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EDITORIAL

BAYRAM ŞAHİN *

This issue of International Journal of Maps in Mathematics is dedicated to the memory of Professor Krishan Lal Duggal. Dr. Krishan Lal Duggal passed away peacefully on December 1 st of 2023 at 92 years of age.

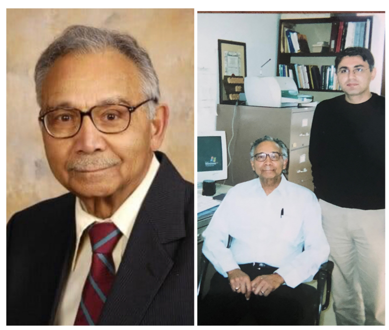

FIGURE 1. Professor Krishan L. Duggal and me at Windsor University

Professor Krishan Lal Duggal was well knownn for his passion of Mathematics and he inspired greatness and guided all he had contact with in achieving their goal of learning. Greatly respected in his field, he authored and published numerous books and research papers. He was an avid musician in his free time singing Bhajans and playing the sitar and harmonium.

I was familiar with Professor Duggal's publications. But I met him face to face in 2003, when I had the opportunity to be his post-doctoral student at the University of Windsor, Canada. We co-authored many research papers with Professor Duggal, and we also co-authored the book titled "Differential Geometry of Lightlike Submanifolds" published by Springer in 2010.

I will remember Professor Duggal with respect and gratitude. I will also remember him saying "The sky is not too high".

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* Editor-in-chief.

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A NEW PARAMETRIZATION OF CARTAN NULL BERTRAND CURVE IN MINKOWSKI 3-SPACE

STUTI TAMTA AND RAM SHANKAR GUPTA *

Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. We define and study a new parametrization of a Bertrand pair $\{\alpha, \alpha^*\}$, where α is a Cartan null Bertrand curve and α^* is a Bertrand partner curve of α in Minkowski 3-space by not taking the principal normal vector of the Cartan null Bertrand curve α parallel to $\overrightarrow{\alpha^* \alpha}$. We characterize both cases when the curve α^* is non-null and the null Bertrand partner of the curve α. Further, we investigate this type of Bertrand pair curve as a helix and a slant helix. Also, we provide some examples.

Keywords: Bertrand curves, general helices, slant helices, Cartan null curve, non-null curve, Minkowski 3-space.

2020 Mathematics Subject Classification: 53B30.

1. Introduction

In 1802, Lancret [14] defined a helix as a curve whose tangent vector makes a constant angle with a fixed straight line called the directrix. Later in 1845, Saint Venant [16] obtained a necessary and sufficient condition for a curve to be a general helix if its ratio of curvature to torsion is constant. In 1995, Scofield studied closed-form arc-length parametrizations for curves of constant precession and slant helices with a constant speed of precession [17]. In 2004, Izumiya and Takeuchi introduced the concept of the slant helix in $\mathbb{E}^3$ saying that the principal normal lines make a constant angle with a fixed direction. They characterized a

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curve as a slant helix if and only if the principal image of the major normal indicatrix has a constant geodesic curvature [10].

In 2010, Kula et al. studied the relationship between slant helices and helices, and they characterized slant helices in $\mathbb{E}^3$ in terms of differential equations [13]. In 2011, Ali and Lopez [1] characterized a non-null spacelike and timelike curve with a spacelike principal normal vector to be a slant helix in $\mathbb{E}^3_1$ if and only if either one of the two functions

$$\left(\frac{\tau}{k}\right)' \frac{k^2}{(k^2-\tau^2)^{3/2}} \quad or \quad \left(\frac{\tau}{k}\right)' \frac{k^2}{(\tau^2-k^2)^{3/2}} \tag{1.1}$$

is a constant function and $\tau^2 - k^2 \neq 0$.

In 2019, Liu and Pei [15] characterized a null Cartan curve α to be a slant helix in $\mathbb{E}^3_1$ if and only if the principal image of the major normal indicatrix has a constant geodesic curvature k_g, i.e.,

$$k_g = \frac{\tau'(s)}{2\sqrt{2}|\tau(s)|^{3/2}} \tag{1.2}$$

is a constant function for a non-zero torsion $\tau(s)$ of the curve.

In the fields of computer-aided design and computer graphics, helices can be used for tool path description, the simulation of kinematic motion, the design of highways, etc. [21]. Also, helix and slant helix play an important role in curve theory with numerous applications in the biological sciences, physics, etc. For instance, in the biological sciences, curves are used in the analysis of Deoxyribonucleic Acid (DNA), and in physics, they are used in characterizing the motion of particles in a magnetic field.

In 1845, Saint Venant [16] posed the question of whether the principal normal of a curve is the principal normal of another curve on the surface generated by the principal normal of the given one. Bertrand [4] gave an answer to this question in 1850 and introduced curves with the property that the principal normal vector of a curve α coincides with the principal normal vector of another curve α^* at their corresponding points. Further, these curves were characterized in $\mathbb{E}^3$ with condition $ak + b\tau = 1$, where a and b are nonzero constants and k and τ are the curvature and torsion of the curve, respectively [7]. Also, Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space (see [2, 3, 9, 11, 20]). In [3], Balgetir et al. studied the Cartan null Bertrand pair curve $\{\alpha, \alpha^*\}$ in $\mathbb{E}^3_1$. Later in 2021, Gokcek and Erdem [8] studied the Cartan null Bertrand curve α with the non-null Bertrand partner curve α^* in $\mathbb{E}^3_1$. In [6], Camci, et al. introduced a new relationship between a Bertrand pair α and α^* in $\mathbb{E}^3$ by not taking the vector $\overrightarrow{\alpha^*\alpha}$ parallel to a normal vector of Bertrand curve α. Using this approach, the present authors studied a new parametrization of Bertrand partner curves and spherical indicatrices in Euclidean 3-space [18, 19].

In view of this, we define and study a new parametrization of a Bertrand pair $\{\alpha,\, \alpha^*\}$, where α is a Cartan null Bertrand curve and α^* is a Bertrand partner curve of α in Minkowski 3-space by not taking the vector $\overrightarrow{\alpha^*\,\alpha}$ parallel to N of α in Minkowski 3-space.

2. Preliminaries

The Lorentz-Minkowski $\mathbb{E}^3_1$ is a space with metric,

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system. With respect to this metric, an arbitrary vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is said to be spacelike if $\langle \alpha, \alpha\rangle > 0$, timelike if $\langle \alpha, \alpha\rangle < 0$, and null if $\langle \alpha, \alpha\rangle = 0$. Similarly, if $\alpha = \alpha(s)$ denotes the position vector of an arbitrary non-null curve in $\mathbb{E}^3_1$, then it is called timelike and spacelike if all of its velocity vectors $\alpha'(s)$ are timelike and spacelike, respectively. The norm of the vector α is given by $||\alpha'|| = \sqrt{|\langle \alpha', \alpha'\rangle|}$. A non-null curve $\alpha(s)$ is parameterized by arc length s if $\langle \alpha'(s), \alpha'(s)\rangle = \pm 1$. A null curve is parameterized by pseudo-arc s if $\langle \alpha''(s),\, \alpha''(s)\rangle = 1$. If a null curve is parameterized by a pseudo-arc function, it is referred to as a Cartan null curve.

Let $\{T,\, N,\, B\}$ be the moving Frenet frame along a curve in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal, and the binormal vector field, respectively. Depending on the causal character of α, the Frenet equations have the following forms:

Case I. If α is a non-null curve, the Frenet formulas are [12]:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_1 k_1 & 0 \\ -\epsilon_0 k_1 & 0 & \epsilon_2\, k_2 \\ 0 & -\epsilon_1 k_2 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.3}$$

where $\langle T, T\rangle = \epsilon_0$, $\langle N, N\rangle = \epsilon_1$, $\langle B, B\rangle = \epsilon_2$, and $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$, and $k(s)$, $\tau(s)$ are curvature and torsion of α.

Case II. If α is a Cartan null curve, the Frenet formulas are [5]:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.4}$$

where $\langle T,\, B\rangle = \langle N,\, N\rangle = 1$, and $\langle T,\, T\rangle = \langle B,\, B\rangle = \langle T,\, N\rangle = \langle N,\, B\rangle = 0$, and $k_1(s)$, $k_2(s)$ are curvature and torsion of α.

In [3, 8], the authors defined the Cartan null Bertrand curve $\alpha : I \to \mathbb{E}^3_1$ with Bertrand partner curve $\alpha^* : I^* \to \mathbb{E}^3_1$ as follows:

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)\, N(s), \tag{2.5}$$

such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ coincides at $s \in I$, $s^* \in I^*$, where $\lambda(s)$ is C^∞-function on I.

Now, we define a new parametrization of a Bertrand pair $\{\alpha,\, \alpha^*\}$, where α is a Cartan null curve and α^* is a Bertrand partner curve of α in $\mathbb{E}^3_1$ such that the vector $\overrightarrow{\alpha\,\alpha^*}$ does not have to be parallel to N, which is given by

$$\alpha^*(s^*) = \alpha(s) + u(s)\, T(s) + v(s)\, N(s) + w(s)\, B(s), \tag{2.6}$$

where $u(s)$, $v(s)$ and $w(s)$ are differentiable functions and $\{T(s), N(s), B(s)\}$ is the Frenet-Serret frame of $\alpha(s)$. If we take $u = w = 0$ in (2.6), we obtain (2.5). Hence, (2.6) is the generalization of Cartan null Bertrand curves in $\mathbb{E}^3_1$.

3. New parametrization of Cartan null Bertrand curve in $\mathbb{E}^3_1$

In this section, we study a pair curve $\{\alpha,\, \alpha^*\}$ in $\mathbb{E}^3_1$ satisfying (2.6), where α is a Cartan null curve with curvature k_1 and torsion k_2, and α^* is a Bertrand partner curve of α with curvature k^*_1 and torsion k^*_2.

Now onwards, we denote the geodesic curvatures of the principal normal indicatrices (images) of a Cartan null Bertrand curve α by Γ, and that of a timelike and spacelike Bertrand partner curve α^* by Γ^*_1 and Γ^*_2, respectively.

Also, we set

$$\mu = \frac{(1 + u' + v\, k_2)}{\frac{ds^*}{ds}}, \quad \nu = \frac{(w' - v\, k_1)}{\frac{ds^*}{ds}}, \quad h = \frac{\mu}{\nu}, \tag{3.7}$$

and

$$\begin{cases} \beta = \frac{(\mu\, k_1 - \nu\, k_2)}{k^*_1\, \frac{ds^*}{ds}} = \pm 1, & \rho(s) = \frac{(w' - v\, k_1)\,(h^2\, k_1^2 - k_2^2)}{2\, k^*_1 \left(\frac{ds^*}{ds}\right)^2}, \\ \eta(s) = -\frac{(w' - v\, k_1)\,(h^2\, k_1^2 - k_2^2)}{2\, h\, k^*_1 \left(\frac{ds^*}{ds}\right)^2}, & \rho(s) = -h\, \eta(s). \end{cases} \tag{3.8}$$

Next, we have:

Theorem 3.1. *Let $\alpha : I \to \mathbb{E}_1^3$ be a Cartan null curve in $\mathbb{E}_1^3$ with curvatures $k_1(s) \neq 0$ and $k_2(s)$ satisfying (2.6).*

(i) If α^* is a timelike curve with $k_1^* \neq 0$, then $\{\alpha, \alpha^*\}$ is a Bertrand pair in $\mathbb{E}_1^3$ if and only if there exist differentiable functions u, v, w, and a real number h satisfying

$$v' + u\,k_1 - w\,k_2 = 0, \quad h < 0, \quad \nu^2 = -\frac{1}{2\,h}, \quad h\,k_1 - k_2 \neq 0. \tag{3.9}$$

(ii) If α^* is a spacelike curve with $k_1^* \neq 0$ and having a spacelike principal normal vector, then $\{\alpha, \alpha^*\}$ is a Bertrand pair in $\mathbb{E}_1^3$ if and only if there exist differentiable functions u, v, w, and a real number h satisfying

$$v' + u\,k_1 - w\,k_2 = 0, \quad h > 0, \quad \nu^2 = \frac{1}{2\,h}, \quad h\,k_1 - k_2 \neq 0. \tag{3.10}$$

Further, in both the cases (i) and (ii), if $\begin{cases} k_2^* \neq 0, & \text{then} \quad h\,k_1 + k_2 \neq 0, \\ k_2^* = 0, & \text{then} \quad h\,k_1 + k_2 = 0. \end{cases}$

Proof. (*i*) Let $\{\alpha, \alpha^*\}$ be a Bertrand pair in $\mathbb{E}_1^3$ satisfying (2.6) such that α is a Cartan null curve and α^* is a timelike curve. Differentiating (2.6) with respect to s and then using (2.3) and (2.4), we obtain

$$T^* \frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (v' + u\,k_1 - w\,k_2)\,N + (w' - v\,k_1)\,B. \tag{3.11}$$

Taking the inner product of (3.11) with N, we get

$$v' + u\,k_1 - w\,k_2 = 0. \tag{3.12}$$

Using (3.12) in (3.11), we obtain

$$T^* \frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (w' - v\,k_1)\,B. \tag{3.13}$$

Using (3.7) in (3.13), we have

$$T^* = \mu\,T + \nu\,B. \tag{3.14}$$

If we take the inner product of the equation (3.14) first with T and then with N, the following results are obtained

$$-1 = 2\,\mu\,\nu, \tag{3.15}$$

which gives $\mu \neq 0$ and $\nu \neq 0$. Consequently, from (3.7), we find that $1 + u' + v\,k_2 \neq 0$ and $w' - v\,k_1 \neq 0$.

Using the third relation of (3.7) in (3.15), we get

$$2\,h\,\nu^2 = -1, \tag{3.16}$$

which gives $h < 0$.

Now, differentiating (3.14) with respect to s and then using (2.3) and (2.4), we obtain

$$k_1^*\,N^*\,\frac{ds^*}{ds} = \mu'\,T + (\mu\,k_1 - \nu\,k_2)\,N + \nu'\,B. \tag{3.17}$$

Taking the inner product of (3.17) with T and B, we find

$$\nu' = 0, \quad \mu' = 0. \tag{3.18}$$

Using (3.18) in (3.17), we obtain

$$k_1^*\,N^*\,\frac{ds^*}{ds} = (\mu\,k_1 - \nu\,k_2)\,N. \tag{3.19}$$

Now, taking the inner product of (3.19) with itself and using (3.16) and the third relation of (3.7), we get

$$(k_1^*)^2\left(\frac{ds^*}{ds}\right)^2 = -\frac{1}{2\,h}\,(h\,k_1 - k_2)^2, \tag{3.20}$$

which gives $(h\,k_1 - k_2) \neq 0$. Now, using the first relation of (3.8), we have

$$N^* = \beta\,N. \tag{3.21}$$

Differentiating (3.21) with respect to s and then using (2.3) and (2.4), we obtain

$$k_2^*\,B^*\,\frac{ds^*}{ds} = \beta\,(k_2\,T - k_1\,B) - \,k_1^*\,T^*\,\frac{ds^*}{ds}. \tag{3.22}$$

Using (3.13) in (3.22), we get

$$k_2^*\,B^*\,\frac{ds^*}{ds} = \big(\beta\,k_2 - k_1^*\,(1 + u' + v\,k_2)\big)\,T - \big(\beta\,k_1 + k_1^*\,(w' - v\,k_1)\big)\,B. \tag{3.23}$$

Using the first relation of (3.8) in (3.23), we have

$$\begin{aligned} k_2^*\,B^*\,\frac{ds^*}{ds} = {} & \Big(\frac{\nu\,k_2\,(h\,k_1 - k_2)}{k_1^*\,\frac{ds^*}{ds}} - k_1^*\,(1 + u' + v\,k_2)\Big)\,T - \Big(\frac{\nu\,k_1\,(h\,k_1 - k_2)}{k_1^*\,\frac{ds^*}{ds}} \\ & + k_1^*\,(w' - v\,k_1)\Big)\,B. \end{aligned} \tag{3.24}$$

Now, using (3.7) in (3.24), we find

$$\begin{aligned} k_2^*\,B^*\,\frac{ds^*}{ds} = (w' - v\,k_1)\,\Big(\Big(&\frac{k_2\,(h\,k_1 - k_2) - k_1^{*2}\,h\,(\frac{ds^*}{ds})^2}{k_1^*\,(\frac{ds^*}{ds})^2}\Big)\,T \\ & - \Big(\frac{k_1\,(h\,k_1 - k_2) + k_1^{*2}\,(\frac{ds^*}{ds})^2}{k_1^*\,(\frac{ds^*}{ds})^2}\Big)\,B\Big). \end{aligned} \tag{3.25}$$

Using (3.20), and the second and third relations of (3.8) in (3.25), we obtain

$$k_2^*\, B^*\, \frac{ds^*}{ds} = \rho(s)\, T(s) + \eta(s)\, B(s). \tag{3.26}$$

Taking the inner product of (3.26) with itself, we get

$$k_2^{*2} \left(\frac{ds^*}{ds}\right)^2 = 2\, \rho(s)\, \eta(s) = -2\, h\, \eta(s)^2. \tag{3.27}$$

From (3.27), depending upon $k_2^* = 0$ or $k_2^* \neq 0$, we find that $h\, k_1 + k_2 = 0$ or $h\, k_1 + k_2 \neq 0$.

