat{X}}\mathbb{P}\psi\mathbb{Q}\hat{Z}, \hat{W}) \\ &= g_1(\mathscr{T}_{\hat{X}}\zeta\mathbb{Q}\hat{Z}, \psi\hat{W}) - \sin^2\theta g_1([\hat{Z}, \hat{X}], \hat{W}) - 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z}, \hat{W}) \\ &\ + g_1(\hat{X}, grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z}, \zeta\hat{W}) + g_1(\zeta\mathbb{Q}\hat{Z}, grad\ln\lambda)g_1(\hat{X}, \zeta\hat{W}) \\ &\ - g_1(\zeta\hat{W}, grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z}, \zeta\hat{W}), \end{aligned} \tag{4.36}$$

*for any vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta), \hat{U} \in \Gamma(\mathfrak{D})$ and $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$.*

*Proof.* Let us consider for any vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta)$ and $\hat{U} \in \Gamma(\mathfrak{D})$. In light of equations (2.11), (2.12) and (2.13) after operating almost complex structure $J$ on both side, we have

$$g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) = g_1(\nabla_{\hat{Z}}J\hat{W}, J\hat{U}).$$

By using decomposition (3.14), $g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) = g_1(\nabla_{\hat{Z}}J(\mathbb{P}\hat{W} + \mathbb{Q}\hat{W}), J\hat{U})$. Taking account the fact from equation (3.15), we have

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) &= g_1(\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W}, J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{P}\hat{W}, J\hat{U}) \\ &\quad + g_1(\nabla_{\hat{Z}}\psi\mathbb{Q}\hat{W}, J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W}, J\hat{U}). \end{aligned}$$

Considering the equations (2.4) and (2.5) and since $\mathfrak{D}$ is invariant under almost structure $J$, i.e., $J\mathfrak{D} = \mathfrak{D}$, we may yields

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) =& g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W}, J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W}, J\hat{U}) \\ &- g_1(\nabla_{\hat{Z}}\psi^2\mathbb{Q}\hat{W}, \hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W}, J\hat{U}). \end{aligned} \tag{4.37}$$

By using Lemma 3.2 in third term of above equation, which can be write as $-g_1(\nabla_{\hat{Z}}\psi^2\mathbb{Q}\hat{W}, \hat{U}) = g_1(\nabla_{\hat{Z}}(\cos^2\theta)\mathbb{Q}\hat{W}, \hat{U})$. Then the equation (4.37), will takes the form as

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) =& g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W}, J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W}, J\hat{U}) \\ &+ g_1(\nabla_{\hat{Z}}(\cos^2\theta)\mathbb{Q}\hat{W}, \hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W}, J\hat{U}). \end{aligned}$$

Since, $\Pi$ is a $\mathcal{PWSSCS}$ with semi-slant function $\theta$, then we can write the above equation as:

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{U}) = g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W},J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W},J\hat{U}) + g_1(\nabla_{\hat{Z}}(\cos^2\theta)\mathbb{Q}\hat{W},\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W},J\hat{U}).$$

With simple steps of calculations, finally we get

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{U}) = g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W},J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W},J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W},J\hat{U}) + 2\sin\theta\cos\theta\hat{Z}(\theta)g_1(\mathbb{Q}\hat{W},\hat{U}) + \cos^2\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{W},\hat{U}).$$

From which the first part of theorem holds good. For the other part of theorem, let us suppose for any vector fields $\hat{Z},\hat{W}\in\Gamma(\mathfrak{D}^\theta)$ and $\hat{X}\in\Gamma(\ker\Pi_*)^\perp$. We start with considering the term $g_1(\nabla_{\hat{Z}}\hat{W},\hat{X})$, by using basic calcaltions, this term can be write as $g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\nabla_{\hat{X}}\hat{Z},\hat{W})$. By using equation (2.11), (2.12) and (2.13), this term takes the form as

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\nabla_{\hat{X}}J\hat{Z},J\hat{W}).$$

In the light of equations (2.4) and since $\mathfrak{D}$ is invariant under $J$, we get

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},J\hat{W}) - g_1(\nabla_{\hat{X}}\psi\mathbb{Q}\hat{Z},J\hat{W}) - g_1(\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},J\hat{W}).$$

By using equations (2.11), (2.4) and (2.5), we have

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\psi^2Q\hat{Z},\hat{W}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}). \tag{4.38}$$

Since, $\Pi$ is a $\mathcal{PWSSCS}$ with semi-slant function $\theta$, then with simple steps of calculations, the fourth term of above equation take place as

$$g_1(\nabla_{\hat{X}}\psi^2Q\hat{Z},\hat{W}) = -g_1(\nabla_{\hat{X}}(-\cos^2\theta)\mathbb{Q}\hat{Z},\hat{W}) = 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{Z},\hat{W}).$$

By using the above equation in (4.38), we may write as

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) + 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{Z},\hat{W}). \tag{4.39}$$

Now, the first and last term can be write as:

$$-g([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{Z},\hat{W}) = \sin^2\theta g_1([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{X},\hat{W}). \tag{4.40}$$

Since, $\Pi$ is a $\mathcal{P}\hat{\mathcal{W}}\mathcal{SSCS}$, then by using equation (4.40) in (4.39), we can write

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) =& 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) + \sin^2\theta g_1([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{X},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}). \end{aligned} \tag{4.41}$$

Now, using the horizontal conformality of $\Pi$ from Lemma 2.1 and equations (2.1), (2.10) in the last term of above equation, can be written as

$$\begin{aligned} -g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) =& \frac{1}{\lambda^2}g_1(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})) - \frac{1}{\lambda^2}g_1((\nabla\Pi_*)(\hat{X},\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})) \\ & + \frac{1}{\lambda^2}g_1(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})) - g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) \\ & - g_1(\zeta\mathbb{Q}\hat{Z},grad\ln\lambda)g_1(\hat{X},\zeta\hat{W}) + g_1(\zeta\hat{W},grad\ln\lambda)g_1(\hat{X},\zeta\mathbb{Q}\hat{Z}). \end{aligned}$$

Now, by using the above relation, equation (4.41) finally turns into

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) =& 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) + \sin^2\theta g_1([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{X},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) \\ & - g_1(\zeta\mathbb{Q}\hat{Z},grad\ln\lambda)g_1(\hat{X},\zeta\hat{W}) + g_1(\zeta\hat{W},grad\ln\lambda)g_1(\hat{X},\zeta\mathbb{Q}\hat{Z}) \\ & + \frac{1}{\lambda^2}g_1(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})). \end{aligned}$$

Hence, this proves the theorem completely. □

The study of geometry of leaves of horizontal and vertical distributions of $\mathcal{PWSSCS}$ is very important. We start our discussion with necessary and sufficient conditions for vertical distribution $\ker\Pi_*$ is totally geodesic.

**Theorem 4.5.** *Let us suppose that* $\Pi$ *be a* $\mathcal{PWSSCS}$ *with semi-slant function* $\theta$ *from KM* $(\Theta_1, g_1, J)$ *onto a RM* $(\Theta_2, g_2)$. *Then* $\ker\Pi_*$ *is defines totally geodesic foliation if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) + g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) \\ & = \cos^2\theta\, g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{U},\hat{V}) - 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{U},\hat{V}) + g_1([\hat{U},\hat{X}],\hat{V}) \\ & + g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) + g_1(\zeta\mathbb{Q}\hat{U},grad\ln\lambda)g_1(\hat{X},\zeta\hat{V}) \\ & - g_1(\zeta\hat{V},grad\ln\lambda)g_1(\hat{X},\zeta\mathbb{Q}\hat{U}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}), \end{aligned} \tag{4.42}$$

*for any vector fields* $\hat{U},\hat{V}\in\Gamma(\ker\Pi_*)$ *and* $\hat{X}\in\Gamma(\ker\Pi_*)^{\perp}$.

*Proof.* We start the proof of theorem with considering the term $g_1(\nabla_{\hat{U}}\hat{V},\hat{X})$. From simple steps of calculations with basic definition, this turns into $g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = -g_1([\hat{U},\hat{X}],\hat{V}) - g_1(\nabla_{\hat{X}}\hat{U},\hat{V})$. Operating $J$, which is a almost complex structure with using equation (2.11), (2.12) and (2.13) on second term, this will take place

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = -g_1([\hat{U},\hat{X}],\hat{V}) - g_1(\nabla_{\hat{X}}J\hat{U},JV),$$

for any vertical vector fields $\hat{U},\hat{V}$ and horizontal vector field $\hat{X}$. In the light of decomposition (3.14) and (3.15) the second term of above equation, we can write

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = -g_1([\hat{U},\hat{X}],\hat{V}) - g_1(\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},JV) - g_1(\nabla_{\hat{X}}\psi\mathbb{Q}\hat{U},JV) - g_1(\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},JV). \tag{4.43}$$

In the light of equation (3.15) and (2.6), second term of above equation become $-g_1(\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},JV) = g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V})$. Similarly, from equation (2.11), (2.12) and (2.6), third term turns as $-g_1(\nabla_{\hat{X}}\psi\mathbb{Q}\hat{U},JV) = g_1(\nabla_{\hat{X}}\psi^2\mathbb{Q}\hat{U},\hat{V})+g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V})$. In last term, taking account the fact from decomposition (3.15) and equation (2.7), this will take place as $-g_1(\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},JV) = -g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\hat{V})$. Put the values of all these terms in equation (4.43), we get

$$\begin{aligned} g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = &- g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) + g_1(\nabla_{\hat{X}}\psi^2\mathbb{Q}\hat{U},\hat{V}) \\ &+ g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}). \end{aligned}$$

Since, $\Pi$ is a $\mathcal{PWSSCS}$ with semi-slant function $\theta$, using Lemma 3.2, above equation turns into

$$\begin{aligned} g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = &- g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) - g_1(\nabla_{\hat{X}}(\cos^2\theta)\mathbb{Q}\hat{U},\hat{V}) \\ &+ g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}) \\ = &-g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) \\ &+ 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{U},\hat{V}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{U},\hat{V}) \\ &- g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}). \end{aligned} \tag{4.44}$$

Considering equations (2.1) and (2.10), second last term of the above equation will be

$$-g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})) - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})).$$

By using the definition of horizontal conformality of $\Pi$ from Lemma 3.2, we can write

$$-g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})) - \frac{1}{\lambda^2}g_2((\hat{X}(\ln\lambda)\Pi_*(\zeta\mathbb{Q}\hat{U}) + \zeta\mathbb{Q}\hat{U}(\ln\lambda))\Pi_*(\hat{X}) - g_1(\hat{X},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*(\zeta\hat{V})).$$

Now, by using above two equations in (4.44), finally we have

$$\begin{aligned} g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) =& -g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) \\ &+ 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{U},\hat{V}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{U},\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}) \\ &- g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\zeta\mathbb{Q}\hat{U},grad\ln\lambda)g_1(\hat{X},\zeta\hat{V}) \\ &+ g_1(\hat{X},\zeta\mathbb{Q}\hat{U})g_1(\zeta\hat{V},grad\ln\lambda). \end{aligned}$$

□

This completes the proof.

**Theorem 4.6.** *Let $\Pi$ be $\mathcal{PWSSCS}$ from a KM $(\Theta_1, g_1, J)$ onto a RM $(\Theta_2, g_2)$. Then the map $\Pi$ is totally geodesic map if and only if*

(i) $\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\hat{W} + \zeta\psi\hat{Z}(\ln\lambda)\Pi_*\hat{Z} - g_1(\hat{Z},\zeta\psi\hat{W})\Pi_*(grad\ln\lambda),\Pi_*(\hat{X}))$
$= g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X}) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X}))$

(ii) $\cos^2\theta g_1(\mathscr{T}_{\hat{X}}\hat{Y},\hat{Z}) + \frac{1}{\lambda^2}\{g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\psi\hat{Y},\Pi_*(\hat{Z}) - g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\hat{Y},\Pi_*(\mathfrak{C}\hat{Z}))\} = 0$

(iii) $\operatorname{cosec}^2\theta g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W}) + \cot^2\theta\cos^2\theta g_1(\mathfrak{H}\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})$
$= -\frac{1}{\lambda^2}\{g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\mathbb{Q}\hat{U},\Pi_*(\hat{W})) + g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\mathbb{Q}\hat{U},\Pi_*(\mathfrak{C}\hat{W}))\}$,

*for any $\hat{U},\hat{V} \in \Gamma(\mathfrak{D}), \hat{X},\hat{Y} \in \Gamma(\mathfrak{D}^{\theta})$ and $\hat{Z} \in \Gamma(\ker\Pi_*)^{\perp}, \hat{U}_1 \in \Gamma(\ker\Pi_*)$.*

*Proof.* Let us consider $g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X}))$, for any $\hat{Z},\hat{W} \in \Gamma(\mathfrak{D})$ and $\hat{X} \in \Gamma(\ker\Pi_*)^{\perp}$. On using equations (2.10) with definition 2.1, we may obtain $g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = -\lambda^2 g_1(\nabla_{\hat{Z}}\hat{W},\hat{X})$. This relation can be turn into

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}).$$

Taking account the fact that $J\hat{W} = \psi\mathfrak{D}$ if $\hat{W} \in \Gamma(\mathfrak{D})$ and from equations (2.11), (3.15) in the right hand side of above equation, we get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = -g_1(\nabla_{\hat{Z}}\psi\hat{W},J\hat{X}).$$

By using equations (2.4), (2.5) with (3.15), we can get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X}) - g_1(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{X}). \tag{4.45}$$

Since $\Pi$ is $\mathcal{PWSSCS}$, by using definition 2.1, the second term in the right hand side of above equation can be turn into $-g_1(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{X}) = \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\zeta\psi\hat{W}),\Pi_*(\hat{X})) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X}))$. By using this in (4.45), we may have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) =& g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X}) + \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\zeta\psi\hat{W}),\Pi_*(\hat{X}))\\ &- \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X})).\end{aligned}$$

Finally with using Lemma 3.3, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) =& \frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\hat{W} + \zeta\psi\hat{W}(\ln\lambda)\Pi_*\hat{Z}\\ &- g_1(\hat{Z},\zeta\psi\hat{W})\Pi_*(grad\ln\lambda)) + g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X})\\ &- \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X})),\end{aligned}$$

which is part $(i)$. For part $(ii)$, take into consideration $g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z}))$, for any $\hat{X},\hat{Y}\in\Gamma(\mathfrak{D}^\theta)$ and $\hat{Z}\in\Gamma(\ker\Pi_*)^\perp$. From equations (2.10) with definition 2.1, we can write $g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) = -\lambda^2 g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})$. In the light of relation (2.11), (2.12) and (3.15), we get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) = -g_1(\nabla_{\hat{X}}\psi\hat{Y},J\hat{Z}) - g_1(\nabla_{\hat{X}}\zeta\hat{Y},J\hat{Z}).$$

By using equations (2.11), (2.12), (3.17), above equations turn into

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) = g_1(\nabla_{\hat{X}}J\psi\hat{Y},\hat{Z}) - g_1(\nabla_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}+\mathfrak{C}\hat{Z}).$$

By using equation (2.5), we can write

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) =& g_1(\nabla_{\hat{X}}\psi^2\hat{Y},\hat{Z}) + g_1(\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z})\\ &- g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\hat{Y},\mathfrak{C}\hat{Z}) - g_1(\mathscr{T}_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}).\end{aligned}$$

Taking account the fact from equation (2.5) with Lemma 3.3, we may have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) =& g_1(\nabla_{\hat{X}}(\cos^2\theta)\hat{Y},\hat{Z}) + g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z})\\ &- g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\hat{Y},\mathfrak{C}\hat{Z}) - g_1(\mathscr{T}_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}).\end{aligned} \tag{4.46}$$

Since $\Pi$ is a $\mathcal{PWSSCS}$ from a KM $\Theta_1$, the the first term of equation (4.46) turn into as $g_1(\nabla_{\hat{X}}(\cos^2\theta)\hat{Y},\hat{Z}) = 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\hat{Y},\hat{Z}) + \cos^2\theta g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})$, where the second term as $g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z}) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\psi\hat{Y},\Pi_(*\hat{Z})) - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\psi\hat{Y}),\Pi_*(\hat{Z}))$ and third term as $g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z}) = -\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\hat{Y},\Pi_*(\mathfrak{C}\hat{Z})) + \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\hat{Y}),\Pi_*(\mathfrak{C}\hat{Z}))$ by using equation (2.10) and definition 2.1. □

With all these facts using in equation (4.46), we can write

$$
\begin{aligned}
&\frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{X}, \hat{Y}), \Pi_*(\hat{Z}))\\
&= 2\sin\theta\cos\theta \hat{X}(\theta) g_1(\hat{Y}, \hat{Z}) + \cos^2\theta g_1(\nabla_{\hat{X}} \hat{Y}, \hat{Z}) + \frac{1}{\lambda^2} g_2(\nabla^{\Pi}_{\hat{X}} \Pi_* \zeta\psi\hat{Y}, (\Pi_*\hat{Z}))\\
&\quad - \frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{X}, \zeta\psi\hat{Y}), \Pi_*(\hat{Z})) - \frac{1}{\lambda^2} g_2(\nabla^{\Pi}_{\hat{X}} \Pi_* \zeta\hat{Y}, \Pi_(*\mathfrak{C}\hat{Z}))\\
&\quad + \frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{X}, \zeta\hat{Y}), \Pi_*(\mathfrak{C}\hat{Z})) - g_1(\mathscr{T}_{\hat{X}} \zeta\hat{Y}, \mathfrak{B}\hat{Z}).
\end{aligned}
$$

Finally, by using the Lemma 3.3 in fourth and fifth terms, the above equations takes the form

$$
\begin{aligned}
&\frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{X}, \hat{Y}), \Pi_*(\hat{Z}))\\
&= \frac{1}{\lambda^2} g_2(\hat{X}(\ln\lambda)\Pi_*\zeta\hat{Y} + \zeta\hat{Y}(\ln\lambda)\Pi_*\hat{X} - g_1(\hat{X}, \zeta\hat{Y})\Pi_*(grad \ln\lambda), \Pi_*(\mathfrak{C}\hat{Z}))\\
&- \frac{1}{\lambda^2} g_2(\hat{X}(\ln\lambda)\Pi_*\zeta\psi\hat{Y} + \zeta\psi\hat{Y}(\ln\lambda)\Pi_*\hat{X} - g_1(\hat{X}, \zeta\psi\hat{Y})\Pi_*(grad \ln\lambda), \Pi_*(\hat{Z}))\\
&- \frac{1}{\lambda^2} g_2(\nabla^{\Pi}_{\hat{X}} \Pi_* \zeta\hat{Y}, \Pi_(*\mathfrak{C}\hat{Z})) - g_1(\mathscr{T}_{\hat{X}} \zeta\hat{Y}, \mathfrak{B}\hat{Z})\\
&+ \cos^2\theta g_1(\nabla_{\hat{X}} \hat{Y}, \hat{Z}) + \frac{1}{\lambda^2} g_2(\nabla^{\Pi}_{\hat{X}} \Pi_* \zeta\psi\hat{Y}, \Pi_*(\hat{Z})).
\end{aligned}
$$

This is the proof of part $(ii)$. For $(iii)$ part, we consider

$$
\frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{Z}, \hat{U}_1), \Pi_*\hat{W}) = -g_1(\Pi_*\nabla_{\hat{Z}} \hat{U}_1, \Pi_*\hat{W}),
$$

for any $\hat{U}_1 \in \Gamma(\ker \Pi_*)$ and $\hat{Z}, \hat{W} \in \Gamma(\ker \Pi_*)^{\perp}$. By using equations (2.11), (2.12), (3.14) and (3.15), we can write

$$
\begin{aligned}
\frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{Z}, \hat{U}_1), \Pi_*\hat{W}) =& - g_1(\nabla_{\hat{Z}} \mathbb{P}\hat{U}, \hat{W}) + g_1(\nabla_{\hat{Z}} \psi^2 Q\hat{U}, \hat{W}) - g_1(\nabla_{\hat{Z}} \zeta\psi\mathbb{Q}\hat{U}, \hat{W})\\
&- g_1(\mathfrak{H}\nabla_{\hat{Z}} \zeta\mathbb{Q}\hat{U}, \mathfrak{C}\hat{W}) - g_1(\mathfrak{A}_{\hat{Z}} \zeta\mathbb{Q}\hat{U}, \mathfrak{B}\hat{W}).
\end{aligned}
$$

Since $\mathcal{PWSSCS}$, then by using Lemma 3.3 and definition of horizontal conformality 2.1, the above equation turn into

$$
\begin{aligned}
\frac{1}{\lambda^2} g_2((\nabla \Pi_*)(\hat{Z}, \hat{U}_1), \Pi_*\hat{W}) =& - g_1(\mathfrak{A}_{\hat{Z}} \mathbb{P}\hat{U}, \hat{W}) - \sin 2\theta \hat{Z}(\theta) g_1(\mathbb{Q}\hat{U}, \hat{W}) + \cos^2\theta g_1(\nabla_{\hat{Z}} \mathbb{Q}\hat{U}, \hat{W})\\
&- \frac{1}{\lambda^2} g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}} \zeta\psi\mathbb{Q}\hat{U}), \Pi_*(\hat{W})) - \frac{1}{\lambda^2} g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}} \zeta\mathbb{Q}\hat{U}), \Pi_*(\mathfrak{C}\hat{W}))\\
&- g_1(\mathfrak{A}_{\hat{Z}} \zeta\mathbb{Q}\hat{U}, \mathfrak{B}\hat{W}).
\end{aligned}
\tag{4.47}
$$

The second term on right hand side of above equations become 0 since $\mathbb{Q}\hat{U}$ and $\hat{W}$ both are orthogonal, whereas the third term reduces with equations (2.11), (2.12) and Lemma 3.3 as,

$-\cos^2\theta(g_1(\nabla_{\hat{Z}}\psi^2 Q\hat{U},\hat{W})+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W}))+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}+\mathfrak{C}\hat{W})$. With this value equation (4.47) reduces to

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W})\\
&=-g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W})-\cos^2\theta g_1(\nabla_{\hat{Z}}\psi^2 Q\hat{U},\hat{W})+\cos^4\theta g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W})\\
&\quad+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}+\mathfrak{C}\hat{W})+\cos^2\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})\\
&\quad-\frac{1}{\lambda^2}g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U}),\Pi_*(\hat{W}))-\frac{1}{\lambda^2}g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U}),\Pi_*(\mathfrak{C}\hat{W}))\\
&\quad-g_1(\mathfrak{A}_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}).
\end{aligned}\tag{4.48}$$

Since $\Pi$ is a $\mathcal{PWSSCS}$ from KM onto RM, by using the formula of second fundamental form of $\Pi$ and Lemma 3.3, sixth term in the right hand side of above equations reduces to $\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}\hat{U}+\zeta\psi\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\psi\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\hat{W})$ and the seventh term as $\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\mathbb{Q}\hat{U}+\zeta\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\mathfrak{C}\hat{W})$. Putting these values in equation (4.48), we have

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W})\\
=&\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}\hat{U}+\zeta\psi\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\psi\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\hat{W})\\
&+\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\mathbb{Q}\hat{U}+\zeta\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\mathfrak{C}\hat{W})\\
&-g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W})-\cos^2\theta g_1(\nabla_{\hat{Z}}\psi^2 Q\hat{U},\hat{W})+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W})-g_1(\mathfrak{A}_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W})\\
&+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}+\mathfrak{C}\hat{W})+\cos^4\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})\\
&-\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\mathbb{Q}\hat{U},\Pi_*\hat{W})-\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\mathbb{Q}\hat{U},\Pi_*\mathfrak{C}\hat{W}).
\end{aligned}$$

Finally, by using definition of horizontal conformality with Lemma 3.3 and equation (2.7), we can write

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W})\\
&=\sin^2\theta\{\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}\hat{U}+\zeta\psi\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\psi\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\hat{W})\\
&+\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\mathbb{Q}\hat{U}+\zeta\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\mathfrak{C}\hat{W})\}\\
&-\sin^2\theta\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\mathbb{Q}\hat{U},\Pi_*\hat{W})-\sin^2\theta\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\mathbb{Q}\hat{U},\Pi_*\mathfrak{C}\hat{W})\\
&-g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W})+\cos^4\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W}),
\end{aligned}$$

from which we can get part $(iii)$ of Theorem.

## 5. Pluriharmonicity

In this section, we discussed the concept of $J$-pluriharmonicity on AHMs which was once studied and defined by Y. Ohnita [26]. Let $\Pi$ be a $\mathcal{PWSSCS}$ from KM $(\Theta_1, g_1, J)$ onto a RM $(\Theta_2, g_2)$. Then $\mathcal{PWSSCS}$ is $J$-pluriharmonic, $\mathfrak{D}$-$J$-pluriharmonic, $\mathfrak{D}^\theta$-$J$-pluriharmonic, $(\mathfrak{D}-\mathfrak{D}^\theta)$-$\phi$ pluriharmonic, $\ker \Pi_*$-$J$-pluriharmonic, $(\ker \Pi_*)^\perp$-$J$-pluriharmonic and $((\ker \Pi_*)^\perp - \ker \Pi_*)$-$\phi$-pluriharmonic if

$$(\nabla \Pi_*)(\hat{X}, \hat{Y}) + (\nabla \Pi_*)(J\hat{X}, J\hat{Y}) = 0, \tag{5.49}$$

for any $\hat{X}, \hat{Y} \in \Gamma(\mathfrak{D})$, for any $\hat{X}, \hat{Y} \in \Gamma(\mathfrak{D}^\theta)$, for any $\hat{X} \in \Gamma(\mathfrak{D}), \hat{Y} \in \Gamma(\mathfrak{D}^\theta)$, for any $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)$, for any $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^\perp$ and for any $\hat{X} \in \Gamma(\ker \Pi_*)^\perp, \hat{Y} \in \Gamma(\ker \Pi_*)$.

**Theorem 5.1.** *Let $\Pi$ be a $\mathcal{PWSSCS}$ from KM $(\Theta_1, g_1, J)$ onto a RM $(\Theta_2, g_2)$. Suppose that $\Pi$ is $\mathfrak{D}^\theta$-$J$-pluriharmonic. Then $\mathfrak{D}^\theta$ defines totally geodesic foliation on $\Theta_1$ if and only if*

$$\begin{aligned}\nabla^{\Pi}_{J\hat{X}_1}\Pi_* J\hat{Y}_1 + \nabla^{\Pi}_{\zeta\hat{X}_1}\Pi_*\zeta\hat{Y}_1 =& \Pi_*(\mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \mathfrak{A}_{\zeta\hat{X}_1}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{P}\psi\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{P}\psi\hat{Y}_1)\\ &+ \Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{Q}\psi\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{Q}\psi\hat{Y}_1)\\ &- \cos^2\theta\,\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi\zeta\mathbb{Q}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\mathbb{Q}\psi\hat{Y}_1),\end{aligned}$$

*for any $\hat{\hat{X}}_1, \hat{Y}_1 \in \Gamma(\mathfrak{D}^\theta)$.*

*Proof.* For any $\hat{\hat{X}}_1, \hat{Y}_1 \in \Gamma(\mathfrak{D}^\theta)$ and using the pluriharmonicity of $J$ with equation (2.10), we get

$$\begin{aligned}0 &= (\nabla \Pi_*)(\hat{X}_1, \hat{Y}_1) + (\nabla \Pi_*)(J\hat{X}_1, J\hat{Y}_1)\\ &= -\Pi_*\nabla_{\hat{\hat{X}}_1}\hat{Y}_1 + \nabla^{\Pi}_{J\hat{X}_1}\Pi_* J\hat{Y}_1 - \Pi_*\nabla_{J\hat{X}_1}J\hat{Y}_1.\end{aligned}$$

Now, from the above equation, we can write

$$\Pi_*\nabla_{\hat{\hat{X}}_1}\hat{Y}_1 = \nabla^{\Pi}_{J\hat{X}_1}\Pi_* J\hat{Y}_1 - \Pi_*\nabla_{J\hat{X}_1}J\hat{Y}_1.$$

The second term in the right hand side of above equation with using equation (3.15), takes the form as $\Pi_*\nabla_{\psi\hat{X}_1}\psi\hat{Y}_1 + \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \Pi_*\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1 + \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1$. Now, equation (5) can be write as

$$\begin{aligned}\Pi_*\nabla_{\hat{\hat{X}}_1}\hat{Y}_1 =& \nabla^{\Pi}_{J\hat{X}_1}\Pi_* J\hat{Y}_1 - \Pi_*\nabla_{\psi\hat{X}_1}\psi\hat{Y}_1 - \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1\\ &- \Pi_*\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1 - \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1.\end{aligned}$$

Taking account the fact that $\Pi$ is $\mathcal{PWSSCS}$ with using equations (2.5), (2.6), (2.10) and (3.14), we have

$$\begin{aligned}\Pi_*\nabla_{\hat{X}_1}\hat{Y}_1 =& -\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \mathfrak{A}_{\zeta\hat{X}_1}\psi\hat{Y}_1 + \mathscr{V}\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1)\\ &+ \{\zeta\hat{X}_1(\ln\lambda)\Pi_*\zeta\hat{Y}_1 + \zeta\hat{Y}_1(\ln\lambda)\Pi_*\zeta\hat{X}_1 - g_1(\zeta\hat{X}_1,\zeta\hat{Y}_1)\Pi_*(grad\ln\lambda)\}\\ &- \nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1 - \nabla^{\Pi}_{\zeta\hat{X}_1}\Pi_*\zeta\hat{Y}_1 + \Pi_*(J\nabla_{\psi\hat{X}_1}J(\mathbb{P}\psi\hat{Y}_1 + \mathbb{Q}\psi\hat{Y}_1)).\end{aligned}$$

Operating $J$ in the last term in the right hand side of above equation with Lemma 3.2 and equations (2.5) and (2.6), we may have

$$\begin{aligned}\Pi_*\nabla_{\hat{X}_1}\hat{Y}_1 =& \{\zeta\hat{X}_1(\ln\lambda)\Pi_*\zeta\hat{Y}_1 + \zeta\hat{Y}_1(\ln\lambda)\Pi_*\zeta\hat{X}_1 - g_1(\zeta\hat{X}_1,\zeta\hat{Y}_1)\Pi_*(grad\ln\lambda)\}\\ &+ \Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{P}\psi\hat{Y}_1 + \mathscr{V}\nabla_{\psi\hat{X}_1}\psi^2\mathbb{P}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\zeta\psi\mathbb{P}\psi\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{P}\psi\hat{Y}_1)\\ &+ \Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{Q}\psi\hat{Y}_1 + \mathscr{V}\nabla_{\psi\hat{X}_1}\psi^2\mathbb{Q}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\zeta\psi\mathbb{Q}\psi\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{Q}\psi\hat{Y}_1)\\ &- \cos^2\theta\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi\zeta\mathbb{Q}\psi\hat{Y}_1 + \mathscr{V}\nabla_{\psi\hat{X}_1}\psi\zeta\mathbb{Q}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\mathbb{Q}\psi\hat{Y}_1 + \mathscr{V}\nabla_{\psi\hat{X}_1}\mathbb{Q}\psi\hat{Y}_1)\\ &+ \Pi_*(\mathscr{T}_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \mathfrak{A}_{\zeta\hat{X}_1}\psi\hat{Y}_1 + \mathscr{V}\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1)\\ &- \nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1 - \nabla^{\Pi}_{\zeta\hat{X}_1}\Pi_*\zeta\hat{Y}_1.\end{aligned}$$

□

**Theorem 5.2.** *Let $\Pi$ be a $\mathcal{PWSSCS}$ from KM $(\Theta_1, g_1, J)$ onto a RM $(\Theta_2, g_2)$. Suppose that $\Pi$ is $((\ker\Pi_*)^{\perp} - \ker\Pi_*)$-$J$-pluriharmonic. Then the horizontal distribution $(\ker\Pi_*)^{\perp}$ defines totally geodesic foliation on $\Theta_1$ if and only if*

$$\begin{aligned}&\Pi_*\nabla_{\hat{Y}_1}\hat{W} - \nabla^{\Pi}_{J\hat{X}_1}\Pi_*(J\hat{W}) + \cos^4\theta\Pi_*\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\mathbb{Q}\hat{W} - \cos^2\theta\Pi_*\zeta\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\zeta\mathbb{Q}\hat{W}\\ &= \sin^2\theta\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \zeta\psi\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1) - g_1(\zeta\psi\mathbb{Q}\hat{W},\mathfrak{C}\hat{Y}_1)\Pi_*grad\ln\lambda\}\\ &+ \cos^2\theta J\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\mathbb{Q}\hat{W}) + \zeta\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1) - g_1(\zeta\mathbb{Q}\hat{W},\mathfrak{C}\hat{Y}_1)\Pi_*grad\ln\lambda\}\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} + \mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W})\\ &- \sin^2\theta\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \cos^2\theta J\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\mathbb{Q}\hat{W}),\end{aligned}$$

*for any $\hat{Y}_1 \in \Gamma(\ker\Pi_*)^{\perp}$ and $\hat{W} \in \Gamma(\ker\Pi_*)$.*

*Proof.* For any $\hat{Y}_1 \in \Gamma(\ker\Pi_*)^{\perp}, \hat{W} \in \Gamma(\ker\Pi_*)$ and using equations (2.10), (3.14), (3.15) with considering the fact that the pluriharminicity of $J$, we can write

$$\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W} = -\Pi_*(\nabla_{\mathfrak{B}\hat{Y}_1}\psi\hat{W} + \nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \nabla_{\mathfrak{C}\hat{Y}_1}\psi\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W}.$$

Now, by using equations (2.4), (2.6), (3.14), (3.15) and from the Lemma 3.2, above equation can takes the form as

$$\begin{aligned}\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W} = &-\Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W} - \Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\psi\mathbb{Q}\hat{W}\\ &+ \Pi_*(J\nabla_{\mathfrak{B}\hat{Y}_1}J\psi\hat{W} + J\nabla_{\mathfrak{C}\hat{Y}_1}J\psi\hat{W}) - \cos^2\theta\Pi_*(J\nabla_{\mathfrak{C}\hat{Y}_1}JQ\hat{W})\\ &+ \Pi_*(-\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} - \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} - \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W})\\ &+ \Pi_*(-\mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} - \mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} - \mathscr{V}\nabla_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}).\end{aligned}$$

By using the horizontal conformality of Π, Lemma 3.2, equations (3.15) and (2.10) with some simple steps of calculations, we may have

$$\begin{aligned}\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W} = &-\cos^4\theta\Pi_*(\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\mathbb{Q}\hat{W}) + \cos^2\theta\Pi_*(\zeta\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\zeta\mathbb{Q}\hat{W}) + \sin^2\theta(\nabla\Pi_*)(\mathfrak{C}\hat{Y}_1,\zeta\psi\mathbb{Q}\hat{W})\\ &- \sin^2\theta\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \cos^2\theta J(\nabla\Pi_*)(\mathfrak{C}\hat{Y}_1,\zeta\mathbb{Q}\hat{W}) + \cos^2\theta J\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\mathbb{Q}\hat{W})\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W}\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W})\\ &- \Pi*(\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} - \mathscr{V}\nabla_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}).\end{aligned}$$

Finally, by using the Lemma 2.1, the above equation takes the form

$$\begin{aligned}&\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W}\\ &=\sin^2\theta\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \zeta\psi\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1) - g_1(\mathfrak{C}\hat{Y}_1,\zeta\psi\mathbb{Q}\hat{W})\Pi_*(grad\ln\lambda)\}\\ &\quad\cos^2\theta J\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\mathbb{Q}\hat{W}) + \zeta\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1) - g_1(\mathfrak{C}\hat{Y}_1,\zeta\mathbb{Q}\hat{W})\Pi_*(grad\ln\lambda)\}\\ &\quad- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W} - \cos^4\theta\Pi_*(\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\mathbb{Q}\hat{W})\\ &\quad- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W}) + \cos^2\theta\Pi_*(\zeta\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\zeta\mathbb{Q}\hat{W})\\ &\quad- \Pi*(\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} - \mathscr{V}\nabla_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}) - \sin^2\theta\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \cos^2\theta J\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\mathbb{Q}\hat{W}).\end{aligned}$$

From the above equation, we can get the proof of Theorem 5.2. □

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# RULED SURFACES WITH $T_1N_1B_1$-SMARANDACHE BASE CURVE OBTAINED FROM THE SUCCESSOR FRAME

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Abstract. In this study, ruled surfaces formed by the movement of the Frenet vectors of the successor curve along the Smarandache curve obtained from the tangent and principal normal vectors of the successor curve of a curve are definened. Then, the Gaussian and mean curvatures of each ruled surface were calculated. It has been shown that the ruled surface formed by the tangent vector of the successor curve moving along the Smarandache curve is a developable ruled surface. In addition, it was found that the surface formed by the principal normal vector of the succesor curve along the Smarandache curve is a minimal developable ruled surface if the principal curve is planar. Conditions are given for other surfaces to be developable or minimal surfaces.

**Keywords**: Smarandache ruled surfaces, Sucessor curve, mean curvature, Gaussian curvature.

**2010 Mathematics Subject Classification**: 53A04, 53A05.

## 1. Introduction

The image of a function with two real variables in three-dimensional space is a surface. Surfaces are used in many fields, such as architecture and engineering [26]. In 1795, Monge defined the ruled surface as the surface formed by the movement of the line along the curve. Any ruled surface is formed as a result of the continuous movement of a line along any curve. These curves are called the base curve and the director curve, respectively. The curvature

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of surfaces was defined by Gauss in the 19th century, and therefore it was named Gaussian curvature [19]. Gaussian curvatures are related to the dimensions of the surface [27]. Since the average curvature of the surface is a ratio, it is independent of the size of the surface. Thus far, many studies [1, 3, 6, 7, 8, 9, 10, 12, 13] on the Gaussian curvatures of surfaces have been conducted.

There are many special curves in differential geometry. One of them is the successor curve. This curve is defined as, there is a new curve, such that the tangent of one curve the principal normal of the other curve, by Menninger [14] in 2014. Later, Masal [11] investigated the relationships between the position vectors of this curve and defined Successor planes. Thus far, many studies have been conducted on this concept [5, 30]. Another special curve is the Smarandache curve defined in Minkowski space [2, 21, 28, 29].

In recent years, many studies have been carried out on ruled surfaces whose base curve is Smarandache curve. Some of these studies can be accessed from [4, 16, 17, 18, 22, 23, 24, 25].

In this paper, we present some special ruled surfaces with $T_1N_1B_1$-Smarandache curves obtained from their successor frames. We then investigate the properties of these ruled surfaces by means of Gaussian and mean curvatures. We obtain the conditions that which of these surfaces developable and which of these minimal. At the end, we visualise the main idea by providing four examples.

## 2. Preliminaries

This section provides some basic notions needed to be the following sections. Throughout this paper, let $\alpha = \alpha(s)$ and $\beta = \beta(s)$ be two differentiable unit speed curve in $E^3$ and their Frenet aparatus be $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$, respectively. Then,

$$\begin{aligned} T &= \alpha', \ N = \frac{\alpha''}{\|\alpha''\|}, \ B = T \wedge N, \ \kappa = \|\alpha''\|, \ \tau = \langle N', B \rangle, \\ T' &= \kappa N, \ N' = -\kappa T + \tau B, \ B' = -\tau N. \end{aligned}$$

The surface formed by a line moving depending on the parameter of a curve is called a ruled surface, and its parametric expression is $X(s,\nu) = \alpha(s) + \nu r(s)$. Here, $\nu$ is a constant. Besides, $\alpha$ and $r$ are referred to as the base curve and the director curve of $X$, respectively. The normal vector field $N_X$, the Gaussian curvature $K_X$, and the mean curvature $H_X$ of $X(s,\nu)$ are as follows:

$$N_X = \frac{X_s \wedge X_\nu}{\|X_s \wedge X_\nu\|}, \tag{2.1}$$

$$K_X = \frac{eg - f^2}{EG - F^2}, \quad H_X = \frac{Eg - 2fF + eG}{2(EG - F^2)}, \tag{2.2}$$

Here,

$$E = \langle X_s, X_s \rangle, \ \ F = \langle X_s, X_\nu \rangle, \ \ G = \langle X_\nu, X_\nu \rangle, \tag{2.3}$$

$$e = \langle X_{ss}, N_X \rangle, \ \ f = \langle X_{s\nu}, N_X \rangle, \ \ g = \langle X_{\nu\nu}, N_X \rangle. \tag{2.4}$$

**Definition 2.1.** [11, 14] *If the unit tangent vector of $\alpha$ is the principal normal vector of $\beta$, then $\beta$ is called Successor curve of $\alpha$.*

**Theorem 2.1.** [11, 14] *Let $\beta$ be the Successor curve of $\alpha$. Frenet apparatus of $\beta$ curve is as follows:*

$$T_1 = -\cos\theta N + \sin\theta B, \quad N_1 = T, \quad B_1 = \sin\theta N + \cos\theta B, \quad \kappa_1 = \kappa\cos\theta, \ \tau_1 = \kappa\sin\theta.$$

*where, $\theta$ is the angle between binormal vectors $B$ and $B_1$ and $\theta(s) = \theta_0 + \int \tau(s)ds$.*

**Definition 2.2.** [29] *A regular curve in Minkowski space, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Let $\beta$ be the Successor curve of $\alpha$. It can be observed that the unit curve $\gamma$, inspired in [11], produces Smarandache curves, for all $s \in I \subseteq \mathbb{R}$, such that

$$\gamma(s) = \frac{aT + bN + cB}{\sqrt{a^2 + b^2 + c^2}}, \ \ a, b, c \in \mathbb{R}.$$

Here, if $a$, $b$, and $c$ are nonzero the Smarandache curves produced by $\gamma(s)$ are denoted by $\{T\,N\,B\}$-Smarandache Curves. This paper consider $\{T\,N\,B\}$-Smarandache Curves.

## 3. Ruled Surfaces with $T_1N_1B_1$-Smarandache Base Curve Obtained From the Successor Frame

In this section, firstly we define some special ruled surfaces with $T_1N_1B_1$-Smarandache base curve obtained from the successor frame. Then we examine the properties of these ruled surfaces by means of Gaussian and mean curvatures. And we give the conditions of being developable or minimal surface.

**Definition 3.1.** *Let the successor curve of the $\alpha$ curve be $\beta$. The ruled surface formed by targent vector $T_1$ the vector along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as follows:*

$$\begin{aligned} \Phi(s,v) &= \tfrac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vT_1 \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B) + v(-\cos\theta N + \sin\theta B). \end{aligned} \tag{3.5}$$

**Theorem 3.1.** *Let the successor curve of the curve $\alpha$ be $\beta$. The Gaussian and mean curvature of the $\Phi(s,v)$ ruled surface are as follows:*

$$K_\Phi = \frac{-3\cos^2\theta\sin^2\theta\big(((1+v\sqrt{3})\cos\theta-\sin\theta)^2+\sin^2\theta\big)}{\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$H_\Phi = \frac{\sqrt{3}\kappa\sin\theta\Big(2\cos^2\theta-\big(((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2+1)\Big)-\sqrt{3}\tau(1+v\sqrt{3})}{2\kappa\big(\sin^2\theta+((1+v\sqrt{3})^2\cos\theta-\sin\theta)^2\big)^{\frac{3}{2}}}.$$

*Proof.* Partial derivatives of equation (3.5) are,

$$\Phi_s = \frac{\kappa}{\sqrt{3}}\Big(((1+v\sqrt{3})\cos\theta-\sin\theta\big)T+N\Big),\quad \Phi_v = -\cos\theta N+\sin\theta B,\quad \Phi_{sv}=\kappa\cos\theta T,$$

$$\Phi_{ss} = \frac{\begin{array}{l}\Big(\kappa'\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)-\kappa^2-\kappa\tau\big((1+v\sqrt{3})\sin\theta+\cos\theta\big)\Big)T\\ +\Big(\kappa'+\kappa^2\big(\sin\theta-(1-v\sqrt{3})\cos\theta\big)\Big)N+\kappa\tau B\end{array}}{\sqrt{3}},\quad \Phi_{vv}=0.$$

Thus, from equation (2.1) the normal of the surface $N_\Phi$ is given as

$$N_\Phi = \frac{\sin\theta T-\sin\theta\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)N-\cos\theta\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)B}{\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\Phi = \frac{\kappa^2}{3}\big(((1+v\sqrt{3})\cos\theta-\sin\theta)^2+1\big),\quad F_\Phi = -\frac{\kappa\cos\theta}{\sqrt{3}},\quad G_\Phi = 1,$$

$$e_\Phi = -\frac{\kappa^2\sin\theta\big(\sin\theta-(1+v\sqrt{3})\cos\theta+1\big)+\kappa\tau(1+v\sqrt{3})}{\sqrt{3}\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_\Phi = \frac{\kappa\cos\theta\sin\theta}{\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}},\quad g_\Phi = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Corollary 3.1.** *Let the successor curve of the $\alpha$ curve be $\beta$. If $\alpha$ curve is planar and $\theta=\pi+k\pi$ $(k\in\mathbb{N})$, the ruled surface $\Phi(s,v)$ is the minimal developable surface.*

**Definition 3.2.** *Let the successor curve of the $\alpha$ curve be $\beta$. The ruled surface formed by principal normal normal vector $N_1$ along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as* *follows:*

$$\begin{aligned} Q(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1)+vN_1 \\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)+vT. \end{aligned} \tag{3.6}$$

**Theorem 3.2.** *Let the successor curve of the $\alpha$ curve be $\beta$. The Gaussian and mean curvature of the $Q(s,v)$ ruled surface are as follows:*

$$K_Q=0, \quad H_Q=\frac{\sqrt{3}\tau}{2\kappa(1+v\sqrt{3})}.$$

*Proof.* Partial derivatives of equation (3.6) are,

$$Q_s=\frac{\kappa}{\sqrt{3}}\big((\cos\theta-\sin\theta)T+(1+v\sqrt{3})N\big), \qquad Q_v=T, \qquad Q_{sv}=\kappa N \qquad Q_{vv}=0,$$

$$Q_{ss}=\frac{\big(\kappa'(\cos\theta-\sin\theta)-\kappa\tau(\cos\theta+\sin\theta)-\kappa^2(1+v\sqrt{3})\big)T+\big(\kappa^2(\cos\theta-\sin\theta)+\kappa'(1+v\sqrt{3})\big)N+\kappa\tau(1+v\sqrt{3})B}{\sqrt{3}}.$$

Thus, from equation (2.1) the normal of the surface $N_Q$ is given as $N_Q=-B$. Moreover, in equaitons (2.3) and (2.4) the coefficients of fundamental forms are

$$\begin{aligned} &E_Q=\frac{\kappa^2}{3}\big((\cos\theta-\sin\theta)^2+(1+v\sqrt{3})^2\big), \quad F_Q=\frac{\kappa}{\sqrt{3}}(\cos\theta-\sin\theta), \quad G_Q=1, \\ &e_Q=-\frac{\kappa\tau}{\sqrt{3}}(1+v\sqrt{3}), \quad f_Q=g_Q=0. \end{aligned}$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Corollary 3.2.** *Let the successor curve of the $\alpha$ curve be $\beta$. If $\alpha$ curve is planar, the ruled surface $Q(s,v)$ is the minimal developable surface.*

**Definition 3.3.** *Let the successor curve of the $\alpha$ curve be $\beta$. The ruled surface formed by binormal vector $B_1$ the vector along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as follows:*

$$\begin{aligned} M(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1)+vB_1 \\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)+v(\sin\theta N+\cos\theta B). \end{aligned} \tag{3.7}$$

**Theorem 3.3.** *Let the successor curve of the $\alpha$ curve be $\beta$. The Gaussian and mean curvature of the $M(s,v)$ ruled surface are as follows:*

$$\begin{aligned} K_M &= \frac{-3\sin^2\theta\cos^2\theta}{\Big(\cos^2\theta+\big(\cos\theta-(1+v\sqrt{3})\sin\theta\big)^2\Big)^2}, \\ H_M &= \frac{-\sqrt{3}\kappa\cos\theta\big(2\sin^2+(\cos\theta-(1+v\sqrt{3})\sin\theta)^2+1\big)-\sqrt{3}\tau(1+v\sqrt{3})}{2\kappa\Big(\cos^2\theta+\big(\cos\theta-(1+v\sqrt{3})\sin\theta\big)\Big)^{\frac{3}{2}}}. \end{aligned}$$

*Proof.* Partial derivatives of equation (3.7) are,

$$M_s = \frac{\kappa}{\sqrt{3}}\big((\cos\theta - (1+v\sqrt{3})\sin\theta)T + N\big), \quad M_v = \sin\theta N + \cos\theta B,$$

$$M_{sv} = -\kappa\sin\theta T, \quad M_{vv} = 0,$$

$$M_{ss} = \frac{\begin{array}{l}\big(\kappa'(\cos\theta - (1+v\sqrt{3})\sin\theta) - \kappa\tau(\sin\theta + (1-v\sqrt{3})\cos\theta) - \kappa^2\big)T \\ +\big(\kappa' + \kappa^2(\cos\theta - (1+v\sqrt{3})\sin\theta)\big)N + \kappa\tau B\end{array}}{\sqrt{3}}.$$

Thus, from equation (2.1) the normal of the surface $N_M$ is given as

$$N_M = \frac{\cos\theta T - \cos\theta\big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)N + \sin\theta\big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)B}{\Big(\cos^2\theta + \big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_M = \frac{\kappa^2}{3}\big((\cos\theta - (1+v\sqrt{3})\sin\theta)^2 + 1\big), \quad F_M = \frac{\kappa\sin\theta}{\sqrt{3}}, \quad G_M = 1,$$

$$e_M = \frac{-\kappa\tau(1+v\sqrt{3}) - \kappa^2\cos\theta\big((\cos\theta - (1+v\sqrt{3})\sin\theta)^2 + 1\big)}{\sqrt{3}\Big(\cos^2\theta + \big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_M = \frac{-\kappa\sin\theta\cos\theta}{\Big(\cos^2\theta + \big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)^2\Big)^{\frac{1}{2}}}, \quad g_M = 0.$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Corollary 3.3.** *If the $\theta = \pi + k\pi \ (k \in \mathbb{N})$, the ruled surface $M(s,v)$ is a developable surface.*

**Definition 3.4.** *Let the successor curve of the $\alpha$ curve be $\beta$. The ruled surface formed by $T_1N_1$ the vector along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as follows:*

$$\begin{aligned}\mu(s,v) &= \tfrac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \tfrac{v}{\sqrt{2}}(T_1 + N_1) \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B) + \tfrac{v}{\sqrt{2}}(T - \cos\theta N + \sin\theta B).\end{aligned} \tag{3.8}$$

**Theorem 3.4.** *Let the successor curve of the $\alpha$ curve be $\beta$. The Gaussian and mean curvature of the $\mu(s,v)$ ruled surface are as follows:*

$$K_\mu = \frac{-6\sin^4\theta}{\begin{array}{l}\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big(-\sqrt{2}\sin^2\theta + (\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta\big)^2\big(\sqrt{2}\cos\theta\sin\theta - (\sqrt{2}+v\sqrt{3})(\cos^2\theta+1)\big)^2\Big). \\ \Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2 - \sin^2\theta\Big)\end{array}},$$

$$H_\mu = \frac{2\sqrt{6}\kappa\sin^3\theta - \sqrt{6}\kappa\sin\theta\Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2\Big) - 2\sqrt{6}\tau(\sqrt{2}+v\sqrt{3})^2}{\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\cos^2\theta\big)^2\Big)^{\frac{1}{2}}.}.$$
$$2\kappa\Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2 - \sin^2\theta\Big)$$

*Proof.* Partial derivatives of equation (3.8) are,

$$\mu_s = \frac{\kappa}{\sqrt{6}}\big(((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta)T + (\sqrt{2}+v\sqrt{3})N\big), \quad \mu_{sv} = \frac{\kappa}{\sqrt{2}}(\cos\theta T + N),$$
$$\mu_v = \frac{1}{\sqrt{2}}(T - \cos\theta N + \sin\theta B), \quad \mu_{vv} = 0,$$
$$\mu_{ss} = \frac{\begin{array}{c}\Big((\sqrt{2}+v\sqrt{3})\big(\kappa'\cos\theta - \kappa\tau\sin\theta - \kappa^2\big) - \sqrt{2}\big(\kappa'\sin\theta + \kappa\tau\cos\theta\big)\Big)T \\ +\Big(\kappa^2\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big) + (\sqrt{2}+v\sqrt{3})\kappa'\Big)N + (\sqrt{2}+v\sqrt{3})\kappa\tau B\end{array}}{\sqrt{6}}.$$

Thus, from equation (2.1) the normal of the surface $N_\mu$ is given as

$$N_\mu = \frac{(\sqrt{2}+v\sqrt{3})\sin\theta T - \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)N - \big((\sqrt{2}+v\sqrt{3})(\cos^2\theta+1) - \sqrt{2}\cos\theta\sin\theta\big)B}{\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})(\cos^2\theta+1) - \sqrt{2}\cos\theta\sin\theta\big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\mu = \frac{\kappa^2}{6}\big(((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta)^2 + (\sqrt{2}+v\sqrt{3})^2\big), \quad F_\mu = -\frac{\kappa\sin\theta}{\sqrt{6}}, \quad G_\mu = 1, \quad g_\mu = 0,$$

$$e_\mu = -\frac{2\kappa\tau(\sqrt{2}+v\sqrt{3})^2 + \kappa^2\sin\theta\Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2\Big)}{\sqrt{6}\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\cos^2\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_\mu = \frac{\kappa\sin^2\theta}{\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\cos^2\theta\big)^2\Big)^{\frac{1}{2}}.}.$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Corollary 3.4.** *If the $\theta = k\pi$ $(k \in \mathbb{N})$, the ruled surface $\mu(s,v)$ is a developable surface.*

**Definition 3.5.** *Let the successor curve of the $\alpha$ curve be $\beta$. The ruled surface formed by $T_1B_1$ the vector along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as follows:*

$$\begin{aligned}\psi(s,v) &= \tfrac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \tfrac{v}{\sqrt{2}}(T_1 + B_1) \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B) + \tfrac{v}{\sqrt{2}}\big((\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B\big).\end{aligned} \tag{3.9}$$

**Theorem 3.5.** *Let the successor curve of the $\alpha$ curve be $\beta$. The Gaussian and mean curvature of the $\psi(s,v)$ ruled surface are as follows:*

$$K_\psi = \frac{-6(\cos^2\theta-\sin^2\theta)^2}{\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big).\\ (1-\sin 2\theta)\big((\sqrt{2}+v\sqrt{3})^2-1)+2\big)},$$

$$H_\psi = \frac{3\kappa(\sin\theta+\cos\theta)\big((\sqrt{2}+v\sqrt{3})^2(\sin 2\theta-1)+2(\sin\theta-\cos\theta)^2-2\big)\\ -3\tau(\sqrt{2}+v\sqrt{3})}{\sqrt{6}\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big).\Big)^{\frac{1}{2}}\\ (1-\sin 2\theta)\big((\sqrt{2}+v\sqrt{3})^2-1)+2\big)}.$$

*Proof.* Partial derivatives of equation (3.8) are,

$$\psi_s = \frac{\kappa}{\sqrt{6}}\big((\sqrt{2}+v\sqrt{3})(\cos\theta-\sin\theta)T+\sqrt{2}N\big), \quad \psi_{sv} = \frac{\kappa}{\sqrt{2}}(\cos\theta-\sin\theta)T,$$

$$\psi_v = \frac{1}{\sqrt{2}}(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B, \quad \psi_{vv}=0,$$

$$\psi_{ss} = \frac{\Big(\big((\sqrt{2}+v\sqrt{3})(\kappa'(\cos\theta-\sin\theta)-\kappa\tau(\sin\theta+\cos\theta)\big)-\sqrt{2}\kappa^2\Big)T\\ +\Big(\sqrt{2}\kappa'+\kappa^2(\sqrt{2}+v\sqrt{3})(\cos\theta-\sin\theta)\Big)N+\sqrt{2}\kappa\tau B}{\sqrt{6}}.$$

Thus, from equation (2.1) the normal of the surface $N_\psi$ is given as

$$N_\psi = \frac{\sqrt{2}(\sin\theta+\cos\theta)T+(\sqrt{2}+v\sqrt{3})(\sin^2\theta-\cos^2\theta)N+(\sqrt{2}+v\sqrt{3})(\sin 2\theta-1)B}{\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big)^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\psi = \frac{\kappa^2}{6}\big((\sqrt{2}+v\sqrt{3})^2(1-\sin 2\theta)+2\big), \quad F_\psi = \frac{\kappa}{\sqrt{6}}(\sin\theta-\cos\theta), \quad G_\psi = 1,$$

$$e_\psi = -\frac{\kappa\tau(4-2v\sqrt{6})+\kappa^2(\sin\theta+\cos\theta)\big(2+(\sqrt{2}+v\sqrt{3})^2(1-\sin 2\theta)\big)}{\sqrt{6}\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big)^{\frac{1}{2}}},$$

$$f_\psi = -\frac{-\kappa(\cos^2\theta-\sin^2\theta)}{\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big)^{\frac{1}{2}}}, \quad g_\psi = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Corollary 3.5.** *If the $\theta = \frac{\pi}{4}+\frac{k\pi}{2}$ $(k\in\mathbb{Z})$, the ruled surface $\psi(s,v)$ is a developable surface.*

**Definition 3.6.** *Let the successor curve of the $\alpha$ curve be $\beta$. The ruled surface formed by $N_1B_1$ the vector along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as follows:*

$$\begin{aligned}\eta(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1)+\tfrac{v}{\sqrt{2}}(N_1+B_1)\\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)+\tfrac{v}{\sqrt{2}}(T+\sin\theta N+\cos\theta B).\end{aligned} \tag{3.10}$$