Conversely, let α be a Cartan null curve with curvatures $k_1 \neq 0$ and k_2 in $\mathbb{E}_1^3$ satisfying (3.10). Then, we can define the curve α^* as (2.6). Differentiating (2.6) with respect to s and then using (2.4), we obtain

$$T^* = \mu\, T + \nu\, B. \tag{3.28}$$

Using the third relation of (3.7) and the third relation of (3.9) in (3.28), we get

$$T^* = \frac{1}{\sqrt{-2h}}\, (h\, T + B), \quad \langle T^*, T^* \rangle = -1. \tag{3.29}$$

Now, differentiating (3.29) with respect to s and then using (2.4), we get

$$\frac{dT^*}{ds} = \frac{1}{\sqrt{-2\, h}}\, (h\, k_1 - k_2)\, N, \tag{3.30}$$

which gives

$$k_1^* = ||\frac{dT^*}{ds^*}|| = \frac{\xi_1\, (h\, k_1 - k_2)}{\sqrt{-2h}\, \frac{ds^*}{ds}}, \tag{3.31}$$

where $\xi_1 = \pm 1$. Now, N^* can be obtained as

$$N^* = \xi_1\, N, \quad \langle N^*, N^* \rangle = 1. \tag{3.32}$$

Differentiating (3.32) with respect to s and using $\epsilon_0 = -1$, $\epsilon_2 = 1$, (2.3) and (2.4), we obtain

$$(k_1^*\, T^* + k_2^*\, B^*)\, \frac{ds^*}{ds} = \xi_1\, (k_2\, T - k_1\, B). \tag{3.33}$$

Taking the inner product of (3.33) with itself, we get

$$(-k_1^{*2} + k_2^{*2}) \left(\frac{ds^*}{ds}\right)^2 = -2k_1\, k_2. \tag{3.34}$$

Using (3.31) in (3.34), we get

$$k_2^* = \frac{\xi_2\, (h\, k_1 + k_2)}{\sqrt{-2h}\, \frac{ds^*}{ds}}, \tag{3.35}$$

where $\xi_2 = \pm 1$.

Using (3.29), (3.31), and (3.35) in (3.33), we find

$$B^* = \frac{\xi_1\,\xi_2}{\sqrt{-2\,h}}\,(-h\,T + B), \quad \langle B^*, B^*\rangle = 1. \tag{3.36}$$

Then, α^* is a timelike curve, and the Bertrand partner curve of the null Cartan curve α. Thus, α is a Bertrand curve.

(ii) Let $\{\alpha, \alpha^*\}$ be a Bertrand pair in $\mathbb{E}^3_1$ satisfying (2.6) such that α is a Cartan null curve and α^* is a spacelike curve. Differentiating (2.6) with respect to s, and using (2.3) and (2.4), we get

$$T^* \frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (v' + u\,k_1 - w\,k_2)\,N + (w' - v\,k_1)\,B. \tag{3.37}$$

Taking the inner product of (3.37) with N, we obtain

$$v' + u\,k_1 - w\,k_2 = 0. \tag{3.38}$$

Using (3.38) in (3.37), we obtain

$$T^* \frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (w' - v\,k_1)\,B. \tag{3.39}$$

Using (3.7) in (3.39), we have

$$T^* = \mu\,T + \nu\,B. \tag{3.40}$$

Taking the inner product of (3.40) with itself, we find

$$1 = 2\,\mu\,\nu, \tag{3.41}$$

which gives $\nu \neq 0$ and $\mu \neq 0$. Consequently, from (3.7), we find that $1 + u' + v\,k_2 \neq 0$ and $w' - v\,k_1 \neq 0$.

Using the third relation of (3.7) in (3.41), we obtain

$$2\,h\,\nu^2 = 1, \tag{3.42}$$

which gives $h > 0$.

Now, differentiating (3.40) with respect to s and then using (2.3) and (2.4), we get

$$k_1^*\,N^* \frac{ds^*}{ds} = \mu'\,T + (\mu\,k_1 - \nu\,k_2)\,N + \nu'\,B. \tag{3.43}$$

If we take the inner product of the equation (3.43) first with T and then with B, the following results are obtained

$$\nu' = 0, \quad \mu' = 0. \tag{3.44}$$

Using (3.44) in (3.43), we obtain

$$k_1^* N^* \frac{ds^*}{ds} = (\mu k_1 - \nu k_2) N. \tag{3.45}$$

Now, taking the inner product of (3.45) with itself and using (3.42) and the third relation of (3.7), we get

$$(k_1^*)^2 \left(\frac{ds^*}{ds}\right)^2 = \frac{1}{2h} (h k_1 - k_2)^2, \tag{3.46}$$

which gives $(h k_1 - k_2) \neq 0$. Now, using the first relation of (3.8), we have

$$N^* = \beta N. \tag{3.47}$$

Differentiating (3.47) with respect to s and then using (2.3) and (2.4), we get

$$-k_2^* B^* \frac{ds^*}{ds} = \beta (k_2 T - k_1 B) + k_1^* T^* \frac{ds^*}{ds}. \tag{3.48}$$

Using (3.39) in (3.48), we get

$$-k_2^* B^* \frac{ds^*}{ds} = \beta (k_2 T - k_1 B) + k_1^* \Big((1 + u' + v k_2) T + (w' - v k_1) B\Big). \tag{3.49}$$

Using (3.7), (3.8) and (3.46) in (3.49), we get

$$-k_2^* B^* \frac{ds^*}{ds} = \rho(s) T(s) - \eta(s) B(s). \tag{3.50}$$

Taking the inner product of (3.50) with itself, we get

$$k_2^{*2} \left(\frac{ds^*}{ds}\right)^2 = 2 \rho(s) \eta(s) = -2 h \eta(s)^2. \tag{3.51}$$

From (3.51), depending upon $k_2^* = 0$ or $k_2^* \neq 0$, we find that $h k_1 + k_2 = 0$ or $h k_1 + k_2 \neq 0$.

Conversely, let α be a Cartan null curve in E_1^3 with curvatures $k_1 \neq 0$ and k_2 satisfying (3.10). Then, we can define the curve α^* as (2.6). Now, differentiating (2.6) with respect to s and then using (2.4), we get

$$T^* = \mu T + \nu B. \tag{3.52}$$

Using the third relation of (3.7) and the third relation of (3.10) in (3.52), we obtain

$$T^* = \frac{1}{\sqrt{2h}} (h T + B), \quad \langle T^*, T^* \rangle = 1. \tag{3.53}$$

Now, differentiating (3.53) with respect to s and then using (2.4), we get

$$\frac{dT^*}{ds} = \frac{1}{\sqrt{2 h}} (h k_1 - k_2) N, \tag{3.54}$$

which gives

$$k_1^* = ||\frac{dT^*}{ds^*}|| = \frac{\xi_3\,(h\,k_1 - k_2)}{\sqrt{2h}\,\frac{ds^*}{ds}}, \tag{3.55}$$

where $\xi_3 = \pm 1$. Now, N^* can be obtained as

$$N^* = \xi_3\,N, \quad \langle N^*,\, N^*\rangle = 1. \tag{3.56}$$

Differentiating (3.56) with respect to s and using $\epsilon_0 = 1$, $\epsilon_2 = -1$, (2.3) and (2.4), we obtain

$$(-k_1^*\,T^* - k_2^*\,B^*)\,\frac{ds^*}{ds} = \xi_3\,(k_2\,T - k_1\,B). \tag{3.57}$$

Taking the inner product of (3.57) with itself, we get

$$(k_1^{*2} - k_2^{*2})\Big(\frac{ds^*}{ds}\Big)^2 = -2k_1\,k_2. \tag{3.58}$$

Using (3.55) in (3.58), we obtain

$$k_2^* = \frac{\xi_4\,(h\,k_1 + k_2)}{\sqrt{2h}\,\frac{ds^*}{ds}}, \tag{3.59}$$

where $\xi_4 = \pm 1$.

Using (3.53), (3.55), and (3.59) in (3.57), we find

$$B^* = \frac{-\xi_3\,\xi_4}{\sqrt{2\,h}}\,(h\,T - B), \quad \langle B^*,\, B^*\rangle = -1. \tag{3.60}$$

Then, α^* is a spacelike curve with a spacelike principal normal vector and the Bertrand partner curve of α. As a result, α is a Bertrand curve, and the proof of the Theorem is complete. □

Now, from Theorem 3.1, we have:

Corollary 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with non-zero curvature $k_1 \neq 0$, k_2, and the curve α^* given in (2.6) be a non-null Bertrand partner curve of α with the non zero curvatures k_1^*, k_2^*. Then α^* is a general helix if and only if α is a general helix.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with the curvatures k_1, k_2 and the curve α^* is a Bertrand partner curve of α.

(i) If α^* is a timelike curve, then from (3.31) and (3.35), we have

$$\frac{k_1^*}{k_2^*} = \xi_1\,\xi_2\,\frac{h\,\frac{k_1}{k_2} - 1}{h\,\frac{k_1}{k_2} + 1}. \tag{3.61}$$

(ii) If α^* is a spacelike curve, then from (3.55) and (3.59), we have

$$\frac{k_1^*}{k_2^*} = \xi_3\,\xi_4\,\frac{h\,\frac{k_1}{k_2} - 1}{h\,\frac{k_1}{k_2} + 1}. \tag{3.62}$$

Combining (3.61) and (3.62), the proof is complete. □

Corollary 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with non-zero curvatures $k_1 = 1$, k_2, and the curve α^* be a non-null Bertrand partner curve of α with the non zero curvatures k_1^*, k_2^* satisfying (2.6). Then α^* is a slant helix if and only if α is a slant helix. Moreover, we have*

$$\Gamma_1^* = -\xi_1\,\xi_2\,\Gamma, \quad \Gamma_2^* = \xi_3\,\xi_4\,\Gamma. \tag{3.63}$$

Proof. Assume that $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ is a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with curvature $k_1 \neq 0$, k_2 and the curve α^* is a Bertrand partner curve of α with the non-zero curvatures k_1^*, k_2^* satisfying (2.6). Now, if the curve α is a slant helix, and then for the principal normal vector N of α and a constant vector field U, we have

$$\langle N, U \rangle = constant. \tag{3.64}$$

Since N is parallel to N^* from (3.64) we find

$$\langle N^*, U \rangle = constant, \tag{3.65}$$

which implies α^* is also a slant helix and converse is easy to prove. Further, we have:

(i) If α^* is a timelike curve, then using k_1^* and k_2^* from (3.31) and (3.35) in (1.1), we have

$$\Gamma_1^* = -\xi_1\,\xi_2\,\frac{k_2'}{2\sqrt{2}\,k_2^{3/2}}. \tag{3.66}$$

(ii) If α^* is a spacelike curve, then using k_1^* and k_2^* from (3.55) and (3.59) in (1.1), we have

$$\Gamma_2^* = \xi_3\,\xi_4\,\frac{k_2'}{2\sqrt{2}\,k_2^{3/2}}. \tag{3.67}$$

Then, using (1.2), (3.66), and (3.67), we have (3.63). Thus, the proof is complete. □

Now, we have:

Theorem 3.2. *Let α and α^* be a Cartan null curves in $\mathbb{E}_1^3$. Then, α^* is a Bertrand partner curve of the Bertrand curve α if*

(i) *there exist differentiable functions u, v, and w satisfying*

$$v' + u\,k_1 - w\,k_2 = 0, \quad w' - v\,k_1 = 0, \tag{3.68}$$

and its Cartan null frames are related by

$$T^* = \mu\, T, \quad N^* = \xi_5\, N, \quad B^* = \frac{1}{\mu}\, B, \tag{3.69}$$

or

(ii) *there exist differentiable functions* u, v, *and* w *satisfying*

$$v' + u\, k_1 - w\, k_2 = 0, \quad 1 + u' + v\, k_2 = 0, \tag{3.70}$$

and its Cartan null frames are related by

$$T^* = \nu\, B, \quad N^* = -\xi_6\, N, \quad B^* = \frac{1}{\nu}\, T, \tag{3.71}$$

where $\xi_5 = \pm 1$, $\xi_6 = \pm 1$.

Proof. Let α is a Cartan null Bertrand curve in $\mathbb{E}^3_1$ with $k_1 \neq 0$, k_2 and the curve α^* is the Cartan null Bertrand partner curve of the curve α satisfying (2.6). Now, differentiating (2.6) with respect to s and then using (2.4), we get

$$T^*\, \frac{ds^*}{ds} = (1 + u' + v\, k_2)\, T + (v' + u\, k_1 - w\, k_2)\, N + (w' - v\, k_1)\, B. \tag{3.72}$$

Taking the inner product of (3.72) with N, we obtain

$$v' + u\, k_1 - w\, k_2 = 0. \tag{3.73}$$

Using (3.73) in (3.72), we get

$$T^*\, \frac{ds^*}{ds} = (1 + u' + v\, k_2)\, T + (w' - v\, k_1)\, B. \tag{3.74}$$

Using (3.7) in (3.74), we have

$$T^* = \mu\, T + \nu\, B. \tag{3.75}$$

Taking the inner product of (3.75) with itself, we find

$$0 = 2\, \mu\, \nu. \tag{3.76}$$

Now, we have two cases:

Case (i) If $\nu = 0$, then we have

$$T^* = \mu T. \tag{3.77}$$

Now, differentiating (3.77) with respect to s and then using (2.4), we get

$$k_1^*\, N^*\, \frac{ds^*}{ds} = \mu'\, T + \mu\, k_1\, N. \tag{3.78}$$

Taking the inner product of (3.78) with B, we find

$$\mu' = 0. \tag{3.79}$$

Using (3.79) in (3.78), we obtain

$$k_1^* \, N^* \, \frac{ds^*}{ds} = \mu \, k_1 \, N. \tag{3.80}$$

Taking the inner product of (3.80) with itself, we get

$$(k_1^*)^2 \left(\frac{ds^*}{ds} \right)^2 = \mu^2 \, k_1^2, \tag{3.81}$$

which gives

$$k_1^* = \frac{\xi_5 \, \mu \, k_1}{\frac{ds^*}{ds}}. \tag{3.82}$$

Using (3.82) in (3.80), we obtain

$$N^* = \xi_5 \, N. \tag{3.83}$$

Differentiating (3.83) with respect to s and then using (2.4), we get

$$(k_2^* \, T^* - k_1^* \, B^*) \, \frac{ds^*}{ds} = \xi_5 \, (k_2 \, T - k_1 \, B). \tag{3.84}$$

Taking the inner product of (3.84) with itself, we get

$$k_1^* \, k_2^* \left(\frac{ds^*}{ds} \right)^2 = k_1 \, k_2. \tag{3.85}$$

Using (3.82) in (3.85), we obtain

$$k_2^* = \frac{\xi_5 \, k_2}{\mu \, \frac{ds^*}{ds}}. \tag{3.86}$$

Using (3.77), (3.82), and (3.86) in (3.84), we get

$$B^* = \frac{1}{\mu} \, B. \tag{3.87}$$

Case (ii) If $\mu = 0$, then we have

$$T^* = \nu \, B. \tag{3.88}$$

Now, differentiating (3.88) with respect to s and then using (2.4), we get

$$k_1^* \, N^* \, \frac{ds^*}{ds} = \nu' \, B - \nu \, k_2 \, N. \tag{3.89}$$

Taking the inner product of (3.89) with T, we find

$$\nu' = 0. \tag{3.90}$$

Using (3.90) in (3.89), we obtain

$$k_1^* \, N^* \, \frac{ds^*}{ds} = -\nu \, k_2 \, N. \tag{3.91}$$

Now, taking the inner product of (3.91) with itself, we get

$$(k_1^*)^2 \left(\frac{ds^*}{ds}\right)^2 = \nu^2 \, k_2^2, \tag{3.92}$$

which gives

$$k_1^* = \frac{\xi_6 \, \nu \, k_2}{\frac{ds^*}{ds}}. \tag{3.93}$$

Using (3.93) in (3.91), we obtain

$$N^* = -\xi_6 \, N. \tag{3.94}$$

Differentiating (3.94) with respect to s and then using (2.4), we get

$$(k_2^* \, T^* - k_1^* \, B^*) \, \frac{ds^*}{ds} = -\xi_6 \, (k_2 \, T - k_1 \, B). \tag{3.95}$$

Taking the inner product of (3.95) with itself, we get

$$k_1^* \, k_2^* \left(\frac{ds^*}{ds}\right)^2 = k_1 \, k_2. \tag{3.96}$$

Using (3.93) in (3.96), we obtain

$$k_2^* = \frac{\xi_6 \, k_1}{\nu \, \frac{ds^*}{ds}}. \tag{3.97}$$

Using (3.88), (3.93), and (3.97) in (3.95), we get

$$B^* = \frac{1}{\nu} \, T. \tag{3.98}$$

Thus, the proof is complete. □

4. Examples

Example 4.1. *Let $\alpha(s)$ be a Cartan null curve in $\mathbb{E}_1^3$ given by*

$$\alpha(s) = \Big(\frac{1}{\sqrt{2}} \, \sinh(\sqrt{2} \, s) + \frac{1}{2} \, \cosh(\sqrt{2} \, s), \frac{1}{\sqrt{2}} \, \cosh(\sqrt{2} \, s) + \frac{1}{2} \, \sinh(\sqrt{2} \, s), \, \frac{1}{\sqrt{2}} \, s\Big),$$

with curvature $k_1 = 1$ and torsion $k_2 = 1$.

The Frenet frame of $\alpha(s)$ is given by

$$\begin{cases} T &= \Big(\cosh(\sqrt{2}\,s)+\frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,s),\,\sinh(\sqrt{2}\,s)+\frac{1}{\sqrt{2}}\cosh(\sqrt{2}\,s),\,\frac{1}{\sqrt{2}}\Big),\\ N &= \big(\sqrt{2}\sinh(\sqrt{2}s)+\cosh(\sqrt{2}s),\,\sqrt{2}\cosh(\sqrt{2}s)+\sinh(\sqrt{2}s),\,0\big),\\ B &= -\Big(\cosh(\sqrt{2}\,s)+\frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,s),\,\sinh(\sqrt{2}\,s)+\frac{1}{\sqrt{2}}\cosh(\sqrt{2}\,s),\,-\frac{1}{\sqrt{2}}\Big).\end{cases}$$

If we take $u=2\,s$, $v=\frac{5\,s^2}{2}$, $w=-3\,s$ in (2.6), we find the Bertrand partner curve $\alpha^*(s^*)$ as:

$$\alpha^*=\Big(\sinh(\sqrt{2}\,s)\,A(s)+\cosh(\sqrt{2}\,s)\,B(s),\,\cosh(\sqrt{2}\,s)\,A(s)+\sinh(\sqrt{2}\,s)\,B(s),\,0\Big),$$

where $A(s)=\frac{1+5\,s-5\,s^2}{\sqrt{2}}$, $B(s)=\frac{1+10\,s-5\,s^2}{2}$.

By computing the curvature and torsion of α^*, we get

$$k_1^*=\frac{2}{\sqrt{6-5\,s^2}},\quad k_2^*=0.$$

Further, the Frenet frame of α^* is given by

$$\begin{cases} T^* = \big(\sinh(\sqrt{2}\,s)+\sqrt{2}\cosh(\sqrt{2}\,s),\,\cosh(\sqrt{2}\,s)+\sqrt{2}\sinh(\sqrt{2}\,s),\,0\big),\\ N^* = \big(\sqrt{2}\sinh(\sqrt{2}s)+\cosh(\sqrt{2}s),\,\sqrt{2}\cosh(\sqrt{2}s)+\sinh(\sqrt{2}s),\,0\big),\\ B^* = \big(0,\,0,\,1\big).\end{cases}$$

Thus, $\alpha^*(s^*)$ is a timelike Bertrand partner curve of the curve $\alpha(s)$.

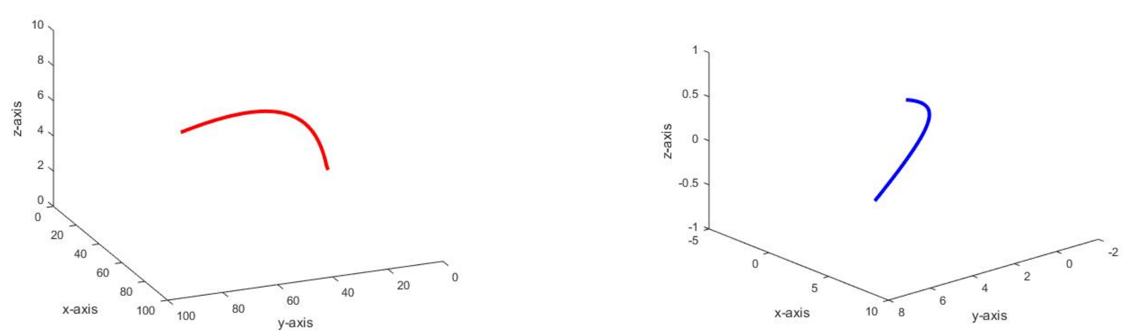

FIGURE 1. Curve α (red) and α^* (blue) in $\mathbb{E}_1^3$

Example 4.2. *Let $\alpha_1(s)$ be a Cartan null curve in $\mathbb{E}_1^3$ given by*

$$\alpha_1(s)=\Big(\sinh(s),\,\cosh(s),\,s\Big),$$

with curvature $k_1=1$ and torsion $k_2=1/2$.

The Frenet frame of $\alpha_1(s)$ is given by

$$\begin{cases} T_1 = \big(\cosh(s), \sinh(s), 1\big), \\ N_1 = \big(\sinh(s), \cosh(s), 0\big), \\ B_1 = \Big(\dfrac{-\cosh(s)}{2}, \dfrac{-\sinh(s)}{2}, \dfrac{1}{2}\Big). \end{cases}$$

If we take $u = \frac{s}{2}$, $v = -\frac{1}{3}$, $w = s$ in (2.6), we find the Bertrand partner curve $\alpha_1^*(s^*)$ as:

$$\alpha_1^* = \Big(\frac{2}{3}\sinh(s), \frac{2}{3}\cosh(s), 2\,s\Big).$$

By computing the curvature and torsion of α_1^*, we get

$$k_1^* = \frac{3}{16}, \quad k_2^* = \frac{9}{16}.$$

Further, the Frenet frame of α_1^* is given by

$$\begin{cases} T_1^* &= \frac{1}{2\sqrt{2}}\Big(\cosh(s), \sinh(s), 3\Big), \\ N_1^* &= \big(\sinh(s), \cosh(s), 0\big), \\ B_1^* &= -\frac{1}{2\sqrt{2}}\Big(3\cosh(s), 3\sinh(s), 1\Big). \end{cases}$$

Thus, $\alpha_1^*(s^*)$ is a spacelike Bertrand partner curve of the curve $\alpha_1(s)$.

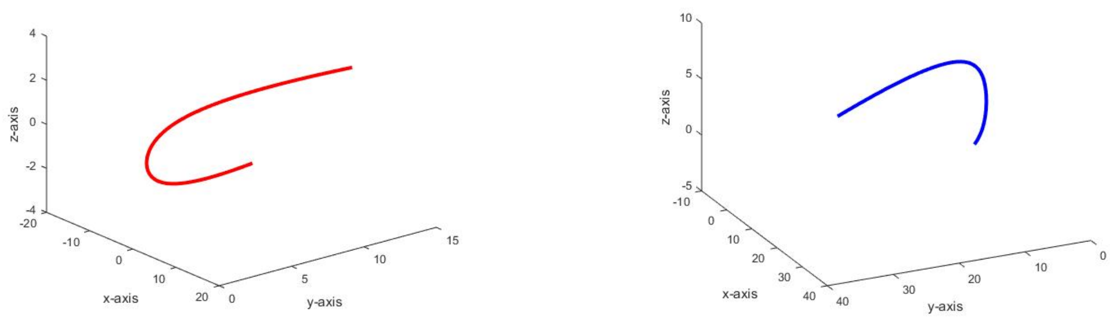

FIGURE 2. Curve α_1 (red) and α_1^* (blue) in $\mathbb{E}_1^3$

Example 4.3. *If we take $u = \frac{s}{4}$, $v = \frac{1}{2}$, $w = \frac{s}{2}$ in (2.6) for the Cartan null curve $\alpha_1(s)$ in Example 4.2, we find the Bertrand partner curve $\alpha_2^*(s^*)$ as:*

$$\alpha_2^* = \frac{1}{2}\Big(3\sinh(s), 3\cosh(s), 3\,s\Big).$$

By computing the curvature and torsion of α_2^*, we get

$$k_1^* = 1, \quad k_2^* = \frac{1}{3}.$$

Further, the Frenet frame of α_2^* *is given by*

$$\begin{cases} T_2^* &= \sqrt{\frac{3}{2}}\Big(\cosh(s),\, \sinh(s),\, 1\Big), \\ N_2^* &= \big(\sinh(s),\, \cosh(s),\, 0\big), \\ B_2^* &= \frac{1}{\sqrt{6}}\Big(-\cosh(s),\, -\sinh(s),\, 1\Big). \end{cases}$$

Thus, $\alpha_2^*(s^*)$ *is a Cartan null Bertrand partner curve of the curve* $\alpha_1(s)$*.*

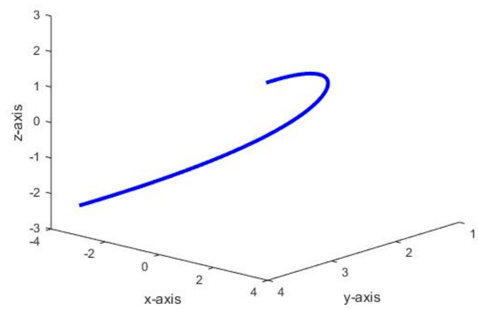

FIGURE 3. Cartan null Bertrand partner curve α_2^* of a null Cartan curve α_1 in $\mathbb{E}_1^3$

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SOME RESULTS ON β-KENMOTSU MANIFOLDS WITH A NON-SYMMETRIC NON-METRIC CONNECTION

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. The object of the present paper is to study some results on a β-Kenmotsu manifold with a non-symmetric non-metric connection. We obtain the condition for the manifold with a non-symmetric non-metric connection to be projectively flat and conformally flat. Also, it has been demonstrated that the manifold satisfying the condition $\breve{\mathcal{R}}^{\dagger} \cdot \breve{\mathcal{S}}^{\dagger}=0$ is an Einstein manifold. Further, by virtue of this result, we found the condition of Ricci soliton in β-Kenmotsu manifold to be expanding.

Keywords: Non-symmetric non-metric connection, β-Kenmotsu manifold, conformal curvature tensor, Ricci soliton, Einstein manifold, Ricci semi-symmetric.

2010 Mathematics Subject Classification: 53C25, 53D15, 53D10.

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1. Introduction

K. Kenmotsu [14] studied a class of almost contact manifolds and identified it as a Kenmotsu manifold. The fundamental properties of local structure of these manifolds were studied by him [14]. Trans-Sasakian manifolds were introduced by J. A. Oubiña [16], which

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generalizes forms of Sasakian, Kenmotsu and cosymplectic manifolds. A trans-Sasakian manifold of type $(0,0)$, $(\alpha,0)$ and $(0,\beta)$ are Cosymplectic, α-Sasakian and β-Kenmotsu manifolds respectively, where α,β are smooth functions. In particular, a trans-Sasakian manifold will be Kenmotsu and Sasakian manifold, if $\alpha=0,\beta=1$ and $\alpha=1,\beta=0$ respectively. β-Kenmotsu manifold provides a large variety of Kenmotsu manifolds. Recently, Kenmotsu manifolds have been studied by several authors (cf. [8, 6, 11, 13, 23, 24]).

On differentiable manifolds, A. Friedmann and J. A. Schouten [12] first proposed a semi-symmetric linear connection. On Riemannian manifolds, semi-symmetric metric connection was first systematically examined by K. Yano [25], which was further studied by authors, including S. Ahmad and S. I. Hussain [21], M. M. Tripathi [22] and others. Semi-symmetric non-metric connection was established in a Riemannian manifold by N. S. Agashe and M. R. Chafle [1]. In line with this, S. K. Chaubey et al. [2] introduced the notion of non-symmetric non-metric connection. It has been further studied in [4, 5, 7, 17, 18, 19].

A torsion tensor of a connection is a mapping $\mathcal{T}':\chi(\Omega)\times\chi(\Omega)\to\chi(\Omega)$ defined by

$$\mathcal{T}'(\mathcal{X}_1,\mathcal{X}_2) = \hat{\nabla}_{\mathcal{X}_1}\mathcal{X}_2-\hat{\nabla}_{\mathcal{X}_2}\mathcal{X}_1-[\mathcal{X}_1,\mathcal{X}_2]. \tag{1.1}$$

A connection $\hat{\nabla}$ is symmetric if $\mathcal{T}'=0$ and it is non-symmetric if $\mathcal{T}'\neq 0$. The connection $\breve{\nabla}$ is metric if $\breve{\nabla}_{\mathcal{X}}\hat{g}=0$ and it is non-metric if $\breve{\nabla}_{\mathcal{X}}\hat{g}\neq 0$. It was further studied by several geometers [10, 9].

In a Riemannian manifold $(\Omega^{2n+1},\hat{g}),\hat{g}$ is a Ricci soliton if

$$(\pounds_{\mathcal{V}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)+2\mathcal{S}^\dagger(\mathcal{X}_1,\mathcal{X}_2)+2\Theta\hat{g}(\mathcal{X}_1,\mathcal{X}_2)=0, \tag{1.2}$$

$\forall$ $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{V}$ on Ω^{2n+1}, where $\pounds_{\mathcal{V}}$ denote the Lie-derivative along the vector field $\mathcal{V}$, $\mathcal{S}^\dagger$ is Ricci tensor and Θ is a constant. The Ricci soliton is shrinking, steady and expanding if $\Theta<0$, $\Theta=0$ and $\Theta>0$ respectively.