**Theorem 3.6.** *Let the successor curve of the $\alpha$ curve be $\beta$. The Gaussian and mean curvature of the $\eta(s,v)$ ruled surface are as follows:*

$$K_\eta = \frac{-6\cos^4\theta}{\begin{pmatrix}((\sqrt{2}+v\sqrt{3})\cos\theta)^2-(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta)^2\\ +(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1))^2\end{pmatrix}^{\frac{1}{2}}\cdot\Big((\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta)^2+(\sqrt{2}+v\sqrt{3})^2-\cos^2\theta\Big)},$$

$$H_\eta = -\frac{6\tau(\sqrt{2}+v\sqrt{3})^2+3\kappa\cos\theta\Big((\sqrt{2}+v\sqrt{3})^2+(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta)^2-2\cos^2\theta\Big)}{\sqrt{6}\kappa\begin{pmatrix}((\sqrt{2}+v\sqrt{3})\cos\theta)^2-(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta)^2\\ +(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1))^2\end{pmatrix}^{\frac{1}{2}}\cdot\Big((\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta)^2+(\sqrt{2}+v\sqrt{3})^2-\cos^2\theta\Big)}.$$

*Proof.* Partial derivatives of equation (3.10) are,

$$\eta_s = \frac{\kappa}{\sqrt{6}}\Big((\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta)T+(\sqrt{2}+v\sqrt{3})N\Big), \quad \eta_{sv} = -\frac{\kappa}{\sqrt{2}}(\sin\theta T-N),$$
$$\eta_v = \frac{1}{\sqrt{2}}(T+\sin\theta N+\cos\theta B), \quad \eta_{vv}=0,$$

$$\eta_{ss} = \frac{\begin{aligned}&\Big(\kappa'(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta)-\kappa\tau((\sqrt{2}+v\sqrt{3})\cos\theta+\sqrt{2}\sin\theta)-\kappa^2(\sqrt{2}+v\sqrt{3})\Big)T\\ &+\Big(\kappa'(\sqrt{2}+v\sqrt{3})+\kappa^2(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta)\Big)N+\kappa\tau(\sqrt{2}+v\sqrt{3})B\end{aligned}}{\sqrt{6}}.$$

Thus, from equation (2.1) the normal of the surface $N_\eta$ is given as

$$N_\eta = \frac{\begin{aligned}&((\sqrt{2}+v\sqrt{3})\cos\theta)T-(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta)N\\ &+(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1))B\end{aligned}}{\begin{pmatrix}((\sqrt{2}+v\sqrt{3})\cos\theta)^2+(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta)^2\\ +(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1))^2\end{pmatrix}^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\eta = \frac{\kappa^2}{6}\Big(\big(\sqrt{2}\cos\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2\Big), \quad F_\eta = \frac{\kappa\cos\theta}{\sqrt{6}}, \quad G_\eta = 1,$$

$$e_\eta = \frac{-2\kappa\tau(\sqrt{2}+v\sqrt{3})^2 - \kappa^2\cos\theta\Big((\sqrt{2}+v\sqrt{3})^2 + \big(\sqrt{2}\cos\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\big)^2\Big)}{\sqrt{6}\begin{pmatrix}\big((\sqrt{2}+v\sqrt{3})\cos\theta\big)^2 + \big(\sqrt{2}\cos^2\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2 \\ +\big(\sqrt{2}\cos\theta\sin\theta - (\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}},$$

$$f_\eta = \frac{-\kappa\cos^2\theta}{\begin{pmatrix}\big((\sqrt{2}+v\sqrt{3})\cos\theta\big)^2 + \big(\sqrt{2}\cos^2\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2 \\ +\big(\sqrt{2}\cos\theta\sin\theta - (\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}}, \quad g_\eta = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Corollary 3.6.** *If the $\theta = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{N})$, the ruled surface $\eta(s,v)$ is a minimal developable surface.*

**Definition 3.7.** *Let $\beta$ be the Successor curve of $\alpha$. The ruled surface formed by $T_1N_1B_1$ the vector along the $T_1N_1B_1$ Smarandache curve obtained from the $T_1$ targent vector, $N_1$ principal normal vector and $B_1$ binormal vector of the $\beta$ curve as follows:*

$$\begin{aligned}\Gamma(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1) + \tfrac{v}{\sqrt{3}}(T_1+N_1+B_1) \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta-\cos\theta)N + (\sin\theta+\cos\theta)B) + \tfrac{v}{\sqrt{3}}(T + (\sin\theta-\cos\theta)N + (\sin\theta+\cos\theta)B)\end{aligned} \tag{3.11}$$

**Theorem 3.7.** *Let $\beta$ be the Successor curve of $\alpha$. The Gaussian and mean curvature of the $\Gamma(s,v)$ ruled surface are as follows:*

$$K_\Gamma = 0, \quad H_\Gamma = -\frac{\sqrt{3}\tau + \sqrt{3}\kappa\big((\sin\theta+\cos\theta)(\sin\theta-\cos\theta)\cos 2\theta}{2\kappa(1+v)(2-\sin 2\theta)\sqrt{2+\sin 2\theta}}.$$

*Proof.* Partial derivatives of equation (3.11) are,

$$\Gamma_s = \frac{\kappa(1+v)}{\sqrt{3}}\big((\cos\theta-\sin\theta)T + N\big), \quad \Gamma_{sv} = \frac{\kappa}{\sqrt{3}}\big((\cos\theta-\sin\theta)T + N\big),$$

$$\Gamma_v = \frac{1}{\sqrt{3}}\big(T + (\sin\theta-\cos\theta)N + (\sin\theta+\cos\theta)B\big), \quad \Gamma_{vv} = 0,$$

$$\Gamma_{ss} = \frac{(1+v)\big(\kappa'(\cos\theta-\sin\theta) - \kappa\tau(\sin\theta+\cos\theta) - \kappa^2\big)T + \big(\kappa' + \kappa^2(\sin\theta-\cos\theta)\big)N + \kappa\tau B}{\sqrt{3}}.$$

Thus, from equation (2.1) the normal of the surface $N_\Gamma$ is given as

$$N_\Gamma = \frac{(\sin\theta+\cos\theta)T - \cos 2\theta N + \sin 2\theta B}{\sqrt{2+\sin 2\theta}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\Gamma = \frac{\kappa^2(1+v)^2(2-\sin 2\theta)}{3}, \quad F_\Gamma = 0, \quad G_\Gamma = 1,$$
$$e_\Gamma = -\frac{\kappa\tau(1+v) + \kappa^2(1+v)\big((\sin\theta+\cos\theta)(\cos\theta-\sin\theta)\cos 2\theta\big)}{\sqrt{6+\sqrt{3}\sin 2\theta}}, \quad f_\Gamma = 0, \quad g_\Gamma = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

**Example 3.1.** *Let $\beta$ Salkowski curve* [15] *be the Successor curve of $\alpha$. The equation of this curve for $m = \frac{1}{3}$ is as follows:*

$$\beta(s) = \frac{3}{\sqrt{10}} \begin{pmatrix} -\frac{\sqrt{10}-1}{4\sqrt{10}+8}\big(\sin(\frac{\sqrt{10}+2}{\sqrt{10}})s\big) - \frac{\sqrt{10}-1}{4\sqrt{10}-8}\big(\sin(\frac{\sqrt{10}-2}{\sqrt{10}})s\big) - \frac{1}{2}\sin s, \\ -\frac{\sqrt{10}-1}{4\sqrt{10}+8}\big(\cos(\frac{\sqrt{10}+2}{\sqrt{10}})s\big) + \frac{\sqrt{10}-1}{4\sqrt{10}-8}\big(\cos(\frac{\sqrt{10}-2}{\sqrt{10}})s\big) + \frac{1}{2}\cos s, \frac{3}{4}\cos(\frac{2s}{\sqrt{10}}) \end{pmatrix}$$

*The Successor frames of $\beta$ curve $\{T_1, N_1, B_1\}$ are as follows:*

$$T_1(s) = \left( -\cos s\cos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}, \ -\sin s\cos\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}}, \ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}} \right),$$

$$N_1(s) = \left( \frac{3}{\sqrt{10}}\sin s, \ -\frac{3}{\sqrt{10}}\cos s, \ -\frac{1}{\sqrt{10}} \right),$$

$$B_1(s) = \left( -\cos s\sin\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}, \ -\sin s\cos\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}, \ \frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}} \right)$$

*The graphs of the ruled surfaces obtained from these frames for $s \in [-\pi, \pi]$ and $v \in [-1, 1]$ are shown figures 1- 7;*

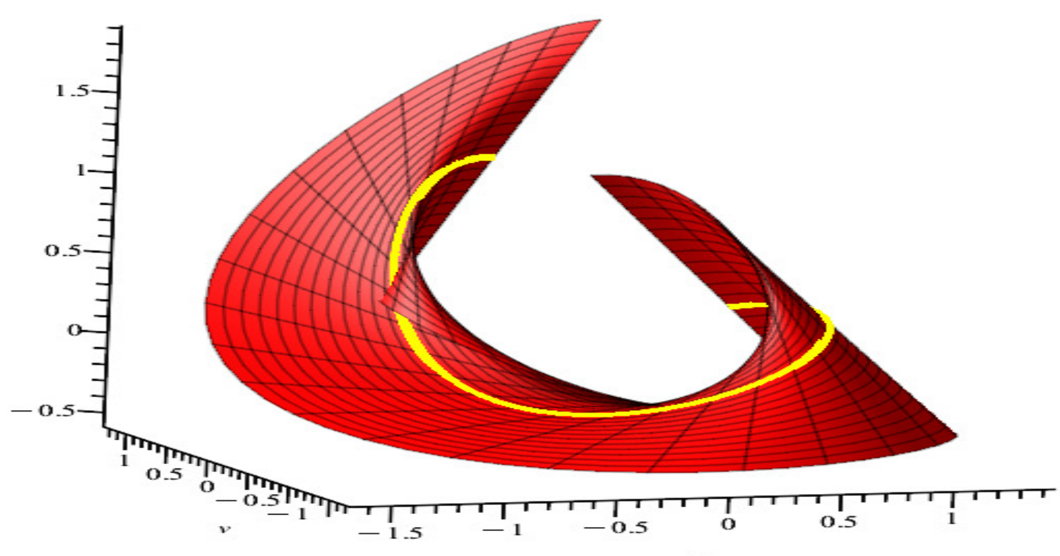


FIGURE 1. The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vT_1$

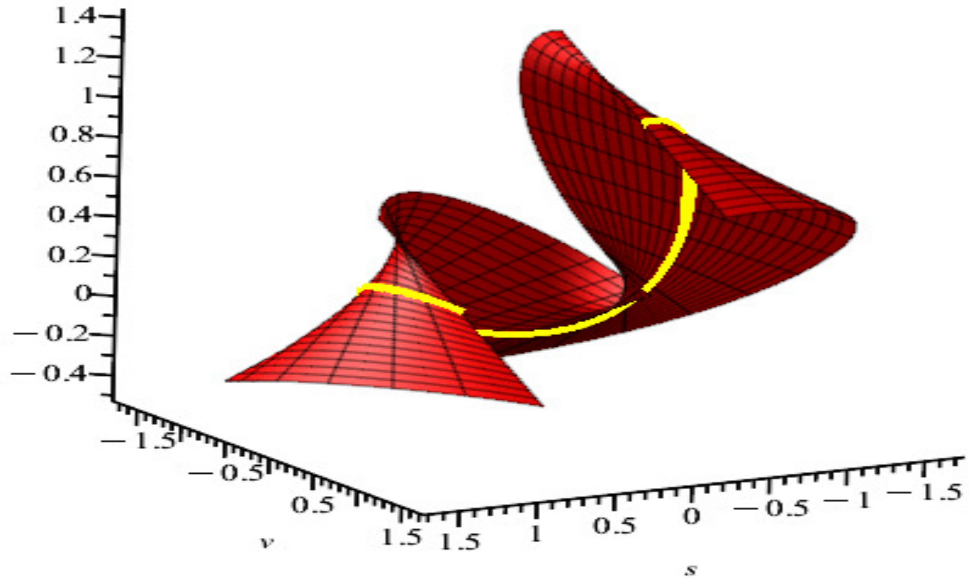


FIGURE 2. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

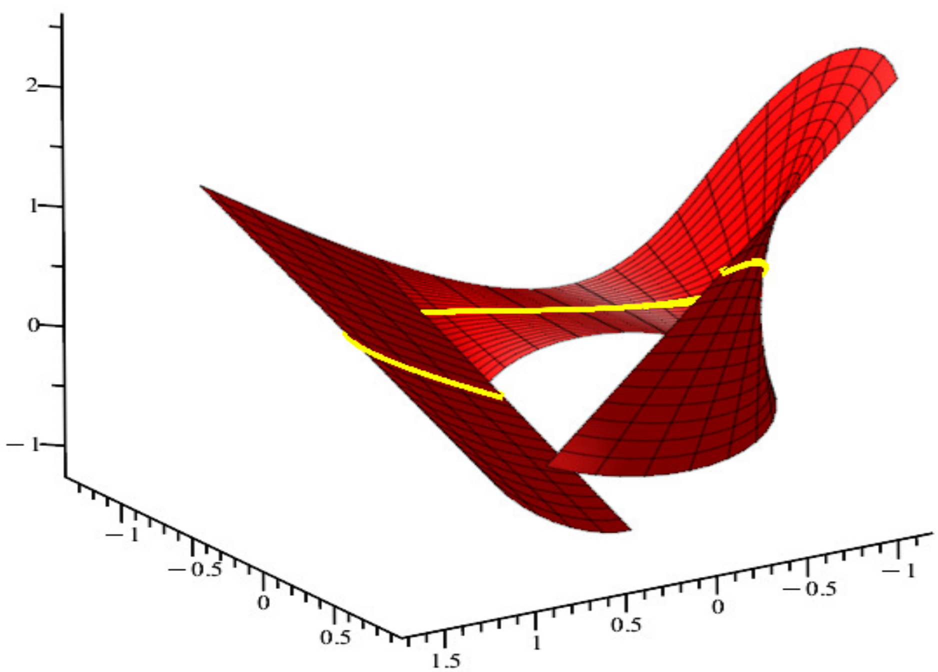


FIGURE 3. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

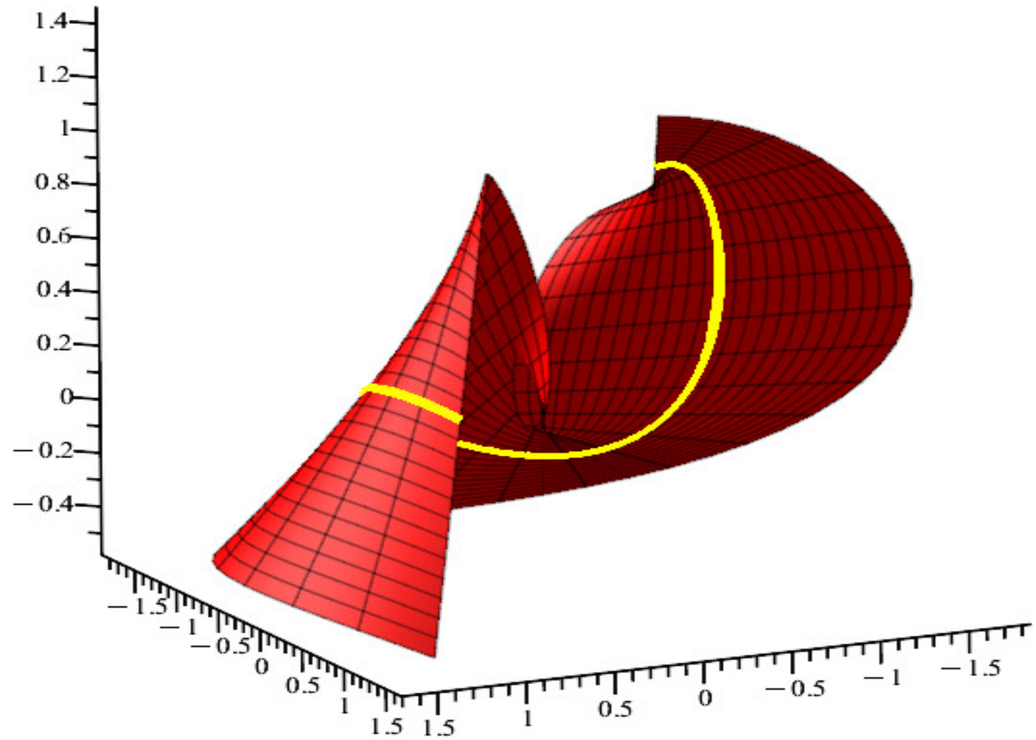

FIGURE 4. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

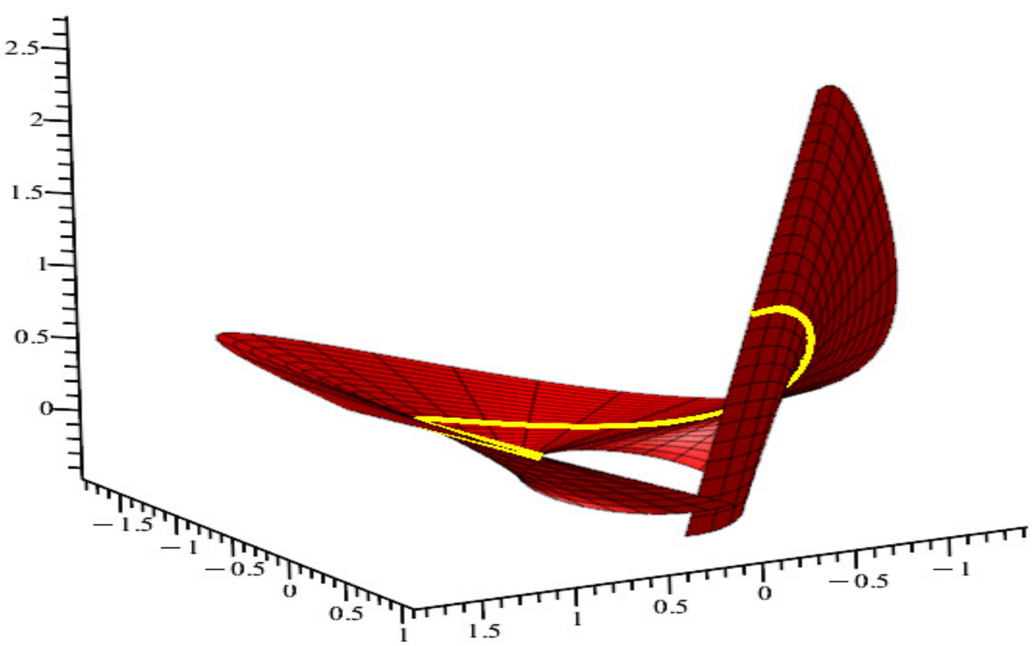

FIGURE 5. The ruled surface $\psi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

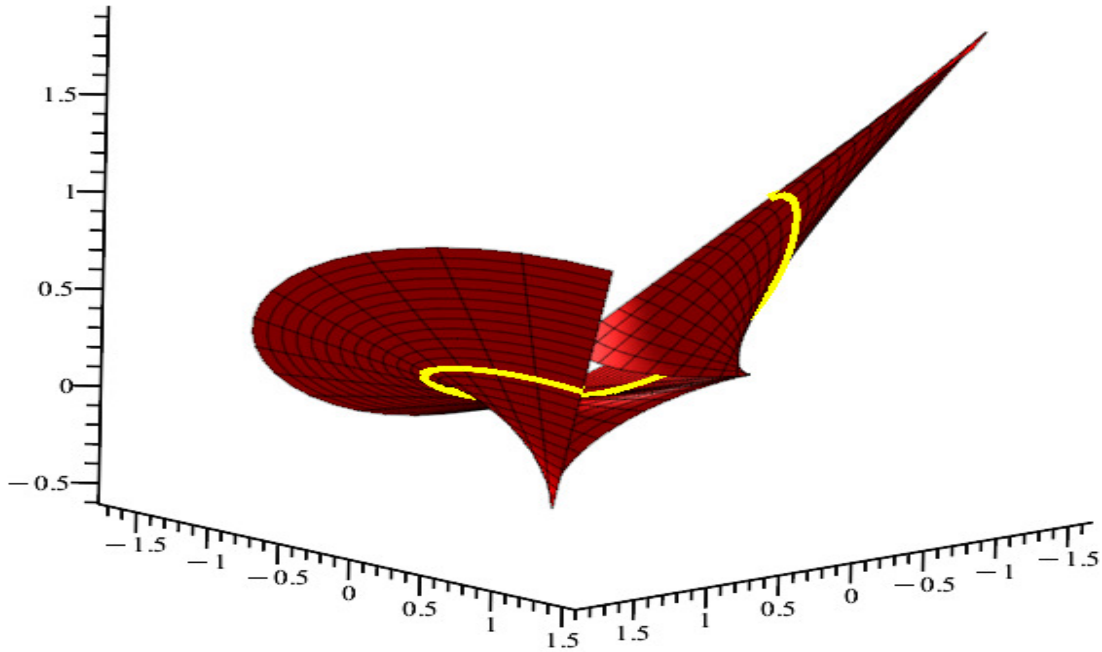


FIGURE 6. The ruled surface $\eta(s,v)\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

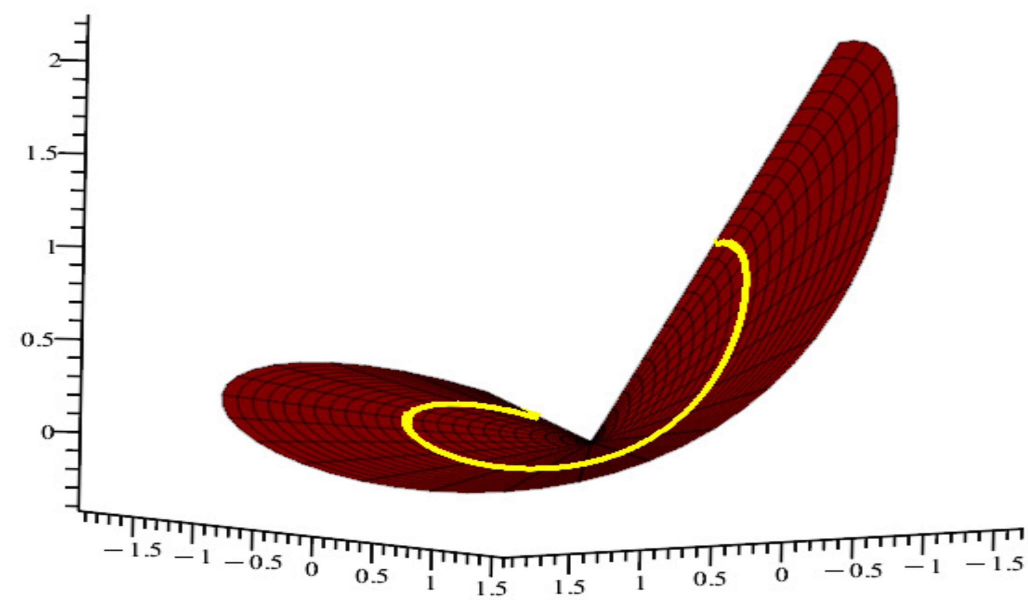


FIGURE 7. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

**Example 3.2.** *Let the Salkowski curve in Example 3.1 be the main curve. From* [15] *and Theorem 2.1 the Successor frames are as follows:*

$$T_1(s)=\begin{pmatrix} -\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right)+\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\cos s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right),\\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right)-\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\sin s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right),\\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}}+\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right)\end{pmatrix},$$

$$N_1(s)=\begin{pmatrix} -\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}, -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}, \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\end{pmatrix},$$

$$B_1(s)=\begin{pmatrix} \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right)-\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right),\\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right)-\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right),\\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}}+\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right)\end{pmatrix}.$$

*The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 8-14;*

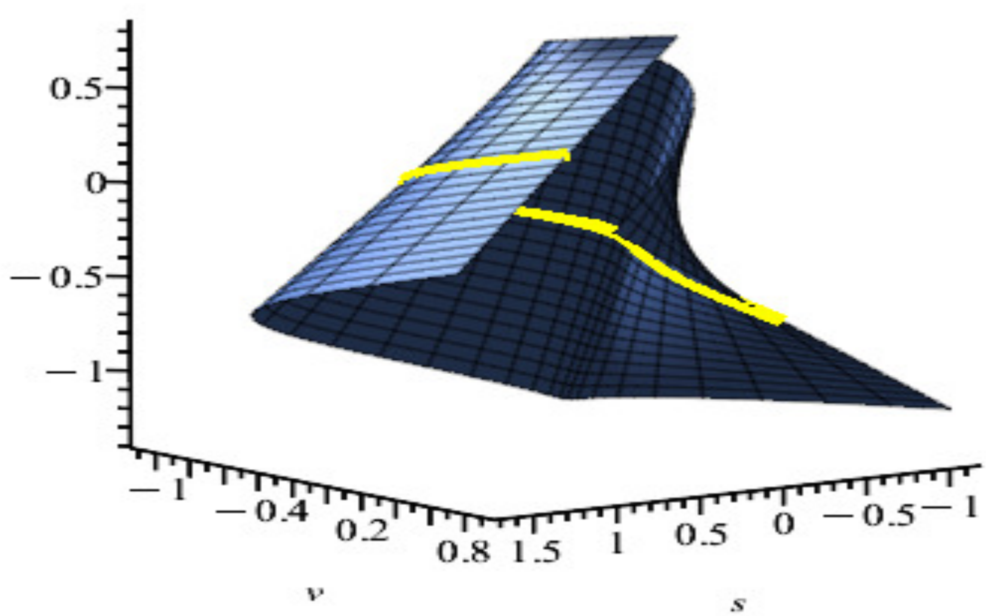


FIGURE 8. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+vT_1$

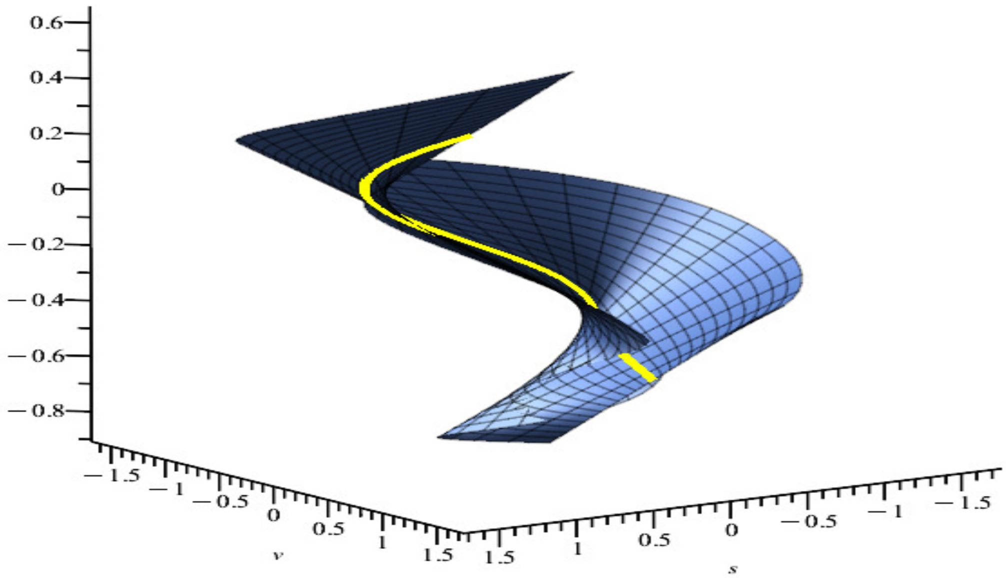


FIGURE 9. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

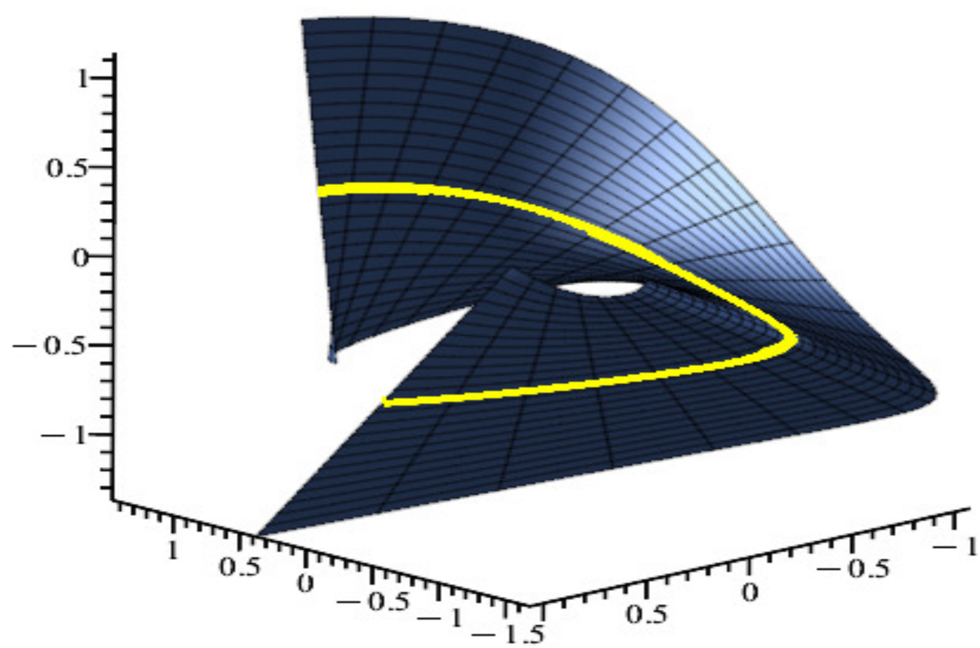


FIGURE 10. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

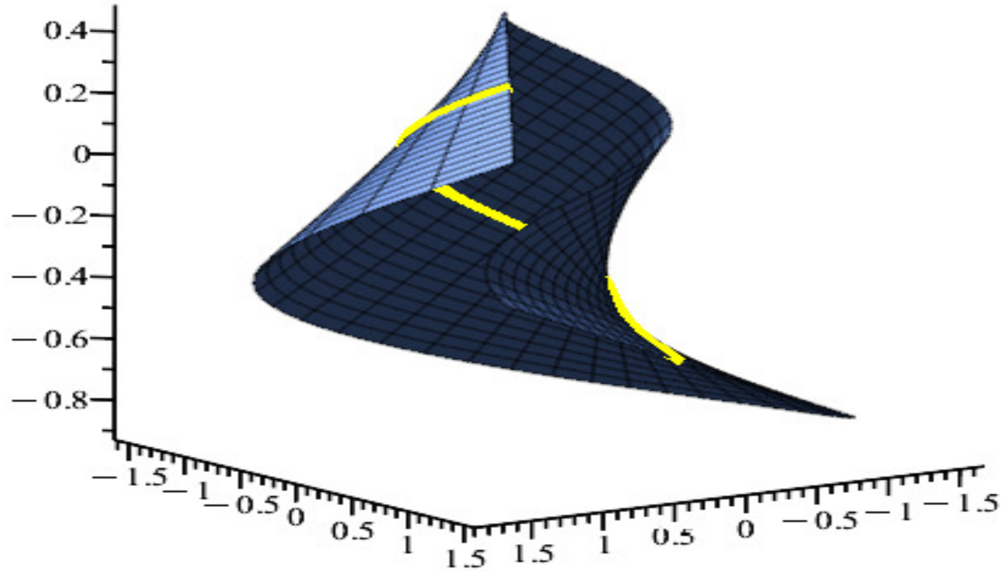


FIGURE 11. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

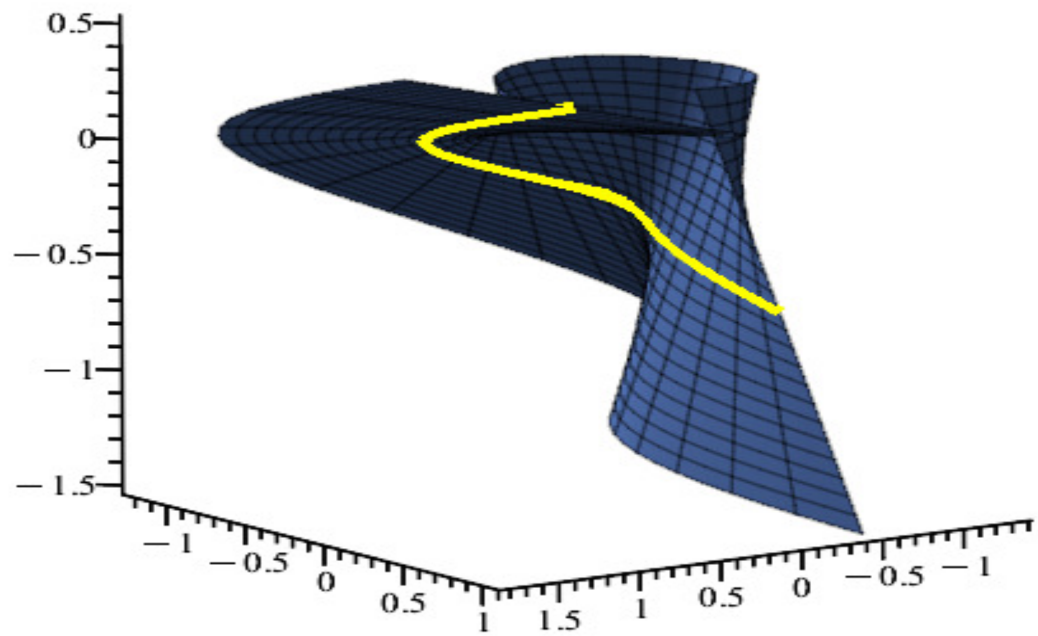


FIGURE 12. The ruled surface $\psi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

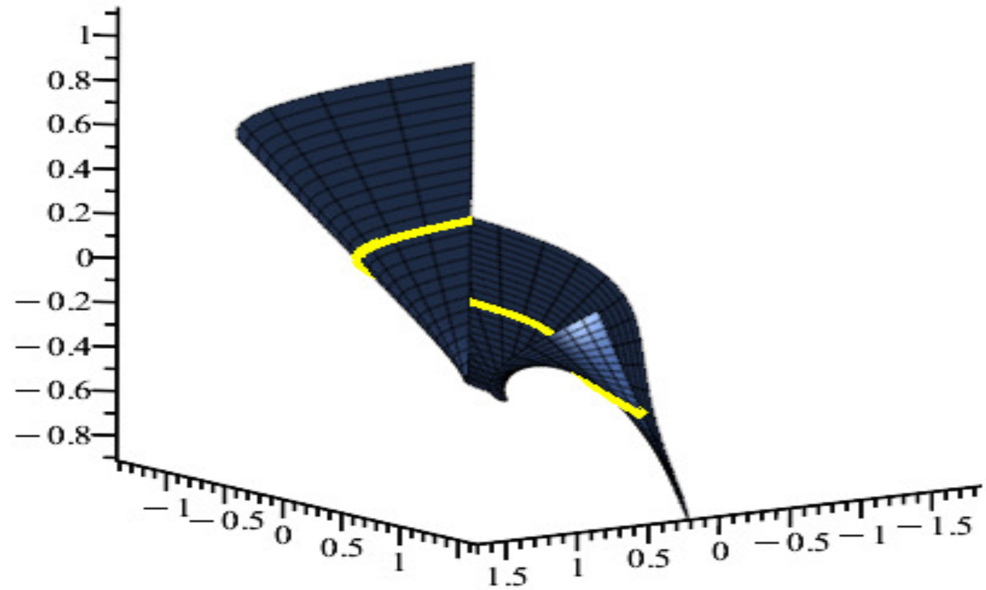


FIGURE 13. The ruled surface $\eta(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(N_1 + B_1)$

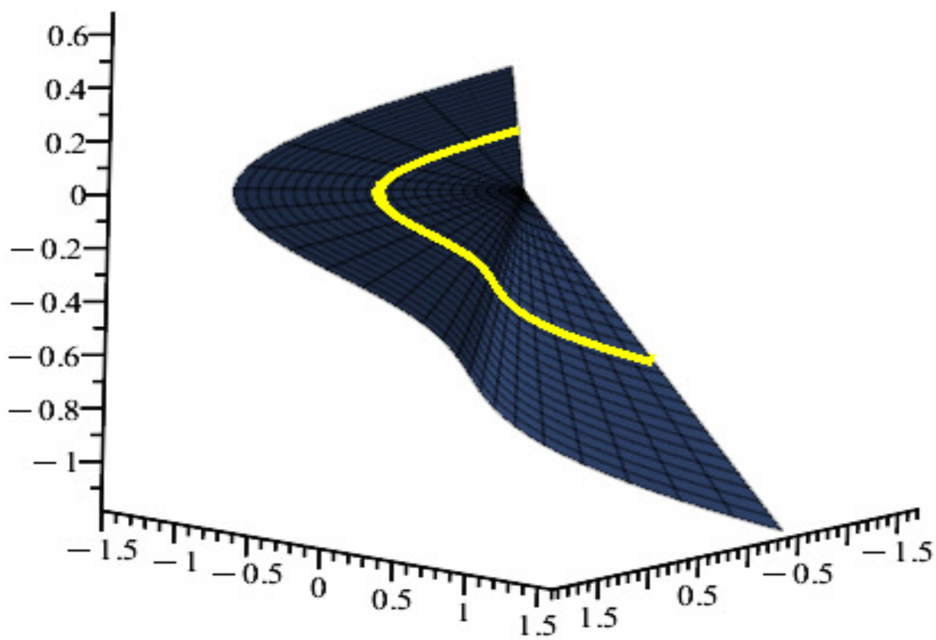


FIGURE 14. The ruled
surface $\Gamma(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{3}}(T_1 + N_1 + B_1)$

**Example 3.3.** *Let $\beta^*$ anti Salkowski curve* [15] *be the Successor curve of $\alpha$. The equation of this curve for $m=\frac{1}{3}$ is as follows:*

$$\beta^*(s)=\frac{\sqrt{10}}{40}\begin{pmatrix} -\frac{5}{2\sqrt{10}}\big(\frac{3}{\sqrt{10}}\cos(\frac{1}{5}+\cos(\frac{2}{\sqrt{10}})s)\big)+\frac{6}{5}\sin s\sin\frac{2}{\sqrt{10}}s, \\ -\frac{5}{2\sqrt{10}}\big(\frac{3}{\sqrt{10}}\sin(\frac{1}{5}+\cos(\frac{2}{\sqrt{10}})s)\big)+\frac{6}{5}\cos s\sin\frac{2}{\sqrt{10}}s,\ \ -\frac{9\sqrt{10}}{40}\big(\frac{2}{\sqrt{10}}s+\sin(\frac{2}{\sqrt{10}})s\big)\end{pmatrix}.$$

*The Successor frames of $\beta^*$ curve $\{T_1^*,N_1^*,B_1^*\}$ are as follows:*

$$T_1^*(s)=\left(\ -\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}},\ -\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}},\ -\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\ \right),$$

$$N_1^*(s)=\left(\ \frac{3}{\sqrt{10}}\sin s,\quad -\frac{3}{\sqrt{10}}\cos s,\ \ \frac{1}{\sqrt{10}}\ \right),$$

$$B_1^*(s)=\left(\ -\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}},\ \ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\ \right).$$

*The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figure {15-21};*

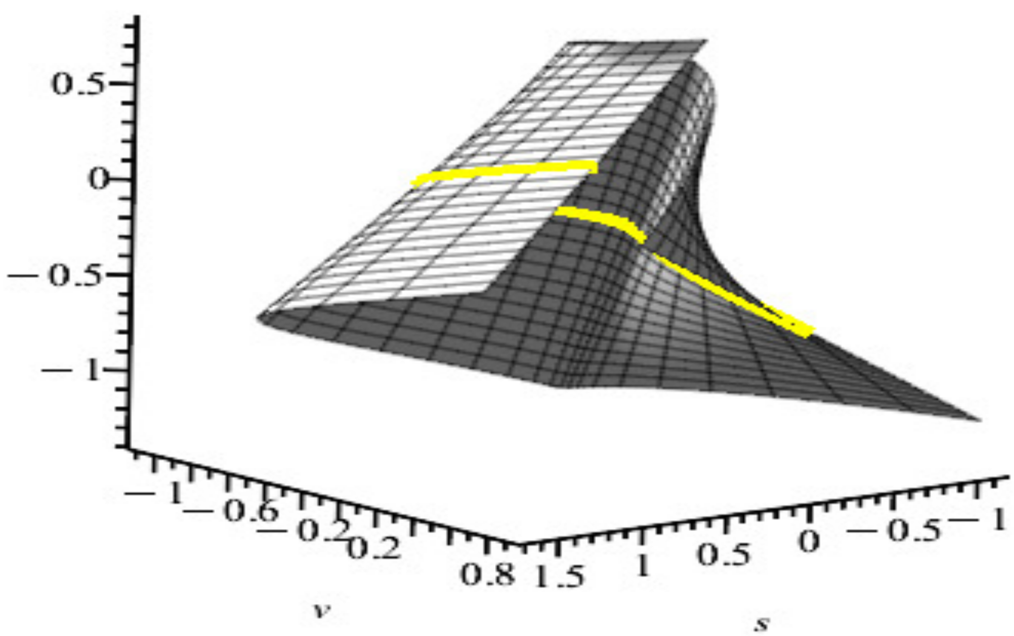


FIGURE 15. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+vT_1$

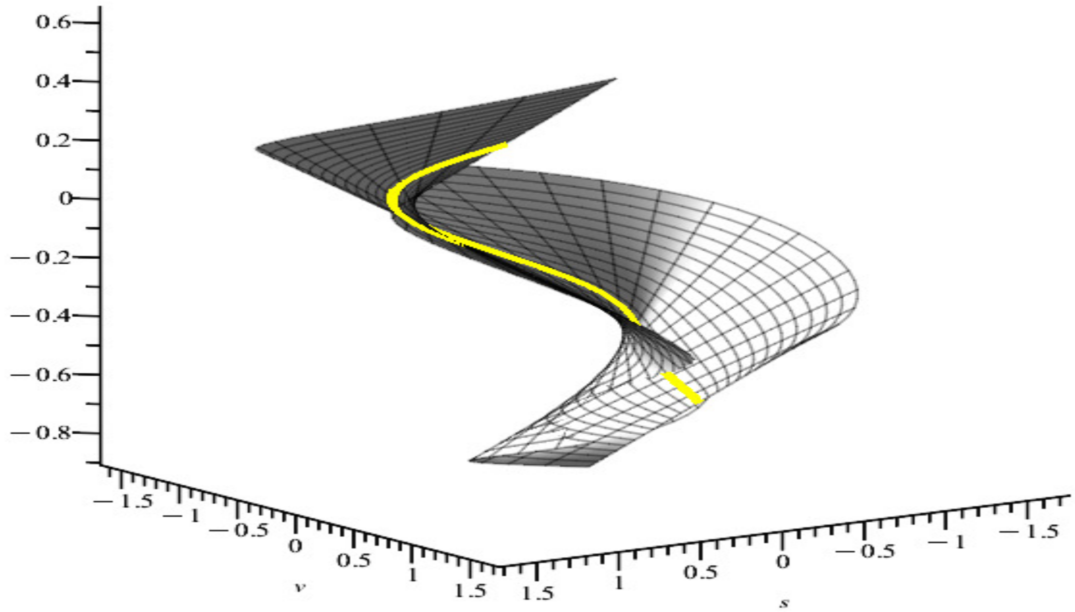


FIGURE 16. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

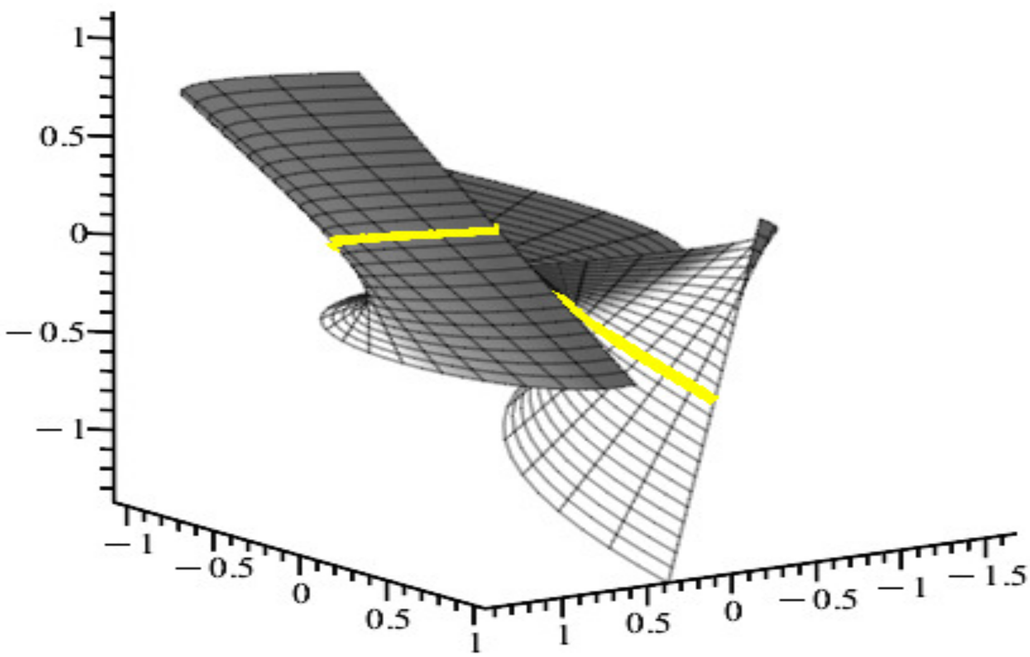


FIGURE 17. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

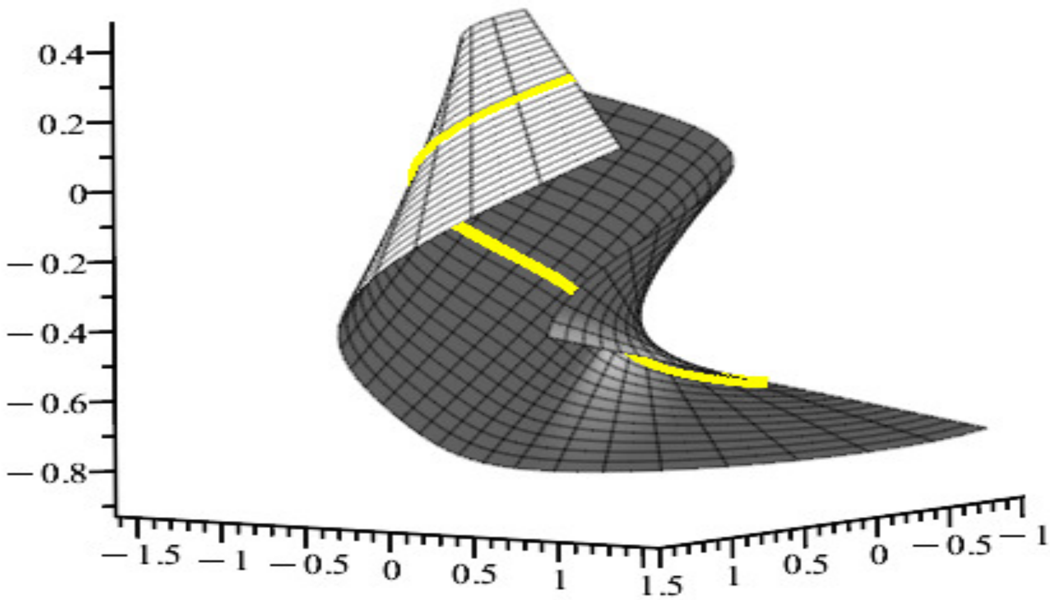


FIGURE 18. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

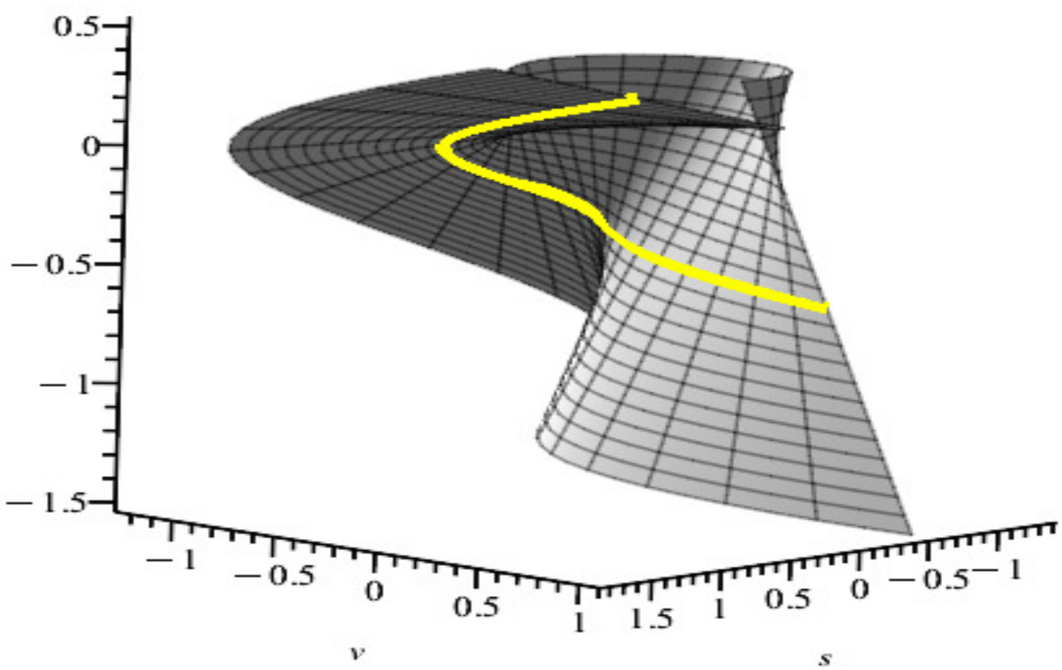


FIGURE 19. The ruled surface $\psi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

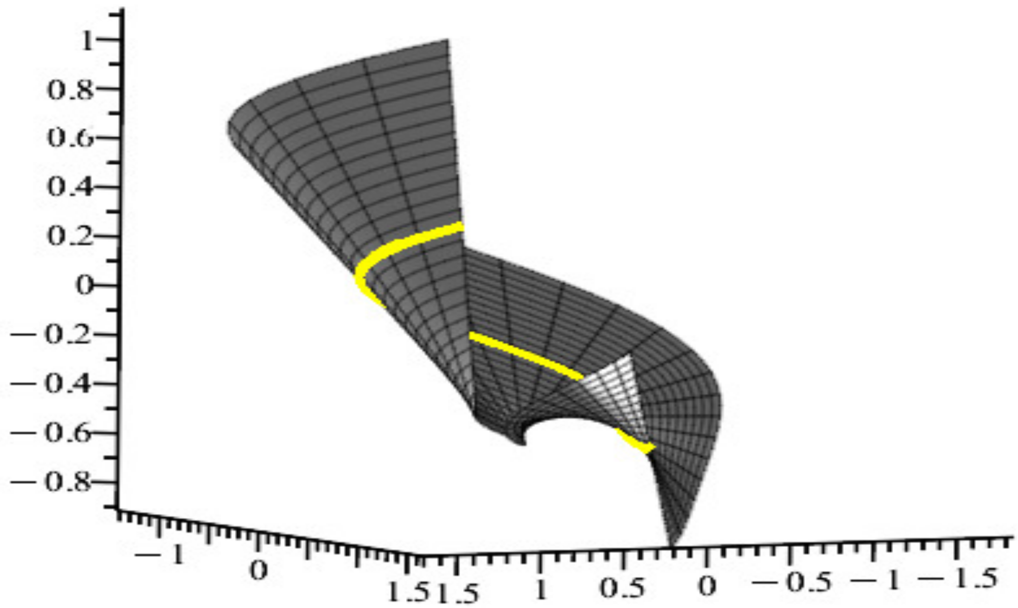


FIGURE 20. The ruled surface $\eta(s,v)\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

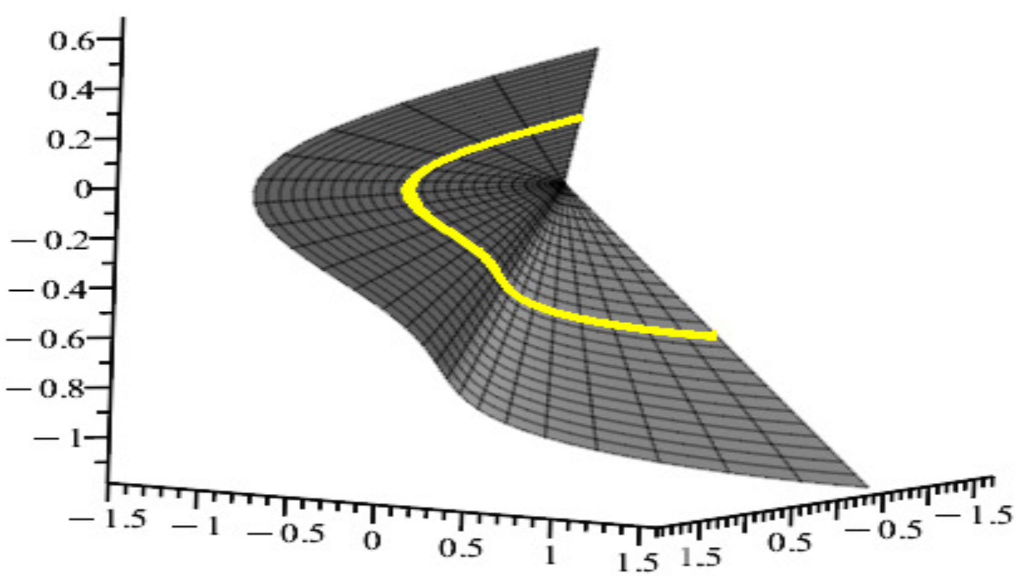


FIGURE 21. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

**Example 3.4.** *Let the Salkowski curve in Example 3.3 be the main curve. From* [15] *and Theorem 2.1 the Successor frames are as follows:*

$$T_1^*(s)=\begin{pmatrix} -\cos(s+c)\big(\frac{3}{\sqrt{10}}\sin s\big)+\sin(s+c)\big(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\big), \\ \cos(s+c)\big(\frac{3}{\sqrt{10}}\cos s\big)+\sin(s+c)\big(-\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}}\big), \\ -\cos(s+c)\frac{1}{\sqrt{10}}+\sin(s+c)\big(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\big) \end{pmatrix},$$

$$N_1^*(s)=\begin{pmatrix} -\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}},-\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{3}{\sqrt{10}}, \\ -\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}} \end{pmatrix},$$

$$B_1^*(s)=\begin{pmatrix} \sin(s+c)\big(\frac{3}{\sqrt{10}}\sin s\big)+\cos(s+c)\big(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\big), \\ \sin(s+c)\big(\frac{3}{\sqrt{10}}\cos s\big)+\cos(s+c)\big(-\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\big), \\ \sin(s+c)\frac{1}{\sqrt{10}}+\cos(s+c)\big(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\big) \end{pmatrix}.$$

*The graphs of the ruled surfaces obtained from these frames for* $s\in[-\pi,\pi]$ *and* $v\in[-1,1]$ *are shown figures* {*22-28*};