This paper is organized as follows: In Section 2, we present an informative introduction of β-Kenmotsu manifold. In Section 3, we define non-symmetric non-metric connection. In Section 4, we find the curvature tensor with non-symmetric non-metric connection. In Section 5, we investigate projectively and conformally flat β-Kenmotsu manifolds with defined connection. In Section 6, we show that the manifold with the defined connection satisfying the condition $\breve{\mathcal{R}}^\dagger\cdot\breve{\mathcal{S}}^\dagger$=0 is an Einstein manifold.

2. Preliminaries

A smooth manifold Ω^{2n+1} is almost contact metric [15] if it admits a $(1,1)$-tensor field $\hat{\varphi}$, an associated vector field $\hat{\zeta}$, a 1-form $\hat{\eta}$ and the Riemannian metric $\hat{g}$ satisfying

$$\hat{\varphi}^2\mathcal{X}_1 = -\mathcal{X}_1 + \hat{\eta}(\mathcal{X}_1)\hat{\zeta}, \quad \hat{\eta}(\hat{\zeta}) = 1, \quad \hat{\varphi}\hat{\zeta} = 0, \quad \hat{\eta}(\hat{\varphi}\mathcal{X}_1) = 0, \tag{2.3}$$

$$\hat{g}(\hat{\varphi}\mathcal{X}_1, \hat{\varphi}\mathcal{X}_2) = \hat{g}(\mathcal{X}_1, \mathcal{X}_2) - \hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2), \quad \hat{g}(\mathcal{X}_1, \hat{\zeta}) = \hat{\eta}(\mathcal{X}_1), \tag{2.4}$$

for all $\mathcal{X}_1, \mathcal{X}_2 \in \mathcal{T}'\Omega$.

An almost contact metric manifold Ω^{2n+1} is a β-Kenmotsu manifold [20] if and only if

$$(\hat{\nabla}_{\mathcal{X}_1}\hat{\varphi})\mathcal{X}_2 = \beta[\hat{g}(\hat{\varphi}\mathcal{X}_1, \mathcal{X}_2)\hat{\zeta} - \hat{\eta}(\mathcal{X}_2)\hat{\varphi}(\mathcal{X}_1)]. \tag{2.5}$$

From (2.5), we have

$$\hat{\nabla}_{\mathcal{X}_1}\hat{\zeta} = \beta[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}], \tag{2.6}$$

$$(\hat{\nabla}_{\mathcal{X}_1}\hat{\eta})\mathcal{X}_2 = \beta\hat{g}(\hat{\varphi}\mathcal{X}_1, \hat{\varphi}\mathcal{X}_2) = \beta[\hat{g}(\mathcal{X}_1, \mathcal{X}_2) - \hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2)]. \tag{2.7}$$

Further, the curvature tensor $\mathcal{R}^\dagger$, Ricci tensor $\mathcal{S}^\dagger$ and Ricci operator $\mathcal{Q}^\dagger$ in β-Kenmotsu manifold with the Levi-Civita connection $\hat{\nabla}$ satisfy [20].

$$\begin{aligned}\mathcal{R}^\dagger(\mathcal{X}_1, \mathcal{X}_2)\hat{\zeta} &= -\beta^2[\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\mathcal{X}_2] + (\mathcal{X}_1\beta)[\mathcal{X}_2 - \hat{\eta}(\mathcal{X}_2)\hat{\zeta}] \\ &\quad -(\mathcal{X}_2\beta)[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}],\end{aligned} \tag{2.8}$$

$$\mathcal{R}^\dagger(\hat{\zeta}, \mathcal{X}_1)\mathcal{X}_2 = (\beta^2 + \hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1 - \hat{g}(\mathcal{X}_1, \mathcal{X}_2)\hat{\zeta}], \tag{2.9}$$

$$\mathcal{R}^\dagger(\hat{\zeta}, \mathcal{X}_1)\hat{\zeta} = (\beta^2 + \hat{\zeta}\beta)[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}], \tag{2.10}$$

$$\mathcal{S}^\dagger(\mathcal{X}_1, \hat{\zeta}) = -(2n\beta^2 + \hat{\zeta}\beta)\hat{\eta}(\mathcal{X}_1) - (2n-1)(\mathcal{X}_1\beta), \tag{2.11}$$

$$\mathcal{S}^\dagger(\hat{\zeta}, \hat{\zeta}) = -(2n\beta^2 + \hat{\zeta}\beta), \tag{2.12}$$

$$\mathcal{Q}^\dagger\hat{\zeta} = -(2n\beta^2 + \hat{\zeta}\beta)\hat{\zeta} - (2n-1)grad\beta. \tag{2.13}$$

Definition 2.1. *A β-Kenmotsu manifold Ω^{2n+1} is known as a generalized η-Einstein manifold if its Ricci tensor $\mathcal{S}^\dagger$ of type $(0,2)$ satisfies*

$$\mathcal{S}^\dagger = \lambda_1\hat{g} + \lambda_2\hat{\eta}\otimes\hat{\eta} + \lambda_3[\hat{\eta}\otimes\omega + \omega\otimes\hat{\eta}], \tag{2.14}$$

where, λ_1, λ_2 and λ_3 are smoth functions, ω is a 1-form defined by $\omega(\mathcal{X}_1) = \hat{g}(\mathcal{X}_1, \rho)$ $\forall$ $\mathcal{X}_1$, ρ and $\hat{\zeta}$ are mutually orthogonal to each other.

Definition 2.2. *The projective curvature tensor of a $(2n+1)$-dimensional β-Kenmotsu manifold Ω is given by* [4]

$$\mathcal{P}^\flat(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3=\mathcal{R}^\dagger(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3-\frac{1}{2n}[\mathcal{S}^\dagger(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\mathcal{S}^\dagger(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2]. \tag{2.15}$$

Definition 2.3. *The conformal curvature tensor $\mathcal{C}^\flat$ of a $(2n+1)$-dimensional β-Kenmotsu manifold Ω* [20] *is given by*

$$\begin{aligned}\mathcal{C}^\flat(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3 \;=\;& \mathcal{R}^\dagger(\mathcal{X}_1,\mathcal{X}_2)\mathcal{X}_3-\frac{1}{2n-1}[\mathcal{S}^\dagger(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\mathcal{S}^\dagger(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2\\ &+\hat{g}(\mathcal{X}_2,\mathcal{X}_3)\mathcal{Q}^\dagger\mathcal{X}_1-\hat{g}(\mathcal{X}_1,\mathcal{X}_3)\mathcal{Q}^\dagger\mathcal{X}_2]\\ &+\frac{k}{2n(2n-1)}[\hat{g}(\mathcal{X}_2,\mathcal{X}_3)\mathcal{X}_1-\hat{g}(\mathcal{X}_1,\mathcal{X}_3)\mathcal{X}_2]\end{aligned} \tag{2.16}$$

where $\mathcal{R}^\dagger$, $\mathcal{S}^\dagger$, $\mathcal{Q}^\dagger$ and k is the curvature tensor, Ricci tensor, Ricci opretor and scalar curvature respectively with $\hat{\nabla}$.

3. Non-symmetric non-metric connection

The relation between non-symmetric non-metric connection $\breve{\nabla}$ and the Levi-Civita connection $\hat{\nabla}$ [2, 3] is given as

$$\breve{\nabla}_{\mathcal{X}_1}\mathcal{X}_2=\hat{\nabla}_{\mathcal{X}_1}\mathcal{X}_2+\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta}, \tag{3.17}$$

which satisfies

$$\breve{\mathcal{T}}'(\mathcal{X}_1,\mathcal{X}_2)=2\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta} \tag{3.18}$$

and

$$(\breve{\nabla}_{\mathcal{X}_1}\hat{g})(\mathcal{X}_2,\mathcal{X}_3)=-\hat{\eta}(\mathcal{X}_3)\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)-\hat{\eta}(\mathcal{X}_2)\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_3) \tag{3.19}$$

for arbitrary vector fields $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}_3$.

Let Ω^{2n+1} be a β-Kenmotsu manifold with a non-symmetric non-metric connection $\breve{\nabla}$, then

$$(\breve{\nabla}_{\mathcal{X}_1}\hat{\varphi})(\mathcal{X}_2)=(\hat{\nabla}_{\mathcal{X}_1}\hat{\varphi})(\mathcal{X}_2)+\hat{g}(\hat{\varphi}\mathcal{X}_1,\hat{\varphi}\mathcal{X}_2)\hat{\zeta}, \tag{3.20}$$

$$(\breve{\nabla}_{\mathcal{X}_1}\hat{\eta})(\mathcal{X}_2)=(\hat{\nabla}_{\mathcal{X}_1}\hat{\eta})(\mathcal{X}_2)-\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2), \tag{3.21}$$

$$\breve{\nabla}_{\mathcal{X}_1}\hat{\zeta}=\hat{\nabla}_{\mathcal{X}_1}\hat{\zeta}. \tag{3.22}$$

From (3.22), the following theorem yields:

Theorem 3.1. *The vector field $\hat{\zeta}$ is invariant with respect to the connections $\hat{\nabla}$ and $\breve{\nabla}$* [18].

4. Curvature tensor on a β-Kenmotsu manifold with non-symmetric non-metric connection

If $\mathcal{R}^\dagger$ and $\breve{\mathcal{R}}^\dagger$ are the curvature tensors of connections $\hat{\nabla}$ and $\breve{\nabla}$ respectively, we have

$$\breve{\mathcal{R}}^\dagger\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3=\breve{\nabla}_{\mathcal{X}_1}\breve{\nabla}_{\mathcal{X}_2}\mathcal{X}_3-\breve{\nabla}_{\mathcal{X}_2}\breve{\nabla}_{\mathcal{X}_1}\mathcal{X}_3-\breve{\nabla}_{[\mathcal{X}_1,\mathcal{X}_2]}\mathcal{X}_3, \tag{4.23}$$

from (2.5), (2.6) and (3.17), we have

$$\begin{aligned}\breve{\mathcal{R}}^\dagger\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 &= \mathcal{R}^\dagger\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3+\beta[2\hat{g}\left(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2\right)\hat{\eta}(\mathcal{X}_3)\hat{\zeta}\\ &\quad+\hat{g}\left(\hat{\varphi}\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\hat{g}\left(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2].\end{aligned} \tag{4.24}$$

Putting $\mathcal{X}_1=e_i$ in (4.24) and summing over $1\leq i\leq(2n+1)$, we get

$$\breve{\mathcal{S}}^\dagger\left(\mathcal{X}_2,\mathcal{X}_3\right)=\mathcal{S}^\dagger\left(\mathcal{X}_2,\mathcal{X}_3\right)+2n\beta\hat{g}\left(\hat{\varphi}\mathcal{X}_2,\mathcal{X}_3\right), \tag{4.25}$$

$$\breve{\mathcal{Q}}^\dagger\left(\mathcal{X}_2\right)=\mathcal{Q}^\dagger\left(\mathcal{X}_2\right)+2n\beta\left(\hat{\varphi}\mathcal{X}_2\right). \tag{4.26}$$

Thus we state the following theorem:

Theorem 4.1. *In a β-Kenmotsu manifold, Ricci tensor and Ricci operator are defined by the equations* (4.25) *and* (4.26) *respectively endowed with $\breve{\nabla}$ and $\hat{\nabla}$.*

Contracting (4.25), it follows that

$$\breve{k}=k. \tag{4.27}$$

Here $\breve{\mathcal{R}}^\dagger$, $\breve{\mathcal{S}}^\dagger$, $\breve{\mathcal{Q}}^\dagger$ and $\breve{k}$ is the curvature tensor, Ricci tensor, Ricci operator and scalar curvature respectively with $\breve{\nabla}$.
Thus with the help of (4.27), we have following theorem:

Theorem 4.2. *If a β-Kenmotsu manifold Ω^{2n+1} admits $\breve{\nabla}$, then the scalar curvatures corresponding to $\breve{\nabla}$ and $\hat{\nabla}$ coincide.*

By replacing $\mathcal{X}_3=\hat{\zeta}$, in (4.24) and in view of (2.3), (2.4) and (2.8), we get

$$\begin{aligned}\breve{\mathcal{R}}^\dagger\left(\mathcal{X}_1,\mathcal{X}_2\right)\hat{\zeta} &= \beta^2\left(\hat{\eta}(\mathcal{X}_1)\mathcal{X}_2-\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1\right)+2\beta\hat{g}\left(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2\right)\hat{\zeta}\\ &\quad+(\mathcal{X}_1\beta)[\mathcal{X}_2-\hat{\eta}(\mathcal{X}_2)\hat{\zeta}]-(\mathcal{X}_2\beta)[\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\hat{\zeta}].\end{aligned} \tag{4.28}$$

From (2.3), (2.9) and (4.24), we get

$$\breve{\mathcal{R}}^\dagger(\hat{\zeta},\mathcal{X}_2)\mathcal{X}_3 = (\beta^2+\hat{\zeta}\beta)[\hat{\eta}\left(\mathcal{X}_3\right)\mathcal{X}_2-\hat{g}\left(\mathcal{X}_2,\mathcal{X}_3\right)\hat{\zeta}]+\beta\hat{g}\left(\hat{\varphi}\mathcal{X}_2,\mathcal{X}_3\right)\hat{\zeta}. \tag{4.29}$$

By using (2.3), (2.4), (2.10) and (4.24), we get

$$\begin{aligned} \breve{\mathcal{R}}^{\dagger}(\hat{\zeta},\mathcal{X}_1)\hat{\zeta} &= \mathcal{R}^{\dagger}(\hat{\zeta},\mathcal{X}_1)\hat{\zeta} \\ &= (\beta^2+\hat{\zeta}\beta)[\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\hat{\zeta}]. \end{aligned} \tag{4.30}$$

Putting $\mathcal{X}_3=\hat{\zeta}$ in (4.25) and using (2.11), we get

$$\begin{aligned} \breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2,\hat{\zeta}) &= \mathcal{S}^{\dagger}(\mathcal{X}_2,\hat{\zeta}) \\ &= -(2n\beta^2+\hat{\zeta}\beta)\hat{\eta}\left(\mathcal{X}_2\right)-(2n-1)(\mathcal{X}_2\beta) \end{aligned} \tag{4.31}$$

and

$$\breve{\mathcal{Q}}^{\dagger}\left(\mathcal{X}_2\right)=-(2n\beta^2+\hat{\zeta}\beta)\hat{\zeta}-(2n-1)grad\beta. \tag{4.32}$$

5. PROJECTIVELY CURVATURE TENSOR ON β-KENMOTSU MANIFOLD WITH NON-SYMMETRIC NON-METRIC CONNECTION

From Definition 2.2, we have

$$\breve{\mathcal{P}}^{\flat}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 = \breve{\mathcal{R}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3-\frac{1}{2n}[\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2]. \tag{5.33}$$

Using (4.24), (4.25) in (5.33), we acquire

$$\breve{\mathcal{P}}^{\flat}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 = \mathcal{P}^{\flat}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3+2\beta\hat{g}\left(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2\right)\hat{\eta}(\mathcal{X}_3)\hat{\zeta}. \tag{5.34}$$

Thus, we have the following results:

Theorem 5.1. *If a β-Kenmotsu manifold Ω^{2n+1} admits $\breve{\nabla}$, then the projective curvature tensors corresponding to $\breve{\nabla}$ and $\hat{\nabla}$ are related by the equation* (5.34).