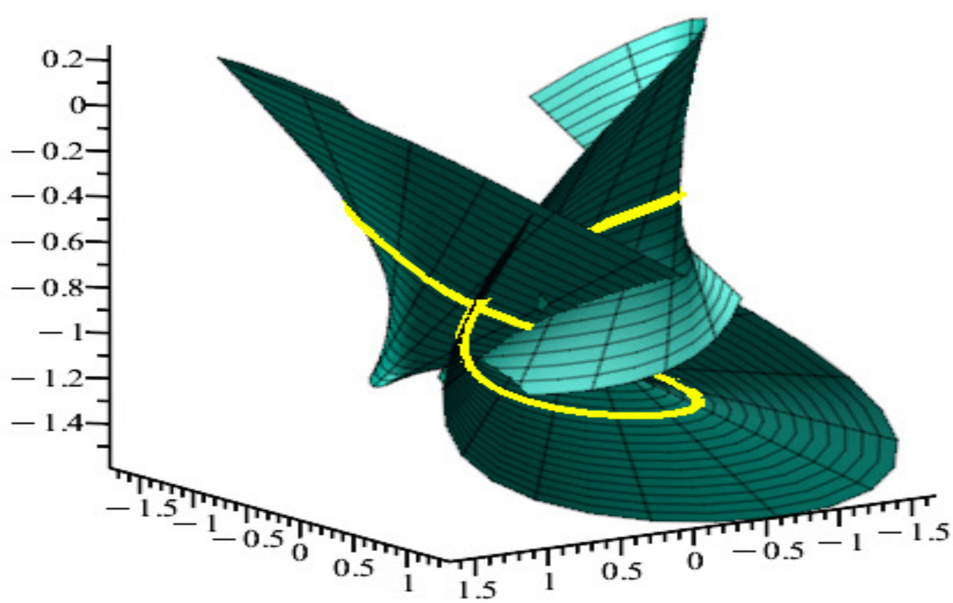


FIGURE 22. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+vT_1$

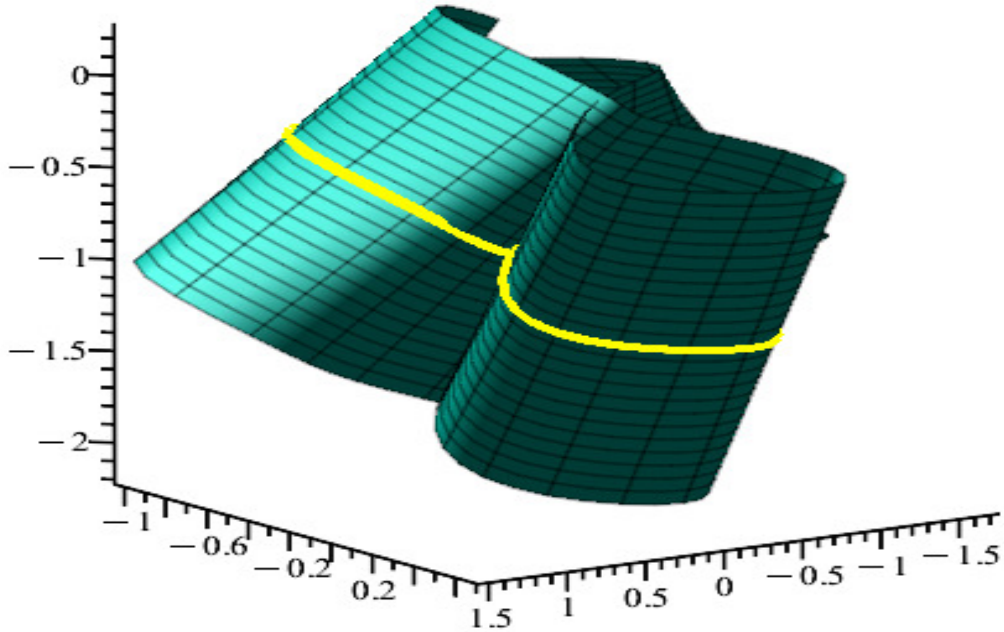


FIGURE 23. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

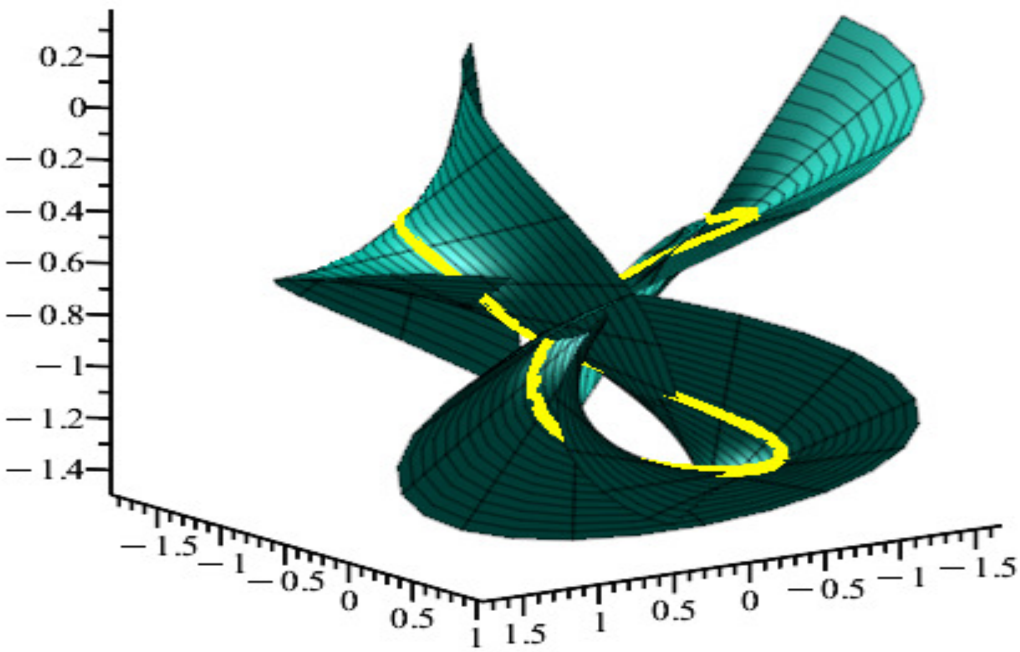


FIGURE 24. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

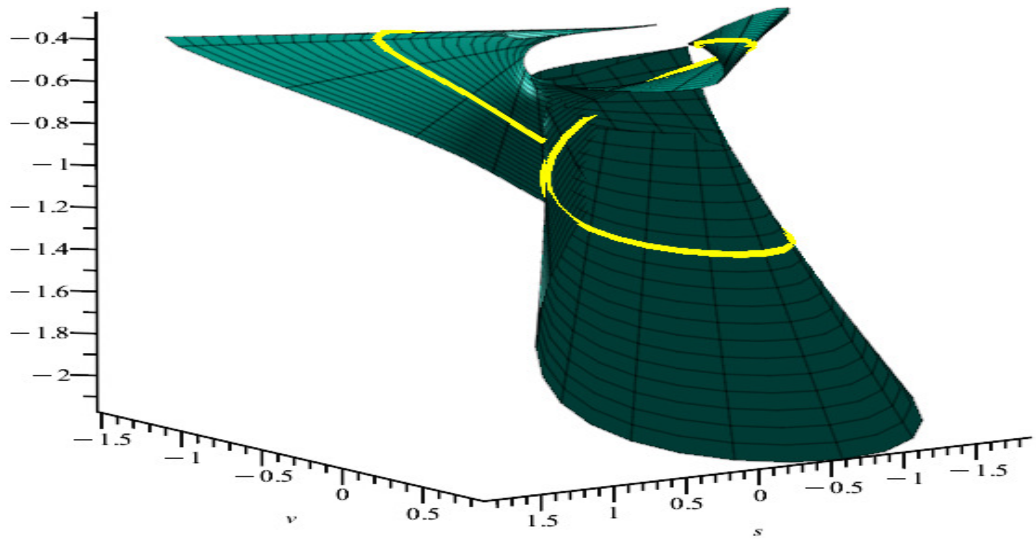


FIGURE 25. The ruled surface $\mu(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+N_1)$

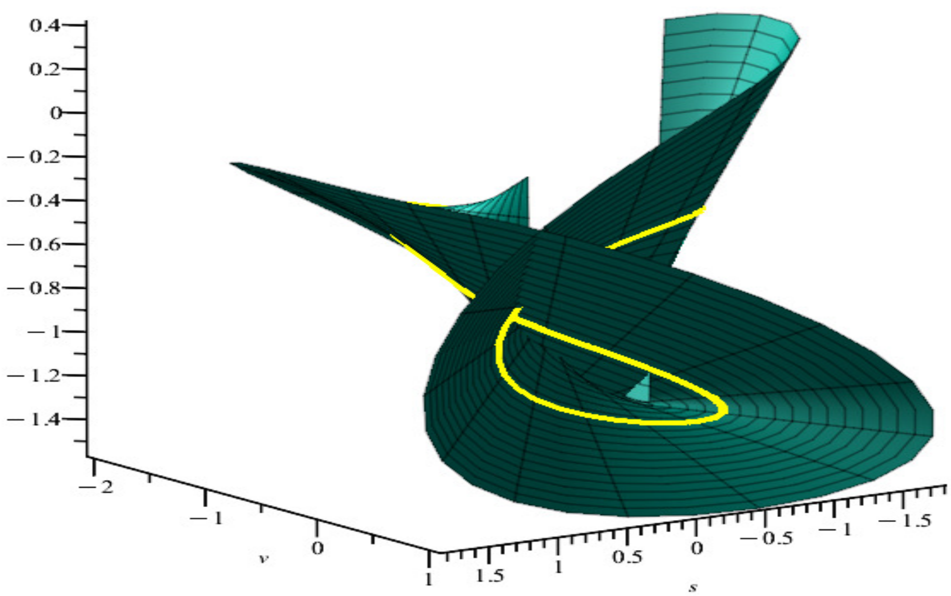


FIGURE 26. The ruled surface $\psi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

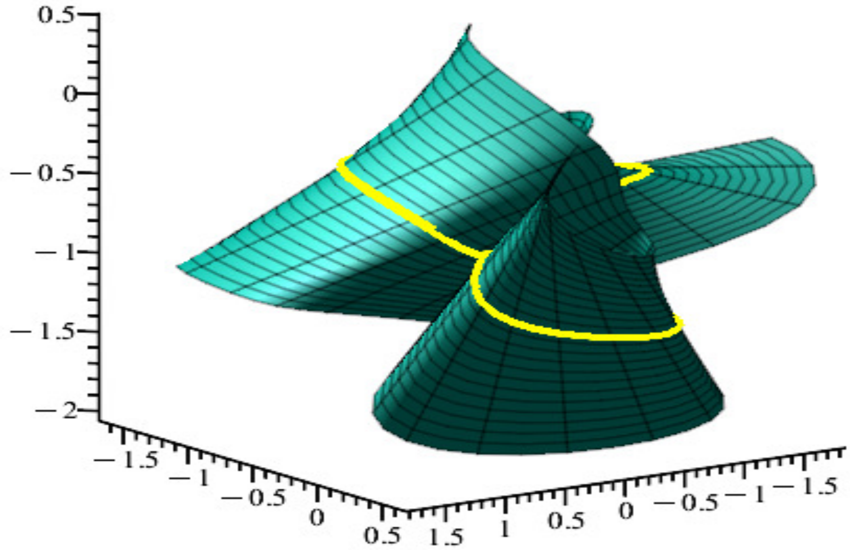


FIGURE 27. The ruled surface $\eta(s,v)\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

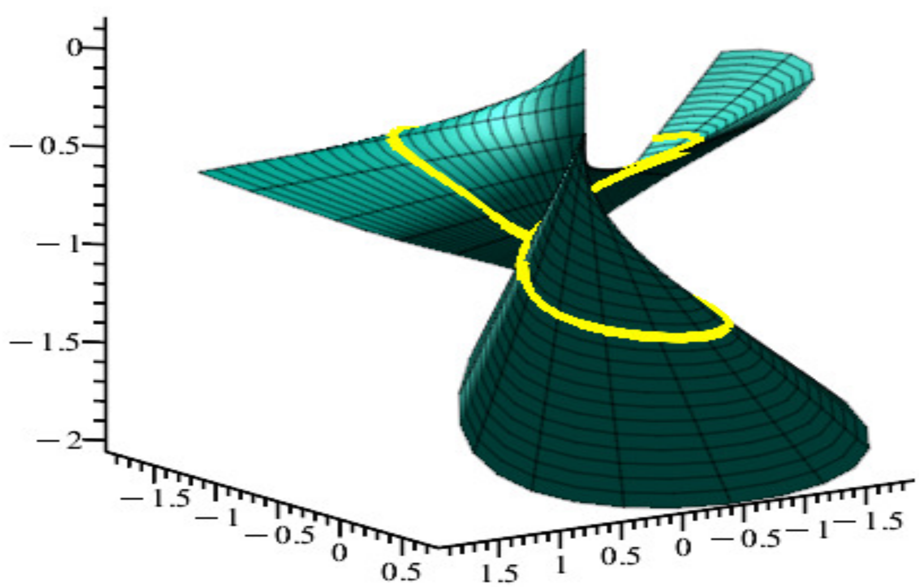


FIGURE 28. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

## 4. Conclusion

This study defined ruled surfaces which one their base curve are $T_1N_1B_1$-Smarandache curve. There base curves targent vector, normal vector and binormal vector is successor curves Frenet aparatus. The Gaussian and mean curvatures of the surfaces were obatined using the coefficients of the first and the second fundamental forms. The conditions for the surfaces to be developable and minimal were given. These surfaces were drawn. This paper can be studied in Euclidean, Lorentz and dual space. New ruled surfaces can be defined and similar work can be done, by changing the base curve. Also, the singularity of surfaces can be examined.

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# PROJECTIVE AND PROJECTIVE-PERMUTATION INVARIANTS FOR POINT SHAPE RECOGNITION

DJAVVAT KHADJIEV AND İDRİS ÖREN *

Abstract. In the present paper, the complete system of projective invariants of a point shape and the complete system of invariants under simultaneous projective and permutation transformations of a point shape are obtained.

**Keywords**: projective invariant, Cross ratio, Projective-permutation invariant.
**2010 Mathematics Subject Classification**: 14N05, 20B07,16W20.

## 1. Introduction

Projective invariants and projective-permutation invariants have an important role in computer vision for recognition of shapes (see books [6, 13, 14, 17, 18] and papers [2, 4, 5, 7, 8, 10, 11, 16] ). The projectively invariant descriptors of objects in the object recognition problems can be computed from relations between points, lines and conics that are coplanar on object surfaces in 3D. (see [4]). By [1, Corollary 6.1.4] and [6, Lemma 5.8.2], the cross-ratio is a complete system of projective invariants of a regular point shape of size 4. The volume cross ratios of points and theirs invariants in the projective space are introduced in [23, Section 27].

An extension of the cross-ratio (an harmonic ratio) to $n$-space is given in the paper [2]. In the paper Burns, Weiss and Riseman [3], it is proved that there is no a non-trivial function that is view-invariant for all possible (non-degenerate) $3D$ point sets of size $n$ for any $n$. The

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non-existence of such a general-case view invariant is shown for the true perspective, weak perspective and orthographic models. Moreover, complete classifications of joint invariants of points for the groups in the Euclidean,affine and projective spaces are given in [15].

Let $n, m$ be natural numbers such that $n < m$. In the present paper, we give a definition of a regular $nD$ point set of size $m$ and obtain a complete system of projective invariants for the system of all regular $nD$ point sets of size $m$. We investigate fundamental relations between elements of the complete system of projective invariants. Similar results have obtained for the complete system of invariants under simultaneous projective and permutation transformations ($p^2$-invariants, for short) of a $2D$ and $3D$ point set of size $m$. The problem on complete systems of $p^2$-invariants of a $nD$ point set of size $m$ in computer vision is considered in papers ([5, 7, 12, 21, 22]). This problem investigated also in projective geometry, algebraic geometry (theory of hyperelliptic curves) and the invariant theory of binary forms (see [25]).

Our paper is organized as follows. In section 2, we give the definition of a regular $nD$ point of size $m$ and obtain the complete system of projectively invariants for the system of all regular $nD$ points of size $m$ (Theorem 1). We describe the system of fundamental relations between elements of the complete system of projectively invariants (Theorem 2). We prove that the complete system is a minimal complete system of projectively invariants. In section 3, we obtain the complete system of $p^2$-invariants for the system of all regular $nD$ points of size $m$ (Theorem 3).

## 2. Projectively invariants of a point shape and their complete and the minimal complete systems

Let $\mathbb{R}$ be the field of real numbers, $n$ and $m$ are natural numbers, $n \geq 2, m > n+1$. The general linear group $GL(n, \mathbb{R})$ is the set $n \times n$ invertible matrices with elements in $\mathbb{R}$. The special linear group $SL(n, \mathbb{R})$ is the set $n \times n$ matrices with determinant 1. $\mathbb{R}_*$ be a group with respect to the multiplication in $\mathbb{R}$. Let $(\mathbb{R}_*)^m$ be the $m$ time direct product of the group $\mathbb{R}_*$. We denote the direct product of groups $(\mathbb{R}_*)^m$ and $GL(n, \mathbb{R})$ by $P(m, n)$. Let $(\mathbb{R}^n)^m$ be the $m$ time direct sum of the $n$-dimensional real linear space $\mathbb{R}^n$. We define an action $\Psi$ of the group $P(m, n)$ on the space $(\mathbb{R}^n)^m$ by the following: for $q = ((r_1, r_2, \ldots, r_m), g) \in P(m, n)$, $r_i \in \mathbb{R}, g \in GL(n, \mathbb{R})$, and $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$, we put

$$\Psi(q, X) = ((r_1, r_2, \ldots, r_m), g), X) = (r_1 g x_1, r_2 g x_2, \ldots, r_m g x_m).$$

The following definitions 1-5 and proposition 2 are known in the literature. (See some papers ([9, 19], [20, p.11] ). Proposition 1 is given in [18].

**Definition 2.1.** *Let $\Gamma, \Omega \in (\mathbb{R}^n)^m$. If there exists $q \in P(m,n)$ such that $\Omega = \Psi(q,\Gamma)$, then the elements $\Gamma$ and $\Omega$ are called $P(m,n)$-equivalent, a relationship which is written symbolically in this paper as $\Gamma \overset{P(m,n)}{\sim} \Omega$.*

**Definition 2.2.** *A real rational function $f(x_1, x_2, \ldots, x_k)$ of elements $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$ is called projectively invariant if*

$$f(\Psi(q,X)) = f(X).$$

*for all $q \in P(m,n)$.*

**Definition 2.3.** *A set $M \subseteq (R^n)^m$ is $P(m,n)$-invariant if $\Psi(q,X) \in M$ for all $X \in M$ and for all $q \in P(m,n)$.*

**Definition 2.4.** *Let $M$ be a $P(m,n)$-invariant subset of $(\mathbb{R}^n)^m$. Let $f_i : M \to R$ for $i = 1, 2, \ldots, k$ be the projectively invariant rational functions.*

*A system $\{f_1, f_2, \ldots, f_k\}$ of is called a complete system of $P(m,n)$-invariants on the set $M$ if $f_i(\Gamma) = f_i(\Omega)$ for all $i \in \{1, 2, \ldots, k\}$ and for $\Gamma, \Omega \in M$ imply $\Gamma \overset{P(m,n)}{\sim} \Omega$.*

**Proposition 2.1.** *Let $M$ be a $P(m,n)$-invariant set of $(\mathbb{R}^n)^m$. Then every projectively invariant rational function on $M$ if a function of the system $\{f_1, f_2, \ldots, f_k\}$.*

**Definition 2.5.** *A complete system of projectively invariant rational functions*

*$W = \{f_1, f_2, \ldots, f_k\}$ is called a minimal complete system of projectively invariant rational functions if $W \setminus \{f_i\}$ is not complete for any $i \in \{1, 2, \ldots, k\}$.*

**Proposition 2.2.** *$W = \{f_1, f_2, \ldots, f_k\}$ is a minimal complete system iff $f_i$ is not function of the subsystem $\{f_1, f_2, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m\}$ for all $i = 1, 2, \ldots, k$.*

Let $[x_1 x_2 \cdots x_n]$ be the determinant of vectors $x_1, x_2, \cdots, x_n \in \mathbb{R}^n$. Assume that $x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2} \in \mathbb{R}^n$ vectors such that $[x_1 x_2 \cdots x_{n-1} x_k] \neq 0$ for all $i = n, n+1, n+2$ and $[x_2 x_3 \cdots x_n x_j] \neq 0$ for $j = n+1, n+2$. Consider the following cross-invariant of vectors $x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2} \in \mathbb{R}^n$:

$$\frac{[x_1 x_2 \cdots x_{n-1} x_{n+1}][x_2 x_3 \cdots x_n x_{n+2}]}{[x_1 x_2 \cdots x_{n-1} x_{n+2}][x_2 x_3 \cdots x_n x_{n+1}]}.$$

We denote it by $\langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \rangle$. It is known that it is projectively invariant.

**Definition 2.6.** *$X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$ is called regular if $[x_{p_1} x_{p_2} \cdots x_{p_{n-1}} x_{p_n}] \neq 0$ for all natural numbers $p_1, p_2, \ldots, p_n$ such that $1 \leq p_1 < p_2 < \cdots < p_n \leq m$.*

If $X = (x_1, x_2, \ldots, x_m)$ is regular, from $X \overset{P(m,n)}{\sim} Y$, we have $Y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^n)^m$ . Hence $Y = (y_1, y_2, \ldots, y_m)$ is also regular. Hence the set of all regular elements is a $P(m, n)$-invariant subset of $(\mathbb{R}^n)^m$.

The cross-ratio obtained from $\langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \rangle$ by transposition of elements $x_1$ and $x_j$, where $1 \leq j \leq n-1$, will be denoted by $T_j \langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \rangle$. Thus

$$T_j \langle x_1 x_2 x_3 \cdots x_{j-1} x_j x_{j+1} \ldots x_n x_{n+1} x_{n+2} \rangle = \langle x_j x_2 x_3 \cdots x_{j-1} x_1 x_{j+1} \ldots x_n x_{n+1} x_{n+2} \rangle$$

for all $j = 1, 2, \ldots, n-1$.

If $X = (x_1, x_2, \ldots, x_m)$ is regular, then $T_j \langle x_1 x_2 x_3 \ldots x_n x_{n+1} x_{n+2} \rangle \neq 0$ and $T_j \langle x_1 x_2 x_3 \ldots x_n x_{n+1} x_{n+2} \rangle \neq \infty$ for all $j = 1, 2, \ldots, n-1$.

**Theorem 2.1.** *Regular elements $X = (x_1, x_2, \ldots, x_m), Y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^2)^m$ are $P(m, n)$-equivalent if and only if*

$$T_j \langle x_1 x_2 x_3 \ldots x_{n-1} x_n x_{n+1} x_k \rangle = T_j \langle y_1 y_2 x_3 \ldots y_{n-1} y_n y_{n+1} y_k \rangle \tag{2.1}$$

*for all $j = 1, 2, \ldots n-1$ and for all $k = n+2, \ldots, m$.*

*Proof.* Since the function $\langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_k \rangle$ is projectively invariant, $P(m, n)$-equivalence of $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_m)$ implies (2.1). Prove the converse assertion.

Assume that (2.1) holds. We consider vectors $e_1, e_2, \ldots, e_n, e_{n+1} \in \mathbb{R}^n$, where

$$e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots e_n = (0, 0, \ldots, 1), e_{n+1} = (1, 1, \ldots, 1).$$

By the fundamental theorem of projective geometry ([1, p.97]), elements $g \in GL(n, \mathbb{R})$ and $r_1, r_2, \ldots, r_n, r_{n+1} \in \mathbb{R}_*$ exist such that $r_i g x_i = e_i, i = 1, 2, \ldots n, n+1$. Similarly, elements $h \in GL(n, \mathbb{R})$ and $q_1, q_2, \ldots, q_n, q_{n+1} \in \mathbb{R}_*$ exist such that $q_i h y_i = e_i, i = 1, 2, \ldots, n, n+1$. Since the function $\langle x_1 x_2 x_3 \ldots x_{n-1} x_n x_{n+1} x_k \rangle$ is projectively invariant, we have

$$\langle (r_1 g x_1)(r_2 g x_2) \cdots (r_n g x_n)(r_{n+1} g x_{n+1})(g x_k) \rangle = \langle e_1 e_2 \cdots e_n e_{n+1} (g x_k) \rangle = \langle x_1 x_2 \cdots x_n x_{n+1} x_k \rangle =$$

$$\langle y_1 y_2 \cdots y_n y_{n+1} y_k \rangle = \langle (q_1 h y_1)(q_2 h y_2) \cdots (q_n h y_n)(q_{n+1} h y_{n+1})(h y_k) \rangle = \langle e_1 e_2 \cdots e_n e_{n+1} (h y_k) \rangle$$

for all $k = n+2, \ldots m$. Hence $\langle e_1 e_2 \cdots e_n e_{n+1} (g x_k) \rangle = \langle e_1 e_2 \cdots e_n e_{n+1} (h y_k) \rangle$. Similarly, we obtain

$$T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (g x_k) \rangle = T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (h y_k) \rangle \tag{2.2}$$

for all $j = 1, 2, \dots n-1$ and for all $k = n+2, \dots, m$.

Let $gx_k = (a_{1k}, a_{2k}, \dots a_{nk}), hy_k = (b_{1k}, b_{2k}, \dots b_{nk}), k = n+2 \dots, m$. Using regularity of $(r_1 gx_1, \dots r_{n+1} gx_{n+1}, gx_{n+2}, \dots, gx_m)$ and $(q_1 hy_1, \dots q_{n+1} hy_{n+1}, hy_{n+2}, \dots, hy_m)$, we obtain $[e_1 e_2 \dots e_{n-1} gx_k] = a_{nk} \neq 0$ and $[e_1 e_2 \dots e_{n-1} gy_k] = b_{nk} \neq 0$ for all $k = n+2, \dots m$. It is easy to see that

$$T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (gx_k) \rangle = \frac{a_{jk}}{a_{nk}} \tag{2.3}$$

and

$$T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (gy_k) \rangle = \frac{b_{jk}}{b_{nk}}, \tag{2.4}$$

for all $j = 1, 2, \dots n-1$ and for all $k = n+2, \dots, m$.

Equations (2.2), (2.3) and (2.4) imply $\frac{a_{jk}}{a_{nk}} = \frac{b_{jk}}{b_{nk}}$ for all $j = 1, 2, \dots n-1; k = n+2, \dots m$. Put $d_{jk} = \frac{a_{jk}}{a_{nk}} = \frac{b_{jk}}{b_{nk}}$ for all $j = 1, 2, \dots n-1; k = n+2, \dots m$. Then $gx_k = a_{nk}(d_{1k}, d_{2k}, \dots, d_{n-1k}, 1)$ and $hy_k = b_{nk}(d_{1k}, d_{2k}, \dots, d_{n-1k}, 1)$ for all $k = n+2, \dots m$.

This means that

$(e_1, e_2, \dots e_n, e_{n+1}, gx_{n+2}, \dots, gx_m) \overset{P(m,n)}{\sim} (e_1, e_2, \dots e_n, e_{n+1}, hy_{n+2}, \dots, hy_m)$.

Using $(x_1, x_2, \dots x_n, x_{n+1}, x_{n+2}, \dots, x_m) \overset{P(m,n)}{\sim} (e_1, e_2, \dots e_n, e_{n+1}, gx_{n+2}, \dots, gx_m)$, $(e_1, e_2, \dots e_n, e_{n+1}, gx_{n+2}, \dots, gx_m) \overset{P(m,n)}{\sim} (e_1, e_2, \dots e_n, e_{n+1}, hy_{n+2}, \dots, hy_m)$ and $(e_1, e_2, \dots e_n, e_{n+1}, hy_{n+2}, \dots, hy_m) \overset{P(m,n)}{\sim} (y_1, y_2, \dots y_n, y_{n+1}, y_{n+2}, \dots, y_m)$, we obtain $X \overset{P(m,n)}{\sim} Y$. □

**Remark 2.1.** *Theorem* 2.1 *means that the system of projectively invariants*

$$T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle, \tag{2.5}$$

*for all* $j = 1, 2, \dots n-1$ *and for all* $k = n+2, \dots m$ *is a complete system of projectively invariants on the set of all regular elements of* $(\mathbb{R}^n)^m$.

**Corollary 2.1.** *Every projectively invariant function* $f(x_1, x_2, \dots, x_m)$ *on the set of all regular elements* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ *is a function of elements of the system* (2.5)

*Proof.* It follows from ([20, Theorem 1 and 1.1]). □

Now we find all fundamental relations between elements of the complete system (2.5) If $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is regular, then

$$T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle \neq 0 \tag{2.6}$$

for all $j = 1, 2, \dots n-1$ and for all $k = n+2, \dots m$.

**Theorem 2.2.** *Let* $\{c_{jk}, j = 1, 2, \dots n-1; k = n+2, \dots, m\}$ *be a system of real numbers such that* $c_{jk} \neq 0$ *for all* $j = 1, 2, \dots n-1; k = n+2, \dots, m$. *Then a regular element* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ *exists such that*

$$T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle = c_{jk} \tag{2.7}$$

*for all* $j = 1, 2, \dots n-1$ *and for all* $k = n+2, \dots m$

*Proof.* Let $e_1, e_2, \dots, e_n, e_{n+1} \in \mathbb{R}^n$ be vectors in Theorem (2.1). Consider the element $X = (x_1, x_2, \dots, x_n, x_{n+1}, \dots x_m) \in (\mathbb{R}^n)^m$, where

$x_1 = e_1 = (1, 0, 0, \dots, 0), x_2 = e_2 = (0, 1, 0, \dots 0), \dots, x_n = e_n = (0, 0, 0, \dots 1), x_{n+1} = e_{n+1} = (1, 1, 1, \dots, 1)$, $x_k = (c_{1k}, c_{2k}, \dots, c_{n-1k}, 1)$ for all $k = n+2, \dots m$. It is easy to see that (2.7) hold for $X$. Since $c_{jk} \neq 0$ for all $j = 1, 2, \dots n-1; k = n+2, \dots, m$, (2.7) implies that $X$ is regular. □

**Corollary 2.2.** *The system* (2.5) *is a system of functionally independent projectively invariants on the set of of all regular elements* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$.

*Proof.* It follows from Theorem 2.2 □

**Corollary 2.3.** *The system* (2.5) *is a minimal complete system of projectively invariants on the set of of all regular elements* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$.

*Proof.* It follows from Proposition 2.1 and Corollary 2.2. □

## 3. Projective-Permutation invariants of a point shape

Let $S(n, m)$ be the group of all permutations of the numbers $n+2, n+3, \dots, m$ and $P(m, n) \times S(n, m)$ is a direct product of groups $P(m, n)$ and $S(n, m)$. We define an action $\beta$ of the group $P(m, n) \times S(n, m)$ on the space $(\mathbb{R}^n)^m$ as follows: for $q = ((r_1, r_2, \dots, r_m), g, h) \in P(m, n) \times S(n, m)$, $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$, $h \in S(n, m)$,

$$h = \begin{pmatrix} 1 & 2 & \dots & n+1 & n+2 \dots & m \\ 1 & 2 & \dots & n+1 & h(n+2) \dots & h(m) \end{pmatrix}, \tag{3.8}$$

we put $\beta(q, X) = ((r_1, \dots, r_m), g, h), X) = (r_1 g x_{h(1)}, r_2 g x_{h(2)}, \dots, r_m g x_{h(m)})$, where $h(j) = j$ for $j = 1, 2, \dots, n+1$.

**Definition 3.1.** *Elements* $A, B \in (\mathbb{R}^n)^m$ *is called* $P(m, n) \times S(n, m)$*-equivalent if there exists* $q \in P(m, n) \times S(n, m)$ *such that* $B = \beta(q, A)$. *In this case, we write* $A \overset{P(m,n)\times S(n,m)}{\sim} B$.

**Definition 3.2.** *A rational function $f(x_1,\dots,x_m)$ of $X=(x_1,\dots,x_m)\in(\mathbb{R}^n)^m$ is called $P(m,n)\times S(n,m)$-invariant if $f(\alpha(q,X))=f(X)$ for all $q\in P(m,n)\times S(n,m)$.*

For $j=1,2,\dots n-1; k=n+2,\dots m$, we put $T_{jk}(X)=T_j\langle x_1x_2x_3\dots x_{n-1}x_nx_{n+1}x_k\rangle$. We denote the algebra of polynomials of all $T_{jk}(X)$ by $A(n,m)$. Let $\{z_j : j=1,2,\dots n-1\}$ be independent variables. We consider the following function

$$\prod_{k=1}^{m-n-1}\left(1+\sum_{i=1}^{n-1}T_k(X)z_i\right)$$

and define the polynomials $U_{r_1r_2\dots r_{n-1}}(X)\in A(n,m)$ by the following equality

$$\prod_{k=1}^{m-n-1}\left(1+\sum_{i=1}^{n-1}T_k(X)z_i\right)=1+\sum_{1\le\sum_{i=1}^{n-1}r_i\le m-n-1}U_{r_1r_2\dots r_{n-1}}(X)z_1^{r_1}z_2^{r_2}\cdots z_{n-1}^{r_{n-1}}. \tag{3.9}$$

It is obvious that every function $U_{r_1r_2\dots r_{n-1}}(X)$ is $P(m,n)\times S(n,m)$-invariant.

**Theorem 3.1.** *Regular elements $X=(x_1,x_2,\dots,x_m), Y=(y_1,y_2,\dots,y_m)\in(\mathbb{R}^n)^m$ are $P(m,n)\times S(n,m)$-equivalent if and only if*

$$U_{r_1r_2\dots r_{n-1}}(X)=U_{r_1r_2\dots r_{n-1}}(Y) \tag{3.10}$$

*for $1\le\sum_{i=1}^{n-1}r_i\le m-n-1$.*

*Proof.* Since $U_{r_1r_2\dots r_{n-1}}(X)$ is $P(m,n)\times S(n,m)$-invariant, $P(m,n)\times S(n,m)$-equivalence of $X=(x_1,x_2,\dots,x_m)$ and $Y=(y_1,y_2,\dots,y_m)$ implies (3.10). Prove the converse assertion.

Assume that (3.10) holds. Then (3.9) and (3.10) imply the equation

$$\prod_{k=1}^{m-n-1}(1+\sum_{i=1}^{n-1}T_k(X)z_i)=\prod_{k=1}^{m-n-1}(1+\sum_{i=1}^{n-1}T_k(Y)z_i). \tag{3.11}$$

By the theorem on the unique factorization in the algebra $A(n,m)$ (see [24, p.91-94]), a permutation (3.8) exists such that

$$\sum_{i=1}^{n-1}T_k(X)z_i=\sum_{i=1}^{n-1}T_k(Y)z_i$$

for all $k=n+2,\dots,m$.

This equality implies

$$\begin{cases} T_j\langle x_1x_2x_3\dots x_{n-1}x_nx_{n+1}x_k\rangle=T_j\left\langle y_1y_2y_3\dots y_{n-1}y_ny_{n+1}y_{h(k)}\right\rangle, \\ j=1,2,\dots n-1; k=n+2,\dots,m. \end{cases}$$

By Theorem 1, these equalities imply $P(m,n)$-equivalence of elements $X=(x_1,x_2,\dots,x_m)$and $hY=(y_{h(1)},y_{h(2)},\dots,y_{h(m)})$, where $h(j)=j$ for $j=1,2,\dots\dots,n+1$. This means $P(m,n)\times S(n,m)$-equivalence of elements $X=(x_1,x_2,\dots,x_m)$ and $Y=(y_1,y_2,\dots,y_m)$. □

Let $S(m)$ be the group of all permutations of the numbers $1, 2, \dots, m$ and $P(m,n)\times S(m)$ is a direct product of groups $P(m,n)$ and $S(m)$. We define an action $\beta$ of the group $P(m,n)\times S(m)$ on the space $(R^n)^m$ as follows: for $q = ((r_1, r_2, \dots, r_m), g, h) \in P(m,n) \times S(m)$, $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$, where

$$h = \begin{pmatrix} 1 & 2 & \dots & m \\ h(1) & h(2) & \dots & h(m) \end{pmatrix}, \tag{3.12}$$

we put

$$\beta(q, X) = ((r_1, r_2, \dots, r_m), g, h), X) = (r_1 g x_{h(1)}, r_2 g x_{h(2)}, \dots, r_m g x_{h(m)}).$$

**Definition 3.3.** *Elements $A, B \in (\mathbb{R}^n)^m$ is called $P(m,n) \times S(m)$-equivalent if there exists $q \in P(m,n) \times S(m)$ such that $B = \beta(q, A)$. In this case, we write $A \overset{P(m,n)\times S(m)}{\sim} B$.*

**Definition 3.4.** *A rational function $f(x_1, x_2, \dots, x_m)$ of elements $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is called $P(m,n) \times S(m)$-invariant if $f(\alpha(q, X)) = f(X)$ for all $q \in P(m,n) \times S(m)$.*

**Definition 3.5.** *$X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is called strongly regular if $[x_{p_1} x_{p_2} \cdots x_{p_n}] \neq 0$ for all natural numbers $p_1, p_2, \dots, p_n$ such that $1 \le p_1 < p_2 < \dots < p_n \le m$.*

If $X = (x_1, x_2, \dots, x_m)$ is strongly regular, $Y = (y_1, y_2, \dots, y_m) \in (\mathbb{R}^n)^m$ and $X \overset{P(m,n)}{\sim} Y$ then $Y = (y_1, y_2, \dots, y_m)$ is also strongly regular. Hence the set of all strongly regular elements is a $P(m,n)$-invariant subset in $(\mathbb{R}^n)^m$.

For $j = 1, 2, \dots n-1; k = n+2, \dots m$, we put $T_{jk}(X) = T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle$, where $X = (x_1, x_2, \dots, x_m)$. We denote by $A_{nm}$ the algebra of real polynomials of $T_{jk}(X); j = 1, 2, \dots n-1; k = n+2, \dots m$. Let $t$ and $\{z_{jk} | j = 1, 2, \dots n-1; k = n+2, \dots m\}$ be independent variables. We consider the following function

$$\prod_{h \in S(m)} \left[ t - \sum_{k=n+2}^{m} (\sum_{i=1}^{n-1} T_{ik}(h(X)) z_{ik}) \right].$$

We define the polynomials $U_{r_1 r_2 \dots r_{m-n-1}}(X) \in A_{nm}$ by the following equality

$$\prod_{h \in S(m)} \left[ t - \sum_{k=n+2}^{m} (\sum_{i=1}^{n-1} T_{ik}(h(X)) z_{ik}) \right] = \tag{3.13}$$

$$t^{m!} + \sum_{l=1}^{m!-1} (-1)^l t^{m!-l} \sum_{\sum_{j=1}^{m-n-1} (\sum_{i=1}^{n-1} r_{ij} = l)} U_{r_{ij}}(X) \cdot \prod_{k=n+2}^{m} (z_{1k}^{r_{1k}} z_{2k}^{r_{2k}} \cdots z_{n-1k}^{r_{n-1k}}),$$

where $m! = 1 \cdot 2 \cdot \dots \cdot m$ and $i = 1, \dots, n-1; j = 1, \dots, m-n-1$. It is obvious that every function $U_{r_{ij}}(X)$ is $P(m,n) \times S(m)$-invariant.

**Theorem 3.2.** *Strongly regular elements* $X = (x_1, x_2, \ldots, x_m), Y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^n)^m$ *are* $P(m,n) \times S(m)$*-equivalent if and only if*

$$\begin{cases} U_{r_{ij}}(X) = U_{r_{ij}}(Y), \\ 1 \leq r_1 + r_2 \cdots + r_{n-1} \leq m - n - 1; i = 1, \ldots, n-1; j = 1, \ldots, m-n-1. \end{cases} \tag{3.14}$$

*Proof.* A proof is similar to the proof of Theorem 3.1. □

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# RICCI-YAMABE SOLITONS ON THE LIE GROUP $H_2 \times \mathbb{R}$

MURAT ALTUNBAŞ *

Abstract. In this paper, we study Ricci-Yamabe and gradient Yamabe solitons on the Lie group $H_2 \times \mathbb{R}$ with a left-invariant metric. We prove that the kind of the Ricci-Yamabe soliton is only related with one variable existing in the definition.

**Keywords**: $H_2 \times \mathbb{R}$ Lie group, Ricci-Yamabe soliton, gradient Ricci-Yamabe soliton.

**2010 Mathematics Subject Classification**: 58B30, 58C30.

## 1. Introduction

Homogenous geometries play crucial role in the theory of manifolds. Their importance come from the well-known Thurston conjecture. This conjecture says that every compact orientable 3-dimensional manifold has a canonical decomposition into parts, each of which involves a canonical geometric structure from among the eight maximal simple connected homogenous Riemannian three-dimensional geometries [8]. These model spaces are $E^3$, $H^3$, $S^3$, $S^2 \times \mathbb{R}$, $Nil$, $\widetilde{SL_2\mathbb{R}}$, $Sol$ and $H_2 \times \mathbb{R}$. In this paper, we deal with the latter one.

The geometry of different types of solitons on manifolds has been the focus of attention of many mathematicians during the last years (see for examples [1],[2],[3]). Yamabe flow was introduced by Hamilton and Yamabe solitons are special solutions of the Yamabe flow, [5]. Given an $n(n \geq 2)$, dimensional Riemannian manifold $(M, g)$ such that $\{g(t)\}$ is the 1-parameter family of metrics and $r(t)$ is its scalar curvature. In this case, the equation of

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Yamabe flow is defined by

$$\frac{\partial g(t)}{\partial t} = -r(t)g(t).$$

Similarly, the equation of Ricci flow is given by

$$\frac{\partial g(t)}{\partial t} = -2Ric(t)g(t),$$

where $Ric(t)$ is the Ricci tensor [6]. In 2019, scalar combination of the Yamabe and Ricci flow was introduced by Güler and Crasmareanu as follows:

$$\frac{\partial g(t)}{\partial t} = \beta_2 r(t)g(t) - 2\beta_1 Ric(t)g(t).$$

This equation is known as Ricci-Yamabe flow and special solutions of the Ricci-Yamabe flow are famous as Ricci-Yamabe solitons [7]. Due to the sign of the scalars $\beta_1$ and $\beta_2$ the Ricci-Yamabe flow becomes a Riemannian, a semi-Riemannian or a singular Riemannian flow.

In this paper, we show that the Lie group $H_2 \times \mathbb{R}$ involves a vector field satisfying a Ricci-Yamabe soliton. We also prove that there does not exist a gradient Ricci-Yamabe soliton. Throughout the paper, all geometric objects (curves, manifolds, vector fields, functions etc.) are assumed to be smooth.

## 2. Preliminaries

2.1. **Ricci-Yamabe and Gradient Ricci-Yamabe Solitons.** A connected Riemannian manifold $(M, g, \beta_1, \beta_2, \beta_3)$ of dimension $n$ $(n \geq 2)$ is said to be a Ricci-Yamabe soliton if it satisfies

$$L_X g + 2\beta_1 Ric = -(2\beta_3 - \beta_2 r)g, \tag{2.1}$$

where $L_X g$ is the Lie derivative of the metric $g$ in the direction of the vector field $X$, $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, $Ric$ and $r$ denote the Ricci tensor and the scalar curvature of $M$, respectively. A Ricci-Yamabe soliton $(M, g, \beta_1, \beta_2, \beta_3)$ is said to be steady, expanding or shrinking and Ricci-Yamabe soliton if $\beta_3 = 0$ , $\beta_3 > 0$ and $\beta_3 < 0$ respectively. If a function $f : M \to \mathbb{R}$ satisfies $X = gradf$, then we say that the Ricci-Yamabe soliton is a gradient Ricci-Yamabe soliton.

2.2. **The Lie group $H_2 \times \mathbb{R}$.** We recall fundamental information about the Lie group $H_2 \times \mathbb{R}$ from [4]. Let $H_2 = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ denotes the upper half model of the hyperbolic plane equipped with the metric $g_{H_2} = \frac{1}{v^2}(du^2 + dv^2)$. The hyperbolic space $H_2$ with the group structure occured by the composition of proper affine maps is a Lie group with the left invariant metric

$$g = \frac{1}{v^2}(du^2 + dv^2) + dw^2. \tag{2.2}$$

A left-invariant orthonormal frame field $(e_1, e_2, e_3)$ is given by

$$e_1 = v\frac{\partial}{\partial u},\ e_2 = v\frac{\partial}{\partial v},\ e_3 = \frac{\partial}{\partial w}.$$

The Levi-Civita connection $\nabla$ with respect to the this orthonormal frame is given by

$$\begin{aligned} &\nabla_{e_1}e_1 = e_2,\ \nabla_{e_1}e_2 = -e_1,\ \nabla_{e_1}e_3 = 0,\\ &\nabla_{e_2}e_1 = 0,\ \nabla_{e_2}e_2 = 0,\ \nabla_{e_2}e_3 = 0,\\ &\nabla_{e_3}e_1 = 0, \nabla_{e_3}e_2 = 0, \nabla_{e_3}e_3 = 0. \end{aligned} \tag{2.3}$$

The Lie brackets are obtained as

$$[e_2, e_3] = 0,\ [e_1, e_3] = 0,\ [e_1, e_2] = -e_1.$$

The non-zero components of the curvature tensor field $R$ and the Ricci tensor $Ric$ are

$$\begin{aligned} R(e_1, e_2)e_1 &= e_2,\ R(e_1, e_2)e_2 = -e_1,\\ Ric(e_1, e_1) &= Ric(e_2, e_2) = -1. \end{aligned}$$

The scalar curvature $r$ of $H_2 \times \mathbb{R}$ is

$$r = -2.$$

## 3. Main results

We start this section by considering

$$X = t_1e_1 + t_2e_2 + t_3e_3 \tag{3.4}$$

is a potential vector field on $M$, where $t_1,\ t_2$ and $t_3$ are differentiable functions of $u, v$ and $w$. We label the coordinate basis by $\{\partial_u, \partial_v, \partial_w\}$.

**Theorem 3.1.** *Let us consider the Lie group $H_2 \times \mathbb{R}$ with the metric (2.2). Then the space $H_2 \times \mathbb{R}$ admits an expanding, steady or shrinking Ricci Yamabe soliton if and only if $\beta_1 > 0,\ \beta_1 < 0$ or $\beta_1 = 0$.*

*Proof.* From (2.2), (2.3) and (3.4) the Lie derivative of the metric tensor $g$ is computed as

$$\begin{aligned} L_X g(e_1,e_1) &= -2(t_2-\partial_u t_1), \\ L_X g(e_1,e_2) &= t_1+\partial_v t_1+\partial_u t_2, \\ L_X g(e_1,e_3) &= \partial_w t_1+\partial_u t_3, \\ L_X g(e_2,e_2) &= 2\partial_v t_2, \\ L_X g(e_2,e_3) &= \partial_w t_2+\partial_v t_3, \\ L_X g(e_3,e_3) &= 2\partial_w t_3. \end{aligned}$$

Putting (2.2), (2.3) and (3.4) in (2.1), we have

$$-(t_2-\partial_u t_1) = \beta_1-(\beta_3+\beta_2), \tag{3.5}$$

$$t_1+\partial_v t_1+\partial_u t_2 = 2\beta_2, \tag{3.6}$$

$$\partial_w t_1+\partial_u t_3 = 2\beta_2, \tag{3.7}$$

$$\partial_v t_2 = \beta_1-(\beta_3+\beta_2), \tag{3.8}$$

$$\partial_w t_2+\partial_v t_3 = 2\beta_2, \tag{3.9}$$

$$\partial_w t_3 = -(\beta_3+\beta_2). \tag{3.10}$$

Equation (3.10) gives $\partial_u\partial_w t_3=0$. Using this equation and deriving (3.7) with respect to $w$, we get

$$\partial_w^2 t_1=0. \tag{3.11}$$

From (3.11), we occur

$$t_1=\varphi(u,v)w+\psi(u,v),$$

where $\varphi$ and $\psi$ are functions. Equation (3.5) gives $-\partial_v t_2+\partial_u\partial_v t_1=0$. Having in mind this relation and taking derivative in (3.5) with respect to $v$ and using (3.8), we obtain

$$\partial_u\partial_v t_1=\beta_1-(\beta_3+\beta_2). \tag{3.12}$$

Taking derivative in (3.6) with respect to $v$, we find $\partial_v t_1+\partial_v^2 t_1+\partial_v\partial_u t_2=0$. Using this relation in (3.12), we find

$$\partial_v t_1+\partial_v^2 t_1=0. \tag{3.13}$$

Substituting $t_1$ in equations (3.12) and (3.13), we get

$$\begin{cases} \partial_u \partial_v \varphi w + \partial_u \partial_v \psi = \beta_1 - (\beta_3 + \beta_2), \\ (\partial_v \varphi + \partial_v^2 \varphi) w + \partial_v \psi + \partial_v^2 \psi = 0. \end{cases} \tag{3.14}$$

Taking derivative in (3.14) with respect to $w$, we get

$$\begin{cases} \partial_v \varphi + \partial_v^2 \varphi = 0, \\ \partial_v \psi + \partial_v^2 \psi = 0, \\ \partial_u \partial_v \varphi = 0, \\ \partial_u \partial_v \psi = \beta_1 - (\beta_3 + \beta_2). \end{cases} \tag{3.15}$$

Now, if we take derivative in $(3.15)_2$ with respect to $u$ and use in $(3.15)_4$, we find

$$\beta_1 = \beta_3 + \beta_2. \tag{3.16}$$

Integrating $(3.15)_1$ and $(3.15)_2$ give us

$$\begin{cases} \varphi(u,v) = \gamma_1 e^{-v} + \varphi_1(u), \\ \psi(u,v) = \gamma_2 e^{-v} + \psi_1(u), \end{cases} \tag{3.17}$$

where $\gamma_i \in \mathbb{R}$ and $\varphi_1$, $\psi_1$ are functions. Therefore,

$$t_1 = (\gamma_1 e^{-v} + \varphi_1(u))w + \gamma_2 e^{-v} + \psi_1(u). \tag{3.18}$$

Substituting $t_1$ in (3.6), we deduce

$$(\varphi_1(u) + \varphi_1''(u))w + \psi_1(u) + \psi_1''(u) = 2\beta_2. \tag{3.19}$$

Deriving equation (3.19) with respect to $w$, we find

$$\begin{cases} \varphi_1(u) + \varphi_1''(u) = 0, \\ \psi_1(u) + \psi_1''(u) = 0. \end{cases} \tag{3.20}$$

Integrating (3.20) with respect to $u$, we find

$$\begin{cases} \varphi_1(u) = \gamma_3 \cos u + \gamma_4 \sin u, \\ \psi_1(u) = \gamma_5 \cos u + \gamma_6 \sin u, \end{cases} \tag{3.21}$$

where $\gamma_i \in \mathbb{R}$. Hence,

$$t_1 = (\gamma_1 e^{-v} + \gamma_3 \cos u + \gamma_4 \sin u)w + \gamma_2 e^{-v} + \gamma_5 \cos u + \gamma_6 \sin u. \tag{3.22}$$

From (3.5), we have

$$t_2 = (-\gamma_3 \sin u + \gamma_4 \cos u)w - \gamma_5 \sin u + \gamma_6 \cos u. \tag{3.23}$$

Equation (3.10) yields

$$t_3 = -(\beta_3 + \beta_2)w + \xi(u, v),$$

where $\xi$ is a function. Substituting $t_1,\ t_2,\ t_3$ in (3.7) and (3.9) lead to

$$\begin{cases} \partial_u \xi = 2\beta_2 - (\gamma_1 e^{-v} + \gamma_3 \cos u + \gamma_4 \sin u) \\ \partial_v \xi = 2\beta_2 - (\gamma_3 \sin u + \gamma_4 \cos u). \end{cases} \tag{3.24}$$

Integrating $(3.24)_1$with respect to $u$, we obtain

$$\xi(u, v) = 2\beta_2 u - \gamma_1 u e^{-v} - \gamma_3 \sin u + \gamma_4 \cos u + \gamma_7,$$

and putting $\xi$ in $(3.24)_2$, we see that

$$\gamma_1 u e^{-v} = 2\beta_2 - \gamma_3 \sin u - \gamma_4 \cos u.$$

This gives us

$$\gamma_1 = \gamma_3 = \gamma_4 = \beta_2 = 0.$$

Finally, we find

$$\begin{cases} t_1 = \gamma_2 e^{-v} + \gamma_5 \cos u + \gamma_6 \sin u, \\ t_2 = -\gamma_5 \sin u + \gamma_6 \cos u, \\ t_3 = -\beta_3 w + \gamma_7, \end{cases} \tag{3.25}$$

where $\gamma_i \in \mathbb{R}$.

This shows that $X = t_1 e_1 + t_2 e_2 + t_3 e_3$ given by (3.25) satisfies (2.1). We also found that $\beta_1 = \beta_3 + \beta_2$ and $\beta_2 = 0$. Therefore we proved the theorem. □

**Theorem 3.2.** *Let us consider the Lie group $H_2 \times \mathbb{R}$ with the metric (2.2). Then the space $H_2 \times \mathbb{R}$ does not admit a gradient Ricci-Yamabe soliton.*

*Proof.* Suppose that $X = grady$ is a gradient vector field on $M$ with potential function $y$. Then $X$ is given by

$$grady = v^2 \partial_u y \partial_u + v^2 \partial_v y \partial_v + \partial_w y \partial_w.$$

From (3.25), we see that the Lie group $H_2 \times \mathbb{R}$ is a gradient Yamabe soliton if and only if the function $y$ fulfills the following equations:

$$\begin{cases} \partial_u y = \frac{\gamma_2}{v} e^{-v} + \frac{\gamma_5}{v} \cos u + \frac{\gamma_6}{v} \sin u, \\ \partial_v y = \frac{-\gamma_5}{v} \sin u + \frac{\gamma_6}{v} \cos u, \\ \partial_w y = -\beta_3 w + \gamma_7. \end{cases} \tag{3.26}$$

Integrating above system we find

$$y(u,v,w) = \ln v[\gamma_5 \sin u - \gamma_6 \cos u] - \frac{\beta_3^2}{2}\gamma_7 w^2 + \gamma_8, \ \gamma_i \in \mathbb{R}.$$

But the function $y$ does not fulfill $(3.26)_1$. This ends the proof. □

## 4. Conclusion

Ricci and Yamabe solitons have many applications for several sciences such as differential geometry and theoretical physics. Similar to these solitons, their scalar combinations, which are called Ricci-Yamabe solitons, have been studied increasingly since the work of Güler and Crasmareanu [7]. In this paper, we investigate Ricci-Yamabe solitons in the Lie group $H_2 \times \mathbb{R}$. Our calculations in this paper may provide an insight for further studies about Ricci-Yamabe solitons on other Thurston geometries.

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# ON THE OPERATOR EQUATION $ABA = ACA$ AND ITS GENERALIZATION ON NON-ARCHIMEDEAN BANACH SPACES

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Abstract. Let $X$ and $Y$ be non-Archimedean Banach spaces over a non-Archimedean valued field $\mathbb{K}$. In this paper, we study some properties of $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $ABA = ACA$ and many basic operator properties in common of $AC - I_Y$ and $BA - I_X$ are given. In particular, $N(I_Y - AC)$ is a complemented subspace of $Y$ if and only if $N(I_X - BD)$ is a complemented subspace of $X$. Moreover, the approach is generalized for considering relationships between the properties of $I_Y - AC$ and $I_X - BD$. Finally, several illustrative examples are provided.

**Keywords**: Non-Archimedean Banach spaces, operator equation, operators theory.

**2010 Mathematics Subject Classification**: Primary 47A05, 47A10, Secondary 47S10.

## 1. Introduction

In classical operators theory, the operator equations $ABA = A^2$ and $BAB = B^2$ were studied by sereval researchers, for more details see [10, 11, 12, 14]. Recently, Barnes established some consequences on common operator properties of continuous linear operators $AB$ and $BA$ on complex Banach spaces, for more details, we refer to [1]. Moreover, Corach et al. [2] established some common properties of $AC - I_Y$ and $BA - I_X$ when $ABA = ACA$. In [17], Zeng and Zhong proceeded to examine common properties of $AC$ and $BA$ from the attitude of classical spectral theory considering $A \in \mathcal{L}(X,Y), B, C \in \mathcal{L}(Y,X)$ such that $ABA = ACA$, where $X$ and $Y$ were assumed to be Banach spaces over $\mathbb{C}$. In particular, they gave an affirmative answer to one question raised by the authors [2], by showing that $AC - I_Y$ is of

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closed range if and only if $BA - I_X$ is of closed range. Yan and Fang [16] examined the joint properties of $BD$ and $AC$ in terms of regularity when $A, D \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $ACD = DBD$ and $DBA = ACA$. In non-Archimedean operators theory, Ettayb [4] studied specific properties of operator equations $ABA = A^2$ and $BAB = B^2$ on a non-Archimedean Banach space $X$ and many basic operator properties in common of $I_X - AB$ and $I_X - BA$ were described. In particular, if $X, Y$ are non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$, $A \in \mathcal{L}(X,Y)$ and $B \in \mathcal{L}(Y,X)$, then $N(I_Y - AB)$ is a complemented subspace of $Y$ if and only if $N(I_X - BA)$ is a complemented subspace of $X$. Recently, Ettayb [6] examined the common properties of $AC$ and $BD$ whenever $A, D \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $ACD = DBD$ and $DBA = ACA$ where $X$ and $Y$ were supposed to be non-Archimedean Banach spaces over a non-Archimedean valued field $\mathbb{K}$.

Non-Archimedean spectral theory played a crucial role in non-Archimedean functional analysis which has had numerous applications in non-Archimedean applied mathematics and physics, including non-Archimedean differential, pseudo-differential equations and quantum physics. For additional details see [7, 8, 15]. This work is motivated by many studies on non-Archimedean operators theory and spectral theory of continuous linear operators, e.g. [4, 6, 5, 7, 8, 15]. Throughout this paper, $\mathbb{K}$ is a complete non-Archimedean valued field with a non-trivial valuation $|\cdot|$, $X$ and $Y$ are non-Archimedean Banach spaces over $\mathbb{K}$ and $\mathcal{L}(X,Y)$ will denote the collection of all continuous linear operators from $X$ into $Y$. For $X = Y$, we put $\mathcal{L}(X,X) = \mathcal{L}(X)$. $(\mathbb{Q}_p, |.|_p)$ is the field of $p$-adic numbers. For more details, we refer to [3, 9, 13]. Let $A \in \mathcal{L}(X)$, $R(A)$, $N(A)$, $A^*$, $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the range, the kernel, the adjoint, the spectrum, the point spectrum and the resolvent set of $A$ respectively.

The goal of this work is to develop the theory of some operator equations of continuous linear operators in non-Archimedean Banach spaces.

## 2. Preliminaries

We continue with the following preliminaries.

**Definition 2.1.** [3] *A field $\mathbb{K}$ is said to be non-Archimedean if it is endowed with an absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that:*

(i) $|\alpha| = 0$ *if, and only if,* $\alpha = 0$;

(ii) *For all* $\alpha, \mu \in \mathbb{K}$, $|\alpha\mu| = |\alpha||\mu|$;

(iii) *For each* $\alpha, \mu \in \mathbb{K}$, $|\alpha + \mu| \le \max\{|\alpha|, |\mu|\}$.

**Definition 2.2.** [9] $\mathbb{K}$ *is said to be spherically complete if each sequence of balls* $(B_n)_{n\geq 1}$ *of* $\mathbb{K}$ *such that* $B_{n+1} \subset B_n$ *for every* $n \geq 1$, *we have* $\bigcap_{n\geq 1} B_n \neq \emptyset$.

**Definition 2.3.** [3] *Let* $X$ *be a vector space over* $\mathbb{K}$. *A function* $\|\cdot\| : X \to \mathbb{R}_+$ *is called a non-Archimedean norm if:*

(i) *For each* $u \in X$, $\|u\| = 0$ *if and only if* $u = 0$;
(ii) *For all* $u \in X$ *and* $\lambda \in \mathbb{K}$, $\|\lambda u\| = |\lambda|\|u\|$;
(iii) *For each* $u, y \in X$, $\|u + y\| \leq \max(\|u\|, \|y\|)$.

**Definition 2.4.** [3] *A non-Archimedean Banach space is a complete non-Archimedean normed space.*

**Example 2.1.** [3] *The space* $c_0(\mathbb{K})$ *is the space of all sequences* $(x_i)_{i\in\mathbb{N}}$ *in* $\mathbb{K}$ *such that* $\lim_{i\to\infty} x_i = 0$. *Hence* $(c_0(\mathbb{K}), \|\cdot\|)$ *is a non-Archimedean Banach space where for any* $(x_i)_{i\in\mathbb{N}} \in c_0(\mathbb{K})$, $\| (x_i)_{i\in\mathbb{N}} \| = \sup_{i\in\mathbb{N}} |x_i|$.

**Definition 2.5.** [13] *Let* $P \in \mathcal{L}(X)$. $P$ *is said to be a projection if* $P^2 = P$.

**Remark 2.1.** [13] *If* $P$ *is a projection on* $X$, *then* $R(P)$ *is the kernel of* $I - P$ *so that* $R(P)$ *is a closed linear subspace of* $X$.

The following lemma holds.

**Lemma 2.1.** [13] *Let* $P \in \mathcal{L}(X)$ *be a projection. Hence, we have the following:*

(i) $I_X - P$ *is a projection;*
(ii) *If* $P \neq 0$, *then* $\|P\| \geq 1$;
(iii) *If* $P \neq 0$ *and* $P \neq I_X$, *then* $\|P\| = \|I_X - P\|$;
(iv) *If* $Q$ *is projection such that* $PQ = QP$, *then* $PQ$ *is projection.*

We have the following definition.

**Definition 2.6.** [9] *A subspace* $D$ *of* $X$ *is said to be complemented if there is a continuous projection* $P \in \mathcal{L}(X)$ *such that* $R(P) = D$. *In such case,* $D = R(P)$ *and* $D_1 = N(P)$ *are closed subspaces and* $X = D \oplus D_1$.

**Remark 2.2.** [9] *If* $D$, $D_1$ *are closed subspaces of* $X$ *such that* $X = D \oplus D_1$, *then* $D$ *is complemented in* $X$ *and* $D_1$ *is a complement of* $D$.

For more details, see [9, 13].

**Proposition 2.1.** [6] *Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, we have:*

(i) $N(B) \cap N(I_Y - AB) = \{0\}$;

(ii) $B(N(I_Y - AB)) = N(I_X - BA)$.

The following theorem is valid.

**Theorem 2.1.** [6] *Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$. Hence $R(I_Y - AB)$ is closed if and only if $R(I_X - BA)$ is closed.*

**Theorem 2.2.** [4] *Let $X$ and $Y$ be non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$ and let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, hence $N(I_Y - AB)$ is a complemented subspace of $Y$ if and only if $N(I_X - BA)$ is a complemented subspace of $X$.*

## 3. Main Results

We have the following results.

**Lemma 3.1.** *Let $A \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ with $ABA = ACA$. Hence $R(AC - I_Y)$ is closed in $Y$ if and only if $R(BA - I_X)$ is closed in $X$.*

*Proof.* If $R(AC - I_Y)$ is closed in $Y$, hence let $(x_n)_{n\in\mathbb{N}} \subset R(BA - I_X)$ with $x_n \to x$ for some $x \in X$ as $n \to \infty$. Then there is a sequence $(z_n)_{n\in\mathbb{N}} \subset X$ with $x_n = (BA - I_X)z_n$ for any $n \in \mathbb{N}$. Thus

$$\begin{aligned} Ax &= \lim_{n\to\infty} Ax_n \\ &= \lim_{n\to\infty} A((BA - I_X)z_n) \\ &= \lim_{n\to\infty} (ABA - A)z_n \\ &= \lim_{n\to\infty} (ACA - A)z_n \\ &= \lim_{n\to\infty} (AC - I_Y)Az_n. \end{aligned}$$

From $R(AC-I_Y)$ is closed in $Y$, there is $y \in X$ with $(AC-I_Y)y = Ax$. Thus $y = ACy - Ax$.

$$\begin{aligned} x &= BAx - (BA - I_X)x \\ &= B(AC - I_Y)y - (BA - I_X)x \\ &= (BAC - B)(ACy - Ax) - (BA - I_X)x \\ &= BACACy - BACAx - BACy + BAx - (BA - I_X)x \\ &= BABACy - BABAx - BACy + BAx - (BA - I_X)x \\ &= (BA - I_X)(BACy - BAx - x). \end{aligned}$$

Consequently, $x \in R(BA - I_X)$. Then $R(BA - I_X)$ is closed in $X$. Conversely, assume that $R(BA-I_X)$ is closed in $X$. Then, from Theorem 2.1, $R(BA-I_X)$ is closed in $X$ if and only if $R(AB-I_Y)$ is closed in $Y$. Thus $R(CA-I_X)$ is closed in $X$. By Theorem 2.1, $R(AC-I_Y)$ is closed in $Y$. Consequently, $R(AC - I_Y)$ is closed in $Y$ if and only if $R(BA - I_X)$ is closed in $X$. □

**Lemma 3.2.** *If $A \in \mathcal{L}(X,Y)$, $B, C \in \mathcal{L}(Y,X)$ such that $ABA = ACA$. Hence for each $n \in \mathbb{N}$, $R((AC - I_Y)^n)$ is closed in $Y$ if and only if $R((BA - I_X)^n)$ is closed in $X$.*

*Proof.* Set

$$(\forall n \in \mathbb{N})\ B_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} B(AB)^{k-1}$$

and

$$(\forall n \in \mathbb{N})\ C_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} C(AC)^{k-1}.$$

Since $ABA = ACA$, for each $n \in \mathbb{N}$, $AB_nA = AC_nA$. Moreover, for each $n \in \mathbb{N}$,

$$\begin{aligned} I - AC_n &= I - \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} AC(AC)^{k-1} \\ &= I + \sum_{k=1}^{n+1} (-1)^{k} \binom{n+1}{k} (AC)(AC)^{k-1} \\ &= I + \sum_{k=1}^{n+1} \binom{n+1}{k} (-AC)^{k} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-AC)^{k} \\ &= (I - AC)^{n+1}. \end{aligned}$$

Similarly, we have for all $n \in \mathbb{N}$, $(I - BA)^{n+1} = I - B_n A$. By Lemma 3.1, for each $n \in \mathbb{N}$, $R((AC - I_Y)^n)$ is closed in $Y$ if and only if $R((BA - I_X)^n)$ is closed in $X$. □

**Lemma 3.3.** *If $A \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ with $ABA = ACA$. Hence for each $n \in \mathbb{N}$,*

(i) $AR((BA - I_X)^n) \subset R((AC - I_Y)^n)$;

(ii) $AN((BA - I_X)^n) \subset N((AC - I_Y)^n)$;

(iii) $BACN((AC - I_Y)^n) \subset N((BA - I_X)^n)$;

(iv) $BACR((AC - I_Y)^n) \subset R((BA - I_X)^n)$.

*Proof.* (i) If $x \in R((BA - I_X)^n)$, hence there is $u \in X$ with $x = (BA - I_X)^n u$. Then

$$Ax = A(BA - I_X)^n u = (AC - I_Y)^n Au \in R((AC - I_Y)^n).$$

Thus $AR((BA - I_X)^n) \subset R((AC - I_Y)^n)$.

(ii) Let $x \in N((BA - I_X)^n)$, then $(BA - I_X)^n x = 0$. Hence

$$(AC - I_Y)^n Ax = A(BA - I_X)^n x = 0.$$

Consequently, $Ax \in N((AC - I_Y)^n)$. Thus $AN((BA - I_X)^n) \subset N((AC - I_Y)^n)$.

(iii) Let $y \in N((AC - I_Y)^n)$, then $(AC - I_Y)^n y = 0$. Hence

$$(BA - I_X)^n BACy = BAC(AC - I_Y)^n y = 0.$$

Consequently, $BACy \in N((BA - I_X)^n)$. Then

$$BACN((AC - I_Y)^n) \subset N((BA - I_X)^n).$$

(iv) If $z \in R((AC - I_Y)^n)$. Hence there is $y \in Y$ with $z = (AC - I_Y)^n y$. Then

$$BACz = BAC(AC - I_Y)^n y = (BA - I_X)^n BACy \in R((BA - I_X)^n).$$

Consequently, $BACR((AC - I_Y)^n) \subset R((BA - I_X)^n)$.

□

**Lemma 3.4.** *If $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ with $ABA = ACA$. Hence for each $n \in \mathbb{N}$, $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in $Y$ if and only if $R(BA - I_X) + N((BA - I_X)^n)$ is closed in $X$.*

*Proof.* Assume that $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in $Y$. Let $(x_n)_{n \in \mathbb{N}} \subset R(BA - I_X) + N((BA - I_X)^n)$ with $x_n \to x$ for some $x \in X$ as $n \to \infty$. Hence there are sequences

$(z_n)_{n\in\mathbb{N}} \subset R(BA - I_X)$ and $(w_n)_{n\in\mathbb{N}} \subset N((BA - I_X)^n)$ with $x_n = z_n + w_n$ for any $n \in \mathbb{N}$. Thus

$$\begin{aligned} Ax &= \lim_{n\to\infty} Ax_n \\ &= \lim_{n\to\infty} A(z_n + w_n). \end{aligned}$$

From $(i)$ and $(ii)$ of Lemma 3.3, we get $Az_n \in R(AC - I_Y)$ and $Aw_n \in N((AC - I_Y)^n)$. From $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in $Y$, there are $z \in Y$ and $w \in N((AC - I_Y)^n)$ with $(AC - I_Y)z + w = Ax$. Thus $z = ACz - Ax + w$. Consequently,

$$\begin{aligned} x &= BAx - (BA - I_X)x \\ &= B((AC - I_Y)z + w) - (BA - I_X)x \\ &= (BAC - B)z + Bw - (BA - I_X)x \\ &= (BAC - B)(ACz - Ax + w) + Bw - (BA - I_X)x \\ &= BACACz - BACAx + BACw - BACz + BAx - Bw + Bw - (BA - I_X)x \\ &= BACACz - BACAx + BACw - BACz + BAx - (BA - I_X)x \\ &= BABACz - BABAx + BACw - BACz + BAx - (BA - I_X)x \\ &= (BA - I_X)(BACz - BAx - x) + BACw. \end{aligned}$$

Since $w \in N((AC - I_Y)^n)$ and from $(iii)$ of Lemma 3.3, we have $x \in R(BA - I_X) + N((BA - I_X)^n)$. Then $R(BA - I_X) + N((BA - I_X)^n)$ is closed in $X$. Conversely, assume that $R(BA - I_X) + N((BA - I_X)^n)$ is closed in $X$. By Theorem 2.1, $R(AB - I_Y) + N((AB - I_Y)^n)$ is closed in $Y$. Then $R(CA - I_X) + N((CA - I_X)^n)$ is closed in $X$. From Theorem 2.1, $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in $Y$. □

The following theorem holds.

**Theorem 3.1.** *Let $X$ and $Y$ be non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$. If $A, D \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ with $ACD = DBD$ and $DBA = ACA$. Hence $N(I_Y - AC)$ is complemented in $Y$ if and only if $N(I_X - BD)$ is complemented in $X$.*

*Proof.* Assume that $N(I_Y - AC)$ is a complemented subspace of $Y$, hence there is a bounded projection $Q \in \mathcal{L}(Y)$ with $R(Q) = N(I_Y - AC)$, thus $(I_Y - AC)Q = 0$, hence $Q = ACQ$. Set $P = BQACD \in \mathcal{L}(X)$. By $DBQ = DBACQ = ACACQ = ACQ = Q$, hence

$$P^2 = (BQACD)(BQACD) = BQACQACD = BQACD = P.$$

Note that

$$(I_X - BD)P = (I_X - BD)(BQACD) = BQACD - BDBQACD = 0.$$

Then $R(P) \subseteq N(I_X - BD)$. If $x \in N(I_X - BD)$. Thus $Dx = DBDx = ACDx$, hence $Dx \in N(I_Y - AC) = R(Q)$. Thus $QDx = Dx$, then

$$Px = BQACDx = BQACQDx = BQDx = BDx = x,$$

hence $N(I_X - BD) \subseteq R(P)$. Then $P$ is a projection with $R(P) = N(I_X - BD)$. Conversely, suppose that $W$ is a projection with $R(W) = N(I_X - BD)$. Put $Z = ACDWBACAC$. By $BDW = W$, it follows that

$$\begin{aligned}
Z^2 &= (ACDWBACAC)(ACDWBACAC) \\
&= ACDWB(ACA)CACDWBACAC \\
&= ACDWB(DBA)C(ACD)WBACAC \text{ from } ACA = DBA \text{ and } ACD = DBD \\
&= ACDWBDB(ACD)BDWBACAC \\
&= ACDWBDB(DBD)BDWBACAC \\
&= ACDWBDBDBDBDWBACAC \\
&= ACDWBACAC \\
&= Z.
\end{aligned}$$

From

$$\begin{aligned}
(I_Y - AC)Z &= (I_Y - AC)(ACDWBACAC) \\
&= ACDWBACAC - ACACDWBACAC \\
&= ACDWBACAC - ACDBDWBACAC \\
&= ACDWBACAC - ACDWBACAC \\
&= 0,
\end{aligned}$$

$R(Z) \subseteq N(I_Y - AC)$. Let $x \in N(I_Y - AC)$. Hence $x = ACx$. By $BACx = BACACx = BDBACx$ and $BACx \in N(I_X - BD) = R(W)$, we get $WBACx = BACx$. Thus

$$\begin{aligned} Zx &= ACDWBACACx \\ &= ACDWBACx \\ &= ACDBACx \\ &= ACACACx \\ &= x, \end{aligned}$$

hence $N(I_Y - AC) \subseteq R(Z)$. Then $Z$ is the projection onto $N(I_Y - AC)$. □

**Theorem 3.2.** *Let $X$ and $Y$ be non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$. Let $A, D \in \mathcal{L}(X,Y)$, $B, C \in \mathcal{L}(Y,X)$ such that $ACD = DBD$ and $DBA = ACA$. Hence $R(I_Y - AC)$ is complemented in $Y$ if and only if $R(I_X - BD)$ is complemented in $X$.*

*Proof.* Suppose that $Q$ is the projection with $R(Q) = R(I_Y - AC)$. Put $P = I_X - BAC(I_Y - Q)D$. From $(I_Y - Q)(I_Y - AC) = 0$ and $(I_Y - Q)AC = I_Y - Q$, we get

$$\begin{aligned} P^2 &= [I_X - BAC(I_Y - Q)D][I_X - BAC(I_Y - Q)D] \\ &= I_X - BAC(I_Y - Q)D - BAC(I_Y - Q)D + BAC(I_Y - Q)DBAC(I_Y - Q)D \\ &= I_X - BAC(I_Y - Q)D - BAC(I_Y - Q)D + BAC(I_Y - Q)ACAC(I_Y - Q)D \\ &= I_X - BAC(I_Y - Q)D - BAC(I_Y - Q)D + BAC(I_Y - Q)D \\ &= I_X - BAC(I_Y - Q)D \\ &= P. \end{aligned}$$

Thus $P^2 = P$. From $R(Q) = R(I_Y - AC)$, we get

$$R(BACQD) \subseteq R(BAC(I_Y - AC)) = R((I_X - BD)BAC) \subseteq R(I_X - BD).$$

Moreover,

$$\begin{aligned} P &= I_X - BAC(I_Y - Q)D \\ &= I_X - BACD + BACQD \\ &= I_X - BDBD + BACQD \\ &= (I_X - BD)(I_X + BD) + BACQD, \end{aligned}$$

and thus $R(P) \subseteq R(I_X - BD)$. If $x \in R(I_X - BD)$, hence there is $u \in X$ with $x = (I_X - BD)u$. From $Dx = D(I_X - BD)u = (I_Y - AC)Du \in R(Q)$, we get

$$Px = [I_X - BAC(I_Y - Q)D]x = x,$$

hence $R(I_X - BD) \subseteq R(P)$. Then $R(I_X - BD)$ is complemented in $X$. Conversely, suppose that $Q$ is a projection with $R(Q) = R(I_X - BD)$. Set $W = I_Y - ACD(I_X - Q)BAC$. We demonstrate that $W$ is a projection such that $R(W) = R(I_Y - AC)$. From $(I_X - Q)(I_X - BD) = 0$ and $(I_X - Q)BD = I_X - Q$, hence

$$\begin{aligned} W^2 &= [I_Y - ACD(I_X - Q)BAC]^2 \\ &= I_Y - 2ACD(I_X - Q)BAC + ACD(I_X - Q)BACACD(I_X - Q)BAC \\ &= I_Y - 2ACD(I_X - Q)BAC + ACD(I_X - Q)BDBACD(I_X - Q)BAC \\ &= I_Y - 2ACD(I_X - Q)BAC + ACD(I_X - Q)BDBDBD(I_X - Q)BAC \\ &= I_Y - ACD(I_X - Q)BAC \\ &= W. \end{aligned}$$

Thus $W^2 = W$. One can see that

$$\begin{aligned} W &= I_Y - ACD(I_X - Q)BAC \\ &= I_Y - ACDBAC + ACDQBAC \\ &= I_Y - ACACAC + ACDQBAC, \end{aligned}$$

and

$$\begin{aligned} R(W) &\subseteq R(I_Y - ACACAC + ACDQBAC) \\ &\subseteq R[(I_Y - AC)(I_Y + AC + ACAC)] + R[ACD(I_X - BD)] \\ &\subseteq R(I_Y - AC) + R[(I_Y - AC)ACD] \\ &\subseteq R(I_Y - AC). \end{aligned}$$

For each $y \in R(I_Y - AC)$, there is $w \in Y$ with $y = (I_Y - AC)w$. Hence $BACy = BAC(I_Y - AC)w = (I_X - BD)BACw \in R(Q)$, then

$$Wy = [I_Y - ACD(I_X - Q)BAC]y = y.$$

Thus $R(I_Y - AC) \subseteq R(W)$. Hence $R(I_Y - AC)$ is complemented in $Y$. □

We finish with the following examples.

**Example 3.1.** *Let* $A, B, C \in \mathcal{L}(c_0(\mathbb{K}))$ *be given respectively by*

$$A(x_1, x_2, x_3, x_4, \cdots) = (0, x_2, 0, x_4, \cdots),$$

$$B(x_1, x_2, x_3, x_4, \cdots) = (0, x_1, x_2, x_4, \cdots)$$

*and*

$$C(x_1, x_2, x_3, x_4, \cdots) = (0, 0, x_1, x_4, \cdots).$$

*It is easy to see that* $ABA = ACA$.

**Example 3.2.**

(i) *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

*It is easy to see that* $ABA = ACA = 0_{\mathcal{M}_2(\mathbb{Q}_p)}$.

(ii) *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

*Then* $ABA = ACA$.

(iii) *Let* $a, b, c \in \mathbb{Q}_p$. *Let*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Q}_p).$$

*One can see that if* $a = c = 0$, $b = 1$, *then* $B = C$ *and* $ABA = ACA$.

*Moreover in the case* $B \neq C$, *we have* $ABA = ACA$.

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# STUDY OF SOME CURVES ALONG CONFORMAL SUBMERSION

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Abstract. In this article, we study bi-f-harmonic curves, hyperelastic curves, helices and circles along conformal Riemannian submersion. We investigate the behavior of an arbitrary horizontal curve on the total manifold under the conformal submersion. Moreover, we show that a totally geodesic Riemannian submersion takes a horizontal bi-f-harmonic curve, helix and circle to a bi-f-harmonic curve, helix and circle on target manifold, respectively. In addition, we also find the conditions for which Riemannian submersion takes a horizontal bi-f-harmonic curve, helix and circle to a bi-f-harmonic curve, helix and circle on target manifold, respectively.

**Keywords**: Bi-f-harmonic curve, bi-harmonic curve, helix, circle, hyperelastic curves, elastic curve, totally geodesic conformal submersion and totally umbilical conformal submersion.
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## 1. Introduction

In 1964, J. Eells and J. H. Sampson [6], introduced the concept of bi-harmonic maps by generalizing the harmonic maps. Harmonic maps have important applications in various areas of mathematics and physics with nonlinear partial differential equations. A harmonic map $\alpha : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ between the Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$ is a

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critical point of the energy functional,

$$E(\alpha) = \frac{1}{2}\int_{\Gamma_N} |d\alpha|^2 v_{g_N},$$

where $\Gamma_N$ is some compact domain of $N$ and $\tau(\alpha) = Trace_{g_N}\nabla d\alpha$ is the tension field of $\alpha$. The harmonic map equation is an Euler-Lagrange equation of the functional $\tau(\varphi) \equiv Trace_{g_N}\nabla d\varphi = 0$, where $\tau(\varphi) = Trace_{g_N}\nabla d\varphi$ is the tension field of $\varphi$ [6]. The bi-harmonic map $\alpha$ between the Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$ is a critical point of the bi-energy functional, $E_2(\alpha) = \frac{1}{2}\int_{\Gamma_N} |\tau(\alpha)|^2 v_{g_N}$, where $\Gamma_N$ is a compact domain of $N$. The bi-harmonic map equation is an Euler-Lagrange equation of the functional,

$$\tau_2(\alpha) \equiv Trace_{g_N}(\nabla^\alpha\nabla^\alpha - \nabla^\alpha_{\nabla^N})\tau(\alpha) - Trace_{g_N} R^{\bar{N}}(d\alpha, \tau(\alpha))d\alpha = 0,$$

where $R^{\bar{N}} = [\nabla^{\bar{N}}_X, \nabla^{\bar{N}}_Y]Z - \nabla^{\bar{N}}_{[X,Y]}Z$, is a Riemann curvature tensor of $(\bar{N}, g_{\bar{N}})$ [16]. In 1991 [5], the author introduced the bi-harmonic submanifolds of Euclidean space and stated a conjecture " any bi-harmonic submanifold of Euclidean space is harmonic, thus minimal". If the definition of bi-harmonic maps for Riemannian immersion in Euclidean space is used, then the Chen's definition of a bi-harmonic submanifold coincides with the definition given by the bi-energy functional.

Bi-f-harmonic maps are the generalization of harmonic maps and f-harmonic maps. There are two methods to formalize the link between bi-harmonic maps and f-harmonic maps. In the first method of formalization, the authors extended the bi-energy functional in [32, 39] to the bi-f-energy functional and got bi-f-harmonic maps. Further, for the second formalization, the f-energy functional is extended to the f-bi-energy functional. In [22], the author introduced the f-bi-harmonic maps by generalizing the bi-harmonic maps. The bi-f-harmonic equation for curves in Euclidean space, hyperbolic space, sphere and hypersurfaces of manifolds were studied in [30].

In [34], authors studied the charcterization of submanifold by taking the hyperelastic curves along an immersion. The following properties of Riemannian submersions were studied in [10, 25, 19]. In 1974, the authors proved that if a circle is mapped by immersion from a submanifold to the ambient manifold, then the submanifold is said to be totally umbilical with a parallel mean curvature vector field [26].

In the sixties O' Neill and Gray introduced the concept of Riemannian submersions between Riemannian manifolds [11, 25]. A differential map $G$ between two Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$ is known as a submersion if the rank of $G_*$ is equal to the dimension of the targeted manifold. Also, if the submersion is isometry between $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$, then $G$ is called a Riemannian submersion. Conformal submersion and the fundamental equations of conformal submersion were studied in [28, 12]. In [37, 38], authors study the totally umbilical, geodesic and minimal fibers by using conformal submersions. Horizontally conformal submersion is a generalization of the Riemannian submersion [9, 14]. Horizontally conformal map is useful for the characterization of harmonic morphisms [4] and has many applications in medical imaging (brain imaging) and computer graphics.

Hyperelastic curves in a Riemannian manifold are solutions to a constrained variable problem and are characterized by Euler-Lagrange equations. A parametrized curve by its arclength is said to be a hyperelastic curve if it is a critical point of the following curvature energy action defined on a suitable space of curves in a Riemannian manifold

$$\mathcal{F}^r_\gamma = \int (\kappa^r + \mu) ds, \tag{1.1}$$

where $\kappa$ denotes the curvature of $\gamma$ [3, 36, 31]. If $\mu = 0$, then these curves are called free hyperelastic curves. In 2021, B. Sahin, G. O. Tukel and T. Turhan, studied the effect of hyperelastic curves on the geometry of isometric immersions in [33]. The functional $\mathcal{F}^r_\lambda$ is the classical Euler-Bernoulli's bending (or elastic) energy functional for $r = 2$. Immersed curves which are critical for the bending energy functional satisfying some boundary conditions are said to be elastic curves (or elastica) [20]. The existence, classification or stability problems of elastic curves or their generalizations in Riemannian manifolds attracted the attention of many researchers. There are the following examples in the literature worked by D. Singer et al. [15, 21, 20, 35]. In 1984, J. Langer and D. Singer proved that there exist closed elastic curves of a fixed length in a compact Riemannian manifold [20].

A smooth curve parametrized by its arc-length on a Riemannian manifold $N$ is said to be circle if it satisfies $\nabla^2_{\dot\beta}\dot\beta = -\kappa^2\dot\beta$, where $\kappa$ is a non-negative constant curvature of $\beta$ and $\nabla_{\dot\beta}$ is the covariant differentiation along $\beta$ with respect to the Riemannian connection $\nabla$ on $N$. In [26], Nomizu-Yano proved that $\beta$ is a circle iff the following is satisfies

$$\nabla^2_{\dot\beta}\dot\beta + g(\nabla_{\dot\beta}\dot\beta, \nabla_{\dot\beta}\dot\beta)\dot\beta = 0,$$

where $g$ is the Riemannian metric on $N$ and $\nabla^2_{\dot\beta}\dot\beta = \nabla_{\dot\beta}\nabla_{\dot\beta}\dot\beta$. Many authors studied circles on Riemannian manifolds and they showed that it is possible to obtain certain properties of a submanifolds by observing the extrinsic structure of circles on this submanifold, [34, 2, 7, 13, 17, 23, 24, 27, 29]. In 1963, S. Kobayashi and K. Nomizu showed that an ordinary helix

$c = c(s)$ satisfies the following equation, $\nabla^3_{\dot\beta}\dot\beta + K^2\nabla_{\dot\beta}\dot\beta = 0$, where $K^2 = \kappa^2+\tau^2$ is a positive constant. Conversely, if a curve $c = c(s)$ satisfies the above condition, then it is an ordinary helix or a geodesic, [18].

The structure of the article is as follows: In Section 2, we recall some basic concepts about conformal Riemannian submersion, totally geodesic fibers and the second fundamental form of Riemannian submersion. In Section 3, some conditions are derived for the case where the curve either in the base manifold or in the target manifold is a bi-f-harmonic curve. In section 3, we show that a totally geodesic conformal submersion between two Riemannian manifolds takes a bi-harmonic curve to a bi-harmonic curve. In section 4, we prove that the conformal submersion takes a curve to a helix iff the curve is of constant curvature. In the same section, we also find the conditions for a curve to become a circle in a targeted manifold by conformal submersion. In the final section, we study the hyperelastic curves along the conformal submersions.

## 2. Preliminaries

Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a differentiable map between the Riemannian manifolds $(N, g_N)$ and $(\bar N, g_{\bar N})$ of dimensions $n_1$ and $n_2$, respectively such that $n_1 > n_2$. Then $G$ is said to be a Riemannian submersion if rank of $G$ is maximal and differential $G_*$ preserves the lengths of horizontal vectors. A Riemannian submersion $G : (N, g_N) \to (\bar N, g_{\bar N})$ is said to be a conformal submersion if the restriction of $G_*$ to the horizontal distribution of $G$ is a conformal map, i.e. there exist a smooth function $\lambda : N \to R^+$ such that

$$g_{\bar N}(G_*(X), G_*(Y)) = \lambda^2(p)g_N(X, Y),$$

for all $X, Y \in \Gamma(kerG_*)^\perp$ and $p \in N$.

A curve $\beta : I \to N$ on $(N, g)$ is said to be a bi-f-harmonic curve if and only if $\beta$ satisfies the condition [30],

$$\begin{aligned}(ff''' + f'f'')\dot\beta + (3ff'' + 2f'^2)\nabla_{\dot\beta}\dot\beta + 4ff'\nabla^2_{\dot\beta}\dot\beta + f^2\nabla^3_{\dot\beta}\dot\beta \\ +f^2R(\nabla_{\dot\beta}\dot\beta, \dot\beta)\dot\beta = 0,\end{aligned} \tag{2.2}$$

where $f : I \to (0, \infty)$ is a smooth function, $\nabla$ is a Levi-Civita connection and $R$ is a Riemannian curvature tensor on $N$. Let $G : (N, g) \to (\bar N, \bar g)$ be a Riemannian submersion between $(N, g)$ and $(\bar N, \bar g)$. Then $\beta$ is said to be a horizontal curve if $\dot\beta(t) \in (kerG_*)^\perp$; $\forall\, t \in I$. If $\nabla^{\bar N}$ is the Levi-Civita connection on $(\bar N, \bar g)$, then the second fundamental form of

$G$ is given by

$$(\nabla G_*)(X,Y) = \overset{\bar{N}}{\nabla}{}^G_X G_*(Y) - G_*(\nabla^N_X Y), \quad \forall\, X,Y \in \Gamma(TN), \tag{2.3}$$

where $\overset{\bar{N}}{\nabla}{}^G$ is the pullback connection of $\nabla^{\bar{N}}$. Now, if $X,Y \in \Gamma((kerG_*)^{\perp})$, then the second fundamental form of Riemannian submersion is

$$(\nabla G_*)(X,Y) = 0. \tag{2.4}$$

Also, if $X,Y \in \Gamma((kerG_*)^{\perp})$ and $V \in \Gamma((rangeG_*)^{\perp})$, then

$$\nabla^{\bar{N}}_{G_*(X)} V = -S_V G_*(X) + \nabla^{G\perp}_X V, \tag{2.5}$$

where $S_V G_*(X)$ is the tangential component of $\nabla^{\bar{N}}_{G_*(X)} V$. Since $(\nabla G_*)$ is symmetric and $S_V$ is a symmetric linear transformation of $rangeG_*$, therefore

$$g_{\bar{N}}(S_V G_*(X), G_*(Y)) = g_{\bar{N}}(V, (\nabla G_*)(X,Y)). \tag{2.6}$$

From equations (2.3) and (2.4), we get

$$\begin{aligned} R^{\bar{N}}(G_*(X), G_*(Y))G_*(Z) &= -S_{(\nabla G_*)(Y,Z)} G_*(X) + S_{(\nabla G_*)(X,Z)} G_*(Y) \\ &+ G_*(R^N(X,Y)Z) + (\tilde{\nabla}_X(\nabla G_*))(Y,Z) - (\tilde{\nabla}_Y(\nabla G_*))(X,Z), \end{aligned} \tag{2.7}$$

where $\tilde{\nabla}$ is the covariant derivative of the second fundamental form. The O' Neill tensors [34] $A$ and $T$ are given by

$$A_P P' = h\nabla_{hP} vP' + v\nabla_{hP} hP', \tag{2.8}$$

$$T_P P' = h\nabla_{vP} vP' + v\nabla_{vP} hP', \tag{2.9}$$

for all $P, P' \in \Gamma(TN)$, where $\nabla$ is the Levi-civita connection on $N$. For $P \in \Gamma(TN)$, $T$ is vertical such that $T_P = T_{vP}$ and $A$ is horizontal such that $A_P = A_{hP}$. Also, if $U, W \in \Gamma(kerG_*)$, then we have $T_U W = T_W U$.

From equations (2.8) and (2.9), we get

$$\nabla_V W = T_U V + v\nabla_V W, \tag{2.10}$$

$$\nabla_X V = A_X V + v\nabla_X V, \tag{2.11}$$

$$\nabla_Y Z = A_Y Z + H\nabla_Y Z, \tag{2.12}$$

for all $V, W \in \Gamma(kerG_*)$ and $Y, Z \in \Gamma(kerG_*)^\perp$. The covariant derivative of $\nabla G_*$ and $S$ are

$$(\tilde{\nabla}_X(\nabla G_*))(Y,Z) = \nabla_X^{G\perp}(\nabla G_*)(Y,Z) - (\nabla G_*)(\nabla_X^N Y, Z) - (\nabla G_*)(Y, \nabla_X^N Z), \tag{2.13}$$

and

$$(\tilde{\nabla}_X S)_V G_*(Y) = G_*(\nabla_X^N {}^*G_*(S_V G_*(Y))) - S_{\nabla_X^{G\perp} V} G_*(Y) - S_V Q \overset{\bar{N}}{\nabla}{}_X^G G_*(Y), \tag{2.14}$$

respectively. Here $Q$ is a projection morphism on $rangeG_*$ and ${}^*G_*$ is an adjoint map of $G_*$. From equations (2.13) and (2.14), we obtain

$$g_{\bar{N}}((\tilde{\nabla}_X(\nabla G_*))(Y,Z), V) = g_{\bar{N}}((\tilde{\nabla}_X S)_V G_*(Y), G_*(Z)). \tag{2.15}$$

Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$. Then $G$ is called a conformal submersion with totally geodesic fibers if and only if $T$ vanishes identically.

## 3. Characterization of bi-f-harmonic curves

Let $\beta : I \to N$ be a curve in an $n_1$-dimensional Riemannian manifold $N$ with an orthonormal frame $\{W_0, W_1, ....W_{n_1-1}\}$ in $\Gamma TN$, where $W_0 = T$, $W_1 = N$ and $W_2 = U$ are the unit tangent vector, the unit normal vector and the unit binormal vector of $\alpha$, respectively. Then the Frenet equations are given by

$$\nabla_T W_j = -\kappa_j W_{j-1} + \kappa_{j+1} W_{j+1}, \qquad 0 \leq j \leq m-1, \tag{3.16}$$

where $\kappa_0 = \kappa_{n_1} = 0$, $\kappa_1 = \kappa = ||\nabla_T T||$ is curvature and $\tau = \kappa_2 = -\langle \nabla_T W_1, W_2 \rangle$ is torsion of $\beta$ on $N$, respectively. Next, we introduce the concept horizontal bi-f-harmonic curve.

**Definition 3.1.** *Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a conformal submersion between the Riemannian manifolds $(N, g)$ and $(\bar{N}, \bar{g})$. Then a horizontal curve on $(N, g)$ with (2.2) is said to be a horizontal bi-f-harmonic curve on $(N, g)$.*

**Lemma 3.1.** *Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a conformal submersion between $(N, g)$ and $(\bar{N}, \bar{g})$. Now, if $\bar{\beta} = G \circ \beta$ is a curve on $(\bar{N}, \bar{g})$ and $\beta$ is a horizontal curve on $(N, g)$, then*

$$(i) \qquad \bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta}), \tag{3.17}$$

$$\begin{aligned}(ii) \qquad \bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) &= G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta})\dot{\beta}) - 2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}} \dot{\beta}) \\ &+ (\nabla G_*)(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, A_{\dot{\beta}} \dot{\beta}),\end{aligned} \tag{3.18}$$

*where $\overset{\wedge}{\nabla}$ and $\bar{\nabla}$ are the Levi-Civita connections of $N$ and $\bar{N}$, respectively.*

*Proof.* Let $\beta$ be a horizontal curve with curvature $\kappa$ on Riemannian manifold $(N, g)$ and $\bar{\beta} = G \circ \beta$ is a curve with curvature $\bar{\kappa}$ on $(\bar{N}, \bar{g})$. Then a vector field $G_*(\dot{\beta})$ along $\bar{\beta}$ is defined by

$$G_*(\dot{\beta}) = G_{*\beta}\dot{\beta},$$

where $\dot{\beta}(s) = \dot{\beta}$ is a vector field along $\beta(s) = \beta$.

$(i)$ From equations (2.3), (2.4) and (2.5), we have

$$\bar{\nabla}^2_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}). \tag{3.19}$$

Taking the covariant derivative of (3.22) and using (2.3), (2.4) and (2.5), we get the required condition.

$(ii)$ From equations (2.3), (2.4) and (2.5), we get the required equation. □

**Definition 3.2.** *A Riemannian submersion $G : (N, g) \to (\bar{N}, \bar{g})$ between Riemannian manifolds $(N, g)$ and $(\bar{N}, \bar{g})$ is said to be totally geodesic conformal submersion if second fundamental form of $G$ is identically zero. i.e.*

$$(\nabla G_*)(X, Y) = 0, \ \forall\, X, Y \in \Gamma(TN). \tag{3.20}$$

**Lemma 3.2.** *Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a totally geodesic conformal submersion between Riemannian manifolds $(N, g)$ and $(\bar{N}, \bar{g})$. If $\beta$ is a horizontal curve with curvature $\kappa$ on $(N, g)$ and $\bar{\beta} = G \circ \beta$ is a bi-f-harmonic curve on $(\bar{N}, \bar{g})$, then the curvature of $\bar{\beta}$ is given by*

$$\kappa = \frac{1}{f^{\frac{4}{3}}}\Big(\frac{2}{3}\int f^{\frac{2}{3}}(ff''' + f'f'')ds + C\Big)^{\frac{1}{2}}, \tag{3.21}$$

*where $C$ is some constant.*

*Proof.* Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a conformal submersion between Riemannian manifolds $(N, g)$ and $(\bar{N}, \bar{g})$. Then for any horizontal curve $\beta$ on $(N, g)$ and bi-f-harmonic curve $\bar{\beta} = G \circ \beta$ on $(\bar{N}, \bar{g})$, we have

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta})\\ +f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0.\end{aligned} \tag{3.22}$$

From Lemma 3.1 and equation (3.22), we have

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})\\ +f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0.\end{aligned} \tag{3.23}$$

Now using second part of Lemma (3.1) in equation (3.23), we get

$$(ff''' + f'f'')G_*(\dot\beta) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta)$$
$$+f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot\beta}\dot\beta) + f^2G_*(R(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta)\dot\beta) - 2f^2(\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta)$$
$$+f^2(\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta) + f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta) = 0. \tag{3.24}$$

Taking inner-product of equation (3.24) with $G_*(\dot\beta)$ both sides, we obtain

$$\lambda^2f^2g_N(\overset{\wedge}{\nabla}{}^3_{\dot\beta}\dot\beta,\dot\beta) + \lambda^2f^2g_N(R(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta)\dot\beta,\dot\beta) - 2f^2g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$\lambda^2(ff''' + f'f'') + \lambda^2(3ff'' + 2f'^2)g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta) + \lambda^2 4ff'g_N(\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta,\dot\beta)$$
$$+f^2g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta), G_*(\dot\beta)) + f^2g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.25}$$

Substituting the values of $g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta) = -g_N(A_{\dot\beta}\dot\beta,\dot\beta)$, $g_N(\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta,\dot\beta) = -\kappa^2 - g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta,\dot\beta)$ $- g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta)$ and $g_N(\overset{\wedge}{\nabla}{}^3_{\dot\beta}\dot\beta,\dot\beta) = -3\kappa\kappa' - g_N(v\nabla_{\dot\beta}v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta,\dot\beta) - g_N(v\nabla_{\dot\beta}A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta) -$ $g_N(A_{\dot\beta}\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta,\dot\beta)$ in equation (3.25), we obtain

$$\lambda^2(ff''' + f'f'') - \lambda^2(3ff'' + 2f'^2)g_N(A_{\dot\beta}\dot\beta,\dot\beta) - \lambda^2 4ff'g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta,\dot\beta)$$
$$-\lambda^2 4ff'\kappa^2 - \lambda^2 4ff'g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta) - \lambda^2 3\kappa\kappa' f^2 - \lambda^2f^2g_N(v\nabla_{\dot\beta}v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta,\dot\beta)$$
$$-\lambda^2f^2g_N(v\nabla_{\dot\beta}A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta) - f^2g_N(A_{\dot\beta}\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta,\dot\beta) - 2f^2g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$+f^2g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta), G_*(\dot\beta)) + f^2g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.26}$$

Using the orthogonal condition in equation (3.26), we have

$$(ff''' + f'f'') - 4ff'\kappa^2 - 3\kappa\kappa' f^2 - 2f^2g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$+f^2g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta,\dot\beta), G_*(\dot\beta)) + f^2g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.27}$$

Since $G$ is totally geodesic, then equation (3.27) reduces to (by using mapple),

$$\kappa = \frac{1}{f^{\frac{4}{3}}}\left(\frac{2}{3}\int f^{\frac{2}{3}}(ff''' + f'f'')ds + C\right)^{\frac{1}{2}}. \tag{3.28}$$

□

**Theorem 3.1.** *Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a totally geodesic conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar N, g_{\bar N})$. Then $G$ maps horizontal bi-f-harmonic curve on $(N, g_N)$ to bi-f-harmonic curve on $(\bar N, g_{\bar N})$.*

*Proof.* Substituting the values of $\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$, $\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})$ and $\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ in equation (3.22), we get

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = (ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) \\ &+4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}). \end{aligned} \tag{3.29}$$

Then using second part of Lemma 3.1 in the equation (3.29), we obtain

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = (ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) \\ &+4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &-2f^2(\nabla G_*)(\dot{\beta}, H\nabla_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) + f^2(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \end{aligned} \tag{3.30}$$

Now from equations (2.4) and (3.30), we get

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*\{(ff''' + f'f'')(\dot{\beta}) + (3ff'' + 2f'^2)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) \\ &+4ff'(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + f^2(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta})\} - 2f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &+f^2(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \end{aligned} \tag{3.31}$$

Using the fact that $\beta$ is a horizontal bi-f-harmonic curve on $(N, g_N)$, equation (3.31) reduces to

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + \\ f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) &+ f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(0) - 3f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &+f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \end{aligned} \tag{3.32}$$

Since $G$ is a totally geodesic conformal submersion, therefore

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \end{aligned} \tag{3.33}$$

□

**Theorem 3.2.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$. Now, if $\beta$ is a bi-f-harmonic curve on $(N, g_N)$ and $\bar{\beta} = G \circ \beta$ is a bi-f-harmonic curve on $(\bar{N}, g_{\bar{N}})$, then either*

$$ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' < 0 \text{ or } g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, H'), G_*(\dot{\beta})) \geq 0. \tag{3.34}$$

*Proof.* Let $\bar{\beta}$ be a bi-f-harmonic curve on $(\bar{N}, g_{\bar{N}})$, then

$$(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta})$$
$$+f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{3.35}$$

Substituting the values of $\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$, $\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})$ and $\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ in equation (3.35), we get

$$(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})$$
$$+f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{3.36}$$

Then substituting $\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta})$ $+ (\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta})$ in equation (3.36), we have

$$(ff''' + ff'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$$
$$+f^2G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) + f^2(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})$$
$$+f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}) = 0. \tag{3.37}$$

Taking the inner-product of equation (3.37) with $G_*(\dot{\beta})$ both sides, we obtain

$$(ff''' + f'f'')g_{\bar{N}}(G_*(\dot{\beta}), G_*(\dot{\beta})) + (3ff'' + 2f'^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$$
$$+4ff'g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + f^2g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + f^2g_{\bar{N}}(G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}), G_*(\dot{\beta}))$$
$$-2f^2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + f^2g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta}))$$
$$+f^2g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{3.38}$$

Using the definition of conformal submersion in equation (3.38), we obtain

$$
\begin{aligned}
&(ff''' + f'f'')\lambda^2 + (3ff'' + 2f'^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) + 4ff'\lambda^2 g_N(\overset{\wedge\,2}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) \\
&+ f^2\lambda^2 g_N(\overset{\wedge\,3}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) + f^2\lambda^2 g_N(R(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)\dot\beta, \dot\beta) - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta)) \\
&+ f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \qquad (3.39)
\end{aligned}
$$

Substituting $g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta,$ $g_N(\overset{\wedge\,2}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)$ and $g_N(\overset{\wedge\,3}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)$ in equation (3.39), we obtain

$$
\begin{aligned}
&(ff''' + f'f'')\lambda^2 - 4ff'\kappa^2\lambda^2 - 3\kappa\kappa' f^2\lambda^2 - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta)) \\
&+ f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \qquad (3.40)
\end{aligned}
$$

Then using the definition of totally umbilical i.e. $A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta = g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)H'$,
$A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta = g_N(\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta)H'$ and $A_{\dot\beta}\dot\beta = g_N(\dot\beta, \dot\beta)H'$ in equation (3.40), we have

$$
(ff''' + f'f'')\lambda^2 - 4ff'\kappa^2\lambda^2 - 3\kappa\kappa'\lambda^2 + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) = 0. \qquad (3.41)
$$

Since equation (3.41) is a quadratic equation in $\lambda$, therefore

$$
\lambda = \frac{0 \pm \sqrt{-4(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2) f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta))}}{2(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2)}. \qquad (3.42)
$$

Since $\lambda$ is a positive real valued function, therefore

$$
4(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2) f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) \leq 0. \qquad (3.43)
$$

Thus from equations (3.42) and (3.43), we can conclude that either $(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2) < 0$ and $g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) \geq 0$ or $(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2) > 0$ and $g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) \leq 0$, to make $\lambda$ always positive. □

3.1. **Characterization of bi-harmonic curves.** A bi-harmonic curve (bi-1-harmonic curve) is a special case of bi-f-harmonic curve for $f = 1$. Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar N, g_{\bar N})$ such that $\bar\beta$ is the bi-harmonic curve on $(\bar N, g_{\bar N})$, then

$$
\bar\nabla^3_{G_*(\dot\beta)} G_*(\dot\beta) + \bar R(G_*(\nabla_{\dot\beta}\dot\beta), G_*(\dot\beta))G_*(\dot\beta) = 0.
$$

**Theorem 3.3.** *Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar N, g_{\bar N})$. If $\beta$ is a horizontal curve with curvature $\kappa$ on $(N, g)$ and $\bar\beta = G \circ \beta$ is a bi-harmonic curve on $(\bar N, \bar g)$, then $\kappa$ is constant.*

*Proof.* Let $\bar{\beta}$ is a bi-harmonic curve on $(\bar{N}, g_{\bar{N}})$, then taking $f = 1$ in equation (3.23), we have

$$G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + \bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{3.44}$$

Using second part of Lemma 3.1 in equation (3.44), we get

$$\begin{aligned} &G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &+(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}) = 0. \end{aligned} \tag{3.45}$$

Taking inner-product of equation (3.45) with $G_*(\dot{\beta})$, we obtain

$$\begin{aligned} &g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}(G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}), G_*(\dot{\beta})) - 2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ &+g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \end{aligned} \tag{3.46}$$

Using the definition of conformal submersion and $g_N(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}, \dot{\beta}) = 0$ in equation (3.46), we get

$$\begin{aligned} &\lambda^2 g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - 2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ &+g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \end{aligned} \tag{3.47}$$

Substituting $g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -3\kappa\kappa' - g_N(v\nabla^3_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and $g_N(v\nabla^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0$ in equation (3.47), we obtain

$$\begin{aligned} &-\lambda^2 3\kappa\kappa' - 2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) \\ &+g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \end{aligned} \tag{3.48}$$

Since $G$ be a totally geodesic conformal submersion i.e. second fundamental form is identically zero, therefore equation (3.48) reduces to

$$-\lambda^2 3\kappa\kappa' = 0, \implies \kappa = constant. \tag{3.49}$$

□

**Theorem 3.4.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a totally geodesic conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$. Then $G$ maps horizontal bi-harmonic curve on $(N, g_N)$ to bi-harmonic curve on $(\bar{N}, g_{\bar{N}})$.*

*Proof.* Taking $f = 1$ and substituting $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ in equation (3.33), we get

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = \overset{\wedge}{\nabla}{}^3_{G_*(\dot{\beta})} G_*(\dot{\beta})$$
$$+\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}). \tag{3.50}$$

Using the second part of Lemma 3.1 in equation (3.50), we get

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}$$
$$+R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 3(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \tag{3.51}$$

Using the fact that $\beta$ is a horizontal bi-harmonic curve on $(N, g_N)$, equation (3.51) reduces to

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = -3(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}$$
$$+(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \tag{3.52}$$

Since $G$ is a totally geodesic conformal submersion map, therefore

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0 \tag{3.53}$$

Hence $\bar{\beta}$ is a bi-harmonic curve on $(\bar{N}, g_{\bar{N}})$.

□

## 4. Helices and circles along the conformal submersion

Let $\beta : I \to N$ be a curve, then $\beta$ is said to be a general helix if it satisfies the condition

$$\nabla^3_{\dot{\beta}}\dot{\beta} + K^2\nabla_{\dot{\beta}}\dot{\beta} = 0,$$

where $K^2 = \kappa^2 + \tau^2$ is a positive constant. Conversely, if the curve $\beta = \beta(s)$ satisfies the above condition, then it is an ordinary helix or a geodesic [18].

**Theorem 4.1.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$. Then, $\bar{\beta} = G \circ \beta$ is a helix on $(\bar{N}, g_{\bar{N}})$ iff $\beta$ is a horizontal curve of constant curvature on $(N, g_N)$.*

*Proof.* Let $\bar{\beta}$ be a helix on $(\bar{N}, g_{\bar{N}})$, then

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0. \tag{4.54}$$

Using $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ in equation (4.54), we get

$$G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2+\tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) = 0. \tag{4.55}$$

Taking inner-product of equation (4.55) with $G_*(\dot{\beta})$, we obtain

$$g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + (\kappa^2+\tau^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{4.56}$$

Using the definition of conformal submersion in equation (4.56), we have

$$\lambda^2 g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\kappa^2+\tau^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{4.57}$$

Substituting $g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -3\kappa\kappa' - g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and $g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ in equation (4.57), we get

$$\begin{aligned} &-\lambda^2 3\kappa\kappa' - \lambda^2 g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - \lambda^2 g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) \\ &-\lambda^2 g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - (\kappa^2+\tau^2)\lambda^2 g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \end{aligned} \tag{4.58}$$

Using the condition of orthogonality in equation (4.58), we have

$$\lambda^2 3\kappa\kappa' = 0 \implies \kappa = C(constant).$$

Conversely, assume that $\beta$ be a curve of constant curvature on $(N, g_N)$ and $\bar{\beta} = G \circ \beta$ is a curve on $(\bar{N}, g_{\bar{N}})$, where $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion and using equation (3.17). Then, we have

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) \quad + \quad (\kappa^2+\tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2+\tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}). \tag{4.59}$$

Taking inner-product of equation (4.59) with $G_*(\dot{\beta})$ both sides, we have

$$\begin{aligned} g_{\bar{N}}(\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) \quad + \quad & (\kappa^2+\tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}), G_*(\dot{\beta})) = g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2+\tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ &= g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + (\kappa^2+\tau^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ &= \lambda^2 g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\kappa^2+\tau^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) \\ &= -\lambda^2 3\kappa\kappa' - \lambda^2 g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - \lambda^2 g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) \\ &= -\lambda^2 g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - (\kappa^2+\tau^2)\lambda^2 g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \end{aligned}$$

Therefore

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0.$$

Hence $\bar{\beta}$ is a helix. □

**Theorem 4.2.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a totally geodesic conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$. Then $G$ maps horizontal helix on $(N, g_N)$ to a helix on $(\bar{N}, g_{\bar{N}})$.*

*Proof.* From equation (4.54) and using relation $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$, we get

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta} + (\kappa^2 + \tau^2)\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}). \tag{4.60}$$

Since $\beta$ is a horizontal helix on $(N, g_N)$, therefore equation (4.60) reduces to

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0.$$

Hence, $\bar{\beta}$ is a helix on $(\bar{N}, g_{\bar{N}})$. □

**Corollary 4.1.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion map between two Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$ such that $\beta$ is a helix on $(N, g_N)$. If $\bar{\beta} = G \circ \beta$ is a helix on $(\bar{N}, g_{\bar{N}})$, then $\beta$ is a helix of constant curvature on $(N, g_N)$.*

*Proof.* Since $\bar{\beta}$ is a helix on $(\bar{N}, g_{\bar{N}})$, so

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0. \tag{4.61}$$

Substituting the values of $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ and $\overset{\wedge}{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$ in equation (4.61), we have

$$G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2 + \tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) = 0. \tag{4.62}$$

Taking the inner-product of equation (4.62) with $G_*(\dot{\beta})$ both sides, we get

$$g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + (\kappa^2 + \tau^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{4.63}$$

Using the definition of conformal submersion in equation (4.63), we obtain

$$\lambda^2 + g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\kappa^2 + \tau^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{4.64}$$

Substituting the values of $g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and $g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ in equation (4.64), we get the required result. □

**Theorem 4.3.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion map between two Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$ such that $\beta$ is a circle on $(N, g_N)$. If $\bar{\beta} = G \circ \beta$ is a circle on $(\bar{N}, g_{\bar{N}})$, then curvature $\kappa = \pm 1$, where $\kappa$ is curvature of $\beta$.*

*Proof.* Let $\bar{\beta}$ is a circle on $(\bar{N}, g_{\bar{N}})$, then

$$\bar{\nabla}^2_{G_*(\dot{\beta})} G_*(\dot{\beta}) + g_{\bar{N}}(\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}), \bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta})) G_*(\dot{\beta}) = 0. \tag{4.65}$$

Substituting the values of $\bar{\nabla}^2_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta})$ and $\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta})$ in equation (4.65), we get

$$G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}) + g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}), G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta})) G_*(\dot{\beta}) = 0. \tag{4.66}$$

Using the definition of conformal submersion in equation (4.66), we get

$$G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}) + \lambda^2(p) g_N(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}) G_*(\dot{\beta}) = 0. \tag{4.67}$$

Substituting $g_N(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}) = 1$ in equation (4.67), we obtain

$$G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}) + \lambda^2 G_*(\dot{\beta}) = 0. \tag{4.68}$$

Taking inner-product of equation (4.68) with $G_*(\dot{\beta})$, gives us

$$g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}(G_*(\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{4.69}$$

Again using the definition of conformal submersion in equation (4.69), we have

$$\lambda^2 g_N(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + \lambda^2 g_N(\dot{\beta}, \dot{\beta}) = 0. \tag{4.70}$$

Substituting the values of $g_N(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = -\kappa^2 - g_N(v\nabla_{\dot{\beta}} A_{\dot{\beta}} \dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta})$ and $g_N(\dot{\beta}, \dot{\beta}) = 1$ in equation (4.70), we get

$$-\lambda^2 \kappa^2 - \lambda^2 g_N(v\nabla_{\dot{\beta}} A_{\dot{\beta}} \dot{\beta}, \dot{\beta}) - \lambda^2 g_N(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + \lambda^2 = 0. \tag{4.71}$$

Since $g_N(v\nabla_{\dot{\beta}} A_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = 0$ and $g_N(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = 0$. Thus from equation (4.71), we get the required result.

□

**Theorem 4.4.** *Let $G : (N, g) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion map between two Riemannian manifolds $(N, g)$ and $(\bar{N}, g_{\bar{N}})$. If $\beta$ is a circle on $(N, g)$ and $\bar{\beta} = G \circ \beta$ is a circle on $(\bar{N}, g_{\bar{N}})$, then either $\lambda = \pm\kappa$ or $\lambda = 0$, where $\kappa$ is curvature of $\beta$ on $N$.*

*Proof.* Considering the definition of conformal submersion in equation (4.66), we get

$$G_*(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta) + \lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta) G_*(\dot\beta) = 0. \tag{4.72}$$

Taking inner-product of equation (4.72) with $G_*(\dot\beta)$, gives us

$$g_{\bar N}(G_*(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta), G_*(\dot\beta)) + \lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta) g_{\bar N}(G_*(\dot\beta), G_*(\dot\beta)) = 0. \tag{4.73}$$

Since, $g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta) = 1$ and $g_N(\dot\beta, \dot\beta) = 1$, therefore equation (4.73) reduces to

$$\lambda^2 g_N(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta, \dot\beta) + \lambda^2\lambda^2 = 0. \tag{4.74}$$

Taking $g_N(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta, \dot\beta) = -\kappa^2 - g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) - g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)$ in equation (4.74), then we have

$$-\lambda^2(\kappa^2 + g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) + g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)) + \lambda^2\lambda^2 = 0. \tag{4.75}$$

Substituting $g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) = 0$ and $g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) = 0$ in equation (4.75), we get

$$-\kappa^2\lambda^2 + \lambda^2\lambda^2 = 0. \tag{4.76}$$

As equation (4.76) is quadratic in $\lambda^2$, therefore

$$\lambda^2 = \frac{\kappa^2 \pm \sqrt{\kappa^4}}{2}. \tag{4.77}$$

Thus, from equation (4.77), we can say that either $\lambda = \pm\kappa$ or $\lambda = 0$.

□

## 5. Hyperelastic curve along the conformal submersion

In this section, we study the hyperelastic curve along the conformal submersion.