If Ω^{2n+1} is $\breve{\mathcal{C}}^{\flat}$-flat, then from Definition 2.3 we obtain

$$\begin{aligned} \breve{\mathcal{R}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_2\right)\mathcal{X}_3 &= \frac{1}{2n-1}[\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\breve{\mathcal{S}}^{\dagger}\left(\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2 \\ &\quad +\hat{g}\left(\mathcal{X}_2,\mathcal{X}_3\right)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_1-\hat{g}\left(\mathcal{X}_1,\mathcal{X}_3\right)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_2] \\ &\quad -\frac{\breve{k}}{2n(2n-1)}[\hat{g}\left(\mathcal{X}_2,\mathcal{X}_3\right)\mathcal{X}_1-\hat{g}\left(\mathcal{X}_1,\mathcal{X}_3\right)\mathcal{X}_2]. \end{aligned} \tag{5.35}$$

Putting $\mathcal{X}_3=\hat{\zeta}$ in (5.35) and using (4.25), (4.26), (4.27) and (4.28), we have

$$\begin{aligned} \hat{\eta}(\mathcal{X}_2)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_2 &= (\beta^2+\hat{\zeta}\beta+\frac{k}{2n})[\hat{\eta}(\mathcal{X}_2)\mathcal{X}_1-\hat{\eta}(\mathcal{X}_1)\mathcal{X}_2] \\ &\quad -(2n-1)[(\mathcal{X}_1\beta)\hat{\eta}(\mathcal{X}_2)-(\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_1)]\hat{\zeta} \\ &\quad +2(2n-1)\beta\hat{g}(\hat{\varphi}\mathcal{X}_1,\mathcal{X}_2)\hat{\zeta}. \end{aligned} \tag{5.36}$$

Again putting $\mathcal{X}_2 = \hat{\zeta}$ in (5.36), we obtain

$$\breve{\mathcal{Q}}^{\dagger}\mathcal{X}_1 = (\beta^2 + \hat{\zeta}\beta + \frac{k}{2n})\mathcal{X}_1 - ((2n+1)\beta^2 - (2n-3)\hat{\zeta}\beta + \frac{k}{2n})\hat{\eta}(\mathcal{X}_1)\hat{\zeta} \\ -(2n-1)((\mathcal{X}_1\beta)\hat{\zeta} + \hat{\eta}(\mathcal{X}_1)grad\beta). \tag{5.37}$$

Hence

$$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2) = (\beta^2 + \hat{\zeta}\beta + \frac{k}{2n})\hat{g}(\mathcal{X}_1, \mathcal{X}_2) - (2n-1)((\mathcal{X}_1\beta)\hat{\eta}(\mathcal{X}_2) + (\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_1)) \\ -((2n+1)\beta^2 - (2n-3)\hat{\zeta}\beta + \frac{k}{2n})\hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2). \tag{5.38}$$

Let $\omega(\mathcal{X}_1) = \hat{g}(\mathcal{X}_1, \rho) = (\mathcal{X}_1\beta) = \hat{g}(grad\beta, \mathcal{X}_1)\ \forall\ \mathcal{X}_1$. If ρ and $\hat{\zeta}$ are orthogonal then $\hat{\zeta}\beta = 0$ and (5.38) takes the form of (2.14). Therefore, we have the following theorem:

Theorem 5.2. *A conformally flat β-Kenmotsu manifold endowed with $\breve{\nabla}$ is a generalised η-Einstein manifold equipped with $\breve{\nabla}$.*

6. β-Kenmotsu manifold satisfying $\breve{\mathcal{R}}^{\dagger} \cdot \breve{\mathcal{S}}^{\dagger}=0$

We consider a β-Kenmotsu manifold with $\breve{\nabla}$ connection satisfying

$$\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2).\breve{\mathcal{S}}^{\dagger} = 0. \tag{6.39}$$

Therefore, we get

$$\breve{\mathcal{S}}^{\dagger}(\breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_3, \mathcal{X}_4) + \breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \breve{\mathcal{R}}^{\dagger}(\mathcal{X}_1, \mathcal{X}_2)\mathcal{X}_4) = 0. \tag{6.40}$$

Replacing $\mathcal{X}_1$ by $\hat{\zeta}$ in (6.40), it follows that

$$\breve{\mathcal{S}}^{\dagger}(\breve{\mathcal{R}}^{\dagger}(\hat{\zeta}, \mathcal{X}_2)\mathcal{X}_3, \mathcal{X}_4) + \breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \breve{\mathcal{R}}^{\dagger}(\hat{\zeta}, \mathcal{X}_2)\mathcal{X}_4) = 0. \tag{6.41}$$

In view of (4.29), we have

$$(\beta^2 + \hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_3)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2, \mathcal{X}_4) - \hat{g}(\mathcal{X}_2, \mathcal{X}_3)\breve{\mathcal{S}}^{\dagger}(\hat{\zeta}, \mathcal{X}_4)] \\ + \beta\hat{g}(\hat{\varphi}\mathcal{X}_2, \mathcal{X}_3)\breve{\mathcal{S}}^{\dagger}(\hat{\zeta}, \mathcal{X}_4) + (\beta^2 + \hat{\zeta}\beta)[\hat{\eta}(\mathcal{X}_4)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \mathcal{X}_2) \\ - \hat{g}(\mathcal{X}_2, \mathcal{X}_4)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \hat{\zeta})] + \beta\hat{g}(\hat{\varphi}\mathcal{X}_2, \mathcal{X}_4)\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_3, \hat{\zeta}) = 0. \tag{6.42}$$

Again replacing $\mathcal{X}_3$ by $\hat{\zeta}$ and using (2.3) and (4.31), we have

$$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_2, \mathcal{X}_4) = -(2n\beta^2 + \hat{\zeta}\beta)\hat{g}(\mathcal{X}_2, \mathcal{X}_4) + (2n-1)((\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_4) \\ - (\mathcal{X}_4\beta)\hat{\eta}(\mathcal{X}_2)) + 2n\beta\hat{g}(\hat{\varphi}\mathcal{X}_2, \mathcal{X}_4). \tag{6.43}$$

Using (4.25), we have

$$\begin{aligned}\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_4)=&-(2n\beta^2+\hat{\zeta}\beta)\hat{g}(\mathcal{X}_2,\mathcal{X}_4)+(2n-1)(\mathcal{X}_2\beta)\hat{\eta}(\mathcal{X}_4)\\&-(2n-1)(\mathcal{X}_4\beta)\hat{\eta}(\mathcal{X}_2).\end{aligned} \tag{6.44}$$

Taking $\mathcal{X}_4=\hat{\zeta}$ in (6.44), we get

$$2(\mathcal{X}_2\beta)=(\hat{\zeta}\beta)\hat{\eta}(\mathcal{X}_2). \tag{6.45}$$

Again we take $\mathcal{X}_2=\hat{\zeta}$ in (6.45), we get

$$\hat{\zeta}\beta=0. \tag{6.46}$$

Using (6.45) and (6.46) in (6.44), we have

$$\mathcal{S}^{\dagger}(\mathcal{X}_2,\mathcal{X}_4)=-2n\beta^2\hat{g}(\mathcal{X}_2,\mathcal{X}_4). \tag{6.47}$$

Thus we leads to the theorem:

Theorem 6.1. *A β-Kenmotsu manifold satisfying the condition $\breve{\mathcal{R}}^{\dagger}\cdot\breve{\mathcal{S}}^{\dagger}=0$ with $\breve{\nabla}$ is an Einstien manifold with $\hat{\nabla}$.*

A Ricci soliton in β-Kenmotsu manifold is defined by equation (1.2). Naturally, two cases appear corresponding to the vector field $\mathcal{V}:\mathcal{V}\in Span\hat{\zeta}$ and $\mathcal{V}\perp\hat{\zeta}$. We consider only the case $\mathcal{V}=\hat{\zeta}$. The Ricci soliton $(\hat{g},\hat{\zeta},\Theta)$ on a β-Kenmotsu manifold endowed with $\breve{\nabla}$ is defined as

$$(\breve{\mathcal{L}}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)+2\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)+2\Theta\hat{g}(\mathcal{X}_1,\mathcal{X}_2)=0. \tag{6.48}$$

Here

$$(\breve{\mathcal{L}}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)=(\breve{\nabla}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)+\hat{g}(\breve{\nabla}_{\mathcal{X}_1}\hat{\zeta},\mathcal{X}_2)+\hat{g}(\mathcal{X}_1,\breve{\nabla}_{\mathcal{X}_2}\hat{\zeta}). \tag{6.49}$$

Now using (2.6) and (3.22) in (6.49), we have

$$(\breve{\mathcal{L}}_{\hat{\zeta}}\hat{g})(\mathcal{X}_1,\mathcal{X}_2)=2\beta[\hat{g}(\mathcal{X}_1,\mathcal{X}_2)-\hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2)]. \tag{6.50}$$

Now, from (6.48) and (6.50), we obtain

$$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)=-(\beta+\Theta)\hat{g}(\mathcal{X}_1,\mathcal{X}_2)+\beta\hat{\eta}(\mathcal{X}_1)\hat{\eta}(\mathcal{X}_2). \tag{6.51}$$

Replacing $\mathcal{X}_1,\mathcal{X}_2$ by $\hat{\zeta}$ and using (6.43), we get

$$\Theta=2n(\beta^2+\hat{\zeta}\beta).$$

Since β is some non-zero function, we have $\Theta\neq 0$, so we state the following theorem:

Theorem 6.2. *A Ricci soliton* $(\hat{g}, \hat{\zeta}, \Theta)$ *in* β*-Kenmotsu manifold* Ω^{2n+1} *with* $\breve{\nabla}$ *can not be steady but is expanding if* $\beta^2 + \hat{\zeta}\beta > 0$ *and shrinking if* $\beta^2 + \hat{\zeta}\beta < 0$.

7. Example of β-Kenmotsu Manifold with non-symmetric non-metric connection

Example 7.1. *Let us consider the* 3*-dimensional manifold* $\Omega^{2n+1} = [(x; y; z) \in \mathcal{R}^3 | z \neq 0]$*; where* $(x; y; z)$ *are the standard coordinates in* $\mathcal{R}^3$*. Consider the vector fields*

$$\varrho_1 = z^2 \frac{\partial}{\partial x}, \quad \varrho_2 = z^2 \frac{\partial}{\partial y}, \quad \varrho_3 = \frac{\partial}{\partial z} = \hat{\zeta}.$$

At each point of Ω^{2n+1}, ϱ_1, ϱ_2 and ϱ_3 are linearly independent. Suppose the Riemannian metric $\hat{g}$ is defined as

$$\begin{aligned} &\hat{g}(\varrho_1, \varrho_2) = \hat{g}(\varrho_2, \varrho_3) = \hat{g}(\varrho_3, \varrho_1) = 0, \\ &\hat{g}(\varrho_1, \varrho_1) = \hat{g}(\varrho_2, \varrho_2) = \hat{g}(\varrho_3, \varrho_3) = 1, \end{aligned} \tag{7.52}$$

and $\hat{\varphi}$ is defined by

$$\hat{\varphi}(\varrho_1) = -\varrho_2, \hat{\varphi}(\varrho_2) = \varrho_1, \hat{\varphi}(\varrho_3) = 0. \tag{7.53}$$

According to the Lie bracket definition, we get

$$[\varrho_1, \varrho_2] = 0, \qquad [\varrho_1, \varrho_3] = -\frac{2}{z}\varrho_1, \qquad [\varrho_2, \varrho_3] = -\frac{2}{z}\varrho_2. \tag{7.54}$$

Also

$$\begin{aligned} 2\hat{g}(\hat{\nabla}_{\mathcal{X}_1}\mathcal{X}_2, \mathcal{X}_3) &= \mathcal{X}_1\hat{g}(\mathcal{X}_2, \mathcal{X}_3) + \mathcal{X}_2\hat{g}(\mathcal{X}_3, \mathcal{X}_1) - \mathcal{X}_3\hat{g}(\mathcal{X}_1, \mathcal{X}_2) \\ &\quad + \hat{g}([\mathcal{X}_1, \mathcal{X}_2], \mathcal{X}_3) - \hat{g}([\mathcal{X}_2, \mathcal{X}_3], \mathcal{X}_1) + \hat{g}([\mathcal{X}_3, \mathcal{X}_1], \mathcal{X}_2). \end{aligned} \tag{7.55}$$

Using Koszul's formula, we get

$$\begin{aligned} &\hat{\nabla}_{\varrho_1}\varrho_1 = \frac{2}{z}\varrho_3, \quad \hat{\nabla}_{\varrho_1}\varrho_2 = 0, \qquad \hat{\nabla}_{\varrho_1}\varrho_3 = -\frac{2}{z}\varrho_1, \\ &\hat{\nabla}_{\varrho_2}\varrho_1 = 0, \qquad \hat{\nabla}_{\varrho_2}\varrho_2 = \frac{2}{z}\varrho_3, \quad \hat{\nabla}_{\varrho_2}\varrho_3 = -\frac{2}{z}\varrho_2, \\ &\hat{\nabla}_{\varrho_3}\varrho_1 = 0, \qquad \hat{\nabla}_{\varrho_3}\varrho_2 = 0, \qquad \hat{\nabla}_{\varrho_3}\varrho_3 = 0. \end{aligned} \tag{7.56}$$

Also $\mathcal{X}_1 = \mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3$ and $\hat{\zeta} = \varrho_3$, then we have

$$\begin{aligned} \hat{\nabla}_{\mathcal{X}_1}\hat{\zeta} &= \hat{\nabla}_{\mathcal{X}^1\varrho_1+\mathcal{X}^2\varrho_2+\mathcal{X}^3\varrho_3}\varrho_3 \\ &= \mathcal{X}^1\hat{\nabla}_{\varrho_1}\varrho_3 + \mathcal{X}^2\hat{\nabla}_{\varrho_2}\varrho_3 + \mathcal{X}^3\hat{\nabla}_{\varrho_3}\varrho_3 \\ &= -\frac{2}{z}\left(\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2\right) \end{aligned} \tag{7.57}$$

and

$$\begin{aligned} \hat{\nabla}_{\mathcal{X}_1}\hat{\zeta} &= \beta[\mathcal{X}_1 - \hat{\eta}(\mathcal{X}_1)\hat{\zeta}] \\ &= \beta[\left(\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3\right) - \hat{g}\left(\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3, \varrho_3\right)\varrho_3] \\ &= -\frac{2}{z}[\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2 + \mathcal{X}^3\varrho_3 - \mathcal{X}^3\varrho_3] \\ &= -\frac{2}{z}[\mathcal{X}^1\varrho_1 + \mathcal{X}^2\varrho_2]. \end{aligned} \tag{7.58}$$

From (7.57) and (7.58), the structure $(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{g})$ is a β-Kenmotsu manifold structure. Therefore $\Omega^3(\hat{\varphi}, \hat{\zeta}, \hat{\eta}, \hat{g})$ is a β-Kenmotsu manifold. From (2.3), (2.5), (3.17) and (7.56), we have

$$\begin{aligned} &\breve{\nabla}_{\varrho_1}\varrho_1 = \frac{2}{z}\varrho_3, \quad \breve{\nabla}_{\varrho_1}\varrho_2 = -\varrho_3, \quad \breve{\nabla}_{\varrho_1}\varrho_3 = -\frac{2}{z}\varrho_1, \\ &\breve{\nabla}_{\varrho_2}\varrho_1 = \varrho_3, \quad \breve{\nabla}_{\varrho_2}\varrho_2 = \frac{2}{z}\varrho_3, \quad \breve{\nabla}_{\varrho_2}\varrho_3 = -\frac{2}{z}\varrho_2, \\ &\breve{\nabla}_{\varrho_3}\varrho_1 = 0, \quad \breve{\nabla}_{\varrho_3}\varrho_2 = 0, \quad \breve{\nabla}_{\varrho_3}\varrho_3 = 0. \end{aligned} \tag{7.59}$$

From equations (3.18) and (3.19), we have

$$\breve{\mathcal{T}}'(\varrho_1, \varrho_2) = 2\hat{g}(\hat{\varphi}\varrho_1, \varrho_2) = -2\varrho_3 \neq 0$$

and

$$\begin{aligned} (\breve{\nabla}_{\varrho_1}\hat{g})(\varrho_2, \varrho_3) &= -\hat{\eta}(\varrho_3)\hat{g}(\hat{\varphi}\varrho_1, \varrho_2) - \hat{\eta}(\varrho_2)\hat{g}(\hat{\varphi}\varrho_1, \varrho_3) \\ &= 1 \neq 0. \end{aligned}$$

Consequently, a non-symmetric non-metric connection $\breve{\nabla}$ is defined in (3.17). Also,

$$\begin{aligned} \breve{\nabla}_{\mathcal{X}_1}\hat{\zeta} &= \breve{\nabla}_{\mathcal{X}^1\varrho_1+\mathcal{X}^2\varrho_2+\mathcal{X}^3\varrho_3}\varrho_3 \\ &= \mathcal{X}^1\breve{\nabla}_{\varrho_1}\varrho_3 + \mathcal{X}^2\breve{\nabla}_{\varrho_2}\varrho_3 + \mathcal{X}^3\breve{\nabla}_{\varrho_3}\varrho_3 \\ &= -\frac{2}{z}\mathcal{X}^1\varrho_1 - \frac{2}{z}\mathcal{X}^2\varrho_2, \end{aligned} \tag{7.60}$$

The equation (3.22) can be verified using equations (7.57) and (7.60).

The components of $\mathcal{R}^{\dagger}$ of $\hat{\nabla}$ are defined as

$$\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_1=\frac{4}{z^2}\varrho_2,\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_1=\frac{4}{z^2}\varrho_3,\mathcal{R}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_1=0,$$

$$\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_2=-\frac{4}{z^2}\varrho_1,\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_2=0,\mathcal{R}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_2=\frac{4}{z^2}\varrho_3, \tag{7.61}$$

$$\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_3=0,\mathcal{R}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_1,\mathcal{R}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_2,$$

hence we can verify the equations (2.8), (2.9), (2.10) and (2.12).

Similarly, the components of curvature tensor $\breve{\mathcal{R}}^{\dagger}$ of connection $\breve{\nabla}$ are as under:

$$\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_1=\frac{4}{z^2}\varrho_2-\frac{2}{z}\varrho_1,\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_1=\frac{4}{z^2}\varrho_3,\breve{\mathcal{R}}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_1=\frac{2}{z}\varrho_3,$$

$$\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_2=-\frac{4}{z^2}\varrho_1-\frac{2}{z}\varrho_2,\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_2=-\frac{2}{z}\varrho_3,\breve{\mathcal{R}}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_2=\frac{4}{z^2}\varrho_3, \tag{7.62}$$

$$\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_2\right)\varrho_3=\frac{4}{z}\varrho_3,\breve{\mathcal{R}}^{\dagger}\left(\varrho_1,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_1,\breve{\mathcal{R}}^{\dagger}\left(\varrho_2,\varrho_3\right)\varrho_3=-\frac{4}{z^2}\varrho_2.$$

Thus, we can verify (4.24), (4.28), (4.29) and (4.30).

$\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)$ of connection $\hat{\nabla}$ can be derived by using (7.61) in $\mathcal{S}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)=\sum_{i=1}^{3}\hat{g}\left(\mathcal{R}^{\dagger}\left(\varrho_i,\mathcal{X}_1\right)\mathcal{X}_2,\varrho_i\right)$. It is as under:

$$\mathcal{S}^{\dagger}\left(\varrho_1,\varrho_1\right)=\mathcal{S}^{\dagger}\left(\varrho_2,\varrho_2\right)=\mathcal{S}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{8}{z^2}. \tag{7.63}$$

$\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_1)$ of connection $\breve{\nabla}$ can be derived by using equation (7.62) in $\breve{\mathcal{S}}^{\dagger}(\mathcal{X}_1,\mathcal{X}_2)=\sum_{i=1}^{3}\hat{g}(\breve{\mathcal{R}}^{\dagger}\left(\varrho_i,\mathcal{X}_1\right)\mathcal{X}_2,\varrho_i)$. It is as follows:

$$\breve{\mathcal{S}}^{\dagger}\left(\varrho_1,\varrho_1\right)=\breve{\mathcal{S}}^{\dagger}\left(\varrho_2,\varrho_2\right)=\breve{\mathcal{S}}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{8}{z^2}. \tag{7.64}$$

In view of (7.63) and (7.64), the scalar curvature can be calculated as under:

$$k=\sum_{i=1}^{3}\mathcal{S}^{\dagger}\left(\varrho_i,\varrho_i\right)=\mathcal{S}^{\dagger}\left(\varrho_1,\varrho_1\right)+\mathcal{S}^{\dagger}\left(\varrho_2,\varrho_2\right)+\mathcal{S}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{24}{z^2},$$

$$\breve{k}=\sum_{i=1}^{3}\breve{\mathcal{S}}^{\dagger}\left(\varrho_i,\varrho_i\right)=\breve{\mathcal{S}}^{\dagger}\left(\varrho_1,\varrho_1\right)+\breve{\mathcal{S}}^{\dagger}\left(\varrho_2,\varrho_2\right)+\breve{\mathcal{S}}^{\dagger}\left(\varrho_3,\varrho_3\right)=-\frac{24}{z^2}.$$

Thus we see that the example also verify Theorem 4.2.

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A STUDY OF φ-RICCI SYMMETRIC LP-KENMOTSU MANIFOLDS

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AND VINDHYACHAL SINGH YADAV

Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In the current article we characterize φ-Ricci symmetric (φ-RS) and weakly φ-Ricci symmetric (weakly φ-RS) LP-Kenmotsu m-manifolds ((LP-K)$_m$). We also examine the characteristic of an (LP-K)$_3$ of scalar curvature 6. Moreover, we study (LP-K)$_m$ admitting ω-parallel Ricci tensor. At last, we construct an example of φ-RS (LP-K)$_3$ to verify some of our results.

Keywords: Einstein manifold, φ-Ricci symmetric manifolds, LP-Kenmotsu manifolds, scalar curvature, Ricci tensor.

2010 Mathematics Subject Classification: 53C25, 53C50, 53C80.

1. Introduction

Approximately five decades ago, the notion of Kenmotsu manifold as a class of almost contact metric manifolds was introduced by Kenmotsu [19]. Kenmotsu has proved that a locally Kenmostu manifold is a warped product $\mathcal{I}\times_{\mathfrak{f}} \aleph$ of an interval $\mathcal{I}$ and a K*ä*hler manifold $\aleph$ with warping function $\mathfrak{f}(\mathfrak{t}) = \rho e^{\mathfrak{t}}$, where ρ $(\neq 0)$ is a constant. In 1976, the idea of almost para-contact Riemannian manifolds was proposed by Sato [20]. Then, as a class of almost contact Riemannian manifolds, para-Sasakian and Special para-Sasakian manifolds have been

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defined and studied in [1] by Adati and Matsumoto. In 1989, Matsumoto [14] defined and studied Lorentzian para-Sasakian manifolds. Later, Mihai and Rosca also contribured some remarks on this manifold [16]. The authors Sinha and Prasad [22] studied para-Kenmotsu manifolds. In 2018, the first and second authors proposed and investigated a new class of Lorentzian almost para-contact metric manifolds namely LP-Kenmotsu manifolds [11]. Recently, numerous geometers studied LP-Kenmotsu manifolds in many ways to different point of views such as [2, 17, 12, 9, 15] and many others. Several mathematicians have studied the notion of weakly local symmetric Riemannian manifolds with different approaches in various fields. In 1977, Takahashi [23] introduced the concept of locally φ-symmetric Sasakian manifolds. The φ-symmetric notion in contact geometry was initiated and studied by Vanhecke, Buecken and Boeckx [5]. About two decades ago, the authors De, Shaikh and Biswas have studied φ-recurrent Sasakian manifolds [6] by generalizing the idea of locally φ-symmetric manifolds. In [8], the author studied φ-symmetric Kenmotsu manifolds in which he had given a number of examples. In 2008, De and Sarkar [7] studied φ-RS Sasakian manifolds. Later in 2009, φ-RS Kenmotsu manifold was studied by Shukla and Shukla [21].

This paper is structured in the following manner: Section 2 contains preliminaries, where some basic results are mentioned. In section 3, we study φ-RS (LP-K)$_m$ and prove that an (LP-K)$_m$ is Einstein manifold, if it is φ-symmetric. In section 4, we study of φ-RS (LP-K)$_3$, here we proved that an (LP-K)$_3$ is locally φ-RS, if and only if $\underline{\mathrm{r}}$ is constant. Section 5 is devoted to the study of weakly φ-RS (LP-K)$_m$ and it is proven that a weakly φ-RS (LP-K)$_m$ is an ω-Einstein manifold. Section 6 deals with the study of (LP-K)$_m$ admitting ω-parallel Ricci tensor. At last an example of (LP-K)$_3$ is modeled to inquire some of our findings.

2. Preliminaries

Let $\mathcal{M}^m$ $(\varphi, \zeta, \omega, g)$ be a Lorentzian metric manifold, where φ: $(1,1)$ tensor field, ζ: a characteristic vector field, ω: a 1-form and g: the Lorentz metric. We are well acquainted with the following results [3, 4, 18]:

$$\begin{cases} \varphi\zeta = 0, \\ \omega(\varphi\underline{\mathrm{U}}) = 0, \\ \omega(\zeta) + 1 = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \varphi^2 \underline{U} - \underline{U} - \omega(\underline{U})\zeta = 0, \\ g(\underline{U},\zeta) - \omega(\underline{U}) = 0, \end{cases} \tag{2.2}$$

$$g(\varphi\underline{U}, \varphi\underline{V}) - g(\underline{U}, \underline{V}) = \omega(\underline{U})\omega(\underline{V}), \tag{2.3}$$

$$(\bar{\nabla}_{\underline{U}}\varphi)\underline{V} = -g(\varphi\underline{U}, \underline{V})\zeta - \omega(\underline{V})\varphi\underline{U}, \tag{2.4}$$

$$\bar{\nabla}_{\underline{U}}\zeta = -\underline{U} - \omega(\underline{U})\zeta, \tag{2.5}$$

for all vector fields $\underline{U}, \underline{V}$ on $\mathcal{M}^m$ and $\bar{\nabla}$ represents the Levi-Civita connection of g, then $\mathcal{M}^m$ $(\varphi, \zeta, \omega, g)$ is said to be an (LP-K)$_m$ [11, 10].

In (LP-K)$_m$, the following results hold:

$$(\bar{\nabla}_{\underline{U}}\omega)\underline{V} = -\omega(\underline{U})\omega(\underline{V}) - g(\underline{U}, \underline{V}), \tag{2.6}$$

$$\omega(\underline{R}(\underline{U}, \underline{V})\underline{Z}) = g(\underline{V}, \underline{Z})\omega(\underline{U}) - g(\underline{U}, \underline{Z})\omega(\underline{V}), \tag{2.7}$$

$$\underline{R}(\underline{U}, \underline{V})\zeta = \omega(\underline{V})\underline{U} - \omega(\underline{U})\underline{V}, \tag{2.8}$$

$$\underline{R}(\zeta, \underline{U})\underline{V} = g(\underline{U}, \underline{V})\zeta - \omega(\underline{V})\underline{U}, \tag{2.9}$$

$$\mathcal{S}(\underline{U}, \zeta) = (m-1)\omega(\underline{U}), \qquad \underline{Q}\zeta = (m-1)\zeta, \tag{2.10}$$

$$(\bar{\nabla}_{\underline{Z}}\underline{R})(\underline{U}, \underline{V})\zeta = g(\underline{U}, \underline{Z})\underline{V}g(\underline{V}, \underline{Z})\underline{U} + \underline{R}(\underline{U}, \underline{V})\underline{Z}, \tag{2.11}$$

$$\mathcal{S}(\varphi\underline{U}, \varphi\underline{V}) = \mathcal{S}(\underline{U}, \underline{V}) + (m-1)\omega(\underline{U})\omega(\underline{V}) \tag{2.12}$$

for all vector fields $\underline{U}, \underline{V}, \underline{Z}$ on (LP-K)$_m$, where $\underline{R}$ is the Riemannian curvature tensor, $\mathcal{S}$ is the Ricci tensor and $\underline{Q}$ indicates the Ricci operator such that $\mathcal{S}(\underline{U}, \underline{V}) = g(\underline{Q}\underline{U}, \underline{V})$.

Remark 2.1. [13] *If an (LP-K)$_m$ possesses the constant scalar curvature, then* $r = m(m-1)$.

3. φ-RS (LP-K)$_m$

We start this section with the following definitions:

Definition 3.1. *An (LP-K)$_m$ is called*

(i) φ-RS if

$$\varphi^2((\bar{\nabla}_{\underline{U}}\underline{Q})(\underline{V})) = 0, \tag{3.13}$$

(ii) φ-symmetric if

$$\varphi^2((\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U}, \underline{V})\underline{Z}) = 0 \tag{3.14}$$

for any vector fields $\underline{U}$, $\underline{V}$, $\underline{Z}$, $\underline{K}$ on (LP-K)$_m$. In case, $\underline{U}$, $\underline{V}$ are orthogonal to ζ, then φ-RS (LP-K)$_m$ is named locally φ-RS.

Definition 3.2. *An (LP-K)$_m$ is called Einstein manifold, if its $\mathcal{S}$ is of the form*

$$\mathcal{S}(\underline{U}, \underline{V}) = \lambda g(\underline{U}, \underline{V}),$$

where λ is a constant.

Theorem 3.1. *An (LP-K)$_m$ is φ-RS, iff it is Einstein manifold.*

Proof. Let an (LP-K)$_m$ be φ-RS. Then we have

$$\varphi^2((\bar{\nabla}_{\underline{U}}Q)(\underline{V})) = 0,$$

which by using (2.2) becomes

$$(\bar{\nabla}_{\underline{U}}Q)\underline{V} + \omega((\bar{\nabla}_{\underline{U}}Q)\underline{V})\zeta = 0. \tag{3.15}$$

The inner product of (3.15) with $\underline{Z}$ lead to

$$g((\bar{\nabla}_{\underline{U}}Q)\underline{V}, \underline{Z}) + \omega((\bar{\nabla}_{\underline{U}}Q)\underline{V})\omega(\underline{Z}) = 0,$$

which after simplification takes the form

$$g(\bar{\nabla}_{\underline{U}}(Q\underline{V}), \underline{Z}) - \mathcal{S}(\bar{\nabla}_{\underline{U}}\underline{V}, \underline{Z}) + \omega((\bar{\nabla}_{\underline{U}}Q)\underline{V})\omega(\underline{Z}) = 0. \tag{3.16}$$

By taking $\underline{V} = \zeta$ in (3.16), then using (2.5) and (2.10), we have

$$(m-1)g(\bar{\nabla}_{\underline{U}}\zeta, \underline{Z}) + \mathcal{S}(\underline{U}, \underline{Z}) + \omega(\underline{U})\mathcal{S}(\zeta, \underline{Z}) + \omega((\bar{\nabla}_{\underline{U}}Q)\zeta)\omega(\underline{Z}) = 0. \tag{3.17}$$

Now by virtue of (2.5) and (2.10), (3.17) turns to

$$S(\underline{U}, \underline{Z}) - (m-1)g(\underline{U}, \underline{Z}) + \omega((\bar{\nabla}_{\underline{U}}Q)\zeta)\omega(\underline{Z}) = 0. \tag{3.18}$$

Substituting $\underline{U} \to \varphi\underline{U}$ as well as $\underline{Z} \to \varphi\underline{Z}$ in (3.18), we find

$$\mathcal{S}(\varphi\underline{U}, \varphi\underline{Z}) = (m-1)g(\varphi\underline{U}, \varphi\underline{Z}). \tag{3.19}$$

Keeping in mind (2.3) and (2.12), (3.19) leads to

$$\mathcal{S}(\underline{U}, \underline{Z}) = (m-1)g(\underline{U}, \underline{Z}). \tag{3.20}$$

Conversely, we assume that (LP-K)$_m$ is an Einstein manifold. Therefore, by the Definition 3.2, we have $Q\underline{U} = \lambda\underline{U}$, from which we conclude

$$\varphi^2((\bar{\nabla}_{\underline{U}}Q)(\underline{V})) = 0.$$

This completes the proof. □

Corollary 3.1. *An (LP-K)$_m$ is Einstein manifold, if it is φ-symmetric.*

Proof. Let an $(\text{LP-K})_m$ be φ-symmetric manifold. Then we have

$$\varphi^2((\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\underline{Z}) = 0 \tag{3.21}$$

for any vector fields $\underline{U}$, $\underline{V}$, $\underline{Z}$, $\underline{K}$ on $(\text{LP-K})_m$.