**Theorem 5.1.** *Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar N, g_{\bar N})$. If $\beta$ is a hyperelastic curve on $(N, g_N)$ and $\bar\beta = G \circ \beta$ is a hyperelastic curve on $(\bar N, g_{\bar N})$, then*

$$\begin{aligned}&(-2(r-2)\kappa^{r-1}\kappa' - 3\kappa^{r-1}\kappa')\lambda^2 + \lambda^2(\dot\beta(\tfrac{2r-1}{r}\kappa^r + b))\\ &\quad+\kappa^{r-2} g_{\bar N}((\nabla G_*)(-\tfrac{\lambda^2}{2}(\dot\beta\tfrac{1}{\lambda^2}), \dot\beta), G_*(\dot\beta)) = 0,\end{aligned} \tag{5.78}$$

*where $r \geq 2$ is a natural number.*

*Proof.* Let $\bar{\beta}$ is a hyperelastic curve on $(\bar{N}, g_{\bar{N}})$, then from [33], we have

$$\bar{\nabla}^2_{G_*(\dot{\beta})}(\kappa^{r-2}\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta})) + \bar{\nabla}_{G_*(\dot{\beta})}(\mu G_*(\dot{\beta})) + \kappa^{r-2}\bar{R}(\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{5.79}$$

Substituting $\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$, $\bar{\nabla}^2_{G_*(\dot{\beta})}(\kappa^{r-2}\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta})) = \kappa^{r-2}G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + (r-2)(r-3)\kappa^{r-4}(\kappa')^2G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa''G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa'G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa'G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta})$ and $\bar{\nabla}_{G_*(\dot{\beta})}(\mu G_*(\dot{\beta})) = G_*(\nabla_{\dot{\beta}}\mu\dot{\beta})$ in equation (5.79), we have

$$(r-2)(r-3)\kappa^{r-4}\kappa'^2G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa''G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 2(r-2)\kappa^{r-3}\kappa'G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}) + \kappa^{r-2}G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + \kappa^{r-2}\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) + G_*(\nabla_{\dot{\beta}}\mu\dot{\beta}) = 0. \tag{5.80}$$

Taking inner-product of equation (5.80) with $G_*(\dot{\beta})$ both sides, we get

$$((r-2)(r-3)\kappa^{r-4}\kappa'^2 + (r-2)\kappa^{r-3}\kappa'')g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + 2(r-2)\kappa^{r-3}\kappa'g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}(\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}(G_*(\nabla_{\dot{\beta}}\mu\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{5.81}$$

Substituting $g_{\bar{N}}(\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}), G_*(\dot{\beta})) = -2g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$ and using the definition of conformal submersion in equation (5.81), we get

$$\lambda^2g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})((r-2)(r-3)\kappa^{r-4}\kappa'^2 + \kappa^{r-2}\lambda^2g_N(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (r-2)\kappa^{r-3}\kappa'') + 2(r-2)\kappa^{r-3}\kappa'\lambda^2g_N(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - 2\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + \lambda^2g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\mu\dot{\beta}, \dot{\beta}) + \kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{5.82}$$

Substituting the values of
$g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta})$, $g_N(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -\kappa^2 - g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and
$g_N(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -3\kappa\kappa' - g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ in equation

(5.82), we obtain

$$-2(r-2)\kappa^{r-1}\kappa'\lambda^2 - 3\kappa^{r-1}\kappa'\lambda^2 - 2\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$$
$$+\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}|, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$$
$$+\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\mu\dot{\beta}, \dot{\beta}) = 0. \tag{5.83}$$

Since $\overset{\wedge}{\nabla}_{\dot{\beta}}\mu\dot{\beta} = (\dot{\beta}(\frac{2r-1}{r}\kappa^r + b))\dot{\beta} + (\frac{2r-1}{r}\kappa^r + b)\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}$, where $\mu = \frac{2r-1}{r}\kappa^r + b$, therefore from equation (5.83),

$$-2(r-2)\kappa^{r-1}\kappa'\lambda^2 - 3\kappa^{r-1}\kappa'\lambda^2 - 2\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta}))$$
$$+\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \lambda^2 g_N((\dot{\beta}(\tfrac{2r-1}{r}\kappa^r + b))\dot{\beta}$$
$$+(\tfrac{2r-1}{r}\kappa^r + b)\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{5.84}$$

Using the totally umbilical conditions $A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta} = g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})H'$, $\forall\ \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta},\ \dot{\beta} \in \Gamma(kerG_*)^{\perp}$, where $H' = -\frac{\lambda^2}{2}(\nabla_{\dot{\beta}}\frac{1}{\lambda^2})$ and $A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta} = g_N(\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})H'$ in equation (5.84), we get

$$(-2(r-2)\kappa^{r-1}\kappa' - 3\kappa^{r-1}\kappa')\lambda^2 - \kappa^{r-2}g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})g_{\bar{N}}((\nabla G_*)(H', \dot{\beta}), G_*(\dot{\beta}))$$
$$+\kappa^{r-2}g_N(\dot{\beta}, \dot{\beta})g_{\bar{N}}((\nabla G_*)(H', \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \lambda^2(\dot{\beta}(\tfrac{2r-1}{r}\kappa^r + b))g_N(\dot{\beta}, \dot{\beta})$$
$$+\lambda^2(\tfrac{2r-1}{r}\kappa^r + b)g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{5.85}$$

Substituting $H' = -\frac{\lambda^2}{2}(\dot{\beta}\frac{1}{\lambda^2})$ and $g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0$ in equation (5.85), we obtain

$$-\lambda^2(2(r-2)\kappa^{r-1}\kappa' + 3\kappa^{r-1}\kappa') + \lambda^2(\dot{\beta}(\tfrac{2r-1}{r}\kappa^r + b))$$
$$+\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(-\tfrac{\lambda^2}{2}(\dot{\beta}\tfrac{1}{\lambda^2}), \dot{\beta}), G_*(\dot{\beta})) = 0. \tag{5.86}$$

Hence the proof. □

**Corollary 5.1.** *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds $(N, g_N)$ and $(\bar{N}, g_{\bar{N}})$ such that $\beta$ is a elastic curve on $(N, g_N)$. If $\bar{\beta} = G \circ \beta$ is a elastic curve on $(\bar{N}, g_{\bar{N}})$, then*

$$g_{\bar{N}}((\nabla G_*)(-\frac{\lambda^2}{2}(\dot{\beta}\frac{1}{\lambda^2}), \dot{\beta})G_*(\dot{\beta})) = 0. \tag{5.87}$$

*Proof.* Substituting $r = 2$ in equation (5.78), we have

$$-3\kappa\kappa'\lambda^2 + g_{\bar{N}}((\nabla G_*)(-\frac{\lambda^2}{2}(\dot{\beta}\frac{1}{\lambda^2}), \dot{\beta}), G_*(\dot{\beta})) + \lambda^2(\dot{\beta}(\frac{3}{2}\kappa^2 + b)) = 0. \tag{5.88}$$

Substituting the value of $\dot{\beta}(\frac{3}{2}\kappa^2 + b) = 3\kappa\kappa'$ in equation (5.88), we get the required result. □

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# GENERALIZED TANAKA-WEBSTER CONNECTION ON $\beta$-KENMOTSU MANIFOLDS

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Abstract. This research paper aims to study the postulates of the generalized Tanaka-Webster connection (briefly, gTWc) on $\beta$-Kenmotsu manifolds. We find the curvature properties of a $\beta$-Kenmotsu manifold concerning gTWc, and studied the conditions for the $\phi$-projectively flat, $\phi$-conformally flat and $\phi$-concircular flat $\beta$-Kenmotsu manifolds along with the same connection. Also, we have discussed the $\xi$-flat properties on same curvatures for the $\beta$-Kenmotsu manifold admitting gTWc. At the end we provide an example to verify some of our results.

**Keywords**: $\beta$-Kenmotsu manifold, generalized Tanaka-Webster connection, curvature tensor, $\eta$-Einstein manifold.

**2010 Mathematics Subject Classification**: 53B15, 53C05, 53C25, 53D10.

## 1. Introduction

The generalized Tanaka-Webster connection (gTWc) is a canonical affine connection defined on a non-degenerated pseudo-Hermitian CR-manifold. The gTWc was introduced by Tanno [23] as a generalization of the connections defined at the end of 1976 by Tanaka in [22] and Webster in [25]. These connections coincide with the Tanaka-Webster connection (TWc) if the associated CR-structure is integrable. Many geometers studied some characterizations of the gTWc on various manifolds. Recently, S.Y. Perktas et al. [18], Ghosh and De [5, 7],

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Gautam et al. [6], Ayar and Cavusoglu [2], and many others have studied the properties of this connection on distinct structures. Also, see [12, 16].

Kenmotsu [13], introduced a new class of almost contact Riemannian manifolds, known as the Kenmotsu manifold. As it is well known, odd-dimensional spheres permit Sasakian structures, but odd-dimensional hyperbolic spaces do not admit Sasakian structures but do have Kenmotsu structures. Kenmotsu manifolds are normal almost contact Riemannian manifolds. The basic fundamental properties of the local structure of such manifolds were investigated by many geometers. In general, the Kenmotsu manifolds are locally isometric to warped product spaces with one-dimensional bases. Oubina [17] introduced the notion of trans-Sasakian manifolds of type $(\alpha, \beta)$, which is the generalization of Kenmotsu manifolds and Sasakian manifolds, and are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type $(0,0)$, $(\alpha, 0)$ and $(0, \beta)$ are, respectively called, the cosympletic, $\alpha$-Sasakian and $\beta$-Kenmotsu manifold, where $\alpha$ and $\beta$ be some scalar functions. In particular, if $\alpha = 0$, $\beta = 1$; $\alpha = 0$, $\beta$ is non-zero constant and $\alpha = 1$, $\beta = 0$ then a trans Sasakian manifold will be a Kenmotsu; homothetic Kenmotsu manifold and Sasakian manifold, respectively. $\beta$-Kenmotsu manifolds have been studied by several authors, like Bozdag et al. [3], Hui and Chakraborty [11], Kumar [15], Shaikh and Hui [19] and Mobin et al.[1]. We recommend the papers [8, 9, 10, 20, 21, 24] for more related stidies and references therein.

## 2. Preliminaries

In this section, we review basic definitions and results that are needed to state and prove our results.

A $(2n+1)$-dimensional smooth differentiable manifold $\mathcal{M}$ is said to be an almost contact metric structure $(\phi, \xi, \eta, g)$ if the following conditions are satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.1}$$

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X) \tag{2.4}$$

for all $X, Y, Z$ on $\mathcal{M}$, where $\phi$ is a $(1,1)$-tensor field, $\xi$ is a vector field and $\eta$ is a 1-form. An almost metric manifold $\mathcal{M}$ is said to be a $\beta$-Kenmotsu manifold if it satisfies

$$(\nabla_X \phi)Y = \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.5}$$

$$(\nabla_X \eta)Y = \beta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{2.7}$$

where $\nabla$ is a Levi-Civita connection.

If $\beta = 1$, then $\mathcal{M}$ is called a Kenmotsu manifold, and if $\beta$ is constant then $\mathcal{M}$ are named homothetic Kenmotsu manifolds and provide a large variety of Kenmotsu manifolds. In a $\beta$-Kenmotsu manifold $\mathcal{M}$, the following relations hold:

$$\mathcal{R}(X,Y)\xi = -\beta^2[\eta(Y)X - \eta(X)Y] + (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\},$$

$$\mathcal{R}(\xi, X)Y = \{\beta^2 + \xi\beta\}[\eta(Y)X - g(X,Y)\xi],$$

$$Ric(X,\xi) = -\{2n\beta^2 + \xi\beta\}\eta(X) - (2n-1)(X\beta),$$

$$Ric(\phi X, \phi Y) = Ric(X,Y) + \{2n\beta^2 + \xi\beta\}\eta(X)\eta(Y) + (2n-1)(X\beta)\eta(Y), \tag{2.8}$$

where $X(\beta) = g(X, D\beta)$, $D$ is the gradient operator of $g$.

An $\mathcal{M}$ is said to be $\eta$-Einstein if its Ricci tensor $Ric(\neq 0)$ satisfies

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields $X$ and $Y$ on $\mathcal{M}$, where $a$ and $b$ are smooth functions on $\mathcal{M}$.

The gTWc $\hat{\nabla}$ for a contact metric manifold $\mathcal{M}$ is given by [23],

$$\hat{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi + \eta(X)\phi Y \tag{2.9}$$

for all $X$, $Y$ on $\mathcal{M}$.

## 3. $\beta$-Kenmotsu Manifolds concerning $\hat{\nabla}$

In this section, we prove that the gTWc $\hat{\nabla}$ is a metric connection; and moreover, we obtain an expression of the torsion tensor $\hat{T}$ on the manifold.

Let $\mathcal{M}$ be a $(2n+1)$-dimensional $\beta$-Kenmotsu manifold. The gTWc $\hat{\nabla}$ on an $\mathcal{M}$ is given by

$$\hat{\nabla}_X Y = \nabla_X Y - \beta\eta(Y)X + \beta g(X,Y)\xi + \eta(X)\phi Y, \tag{3.10}$$

where (2.6),(2.7) and (2.9) being used.

Now putting $Y = \xi$ in (3.10) and using (2.1), (2.2) and (2.4), we get

$$\hat{\nabla}_X \xi = 0. \tag{3.11}$$

From (2.9) and (2.3), we find

$$(\hat{\nabla}_X \eta)Y = 0. \tag{3.12}$$

Also, from (2.9) and (2.5), we find

$$(\hat{\nabla}_X g)(Y, Z) = 0. \tag{3.13}$$

Thus, in the view of (3.11), (3.12) and (3.13), we can state the following:

**Proposition 3.1.** *In an $\mathcal{M}$, $\xi$ and $\eta$ are parallel with respect to $\hat{\nabla}$, which is a metric connection.*

**Proposition 3.2.** *In an $\mathcal{M}$, the integral curves of a vector field $\xi$ are geodesic concerning the $gTWc$ $\hat{\nabla}$.*

Now, since the connection $\hat{\nabla}$ is metric, so the torsion tensor $\hat{T}$ of $\hat{\nabla}$ is given by

$$\hat{T}(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X. \tag{3.14}$$

From (3.10) and (3.14), we get

$$\hat{T}(X, Y) = \beta\{\eta(X)Y - \eta(Y)X\} + \eta(X)\phi Y - \eta(Y)\phi X. \tag{3.15}$$

Since, we know

$$\begin{aligned} g(\hat{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2}[g(\hat{T}(X, Y), Z) - g(\hat{T}(X, Z), Y) \\ &\quad - g(\hat{T}(Y, Z), X)]. \end{aligned} \tag{3.16}$$

Using (3.15) in (3.16), we get (3.10). Hence, we can state:

**Theorem 3.1.** *The $gTWc$ $\hat{\nabla}$ associated with the connection $\nabla$ is a unique affine connection, which is metric and its torsion is of the form $\hat{T}(X, Y) = \beta\{\eta(X)Y - \eta(Y)X\} + \eta(X)\phi Y - \eta(Y)\phi X$.*

## 4. Curvature properties of $\beta$-Kenmotsu manifolds concerning $\hat{\nabla}$

In the currect section, we establish the relationships between $R$ and $\hat{\mathcal{R}}$; $Ric$ and $\hat{Ric}$; and $s$ and $\hat{s}$ with respect to $\nabla$ and $\hat{\nabla}$.

The Riemannian curvature tensor with respect to $\hat{\nabla}$ on $\mathcal{M}$ is given by

$$\hat{\mathcal{R}}(X, Y)Z = \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X,Y]} Z. \tag{4.17}$$

By using (3.10), (4.17) takes the form

$$\hat{\mathcal{R}}(X,Y)Z = \mathcal{R}(X,Y)Z + X(\beta)\{g(Y,Z)\xi - \eta(Z)Y\} - Y(\beta)\{g(X,Z)\xi - \eta(Z)X\} + \beta^2\{g(Y,Z)X - g(X,Z)Y\}, \tag{4.18}$$

where $\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

The inner product of (4.18) with $W$ yields

$$\hat{\mathcal{R}}(X,Y,Z,W) = \mathcal{R}(X,Y,Z,W) + X(\beta)\{g(Y,Z)\eta(W) - \eta(Z)g(Y,W)\} - Y(\beta)\{g(X,Z)\eta(W) - \eta(Z)g(X,W)\} + \beta^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\}, \tag{4.19}$$

where $\hat{\mathcal{R}}(X,Y,Z,W) = g(\hat{\mathcal{R}}(X,Y)Z,W)$.

Let $\{e_i,\xi\}_{i=1}^{2n+1}$ be the set of orthonormal basis of tangent space at each point of the manifold, then contracting (4.19) over $X$ and $W$, we get

$$\hat{Ric}(Y,Z) = Ric(Y,Z) + 2n\beta^2 g(Y,Z). \tag{4.20}$$

From (4.20) it follows that

$$\hat{\mathcal{Q}}Z = \mathcal{Q}Z + 2n\beta^2 Z, \tag{4.21}$$

where $\hat{Ric}(Y,Z) = g(\hat{\mathcal{Q}}Y,Z)$.

Also, the scalar curvature $\hat{s}$ is given by,

$$\hat{s} = s + 2n(2n+1)\beta^2. \tag{4.22}$$

Hence, we can state:

**Lemma 4.1.** *In an $\mathcal{M}$ admitting $\hat{\nabla}$ and $\beta$=constant, we have*

- *The curvature tensor $\hat{\mathcal{R}}$ is given by* (4.18)*,*
- *The Ricci tensor $\hat{Ric}$ is given by* (4.20) *and it is symmetric,*
- *The Ricci operator $\hat{\mathcal{Q}}$ is given by* (4.21)*,*
- *The scalar curvature $\hat{s}$ is given by* (4.22)*.*

**Lemma 4.2.** *In an $\mathcal{M}$ admitting $\hat{\nabla}$, we have*

- $\hat{\mathcal{R}}(X,Y)\xi = 0$*,*
- $\hat{\mathcal{R}}(X,Y)Z + \hat{\mathcal{R}}(Y,X)Z = 0$*,*
- $\hat{\mathcal{R}}(X,Y)Z + \hat{\mathcal{R}}(Y,Z)X + \hat{\mathcal{R}}(Z,X)Y = 0$*,*

- $\hat{Ric}(Y,\xi) = 0$ *if* $\beta$ *is constant. Otherwise,* $\hat{Ric}(Y,\xi) = -(\xi\beta)\eta(Y) - (2n-1)(X\beta)$, *for all* $X, Y, Z \in \chi(\mathcal{M})$.

## 5. Projective curvature tensor in $\beta$-Kenmotsu manifolds concerning $\hat{\nabla}$

Let $\mathcal{M}$ be a $(2n+1)$-dimensional Riemannain manifold. If there exists a one to one correspondence between each coordinate neighbourhood of $\mathcal{M}$ and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $\mathcal{M}$ is said to be locally projectively flat. For $n \geq 1$, $\mathcal{M}$ is locally projectively flat if and only if the projective curvature tensor vanishes. The projective curvature tensor $P_1$ with respect to the Levi-Civita connection $\nabla$ is defined by [28]

$$P_1(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{2n}\{Ric(Y,Z)X - Ric(X,Z)Y\}, \tag{5.23}$$

for all $X$, $Y$ on $\mathcal{M}$, where $\mathcal{R}$ are $Ric$ are the Riemannian curvature tensor and the Ricci tensor, respectively.

**Definition 5.1.** *A $\beta$-Kenmotsu manifold $\mathcal{M}$ is said to be $\xi$-projectively flat with respect to $\hat{\nabla}$ if*

$$\hat{P}_1(X,Y)\xi = 0,$$

*where $\hat{P}_1(X,Y)Z$ is the projective curvature tensor of dimension $(2n+1)$ concerning $\hat{\nabla}$ and is given by*

$$\hat{P}_1(X,Y)Z = \hat{\mathcal{R}}(X,Y)Z - \frac{1}{2n}\{\hat{Ric}(Y,Z)X - \hat{Ric}(X,Z)Y\}, \tag{5.24}$$

*for all $X, Y, Z \in \chi(\mathcal{M})$.*

**Theorem 5.1.** *An $\mathcal{M}$ of dimension $(2n+1)$ is $\xi$-projectively flat with respect to $\hat{\nabla}$ if and only if it is $\xi$-projectively flat with respect to $\nabla$, provided $\beta$ is constant.*

*Proof.* From (4.18), (4.20) and (5.24), we have

$$\begin{aligned}\hat{P}_1(X,Y)Z &= P_1(X,Y)Z + X(\beta)\{g(Y,Z)\xi - \eta(Z)Y\} \\ &\quad -Y(\beta)\{g(X,Z)\xi - \eta(Z)X\},\end{aligned} \tag{5.25}$$

where $P_1(X,Y)Z$ is defined in (5.23). Now, putting $Z = \xi$ in (5.25), and considering $\beta$ as a constant, we get

$$\hat{P}_1(X,Y)\xi = P_1(X,Y)\xi.$$

□

**Definition 5.2.** *A $\beta$-Kenmotsu manifold $\mathcal{M}$ satisfying the condition*

$$\phi^2(\hat{P}_1(\phi X, \phi Y)\phi Z) = 0$$

*is called $\phi$-projectively flat with respect to $\hat{\nabla}$. As we know that*

$$\phi^2(\hat{P}_1(\phi X, \phi Y)\phi Z) = 0 \Longleftrightarrow g(\hat{P}_1(\phi X, \phi Y)\phi Z, \phi W) = 0 \tag{5.26}$$

*for all $X, Y, Z, W \in \chi(\mathcal{M})$.*

**Theorem 5.2.** *Let $\mathcal{M}$ be a $(2n+1)$-dimensional $\phi$-projectively flat $\beta$-Kenmotsu manifold with respect to $\hat{\nabla}$ and $\beta$ is constant. Then $\mathcal{M}$ is an $\eta$-Einstein manifold.*

*Proof.* Let $\mathcal{M}$ be a $\phi$-projectively flat $\beta$-Kenmotsu manifold with respect to $\hat{\nabla}$, then (5.26) holds. Thus, from (5.24) and (5.26), we have

$$g(\hat{\mathcal{R}}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n}\{\hat{Ric}(\phi Y, \phi Z)g(\phi X, \phi W) - \hat{Ric}(\phi X, \phi Z)g(\phi Y, \phi W)\},$$

which by using (4.18) and (4.20) turns to

$$\begin{aligned} g(\mathcal{R}(\phi X, \phi Y)\phi Z, \phi W) &= -\beta^2\{g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\} \\ &\quad + \frac{1}{2n}\{Ric(\phi Y, \phi Z)g(\phi X, \phi W) + 2n\beta^2 g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &\quad - Ric(\phi X, \phi Z)g(\phi Y, \phi W) - 2n\beta^2 g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{5.27}$$

Now choosing a set $\{e_i, \phi e_i, \xi\}(1 \leq i \leq 2n)$ as an orthogonal basis of $\mathcal{M}$, by contracting (5.27) over $X$ and $W$, we obtain

$$\begin{aligned} Ric(\phi Y, \phi Z) &= -(2n\beta^2 + \xi\beta)g(\phi Y, \phi Z) \\ &\quad + \frac{1}{2n}\{(2n-1)Ric(\phi Y, \phi Z) + 2n(2n-1)\beta^2 g(\phi Y, \phi Z)\}. \end{aligned}$$

This implies

$$Ric(\phi Y, \phi Z) = -(\beta^2 + \xi\beta)g(\phi Y, \phi Z). \tag{5.28}$$

By using (2.2) and (2.8) in (5.28), we have

$$Ric(Y, Z) = -(\beta^2 + \xi\beta)g(Y, Z) - (2n-1)\beta^2\eta(Y)\eta(Z) - (2n-1)Y(\beta)\eta(Z). \tag{5.29}$$

Now, if $\beta$ is constant, then (5.29) reduces to

$$Ric(Y, Z) = -\beta^2 g(Y, Z) - (2n-1)\beta^2\eta(Y)\eta(Z).$$

Thus $\mathcal{M}$ is an $\eta$-Einstein manifold. $\square$

## 6. Concircular curvature tensor in $\beta$-Kenmotsu manifolds concerning $\hat{\nabla}$

A transformation of a $(2n+1)$-dimensional Riemannian manifold $\mathcal{M}$, which transforms every geodesic circle of $\mathcal{M}$ into a geodesic circle, is called a concircular transformation [14, 27]. A concircular transformation is always a conformal transformation [14]. Here geodesic circle means a curve in $\mathcal{M}$ whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is generalization of inversive geometry in the sense that the change of metric is more general than induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor with respect to the Levi-Civita connection and is defined by

$$P_2(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{s}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{6.30}$$

for all $X$, $Y$ and $Z$ on $\mathcal{M}$, where $s$ is the the scalar curvature with respect to the Levi-Civita connection.

**Definition 6.1.** *A $\beta$-Kenmotsu manifold $\mathcal{M}$ satisfying the condition*

$$\phi^2(\hat{P}_2(\phi X, \phi Y)\phi Z) = 0$$

*is called $\phi$-concircularly flat with respect to $\hat{\nabla}$, where $\hat{P}_2(X,Y)Z$ is the concircular curvature tensor of dimension $(2n+1)$ with respect to $\hat{\nabla}$ and is given by*

$$\hat{P}_2(X,Y)Z = \hat{\mathcal{R}}(X,Y)Z - \frac{\hat{s}}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.31}$$

*As we know that*

$$\phi^2(\hat{P}_2(\phi X, \phi Y)\phi Z) = 0 \Longleftrightarrow g(\hat{P}_2(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{6.32}$$

*for all $X, Y, Z, W$ on $\mathcal{M}$.*

**Theorem 6.1.** *Let $\mathcal{M}$ be a $(2n+1)$-dimensional $\phi$-concircularly flat $\beta$-Kenmotsu manifold with respect to $\hat{\nabla}$ and $\beta$ is constant. Then $\mathcal{M}$ is an $\eta$-Einstein manifold.*

*Proof.* If $\mathcal{M}$ is a $\phi$-concircularly flat with respect to $\hat{\nabla}$, then (6.32) holds. Thus, from (6.31) and (6.32), we have

$$\begin{aligned} g(\hat{\mathcal{R}}(\phi X, \phi Y)\phi Z, \phi W) &= \frac{\hat{s}}{2n(2n-1)}\{g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &\quad - g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{6.33}$$

By using (4.18) and (2.2) in (6.33), we have

$$\begin{aligned} g(\mathcal{R}(\phi X,\phi Y)\phi Z,\phi W) &= -\beta^2\{g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &= \frac{s+2n(2n+1)\beta^2}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &\quad -g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{6.34}$$

Now choosing $\{e_i,\phi e_i,\xi\}(1\le i\le 2n)$ as a set of orthogonal basis of $\mathcal{M}$ and contracting (6.34) over $X$ and $W$, we obtain

$$Ric(\phi Y,\phi Z)=(\frac{s}{2n}+(\beta^2-\xi\beta))g(\phi Y,\phi Z). \tag{6.35}$$

By using (2.2) and (2.8) in (6.35), we have

$$\begin{aligned} Ric(Y,Z) &= (\frac{s}{2n}+(\beta^2-\xi\beta))g(Y,Z)-(\frac{s}{2n}+(2n+1)\beta^2)\eta(Y)\eta(Z) \\ &\quad -(2n-1)Y(\beta)\eta(Z). \end{aligned} \tag{6.36}$$

Now, if $\beta$ is constant, then (6.36) reduces to

$$Ric(Y,Z)=(\frac{s}{2n}+\beta^2)g(Y,Z)-(\frac{s}{2n}+(2n+1)\beta^2)\beta^2\eta(Y)\eta(Z).$$

The above equation shows that $\mathcal{M}$ is an $\eta$-Einstein manifold. □

## 7. Conformal curvature tensor in $\beta$-Kenmotsu manifolds concerning $\hat{\nabla}$

If the Riemannian metric $g$ on a manifold $\mathcal{M}$ is conformally related with a flat Euclidean metric, then $g$ is called conformally flat. A Riemannian manifold equipped with a conformally flat Riemannian metric is named a conformally flat manifold. By using conformal transformation, Weyl [26] introduced a generalized curvature tensor which vanishes whenever the metric is conformally flat. Due to this reason it is called confomal curvature tensor. It is well-known that a Riemannian manifold $\mathcal{M}$ of dimension $(2n+1)$ is conformally flat if and only if the Weyl conformal curvature tensor field $P_3$ vanishes for the dimension $>3$. The conformal curvature tensor $P_3$ in a $(2n+1)$-dimensional Riemannian manifold is defined by

$$\begin{aligned} P_3(X,Y)Z &= \mathcal{R}(X,Y)Z-\frac{1}{2n-1}\{Ric(Y,Z)X-Ric(X,Z)Y+g(Y,Z)\mathcal{Q}X \\ &- g(X,Z)\mathcal{Q}Y\}+\frac{s}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}, \end{aligned} \tag{7.37}$$

for all vector fields $X,Y,Z$ on $\mathcal{M}$, where $\mathcal{R}$, $Ric$, $\mathcal{Q}$, and $s$ be the Riemannian curvature tensor, the Ricci tensor, the Ricci operator, and the scalar curvature, respectively.

**Definition 7.1.** *A $\beta$-Kenmotsu manifold $\mathcal{M}$ is $\xi$-conformally flat with respect to $\hat{\nabla}$ if*

$$\hat{P}_3(X,Y)\xi = 0,$$

*where $\hat{P}_3(X,Y)Z$ is the conformal curvature tensor of dimension $(2n+1)$ with respect to $\hat{\nabla}$ and is given by*

$$\begin{aligned}\hat{P}_3(X,Y)Z &= \hat{\mathcal{R}}(X,Y)Z - \frac{1}{(2n-1)}\{\hat{Ric}(X,Z)X - \hat{Ric}(X,Z)Y + g(Y,Z)\hat{\mathcal{Q}}X \\ &\quad -g(X,Z)\hat{\mathcal{Q}}Y\} + \frac{\hat{s}}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\} \qquad (7.38)\end{aligned}$$

*for all $X,Y,Z$ on $\mathcal{M}$.*

**Theorem 7.1.** *A (2n+1)-dimensional $\beta$-Kenmotsu manifold with respect to $\hat{\nabla}$ is $\xi$-conformally flat iff it is $\xi$-conformally flat with respect to $\nabla$, provided $\beta$ is constant.*

*Proof.* From (4.18), (4.20) and (7.38), we have

$$\begin{aligned}\hat{P}_3(X,Y)Z &= P_3(X,Y)Z + X(\beta)\{g(Y,Z)\xi - \eta(Z)Y\} \qquad (7.39)\\ &\quad -Y(\beta)\{g(X,Z)\xi - \eta(Z)X\},\end{aligned}$$

where $P_3(X,Y)Z$ is defined by (7.37). By putting $Z=\xi$ in (7.39), and considering $\beta$ as a constant, we get

$$\hat{P}_3(X,Y)\xi = P_3(X,Y)\xi.$$

This completes the proof. □

**Definition 7.2.** *A $\beta$-Kenmotsu manifold $\mathcal{M}$ is called $\phi$-conformally flat with respect to $\hat{\nabla}$ if*

$$\phi^2(\hat{P}_3(\phi X,\phi Y)\phi Z) = 0 \Longleftrightarrow g(\hat{P}_3(\phi X,\phi Y)\phi Z,\phi W) = 0, \qquad (7.40)$$

*for all $X,Y,Z,W \in \chi(\mathcal{M})$.*

**Theorem 7.2.** *Let $\mathcal{M}$ be a $(2n+1)$-dimensional $\phi$-conformally flat $\beta$-Kenmotsu manifold with respect to $\hat{\nabla}$ and $\beta$ is constant. Then $\mathcal{M}$ is an $\eta$-Einstein manifold.*

*Proof.* If $\mathcal{M}$ is a $\phi$-conformaly flat, then in the view of equation (7.38) and (7.40), we have

$$\begin{aligned}g(\hat{\mathcal{R}}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{2n}\{\hat{Ric}(\phi Y,\phi Z)g(\phi X,\phi W) - \hat{Ric}(\phi X,\phi Z)g(\phi Y,\phi W) \\ &\quad +g(\phi Y,\phi Z)\hat{Ric}(\phi X,\phi W) - g(\phi X,\phi Z)\hat{Ric}(\phi Y,\phi W)\} \qquad (7.41)\\ &\quad -\frac{\hat{s}}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\}.\end{aligned}$$

By using (4.18) and (4.20), (7.41) takes the form

$$\begin{aligned} g(\mathcal{R}(\phi X,\phi Y)\phi Z,\phi W) &= -\beta^2\{g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &+\frac{1}{2n}\{Ric(\phi Y,\phi Z)g(\phi X,\phi W)+2n\beta^2 g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-Ric(\phi X,\phi Z)g(\phi Y,\phi W)-2n\beta^2 g(\phi X,\phi Z)g(\phi Y,\phi W) \qquad (7.42) \\ &+g(\phi Y,\phi Z)Ric(\phi X,\phi W)+2n\beta^2 g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-g(\phi X,\phi Z)Ric(\phi Y,\phi W)-2n\beta^2 g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &-\frac{s+2n(2n+1)\beta^2}{2n}g(\phi Y,\phi Z). \end{aligned}$$

Now choosing $\{e_i,\phi e_i,\xi\}(1\leq i\leq 2n)$ as a set of orthogonal basis of $\mathcal{M}$ and contracting (7.42) over $X$ and $W$, we obtain

$$Ric(\phi Y,\phi Z)=(\frac{s}{2n}-(2n-1)(\beta^2+\xi\beta))g(\phi Y,\phi Z). \qquad (7.43)$$

Now using (2.2) and (2.8) in (7.43), we have

$$\begin{aligned} Ric(Y,Z) &= (\frac{s}{2n}-(2n-1)(\beta^2+\xi\beta))g(Y,Z) \\ &-(\frac{s}{2n}+\beta^2-2(n-1)(\xi\beta))\eta(Y)\eta(Z)-(2n-1)Y(\beta)\eta(Z). \qquad (7.44) \end{aligned}$$

Now, if $\beta$ is constant, then (7.44) reduces to

$$Ric(Y,Z)=(\frac{s}{2n}-(2n-1)\beta^2)g(Y,Z)-(\frac{s}{2n}+\beta^2)\eta(Y)\eta(Z).$$

The above equation shows that $\mathcal{M}$ is an $\eta$-Einstein manifold. □

## 8. Example

In this section, an example has been stated to verify some results of the paper.

We assume a 3-dimensional manifold $\mathcal{M}=\{(u,v,w)\in\mathbb{R}^3\}$, where $(u,v,w)$ are the usual coordinates in $\mathbb{R}^3$. We choose the linearly independent vector fields at each point of $\mathcal{M}$ as [20]

$$\epsilon_1=w^2\frac{\partial}{\partial u},\ \epsilon_2=w^2\frac{\partial}{\partial v},\ \epsilon_3=\frac{\partial}{\partial w}.$$

Let the Riemannian metric $g$ is defined by

$$g(\epsilon_i,\epsilon_j)=\begin{cases} 1 & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j \end{cases} ;i,j=1,2,3.$$

Let the 1-form $\eta$ is defined by

$$\eta(X)=g(X,\epsilon_3),$$

for any $X$ on $\mathcal{M}$. Let the $(1,1)$-tensor field $\phi$ is defined by

$$\phi(\epsilon_1) = -\epsilon_2, \quad \phi(\epsilon_2) = \epsilon_1, \quad \phi(\epsilon_3) = 0.$$

Using the linearity of $\phi$ and $g$, we have

$$\phi^2 X = -X + \eta(X)\epsilon_3, \quad \eta(\epsilon_3) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any $X, Y$ on $\mathcal{M}$. Thus for $\epsilon_3 = \xi$, the structure $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $\mathcal{M}$. For the connection $\nabla$, we have

$$[\epsilon_1, \epsilon_2] = 0, \quad [\epsilon_1, \epsilon_3] = -\frac{2}{w}\epsilon_1, \quad [\epsilon_2, \epsilon_3] = -\frac{2}{w}\epsilon_2.$$

By using the Koszul's formula, we find

$$\begin{aligned} \nabla_{\epsilon_1}\epsilon_1 &= \tfrac{2}{w}\epsilon_3, & \nabla_{\epsilon_1}\epsilon_2 &= 0, & \nabla_{\epsilon_1}\epsilon_3 &= -\tfrac{2}{w}\epsilon_1, \\ \nabla_{\epsilon_2}\epsilon_1 &= 0, & \nabla_{\epsilon_2}\epsilon_2 &= \tfrac{2}{w}\epsilon_3, & \nabla_{\epsilon_2}\epsilon_3 &= -\tfrac{2}{w}\epsilon_2, \\ \nabla_{\epsilon_3}\epsilon_1 &= 0, & \nabla_{\epsilon_3}\epsilon_2 &= 0, & \nabla_{\epsilon_3}\epsilon_3 &= 0. \end{aligned} \tag{8.45}$$

From the above values, it is clear that $(\phi, \xi, \eta, g)$ is a $\beta$-Kenmotsu structure on $\mathcal{M}$, hence $\mathcal{M}(\phi, \xi, \eta, g)$ is a 3-dimensional $\beta$-Kenmotsu manifold satisfying the conditions (2.5)-(2.7), where $\beta = -\frac{2}{w}$. Using the results from equation (8.45), we can obtain the non-vanishing components of the Riemannian curvature tensor with respect to $\nabla$ as follows:

$$\begin{aligned} \mathcal{R}(\epsilon_1, \epsilon_2)\epsilon_1 &= \tfrac{4}{w^2}\epsilon_2, & \mathcal{R}(\epsilon_1, \epsilon_2)\epsilon_2 &= -\tfrac{4}{w^2}\epsilon_1, & \mathcal{R}(\epsilon_1, \epsilon_3)\epsilon_1 &= \tfrac{4}{w^2}\epsilon_3, \\ \mathcal{R}(\epsilon_1, \epsilon_3)\epsilon_3 &= -\tfrac{4}{w^2}\epsilon_1, & \mathcal{R}(\epsilon_2, \epsilon_3)\epsilon_3 &= -\tfrac{4}{w^2}\epsilon_2, & \mathcal{R}(\epsilon_2, \epsilon_3)\epsilon_2 &= \tfrac{4}{w^2}\epsilon_3. \end{aligned} \tag{8.46}$$

The Ricci tensor concerning to $\nabla$ are

$$Ric(\epsilon_i, \epsilon_i) = \begin{cases} -\frac{8}{w^2}, i = 1, 2, 3, \\ 0, \ otherwise. \end{cases} \tag{8.47}$$

Thus, the scalar curvature $s$ with respect to the $\nabla$ given by

$$s = -\frac{24}{w^2}. \tag{8.48}$$

By using the values of (8.45) in (3.10), we obtain

$$\hat{\nabla}_{\epsilon_i}\epsilon_j = \begin{cases} -\epsilon_2, \ i = 3, \ j = 1, \\ \epsilon_1, \ i = 3, \ j = 2, \\ 0, \ otherwise. \end{cases} \tag{8.49}$$

From the above results given in (8.49), we can easily calculate

$$\hat{\mathcal{R}}(\epsilon_i,\epsilon_j)\epsilon_k=0,\quad \hat{Ric}(\epsilon_i,\epsilon_j)=0,\quad \hat{\mathcal{Q}}=0,\quad \hat{s}=0,\qquad for\quad 1\leq i,j,k\leq 3. \tag{8.50}$$

In view of (8.50), it can be easily seen from (5.24) and (7.38) that

$$\begin{aligned}\hat{P}_1(\epsilon_1,\epsilon_2)\epsilon_3=\hat{P}_1(\epsilon_1,\epsilon_3)\epsilon_3=\hat{P}_1(\epsilon_2,\epsilon_3)\epsilon_3=0,\\ \hat{P}_3(\epsilon_1,\epsilon_2)\epsilon_3=\hat{P}_3(\epsilon_1,\epsilon_3)\epsilon_3=\hat{P}_3(\epsilon_2,\epsilon_3)\epsilon_3=0,\end{aligned} \tag{8.51}$$

respectively.

Also by using (8.46), (8.47) and (8.48) from (5.23) and (7.37), we find

$$\begin{aligned}P_1(\epsilon_1,\epsilon_2)\epsilon_3=P_1(\epsilon_1,\epsilon_3)\epsilon_3=P_1(\epsilon_2,\epsilon_3)\epsilon_3=0,\\ P_3(\epsilon_1,\epsilon_2)\epsilon_3=P_3(\epsilon_1,\epsilon_3)\epsilon_3=P_3(\epsilon_2,\epsilon_3)\epsilon_3=0,\end{aligned} \tag{8.52}$$

respectively.

Thus, the first relations of the equations (8.51) and (8.52) and the second relations of the equations (8.51) and (8.52) verifies Theorem 5.1 and Theorem 7.1, respectively.

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# $g$-$\mathcal{H}$-REGULAR AND $g$-$\mathcal{H}$-NORMAL HEREDITARY SPACES

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ABSTRACT. This paper presents the introduction of the concept of $g$-$\mathcal{H}$-regularity within the context of hereditary spaces. It delves into an exploration of various properties associated with $g$-$\mathcal{H}$-normality, offering proofs for some of these properties. Additionally, the paper investigates the characterization of $g$-$\mathcal{H}$-normality through the application of modified versions of Urysohn's lemma and the famous Tietze Extension Theorem.
**Keywords**: Generalized Topological spaces, Hereditary classes, $\mathcal{H}_g$-closed sets, $g$-$\mathcal{H}$-normal spaces, $g$-$\mathcal{H}$-regular spaces, Urysohn's Lemma, Tietze Extension Theorem.
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## 1. Introduction

The separation axioms play a pivotal role in the examination of topological spaces by enabling us to use topological methods to distinguish between disjoint sets and distinct points. In 2002, Á Császár [7] introduced the concept of generalized topology, which expands upon this framework. For a non-empty set $X$, a family $\mu$ of subsets of $X$ is designated as a generalized topology on $X$ if it satisfies two fundamental properties: it must include the empty set $\emptyset$ and remain closed under arbitrary unions [6]. The pair $(X, \mu)$ is referred to as a generalized topological space, where the elements of $\mu$ are known as $\mu$-open sets, and their complements are designated as $\mu$-closed sets. We define the closure of a set $A$ in this context as $cl_\mu(A)$, given by $\cap\{F \subset X : X - F \in \mu, A \subset F\}$, and the interior of $A$ as $int_\mu(A)$, defined as $\cup\{G \subset X : G \in \mu, G \subset A\}$.

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In 2004, Császár [8] introduced a modified framework for separation axioms ($\mu - T_0$, $\mu - T_1$, $\mu - T_2$, $\mu - S_1$, $\mu - S_2$) tailored specifically for generalized topologies, where the conventional open sets are substituted with $\mu$-open sets. In 2007, he introduced the concept of normality for generalized topological spaces and demonstrated several properties of normal spaces. These properties were characterized using an adapted version of Urysohn's lemma [10]. Sarsak [17] expanded the study of separation axioms by introducing $\mu - D_0$, $\mu - D_1$, $\mu - D_2$ generalized topological spaces. Xun et al. [18] conducted research on generalized topological spaces and provided characterizations for $\mu - T_i$ spaces, for $i = 0, 1, 2, 3, 4$, as well as $\mu - T_D$ spaces and $\mu - R_0$ spaces. Additionally, Min conducted a study on separation axioms within generalized topological spaces in [14].

Also hereditary classes, initially introduced by Császár [9], have been a subject of ongoing exploration by numerous researchers over time. A non-empty family $\mathcal{H}$ consisting of subsets of $X$ is termed a hereditary class on $X$ if, whenever $A$ is a subset of $B$ and $B$ is a member of $\mathcal{H}$, $A$ must also belong to $\mathcal{H}$. The triple $(X, \mu, \mathcal{H})$ is denoted as a hereditary generalized topological space, or simply a hereditary space. Császár [9] defined an operator $cl^*(A) = A \cup A^*$, where $A^* = \{x \in X : U \cap A \notin \mathcal{H} \text{ for each } U \in \mu, x \in U\}$ for $A \subset X$. This operator induces another generalized topology, denoted as $\mu^*$, which is finer than $\mu$, and it is referred to as the $*$-generalized topology. The constituents of $\mu^*$ are known as $*$-open sets and their complements are designated as $*$-closed sets. Additionally, $int^*(A) = \cup\{G \subset X : G \in \mu^*, G \subset A\}$. The exploration of hereditary spaces has been a subject of ongoing research by various authors [12, 1]. In a separate study, the author investigated some generalized separation axioms, such as Hausdorff modulo $\mathcal{H}$ and $\mathcal{H}$-regularity, as outlined in [5].

In 2009, Navaneethakrishnan et al. [16] introduced and examined the notions of $\mathcal{I}_g$-normal and $\mathcal{I}_g$-regular ideal topological spaces, utilizing the concepts of $\mathcal{I}_g$-open and $\mathcal{I}_g$-closed sets [15]. The author later extended these concepts to $\mathcal{H}_g$-normal and $\mathcal{H}_g$-regular hereditary spaces in [4]. Furthermore, the author also introduced and investigated the concept of $g$-$\mathcal{H}$-normal spaces in the same study [4].

Recently, in 2024, Mesfer H. Alqahtani et al. [2] introduced a category of $\aleph$-open sets in topological spaces and discussed its relationships with many different classes of open sets. Additionally, the concepts of continuity of functions and separation axioms have been investigated.

Many authors introduced modified separation axioms for generalized topologies in different set-ups, which motivates the author to investigate further on separation axioms and find the generalizations of well known results for the same. This paper builds upon the previous research to provide further characterizations and a modified version of the well-known Urysohn lemma and Tietze Extension Theorem specifically tailored for $g$-$\mathcal{H}$-normal spaces. Additionally, the concept of $g$-$\mathcal{H}$-regular space is introduced and various properties of this space are investigated.

## 2. Preliminaries

In our study, we will make reference to the following definitions and theorems:

**Definition 2.1.** [5] *The generalized topological space $(X, \mu)$ is said to be $\mu$-regular if, for every point $x$ within $X$ and for every $\mu$-closed set $F$ that does not include $x$, there exist two disjoint $\mu$-open sets denoted as $U$ and $V$ in $X$, satisfying the conditions that $x$ is an element of $U$ and $F$ is entirely contained within $V$.*

**Definition 2.2.** [3] *In the context of a generalized topological space $(X, \mu)$ and any subset $Y$ of $X$, the collection $\{Y \cap G : G \in \mu\}$ is a generalized topology on $Y$, which particularly is referred to as the subspace generalized topology or relative generalized topology and it is denoted by $\mu_Y$. Consequently, when we equip the set $Y$ with this generalized topology $\mu_Y$, it is described as a generalized subspace (or simply subspace) of $X$.*

**Definition 2.3.** [4] *A subset $A$ of a generalized topological space $X$ is said to be $\mathcal{H}_g$-closed when it satisfies the condition that if $U$ is a $\mu$-open set containing $A$, then $A^*$ must be entirely contained within $U$. $A$ is said to be $\mathcal{H}_g$-open if $X - A$ is $\mathcal{H}_g$-closed.*

**Remark 2.1.** [4] *Each $\mu$-open set is also $\mathcal{H}_g$-open and each $\mu$-closed set is also $\mathcal{H}_g$-closed.*

**Definition 2.4.** [4] *A subset $A$ of a generalized topological space $X$ is called $g$-$\mu$-closed when it satisfies the condition that if $U$ is $\mu$-open set cotaining $A$, then $cl_\mu(A)$, the $\mu$-closure of $A$, must be entirely contained within $U$. $A$ is said to be $g$-$\mu$-open if $X - A$ is $g$-$\mu$-closed.*

**Definition 2.5.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is considered to be $\mathcal{H}_g$-regular when, for any point $x$ and a $\mu$-closed set $B \subset X$, provided that $B$ does not contain $x$, there exist two disjoint $\mathcal{H}_g$-open sets $U$ and $V$, within $X$ satisfying $x \in U$ and $B \subset V$.*

**Definition 2.6.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is $\mathcal{H}_g$-normal if, for any two disjoint $\mu$-closed sets $A$ and $B$ in $X$, there exist two disjoint $\mathcal{H}_g$-open sets $U$ and $V$ within $X$, such that $A$ is entirely contained in $U$ and $B$ is entirely contained in $V$.*

**Definition 2.7.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is g-$\mathcal{H}$-normal if, for any two disjoint $\mathcal{H}_g$-closed sets $A$ and $B$ in $X$, there exist two disjoint $\mu$-open sets $U$ and $V$ within $X$ such that $A$ is entirely contained in $U$ and $B$ is entirely contained in $V$.*

**Theorem 2.1.** [9] *For any two subsets $A$ and $B$ of a hereditary space $(X, \mu, \mathcal{H})$, the following properties hold.*

(1) $A^* \subset cl_\mu(A)$.
(2) $A^*$ *is $\mu$-closed set, and therefore* $A^* = cl_\mu(A^*) = cl_\mu(A)$.
(3) $cl^*(A) = A^* = cl_\mu(A) = cl_\mu(A^*)$, *whenever* $A \subset A^*$.
(4) *If* $\mathcal{H} = \{\emptyset\}$, *then* $A^* = cl_\mu(A) = cl^*(A)$.
(5) $(A \cap B)^* \subset A^* \cap B^*$.

**Lemma 2.1.** [10] *Let $\beta$ be any family of subsets of the space $X$. The family $\nu$ of subsets of $X$ consists of the $\emptyset$ and all sets $N$ that can be expressed as the union of sets $\bigcup_{i \in I} B_i$, where $B_i \in \beta$ and $I$ is a non-empty index set, is a generalized topology on $X$, which is referred to as the generalized topology generated by the base $\beta$.*

**Example 2.1.** [10] *Consider $X = \mathbf{R}$ and the family of subsets $\beta = \{(-\infty, t) : t \in \mathbf{R}\} \bigcup \{(t, +\infty) : t \in \mathbf{R}\}$. Then the generalized topology on $\mathbf{R}$ generated by $\beta$, denoted by $\nu$, is known as the usual generalized topology.*

**Lemma 2.2.** [10] *Suppose $\mu$ is a generalized topology on the space $X$ and the generalized topology $\nu$ on another space $Y$ is generated by the base $\beta$. Then a mapping $f : X \to Y$ is considered $(\mu, \nu)$-continuous if and only if the inverse image of each set $B \in \beta$ under the map $f$, denoted as $f^{-1}(B)$, belongs to the generalized topology $\mu$.*

**Theorem 2.2.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is g-$\mathcal{H}$-normal if and only if, for every $\mathcal{H}_g$-closed set $A$ within $X$ and an $\mathcal{H}_g$-open set $B$ that contains $A$, there exists a $\mu$-open set $V$ satisfying $A \subset V \subset cl_\mu(V) \subset B$.*

## 3. $g$-$\mathcal{H}$-regular and $g$-$\mathcal{H}$-normal Spaces

**$g$-$\mathcal{H}$-regular Spaces.** This section will provide an introduction to the concept of $g$-$\mathcal{H}$-regular hereditary spaces and delve into an exploration of the different properties related to these spaces.

**Definition 3.1.** *A hereditary space $(X, \mu, \mathcal{H})$ is defined to be g-$\mathcal{H}$-regular if when, for a point $x$ in $X$ and an $\mathcal{H}_g$-closed set $A$ that does not contain $x$, there exist two disjoint $\mu$-open sets $U$ and $V$ such that $x$ is an element of $U$ and $A$ is entirely contained within $V$.*

**Definition 3.2.** *A generalized topological space $(X, \mu)$ is termed g-$\mu$-regular if, for a point $x$ within $X$ and a g-$\mu$-closed set $A$ that does not include $x$, there exist two disjoint $\mu$-open sets $U$ and $V$ such that $x$ is a member of $U$ and $A$ is entirely contained within $V$.*

**Remark 3.1.** *Every hereditary space that is g-$\mathcal{H}$-regular is also $\mu$-regular because each set that is $\mu$-closed is also $\mathcal{H}_g$-closed, however the converse does not necessarily hold, as illustrated in Example 3.1.*

**Example 3.1.** *Let $X = \{p, q, r\}$, $\mu = \{\emptyset, \{p\}, \{q, r\}, X\}$ and $\mathcal{H} = \{\emptyset\}$. This space is $\mu$-regular, but not g-$\mathcal{H}$-regular, since $\{r\}$ is $\mathcal{H}_g$-closed set that does not contain $q$ and there are no disjoint $\mu$-open sets that contain $q$ and $\{r\}$.*

The following Theorems 3.1 and 3.2 give characterizations of $g$-$\mathcal{H}$-regular spaces.

**Theorem 3.1.** *A hereditary space $(X, \mu, \mathcal{H})$ is g-$\mathcal{H}$-regular if, and only if, for every point $x \in X$ and each $\mathcal{H}_g$-open set $A$ in $X$ that includes $x$, there is a $\mu$-open set $V$ satisfying that $x \in V \subset cl_\mu(V) \subset A$.*

*Proof.* In a $g$-$\mathcal{H}$-regular space $X$, consider a point $x$ and an $\mathcal{H}_g$-open set $A$ containing $x$. Then $X - A$ is $\mathcal{H}_g$-closed set that does not contain $x$. Since $X$ is $g$-$\mathcal{H}$-regular, there exist two disjoint $\mu$-open sets, $V$ and $W$, such that $x$ belongs to $V$ and $(X - A)$ is a subset of $W$. The fact $V \cap W = \emptyset$ implies that $cl_\mu(V) \subset X - W$. Consequently, $x \in V \subset cl_\mu(V) \subset X - W \subset A$. Conversely, suppose $x$ is an element of $X$ and $A$ is any $\mathcal{H}_g$-closed sets in $X$ that does not contain $x$. In this case, $X - A$ is $\mathcal{H}_g$-open set containing $x$. Then there exists a $\mu$-open set $V$ such that $x \in V \subset cl_\mu(V) \subset X - A$. By defining $W = X - cl_\mu(V)$, there will be two disjoint $\mu$-open sets $V$ and $W$ with the properties that $x \in V$ and $A \subset W$. Therefore $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-regular. □

By setting $\mathcal{H} = \{\emptyset\}$ in the above Theorem 3.1, we can derive the following characterization of $g$-$\mu$-regular generalized topological spaces.

**Corollary 3.1.** *A generalized topological space $(X, \mu)$ is g-$\mu$-regular if and only if, for every point $x \in X$ and each g-$\mu$-open set $A$ that contains $x$, there exists a $\mu$-open set $U$ satisfying $x \in U \subset cl_\mu(U) \subset A$.*

**Theorem 3.2.** *A hereditary space $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-regular if and only if, for every $x \in X$ and any $\mathcal{H}_g$-closed set $A$ that does not contain $x$, there exists a $\mu$-open set $V$ that contains $x$ such that $cl_\mu(V)$ is disjoint from $A$.*

*Proof.* The proof of the theorem is straightforward and follows directly from Theorem 3.1. □

The following Corollary 3.2 provides a way to characterize $g$-$\mu$-regular spaces when we take $\mathcal{H} = \{\emptyset\}$ in the Theorem 3.2.

**Corollary 3.2.** *A generalized topological space $(X, \mu)$ is $g$-$\mu$-regular if and only if, for every point $x \in X$ and for any $g$-$\mu$-closed set $A$ that does not include $x$, there exists a $\mu$-open set $V$ containing $x$ such that $cl_\mu(V)$ does not intersect with $A$.*

We have defined $\mathcal{H}_g$-regular hereditary spaces in [4]. Now we establish a relationship between $g$-$\mathcal{H}$-regularity and $\mathcal{H}_g$-regularity of hereditary spaces in the Theorem 3.3.

**Theorem 3.3.** *A hereditary space $(X, \mu, \mathcal{H})$, which is $g$-$\mathcal{H}$-regular, is also $\mathcal{H}_g$-regular.*

*Proof.* The straightforward proof lies in the fact that every $\mu$-open set is $\mathcal{H}_g$-open and every $\mu$-closed set is $\mathcal{H}_g$-closed. □

**Remark 3.2.** *Every $g$-$\mathcal{H}$-regular hereditary space is $\mathcal{H}_g$-regular, as shown in the Theorem 3.3, however the converse does not necessarily hold, as illustrated in Example 3.2.*

**Example 3.2.** *Let $X = \{p, q, r\}$, $\mu = \{\emptyset, \{p\}, \{p, q\}, \{p, r\}, X\}$ and $\mathcal{H} = \{\emptyset, \{p\}\}$. Every $\mu$-open subset of $X$ is $*$-closed, therefore every subset of $X$ is $\mathcal{H}_g$-open, which makes the space $(X, \mu, \mathcal{H})$, $\mathcal{H}_g$-regular. $\{r\}$ is $\mathcal{H}_g$-closed set that does not contain $q$ and there are no disjoint $\mu$-open sets that contain $q$ and $\{r\}$. Therefore $(X, \mu, \mathcal{H})$ is not $g$-$\mathcal{H}$-regular.*

**$g$-$\mathcal{H}$-normal Spaces.** The notion of $g$-$\mathcal{H}$-normal hereditary spaces was originally introduced in the reference [4]. In this context, we will explore a range of properties and characterizations of these $g$-$\mathcal{H}$-normal hereditary spaces.

**Theorem 3.4.** *Let $X$ be $g$-$\mathcal{H}$-normal space. Then a $\mu$-closed subspace of $X$ is $g$-$\mathcal{H}$-normal.*

*Proof.* In a $\mu$-closed subspace $Y$ of $X$, if $A$ and $B$ be two disjoint $\mathcal{H}_g$-closed sets, then, by Theorem 3.6, $A$ and $B$ are disjoint $\mathcal{H}_g$-closed subsets of the space $X$. Since $X$ is $g$-$\mathcal{H}$-normal, there exist two disjoint $\mu$-open sets $U$ and $V$ in $X$ such that $A$ is contained in $U$ and $B$ is contained in $V$. Then $U \cap Y$ and $V \cap Y$ are two disjoint $\mu_Y$-open sets in $Y$ such that $A = (A \cap Y) \subset (U \cap Y)$ and $B = (B \cap Y) \subset (V \cap Y)$. Hence $Y$ is $g$-$\mathcal{H}$-normal space. □

The Theorem 3.5 discussed below establishes a relationship between the notions of $g$-$\mathcal{H}$-normality and $\mathcal{H}_g$-normality within the context of hereditary spaces.