By using (2.2) in (3.21), it yields

$$(\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\underline{Z} - g((\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\zeta,\underline{Z})\zeta = 0. \tag{3.22}$$

Now in view of (2.11), (3.22) takes the form

$$\begin{aligned}(\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\underline{Z} - g(\underline{U},\underline{K})g(\underline{V},\underline{Z})\zeta \\ +g(\underline{V},\underline{K})g(\underline{U},\underline{Z})\zeta - g(\underline{R}(\underline{U},\underline{V})\underline{K},\underline{Z})\zeta = 0.\end{aligned} \tag{3.23}$$

On contracting (3.23), we obtain

$$(\bar{\nabla}_{\underline{K}}\mathcal{S})(\underline{V},\underline{Z}) - g(\underline{V},\underline{Z})\omega(\underline{K}) + g(\underline{V},\underline{K})\omega(\underline{Z}) + \omega(\underline{R}(\underline{K},\underline{Z})\underline{V}) = 0. \tag{3.24}$$

By virtue of (2.7), equation (3.24) reduces to

$$(\bar{\nabla}_{\underline{K}}\mathcal{S})(\underline{V},\underline{Z}) = 0. \tag{3.25}$$

Consequently, we obtain

$$\varphi^2((\bar{\nabla}_{\underline{K}}\mathcal{S})(\underline{V},\underline{Z})) = 0. \tag{3.26}$$

Thus φ-symmetric $(\text{LP-K})_m$ is φ-RS. And hence Corollary 3.1 follows from Theorem 3.1. □

4. φ-RS $(\text{LP-K})_3$

Theorem 4.1. *In case, the scalar curvature $\underline{r}$ of an $(LP\text{-}K)_3$ is 6, then $(LP\text{-}K)_3$ is φ-RS.*

Proof. In an $(\text{LP-K})_3$, the curvature tensor $\underline{R}$ is given by [11, 24]

$$\begin{aligned}\underline{R}(\underline{U},\underline{V})\underline{Z} &= (\frac{\underline{r}}{2}-2)[g(\underline{V},\underline{Z})\underline{U} - g(\underline{U},\underline{Z})\underline{V}] \\ &+(\frac{\underline{r}}{2}-3)[g(\underline{V},\underline{Z})\omega(\underline{U})\zeta - g(\underline{U},\underline{Z})\omega(\underline{V})\zeta] \\ &+(\frac{\underline{r}}{2}-3)[\omega(\underline{V})\omega(\underline{Z})\underline{U} - \omega(\underline{U})\omega(\underline{Z})\underline{V}]\end{aligned} \tag{4.27}$$

for all vector fields $\underline{U},\underline{V},\underline{Z}$ on $(\text{LP-K})_3$.

The inner product of (4.27) with $\underline{K}$ leads to

$$\begin{aligned}g(\underline{R}(\underline{U},\underline{V})\underline{Z},\underline{K}) &= (\frac{\underline{r}}{2}-2)[g(\underline{V},\underline{Z})g(\underline{U},\underline{K}) - g(\underline{U},\underline{Z})g(\underline{V},\underline{K})] \\ &+(\frac{\underline{r}}{2}-3)[g(\underline{V},\underline{Z})\omega(\underline{U})\omega(\underline{K}) - g(\underline{U},\underline{Z})\omega(\underline{V})\omega(\underline{K})] \\ &+(\frac{\underline{r}}{2}-3)[\omega(\underline{V})\omega(\underline{Z})g(\underline{U},\underline{K}) - \omega(\underline{U})\omega(\underline{Z})g(\underline{V},\underline{K})].\end{aligned} \tag{4.28}$$

Let $\{\underline{l}_1, \underline{l}_2, \underline{l}_3\}$ be the orthonormal basis of the tangent space at every point of $(\text{LP-K})_3$. Now setting $\underline{\text{U}} = \underline{\text{K}} = \underline{l}_i$ as well as proceeding for sum from $i = 1$ to 3 in equation (4.28), it provides

$$\mathcal{S}(\underline{\text{V}}, \underline{\text{Z}}) = (\frac{\underline{\text{r}}}{2} - 1)g(\underline{\text{V}}, \underline{\text{Z}}) + (\frac{\underline{\text{r}}}{2} - 3)\omega(\underline{\text{V}})\omega(\underline{\text{Z}}). \tag{4.29}$$

From (4.29) it follows that

$$\underline{\text{Q}}\underline{\text{V}} = (\frac{\underline{\text{r}}}{2} - 1)\underline{\text{V}} + (\frac{\underline{\text{r}}}{2} - 3)\omega(\underline{\text{V}})\zeta. \tag{4.30}$$

Differentiating (4.30) covariantly along $\underline{\text{K}}$, we have

$$\begin{aligned}(\bar{\nabla}_{\underline{\text{K}}}\underline{\text{Q}})\underline{\text{V}} + \underline{\text{Q}}(\bar{\nabla}_{\underline{\text{K}}}\underline{\text{V}}) &= (\frac{\underline{\text{r}}}{2} - 1)\bar{\nabla}_{\underline{\text{K}}}\underline{\text{V}} + \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\underline{\text{V}} + \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\omega(\underline{\text{V}})\zeta + (\frac{\underline{\text{r}}}{2} - 3)(\bar{\nabla}_{\underline{\text{K}}}\omega)(\underline{\text{V}})\zeta \\ &\quad + (\frac{\underline{\text{r}}}{2} - 3)\omega(\bar{\nabla}_{\underline{\text{K}}}\underline{\text{V}})\zeta + (\frac{\underline{\text{r}}}{2} - 3)\omega(\underline{\text{V}})\bar{\nabla}_{\underline{\text{K}}}\zeta.\end{aligned} \tag{4.31}$$

By virtue of (4.30), (4.31) takes the form

$$\begin{aligned}(\bar{\nabla}_{\underline{\text{K}}}\underline{\text{Q}})\underline{\text{V}} &= \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\underline{\text{V}} + \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\omega(\underline{\text{V}})\zeta + (\frac{\underline{\text{r}}}{2} - 3)(\bar{\nabla}_{\underline{\text{K}}}\omega)(\underline{\text{V}})\zeta \\ &\quad + (\frac{\underline{\text{r}}}{2} - 3)\omega(\underline{\text{V}})\bar{\nabla}_{\underline{\text{K}}}\zeta.\end{aligned} \tag{4.32}$$

By using (2.5) and (2.6) in (4.32), we have

$$\begin{aligned}(\bar{\nabla}_{\underline{\text{K}}}\underline{\text{Q}})\underline{\text{V}} &= \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\underline{\text{V}} + \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\omega(\underline{\text{V}})\zeta - (\frac{\underline{\text{r}}}{2} - 3)g(\underline{\text{V}}, \underline{\text{K}})\zeta \\ &- (\frac{\underline{\text{r}}}{2} - 3)\omega(\underline{\text{V}})\omega(\underline{\text{K}})\zeta - (\frac{\underline{\text{r}}}{2} - 3)[\omega(\underline{\text{V}})\underline{\text{K}} + \omega(\underline{\text{V}})\omega(\underline{\text{K}})\zeta].\end{aligned} \tag{4.33}$$

By operating φ^2 on both the sides of (4.33), then using (2.1) and (2.2), we arrive at

$$\varphi^2((\bar{\nabla}_{\underline{\text{K}}}\underline{\text{Q}})\underline{\text{V}}) = \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}[\underline{\text{V}} + \omega(\underline{\text{V}})\zeta] - (\frac{\underline{\text{r}}}{2} - 3)[\omega(\underline{\text{V}})(\underline{\text{K}} + \omega(\underline{\text{K}})\zeta)]. \tag{4.34}$$

Since $\underline{\text{r}} = 6$, therefore, from (4.34) it follows that

$$\varphi^2((\bar{\nabla}_{\underline{\text{K}}}\underline{\text{Q}})\underline{\text{V}}) = 0. \tag{4.35}$$

Hence, this completes the proof. □

Corollary 4.1. *An (LP-K)$_3$ is locally φ-RS, if and only if $\underline{\text{r}}$ is constant.*

Proof. By taking $\underline{\text{V}}$ as orthogonal to ζ, then (4.34) provides

$$\varphi^2((\bar{\nabla}_{\underline{\text{K}}}\underline{\text{Q}})\underline{\text{V}}) = \frac{d\underline{\text{r}}(\underline{\text{K}})}{2}\underline{\text{V}}. \tag{4.36}$$

The result follows from (4.36) and Theorem 4.1. □

5. Weakly φ-RS (LP-K)$_m$

Definition 5.1. *An $(LP\text{-}K)_m$ is called weakly φ-RS if its Ricci operator Q satisfies*

$$\varphi^2((\bar{\nabla}_{U}Q)(V)) = A(U)\varphi^2(Q(V)) + B(V)\varphi^2(Q(U)) + \mathcal{S}(V, U)\varphi^2(\rho), \tag{5.37}$$

where $U, V \in (LP\text{-}K)_m$. A, B, D are 1-forms and ρ is a vector field associated with 1-form D, i.e., $g(\rho, Z) = D(Z)$.

If the 1-forms $A = B = D = 0$, then the relation (5.37) reduces to the concept of φ-RS given by

$$\varphi^2((\nabla_{U}Q)(V)) = 0. \tag{5.38}$$

This concept was initiated by Shukla and Shukla [21].

Now, we consider an (LP-K)$_m$, which is weakly φ Ricci symmetric. Consequently, the relation (5.37) together with (2.2) gives

$$\begin{aligned}(\bar{\nabla}_{U}Q)(V) + \omega((\bar{\nabla}_{U}Q)(V))\zeta \;&=\; A(U)[QV + \omega(QV)\zeta] + B(V)[QU + \omega(QU)\zeta]\\ &\quad+\mathcal{S}(V, U)[\rho + \omega(\rho)\zeta],\end{aligned}$$

which can be written as

$$\begin{aligned}&\bar{\nabla}_{U}(QV) - Q(\bar{\nabla}_{U}V) + \omega(\bar{\nabla}_{U}(QV) - Q(\nabla_{U}V))\zeta = A(U)QV\\ &\quad + A(U)\omega(QV)\zeta + B(V)[QU + \omega(QU)\zeta] + \mathcal{S}(V, U)\rho + \mathcal{S}(V, U)\omega(\rho)\zeta.\end{aligned} \tag{5.39}$$

Taking the inner product of (5.39) with Z and using (2.2), we have

$$\begin{aligned}&g(\bar{\nabla}_{U}(QV), Z) - g(Q(\nabla_{U}V), Z) + \omega(\bar{\nabla}_{U}(QV) - Q(\nabla_{U}V))\omega(Z)\\ &= A(U)g(QV, Z) + A(U)\omega(QV)\omega(Z) + B(V)[g(QU, Z)\\ &+\omega(QU)\omega(Z)] + \mathcal{S}(V, U)D(Z) + \mathcal{S}(V, U)\omega(\rho)\omega(Z),\end{aligned} \tag{5.40}$$

where $g(\rho, Z) = D(Z)$.

Setting $V = \zeta$ in (5.40), it yields

$$\begin{aligned}&g(\bar{\nabla}_{U}(Q\zeta), Z) - g(Q(\bar{\nabla}_{U}\zeta), Z) + \omega(\bar{\nabla}_{U}(Q\zeta) - (Q\bar{\nabla}_{U}\zeta))\omega(Z)\\ &= A(U)g(Q\zeta, Z) + A(U)\omega(Q\zeta)\omega(Z) + B(\zeta)[g(QU, Z)\\ &+\omega(QU)\omega(Z)] + \mathcal{S}(\zeta, U)D(Z) + \mathcal{S}(\zeta, U)\omega(\rho)\omega(Z).\end{aligned} \tag{5.41}$$

By using (2.5) and (2.10) in (5.41), it gives

$$\begin{aligned}\mathcal{S}(\underline{U},\underline{Z})[1-B(\zeta)] &= (m-1)[g(\underline{U},\underline{Z})+\omega(\underline{U})D(\underline{Z})] \\ &\quad +(m-1)[B(\zeta)+\omega(\rho)]\omega(\underline{U})\omega(\underline{Z}).\end{aligned} \tag{5.42}$$

Applying $\underline{U} \longrightarrow \varphi\underline{U}$ and $\underline{Z} \longrightarrow \varphi\underline{Z}$ in (5.42), then using relation (2.1), (2.3) and (2.12), we lead to

$$[1-B(\zeta)]\mathcal{S}(\underline{U},\underline{Z})+(m-1)[1-B(\zeta)]\omega(\underline{U})\omega(\underline{Z})=(m-1)[g(\underline{U},\underline{Z})+\omega(\underline{U})\omega(\underline{Z})],$$

which is of the form

$$\mathcal{S}(\underline{U},\underline{Z})=\mu g(\underline{U},\underline{Z})+\nu\omega(\underline{U})\omega(\underline{Z}), \tag{5.43}$$

where $\mu = \dfrac{(m-1)}{1-B(\zeta)}$ and $\nu = \dfrac{(m-1)B(\zeta)}{1-B(\zeta)}$, provided, $1-B(\zeta)\neq 0$. Thus, we state the following theorem:

Theorem 5.1. *A weakly φ-RS $(LP\text{-}K)_m$ is an ω-Einstein manifold.*

6. $(\text{LP-K})_m$ admitting ω-Parallel Ricci tensor

Definition 6.1. *The Ricci tensor of an $(LP\text{-}K)_m$ is said to be ω-parallel if it satisfies*

$$(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z})=0, \tag{6.44}$$

for all vector fields $\underline{U}$, $\underline{V}$, $\underline{Z}$ on $(LP\text{-}K)_m$.