**Theorem 3.5.** *If a hereditary space $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-normal, then it is $\mathcal{H}_g$-normal.*

*Proof.* The proof can be immediately established by the fact that every $\mu$-open set is $\mathcal{H}_g$-open and every $\mu$-closed set is $\mathcal{H}_g$-closed. □

**Remark 3.3.** *Every $g$-$\mathcal{H}$-normal hereditary space is $\mathcal{H}_g$-normal, as shown in the Theorem 3.5, however the converse does not necessarily hold, as illustrated in Example 3.3.*

**Example 3.3.** *Consider the hereditary space $X = \{p, q, r\}$, $\mu = \{\emptyset, \{p\}, \{p, q\}, \{p, r\}, X\}$ and $\mathcal{H} = \{\emptyset, \{p\}\}$. In this space, every $\mu$-open set is essentially $*$-closed, making every subset of $X$, $\mathcal{H}g$-open. Consequently, $(X, \mu, \mathcal{H})$ is $\mathcal{H}_g$-normal. However, $\{q\}$ and $\{r\}$ are two disjoint $\mathcal{H}_g$-closed sets which can not be separated by disjoint $\mu$-open sets and therefore $(X, \mu, \mathcal{H})$ is not $g$-$\mathcal{H}$-normal.*

**Theorem 3.6.** *Consider a generalized subspace $Y$ of the space $X$. If a subset $A \subset Y$ is $\mathcal{H}_g$-closed within $Y$, then $A$ is $\mathcal{H}_g$-closed in $X$.*

*Proof.* Let $U$ be a $\mu$-open set containing $A$. Then $(U \cap Y) \in \mu_Y$ and $A \subset (U \cap Y)$. Since $A$ is $\mathcal{H}_g$-closed in $Y$, $A^* \subset (U \cap Y) \subset U$. Therefore $A$ is $\mathcal{H}_g$-closed in $X$. □

**Corollary 3.3.** *Consider a generalized subspace $Y$ of the space $X$. If a subset $A \subset Y$ is $\mu_Y$-closed within $Y$ then $A$ is $\mathcal{H}_g$-closed in $X$.*

**Theorem 3.7.** *Let $Y$ be a generalized subspace of $X$. If a set $A$ is $\mathcal{H}_g$-closed within the space $X$ and $Y$ is $\mu$-closed within $X$, then the intersection $A \cap Y$ is $\mathcal{H}_g$-closed within $Y$.*

*Proof.* Consider $(A \cap Y) \subset U$ with $U \in \mu_Y$. Then $U$ can be expressed as $U = (G \cap Y)$ for some $G \in \mu$. Then $A = (A \cap Y) \cup (A \cap (X - Y)) \subset (U \cup (X - Y)) = (G \cap Y) \cup (X - Y) = (G \cup (X - Y)) \in \mu$, since $Y$ is $\mu$-closed in $X$. Also, $A$ is $\mathcal{H}_g$-closed set within $X$, $A^* \subset (G \cup (X - Y))$. Then $(A \cap Y)^* \subset (A^* \cap Y^*) \subset (A^* \cap Y) \subset ((G \cup (X - Y)) \cap Y) = G \cap Y = U$. Therefore $A$ is $\mathcal{H}_g$-closed in $Y$. □

**Theorem 3.8.** *If a set $A$ is $\mathcal{H}_g$-closed and set $B$ is $\mu$-closed, then their intersection $A \cap B$ is $\mathcal{H}_g$-closed.*

*Proof.* The proof can be deduced from the Theorems 3.6 and 3.7. □

*Urysohn's Lemma.* We will now provide a proof for the following variation of Urysohn's Lemma adapted for $g$-$\mathcal{H}$-normal hereditary spaces:

**Theorem 3.9.** *Necessary Condition for $g$-$\mathcal{H}$-Normality in Hereditary Space: Let $(X, \mu, \mathcal{H})$ be a $g$-$\mathcal{H}$-normal hereditary space and let $A$, $B$ be disjoint $\mathcal{H}_g$-closed subsets of $X$. Then there exist a function $f : X \to [0, 1]$ that is $(\mu, \nu)$-continuous where $\nu$ is the standard generalized topology on the interval $[0, 1]$, such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

*Proof.* Consider the collection $D$ of dyadic fractions in the interval [0,1], defined as $\frac{m}{2^n}$, where $m = 0, 1, 2, .....2^n$ and $n = 0, 1, 2, 3, .....$ For each $r \in D$, we construct $\mu$-open sets $U_r$ and $\mu$-closed sets $F_r$ in such a way that:

(1) $U_r \subset F_r$ for each $r \in D$.
(2) If $r$ and $s$ are in $D$ and $r < s$, then $F_r \subset U_s$.

Start by setting $F_0 = A$ and $U_1 = X - B$. As $A$ and $B$ are disjoint, $A \subset (X - B)$. Using Theorem 2.2, since $A$ is $\mathcal{H}_g$-closed and $X - B$ is $\mathcal{H}_g$-open, there exists a $\mu$-open set and therefore $\mathcal{H}_g$-open $U_{\frac{1}{2}}$ such that $A \subset U_{\frac{1}{2}} \subset cl_\mu(U_{\frac{1}{2}}) \subset X - B$. Continuing this construction, we can obtain $U_r$ and $F_r$ for each $r \in D$, ensuring that $U_r \subset F_r$ and $F_r \subset U_s$ for $r < s$.
Define a function $f : X \to [0, 1]$ as $f(x) = inf\{r \in D : x \in F_r\}$. Then $f(x) = 0$ for $x \in F_0 = A$ and $f(x) = 1$ for $x \in B$. To show that $f$ is $(\mu, \nu)$-continuous, it is sufficient to prove that $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are $\mu$-open sets in $X$. $f^{-1}([0, a)) = \cup\{U_r : r < a\}$ and $f^{-1}((b, 1]) = \cup\{X - F_r : r > b\}$, ensuring that both sets are $\mu$-open, making $f$, $(\mu, \nu)$-continuous. □

The following Theorem 3.10 provides a sufficient condition for $g$-$\mathcal{H}$-normality in hereditary spaces.

**Theorem 3.10.** *Sufficient Condition for $g$-$\mathcal{H}$-Normality in Hereditary Space: If $(X, \mu, \mathcal{H})$ is a hereditary space with the property that for any two disjoint $\mathcal{H}_g$-closed subsets $A$ and $B$ of $X$, there exist a function $f : X \to [0, 1]$ that is $(\mu, \nu)$-continuous, where $\nu$ is the standard generalized topology on the interval $[0, 1]$, such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$, then $X$ is $g$-$\mathcal{H}$-normal.*

*Proof.* Consider the sets $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$. These sets are disjoint $\mu$-open sets in $X$ containing $A$ and $B$, respectively. Consequently, $X$ is $g$-$\mathcal{H}$-normal. □

*Tietze Extension Theorem.* A modified version of the Tietze Extension Theorem has been established for $g$-$\mathcal{H}$-normal hereditary spaces, as outlined in Theorem 3.11.

**Theorem 3.11.** *Let $(X, \mu, \mathcal{H})$ be a $g$-$\mathcal{H}$-normal hereditary space and let $f : F \to \mathbf{R}$ be a $(\mu_F, \nu)$-continuous mapping, where $F$ is an $\mathcal{H}_g$-closed subset of $X$. Then there exist a $(\mu, \nu)$-continuous mapping $g : X \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in F$, where $\nu$ is usual generalized topology on $\mathbf{R}$.*

*Proof.* We first assume that $f$ is bounded function with $c = sup\{|f(y)| : y \in F\}$. We define sets $A_0 = \{y \in F : f(y) \leq -c/3\}$ and $B_0 = \{y \in F : f(y) \geq c/3\}$. These sets are disjoint $\nu$-closed sets in the interval $[-c, c]$. Since $f$ is $(\mu_F, \nu)$-continuous mapping, $f^{-1}(A_0)$ and $f^{-1}(B_0)$ are disjoint $\mu_F$-closed sets and consequently $\mathcal{H}_g$-closed sets in $X$. By Theorem 3.9, there exists a $(\mu, \nu)$-continuous function $g_0 : X \to [-c/3, c/3]$ such that $g_0(A_0) = -c/3$ and $g_0(B_0) = c/3$. This function satisfies $|g_0| \leq c/3$ and $|f - g_0| \leq 2c/3$ on $F$. We then define sets $A_1 = \{y \in F : (f - g_0)(y) \leq -2c/9\}$ and $B_1 = \{y \in F : (f - g_0)(y) \geq 2c/9\}$. These sets are again disjoint $\nu$-closed sets in $[-c, c]$ and therefore $(f - g_0)^{-1}(A_1)$ and $(f - g_0)^{-1}(B_1)$ are disjoint $\mu$-closed sets in $X$, making them $\mathcal{H}_g$-closed sets in $X$. By applying Theorem 3.9, we obtain a $(\mu, \nu)$-continuous function $g_1 : X \to [-2c/9, 2c/9]$ such that $g_1(A_1) = -2c/9$ and $g_1(B_1) = 2c/9$ and $|g_1| \leq 2c/9$, $|f - g_0 - g_1| \leq 4c/9$ on $F$. This process is continued, producing a sequence $\{g_n\}$ of $(\mu, \nu)$-continuous functions defined on $X$ such that $|g_n| \leq \frac{2^n c}{3^{n+1}}$ and $|f - g_0 - g_1 ...... - g_n| \leq \frac{2^{n+1} c}{3^{n+1}}$ on $F$.

We define $h_n = g_0 + g_1 + ...... + g_n$ for $n \geq 1$. This is a sequence of $(\mu, \nu)$-continuous functions on $X$. For $n \geq m$, $|h_n - h_m|$ is bounded by $(\frac{2}{3})^{m+1} c$. Therefore, $\{h_n\}$ is a Cauchy sequence and converges uniformly to a real valued function $h$ on $X$. This limit function $h = \lim_{n\to\infty} h_n = \lim(g_0 + g_1 + ...... + g_n) = \sum_{n=0}^{\infty} g_n$ and therefore $h(x) = f(x)$ on $F$.

To complete the proof, we prove that $h$ is $(\mu, \nu)$-continuous function. Let $x \in X$ and $V$ be a $\nu$-open set in $\mathbf{R}$ containing $h(x)$. Since $h_n(x)$ converges uniformly to $h$, for any given $\epsilon > 0$, there exists an integer $N$ such that $h_n(x) \in V$ for all $n \geq N$. Since $h_n$ is $(\mu, \nu)$-continuous, there exists a $\mu$-open set $U$ in $X$ containing $x$ such that $h_n(U) \subset V$. Therefore, $h(U) = \lim_{n\to\infty} h_n(U) \subset V$. Thus, we have established that $h$ is $(\mu, \nu)$-continuous, concluding the proof. □

## 4. Conclusion

This research primarily delves into two distinct areas within the realm of hereditary generalized topological spaces. The first area explores the concept of $g$-$\mathcal{H}$-regularity in hereditary spaces, which provides generalized versions of fundamental properties typically associated with regular topological spaces. In the second area, the focus shifts to the generalization of renowned results such as Urysohn's lemma and the Tietze Extension Theorem, specifically within the context of $g$-$\mathcal{H}$-normal hereditary spaces.

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# ON SHEFFER STROKE BH-ALGEBRAS

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Abstract. In this study, a Sheffer stroke BH-algebra is introduced and its features are examined. After showing that the axioms of a Sheffer stroke BH-algebra are independent, the connection between a Sheffer stroke BH-algebra and a BH-algebra is stated. After describing a subalgebra and a normal subset of a Sheffer stroke BH-algebra, the relationship between these structures is shown. A filter of a Sheffer stroke BH-algebra is defined and the quotient of a Sheffer stroke BH-algebra is constructed. Then a homomorphism between Sheffer stroke BH-algebras is introduced and its properties are studied.

**Keywords**: (Sheffer stroke) BH-algebra, Sheffer stroke, homomorphism

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## 1. Introduction

The notion of BCK-algebras was first formulated in 1966 [9] by Y. Imai and K. Iséki as a generalization of the notation of set-theoretic difference and propositional calculus, where this notion was originated from two different ways: one of the motivation was set theory, another motivation was classical and theory, the other from classical and non-classical propositional calculus. In the same year, K. Iséki introduced the notion of a BCI–algebra [10]. It is known that the BCI-algebra is a generalization of a BCK-algebra. Q. P. Hu and X. Li introduced a

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large class of abstract algebras: BCH-algebras [7, 8]. The class of BCI-algebras is a proper subclass of the class of BCH-algebras. In 1998, Y. B. Jun, E. H. Roh and H.S. Kim introduced a new concept, called a BH-algebra, which is generalization of BCH/BCI/BCK-algebras [11]. They argued further properties of BH-algebras. In 2011, H. H. Abbass and H. M. Saeed introduced the notions of (a closed ideal and closed BCH-algebra) with respect to an element of a BCH-algebra[3]. In 2012, H. H. Abbass and H. A. Dahham introduced the notion of completely closed ideal of a BH-algebra [2]. In 2014, H. H. Abbass and S. A. Neamah introduced the notion of a fuzzy implicative ideal with respect to an element of a BH-algebra [4].

The Sheffer stroke operation was originally introduced by H. M. Sheffer [20]. Since any Boolean function or operation can be stated by only this operation [12], it attracts the attention of many researchers. It also leads to reduction of axiom systems of many structures. Also, some applications of this operation have appeared in algebraic structures such as Sheffer stroke BG-algebras [13], Sheffer stroke BCK-algebras [14], the Sheffer stroke operation reducts of basic algebra [15], a construction of very true operator on Sheffer stroke MLT-algebras [17], congruences of Sheffer Stroke Basic Algebras [16], a view on state operators in Sheffer stroke basic algebras [18] and Bosbach state operators on Sheffer stroke MTL-algebras [19].

After giving main definitions and concepts of a Sheffer stroke and a BH-algebra, a Sheffer stroke BH-algebra is defined. It is proved that the axiom system of a Sheffer stroke BH-algebra is independent. By presenting fundamental notions about this algebraic structure, the connection between a Sheffer stroke BH-algebras is a BH-algebra is given. It is shown that Cartesian product of two Sheffer stroke BH-algebras is a Sheffer stroke BH-algebra. After defining a subalgebra and a normal subset, the relationship between a subalgebra and a normal subset on a Sheffer stroke BH-algebra is shown. A filter in a Sheffer stroke BH-algebra is defined. It is proved that the family of all filters of a Sheffer stroke BH-algebra forms a complete lattice. Then a homomorphism on a Sheffer stroke BH-algebra is defined and it is shown that the notion of a filter on a Sheffer stroke BH-algebra is preserved under the homomorphism. It is presented that a quotient of a Sheffer stroke BH-algebra is a Sheffer stroke BH-algebra. furthermore, a kernel of a homomorphism is constructed and proved that the kernel is a filter under a condition.

## 2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a BH-algebra.

**Definition 2.1.** [5] *Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke if it satisfies the following conditions:*
*(S1)* $\breve{a}_1|\breve{a}_2 = \breve{a}_2|\breve{a}_1$,
*(S2)* $(\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_2) = \breve{a}_1$,
*(S3)* $\breve{a}_1|((\breve{a}_2|\breve{a}_3)|(\breve{a}_2|\breve{a}_3)) = ((\breve{a}_1|\breve{a}_2)|(\breve{a}_1|\breve{a}_2))|\breve{a}_3$,
*(S4)* $(\breve{a}_1|((\breve{a}_1|\breve{a}_1)|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|((\breve{a}_1|\breve{a}_1)|(\breve{a}_2|\breve{a}_2))) = \breve{a}_1$.

**Definition 2.2.** [11] *A BH-algebra is an algebra $(A, *, 0)$ of type (2, 0) satisfying the following conditions:*
$(BH.1)$ $\breve{a}_1 * \breve{a}_1 = 0$,
$(BH.2)$ $\breve{a}_1 * \breve{a}_2 = 0$ *and* $\breve{a}_2 * \breve{a}_1 = 0$ *imply* $\breve{a}_1 = \breve{a}_2$,
$(BH.3)$ $\breve{a}_1 * 0 = \breve{a}_1$
*for all* $\breve{a}_1, \breve{a}_2 \in A$.

A BH-algebra is called bounded if it has the greatest element.

**Definition 2.3.** [11] *A nonempty subset $S$ of a BH-algebra $A$ is called a BH-subalgebra if $\breve{a}_1 * \breve{a}_2 \in S$, for all $\breve{a}_1, \breve{a}_2 \in S$.*

**Definition 2.4.** [1] *Let $A$ be a BH-algebra. A nonempty subset $N$ of $A$ is said to be normal if $(\breve{a}_1 * x) * (\breve{a}_2 * y) \in N$, for any $\breve{a}_1 * \breve{a}_2$, $x * y \in N$.*

**Definition 2.5.** [1] *A filter of a BH-algebra $A$ is a non-empty subset $F$ of $A$ satisfying the following conditions:*
$(F_1)$ *If* $\breve{a}_1 \in F$ *and* $\breve{a}_2 \in F$*, then* $\breve{a}_2 * (\breve{a}_2 * \breve{a}_1) \in F$ *and* $\breve{a}_1 * (\breve{a}_1 * \breve{a}_2) \in F$,
$(F_2)$ *If* $\breve{a}_1 \in F$ *and* $\breve{a}_1 * \breve{a}_2 = 0$ *then* $\breve{a}_2 \in F$.

**Definition 2.6.** [6] *A BH-algebra $A$ is called an associative BH-algebra if $(\breve{a}_1 * \breve{a}_2) * \breve{a}_3 = \breve{a}_1 * (\breve{a}_2 * \breve{a}_3)$, for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$.*

**Definition 2.7.** [3] *Let $A$ be a BH-algebra. Then the set $A_+ = \{\breve{a}_1 \in A | 0 * \breve{a}_1 = 0\}$ is called the BCA-part of $A$.*

## 3. Sheffer stroke BH-algebras

In this section, we provide fundamental definitions and concepts regarding a Sheffer stroke and a BH-algebra.

**Definition 3.1.** *A Sheffer stroke BH-algebra is a structure* $(A, |, 0)$ *of type* $(2, 0)$*, where* $0$ *is the constant on* $A$ *and the following axioms hold for all* $\breve{a}_1, \breve{a}_2 \in A$*:*
$(sBH.1)$ $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = 0,$
$(sBH.2)$ $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ *and* $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = 0$ *imply* $\breve{a}_1 = \breve{a}_2$*.*

Let $A$ be a Sheffer stroke BH-algebra unless otherwise stated.

**Lemma 3.1.** *The axioms (sBH.1) and (sBH.2) are independent:*

*Proof.* Consider the groupoid $(\{0, 1\}, |_p)$.
(1) Independence of (sBH.1):

Table 1. Operation table for independence of (sBH.1)

| $\|_p$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 0 | 0 |

Then $|_p$ satisfies (sBH.2) but not (sBH.1) since $(1|_p(1|_p1))|_p(1|_p(1|_p1)) = (0|_p0) = 1 \neq 0$.

(2) Independence of (sBH.2):

Table 2. Operation table for independence of (sBH.2)

| $\|_q$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

Then $|_q$ satisfies (sBH.1) but not (sBH.2) since $(0|_q(1|_q1))|_q(0|_q(1|_q1)) = 0|_q0 = 0$ and $(1|_q(0|_q0))|_q(1|_q(0|_q0)) = 1|_q1 = 0$ but $1 \neq 0$.

□

**Example 3.1.** *Consider a set $A = \{0, x, y, 1\}$, and define a Sheffer stroke $|$ by Table 3 and its Hasse diagram is given in Figure 1.*

TABLE 3

| $\vert$ | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ |
| $x$ | $1$ | $y$ | $1$ | $y$ |
| $y$ | $1$ | $1$ | $x$ | $x$ |
| $1$ | $1$ | $y$ | $x$ | $0$ |

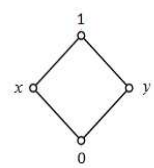


FIGURE 1. Hasse diagram

*Then $(A, |)$ is a Sheffer stroke BH-algebra.*

**Lemma 3.2.** *Let $A$ be a Sheffer stroke BH-algebra. Then the following features hold for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$:*

(1) $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|\breve{a}_1) = \breve{a}_1$,

(2) $(0|0)|(\breve{a}_1|\breve{a}_1) = \breve{a}_1$,

(3) $(\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) = \breve{a}_1$,

(4) $\breve{a}_1|0 = 0|0$,

(5) $\breve{a}_1|((\breve{a}_2|(\breve{a}_3|\breve{a}_3))|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))) = \breve{a}_2|((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))$,

(6) $(\breve{a}_1|((\breve{a}_2|(\breve{a}_3|\breve{a}_3))|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))))|((\breve{a}_2|(\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))) = 0|0$,

(7) $\breve{a}_1|(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2))) = 0|0$,

(8) $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|\breve{a}_1) = 0|0$,

(9) $((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) = 0|0$.

*Proof.* (1) Substituting $[\breve{a}_2 := (\breve{a}_1|\breve{a}_1)]$ in (S2), we obtain

$$(\breve{a}_1|\breve{a}_1)|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = \breve{a}_1.$$

Then we have $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|\breve{a}_1) = \breve{a}_1$ from (S1).

(2) By (sBH.1), (S2) and (1), we obtain

$$\begin{aligned}(0|0)|(\breve{a}_1|\breve{a}_1) &= (((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1))))|(\breve{a}_1|\breve{a}_1)\\ &= (\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|\breve{a}_1)\\ &= \breve{a}_1.\end{aligned}$$

(3) By (S1), (S2) and (2), we have

$$\begin{aligned}(\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) &= (((\breve{a}_1|(\breve{a}_1)|(\breve{a}_1|\breve{a}_1))|(0|0))|((\breve{a}_1|(\breve{a}_1)|(\breve{a}_1|\breve{a}_1))|(0|0)))\\ &= ((0|0)|((\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_1)))|((0|0)|((\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_1)))\\ &= (\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_1)\\ &= \breve{a}_1.\end{aligned}$$

(4) By (S1), (S2) and (2), it is implied that

$$\begin{aligned}\breve{a}_1|0 &= \breve{a}_1|((0|0)|(0|0))\\ &= ((0|0)|(\breve{a}_1|\breve{a}_1))|((0|0)|(0|0))\\ &= ((0|0)|(0|0))|((0|0)|(\breve{a}_1|\breve{a}_1))\\ &= (0|0).\end{aligned}$$

(5) By (S1) and (S3), we have

$$\begin{aligned}\breve{a}_1|((\breve{a}_2|(\breve{a}_3|\breve{a}_3))|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))) &= (((\breve{a}_1|\breve{a}_2)|(\breve{a}_1|\breve{a}_2))|(\breve{a}_3|\breve{a}_3))\\ &= (((\breve{a}_2|\breve{a}_1)|(\breve{a}_2|\breve{a}_1))|(\breve{a}_3|\breve{a}_3))\\ &= \breve{a}_2|((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3))).\end{aligned}$$

(6) It is obtained from (sBH.1) and (5).

(7) In (S3), by substituting $[\breve{a}_2 := \breve{a}_1|(\breve{a}_2|\breve{a}_2)]$ and $[\breve{a}_3 := \breve{a}_2|\breve{a}_2]$ and applying (S1), (S3) and (sBH.1), we obtain

$$\begin{aligned}
\breve{a}_1|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2)) &= \breve{a}_1|(((\breve{a}_2|\breve{a}_2)|\breve{a}_1|(\breve{a}_2|\breve{a}_2))| \\
&\quad ((\breve{a}_2|\breve{a}_2)|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))) \\
&= ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \\
&= (\breve{a}_1|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\
&= 0|0.
\end{aligned}$$

(8) By (S1), (S3), (sBH.1) and (4), we have

$$\begin{aligned}
((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|\breve{a}_1) &= (\breve{a}_1|\breve{a}_1)|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\
&= (((\breve{a}_1|\breve{a}_1)|\breve{a}_1)|((\breve{a}_1|\breve{a}_1)|\breve{a}_1))|(\breve{a}_2|\breve{a}_2) \\
&= ((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|\breve{a}_2) \\
&= 0|(\breve{a}_2|\breve{a}_2) \\
&= 0|0.
\end{aligned}$$

(9) By (S1), (S3) and (sBH.1), we get

$$\begin{aligned}
((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) &= (\breve{a}_2|\breve{a}_2)|((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))| \\
&\quad (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))) \\
&= (((\breve{a}_2|\breve{a}_2)|\breve{a}_1)|((\breve{a}_2|\breve{a}_2)|\breve{a}_1))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \\
&= (\breve{a}_1|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\
&= 0|0.
\end{aligned}$$

□

**Theorem 3.1.** *Let $(A, |, 0)$ be a Sheffer stroke BH-algebra. If we define*

$\breve{a}_1 * \breve{a}_2 := (\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)),$

*then* $(A,*,0)$ *is a BH-algebra.*

*Proof.* By using (sBH.1), (sBH.2), Lemma 3.2 (3), we have

$(BH.1)$ : $\breve{a}_1 * \breve{a}_1 = (\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = 0$.

$(BH.2)$: $\breve{a}_1 * \breve{a}_2 = (\breve{a}_1|(\breve{a}_2|(\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|(\breve{a}_2)) = 0$ and $\breve{a}_2 * \breve{a}_1 = (\breve{a}_2|(\breve{a}_1|(\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|(\breve{a}_1)) = 0$ imply $\breve{a}_1 = \breve{a}_2$.

$(BH.3)$: $\breve{a}_1 * 0 = (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) = \breve{a}_1$.

Then $(A,*,0)$ is a BH-algebra. □

**Example 3.2.** *Consider the Sheffer stroke BH-algebra* $(A,|,0)$ *in Example 3.1 and define the binary operation* $*$ *by Table 4.*

TABLE 4

| $*$ | $0$ | $x$ | $y$ | $1$ |
|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ |
| $x$ | $x$ | $0$ | $x$ | $0$ |
| $y$ | $y$ | $y$ | $0$ | $0$ |
| $1$ | $1$ | $y$ | $x$ | $0$ |

*Then* $(A,*,0)$ *is a BH-algebra.*

**Theorem 3.2.** *Let* $(A,*,0,1)$ *be a bounded BH-algebra. If we define* $\breve{a}_1|\breve{a}_2 = (\breve{a}_1 * \breve{a}_2^0)^0$ *and* $\breve{a}_1^0 = 1 * \breve{a}_1$, *where* $\breve{a}_1 * (1 * \breve{a}_1) = \breve{a}_1$ *and* $1 * (1 * \breve{a}_1) = \breve{a}_1$, *then* $(A,|,0)$ *is a Sheffer stroke BH-algebra.*

*Proof.* $(sBH.1)$: By using (BH.1), we have

$$
\begin{aligned}
(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) &= (\breve{a}_1 * \breve{a}_1)^0|(\breve{a}_1 * \breve{a}_1)^0 \\
&= ((\breve{a}_1 * \breve{a}_1)^0)^0 \\
&= \breve{a}_1 * \breve{a}_1 \\
&= 0.
\end{aligned}
$$

$(sBH.2)$: By using (BH.2), we obtain

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = \breve{a}_1 * \breve{a}_2 = 0$$

and $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = \breve{a}_2 * \breve{a}_1 = 0$ imply $\breve{a}_1 = \breve{a}_2$.

Then $(A,|,0)$ is a Sheffer stroke BH-algebra. □

**Example 3.3.** *Consider a set* $A = \{0, x, y, z, t, u, v, 1\}$*, and define the binary operation by Table 5 and the Sheffer stroke "*$|$*" by Table 6. Then* $(A, *, 0, 1)$ *is a a bounded BH-algebra and* $(A, |, 1)$ *is a Sheffer stroke BH-algebra.*

TABLE 5

| $*$ | $0$ | $x$ | $y$ | $z$ | $t$ | $u$ | $v$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ |
| $x$ | $x$ | $0$ | $x$ | $x$ | $0$ | $0$ | $x$ | $0$ |
| $y$ | $y$ | $y$ | $0$ | $y$ | $0$ | $y$ | $0$ | $0$ |
| $z$ | $z$ | $z$ | $z$ | $0$ | $z$ | $0$ | $0$ | $0$ |
| $t$ | $t$ | $y$ | $x$ | $t$ | $0$ | $y$ | $x$ | $0$ |
| $u$ | $u$ | $z$ | $u$ | $x$ | $z$ | $0$ | $x$ | $0$ |
| $v$ | $v$ | $v$ | $z$ | $y$ | $z$ | $y$ | $0$ | $0$ |
| $1$ | $1$ | $v$ | $u$ | $t$ | $z$ | $y$ | $x$ | $0$ |

TABLE 6

| $\vert$ | $0$ | $x$ | $y$ | $z$ | $t$ | $u$ | $v$ | $1$ |
|---|---|---|---|---|---|---|---|---|
| $0$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ | $1$ |
| $x$ | $1$ | $v$ | $1$ | $1$ | $v$ | $v$ | $1$ | $v$ |
| $y$ | $1$ | $1$ | $u$ | $1$ | $u$ | $1$ | $u$ | $u$ |
| $z$ | $1$ | $1$ | $1$ | $t$ | $1$ | $t$ | $t$ | $t$ |
| $t$ | $1$ | $v$ | $u$ | $1$ | $z$ | $v$ | $u$ | $z$ |
| $u$ | $1$ | $v$ | $1$ | $t$ | $v$ | $y$ | $t$ | $y$ |
| $v$ | $1$ | $1$ | $u$ | $t$ | $u$ | $t$ | $x$ | $x$ |
| $1$ | $1$ | $v$ | $u$ | $t$ | $z$ | $y$ | $x$ | $0$ |

**Theorem 3.3.** *Let* $(A, |_A, 0_A)$ *and* $(B, |_B, 0_B)$ *be Sheffer stroke BH-algebras. Then,* $(A \times B, |_{A\times B}, 0_{A\times B})$ *is a Sheffer stroke BH-algebra where the set* $A \times B$ *is the Cartesian product of* $A$ *and* $B$*, the operation* $|_{A\times B}$ *is defined by* $(\breve{a}_1, b_1)|_{A\times B}(\breve{a}_2, b_2) = (\breve{a}_1|_A\breve{a}_2, b_1|_B b_2)$ *and* $0_{A\times B} = (0_A, 0_B)$.

**Definition 3.2.** *A Sheffer stroke BH-algebra* $A$ *is called an associative Sheffer stroke BH-algebra if* $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|\breve{a}_3) = (\breve{a}_1|(\breve{a}_2|(\breve{a}_3|\breve{a}_3)))$ *holds for all* $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$.

**Theorem 3.4.** *Let A be an associative Sheffer stroke BH-algebra. Then the following properties are hold:*

(1) $(0|(\breve{a}_1|\breve{a}_1))|(0|(\breve{a}_1|\breve{a}_1)) = \breve{a}_1$,

(2) $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = \breve{a}_2$,

(3) $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))$,

(4) $((\breve{a}_3|(\breve{a}_1|\breve{a}_1))|(\breve{a}_3|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) = \breve{a}_1|(\breve{a}_2|\breve{a}_2)$,

(5) $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ *implies* $\breve{a}_1 = \breve{a}_2$,

(6) $((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) = 0|0$,

(7) $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|\breve{a}_3) = ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|\breve{a}_2)$,

(8) $((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(a_4|a_4)) = ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|(a_4|a_4))$.

*Proof.* (1) By Lemma 3.2 (3), (S2) and (sBH.1), we have

$$\begin{aligned}(0|(\breve{a}_1|\breve{a}_1))|(0|(\breve{a}_1|\breve{a}_1)) &= (((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|\breve{a}_1))|(((\breve{a}_1|(\breve{a}_1|\breve{a}_1))| \\ &\quad (\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|\breve{a}_1)) \\ &= (\breve{a}_1|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_1|\breve{a}_1))) \\ &= (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) \\ &= \breve{a}_1.\end{aligned}$$

(2) By (sBH.1) and (1), we obtain

$$\begin{aligned}(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) &= ((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))(\breve{a}_2|\breve{a}_2)| \\ &\quad ((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))(\breve{a}_2|\breve{a}_2) \\ &= (0|(\breve{a}_2|\breve{a}_2))|(0|(\breve{a}_2|\breve{a}_2)) \\ &= \breve{a}_2.\end{aligned}$$

(3) By (S2), (sBH.1) and (2),

$$\begin{aligned}&((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\ &|((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\ &= ((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\ &= (\breve{a}_2|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))) \\ &= (\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2)) \\ &= 0.\end{aligned}$$

Similarly, we get

$$((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) = 0.$$

Therefore, we obtain from (sBH.2) that $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))$.

(4) By (sBH.1), (1) and (3),

$$\begin{aligned}
((\breve{a}_3|(\breve{a}_1|\breve{a}_1))|(\breve{a}_3|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) &= ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \\
&= \breve{a}_1|(\breve{a}_3|(\breve{a}_3|(\breve{a}_2|\breve{a}_2))) \\
&= \breve{a}_1|(((\breve{a}_3|(\breve{a}_3|\breve{a}_3))|(\breve{a}_3|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|\breve{a}_2)) \\
&= \breve{a}_1|(0|(\breve{a}_2|\breve{a}_2)) \\
&= \breve{a}_1|(\breve{a}_2|\breve{a}_2).
\end{aligned}$$

(5) Let $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$. Then $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = 0$ from (3). Thus $\breve{a}_1 = \breve{a}_2$ from (sBH.2).

(6) $((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) = (\breve{a}_2|(\breve{a}_2|\breve{a}_2)) = 0|0$ from (2) and (sBH.1).

(7) By (3), we have

$$\begin{aligned}
((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|\breve{a}_3) &= (\breve{a}_1|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))) \\
&= (\breve{a}_1|(\breve{a}_3|(\breve{a}_2|\breve{a}_2))) \\
&= ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|\breve{a}_2).
\end{aligned}$$

(8) By (3) and (7), we have

$$\begin{aligned}
((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(a_4|a_4)) &= ((\breve{a}_1|(\breve{a}_2|(a_4|a_4)))|(\breve{a}_1|(\breve{a}_2|(a_4|a_4))))|(\breve{a}_3|\breve{a}_3) \\
&= (((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(a_4|a_4)| \\
&\quad ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(a_4|a_4))| \\
&\quad (\breve{a}_3|\breve{a}_3) \\
&= ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(a_4|(\breve{a}_3|\breve{a}_3)) \\
&= ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|(a_4|a_4)).
\end{aligned}$$

□

**Definition 3.3.** *A non-empty subset $S$ of a Sheffer stroke BH-algebra $A$ is called a Sheffer stroke BH-subalgebra of $A$ if $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in S$, for all $\breve{a}_1, \breve{a}_2 \in S$.*

**Example 3.4.** *In Example 3.1,* $S_1 = \{0, x\}$, $S_2 = \{0, y\}$ *and* $S_3 = \{0, x, y\}$ *are subalgebras of* $A$.

**Theorem 3.5.** *Let* $(A, |, 0)$ *be a Sheffer stroke BH-algebra and* $\emptyset \neq S \subseteq A$. *Then the following are equivalent:*

$(a)$ $S$ *is a subalgebra of* $A$,

$(b)$ $(\breve{a}_1|(\breve{a}_2|(0|0)))|(\breve{a}_1|(\breve{a}_2|(0|0))) \in S$ *for any* $\breve{a}_1, \breve{a}_2 \in S$.

*Proof.* $(a) \Rightarrow (b)$: Since $S \neq \emptyset$, there exists an element $\breve{a}_1 \in S$ and

$$0 = (\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) \in S.$$

Since $S$ is closed under $|$, $(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0)) \in S$ and thus
$(\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))))|(\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))))$
$= (\breve{a}_1|(\breve{a}_2|(0|0)))|(\breve{a}_1|(\breve{a}_2|(0|0))) \in S.$

$(b) \Rightarrow (a)$: By using Lemma 3.2 (3), we get $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))))|(\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0)))) = (\breve{a}_1|(\breve{a}_2|(0|0)))|(\breve{a}_1|(\breve{a}_2|(0|0))) \in S$ for any $\breve{a}_1, \breve{a}_2 \in S$. □

**Definition 3.4.** *The set* $A_+ = \{\breve{a}_1 \in A | (0|(\breve{a}_1|\breve{a}_1))|(0|(\breve{a}_1|\breve{a}_1)) = 0\}$ *is called a BCA-part of* $A$.

**Example 3.5.** *Given the Sheffer stroke BH-algebra in Example 3.1. Then it is obvious that the set* $A_+ = \{0, x, y, 1\}$ *is a BCA-part of* $A$.

**Theorem 3.6.** *Let* $A$ *be a Sheffer stroke BH-algebra. Then* $A_+$ *is a subalgebra of* $A$.

*Proof.* Clearly, $0 \in A_+$ and so $A_+$ is nonempty. Let $\breve{a}_1, \breve{a}_2 \in A_+$. By (S2) and Lemma 3.2 (4), we have
$(0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))$
$= (0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))))$
$= (0|0)|(0|0)$
$= 0.$
Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in A_+$ and so $A_+$ is a subalgebra of $A$. □

**Definition 3.5.** *A non-empty subset $N$ of $A$ is said to be normal subset of $A$ if*

$$(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b)))|(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b))) \in N,$$

*for any* $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)), (a|(b|b))|(a|(b|b)) \in N$.

**Example 3.6.** *In Example 3.1,* $N = \{0, x\}$ *is a normal subset of* $A$ *when* $[\breve{a}_1 := 0], [\breve{a}_2 := 1], [a := x]$, *and* $[b := y]$. *Since* $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (0|(1|1))|(0|(1|1)) = 0 \in N$ *and* $(a|(b|b))|(a|(b|b)) = (x|(y|y))|(x|(y|y)) = x \in N$, *we have*
$$(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b)))|(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b)))$$
$$= (((0|(x|x))|(0|(x|x)))|(1|(y|y)))|(((0|(x|x))|(0|(x|x)))|(1|(y|y))) = 1|1 = 0 \in N.$$

**Theorem 3.7.** *Every normal subset $N$ of a Sheffer stroke BH-algebra $A$ is a Sheffer stroke subalgebra of $A$.*

*Proof.* If $\breve{a}_1, \breve{a}_2 \in N$ then $(\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)), (\breve{a}_2|(0|0))|(\breve{a}_2|(0|0)) \in N$. Since $N$ is normal , then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(0|(0|0))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1| (\breve{a}_2|\breve{a}_2)))|(0|(0|0)) \in N$. Therefore, $N$ is a Sheffer stroke subalgebra of $A$. □

**Remark 3.1.** *The converse of Theorem 3.7 does not hold. In Example 3.1,* $N = \{0, x, y\}$ *is a subalgebra of $A$, but it is not normal, since* $(1|(y|y))|(1|(y|y)) = x \in N$, $(0|(1|1))|(0|(1|1)) = 0 \in N$, *while* $(((1|(0|0))|(1|(0|0)))|(y|(1|1)))|(((1|(0|0))|(1|(0|0)))|(y|(1|1))) = 0|0 = 1 \notin N$.

**Proposition 3.1.** *Let $N$ be a Sheffer stroke normal subalgebra of $A$. If* $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ *for* $\breve{a}_1, \breve{a}_2 \in N$, *then* $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) \in N$.

*Proof.* Let $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ . Since $(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2)) = 0 \in N$ and $N$ is a normal subalgebra,
$(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = ((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))| (\breve{a}_2|(\breve{a}_2|\breve{a}_2)))|((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))) \in N$ from (sBH.1) and Lemma 3.2 (3). □

## 4. On Filters of Sheffer stroke BH-algebras

We introduce the notion of filter in a Sheffer stroke BH-algebra in this section.

**Definition 4.1.** *A filter of $A$ is a nonempty subset $F \subseteq A$ satisfying*

$(SF.1)$ *If* $\breve{a}_1, \breve{a}_2 \in F$, *then*

$$(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) \in F$$

*and*

$$(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in F.$$

$(SF.2)$ *If* $\breve{a}_1 \in F$ *and* $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$*, then* $\breve{a}_2 \in F$*.*

**Example 4.1.** *Consider the Sheffer stroke BH-algebra in Example 3.3. Then it is obvious that* $\{t, 1\}$ *is a filter of* $A$*.*

**Theorem 4.1.** *The family* $K_A$ *of all filters in* $A$ *forms a complete lattice.*

*Proof.* Let $\{F_i\}_{i\in I}$ be a family of filters of $A$. If $\breve{a}_1, \breve{a}_2 \in \bigcap_{i\in I} F_i$, then $\breve{a}_1, \breve{a}_2 \in F_i$, for all $i \in I$. Since $F_i$ is a filter of $A$, then $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))$, $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in F_i$. Thus,

$$(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))), (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in \bigcap_{i\in I} F_i.$$

- $(i)$ Suppose that $\breve{a}_1 \in \bigcap_{i\in I} F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in \bigcap_{i\in I} F_i$ hold for $\breve{a}_1, \breve{a}_2 \in A$, that is $\breve{a}_1 \in F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ hold for all $i \in I$. Then it is obtained from (SF.2) that $\breve{a}_2 \in F_i$ for all $i \in I$. Then $\breve{a}_2 \in \bigcap_{i\in I} F_i$.

- $(ii)$ Let $\eta$ be the family of all filters of $A$ containing the union $\bigcup_{i\in I} F_i$. Then $\bigcap \eta$ is a filter of $A$ from (i). If $\bigwedge_{i\in I} F_i = \bigcap_{i\in I} F_i$ and $\bigvee_{i\in I} F_i = \bigcap \eta$, then $(K_A, \bigwedge, \bigvee)$ is a complete lattice.

□

**Corollary 4.1.** *Let* $B$ *be a subset of* $A$*. Then there is the minimal filter* $\langle B\rangle$ *containing the subset* $B$*.*

*Proof.* Let $\varepsilon = \{F : F \textit{ is a filter of } A \textit{ containing } B\}$. Then $\langle B\rangle = \{x \in A : x \in \bigcap_{F\in\varepsilon} F\}$ is the minimal filter of $A$ containing $B$. □

**Theorem 4.2.** *Let* $S$ *be a subalgebra of* $A$*. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

*holds for all* $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$*, then* $S$ *is a filter of* $A$*.*

*Proof.* $(SF.1)$ Let $S$ be a subalgebra of $A$ and $\breve{a}_1, \breve{a}_2 \in S$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in S$ and $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) \in S$. So $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in S$ and $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) \in S$.

$(SF.2)$ Let $\breve{a}_1 \in S$, $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))$. We obtain $\breve{a}_1 = \breve{a}_2$. Thus, $\breve{a}_2 \in S$. Therefore, $S$ is a filter of $A$. □

**Corollary 4.2.** *Let $S$ be a normal subalgebra of $A$. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

*holds for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then $S$ is a filter of $A$.*

*Proof.* It is obtained from Theorem 3.7 and Theorem 4.2. □

**Theorem 4.3.** *Let $A$ be a Sheffer stroke BH-algebra. If*

$$(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = \breve{a}_2$$

*holds for all $\breve{a}_1, \breve{a}_2 \in A$, then every non-empty subset $S$ of $A$ is a filter of $A$.*

*Proof.* Let $S$ be a non-empty subset of $A$.
$(SF.1)$ Let $\breve{a}_1, \breve{a}_2 \in S$. Then $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) = \breve{a}_1 \in S$. Similarly, $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = \breve{a}_2 \in S$.
$(SF.2)$ Let $\breve{a}_1 \in S$, $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$. Then $\breve{a}_2 = (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) = \breve{a}_1$ from Lemma 3.2 (3). Thus, $\breve{a}_2 \in S$. Therefore, $S$ is a filter of $A$. □

**Proposition 4.1.** *Let $\{Fi, i \in \lambda\}$ be a family of Sheffer stroke BH-filters of $A$. Then $\bigcap_{i\in\lambda} F_i$ is a Sheffer stroke BH-filter of $A$.*

*Proof.* Let $\{F_i, i \in \lambda\}$ be a family of Sheffer stroke BH-filters of $A$.
$(SF.1)$ If $\breve{a}_1, \breve{a}_2 \in \bigcap_{i\in\lambda} F_i$, then $\breve{a}_1, \breve{a}_2 \in F_i$, for all $i \in \lambda$. Since $F_i$ is a filter of $A$, we have

$$(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))), (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in F_i.$$

Then $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))), (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in \bigcap_{i\in\lambda} F_i$.
$(SF.2)$ Suppose that $\breve{a}_1 \in \bigcap_{i\in\lambda} F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in \bigcap_{i\in\lambda} F_i$ hold for $\breve{a}_1, \breve{a}_2 \in A$, that is $\breve{a}_1 \in F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ hold for all $i \in \lambda$. Then $\breve{a}_2 \in F_i$ for all $i \in \lambda$. Then $\breve{a}_2 \in \bigcap_{i\in\lambda} F_i$. Therefore $\bigcap_{i\in\lambda} F_i$ is a Sheffer stroke BH-filter of $A$. □

The union of Sheffer stroke BH-filters of Sheffer stroke BH-algebras may not be a Sheffer stroke BH-filter as in the following example.

**Example 4.2.** *Consider the Sheffer stroke BH-algebra in Example 3.3. Then it is obvious that $F_1 = \{t, 1\}$ and $F_2 = \{u, 1\}$ are two filters of $A$. The union of that filters is not a Sheffer stroke BH-filter of $A$. Since $u, t \in F_1 \bigcup F_2$, but $(u|(u|(t|t)))|(u|(u|(t|t))) = x \notin F_1 \bigcup F_2$.*

**Proposition 4.2.** *Let $\{F_i, i \in \lambda\}$ be a chain of Sheffer stroke BH-filters of $A$. Then $\bigcup_{i\in\lambda} F_i$ is a Sheffer stroke BH-filter of $A$.*

## 5. Homomorphisms on Sheffer stroke BH-algebras

In this section, we present some definitions and concepts about homomorphism between Sheffer stroke BH-algebras.

**Definition 5.1.** *Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras. A mapping $f : A \longrightarrow B$ is called a homomorphism if*

$$f(\breve{a}_1|_A\breve{a}_2) = f(\breve{a}_1)|_B f(\breve{a}_2),$$

*for all $\breve{a}_1, \breve{a}_2 \in A$.*

A Sheffer stroke BH-homomorphism $f$ is called a Sheffer stroke BH-monomorphism if it is injective.

**Lemma 5.1.** *Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a monomorphism. Then if $F$ is a filter of $A$, $f(F)$ is a filter of $B$.*

*Proof.* Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a monomorphism.

- $(i)$ Let $F$ be a Sheffer stroke BH-filter of $A$ and $\breve{a}_1, \breve{a}_2 \in f(F)$. Then there exist $x, y \in F$ such that $\breve{a}_1 = f(x), \breve{a}_2 = f(y)$. Since $F$ is a filter, then $(\breve{a}_2|_B(\breve{a}_2|_B(\breve{a}_1|_B\breve{a}_1)))|_B(\breve{a}_2|_B (\breve{a}_2|_B(\breve{a}_1|_B\breve{a}_1))) = (f(y)|_B(f(y)|_B(f(x)|_B f(x))))|_B(f(y)|_B(f(y)|_B(f(x)|_B f(x)))) = f((y|_A(y|_A(x|_A x)))|_A(y|_A(y|_A(x|_A x)))) \in f(F)$. Hence $(\breve{a}_2|_B(\breve{a}_2|_B(\breve{a}_1|_B\breve{a}_1)))|_B(\breve{a}_2|_B(\breve{a}_2 |_B(\breve{a}_1|_B\breve{a}_1))) \in f(F)$. Similarly, $(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2)))|_B(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))) \in f(F)$.
- $(ii)$ Let $\breve{a}_1 \in f(F)$ such that $(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2)) = 0_B$. Then there exist $x, y \in F$ such that $\breve{a}_1 = f(x)$ and $\breve{a}_2 = f(y)$. In this case, we have $(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))) |_B(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))) = (f(x)|_B(f(y)|_B f(y)))|_B(f(x)|_B(f(y)|_B f(y))) = f((x|_A(y|_A y) )|_A(x|_A(y|_A y))) = 0_B = f(0_A)$. Since $f$ is an injective, then $((x|_A(y|_A y))|_A(x|_A(y|_A y))) = 0_A$. Thus $y \in F$. So, $\breve{a}_2 = f(y) \in f(F)$. Therefore, $f(F)$ is a filter of $B$.

□

**Theorem 5.1.** *Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a homomorphism. If $F$ is a filter of $B$, then $f^{-1}(F)$ is a filter of $A$.*

*Proof.* Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a homomorphism. Suppose that $F$ is a filter of $A$.

• Let $\breve{a}_1, \breve{a}_2 \in f^{-1}(F)$. Then $f(\breve{a}_1), f(\breve{a}_2) \in F$. Since $F$ is a filter, then $(f(\breve{a}_2)|_B((f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1)))))|_B(f(\breve{a}_2)|_B((f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1))) = f((\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))) \in F$. Therefore, $(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1))) \in f^{-1}(F)$. Similarly, $(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)))|_A(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) \in f^{-1}(F)$.

• Let $\breve{a}_1 \in f^{-1}(F)$ such that $(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)) = 0_A$. Then $f(\breve{a}_1) \in F$ and $f((\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) = (f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2)))|_B(f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2))) = f(0_A) = 0_B$. Hence $f(\breve{a}_2) \in F$. Thus $\breve{a}_2 \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a filter of $A$. □

**Proposition 5.1.** *Let $A$ and $B$ be Sheffer stroke BH-algebras and $f : (A, |_A, 0_A) \to (B, |_B, 0_B)$ be a Sheffer stroke BH-homomorphism. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \text{ implies } \breve{a}_1 = \breve{a}_3$$

*holds for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then $Ker(f)$ is a Sheffer stroke BH-filter of $A$.*

*Proof.* $(SF.1)$ Let $\breve{a}_1, \breve{a}_2 \in Ker(f)$. Then since $f(\breve{a}_1) = 0_B$ and $f(\breve{a}_2) = 0_B$, we have $f((\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))) = (f(\breve{a}_2)|_B(f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1)))|_B(f(\breve{a}_2)|_B(f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1))) = 0_B$. Hence $(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1))) \in Ker(f)$. Similarly, $(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)))|_A(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) \in Ker(f)$.

$(SF.2)$ Let $\breve{a}_1 \in Ker(f)$ and $\breve{a}_2 \in A$ such that $(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)) = 0_A$. Then $f(\breve{a}_1) = 0_B$ and $f((\breve{a}_1)|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) = f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2)))|_B f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2))) = f(0_A)$. We get $(0_B|_B(f(\breve{a}_2)|_B(f(\breve{a}_2))))|_B(0_B|_B(f(\breve{a}_2)|_B(f(\breve{a}_2)))) = (f(\breve{a}_2)|_B(f(\breve{a}_2)|_B(f(\breve{a}_2))))|_B(f(\breve{a}_2)|_B(f(\breve{a}_2)|_B(f(\breve{a}_2))))$. We obtain $f(\breve{a}_2) = 0_B$. Thus $\breve{a}_2 \in Ker(f)$. Therefore, $Ker(f)$ is a filter of $A$. □

**Lemma 5.2.** *Let $N$ be a normal subalgebra of $A$. Define a relation $\sim_N$ on $A$ by $\breve{a}_1 \sim_N \breve{a}_2$ if and only if $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$, where $\breve{a}_1, \breve{a}_2 \in A$. Then $\sim_N$ is an equivalence relation on $A$.*

*Proof.* ▪ Reflexive: Since $0 \in A$, we have $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = 0 \in A$ i.e., $\breve{a}_1 \sim_N \breve{a}_1$ for any $\breve{a}_1 \in A$. This means that $\sim_N$ is reflexive.

▪ Symmetric: Let $\breve{a}_1 \sim_N \breve{a}_2$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ and $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) \in N$ by Proposition 3.1. We obtain $\breve{a}_2 \sim_N \breve{a}_1$ for any $\breve{a}_1, \breve{a}_2 \in A$.

▪ Transitive: Let $\breve{a}_1 \sim_N \breve{a}_2$ and $\breve{a}_2 \sim_N \breve{a}_3$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ and $(\breve{a}_2|(\breve{a}_3|\breve{a}_3))|$

$(\breve{a}_2|(\breve{a}_3|\breve{a}_3)) \in N$. By Proposition 3.1, we have $(\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \in N$. Since $N$ is a normal subalgebra, then $(((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2)))|(((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))) = (((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(0|0))|(((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(0|0)) = ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)) \in N$. We obtain $\breve{a}_1 \sim_N \breve{a}_3$ for any $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$. □

**Lemma 5.3.** *An equivalence relation* $\sim_*$ *is a congruence relation if and only if* $\breve{a}_1 \sim_* \breve{a}_2$ *and* $x \sim_* y$ *imply* $(\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)) \sim_* (\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))$.

**Lemma 5.4.** *Let* $N$ *be a normal subalgebra of* $A$ *and the binary relation defined as Lemma 5.2. Then* $\sim_N$ *is a congruence relation on* $A$.

*Proof.* Let $x, y, \breve{a}_1, \breve{a}_2$ be any elements in $A$ such that $\breve{a}_1 \sim_N \breve{a}_2$ and $x \sim_N y$, i.e., $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ and $(x|(y|y))|(x|(y|y)) \in N$. Since $N$ is a normal subalgebra, we get $(((\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y)))|(((\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y))) \in N$. Then $(\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)) \sim_* (\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))$. Therefore, $\sim_N$ is a congruence relation on $A$. □

Denote the equivalence class containing $\breve{a}_1$ by $[\breve{a}_1]_N$, i.e., $[\breve{a}_1]_N = \{\breve{a}_2 \in N \mid \breve{a}_1 \sim_N \breve{a}_2\}$ and $A/N = \{[\breve{a}_1]_N \mid \breve{a}_1 \in A\}$.

**Theorem 5.2.** *Let* $N$ *be a normal subalgebra of* $A$*. Then* $(A/N, |, [0]_N)$ *is a Sheffer stroke BH-algebra.*

*Proof.* If we define $[\breve{a}_1]_N|[\breve{a}_2]_N := [\breve{a}_1|\breve{a}_2]_N$, then the operation $|$ is well-defined, since if $\breve{a}_1 \sim_N p$ and $\breve{a}_2 \sim_N q$, then $(\breve{a}_1|(p|p))|(\breve{a}_1|(p|p)) \in N$ and $(\breve{a}_2|(q|q))|(\breve{a}_2|(q|q)) \in N$ implies $(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(p|(q|q)))|(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(p|(q|q))) \in N$ by normality of $N$. Then we have $(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \sim_N (((p|(q|q))|(p|(q|q)))$ and so $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))_N = (((p|(q|q))|(p|(q|q))))_N$. Note that $[0]_N = \{\breve{a}_1 \in A \mid \breve{a}_1 \sim_N 0\} = \{\breve{a}_1 \in A \mid (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) \in N\} = \{\breve{a}_1 \in A \mid \breve{a}_1 \in N\} = N$. □

$(sBH.1)$ $([\breve{a}_1]_N|([\breve{a}_1]_N|[\breve{a}_1]_N)|([\breve{a}_1]_N|([\breve{a}_1]_N|[\breve{a}_1]_N) = [0]_N$ ,
$(sBH.2)$ $([\breve{a}_1]_N|([\breve{a}_2]_N|[\breve{a}_2]_N)|([\breve{a}_1]_N|([\breve{a}_2]_N|[\breve{a}_2]_N) = [0]_N$ and $([\breve{a}_2]_N|([\breve{a}_1]_N|[\breve{a}_1]_N)|([\breve{a}_2]_N|([\breve{a}_1]_N|[\breve{a}_1]_N) = [0]_N$ imply $[\breve{a}_1]_N = [\breve{a}_2]_N$.

The Sheffer stroke BH-algebra $A/N$ discussed in Theorem 5.2 is called the quotient Sheffer stroke BH-algebra of $A$ by $N$.

**Example 5.1.** *Consider the Sheffer stroke BH-algebra in Example 3.3. For the normal subalgebra* $F = \{0, t\}$ *of* $A$*,* $\beta_F = \{(0,0), (x,x), (y,y), (z,z), (t,t), (u,u), (v,v), (1,1), (0,t), (t,0), (z,$

*1), (1, z)} is a congruence on A defined by F. Then* $(A/F, |_{\beta_F}, [0]_{\beta_F})$ *is a Sheffer stroke BH-algebra with the following Hasse diagram in which the quotient set is* $A/F = \{[0]_{\beta_F}, [x]_{\beta_F}, [y]_{\beta_F}, [1]_{\beta_F}\}$*:*

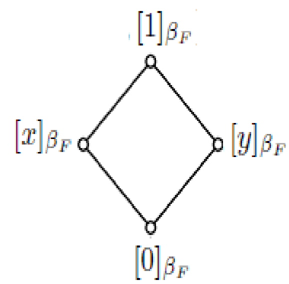


FIGURE 2. Hasse diagram

*The binary operation* $|_{\beta_F}$ *on* $A/F$ *has Cayley table in Table 7.*

TABLE 7

| $\vert_{\beta_F}$ | $[0]_{\beta_F}$ | $[x]_{\beta_F}$ | $[y]_{\beta_F}$ | $[1]_{\beta_F}$ |
|---|---|---|---|---|
| $[0]_{\beta_F}$ | $[1]_{\beta_F}$ | $[1]_{\beta_F}$ | $[1]_{\beta_F}$ | $[1]_{\beta_F}$ |
| $[x]_{\beta_F}$ | $[1]_{\beta_F}$ | $[y]_{\beta_F}$ | $[1]_{\beta_F}$ | $[y]_{\beta_F}$ |
| $[y]_{\beta_F}$ | $[1]_{\beta_F}$ | $[1]_{\beta_F}$ | $[x]_{\beta_F}$ | $[x]_{\beta_F}$ |
| $[1]_{\beta_F}$ | $[1]_{\beta_F}$ | $[y]_{\beta_F}$ | $[x]_{\beta_F}$ | $[0]_{\beta_F}$ |

**Theorem 5.3.** *Let N be a normal subalgebra of A. Then* $[0]_N$ *is a normal subalgebra of A.*

*Proof.* Since $0_A \in [0]_N$, $[0]_N$ is non-empty. Let $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)), (\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2)) \in [0]_N$. Then $(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = 0 \in [0]_N$. By Theorem 3.7, $[0]_N$ is a normal subalgebra of $A$. □

**Theorem 5.4.** *Let N be a filter of A and* $(A/N, |', [0]_N)$ *be a Sheffer stroke BH-algebra. If F is a filter of A such that* $N \subseteq F$*, then* $F/N$ *is a Sheffer stroke BH-filter.*

*Proof.* Let $F$ be a Sheffer stroke BH-filter of $A$.

• Let $[\breve{a}_1]_N, [\breve{a}_2]_N \in F/N$, then $([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N)))|'([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N))) = [(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))]_N$. Hence $([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N)))|'([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N))) \in F/N$. (Since $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) \in F$, F is a filter of $A$).Similarly,

$([\breve{a}_1]_N|'([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N)))|'([\breve{a}_1]_N|'([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N))) \in F/N$.

- Let $\breve{a}_1 \in F/N$ such that $([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N))|'([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N)) = [0]_N$. Then $[(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))]_N = [0]_N$. Hence $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(0|0)|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(0|0) \in N$. Since $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in N$, we have $\breve{a}_2 \in F/N$. We obtain $[\breve{a}_2]_N = [\breve{a}_1]_N$, then $[\breve{a}_2]_N \in F/N$. Therefore, $F/N$ is a filter of $A$. □

**Theorem 5.5.** *Let $N$ be a normal subalgebra of $A$. Then the mapping $\gamma : A \to A/N$ given by $\gamma(\breve{a}_1) := [\breve{a}_1]_N$ is a surjective Sheffer stroke BH-homomorphism and $Ker\gamma = N$.*

The mapping $\gamma$ discussed in above theorem is called the natural(or canonical) homomorphism of $A$ onto $A/N$.

**Theorem 5.6.** *Let $\varphi : A \to B$ be a Sheffer stroke BH-homomorphism. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

*hold for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then $\varphi$ is injective if and only if $Ker\varphi = \{0_A\}$.*

*Proof.* Let $\breve{a}_1, \breve{a}_2 \in A$ with $\varphi(\breve{a}_1) = \varphi(\breve{a}_2)$. Then from (sBH.1), we obtain $(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2)))|(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2))) = 0_B$. So $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in Ker\varphi$. Since $Ker\varphi = \{0_A\}$, $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0_A$=$(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))$. Then $\breve{a}_1 = \breve{a}_2$. Hence $\varphi$ is injective. The converse is trivial. □

**Theorem 5.7.** *Let $\varphi : A \to B$ be a Sheffer stroke BH-homomorphism. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

*hold for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then $Ker\varphi$ is a normal subalgebra of $A$.*

*Proof.* Since $0_A \in Ker\varphi$, Ker $\varphi \neq \emptyset$. Let $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)), (x|(y|y))|(x|(y|y)) \in Ker\varphi$. Then $(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2)))|(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2))) = 0 = (\varphi(x)|(\varphi(y)|\varphi(y)))|(\varphi(x)|(\varphi(y)|\varphi(y)))$. Since $\varphi(\breve{a}_1) = \varphi(\breve{a}_2)$ and $\varphi(x) = \varphi(y)$, we obtain $\varphi((((\breve{a}_1)|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y)))|((((\breve{a}_1)|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y)))$=$(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(\varphi(\breve{a}_2)|(\varphi(y)|\varphi(y))))|(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(\varphi(\breve{a}_2)|(\varphi(y)|\varphi(y))))$=$(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))) = 0$. Then we have $((((\breve{a}_1)|(x|x))|(\breve{a}_1|(x|x)))|((\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))|(\breve{a}_2|(y|y)))) \in Ker\varphi$. Hence $Ker\varphi$ is a normal subalgebra of $A$. □

By Theorem 5.5 and 5.7, if $\varphi : A \to B$ is a Sheffer stroke BH-homomorphism, then $A/Ker\varphi$ is a Sheffer stroke BH-algebra.

**Theorem 5.8.** *Let $\varphi : A \to B$ be a Sheffer stroke BH-homomorphism. Then $A/Ker\varphi \cong Im\varphi$. In particular, if $\varphi$ is surjective, then $A/Ker \cong B$.*

**Theorem 5.9.** *Let $N$ and $K$ be normal subalgebra of $A$, and $K \subseteq N$. Then $A/N \cong (A/K)/(N/K)$.*

**Theorem 5.10.** *Let $A$, $B$ and $C$ be Sheffer stroke BH-algebras, and $h : A \to B$ be a Sheffer stroke BH-epimorphism and $g : A \to C$ be a Sheffer stroke BH-homomorphism. If $Ker(h) \subseteq Ker(g)$, then there exists a unique Sheffer stroke BH-homomorphism $f : A \to B$ satisfying $f \circ h = g$.*

**Theorem 5.11.** *Let $A$, $B$ and $C$ be Sheffer stroke BH-algebras, and $h : B \to C$ be a Sheffer stroke BH-homomorphism and $g : A \to C$ be a Sheffer stroke BH-monomorphism. If $Im(g) \subseteq Im(h)$, then there exists a unique Sheffer stroke BH-homomorphism $f : A \to B$ satisfying $h \circ f = g$.*

*Proof.* For each $\breve{a}_1 \in A, g(\breve{a}_1) \in Im(g) \subseteq Im(h)$. Since $h$ is a Sheffer stroke BH-monomorphism, there exists a unique $\breve{a}_2 \in B$ such that $h(\breve{a}_2) = g(\breve{a}_1)$. Define a map $f : A \to B$ by $f(\breve{a}_1) = \breve{a}_2$. Then $h \circ f = g$. Let $\breve{a}_3, a_4 \in A$, then $g((\breve{a}_3|(a_4|a_4))|(\breve{a}_3|(a_4|a_4))) = h(f((\breve{a}_3|(a_4|a_4))|(\breve{a}_3|(a_4|a_4))))$. Since h is injective, $f((\breve{a}_3|(a_4|a_4))|(\breve{a}_3|(a_4|a_4))) = f(\breve{a}_3|(a_4|a_4))$ $|f((\breve{a}_3|(a_4|a_4)))$=$f(\breve{a}_3|f(a_4)|f(a_4))|f((\breve{a}_3)|f(a_4)|f(a_4))$. Therefore, $f$ is a Sheffer stroke BH-homomorphism. The uniqueness of $f$ follows from the fact that $h$ is a Sheffer stroke BH-monomorphism. □

**Theorem 5.12.** *Let $A$ and $B$ be Sheffer stroke BH-algebras and $f : A \to B$ be a Sheffer stroke BH-homomorphism. If $N$ is a normal subalgebra of $A$ such that $N \subseteq Ker(f)$, then $\bar{f} : A/N \to B$ defined by $\bar{f}([\breve{a}_1]_N) := f(\breve{a}_1)$ for all $\breve{a}_1 \in A$ is a unique Sheffer stroke BH-homomorphism such that $\bar{f} \circ \gamma = f$ where $\gamma : A \to A/N$ is natural Sheffer stroke BH-homomorphism.*

**Corollary 5.1.** *Let $A$ and $B$ be Sheffer stroke BH-algebras and $f : A \to B$ be a Sheffer stroke BH-homomorphism. If $N$ is a normal subalgebra of $A$ such that $N \subseteq Ker(f)$, then the following are equivalent:*
*$(i)$ there exists a unique Sheffer stroke BH-homomorphism $\bar{f} : A/N \to B$ such that $\bar{f} \circ \gamma = f$ where $\gamma : A \to A/N$ is the natural Sheffer stroke BH-homomorphism;*
*$(ii)$ $N \subseteq Ker(f)$.*
*Furthermore, $\bar{f}$ is a Sheffer stroke BH-monomorphism if and only if $N = Ker(f)$.*

*Proof.* $(ii) \Rightarrow (i)$: It is obtained from Theorem 5.11.
$(i) \Rightarrow (ii)$: If $\breve{a}_1 \in N$, then $f(\breve{a}_1) = (\bar{f} \circ \gamma)(\breve{a}_1) = \bar{f}([\breve{a}_1]_N) = \bar{f}([0]_N) = f(0) = 0$. Thus, $\breve{a}_1 \in Ker(f)$.
Furthermore, $\bar{f}$ is a monomorphism if and only if $Ker\bar{f} = \{N\}$ if and only if $f(\breve{a}_1) = 0$ implies $[\breve{a}_1]_N = [0]_N = N$ if and only if $Ker(f) \subseteq N$. □

## 6. Conclusion

In this study, we have given a Sheffer Stroke BH-algebra, and study a Cartesian product, a filter, a homomorphism between Sheffer stroke BH-algebras, kernel and many features in Sheffer stroke BH-algebras. After giving basic definitions and concepts about Sheffer stroke operation and a BH-algebra, we describe a Sheffer stroke BH-algebra and present basic notions about this algebraic structure. We show that a Sheffer stroke BH-algebra is a BH-algebra and that a Cartesian product of two Sheffer stroke BH-algebras is a Sheffer Stroke BH-algebra. After defining a subalgebra and a normal subset, we introduce the relation between a subalgebra and a normal subset on Sheffer stroke BH-algebra. We define a filter of a Sheffer stroke BH-algebra. Finally, a homomorphism between two Sheffer stroke BH-algebras is described and it is stated that mentioned notions are preserved under this homomorphism. It is shown that a kernel of a homomorphism is a filter of Sheffer stroke BH-algebra under one condition.

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# ON $e^*$-TOPOLOGICAL RINGS

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Abstract. The main purpose of this manuscript is to introduce the concept of $e^*$-topological ring. This class appears as a generalised version of the class of $\beta$-topological rings. In addition, we have discussed the relation between the concept of $e^*$-topological ring and some other types of topological rings existing in the literature. Also, some fundamental results about $e^*$-topological rings are revealed. Moreover, we give some counterexamples regarding our results.

**Keywords**: topological ring, $e^*$-open, $e^*$-topological ring, $e^*$-continuous

**2010 Mathematics Subject Classification**: 54H13

## 1. Introduction

In topology, it is sometimes necessary to use algebra to find solutions to some problems, such as determining whether two topological spaces are homeomorphic. For instance, if the fundamental groups of two topological spaces are not isomorphic, then the topological spaces can not be homeomorphic. Thanks to fundamental groups of topological spaces, we can decide that two topological spaces are not homeomorphic but not all. This situation leads to the definition of different concepts in the related field. One of these concepts is the concept of topological ring. To better understand topological rings, the concept of topological groups should be well known. A topological group is a group $\mathbb{X}$ that is also a topological space such that the addition and the inversion are continuous as functions $\psi : \mathbb{X} \to \mathbb{X}, x \mapsto -x$ and

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$\varphi : \mathbb{X} \times \mathbb{X} \to \mathbb{X}, (x, y) \mapsto x + y$, where $\mathbb{X} \times \mathbb{X}$ carries the product topology. The concept of topological ring was first introduced in [4, 5] by Kaplansky. A topological ring is a ring $\mathbb{X}$ that is also a topological space such that both the addition and the multiplication are continuous as functions $\varphi : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, where $\mathbb{X} \times \mathbb{X}$ carries the product topology. That means $\mathbb{X}$ is an additive topological group and a multiplicative topological semigroup.

The types of open sets in the literature such as $\alpha$-open [9], semi-open [8], pre-open [10], $\beta$-open [1], etc. allow a generalization of the notion of topological ring. Studying the features of these generalised versions and investigating their relations with topological rings are just some of the different advances in the literature. Some of the recent advancements in this direction are $\beta$-topological rings [2], irresolute topological rings [12] and $\alpha$-irresolute topological rings [11].

In 2021, Billawaria et al. studied $\beta$-topological ring which is a more general notion than the notion of topological ring [2]. They have revealed some fundamental properties of $\beta$-topological rings. Also, the authors gave some other useful results on $\beta$-topological rings.

In this paper, we introduce the notion of $e^*$-topological ring by utilizing $e^*$-open sets defined by Ekici in [3]. Also, we obtain some of its fundamental properties. Moreover, we compare between this notion and some notions existing in the literature. In addition, we give some counterexamples regarding our results obtained in the scope of this study. Furthermore, we provide an example of $e^*$-topological ring which is not a $\beta$-topological ring.

## 2. Preliminaries

Throughout this paper, $(\mathbb{X}, \mu)$ and $(\mathbb{Y}, \rho)$ (or briefly $\mathbb{X}$ and $\mathbb{Y}$) always mean topological spaces. For a subset $E$ of a topological space $\mathbb{X}$, the interior of $E$ and the closure of $E$ are denoted by $int(E)$ and $cl(E)$, respectively. The family of all open (resp. closed) sets of $\mathbb{X}$ will be denoted by $O(\mathbb{X})$ (resp. $C(\mathbb{X})$). In addition, the family of all open sets of $\mathbb{X}$ containing a point $a$ of $\mathbb{X}$ is denoted by $O(\mathbb{X}, a)$. Recall that a subset $E$ of a space $\mathbb{X}$ is called regular open [13] (resp. regular closed [13]) if $E = int(cl(E))$ (resp. $E = cl(int(E))$). The family of all regular open subsets of $\mathbb{X}$ is denoted by $RO(\mathbb{X})$. The family of all regular open sets of $\mathbb{X}$ containing a point $a$ of $\mathbb{X}$ is denoted by $RO(\mathbb{X}, a)$.

The union of all regular open sets of $\mathbb{X}$ contained in $E$ is called the $\delta$-interior [14] of $E$ and is denoted by $\delta$-$int(E)$. A subset $E$ of a space $\mathbb{X}$ is said to be $\delta$-open [14] if $A = \delta$-$int(A)$. Also, a subset $E$ of a space $\mathbb{X}$ is said to be $\delta$-closed if its complement is $\delta$-open. The intersection of all regular closed sets of $\mathbb{X}$ containing $E$ is called the $\delta$-closure [14] of $E$ and is denoted by $\delta$-$cl(E)$.

A subset $E$ of a space $\mathbb{X}$ is called $e^*$-open if $E \subseteq cl(int(\delta\text{-}cl(E)))$. The complement of an $e^*$-open set is called $e^*$-closed. The intersection of all $e^*$-closed sets of $\mathbb{X}$ containing $E$ is called the $e^*$-closure of $E$ and is denoted by $e^*$-$cl(E)$. Dually, the union of all $e^*$-open sets of $\mathbb{X}$ contained in $E$ is called the $e^*$-interior of $E$ and is denoted by $e^*$-$int(E)$. The family of all $e^*$-open subsets (resp. $e^*$-closed) $\mathbb{X}$ denoted by $e^*O(\mathbb{X})$ (resp. $e^*C(\mathbb{X})$). The family of all $e^*$-open (resp. $e^*$-closed) sets of $\mathbb{X}$ containing a point $a$ of $\mathbb{X}$ denoted by $e^*O(\mathbb{X}, a)$ (resp. $e^*C(\mathbb{X}, a)$).

**Definition 2.1.** [4] *Let $(\mathbb{X}, +, \cdot)$ be a ring and $\mu$ be a topology on $\mathbb{X}$. The quadruple $(\mathbb{X}, +, \cdot, \mu)$ is called a topological ring if the following three conditions hold:*

*i) For every $a, b \in \mathbb{X}$ and every open set $M \in O(\mathbb{X}, a + b)$, there exist $K \in O(\mathbb{X}, a)$ and $L \in O(\mathbb{X}, b)$ such that $K + L \subseteq M$,*

*ii) For every $a \in \mathbb{X}$ and every $L \in O(\mathbb{X}, -a)$, there exists $K \in O(\mathbb{X}, a)$ such that $-K \subseteq L$,*

*iii) For every $a, b \in \mathbb{X}$ and every $M \in O(\mathbb{X}, ab)$, there exist $K \in O(\mathbb{X}, a)$ and $L \in O(\mathbb{X}, b)$ such that $KL \subseteq M$.*

**Definition 2.2.** [2] *Let $(\mathbb{X}, +, \cdot)$ be a ring and $\mu$ be a topology on $\mathbb{X}$. The quadruple $(\mathbb{X}, +, \cdot, \mu)$ is called an $\beta$-topological ring if the following three conditions hold:*

*i) For every $a, b \in \mathbb{X}$ and every $M \in O(\mathbb{X}, a+b)$, there exist $K \in \beta O(\mathbb{X}, a)$ and $L \in \beta O(\mathbb{X}, b)$ such that $K + L \subseteq M$,*

*ii) For every $a \in \mathbb{X}$ and every $L \in O(\mathbb{X}, -a)$, there exists $K \in \beta O(\mathbb{X}, a)$ such that $-K \subseteq L$,*

*iii) For every $a, b \in \mathbb{X}$ and every $M \in O(\mathbb{X}, ab)$, there exist $K \in \beta O(\mathbb{X}, a)$ and $L \in \beta O(\mathbb{X}, b)$ such that $KL \subseteq M$.*

**Definition 2.3.** [3] *A function $f : (\mathbb{X}, \mu) \to (\mathbb{Y}, \rho)$ is said to be $e^*$-continuous if $f^{-1}[G] \in e^*O(\mathbb{X})$ for every $G \in O(\mathbb{Y})$.*

**Lemma 2.1.** [3] *A function $f : (\mathbb{X}, \mu) \to (\mathbb{Y}, \rho)$ is $e^*$-continuous if and only if for every $a \in \mathbb{X}$ and for every $H \in O(\mathbb{Y}, f(a))$, there exists $G \in e^*O(\mathbb{X}, a)$ such that $f[G] \subseteq H$.*

**Definition 2.4.** [3] *Let $(\mathbb{X}, \mu)$ be a topological space and $E \subseteq \mathbb{X}$. Then, the following statements hold:*

*a) $E$ is $e^*$-open if and only if $E = e^*$-$int(E)$,*

*b) $E$ is $e^*$-closed if and only if $E = e^*$-$cl(E)$.*

**Lemma 2.2.** *Let $(\mathbb{X},\mu)$ and $(\mathbb{Y},\rho)$ be two topological spaces. If $E\in e^*O(\mathbb{X})$ and $F\in e^*O(\mathbb{Y})$, then $E\times F\in e^*O(\mathbb{X}\times\mathbb{Y},\mu\star\rho)$.*

*Proof.* Let $E\in e^*O(\mathbb{X})$ and $F\in e^*O(\mathbb{Y})$.

$$\left.\begin{array}{l} E\in e^*O(\mathbb{X})\Rightarrow E\subseteq cl(int(\delta\text{-}cl(E))) \\ F\in e^*O(\mathbb{Y})\Rightarrow F\subseteq cl(int(\delta\text{-}cl(F))) \end{array}\right\}\Rightarrow E\times F\subseteq cl(int(\delta\text{-}cl(E)))\times cl(int(\delta\text{-}cl(F)))$$

$$\begin{aligned} \Rightarrow E\times F &\subseteq cl(int(\delta\text{-}cl(E)))\times cl(int(\delta\text{-}cl(F))) \\ &= cl[int(\delta\text{-}cl(E))\times int(\delta\text{-}cl(F))] \\ &= cl(int[\delta\text{-}cl(E)\times\delta\text{-}cl(F)]) \\ &= cl(int(\delta\text{-}cl(E\times F))) \end{aligned}$$

This means $E\times F\in e^*O(\mathbb{X}\times\mathbb{Y})$. □

## 3. $e^*$-Topological Rings

Now, we introduce and study the concept of $e^*$-topological ring by utilizing $e^*$-open sets.

**Definition 3.1.** *Let $(\mathbb{X},+,\cdot)$ be a ring and $\mu$ be a topology on $\mathbb{X}$. The quadruple $(\mathbb{X},+,\cdot,\mu)$ is called an e*-topological ring if the following three conditions hold:*

*i) For every $a,b\in\mathbb{X}$ and every open set $M\in O(\mathbb{X},a+b)$, there exist $K\in e^*O(\mathbb{X},a)$ and $L\in e^*O(\mathbb{X},b)$ such that $K+L\subseteq M$,*

*ii) For every $a\in\mathbb{X}$ and every open set $L\in O(\mathbb{X},-a)$, there exists $K\in e^*O(\mathbb{X},a)$ such that $-K\subseteq L$,*

*iii) For every $a,b\in\mathbb{X}$ and every open set $M\in O(\mathbb{X},ab)$, there exist $K\in e^*O(\mathbb{X},a)$ and $L\in e^*O(\mathbb{X},b)$ such that $KL\subseteq M$.*

**Remark 3.1.** *It is clear that every $\beta$-topological ring is an e*-topological ring since every $\beta$-open set is an e*-open set. Nevertheless, the converse need not always to be true as shown in the following example.*

**Example 3.1.** *Let $\mathbb{X}=\{k,l,m,n\}$ and $\mu=\{\emptyset,\mathbb{X},\{k\},\{k,l\}\}$. Let the addition and the multiplication operations on $\mathbb{X}$ be as given in the following tables:*

| $+$ | $k$ | $l$ | $m$ | $n$ |
|---|---|---|---|---|
| $k$ | $k$ | $l$ | $m$ | $n$ |
| $l$ | $l$ | $m$ | $n$ | $k$ |
| $m$ | $m$ | $n$ | $k$ | $l$ |
| $n$ | $l$ | $k$ | $l$ | $m$ |

| $\cdot$ | $k$ | $l$ | $m$ | $n$ |
|---|---|---|---|---|
| $k$ | $k$ | $k$ | $k$ | $k$ |
| $l$ | $k$ | $m$ | $k$ | $m$ |
| $m$ | $k$ | $k$ | $k$ | $k$ |
| $n$ | $k$ | $m$ | $k$ | $m$ |

In this topological space, simple calculations show that $e^*O(\mathbb{X}) = 2^{\mathbb{X}}$ and $\beta O(\mathbb{X}) = \{\emptyset, \mathbb{X}, \{k\}, \{k,n\}, \{k,m\}, \{k,l\}, \{k,l,n\}, \{k,l,m\}, \{k,m,n\}\}$. Then, it is clear that $(\mathbb{X}, +, \cdot, \mu)$ is an $e^*$-topological ring but it is not a $\beta$-topological ring.

**Example 3.2.** *Let $(\mathbb{R}, +, \cdot)$ be the ring of real numbers and let $\mathcal{U}$ the usual topology on $\mathbb{R}$. Then, $(\mathbb{R}, +, \cdot, \mathcal{U})$ is an $e^*$-topological ring.*

**Example 3.3.** *Let $(\mathbb{X}, +, \cdot)$ be any ring and let $\mu$ the discrete topology on $\mathbb{X}$. Then, $(\mathbb{X}, +, \cdot, \mu)$ is an $e^*$-topological ring.*

**Theorem 3.1.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring. Then, the following functions are $e^*$-continuous.*

*a) $+ : \mathbb{X}^2 \to \mathbb{X}$ defined by $+(x,y) = x + y$ for all $(x,y) \in \mathbb{X}^2$,*

*b) $\cdot : \mathbb{X}^2 \to \mathbb{X}$ defined by $\cdot(x,y) = xy$ for all $(x,y) \in \mathbb{X}^2$,*

*c) $- : \mathbb{X} \to \mathbb{X}$ defined by $-(x) = -x$ for all $x \in \mathbb{X}$.*

*Proof.* $a)$ Let $(x,y) \in \mathbb{X}^2$ and $W \in O(\mathbb{X}, x+y)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, x+y) \Rightarrow (\exists U \in e^*O(\mathbb{X}, x))(\exists V \in e^*O(\mathbb{X}, y))(U + V \subseteq W) \\ O := U \times V \end{array}\right\} \overset{\text{Lemma 2.2}}{\Rightarrow}$$

$$\Rightarrow (O \in e^*O(\mathbb{X}^2, (x,y)))(+[O] = +[U \times V] = U + V \subseteq W).$$

$b)$ Let $(x,y) \in \mathbb{X}^2$ and $W \in O(\mathbb{X}, xy)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, xy) \Rightarrow (\exists U \in e^*O(\mathbb{X}, x))(\exists V \in e^*O(\mathbb{X}, y))(UV \subseteq W) \\ O := U \times V \end{array}\right\} \overset{\text{Lemma 2.2}}{\Rightarrow}$$

$$\Rightarrow (O \in e^*O(\mathbb{X}^2, (x,y)))(\cdot[O] = \cdot[U \times V] = UV \subseteq W).$$

$c)$ Let $V \in O(\mathbb{X})$. Our aim is to show that $-^{-1}[V] \in e^*O(\mathbb{X})$.

$$\left.\begin{array}{r} -^{-1}[V] = \{x \in \mathbb{X} : -(x) \in V\} = \{x \in \mathbb{X} : -x \in V\} = -V \\ V \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Teorem 3.2}}{\Rightarrow} -^{-1}[V] \in e^*O(\mathbb{X}).$$

□

**Theorem 3.2.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring. Then, the following properties hold.*

*a) If $G \in O(\mathbb{X})$, then $-G \in e^*O(\mathbb{X})$,*

*b) If $G \in O(\mathbb{X})$ and $a \in \mathbb{X}$, then $a + G \in e^*O(\mathbb{X})$,*

*c) If $G \in O(\mathbb{X})$ and $a \in \mathbb{X}$, then $G + a \in e^*O(\mathbb{X})$.*

*Proof.* $a)$ Let $G \in O(\mathbb{X})$.

$G \in O(\mathbb{X}) \Rightarrow -G \subseteq \mathbb{X} \Rightarrow e^*\text{-}int(-G) \subseteq -G \ldots (1)$

Now, let $b \in -G$. Our purpose is to show that $b \in e^*\text{-}int(-G)$.

$$\left.\begin{array}{r} b \in -G \Rightarrow -b \in G \\ G \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Definition 3.1}}{\Rightarrow} (\exists U \in e^*O(\mathbb{X}, b))(-U \subseteq G)$$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(U \subseteq -G)$

$\Rightarrow b \in e^*\text{-}int(-G)$

Then, we have $-G \subseteq e^*\text{-}int(-G) \ldots (2)$

$(1), (2) \Rightarrow e^*\text{-}int(-G) = -G \Rightarrow -G \in e^*O(\mathbb{X})$.

$b)$ Let $G \in O(\mathbb{X})$ and $a \in \mathbb{X}$. Our purpose is to show that $a + G \in e^*O(\mathbb{X})$. For this, we will show that $a + G = e^*\text{-}int(a + G)$. Now, let $b \in a + G$. If we prove $b \in e^*\text{-}int(a + G)$, then the proof complete.

$$\left.\begin{array}{r} b \in a + G \Rightarrow (\exists c \in G)(b = a + c) \\ G \in O(\mathbb{X}) \end{array}\right\} \Rightarrow -a + b \in G \in O(\mathbb{X})$$

$\overset{\text{Definition 3.1}}{\Rightarrow} (\exists U \in e^*O(\mathbb{X}, -a))(\exists V \in e^*O(\mathbb{X}, b))(-a + V \subseteq U + V \subseteq G)$

$\Rightarrow (\exists V \in e^*O(\mathbb{X}, b))(-a + V \subseteq G)$

$\Rightarrow (\exists V \in e^*O(\mathbb{X}, b))(V \subseteq a + G)$

$\Rightarrow b \in e^*\text{-}int(a + G)$.

$c)$ This follows $(b)$ since the addition is commutative.

□

**Corollary 3.1.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G \subseteq \mathbb{X}$. Then, the following statements hold.*

*a) If $G \in O(\mathbb{X})$, then $-G \subseteq cl(int(\delta\text{-}cl(-G)))$,*

*b) If $G \in O(\mathbb{X})$, then $a + G \subseteq cl(int(\delta\text{-}cl(a + G)))$ for every $a \in \mathbb{X}$,*

*c) If $G \in O(\mathbb{X})$, then $G + a \subseteq cl(int(\delta\text{-}cl(G + a)))$ for every $a \in \mathbb{X}$.*

**Theorem 3.3.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G \subseteq \mathbb{X}$. Then, the following properties hold.*

*a) If $G \in C(\mathbb{X})$, then $-G \in e^*C(\mathbb{X})$,*

*b) If $a \in \mathbb{X}$ and $G \in C(\mathbb{X})$, then $a + G \in e^*C(\mathbb{X})$,*

*c) If $a \in \mathbb{X}$ and $G \in C(\mathbb{X})$, then $G + a \in e^*C(\mathbb{X})$.*

*Proof.* $a)$ Let $G \in C(\mathbb{X})$. Our purpose is to show that $-G \in e^*C(\mathbb{X})$. Now, let $b \in e^*\text{-}cl(-G)$. We will show that $b \in -G$, i.e. $-b \in G$. Let $W \in O(\mathbb{X}, -b)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, -b) \Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(-U \subseteq W) \\ b \in e^*\text{-}cl(-G) \end{array}\right\} \Rightarrow (U \subseteq -W)(U \cap (-G) \neq \emptyset)$$

$\Rightarrow \emptyset \neq U \cap (-G) \subseteq (-W) \cap (-G)$

$\Rightarrow W \cap G \neq \emptyset$

Then, we get $-b \in cl(G)$. Since $G \in C(\mathbb{X})$, we have $-b \in G$, i.e. $b \in -G$. Thus, we have $-G \subseteq e^*\text{-}cl(-G) \subseteq -G$, i.e. $-G = e^*\text{-}cl(-G)$. This means $-G \in e^*C(\mathbb{X})$.

$b)$ Let $b \in \mathbb{X}$ and $G \in C(\mathbb{X})$. Our purpose is to show that $a + G \in e^*C(\mathbb{X})$. Now, let $b \in e^*\text{-}cl(a+G)$. We will prove that $b \in a+G$, i.e. $-a+b \in G$. Let $W \in O(\mathbb{X}, -b+a)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, -a+b) \Rightarrow (\exists U \in e^*O(\mathbb{X}, -a))(\exists V \in e^*O(\mathbb{X}, y))(U+V \subseteq W) \\ b \in e^*\text{-}cl(a+G) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U+V \subseteq W)(V \cap (a+G) \neq \emptyset)$

$\Rightarrow \emptyset \neq (-a+V) \cap G \subseteq (U+V) \cap G \subseteq W \cap G$

$\Rightarrow W \cap G \neq \emptyset$

Then, we have $-a+b \in cl(G)$. Since $G \in C(\mathbb{X})$, we get $-a+b \in G$. Hence, $b \in a+G$.

$c)$ This follows $(b)$ since the addition is commutative. □

**Corollary 3.2.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G \subseteq \mathbb{X}$. Then, the following statements hold.*

*$a)$ If $G \in C(\mathbb{X})$, then $int(cl(int(-G))) \subseteq -G$,*

*$b)$ If $G \in C(\mathbb{X})$, then $int(cl(int(a+G))) \subseteq a+G$ for all $a \in \mathbb{X}$,*

*$c)$ If $G \in C(\mathbb{X})$, then $int(cl(int(G+a))) \subseteq G+a$ for all $a \in \mathbb{X}$.*

## 4. MAIN RESULTS

In this section, we obtain some basic properties of $e^*$-topological ring. In addition, this section contains the definition of $e^*$-topological rings with unit and many fundamental results on this new notion.

**Theorem 4.1.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring. Then, the following functions are $e^*$-continuous:*

*$a)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a+b$ for all $b \in \mathbb{X}$,*

*$b)$ $f : \mathbb{X} \to \mathbb{X}$ defined by $f(a) = -a$ for all $a \in \mathbb{X}$,*

*$c)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = b+a$ for all $b \in \mathbb{X}$,*

*$d)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a+b+a$ for all $b \in \mathbb{X}$.*

*Proof.* $a$) Let $H \in O(\mathbb{X})$. Our aim is to show that $f_a^{-1}[H] \in e^*O(\mathbb{X})$.

$$\left.\begin{array}{r} f_a^{-1}[H] = \{b \in \mathbb{X} | f_a(b) \in H\} = \{b \in \mathbb{X} | b \in -a + H\} = -a + H \\ H \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow}$$

$\Rightarrow f_a^{-1}[H] \in e^*O(\mathbb{X})$.

$b$) Let $H \in O(\mathbb{X})$. Our purpose is to show that $f^{-1}[H] \in e^*O(\mathbb{X})$.

$$\left.\begin{array}{r} f^{-1}[H] = \{a \in \mathbb{X} | f(a) \in H\} = \{a \in \mathbb{X} | -a \in H\} = -H \\ H \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} f^{-1}[H] \in e^*O(\mathbb{X}).$$

$c$) This follows ($b$) since the addition is commutative.

$d$) This follows ($b$) and ($c$) since the addition is commutative.

□

**Definition 4.1.** *A bijective function $f : (\mathbb{X}, \mu) \to (\mathbb{Y}, \rho)$ which is $e^*$-continuous and whose inverse is $e^*$-continuous is called an $e^*$-homeomorphism.*

**Corollary 4.1.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring. Then, the following functions are $e^*$-homeomorphism.*

*$a$) For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a + b$ for all $b \in \mathbb{X}$,*

*$b$) $f : \mathbb{X} \to \mathbb{X}$ defined by $f(a) = -a$ for all $a \in \mathbb{X}$,*

*$c$) For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = b + a$ for all $b \in \mathbb{X}$,*

*$d$) For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a + b + a$ for all $b \in \mathbb{X}$.*

**Definition 4.2.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring. If $(\mathbb{X}, +, \cdot)$ is a ring with unit, then $(\mathbb{X}, +, \cdot, \mu)$ is said to be an $e^*$-topological ring with unit. The notation $\mathbb{X}^*$ will be used to denote the set of all invertible elements in $(\mathbb{X}, +, \cdot)$.*

**Theorem 4.2.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring with unit and $G \subseteq \mathbb{X}$. Then, the following properties hold.*

*$a$) If $G \in O(\mathbb{X})$, then $Gs$ is $e^*$-open in $\mathbb{X}$ for each $s \in \mathbb{X}^*$,*

*$b$) If $G \in O(\mathbb{X})$, then $sG$ is $e^*$-open in $\mathbb{X}$ for each $s \in \mathbb{X}^*$.*

*Proof.* $a$) Let $G \in O(\mathbb{X})$ and $s \in \mathbb{X}^*$. We will prove $Gs \in e^*O(\mathbb{X})$. If we prove $Gs \subseteq e^*$-$int(Gs)$, then the proof complete. Let $b \in Gs$.

$$\left.\begin{array}{r} \left.\begin{array}{r} b \in Gs \Rightarrow (\exists k \in G)(b = ks) \\ s \in T^* \end{array}\right\} \Rightarrow (\exists k \in G)(bs^{-1} = k) \\ G \in O(\mathbb{X}) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(\exists V \in e^*O(\mathbb{X}, s^{-1}))(Us^{-1} \subseteq UV \subseteq G)$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(U \subseteq Gs)$

$\Rightarrow y \in e^*\text{-}int(Gs)$

Then, we have $Gs \subseteq e^*\text{-}int(Gs) \subseteq Gs$ which means $Gs \in e^*O(\mathbb{X})$.

$b)$ It is proved similarly to $(a)$. □

**Theorem 4.3.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring with unit and $G \subseteq \mathbb{X}$. Then, the following properties hold.*

*a) If $G \in C(\mathbb{X})$, then $Gs \in e^*C(\mathbb{X})$ for each $s \in \mathbb{X}^*$,*

*b) If $G \in C(\mathbb{X})$, then $sG \in e^*C(\mathbb{X})$ for each $s \in \mathbb{X}^*$.*

*Proof.* Let $G \in C(\mathbb{X})$ and $s \in \mathbb{X}^*$.

$$\left.\begin{array}{r} b \notin sG \Rightarrow (\forall k \in G)(b \neq sk) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\forall k \in cl(G))(b \neq sG)$$

$\Rightarrow (\forall k \in cl(G))(s^{-1}b \neq k)$

$\Rightarrow s^{-1}y \notin cl(G)$

$\Rightarrow (\exists U \in O(\mathbb{X}, s^{-1}b))(U \cap G = \emptyset)$

$\Rightarrow (\exists K \in e^*O(\mathbb{X}, s^{-1}))(\exists M \in e^*O(\mathbb{X}, s^{-1}))(s^{-1}M \cap G \subseteq KM \cap G \subseteq U \cap G = \emptyset)$

$\Rightarrow (\exists M \in e^*O(\mathbb{X}, b))(s^{-1}M \cap G = \emptyset)$

$\Rightarrow (\exists M \in e^*O(\mathbb{X}, b))(M \cap sG = \emptyset)$

$\Rightarrow b \notin e^*\text{-}cl(sG)$

Then, we have $sG \subseteq e^*\text{-}cl(sG) \subseteq sG$ which means $sG \in e^*C(\mathbb{X})$.

$b)$ It is proved similarly to $(a)$. □

**Theorem 4.4.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring with unit and $G \subseteq \mathbb{X}$. Then, the following properties hold:*

*a) $s \cdot e^*\text{-}cl(G) \subseteq cl(sG)$ for each $s \in \mathbb{X}$,*

*b) $int(sG) \subseteq s \cdot e^*\text{-}int(G)$ for each $s \in \mathbb{X}$.*

*c) $s \cdot int(G) \subseteq e^*\text{-}int(sG)$ for each $s \in \mathbb{X}^*$,*

*d) $e^*\text{-}cl(sG) \subseteq s \cdot cl(G)$ for each $s \in \mathbb{X}^*$,*

*e) $e^*\text{-}cl(G) \cdot s \subseteq cl(Gs)$ for each $s \in \mathbb{X}$,*

*f) $int(G) \cdot s \subseteq e^*\text{-}int(Gs)$ for each $s \in \mathbb{X}^*$.*

*Proof.* $a)$ Let $a \in s \cdot e^*\text{-}cl(G)$. Our purpose is to show that $a \in cl(sG)$. Now, let $U \in O(\mathbb{X}, a)$.

$$\left.\begin{array}{r} a \in s \cdot e^*\text{-}cl(G) \Rightarrow (\exists b \in e^*\text{-}cl(G))(a = sb) \\ U \in O(\mathbb{X}, a) \end{array}\right\} \Rightarrow$$

$\Rightarrow (b \in e^*\text{-}cl(G))(\exists K \in e^*O(\mathbb{X}, s))(\exists L \in e^*O(\mathbb{X}, b))(KL \subseteq U)$

$\Rightarrow (\exists K \in e^*O(\mathbb{X}, s))(\exists L \in e^*O(\mathbb{X}, b))(KL \subseteq U)(L \cap G \neq \emptyset)$

$\Rightarrow \emptyset \neq KL \cap sG \subseteq U \cap sG$

$\Rightarrow \emptyset \neq U \cap sG$

Then, we have $a \in cl(sG)$.

*b*) Let $a \in int(sG)$. Our purpose is to show that $a \in s \cdot e^*\text{-}int(G)$.

$$\begin{aligned} a \in int(sG) &\Rightarrow (a \in sG)(int(sG) \in O(\mathbb{X}, a)) \\ &\Rightarrow (\exists b \in G)(a = sb)(int(sG) \in O(\mathbb{X}, a)) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, s))(\exists V \in e^*O(\mathbb{X}, b))(sV \subseteq UV \subseteq int(sG) \subseteq sG) \\ &\Rightarrow (\exists V \in e^*O(\mathbb{X}, b))(V \subseteq G) \\ &\Rightarrow b \in e^*\text{-}int(G) \\ &\Rightarrow a = sb \in s \cdot e^*\text{-}int(G). \end{aligned}$$

*c*) Let $a \in s \cdot int(G)$. Our purpose is to show that $a \in e^*\text{-}int(sG)$.

$$\begin{aligned} a \in s \cdot int(G) &\Rightarrow s^{-1}a \in int(G) \\ &\Rightarrow int(G) \in O(\mathbb{X}, s^{-1}a) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, s^{-1}))(\exists V \in e^*O(\mathbb{X}, a))(s^{-1}V \subseteq UV \subseteq int(G) \subseteq G) \\ &\Rightarrow (\exists V \in e^*O(\mathbb{X}, a))(V \subseteq sG) \\ &\Rightarrow a \in e^*\text{-}int(sG). \end{aligned}$$

*d*) Let $a \in e^*\text{-}cl(sG)$ and $W \in O(\mathbb{X}, s^{-1}a)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, s^{-1}a) \Rightarrow (\exists U \in e^*O(\mathbb{X}, s^{-1}))(\exists V \in e^*O(\mathbb{X}, a))(s^{-1}V \subseteq UV \subseteq W) \\ a \in e^*\text{-}cl(sG) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, s^{-1}))(\exists V \in e^*O(\mathbb{X}, a))(s^{-1}V \subseteq UV \subseteq W)(V \cap sG \neq \emptyset)$

$\Rightarrow \emptyset \neq UV \cap A \subseteq W \cap G$

$\Rightarrow W \cap G \neq \emptyset$

Then, we have $s^{-1}a \in cl(G)$ which means $a \in s \cdot cl(G)$.

*e*) Let $a \in e^*\text{-}cl(G) \cdot s$. Our purpose is to show that $a \in cl(Gs)$. Now, let $U \in O(\mathbb{X}, a)$.

$$\left.\begin{array}{r} a \in e^*\text{-}cl(G) \cdot s \Rightarrow (\exists b \in e^*\text{-}cl(G))(a = bs) \\ U \in O(\mathbb{X}, a) \end{array}\right\} \Rightarrow$$

$\Rightarrow (b \in e^*\text{-}cl(G))(\exists K \in e^*O(\mathbb{X}, b))(\exists L \in e^*O(\mathbb{X}, s))(KL \subseteq U)$

$\Rightarrow (\exists K \in e^*O(\mathbb{X}, b))(\exists L \in e^*O(\mathbb{X}, s))(KL \subseteq U)(K \cap G \neq \emptyset)$

$\Rightarrow \emptyset \neq KL \cap Gs \subseteq U \cap Gs$

$\Rightarrow \emptyset \neq U \cap Gs$

Then, we have $a \in cl(Gs)$.

$f$) Let $a \in int(G) \cdot s$. Our purpose is to show that $a \in e^*\text{-}int(Gs)$.

$$\begin{aligned} a \in int(G) \cdot s &\Rightarrow as^{-1} \in int(G) \\ &\Rightarrow int(G) \in O(\mathbb{X}, as^{-1}) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, a))(\exists V \in e^*O(\mathbb{X}, s^{-1}))(Us^{-1} \subseteq UV \subseteq int(G) \subseteq G) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, a))(U \subseteq Gs) \\ &\Rightarrow a \in e^*\text{-}int(Gs). \end{aligned}$$

□

**Theorem 4.5.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring with unit and $s \in \mathbb{X}^*$. Then, the following functions are $e^*$-continuous.*

*a*) *$f_s : \mathbb{X} \to \mathbb{X}$ defined by $f_s(a) = sa$ for all $a \in \mathbb{X}$,*

*b*) *$f_s : \mathbb{X} \to \mathbb{X}$ defined by $f_s(a) = as$ for all $a \in \mathbb{X}$,*

*c*) *$f_s : \mathbb{X} \to \mathbb{X}$ defined by $f_s(a) = sas$ for all $a \in \mathbb{X}$.*

*Proof.* $a$) Let $U \in O(\mathbb{X})$. Our purpose is to show that $f_s^{-1}[U] \in e^*O(\mathbb{X})$. For this, we will prove $f_s^{-1}[U] = e^*\text{-}int(f_s^{-1}[U])$. We have always $e^*\text{-}int(f_s^{-1}[U]) \subseteq f_s^{-1}[U] \ldots (1)$

Now, let $b \in f_s^{-1}[U]$.

$$\left.\begin{array}{r} b \in f_s^{-1}[U] = s^{-1}U \\ U \in O(\mathbb{X}) \Rightarrow U = int(U) \end{array}\right\} \Rightarrow b \in s^{-1} \cdot int(U) \overset{\text{Theorem 4.4}}{\Rightarrow} \left.\begin{array}{r} b \in e^*\text{-}int(s^{-1}U) \\ s^{-1}U = f_s^{-1}[U] \end{array}\right\} \Rightarrow$$

$\Rightarrow b \in f_s^{-1}[U]$

Then, we have $f_s^{-1}[U] \subseteq e^*\text{-}int(f_s^{-1}[U]) \ldots (2)$

$(1), (2) \Rightarrow f_s^{-1}[U] = e^*\text{-}int(f_s^{-1}[U]) \Rightarrow f_s^{-1}[U] \in e^*O(\mathbb{X})$.

$b$) This follows Theorem 4.4.

$c$) This follows ($a$) and ($b$). □

**Corollary 4.2.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring with unit and $s \in \mathbb{X}^*$. Then, the following functions are $e^*$-homeomorphism.*

*a*) *$f_s : \mathbb{X} \to \mathbb{X}$ defined by $f_s(a) = sa$ for all $a \in \mathbb{X}$,*

*b*) *$f_s : \mathbb{X} \to \mathbb{X}$ defined by $f_s(a) = as$ for all $a \in \mathbb{X}$.*

*c*) *$f_s : \mathbb{X} \to \mathbb{X}$ defined by $f_s(a) = sas$ for all $a \in \mathbb{X}$.*

**Theorem 4.6.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G \subseteq \mathbb{X}$. Then, the following properties hold for each $a \in \mathbb{X}$.*

*a*) $a + e^*\text{-}cl(G) \subseteq cl(a + G)$,

*b*) $e^*\text{-}cl(a + G) \subseteq a + cl(G)$,

*c*) $a + int(G) \subseteq e^*\text{-}int(a + G)$, *d)* $int(a+G) \subseteq a + e^*\text{-}int(G)$.

*Proof.* *a)* Let $b \in a + e^*\text{-}cl(G)$. Our purpose is to prove that $b \in cl(a+G)$. Now, let $U \in O(\mathbb{X}, b)$. If we prove $U \cap (a+G) \neq \emptyset$, then the proof complete.

$$\left.\begin{array}{r} b \in a + e^*\text{-}cl(G) \Rightarrow (\exists c \in e^*\text{-}cl(G))(b = a + c) \\ U \in O(\mathbb{X}, b) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists K \in e^*O(\mathbb{X}, a))(\exists L \in e^*O(\mathbb{X}, c))(\emptyset \neq (K+L) \cap (a+G) \subseteq U \cap (a+G))$$

$$\Rightarrow U \cap (a+G) \neq \emptyset.$$

*b)* Let $b \in e^*\text{-}cl(a+G)$. Our purpose is to show that $b \in a + cl(G)$. Now, let $U \in O(\mathbb{X}, -a+b)$.

$$U \in O(\mathbb{X}, -a+b) \Rightarrow (\exists K \in e^*O(\mathbb{X}, -a))(\exists L \in e^*O(\mathbb{X}, b))(-a + L \subseteq K + L \subseteq U)$$

$$\Rightarrow \emptyset \neq (-a+L) \cap G \subseteq U \cap G$$

Therefore, $-a + b \in cl(G)$ which means $b \in a + cl(G)$.

*c)* Let $b \in a + int(G)$. Our purpose is to show that $b \in e^*\text{-}int(a+G)$.

$$\begin{array}{rcl} b \in a + int(G) & \Rightarrow & -a + b \in int(G) \in O(\mathbb{X}) \\ & \Rightarrow & (\exists U \in e^*O(\mathbb{X}, -a))(\exists V \in e^*O(\mathbb{X}, b))(-a + V \subseteq U + V \subseteq int(G) \subseteq G) \\ & \Rightarrow & (\exists V \in e^*O(\mathbb{X}, b))(V \subseteq a + G) \\ & \Rightarrow & b \in e^*\text{-}int(a+G). \end{array}$$

*d)* Let $b \in int(a+G)$. Our purpose is to show that $b \in a + e^*\text{-}int(G)$.

$$\begin{array}{rcl} b \in int(a+G) & \Rightarrow & (\exists U \in O(\mathbb{X}, b))(U \subseteq a + G) \\ & \Rightarrow & (\exists U \in O(\mathbb{X}, b))(-a + U \subseteq G) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & (-a + U \in e^*O(\mathbb{X}, -a+b))(-a + U \subseteq G) \\ & \Rightarrow & -a + b \in e^*\text{-}int(G) \\ & \Rightarrow & b \in a + e^*\text{-}int(G). \end{array}$$

□

**Theorem 4.7.** *Let* $(\mathbb{X}, +, \cdot, \mu)$ *be an* $e^*$*-topological ring and* $G \subseteq \mathbb{X}$*. Then, we have the following properties.*

*a)* $-e^*\text{-}cl(G) \subseteq cl(-G)$,

*b)* $e^*\text{-}cl(-G) \subseteq -cl(G)$,

*c)* $-int(G) \subseteq e^*\text{-}int(-G)$,

*d)* $int(-G) \subseteq -e^*\text{-}int(G)$.

*Proof.* $a$) Let $b \notin cl(-G)$.

$$\begin{aligned} b \notin cl(-G) &\Rightarrow (\exists U \in O(\mathbb{X}, b))(U \cap (-G) = \emptyset) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, -b))((-U) \cap G = \emptyset) \\ &\Rightarrow -b \notin e^*\text{-}cl(G) \\ &\Rightarrow b \notin -e^*\text{-}cl(G). \end{aligned}$$

$b$) Let $b \notin -cl(G)$.

$$\begin{aligned} b \notin -cl(G) &\Rightarrow -b \notin cl(G) \\ &\Rightarrow (\exists U \in O(\mathbb{X}, -b))(U \cap G = \emptyset) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, b))((-U) \cap (-G) = \emptyset) \\ &\Rightarrow b \notin e^*\text{-}cl(-G). \end{aligned}$$

$c$) Let $b \in -int(G)$.

$$\begin{aligned} b \in -int(G) &\Rightarrow -b \in int(G) \\ &\Rightarrow (\exists U \in O(\mathbb{X}, -b))(U \subseteq G) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, b))(-U \subseteq -G) \\ &\Rightarrow b \in e^*\text{-}int(-G). \end{aligned}$$

$d$) Let $b \in int(-G)$.

$$\begin{aligned} b \in int(-G) &\Rightarrow (\exists U \in O(\mathbb{X}, b))(U \subseteq -G) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, -b))(-U \subseteq G) \\ &\Rightarrow -b \in e^*\text{-}int(G) \\ &\Rightarrow b \in -e^*\text{-}int(G). \quad \square \end{aligned}$$

**Theorem 4.8.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G \subseteq \mathbb{X}$. Then, we have the following properties for all $a \in \mathbb{X}$.*

*a)* $a + int(cl(\delta\text{-}int(G))) \subseteq cl(a + G)$,

*b)* $int(cl(\delta\text{-}int(a + G))) \subseteq a + cl(G)$,

*c)* $a + int(G) \subseteq cl(int(\delta\text{-}cl(a + G)))$,

*d)* $int(a + G) \subseteq a + cl(\delta\text{-}cl(G))$.

*Proof.* $a$) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$(G \subseteq X)(a \in \mathbb{X}) \Rightarrow cl(a + G) \in C(\mathbb{X}) \overset{\text{Theorem 3.2}}{\Rightarrow} -a + cl(a + G) \in e^*C(\mathbb{X})$

$\Rightarrow int(cl(\delta\text{-}int(e^*\text{-}cl(G)))) \subseteq int(cl(\delta\text{-}int(-a + cl(a + G)))) \subseteq -a + cl(a + G)$

$\Rightarrow int(cl(\delta\text{-}int(G))) \subseteq int(cl(\delta\text{-}int(e^*\text{-}cl(G)))) \subseteq -a + cl(a + G)$

$\Rightarrow a + int(cl(\delta\text{-}int(G))) \subseteq cl(a + G)$.

*b*) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$\left.\begin{array}{r} G \subseteq \mathbb{X} \Rightarrow cl(G) \in C(\mathbb{X}) \\ a \in \mathbb{X} \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} a + cl(G) \in e^*C(\mathbb{X})$$

$$\Rightarrow int(cl(\delta\text{-}int(a+G))) \subseteq int(cl(\delta\text{-}int(a+cl(G)))) \subseteq a + cl(a+G).$$

*c*) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$\left.\begin{array}{r} G \subseteq \mathbb{X} \Rightarrow int(G) \in O(\mathbb{X}) \\ a \in \mathbb{X} \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} a + int(G) \in e^*O(\mathbb{X})$$

$$\Rightarrow a + int(G) \subseteq cl(int(\delta\text{-}cl(a+int(G)))) \subseteq cl(int(\delta\text{-}cl(a+G))).$$

*d*) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$(G \subseteq \mathbb{X})(a \in \mathbb{X}) \Rightarrow int(a+G) \in O(\mathbb{X}) \overset{\text{Theorem 3.2}}{\Rightarrow} -a + int(a+G) \in e^*O(\mathbb{X})$$

$$\Rightarrow -a + int(a+G) \subseteq cl(int(\delta\text{-}cl(-a+int(a+G)))) \subseteq cl(int(\delta\text{-}cl(G))).$$

□

**Theorem 4.9.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G \subseteq \mathbb{X}$. Then, we have the following properties.*

*a*) $-int(cl(\delta\text{-}int(G))) \subseteq cl(-G)$,

*b*) $int(cl(\delta\text{-}int(-G))) \subseteq -cl(G)$,

*c*) $-int(G) \subseteq cl(int(\delta\text{-}cl(-G)))$,

*d*) $int(-G) \subseteq -cl(int(\delta\text{-}cl(G)))$.

*Proof.* *a*) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{rcl} G \subseteq \mathbb{X} & \Rightarrow & cl(-G) \in C(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -cl(-G) \in e^*C(\mathbb{X}) \\ & \Rightarrow & int(cl(\delta\text{-}int(G))) \subseteq int(cl(\delta\text{-}int(-cl(-G)))) \subseteq -cl(-G) \\ & \Rightarrow & -int(cl(\delta\text{-}int(G))) \subseteq cl(-G). \end{array}$$

*b*) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{rcl} G \subseteq \mathbb{X} & \Rightarrow & cl(G) \in C(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -cl(G) \in e^*C(\mathbb{X}) \\ & \Rightarrow & int(cl(\delta\text{-}int(-G))) \subseteq int(cl(\delta\text{-}int(-cl(G)))) \subseteq -cl(G). \end{array}$$

*c*) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{rcl} G \subseteq \mathbb{X} & \Rightarrow & int(G) \in O(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -int(G) \in e^*O(\mathbb{X}) \\ & \Rightarrow & -int(G) \subseteq cl(int(\delta\text{-}cl(-int(G)))) \subseteq cl(int(\delta\text{-}cl(-G))). \end{array}$$

*d*) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{rcl} G \subseteq \mathbb{X} & \Rightarrow & int(-G) \in O(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -int(-G) \in e^*O(\mathbb{X}) \\ & \Rightarrow & -int(-G) \subseteq cl(int(\delta\text{-}cl(-int(-G)))) \subseteq cl(int(\delta\text{-}cl(G))) \\ & \Rightarrow & int(-G) \subseteq -cl(int(\delta\text{-}cl(G))). \quad \square \end{array}$$

**Theorem 4.10.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and $G, H \subseteq \mathbb{X}$. Then, $e^*\text{-}cl(G) + e^*\text{-}cl(H) \subseteq cl(G + H)$.*

*Proof.* Let $c \in e^*\text{-}cl(G) + e^*\text{-}cl(H)$. Our purpose is to show that $c \in cl(G + H)$. Now, let $W \in O(\mathbb{X}, c)$.

$$\left.\begin{array}{r} c \in e^*\text{-}cl(G) + e^*\text{-}cl(H) \Rightarrow (\exists a \in e^*\text{-}cl(G))(\exists b \in e^*\text{-}cl(H))(c = a + b) \\ W \in O(\mathbb{X}, c) \end{array}\right\} \Rightarrow$$

$$\begin{array}{l} \Rightarrow (\exists U \in O(\mathbb{X}, a))(\exists V \in O(\mathbb{X}, b))(U + V \subseteq W)(U \cap G \neq \emptyset)(V \cap H \neq \emptyset) \\ \Rightarrow (W \in O(\mathbb{X}, c))((U \cap G) + (V \cap H) \neq \emptyset)(U + V \subseteq W) \\ \Rightarrow (W \in O(\mathbb{X}, c))(\exists t \in \mathbb{X})(t \in (U \cap G) + (V \cap H))(U + V \subseteq W) \\ \Rightarrow (W \in O(\mathbb{X}, c))(\exists u \in U \cap G)(\exists v \in V \cap H)(t = u + v)(U + V \subseteq W) \\ \Rightarrow (W \in O(\mathbb{X}, c))(u \in U)(u \in G)(v \in V)(v \in H)(U + V \subseteq W) \\ \Rightarrow (W \in O(\mathbb{X}, c))(u + v \in U + V)(u + v \in G + H)(U + V \subseteq W) \\ \Rightarrow (W \in O(\mathbb{X}, c))(u + v \in (U + V) \cap (G + H) \subseteq W \cap (G + H)) \\ \Rightarrow (W \in O(\mathbb{X}, c))(W \cap (G + H) \neq \emptyset). \end{array}$$

$\square$

**Remark 4.1.** *The following example shows that the converse of inclusion given in Theorem 4.10 need not to be true in general.*

**Example 4.1.** *Let $\mathbb{X} = \{k, l, m, n\}$ and $\mu = \{\emptyset, \mathbb{X}, \{k\}, \{k, l\}\}$. Let the addition and multiplication operations on $\mathbb{X}$ be as given in Example 3.1. For the subsets $G = \{k\}$ and $H = \{m\}$, we have $cl(G + H) = cl(\{m\}) = \{m, n\}$ and $e^*\text{-}cl(G) + e^*\text{-}cl(H) = e^*\text{-}cl(\{k\}) + e^*\text{-}cl(\{m\}) = \{k\} + \{m\} = \{m\}$. It is obvious that $cl(\{m\}) = \{m, n\} \nsubseteq \{m\} = e^*\text{-}cl(G) + e^*\text{-}cl(H)$.*

**Theorem 4.11.** *Let $(\mathbb{X}, +, \cdot, \mu)$ be an $e^*$-topological ring and let $(\mathbb{Y}, +, \cdot, \rho)$ be a topological ring. If $f : \mathbb{X} \to \mathbb{Y}$ is a ring homomorphism and continuous at $0_{\mathbb{X}}$, then $f$ is $e^*$-continuous.*

*Proof.* Let $f$ be a homomorphism and continuous at $0_{\mathbb{X}}$. Our purpose is to show that $f$ is $e^*$-continuous. Now, let $V \in O(\mathbb{X}, f(a))$.

$$\left.\begin{array}{r} V \in O(\mathbb{X}, f(a)) \Rightarrow f(a) = f(a + 0_{\mathbb{X}}) \in V \in O(T) \\ f \text{ is homomorphism} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow f(a) + f(0_{\mathbb{X}}) \in V \in O(\mathbb{X}) \Rightarrow -f(a) + V \in O(\mathbb{X}, f(0_{\mathbb{X}})) \\ f \text{ is continuous in } 0_{\mathbb{X}} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists W \in O(\mathbb{X}, 0_{\mathbb{X}}))(f[W] \subseteq -f(a) + V) \Rightarrow (\exists W \in O(\mathbb{X}, 0_{\mathbb{X}}))(f(a) + f[W] \subseteq V) \\ f \text{ is homomorphism} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists W \in O(\mathbb{X}, 0_{\mathbb{X}}))(f[a + W] \subseteq V) \\ U := a + W \end{array}\right\} \Rightarrow (U \in e^*O(\mathbb{X}, a))(f[U] \subseteq V).$$

□

## 5. Conclusion

The idea of obtaining more general results than those existing in the literature has led mathematicians to introduce new concepts such as topological groups, topological rings, topological fields, and topological vector spaces. In this article, we have introduced a new concept, called $e^*$-topological ring, by utilizing $e^*$-open sets. This new concept comes across as a more general concept than the concept of $\beta$-topological rings. On the other hand, the results given in this study coincide with the results given [2] in regular topological spaces, since the collection of all $\beta$-open sets is equal to the collection of all $e^*$-open sets in regular spaces. We obtained not only many results related to this new notion but also gave some counterexamples. We believe that the results obtained in this study will find an important place in future research on topological rings.

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# AFFINE TRANSLATION SURFACES WITH CONSTANT MEAN CURVATURE IN MINKOWSKI 3-SPACE

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ABSTRACT. In this paper, we obtain classification results for spacelike affine translation surfaces with constant mean curvature in three dimensional Minkowski space $\mathbb{E}_1^3$.

**Keywords**: Minkowski 3-space, affine translation surface, mean curvature, spacelike surface.

**2010 Mathematics Subject Classification**: 53A05; 53A10; 53C42.

## 1. Introduction

In 3-dimensional spaces, a regular surface parameterized as $\Psi(u,v) = (u, v, z(u,v))$ is called a translation surface if usually $z(u,v)$ is of the form

$$z(u,v) = f(u) + g(v),$$

where $f$ and $g$ are differentiable functions of $u$ and $v$, respectively. Scherk [10] discovered the first non-trivial minimal translation surface in Euclidean 3-space $\mathbb{E}^3$, famously known as the Scherk surface, and is given by

$$z(u,v) = \frac{1}{c} \log \left| \frac{\cos(cu)}{\cos(cv)} \right|,$$

where $c(\neq 0)$ is a constant. Planes and Scherk surfaces are the only minimal translation surfaces in $\mathbb{E}^3$. More than a century later, Liu proved that the circular cylinder is the only

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translation surface with non-zero constant mean curvature [5]. The study of constant mean curvature translation surfaces has gathered significant attention. For some of the studies, we refer the reader to see [3, 4, 5, 8, 9, 12]. A natural extension of the translation surface appears in the form of an affine translation surface, which is a surface parameterized by $\Psi(u,v) = (u, v, z(u,v))$, where now

$$z(u,v) = f(u) + g(au + v),$$

and $a(\neq 0)$ is a constant. Liu and Yu proved that the non-trivial minimal affine translation surface in $\mathbb{E}^3$ is given by

$$z(u,v) = \frac{1}{b}\log\left|\frac{\cos\left(b\sqrt{1+a^2}u\right)}{\cos\left(b(v+au)\right)}\right|,$$

where $b(\neq 0)$ is a constant. This surface is known as the affine Scherk surface [7]. For other related works on affine translation surfaces, we refer the reader to see [1, 2, 6, 11].

In connection to the non-zero constant mean curvature of affine translation surfaces, Liu and Jung [6] obtained the classification results in $\mathbb{E}^3$. Now, a Minkowski space is one of the most trivial indefinite space forms, and it marks its great significance as the trivial solution to the vacuum Einstein Field Equations without a cosmological constant. Inspired by the previous developments in the theory of constant mean curvature surfaces, we seek to classify spacelike affine translation surfaces with constant mean curvature in Minkowski 3-space $\mathbb{E}^3_1$.

Consider $\Psi(u,v)$ to be a regular spacelike surface in Minkowski 3-space $\mathbb{E}^3_1$. The coefficients of the $1^{st}$ fundamental form $E, F, G$ of $\Psi(u,v)$ are given by

$$E = \langle \Psi_u, \Psi_u \rangle, F = \langle \Psi_u, \Psi_v \rangle, G = \langle \Psi_v, \Psi_v \rangle,$$

and the coefficients of the $2^{nd}$ fundamental form $L, M, N$ of $\Psi(u,v)$ are given by

$$L = \langle \Psi_{uu}, \hat{n} \rangle, M = \langle \Psi_{uv}, \hat{n} \rangle, N = \langle \Psi_{vv}, \hat{n} \rangle,$$

where $\hat{n}$ is the unit normal vector and $\langle *, * \rangle = du^2 + dv^2 - dz^2$ is the Minkowski metric. The mean curvature $H$ of the surface $\Psi(u,v)$ is given by

$$H(u,v) = \frac{EN - 2FM + GL}{2\big(EG - F^2\big)}. \tag{1.1}$$

For a spacelike surface $\Psi(u,v)$ in $\mathbb{E}^3_1$, we have $EG - F^2 > 0$, and for a timelike surfaces $EG - F^2 < 0$. In regards to the regular surface $\Psi(u,v)$ embedded in $\mathbb{E}^3_1$, following two types of affine translation surfaces exist:

**(i)** Affine translation surface of type 1:

$$\Psi(u,v) = (u, v, z(u,v)) \tag{1.2}$$

such that

$$z(u,v) = f(u) + g(au+v). \tag{1.3}$$

**(ii)** Affine translation surface of type 2:

$$\Psi(u,v) = (u(v,z), v, z) \tag{1.4}$$

such that

$$u(v,z) = h(v) + t(bv+z), \tag{1.5}$$

where $a(\neq 0), b(\neq 0)$ are constants and $f, g, h, t$ are smooth functions. We note that whenever $a = 0$ or $b = 0$, affine translation surfaces reduce simply to translation surfaces.

## 2. Affine translation surfaces with non-zero constant mean curvature

**Theorem 2.1.** *Let $\Psi(u,v) = (u, v, z(u,v))$ be a spacelike affine translation surface of type 1 in $\mathbb{E}^3_1$. If $\Psi(u,v)$ has a non-zero constant mean curvature, then $z(u,v)$ is given by*

$$z(u,v) = \pm\frac{\sqrt{1+a^2-c^2}}{2H\left(1+a^2\right)}\sqrt{1+a^2+4H^2(b-au-v)^2} + \left(\frac{u-av}{1+a^2}\right)c + p$$

*such that $c^2 < 1+a^2$; or*

$$z(u,v) = cv \pm \frac{\sqrt{1-c^2}}{2H}\sqrt{1+4H^2(b-u)^2} + q$$

*such that $c^2 < 1$; where $a, b, c, p, q$ are all constants.*

*Proof.* We know that the mean curvature of a spacelike surface $\Psi(u,v) = (u, v, z(u,v))$ in $\mathbb{E}^3_1$ is given by

$$H(u,v) = \frac{\left(z_u^2-1\right)z_{vv} - 2z_u z_v z_{uv} + \left(z_v^2-1\right)z_{uu}}{2\left(1-{z_u}^2-{z_v}^2\right)^{\frac{3}{2}}}, \tag{2.6}$$

where $z_u, z_v$ denotes partial differentiation of $z$ w.r.t. $u$ and $v$, respectively. We obtain the following partial derivatives of $z(u,v)$ from (1.3)

$$\begin{cases} z_u = f' + ag', \\ z_v = g', \\ z_{uu} = f'' + a^2 g'', \\ z_{vv} = g'', \\ z_{uv} = ag'', \end{cases} \tag{2.7}$$

where $f' = \dfrac{df}{du}$ and $g' = \dfrac{dg}{dx}$ for $x = au + v$. Using (2.7) in (2.6), gives us

$$-f'' - \left(1 + a^2\right)g'' + \left(g'^2 f'' + f'^2 g''\right) = 2HT^3, \tag{2.8}$$

where $T^2 = 1 - \left(f' + ag'\right)^2 - g'^2$ and $H (\neq 0)$ is a constant. Eqn (2.8) writes as

$$-\left(1 - g'^2\right)f'' - \left(1 + a^2 - f'^2\right)g'' = 2HT^3. \tag{2.9}$$

Now, we have the following two cases:

*Case I.* When $f'' = 0$, we have $f' = c$, where $c$ is a constant. Substituting $f' = c$ in (2.9) gives us

$$-\left(1 + a^2 - c^2\right)g'' = 2H\left[1 - \left(c + ag'\right)^2 - g'^2\right]^{\frac{3}{2}}. \tag{2.10}$$

Thus, we have

$$g'' = -\frac{2H\left(1 + a^2\right)^{\frac{3}{2}}}{1 + a^2 - c^2}\left[\frac{1 + a^2 - c^2}{\left(1 + a^2\right)^2} - \left(g' + \frac{ac}{1 + a^2}\right)^2\right]^{\frac{3}{2}}. \tag{2.11}$$

Making the following substitutions in (2.11)

$$\alpha = \frac{2H\left(1 + a^2\right)^{\frac{3}{2}}}{1 + a^2 - c^2}, \quad \beta = \frac{ac}{1 + a^2} \quad \text{and} \quad \gamma^2 = \frac{1 + a^2 - c^2}{\left(1 + a^2\right)^2},$$

results in

$$\frac{g''}{\left[\gamma^2 - \left(g' + \beta\right)^2\right]^{\frac{3}{2}}} = -\alpha. \tag{2.12}$$

Integrating (2.12) and isolating the expression for $g'$ gives us

$$g' = \pm\frac{\gamma^3(c_1 - \alpha x)}{\sqrt{1 + \gamma^4(c_1 - \alpha x)^2}} - \beta, \tag{2.13}$$

where $c_1$ is a constant. By integrating (2.13), we obtain

$$g(x) = \pm\frac{1}{\alpha\gamma}\sqrt{1+\gamma^4(c_1-\alpha x)^2} - \beta x + c_2, \tag{2.14}$$

where $c_2$ is a constant. Substituting the values of $\alpha$, $\beta$ and $\gamma$ in (2.14) gives us

$$g(x) = \pm\frac{\sqrt{1+a^2-c^2}}{2H\big(1+a^2\big)}\sqrt{1+a^2+4H^2(c_3-x)^2} - \left(\frac{ac}{1+a^2}\right)x + c_2, \tag{2.15}$$

where $c_3$ is a constant. Also, $f' = c$ gives us

$$f(u) = cu + c_4, \tag{2.16}$$

where $c_4$ is a constant. Thus from (1.3), (2.15) and (2.16), we have

$$\begin{aligned} z(u,v) = &\pm\frac{\sqrt{1+a^2-c^2}}{2H\big(1+a^2\big)}\sqrt{1+a^2+4H^2(c_3-au-v)^2} \\ &+\left(\frac{u-av}{1+a^2}\right)c + p, \end{aligned} \tag{2.17}$$

where $p$ is a constant and $c^2 < 1 + a^2$.

*Case II.* When $f'' \neq 0$. Differentiating (2.9) w.r.t. $u$ gives us

$$\begin{aligned} &\big(1-g'^2\big)f''' + \big(1+a^2-f'^2\big)ag''' - 2\big(f'+ag'\big)g''f'' \\ &= -6HT\big[\big(f'+ag'\big)\big(f''+a^2g''\big) + ag'g''\big]. \end{aligned} \tag{2.18}$$

Now, differentiating (2.9) w.r.t. $v$ gives us

$$\big(1+a^2-f'^2\big)g''' - 2g'g''f'' = -6HT\big[\big(f'+ag'\big)ag'' + g'g''\big]. \tag{2.19}$$

Eqn's (2.18) and (2.19) yield

$$\big(1-g'^2\big)f''' - 2f'f''g'' = -6HT\big(f'+ag'\big)f''. \tag{2.20}$$

Substituting the value of $g''$ from (2.9) in (2.20), we have

$$\big(1-g'^2\big)f''' - 2f'f''\left[\frac{2HT^3-\big(1-g'^2\big)}{\big(1+a^2-f'^2\big)}\right] = -6HT\big(f'+ag'\big)f''. \tag{2.21}$$

Thus, we obtain

$$\begin{aligned} &\big(1-g'^2\big)\left[\big(1+a^2-f'^2\big)f''' + 2f'f''^2\right] \\ &= -2HT\left[3\big(f'+ag'\big)\big(1+a^2-f'^2\big) - 2T^2f'\right]f''. \end{aligned} \tag{2.22}$$

Squaring both sides of (2.22) and substituting the value of $T^2$ gives us

$$\begin{aligned}&\left(1-g'^2\right)^2\left[\left(1+a^2-f'^2\right)f'''+2f'f''^2\right]^2\\&=4H^2\left[1-\left(f'+ag'\right)^2-g'^2\right]\\&\quad\times\left[3\left(f'+ag'\right)\left(1+a^2-f'^2\right)-2\left\{1-\left(f'+ag'\right)^2-g'^2\right\}f'\right]^2f''^2.\end{aligned}\tag{2.23}$$

We notice that the above expression can be expanded as a polynomial in the powers of $g'$. The coefficients of $g'$ in the above expression are functions of $u$, and the expression itself is identically zero, so each term must be zero. But, the coefficient of $g'$ with the highest degree, i.e., 6 in (2.23), is $-16H^2(1+a^2)^3f'^2f''^2$, which is non-zero. Thus, it follows that $g'$ is a constant (Liu and Jung have used the same argument in [6]). Substituting $g'=c$ in (2.9) yields

$$z(u,v)=cv\pm\frac{\sqrt{1-c^2}}{2H}\sqrt{1+4H^2(b-u)^2}+q,\tag{2.24}$$

where $b,c,q$ are constants and $c^2<1$. Thus, the proof of the theorem is complete. □

**Theorem 2.2.** *Let $\Psi(v,z)=(u(v,z),v,z)$ be a spacelike affine translation surface of type 2 in $\mathbb{E}^3_1$. If $\Psi(v,z)$ has a non-zero constant mean curvature, then $u(v,z)$ is given by*

$$u(v,z)=\pm\frac{\sqrt{1-b^2+c^2}}{2H\left(1-b^2\right)}\sqrt{4H^2\left(a-bv-z\right)^2-\left(1-b^2\right)}+\left(\frac{v+bz}{1-b^2}\right)c+p,$$

*such that $1-b^2>0$; or*

$$u(v,z)=\pm\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2(a-v)^2-1}+cz+q,$$

*where $a,b,c,p,q$ are all constants.*

*Proof.* The mean curvature $H(v,z)$ of a spacelike surface $r(v,z)=(u(v,z),v,z)$ in $\mathbb{E}^3_1$ is given by

$$H(v,z)=\frac{\left(u_v^2+1\right)u_{zz}-2u_vu_zu_{vz}+\left(u_z^2-1\right)u_{vv}}{2\left(u_z^2-u_v^2-1\right)^{\frac{3}{2}}},\tag{2.25}$$

where $u_v, u_z$ denotes partial differentiation of $u$ w.r.t. $v$ and $z$, respectively. We obtain the following partial derivatives of $u(v,z)$ from (1.5)

$$\begin{cases} u_v = h' + bt', \\ u_z = t', \\ u_{vv} = h'' + b^2 t'', \\ u_{zz} = t'', \\ u_{vz} = bt'', \end{cases} \tag{2.26}$$

where $h' = \dfrac{dh}{dv}$ and $t' = \dfrac{dt}{dy}$ for $y = bv + z$. Using (2.26) in (2.25), gives us

$$-h'' + \big(1 - b^2\big)t'' + \big(t'^2 h'' + h'^2 t''\big) = 2HT^3, \tag{2.27}$$

where $T^2 = t'^2 - \big(h' + bt'\big)^2 - 1$ and $H(\neq 0)$ is a constant. Eqn (2.27) writes as

$$\big(-1 + t'^2\big)h'' + \big(1 - b^2 + h'^2\big)t'' = 2HT^3, \tag{2.28}$$

Now, we have the following two cases:

*Case I.* When $h' = c$ is a constant. It follows from (2.28)

$$\big(1 - b^2 + c^2\big)t'' = 2H\left[t'^2 - \big(c + bt'\big)^2 - 1\right]^{\frac{3}{2}}. \tag{2.29}$$

Thus, we have

$$t'' = \frac{2H\big(1-b^2\big)^{\frac{3}{2}}}{1-b^2+c^2}\left[\left(t' - \frac{bc}{1-b^2}\right)^2 - \frac{1-b^2+c^2}{\big(1-b^2\big)^2}\right]^{\frac{3}{2}}. \tag{2.30}$$

Making the following substitutions in (2.30)

$$\alpha = \frac{2H\big(1-b^2\big)^{\frac{3}{2}}}{1-b^2+c^2}, \quad \beta = -\frac{bc}{1-b^2} \quad \text{and} \quad \gamma^2 = \frac{1-b^2+c^2}{\big(1-b^2\big)^2},$$

results in

$$\frac{t''}{\left[\big(t'+\beta\big)^2 - \gamma^2\right]^{\frac{3}{2}}} = \alpha. \tag{2.31}$$

Integrating (2.31) and isolating the expression for $t'$ gives us

$$t' = \pm\frac{\gamma^3(c_1 - \alpha y)}{\sqrt{\gamma^4(c_1 - \alpha y)^2 - 1}} - \beta, \tag{2.32}$$

where $c_1$ is a constant. Thereby integrating (2.32), we obtain

$$t(y) = \pm\frac{1}{\alpha\gamma}\sqrt{\gamma^4(c_1 - \alpha y)^2 - 1} - \beta y + c_2, \tag{2.33}$$

where $c_2$ is a constant. Substituting the values of $\alpha$, $\beta$ and $\gamma$ in (2.33) gives us

$$t(y) = \pm\frac{\sqrt{1-b^2+c^2}}{2H\big(1-b^2\big)}\sqrt{4H^2(c_3-y)^2-\big(1-b^2\big)}+\frac{bc}{1-b^2}y+c_2. \tag{2.34}$$

Also, $h' = c$ gives us

$$h(v) = cv + c_4, \tag{2.35}$$

where $c_4$ is a constant. Thus, from (1.5), (2.34) and (2.35), we have

$$\begin{aligned} u(v,z) = \pm&\frac{\sqrt{1-b^2+c^2}}{2H\big(1-b^2\big)}\sqrt{4H^2\big(c_3-bv-z\big)^2-\big(1-b^2\big)} \\ &+\left(\frac{v+bz}{1-b^2}\right)c+p, \end{aligned} \tag{2.36}$$

where $p$ is a constant.

*Case II.* When $h'' \neq 0$. Differentiating (2.28) w.r.t. $v$ gives us

$$\begin{aligned} &\big(-1+t'^2\big)h''' + \big(1-b^2+h'^2\big)bt''' + 2\big(h'+bt'\big)h''t'' \\ &= 6HT\big[bt't'' - \big(h'+bt'\big)\big(h''+b^2t''\big)\big]. \end{aligned} \tag{2.37}$$

Now, differentiating (2.28) w.r.t. $z$ gives us

$$\big(1-b^2+h'^2\big)t''' + 2h''t't'' = 6HT\big[t't'' - \big(h'+bt'\big)bt''\big]. \tag{2.38}$$

Eqn's (2.37) and (2.38) yield

$$\big(-1+t'^2\big)h''' + 2h'h''t'' = -6HT\big(h'+bt'\big)h''. \tag{2.39}$$

Substituting the value of $t''$ from (2.28) in (2.39), we have

$$\big(-1+t'^2\big)h''' + 2h'h''\left[\frac{2HT^3-\big(-1+t'^2\big)}{\big(1-b^2+h'^2\big)}\right] = -6HT\big(h'+bt'\big)h''. \tag{2.40}$$

Thus, we obtain

$$\begin{aligned} &\big(-1+t'^2\big)\left[\big(1-b^2+h'^2\big)h''' - 2h'h''^2\right] \\ &= 2HT\left[-3\big(h'+bt'\big)\big(1-b^2+h'^2\big) - 2T^2h'\right]h'' \end{aligned} \tag{2.41}$$

Squaring both sides of (2.41) and substituting the value of $T^2$ gives us

$$\begin{aligned} &\big(-1+t'^2\big)^2\left[\big(1-b^2+h'^2\big)h''' - 2h'h''^2\right]^2 \\ &= 4H^2\left[t'^2 - \big(h'+bt'\big)^2 - 1\right] \\ &\times\left[-3\big(h'+bt'\big)\big(1-b^2+h'^2\big) - 2\big\{t'^2 - \big(h'+bt'\big)^2 - 1\big\}h'\right]^2 h''^2. \end{aligned} \tag{2.42}$$

The coefficient of $t'^6$ in (2.42) is $16H^2(1-b^2)^3h'^2h''^2$, which is non-zero and the concluding argument in Theorem 2.1 results in $t'$ being constant. Substituting $t'=c$ in (2.28) yields

$$u(v,z) = \pm\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2(a-v)^2-1}+cz+q, \tag{2.43}$$

where $a, c, q$ is a constant. Thus, the proof of the theorem is complete. □

## 3. Maximal affine translation surfaces

**Theorem 3.1.** *Let $\Psi(u,v)=(u,v,z(u,v))$ be a maximal affine translation surface of type 1 in $\mathbb{E}_1^3$. Then, $\Psi(u,v)$ is either a planar surface or $z(u,v)$ is given by*

$$z(u,v)=\frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{1+a^2}u+c_1\right]}{\cosh\left[c(au+v)+c_2\right]}\right|+c_3,$$

*where $a, c, c_1, c_2, c_3$ are constants and $c\neq 0$.*

*Proof.* Taking $H=0$ in (2.9) gives us

$$-\left(1-g'^2\right)f''-\left(1+a^2-f'^2\right)g''=0, \tag{3.44}$$

which writes as

$$\frac{f''}{1+a^2-f'^2}=\frac{-g''}{1-g'^2}=\lambda, \tag{3.45}$$

where $\lambda$ is a constant and $\left(1+a^2-f'^2\right)\left(1-g'^2\right)\neq 0$.

Depending on $\lambda$, we have the following 2 cases:

*Case I.* $\lambda=0$, gives us

$$f''=0, \quad g''=0, \tag{3.46}$$

which leads to a planar surface in $\mathbb{E}_1^3$.

*Case II.* $\lambda\neq 0$, gives us

$$f(u)=\frac{1}{c}\log\left|2\cosh\left[c\sqrt{1+a^2}u+c_1\right]\right|+c_3, \tag{3.47}$$

$$g(au+v)=\frac{-1}{c}\log\left|2\cosh\left[c(au+v)+c_2\right]\right|+c_4, \tag{3.48}$$

where $a, c, c_1, c_2, c_3, c_4$ are constants. Thus, we have

$$z(u,v)=\frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{1+a^2}u+c_1\right]}{\cosh\left[c(au+v)+c_2\right]}\right|+p, \tag{3.49}$$

where $c\neq 0$ and $p$ is a constant. Thus, the proof of the theorem is complete. □

**Theorem 3.2.** *Let $\Psi(v,z) = (u(v,z), v, z)$ be a maximal generalized affine translation surface of type 2 in $\mathbb{E}^3_1$. Then, $\Psi(v,z)$ is either a planar surface or $u(v,z)$ is given by*

$$u(v,z) = \frac{1}{c}\log\left|\frac{\cos\left[c\sqrt{1-b^2}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \quad b^2 < 1$$

*or*

$$u(v,z) = \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{b^2-1}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \quad b^2 > 1;$$

*where $b, c, c_1, c_2, c_3$ are constants and $c \neq 0$.*

*Proof.* For $H = 0$, it follows from (2.28)

$$\left(-1 + t'^2\right)h'' + \left(1 - b^2 + h'^2\right)t'' = 0. \tag{3.50}$$

Thus, we have

$$\frac{-h''}{1-b^2+h'^2} = \frac{t''}{-1+t'^2} = \lambda, \tag{3.51}$$

where $\lambda$ is a constant and $\left(1-b^2+h'^2\right)\left(-1+t'^2\right) \neq 0$. $\lambda = 0$ leads to a planar surface in $\mathbb{E}^3_1$ and when $\lambda \neq 0$, we have the following cases:

*Case I.* If $b^2 < 1$; (3.51) yields

$$\begin{aligned} u(v,z) &= h(v) + t(bv+z), \\ &= \frac{1}{c}\log\left|\frac{\cos\left[c\sqrt{1-b^2}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \end{aligned} \tag{3.52}$$

where $b, c, c_1, c_2, c_3$ are constants and $c \neq 0$.

*Case II.* If $b^2 > 1$; (3.51) yields

$$\begin{aligned} u(v,z) &= h(v) + t(bv+z), \\ &= \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{b^2-1}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \end{aligned} \tag{3.53}$$

where $b, c, c_1, c_2, c_3$ are constants and $c \neq 0$. Thus, the proof of the theorem is complete. □

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# ALGORITHMIC APPROACH TO BITONIC ALGEBRAS AND THEIR GRAPHS

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Abstract. Under the aim of this paper, we establish the terms of graphs related with bitonic-algebras, which is a bitonic-graph where the vertices are the elements of bitonic algebra and where the edges are the companian of two vertices, that is two elements from bitonic algebra. We designate the upper sets of elements in a bitonic algebra and studied properties of these sets. We state algorithms to check whether the given set is a bitonic algebra or a commutative bitonic algebra or not. Additionally, we mention the codes of these algorithms. Moreover, we associate the algorithms of graphs of a bitonic algebra and state properties of these graphs obtained.

**Keywords**: Bitonic algebras, upper sets, graphs, graphs of algebras

**2010 Mathematics Subject Classification**: 03G25, 05C25, 13P25, 33F99, 68W30

## 1. Introduction

In recent mathematical articles and studies, it has been an important matter that the artificial intelligence is to make a computer simulate a human being in dealing with certainty and uncertainty in information. At this stage, logic plays an important role to act the foundation of this mission. Classical logic is a base for information processing dealing with certain information whereas nonclassical logic including many-valued logic and fuzzy logic

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use classical logic to handle information with various facets of uncertainty, such as fuzziness and randomness. For this reason, for computer science to deal with fuzzy information and uncertain information, non classical logic has been used as a useful and a formal instrument.

As required for this reason, BCK-algebras are considered as a generalization of the notion of algebra of sets with the set subtraction as the only fundamental nonnullary operation and on the other hand the notion of implication algebra. They are defined by Imai and Iseki in [19]. Later, Komori [24] introduced the new class of algebras called BCC-algebras as he gave the name of this work to prove the class of all BCK-algebras does not form a variety. The notion of dual BCC-algebras is a generalization of many different algebras such as DBCK-algebras ([9, 23, 40]), Hilbert-algebras([13, 16, 17, 27]), Heyting-algebras (or Brouwerian lattices)([11, 8]), implication algebras ([1]) and lattice implication algebras ([37, 38])that ensure the property: (P) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$. Lastly, Yon and Özbal in [39] defined the bitonic algebras and with the help of derivations they studied properties of this algebra. Additionally, Özbal studied on filters of bitonic algebras to investigate the relations between filters and upper sets in [5].

With the help of graphs to deeply study algebraic systems, graphs have been considered a significant method and topic in many mathematical papers and studies in recent times. For instance, in 1998 Beck [7] studied rings and algebras in this manner. In this study, by presenting the zero-divisor graph of a commutative ring a correlation between graph theory and commutative ring theory is constructed. Then, many mathematicians extend this graphs in classical structures more definitely, in commutative ring [3], commutative semirings [35] and semigroups [25], near-rings [36], Cayley Vague Graphs [28]. These studies consider graphs in classical and non-classical algebras. The total graph of a commutative ring is studied in [6] and investigated the total graph of a commutative semiring with non-zero identity. Also, the annihilator graph of a commutative ring is considered in [2] and the area of zero-divisor graphs of commutative rings is focused in [12]. Brešar et al.[10] defined the cover incomparability graphs of posets and the directed graphs of lattices is examined in [32]. Nowadays, many mathematicians have focused on graph of logical algebras because of the reason that these algebras are related to information systems and many other different branches of computer sciences. For example, Jun and Lee [20] studied zero-divisor graph in BCK/BCI-algebras whereas Hu and Li [18] obtained some properties on graphs of BCH-algebras. Additionally, Gürsoy et al. introduced an alternative construction of graphs on MV-algebras in [15] and they obtained a suitable representation of MV-algebras by using graphs. Similarly, Kırcalı

Gürsoy focused on the notion of graphs on Wajsberg algebras and stated that commutative W-graphs are also symmetric graphs in [22]. And, many other graph operations are studied in [21], such as coloring of a commutative ring in [4], and these will be applied on graphs of bitonic algebras in the future work.

Motivated by these works, in this paper we study the associated graphs of bitonic algebras that are the generalizations of dual BCC-algebras in a different manner than those mentioned above.

## 2. Preliminaries

During this section, firstly, we give some fundamental definitions, lemmas, theorems about bitonic algebras that will be used as a tool. Secondly, we remind some graph theory concepts used in this study.

### 2.1. Bitonic Algebras.

A *dual BCC-algebra* is an algebraic system $(X, *, 1)$ satisfying the following axioms for all $x, y, z \in X$ :

(D1) $(x * y) * ((y * z) * (x * z)) = 1$,

(D2) $1 * x = x$,

(D3) $x * 1 = 1$,

(D4) $x * x = 1$,

(D5) $x * y = 1$ and $y * x = 1$ imply $x = y$.

**Definition 2.1.** [39] *A bitonic algebra is an algebraic system* $(A, *, 1)$ *satisfying the following axioms for every* $a, b, c \in A$*:*

(B1) $a * 1 = 1$,

(B2) $1 * a = a$,

(B3) $a * b = 1$ *and* $b * a = 1$ *implies* $a = b$,

(B4) $a * b = 1$ *implies* $(c * a) * (c * b) = 1$ *and* $(b * c) * (a * c) = 1$.

*where* $A$ *is a set,* $1$ *an element in* $A$ *and* $*$ *a binary operation on* $A$.

**Lemma 2.1.** [39] *In a bitonic algebra* $(A, *, 1)$ *for all* $a, b, c \in A$ *the followings hold:*

(1) $a * a = 1$,

(2) $a * b = b * c = 1$ *implies* $a * c = 1$,

(3) $a * (b * a) = 1$.

**Corollary 2.1.** [39] *If a binary relation "$\leq$" on $A$ where $(A,*,1)$ be a bitonic algebra is defined by*

$$a \leq b \iff a*b=1$$

*for any $a,b \in A$, then it is clear that by (B3) and Lemma 2.1 $\leq$ is a partial order on $A$.*

**Lemma 2.2.** [39] *Let $(A,*,1)$ be a bitonic algebra. Then for all $a,b,c \in A$ :*

(1) *$a \leq b$ refers $c*a \leq c*b$ and $b*c \leq a*c$,*
(2) *$a \leq b*a$.*

**Example 2.1.** [39] *Let $*$ be a binary operation on $N=\{1,x,y,z,w\}$ with the table given below:*

TABLE 1. Cayley Table for N

| $*$ | $1$ | $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|---|---|
| $1$ | $1$ | $x$ | $y$ | $z$ | $w$ |
| $x$ | $1$ | $1$ | $y$ | $z$ | $w$ |
| $y$ | $1$ | $x$ | $1$ | $z$ | $w$ |
| $z$ | $1$ | $1$ | $1$ | $1$ | $x$ |
| $w$ | $1$ | $1$ | $1$ | $z$ | $1$ |

*$(N,*,1)$ is a bitonic algebra and the Hasse diagram can be given as*

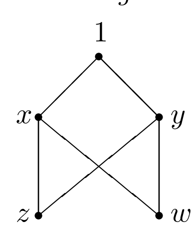


FIGURE 1. Hasse diagram of the bitonic algebra $N$ in Example 2.1

**Definition 2.2.** *Let $(A,*,1)$ be a bitonic algebra. The binary operation "$\diamondsuit$" on $A$ is defined by $a\diamondsuit b=(a*b)*b$ for every $a,b \in A$.*

**Lemma 2.3.** [39] *For the binary operation $\diamondsuit$ on $(A,*,1)$ as a bitonic algebra*

(1) *$b \leq a\diamondsuit b$,*

(2) $a \leq b$ *implies* $a \Diamond b = b$,

(3) $1 \Diamond a = 1$ *and* $a \Diamond 1 = 1$

hold for every $a, b \in A$.

**Definition 2.3.** [39] $S \neq \emptyset$ *as a subset of a bitonic algebra* $A$ *is called a bitonic subalgebra of* $A$ *if "*$x * y \in S$*" for all* $x, y \in S$*, and* $F \neq \emptyset$ *as a subset of* $A$ *is called a filter of* $A$ *if*

(F1) $1 \in F$,

(F2) $x \in F$ *with* $x * y \in F$ *refers* $y \in F$ *for any* $x, y \in F$.

**Definition 2.4.** [39] *A bitonic algebra* $(A, *, 1)$ *is said to be commutative if*

$$(a * b) * b = (b * a) * a \text{ for every } a, b \in A.$$

**Example 2.2.** *Let* $*$ *be a binary operation on* $C = \{1, a, b\}$ *with the table given below then*

TABLE 2. Cayley Table for C

| $*$ | $1$ | $a$ | $b$ |
|---|---|---|---|
| $1$ | $1$ | $a$ | $b$ |
| $a$ | $1$ | $1$ | $b$ |
| $b$ | $1$ | $a$ | $1$ |

$(C, *, 1)$ *is a commutative bitonic algebra.*

2.2. **Some Basic Concepts on Graph Theory.** Here, we consider some basic definitions and concepts in graph theory.

**Definition 2.5.** [14] *A graph is a pair* $G = (V, E)$ *of sets satisfying* $E \subseteq V \times V$*; thus, the elements of* $E$ *are ordered pairs of* $V$ *. To avoid notational ambiguities, we shall always assume tacitly that* $V \cap E = \emptyset$*. The elements of* $V$ *are the vertices (or nodes, or points) of the graph* $G$*, the elements of* $E$ *are its edge edges (or lines). The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge.*

**Example 2.3.** *Let $V = \{1,2,3,4,5\}$ be a vertex set and $E = \{(1,2),(2,3),(3,5),(3,4),(4,5)\}$ be a edge set of $G = (V,E)$. We can illustrate this graph as Figure 2.*

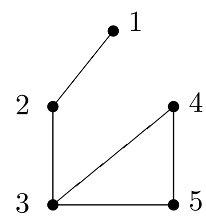


FIGURE 2. The graph of $G = (V,E)$ for Example 2.3

**Definition 2.6.** [14] *A path is a non-empty graph $P = (V,E)$ of the form*

$$V = \{x_0, x_1, \dots, x_k\},\ E = \{(x_0,x_1),(x_1,x_2),\dots,(x_{k-1},x_k)\}$$

*where the $x_i$ are all distinct. The vertices $x_0$ and $x_k$ are linked by $P$ and are called its ends; the vertices $x_1, x_2, \dots, x_{k-1}$ are inner vertices of $P$. The number of edges of a path is its length, and the path of length $k$ ise denoted by $P^k$.*

**Definition 2.7.** [14] *The distance $d_G(x,y)$ in $G$ of two vertices $x,y$ is the length of a shortest $x-y$ path in $G$; if no such path exist, we set $d(x,y)=\infty$. The greatest distance between any two vertices in $G$ is the diameter of $G$, diameter denoted by diam(G). The diameter of $G$ is said to be zero if there is only one vertex in $G$.*

**Definition 2.8.** [26, 34] *A connected graph with more than one vertex has a diameter of one if and only if each pair of distinct vertices forms an edge and it is called a complete graph.*

## 3. Some New Algorithms on Bitonic Algebra

In this section, we will introduce the algorithms that test whether the definitions given in the previous section are ensured by any sets. These algorithms are coded in VB-Script Language in Ms EXCEL Program to make our studies on these algebras easier than usual. After explaining these mentioned algorithms we will share the codes belongs to this language.

**Algorithm 1:** Determining Bitonic Algebra

```
Data: Set A, * operation, operator table
Result: (A, *, 1) is a bitonic algebra or not
initialization;
if A ≠ ∅ OR 1 ∉ A then
    Return false
foreach x in A do
    if x * 1 ≠ 1 then
        Return false
    if 1 * x ≠ x then
        Return false
foreach x, y in A do
    if x * y = 1 AND y * x = 1 then
        if x ≠ y then
            Return false
foreach x, y in A do
    if x * y = 1 then
        foreach z in A do
            if (z * x) * (z * y) ≠ 1 OR (y * z) * (x * z) ≠ 1 then
                Return false
```

In Algorithm 1, it is checked whether the given structure is a bitonic algebra or not by using Definition 2.1 of bitonic algebras. The inputs of this algorithm are the set $A$ and the Cayley Table of the operator $*$. Firstly, the algorithm examines whether the given set $A$ is an empty set, and 1 belongs to $A$. If the given set $A$ is empty or does not contain 1, then the algorithm returns $FALSE$. Then, it examines for every $x$ in $A$, whether the conditions $B1$ and $B2$ of Definition 2.1 are satisfied. If any of these conditions are not satisfied then the algorithm returns $FALSE$.

The following step of the algorithm is to check the condition $B3$ of Definition 2.1. According to this step, whenever $x * y = 1$ or $y * x = 1$ is satisfied for any $x, y$ in $A$, it is checked whether these $x$ and $y$ are different elements. If these elements are different than each other then the algorithm returns $FALSE$. The last step of the algorithm checks whether the last part $B4$ of Definition 2.1 is satisfied or not. According to this step, it is examined whether $x*y = 1$ is satisfied or not for any elements $x, y$ in $A$. If this is satisfied for any $x, y$ in $A$, then for any element in $A$ (say $z$ in $A$) the algorithm works out for the solution of $(z * x) * (z * y)$ and $(y * z) * (x * z)$. If one these results is different than 1, then the algorithm $FALSE$.

```
'The Cayley Table is being imported into a two-dimensional array
For i = 2 To m + 1
    For j = 2 To m + 1
        m1(i - 1, j - 1) = Cells(i, j)
    Next j
Next i

'The conditions B1 and B2 are being checked
control = 0
For i = 1 To m
    If m1(1, i) <> Cells(1, i + 1) Then control = 1
    If m1(i, 1) <> 1 Then control = 1
Next i

If control <> 0 Then
    MsgBox "B1 OR B2 are not obtained"
    GoTo bit
End If

'The condition B3 is being checked
For i = 1 To m
    For j = 1 To m
        If m1(i, j) = 1 Then
            If m1(j, i) = 1 Then
                If i <> j Then
                    MsgBox "B3 is not obtained"
```

```
                    control = 1
                    GoTo bit
                End If
            End If
        End If
    Next j
Next i

'The condition B4 is being checked
For i = 1 To m
    For j = 1 To m
        If m1(i, j) = 1 Then
            For k = 1 To m
                If (m1(m1(k, i), m1(k, j)) = 1 And m1(m1(j, k), m1(i, k)) = 1) <> True Then
                    MsgBox "B4 is not appropriate (" & k & "*" & i & ")*(" & k & "*" & j & ") or one of these  (" & j & "*" & k & ")*(" & i & "*" & k & ") is not 1"
                    control = 1
                    GoTo bit
                End If
            Next k
        End If
    Next j
Next i

If control = 0 Then
    Cells(12, 1) = "Bitonic Algebra"
    Else
    Cells(12, 1) = "Not a Bitonic Algebra"
    GoTo bit
End If
bit:
```

Listing 1. Codes for Algorithm 1

We consider a code for this table that will be easily runned in EXCEL to list its entries.

**Algorithm 2:** Generating $\Diamond$-Operation Table on a Bitonic Algebra

**Data:** (A, 1, *) Bitonic Algebra, * Operation Table

**Result:** $\Diamond$-Table[ , ]

```
◊-Table[ , ]=null
foreach x, y in A do
    ◊-Table[x,y]=(x*y)*y
```

The Cayley table of the $\Diamond$ operator is obtained with the help of the Algorithm 2. This time, the input of this algorithm is a bitonic algebra. In the beginning, an empty Cayley table of the $\Diamond$ operator is formed. Then, the solution of $(x * y) * y$ is examined for every element $x, y$ in $A$.

One of the main purpose of this algorithm is that this algorithm is coded in VBScript language and these codes are given in Listing 2.

```
For i = 1 To m
    For j = 1 To m
        m1v(i, j) = m1(m1(i, j), j)
    Next j
Next i
```

Listing 2. Codes for Algorithm 2

**Algorithm 3:** Determining Commutativity of a Bitonic Algebra

**Data:** $\Diamond$-Table[ , ] of (A, 1, *) Bitonic Algebra

**Result:** (A, *, 1) is a Commutative Bitonic Algebra or not

```
initialization;
foreach x, y in A do
    if x ◊ y ≠ y ◊ x then
        Return false
```

In this Algorithm 3, it is checked whether the given bitonic algebra is commutative or not. The inputs of this algorithm are a bitonic algebra and the cayley table of $\Diamond$ operation defined in this bitonic algebra. If this table is symmetric then the given bitonic algebra is commutative. For this reason, it is examined whether $x\Diamond y$ and $y\Diamond x$ are the same or not for any $x, y$ in the given bitonic algebra. If a non-identical condition is detected, then the algorithm outputs $FALSE$.

This algorithm is coded in VBScript language and these codes are given in Listing 3

```
controlv = 0
For i = 1 To m - 1
    For j = i + 1 To m
    If (m1v(i, j) <> m1v(j, i)) Then controlv = 1
    Next j
Next i

If controlv = 0 Then
    Cells(12, 13) = "Commutative Bitonic Algebra"
    Else
    Cells(12, 13) = "Not a Commutative Bitonic Algebra"
    GoTo bit
End If
bit:
```

Listing 3. Codes for Algorithm 3

## 4. Graphs on Bitonic Algebras

To consider graphs of bitonic algebras, firstly we focus on upper sets of these algebras. In this section, initially we give the definition of graph of bitonic algebras and introduce the algorithm lists the entries of adjacency matrix on a bitonic algebra. Moreover, we consider the codes of this algorithm to list the entries of the adjaceny matrix on a bitonic algebra.

**Definition 4.1.** $(A, *, 1)$ *as a commutative bitonic algebra corresponds to an undirected graph* $G(A)$*, where* $V(G(A))$ *consists of the elements of* $A$ *and two distinct elements* $a, b \in A$ *are called adjacent if and only if* $a \Diamond b = 1$*.* $G$ *is said to be a* $A-$*graph under these conditions.*

**Algorithm 4:** Generating Adjacency Matrix on a Bitonic Algebra

**Data:** $\Diamond$-Table[ , ] of (A, 1, *) Bitonic Algebra

**Result:** $Adj(G(A))$[ , ] Adjacency Matrix

initialization;

**foreach** $x$, $y$ **in** $A$ **do**

  **if** $x \Diamond y = 1$ **then**

    $Adj(G(A))[x, y] = 1$

  **else**

    $Adj(G(A))[x, y] = 0$

In this Algorithm 4, the adjacency matrix for any bitonic algebra given according to the Definition 4.1 is created. The inputs of this algorithm are a bitonic algebra and the Cayley table of $\Diamond$ operation defined in this algebra. In the beginning, the solution of $x\Diamond y$ for any elements $x, y$ in the given bitonic algebra is studied. If this solution is 1, then the corresponding element in a two-dimensional array in the adjacency matrix is assigned as 1, otherwise it is assigned as 0.

Coding in this algorithm VBScript language in MS EXCEL programe is one the most important part of this paper and these codes are given is Listing 4.

```
For i = 1 To m
    For j = 1 To m
        If m1v(i, j) = 1 Then
            m1vadj(i, j) = 1
            Else
            m1vadj(i, j) = 0
        End If
    Next j
Next i
```

Listing 4. Codes for Algorithm 4

**Example 4.1.** *Let $(A, *, 1)$ be the bitonic algebra mentione in Example 2.2. Using the Definition 4.1, the adjacency matrix of the graph of $A$ is*

*Adj(G(A))=*

| | 1 | $a$ | $b$ |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| $a$ | 1 | 0 | 1 |
| $b$ | 1 | 1 | 0 |

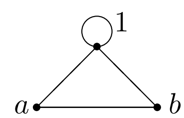


FIGURE 3. The graph of the bitonic algebra $A$ given in Example 2.2 with the Hasse Diagram

**Example 4.2.** *Let $(A, *, 1)$ be the bitonic algebra given in Example 2.1. Using the Definition 4.1, the adjacency matrix of the graph of $A$ is:*

*Adj(G(A))=*

| | 1 | $x$ | $y$ | $z$ | $w$ |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| $x$ | 1 | 0 | 1 | 1 | 1 |
| $y$ | 1 | 1 | 0 | 1 | 1 |
| $z$ | 1 | 0 | 0 | 0 | 0 |
| $w$ | 1 | 0 | 0 | 1 | 0 |

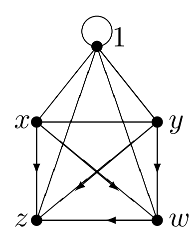


FIGURE 4. The graph of the bitonic algebra $A$ given in Example 2.1

**Lemma 4.1.** *Vertex 1 in a $A$-graph is adjacent to all vertices in $G(A)$.*

*Proof.* By definition of a bitonic algebra $a\Diamond 1 = (a * 1) * 1 = 1 * 1 = 1$ for ever $a \in A$ . So, vertex 1 and vertex $a$ are connected with an edge in every $A$-graph. □

**Lemma 4.2.** *A-graph $G(A)$ is connected by having $diam(G(A)) \leq 2$.*

*Proof.* Let $a, b \in A$ be any two distinct vertices on $G(A)$. Assume that $a \Diamond b = 1$. Therefore, we get $d(a, b) = 1$, so have $diam(G(A)) \leq 2$. Now, assume that $a \Diamond b \neq 1$. For a bitonic algebra we have $a \Diamond 1 = (a * 1) * 1 = 1 * 1 = 1$ and $b \Diamond 1 = (b * 1) * 1$. That is to say that $a$ is adjacent to 1 and $b$ is adjacent to 1. Hence, we get $d(a, b) \leq 2$ meaning that $diam(G(A)) \leq 2$. □

**Definition 4.2.** *$(A, *, 1)$ as a bitonic algebra is said to be $self-distributive$ if $a * (b * c) = (a * b) * (a * c)$ for all $a, b, c \in A$.*

**Example 4.3.** *The bitonic algebra $(A, *, 1)$ given in Example 1 is not a self-distributive algebra. But, if $*$ on $A = \{1, x\}$ is defined as a binary relation whose table is*

| $*$ | 1 | $x$ |
|---|---|---|
| 1 | 1 | $x$ |
| $x$ | 1 | 1 |

*It is easy to see that $(A, *, 1)$ a self-distributive bitonic algebra.*

We will consider the upper set of $a$ as an element in a bitonic algebra $A$ by

$$U(1, a) = \{x \in A | 1 * (a * x) = 1\}$$

for each $a \in A$.

For the rest of the paper $(A, *, 1) = A$ is given as a self-distributive bitonic algebra.

**Proposition 4.1.** *For any $a \in A$, the upper set $U(1, a)$ is a filter of $A$.*

*Proof.* Let $a \in A$. Then by $(B1)$, $(B2)$ we have for all $a \in A$ $a * 1 = 1$ and $1 * (a * 1) = 1$. Therefore, $1 \in U(1, a)$. Now, let $y \in U(1, a)$ and $y * x \in U(1, a)$ for any $x, y \in A$. Then we have $a * y = 1$ and $a * (y * x) = 1$. Since $A$ is a self-distributive bitonic algebra we have

$$a * (y * x) = (a * y) * (a * x) = 1 * (a * x) = (a * x) = 1.$$

Therefore, $x \in U(1, a)$. Hence we get $U(1, a)$ is a filter of $A$. □

**Proposition 4.2.** *Let $B, C \subseteq A$, then*

*(1) If $B \subseteq C$ then $U(1, C) \subseteq U(1, B)$,*

*(2) $U(1, B \cup C) = U(1, B) \cap U(1, C)$,*

*(3)* $U(1,B)\cup U(1,C)\subseteq U(1,B\cap C)$.

*Proof.* Let $B,C\subseteq A$.

(1) Let $B\subseteq C$ and suppose that $x\in U(1,C)$. So we have $1*(c*x)=1$ and $c*x=1$ for all $c\in C$. But we know that $B\subseteq C$ so every element of $B$ is in $C$ therefore, $b*x=1*(b*x)=1$ for all $b\in B\subseteq C$. Therefore, $x\in U(1,B)$ that is to say that $U(1,C)\subseteq U(1,B)$.

(2) We know that $B\subseteq B\cup C$ and $C\subseteq B\cup C$, and by part 1 we have $U(1,B\cup C)\subseteq U(1,B)$ and $U(1,C)$. Then $U(1,B\cup C)\subseteq U(1,B)\cap U(1,C)$.

Now, conversely, let $x\in U(1,B)\cap U(1,C)$. Then we have $1*(b*x)=1$ that is $b*x=1$ for all $b\in B$ and similarly $1*(c*x)=1$ that is $(c*x)=1$ for all $c\in C$. Therefore, for any $a\in B\cup C$ we have $a\in B$ or $a\in C$, and hence $1*(a*x)=1$ that is $(a*x)=1$ for all $a\in B\cup C$. And so we get $x\in U(1,B\cup C)$ gives us $U(1,B)\cap U(1,C)\subseteq U(1,B\cup C)$. Therefore, $U(1,B\cup C)=U(1,B)\cap U(1,C)$.

(3) We have $B\cap C\subseteq B$ and $B\cap C\subseteq C$ and also by part 1 of this proposition we have $U(1,B)\subseteq U(1,B\cap C)$ and $U(1,C)\subseteq U(1,B\cap C)$. Therefore, we get $U(1,B)\cup U(1,C)\subseteq U(1,B\cap C)$.

□

**Proposition 4.3.** *If $B\neq\emptyset$ is a subset of $A$ then,* $U(1,B)=\bigcap_{b\in B}U(1,b)$.

*Proof.* Let $B$ be a non-empty subset of a self-distributive bitonic algebra $A$. We have $B=\bigcup_{b\in B}\{b\}$, and by Proposition 4.2 (2) we have

$$U(1,B)=U(1,\bigcup_{b\in B}\{b\})=\bigcap_{b\in B}U(1,b).$$

□

**Proposition 4.4.** *If $a\leq b$ for any $a,b\in A$ then $U(1,a)\subseteq U(1,b)$.*

*Proof.* Let $x\in U(1,a)$. Then we get $1*(a*x)=1$ that is $a*x=1$. And, by our assumption we have $a\leq b$ and by Lemma 2.2 (1) $(b*x)\leq(a*x)$. Hence we get $(b*x)=1$ since $(a*x)=1$. Therefore, $x\in U(1,b)$. □

**Definition 4.3.** *Let $A$ be bitonic algebra $(A,*,1)$ that is $self-distributive$. Define a relation $\sim$ on $A$ as $a\sim b\Leftrightarrow U(1,a)=U(1,b)$.*

**Lemma 4.3.** *The relation forms an equivalence relation on a bitonic algebra* $(A, *, 1)$ *that is* $self - distributive$.

**Lemma 4.4.** *For every* $a, b, c \in (A, *, 1)$

$$a * (b * c) = b * (a * c).$$

*Proof.* Let $a, b, c \in A$. Then

$$a * (b * c) = (a * b) * (a * c),$$

and since $b \leq a * b$, $(a * b) * (a * c) \leq b * (a * c)$. This implies $a * (b * c) \leq b * (a * c)$.

Interchanging the role of $a$ and $b$, we can show $b * (a * c) \leq a * (b * c)$. Hence $a * (b * c) = b * (a * c)$. □

On the definition of *upper set*, since $1 * a = a$ for every $a \in A$, we should define the upper set by

$$U(a) = \{x \in A \mid a * x = 1\}.$$

**Definition 4.4.** *Let* $U(x)$ *and* $U(y)$ *be the upper sets of the bitonic algebra* $(A, *, 1)$ *for all* $x, y \in A$. *Then we can define "*$\diamondsuit$*" among the upper sets* $U(x)$ *and* $U(y)$ *as follow*

$$U(x)\diamondsuit U(y) = \{z \in A | a\diamondsuit b = z \ \ for \ \ all \ \ a \in U(x) \ \ and \ \ for \ \ all \ \ b \in U(y)\}.$$

*Then we can define the graph* $G_U(A)$, *where* $V(G_U(A))$ *consists of the elements of* $U(A)$, *and two distinct elements* $U(a), U(b) \in U(A)$ *are called adjacent if and only if* $U(a)\diamondsuit U(b) = \{1\}$.

**Example 4.4.** *Let* $A$ *be the bitonic algebra given in Example 2.1. Then we can consider the upper sets for every elements of* $A$ *as given below.*

$U(1) = \{1\}, U(x) = \{1, x\}, U(y) = \{1, y\}, U(z) = \{1, x, y, z\}, U(w) = \{1, x, y, w\}.$

*Then, we can give the table of* $\diamondsuit$ *among these upper sets as follow.*

| $\diamondsuit$ | $U(1)$ | $U(x)$ | $U(y)$ | $U(z)$ | $U(w)$ |
|---|---|---|---|---|---|
| $U(1)$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ | $\{1\}$ |
| $U(x)$ | $\{1\}$ | $\{1,x\}$ | $\{1\}$ | $\{1,x\}$ | $\{1,x\}$ |
| $U(y)$ | $\{1\}$ | $\{1\}$ | $\{1,y\}$ | $\{1,y\}$ | $\{1,y\}$ |
| $U(z)$ | $\{1\}$ | $\{1,x\}$ | $\{1,y\}$ | $\{1,x,y,z\}$ | $\{1,x,y,w\}$ |
| $U(w)$ | $\{1\}$ | $\{1,x\}$ | $\{1,y\}$ | $\{1,x,y\}$ | $\{1,x,y,w\}$ |

*Then the graph $G_U(A)$, where $V(G_U(A))$ consists of the elements of $U(A)$ and two distinct elements $U(x), U(y) \in U(A)$ are called adjacent if and only if $U(x)\diamondsuit U(y) = \{1\}$ that we can read from this table. Therefore, the adjacency matrix of the graph of $G_U(A)$ can be given as follows:*

$Adj(G_U(A))=$

| | $U(1)$ | $U(x)$ | $U(y)$ | $U(z)$ | $U(w)$ |
|---|---|---|---|---|---|
| $U(1)$ | 1 | 1 | 1 | 1 | 1 |
| $U(x)$ | 1 | 0 | 1 | 0 | 0 |
| $U(y)$ | 1 | 1 | 0 | 0 | 0 |
| $U(z)$ | 1 | 0 | 0 | 0 | 0 |
| $U(w)$ | 1 | 0 | 0 | 0 | 0 |

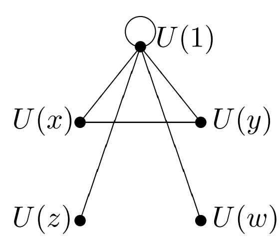


FIGURE 5. The graph of the bitonic algebra $A$ given in Example 2.1 with the Hasse Diagram

**Algorithm 5:** Generating U(1, $x$) Upper Sets on a Bitonic Algebra

**Data:** (A, 1, *) Bitonic Algebra, * Operation Table

**Result:** (U(1, $x$) Upper Sets for all $x \in A$

initialization;

**foreach** $x$ **in** $A$ **do**

  U(1, $x$)=*null*

  **foreach** $y$ **in** $A$ **do**

    **if** *1 * ( x * y ) = 1* **then**

      add element $y$ to U(1, $x$)

In Algorithm 5, the upper sets are examined according to the Definition 4.2 given for a bitonic algebra. The input of this algorithm is a bitonic algebra. Here, according to the given definition, the upper sets of every elements (except 1) are examined separately. Let $x$ be the element whose upper set is examined. In the algorithm, firstly an empty $U(1, x)$ element is created. Then, for every element $y$ in the bitonic algebra given, the operation $1 * (x * y)$ is calculated. If this is equivalent to 1, then $y$ is assigned to the upper set $U(1, x)$.

This algorithm is also coded in MS EXCEL programe in VBScript language and these codes are given in Listing 5.

```

For a = 2 To m
    'the sets are written in a cell in EXCEL"
    Cells(a, 37) = "U(1," & a & ") = "
    counter = 0
    For x = 1 To m
  If m1(1, m1(a, x)) = 1 Then
    If sayac <> 0 Then
        Cells(a, 38) = Cells(a, 38) & "," & x
    End If
    If the counter = 0 Then
        Cells(a, 38) = Cells(a, 38) & x
        counter = counter + 1
    End If
```

```
  End If
    Next x
Next a

```

Listing 5. Codes for Algorithm 5

## 5. Conclusion and Future Works

In this work, firstly we evoke basic knowledge about bitonic algebras and graphs. Then, algorithms are enhanced to check whether any given set is a bitonic algebra or not. These algorithms check the properties or definitions given in preliminaries for a bitonic algebra and additionally these algorithms are coded in VB-Script Language. In the third section, with the help of the operators defined for bitonic algebras, the graphs based on bitonic algebras are defined and some examples are stated. In recent years, the studies consider the relations between algebraic structures and graphs gain very importance. A new point of view is gained on daily life problems by graph modeling of theoretical findings in algebraic structures. Additionally, the Sheffer Stroke Operation that reducts axiom systems of many algebraic structures [30],[31], [29] and fuzzy concepts are used to study different notions of algebraical systems [33]. Because of this reason, our work to search for the relationship between bitonic algebra and different graph types continues based on this study. We will consider fuzzy graphs of bitonic algebras and also extend our work to the graphs of Sheffer stroke bitonic algebras.

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