Let the Ricci tensor of an $(\text{LP-K})_m$ be ω-parallel, therefore (6.44) holds. By the covariant differentiation of $\mathcal{S}(\varphi\underline{V},\varphi\underline{Z})$ along $\underline{U}$, we have

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) &= \bar{\nabla}_{\underline{U}}(\mathcal{S}(\varphi\underline{V},\varphi\underline{Z}))-\mathcal{S}((\bar{\nabla}_{\underline{U}}\varphi)\underline{V},\varphi\underline{Z}) \\ &- \mathcal{S}(\varphi(\bar{\nabla}_{\underline{U}}\underline{V}),\varphi\underline{Z})-\mathcal{S}(\varphi\underline{V},(\bar{\nabla}_{\underline{U}}\varphi)\underline{Z})-\mathcal{S}(\varphi\underline{V},\varphi(\bar{\nabla}_{\underline{U}}\underline{Z})),\end{aligned}$$

which by virtue of (2.12) takes the form

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) &= (\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{V},\underline{Z})+\mathcal{S}(\bar{\nabla}_{\underline{U}}\underline{V},\underline{Z})+\mathcal{S}(\underline{V},\bar{\nabla}_{\underline{U}}\underline{Z}) \\ &+ (n-1)[(\bar{\nabla}_{\underline{U}}\omega)(\underline{V})\omega(\underline{Z})+\omega(\bar{\nabla}_{\underline{U}}\underline{V})\omega(\underline{Z}) \\ &+ \omega(\underline{V})(\bar{\nabla}_{\underline{U}}\omega)(\underline{Z})+\omega(\underline{V})\omega(\bar{\nabla}_{\underline{U}}\underline{Z})]-\mathcal{S}((\bar{\nabla}_{\underline{U}}\varphi)\underline{V},\varphi\underline{Z}) \\ &- \mathcal{S}(\varphi(\bar{\nabla}_{\underline{U}}\underline{V}),\varphi\underline{Z})-\mathcal{S}(\varphi\underline{V},(\bar{\nabla}_{\underline{U}}\varphi)\underline{Z})-\mathcal{S}(\varphi\underline{V},\varphi(\bar{\nabla}_{\underline{U}}\underline{Z})).\end{aligned}$$

In view of (2.4), (2.6), (2.10) and (2.12) the foregoing equation turns to

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) &= (\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{V},\underline{Z}) - (n-1)g(\underline{U},\underline{V})\omega(\underline{Z}) \\ &- (n-1)g(\underline{U},\underline{Z})\omega(\underline{V}) + \mathcal{S}(\underline{U},\underline{Z})\omega(\underline{V}) + \mathcal{S}(\underline{U},\underline{V})\omega(\underline{Z}),\end{aligned}$$

which by virtue of (6.44) gives

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{V},\underline{Z}) &= (n-1)[g(\underline{U},\underline{V})\omega(\underline{Z}) + g(\underline{U},\underline{Z})\omega(\underline{V})] \\ &\quad -[\mathcal{S}(\underline{U},\underline{Z})\omega(\underline{V}) + \mathcal{S}(\underline{U},\underline{V})\omega(\underline{Z})].\end{aligned} \tag{6.45}$$

Let $\{\underline{l}_1, \underline{l}_2, \underline{l}_3......., \underline{l}_m\}$ be the orthonormal basis of the tangent space at every point of (LP-K)$_m$. Now setting $\underline{V} = \underline{Z} = \underline{l}_i$ as well as proceeding for sum from $i = 1$ to m in equation (6.45), it provides

$$\begin{aligned}\sum_{i=1}^{m} \epsilon_i(\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{l}_i,\underline{l}_i) &= (n-1)\sum_{i=1}^{m}\epsilon_i[g(\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta) + g(\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta)] \\ &- \sum_{i=1}^{m}\epsilon_i[g(Q\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta) + g(Q\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta)],\end{aligned} \tag{6.46}$$

where $\epsilon_i = g(e_1, e_i)$. From (6.46) it follows that

$$dr(\underline{U}) = 0. \tag{6.47}$$

Thus, we conclude that $dr = 0$, i.e., r is constant and it is given by $r = m(m-1)$. Moreover, since $\mathcal{S}(\underline{U},\underline{V}) = g(Q\underline{U},\underline{V})$, then we obtain

$$\nabla_U|Q|^2 = 2\sum_{i=1}^{n}\epsilon_i g((\bar{\nabla}_{\underline{U}}Q)e_i, Qe_i). \tag{6.48}$$

By using (6.45) in above equation, we find

$$\nabla_{\underline{U}}|Q|^2 = 2\sum_{i=1}^{n}\epsilon_i g((\bar{\nabla}_{\underline{U}}Q)e_i, Qe_i) = 0. \tag{6.49}$$

This implies that

$$|Q|^2 = constant, \tag{6.50}$$

where Q is the Ricci operator. Hence, the relations (6.47) and (6.50) lead to the following result:

Theorem 6.1. *The scalar curvature of an (LP-K)$_{m>3}$ with the ω-parallel Ricci tensor is constant. Moreover, the norm of the Ricci operator is also constant.*

7. Illustration

We take a 3-dimensional smooth manifold $\mathcal{M}^3 = \{(\underline{u}, \underline{v}, \underline{w}) \in \mathbb{R}^3 : \underline{w} > 0)\}$, where $(\underline{u}, \underline{v}, \underline{w})$ denotes the basic coordinates on a 3-dimensional real space $\mathbb{R}^3$. Consider the vector fields $\{\underline{l}_1, \underline{l}_2, \underline{l}_3\}$, which is linearly independent on $\mathcal{M}^3$ and defined as

$$\underline{l}_1 = (sinh\underline{w} + cosh\underline{w})\frac{\partial}{\partial \underline{u}}, \quad \underline{l}_2 = (sinh\underline{w} + cosh\underline{w})\frac{\partial}{\partial \underline{v}}, \quad \underline{l}_3 = \frac{\partial}{\partial \underline{w}} = \zeta.$$

We define the Lorentz metric g on $\mathcal{M}^3$ as:

$$g_{pq} = g(\underline{l}_p, \underline{l}_q) = \begin{cases} -1 & \text{for } p = q = 3, \\ 0 & \text{for } p \neq q, \\ 1 & p = q = 1, 2. \end{cases}$$

Assume ω be a 1-form corresponding to the Lorentz metric g such that

$$\omega(\underline{U}) = g(\underline{U}, \underline{l}_3)$$

for any $\underline{U} \in \mathfrak{X}(\mathcal{M}^3)$, where $\mathfrak{X}(\mathcal{M}^3)$, denotes the collection of all smooth vector fields on $\mathcal{M}^3$. We define φ as follows

$$\varphi(\underline{l}_1) = \underline{l}_2, \quad \varphi(\underline{l}_2) = \underline{l}_1, \quad \varphi(\underline{l}_3) = 0.$$

Since φ and g have linear nature, so it can be easily proved the following results:

$$\omega(\underline{l}_3) + 1 = 0, \qquad \varphi^2(\underline{U}) - \underline{U} - \omega(\underline{U})\underline{l}_3 = 0, \qquad g(\varphi\underline{U}, \varphi\underline{V}) - g(\underline{U}, \underline{V}) - \omega(\underline{U})\omega(\underline{V}) = 0$$

for all $\underline{U}, \underline{V} \in \mathfrak{X}(\mathcal{M}^3)$. This implies that for $\underline{l}_3 = \zeta$, the structure $(\varphi, \zeta, \omega, g)$ defines a Lorentzian paracontact structure and $(\mathcal{M}^3, \varphi, \zeta, \omega, g)$ is a Lorentzian paracontact manifold of dimension 3. The non-zero constituents of the Lie bracket are given as

$$[\underline{l}_3, \underline{l}_p] = \begin{cases} \underline{l}_p, p = 1, 2, \\ 0, \text{otherwise}. \end{cases}$$

The well-known Koszul's formula provides

$$\bar{\nabla}_{\underline{l}_p}\underline{l}_q = \begin{cases} -\underline{l}_3, p = q = 1, 2, \\ -\underline{l}_p, p = 1, 2, q = 3, \\ 0, \text{otherwise}. \end{cases}$$

From the above equations, it can be easily verified that $\bar{\nabla}_{\underline{U}}\underline{l}_3 = -\{\underline{U} + \omega(\underline{U})\underline{l}_3\}$ and $(\bar{\nabla}_{\underline{U}}\varphi)\underline{V} = -g(\varphi\underline{U}, \underline{V})\zeta - \omega(\underline{V})\varphi\underline{U}$ holds for each $\underline{U}, \underline{V} \in \mathfrak{X}(\mathcal{M}^3)$. Hence the Lorentzian paracontact manifold is an $(\text{LP-K})_3$. From the above equations, the non-zero constituents of $\underline{R}$ are evaluated as follows

$$\underline{R}(\underline{l}_2,\underline{l}_1)\underline{l}_2 = -\underline{l}_1, \quad \underline{R}(\underline{l}_2,\underline{l}_3)\underline{l}_2 = -\underline{l}_3, \quad \underline{R}(\underline{l}_3,\underline{l}_1)\underline{l}_3 = \underline{l}_1,$$

$$\underline{R}(\underline{l}_2,\underline{l}_3)\underline{l}_3 = -\underline{l}_2, \quad \underline{R}(\underline{l}_2,\underline{l}_1)\underline{l}_1 = \underline{l}_2, \quad \underline{R}(\underline{l}_1,\underline{l}_3)\underline{l}_1 = -\underline{l}_3.$$

Thus we have

$$\underline{R}(\underline{U},\underline{V})\underline{Z} = -g(\underline{U},\underline{Z})\underline{V} + g(\underline{V},\underline{Z})\underline{U}, \tag{7.51}$$

which is a space of constant curvature 1.

The matrix representation of $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Thus we find $\underline{r} = 6$. From (7.51) it follows that $\mathcal{S}(\underline{U},\underline{V}) = 2g(\underline{U},\underline{V}) \implies Q\underline{U} = 2\underline{U}$, which implies that $\varphi^2((\bar{\nabla}_{\underline{W}}Q)\underline{U}) = 0$. As we see that $\mathcal{M}^3$ is φ-RS with the scalar curvature 6. Thus this illustration proves Theorem 4.1. Since $\mathcal{M}^3$ is φ-RS and Einstein, this illustration also admits Theorem 3.4 for three dimensional case.

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YAMABE SOLITONS ON Sol_3 SPACE

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. The aim of this work is to find the existence/non-existence of Yamabe solitons and gradient Yamabe solitons of Sol_3 space with left-invariant Riemannian and Lorentzian metric. We show that there exists an expanding Yamabe soliton and a gradient Yamabe soliton with a constant potential function on Sol_3 space.

Keywords: Solvable Lie group, Yamabe soliton, gradient Yamabe soliton, Riemannian metric, Lorentzian metric, left-invariant metric.

2010 Mathematics Subject Classification: 53B30, 53C50, 53C30.

1. Introduction

Thurston [12] gave a classification of 3-dimensional homogeneous manifolds into eight model spaces, which are real space forms having groups of isometries of dimension 6, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil_3 the Heisenberg group, the universal covering $\widetilde{SL_2}\mathbb{R}$ of $SL_2\mathbb{R}$ having a group of isometries of dimension 4, and Sol_3 space with a group of isometries of dimension 3. The Sol_3 space is a simply connected homogeneous 3-dimensional manifold having the smallest number of isometries. The Poincaré conjecture is a special case of the Thurston conjecture, which states that every compact orientable 3-manifold has a canonical decomposition into pieces that each have one of the eight types of geometric structures. In the last three decades, there have been extensive studies to understand this problem; however, the most important

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efforts are due to R. Hamilton [5]. G. Perelman gave a proof of the Thurston conjecture using Ricci flow. Henceforth, this technique attracted the attention of researchers to study problems in homogeneous spaces. Although most of the investigation has been done in the case of the Riemannian setting using the Ricci flow technique (see [9, 10, 8] and references therein). However, in the Lorentzian setting, it has been studied in the last decade (see [2, 7]). Yamabe flows are well-posed in the Riemannian setting, which may not be true in the Lorentzian case due to the non-existence of short-time solutions in general because of the lack of parabolicity. In this paper, we study Yamabe soliton and gradient Yamabe soliton on Sol_3 space with left-invariant Riemannian and Lorentzian metrics.

A Yamabe soliton on a complete Riemannian manifold satisfies [5]:

$$\frac{1}{2}\mathcal{L}_V g = (\nu - r)g, \tag{1.1}$$

where $\mathcal{L}_V$ is the Lie-derivative along the smooth potential field V, g is the Riemannian metric, ν a real scalar, and r is the scalar curvature of g. Also, Yamabe solitons serve as solutions of the Yamabe flow of Hamilton [5], which develops along the symmetries of the flow. The soliton is steady, shrinking or expanding if $\nu = 0$, > 0, or < 0, respectively. If $V = \operatorname{grad} F$ for some real-valued function $F \in C^\infty(M)$, then it is called the gradient Yamabe soliton.

On a smooth Riemannian manifold (M, g_0), the evolution of the metric g_0 in time t to $g = g(t)$ through the equation

$$\frac{\partial}{\partial t} g_t = -rg,\; g(0) = g_0,$$

is known as the Yamabe flow [5]. Yamabe flow is significant as it is a natural geometric deformation to metrics of constant scalar curvature. In mathematical physics, Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation). A Yamabe soliton is a special solution of the Yamabe flow. If V is Killing, then Yamabe soliton is called trivial Yamabe soliton.

In 2012, Calviño-Louzao et al. [3] gave a geometric characterization of Yamabe solitons on three-dimensional homogeneous Lorentzian manifolds. In 2013, Daskalopoulos and Sesum [4] classified the locally conformally flat gradient Yamabe solitons with positive sectional curvature. In 2017, Neto and Tenenblat [6] investigated gradient Yamabe solitons, conformal to an $n-$dimensional pseudo-Euclidean space. Recently, Shaikh et al. [11] examined a gradient Yamabe soliton with some additional conditions and proved that it must be of constant scalar curvature.

The article is organised as follows: In Section 2, we recall the group structure, connection, and curvature of the Sol_3 group. In Section 3, we investigate Yamabe and gradient Yamabe solitons on Sol_3 space with the Riemannian metric. In Section 4, we examine Yamabe and gradient Yamabe solitons on Sol_3 space with the Lorentzian metric.

2. Preliminaries

In this section we recall some basic facts on Sol_3 given in [1].

The Sol_3 space is defined as a group of 3×3 matrices

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix},$$

with the group structure given by

$$(x', y', z') \star (x, y, z) = (e^{-z'}x + x', e^{z'}y + y', z + z'),$$

where $(x, y, z) \in \mathbb{R}^3$.

We denote by ∇ and R the Levi-Civita connection and the Riemann curvature tensor of (Sol_3, g), respectively, such that R is given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and by Ric the Ricci tensor of (Sol_3, g), which is defined by

$$Ric(X, Y) = \sum_{k=1}^{3} g(E_k, E_k) g(R(E_k, X)Y, E_k),$$

where $\{E_k\}_{k=1,\ldots,3}$ is an orthonormal basis.

3. Yamabe and gradient Yamabe solitons on Sol_3 space with Riemannian metric

In this section, we examine the existence of Yamabe and gradient Yamabe solitons on a three-dimensional solvable Lie group (Sol_3, g) with the Riemannian metric.

We consider Sol_3 space with a left-invariant Riemannian metric

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{3.2}$$

with a left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}, \tag{3.3}$$

where $(x, y, z) \in \mathbb{R}^3$.

The non-vanishing Lie brackets are

$$[E_1, E_3] = E_1,\ [E_2, E_3] = -E_2. \tag{3.4}$$

Using (3.2), (3.3), and (3.4), the Levi-Civita connection ∇ is given by

$$(\nabla_{E_i}E_j) = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

where $i,\ j = 1, 2, 3$.

The non-vanishing components of Riemann curvature tensor and Ricci tensor are

$$\begin{cases} R(E_1, E_2)E_1 = -E_2 = R(E_2, E_3)E_3,\ R(E_j, E_3)E_j = E_3, \text{ for } j = 1, 2, \\ R(E_1, E_3)E_3 = -E_1,\ R(E_1, E_2)E_2 = E_1,\ S(E_3, E_3) = -2. \end{cases} \tag{3.6}$$

The scalar curvature r of the Riemannian Sol_3 Lie group is

$$r = \sum_{i=1}^{3} g(E_i, E_i)S(E_i, E_i) = -2. \tag{3.7}$$

Let

$$V = f_1E_1 + f_2E_2 + f_3E_3, \tag{3.8}$$

be an arbitrary potential vector field on (Sol_3, g), where f_1, f_2 and f_3 are smooth functions of x, y and z. We denote the coordinate basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ by $\{\partial_x, \partial_y, \partial_z\}$.

Now, we have

Theorem 3.1. *The Sol_3 space with a left-invariant Riemannian metric given by (3.2) satisfies a Yamabe soliton equation*

$$\mathcal{L}_V g = 2(-2 - r)g,$$

with

$$V = (\alpha_1 - \delta_4 x)\partial_x + (\alpha_2 + \delta_4 y)\partial_y + \delta_4\partial_z,$$

where α_1, α_2, $\delta_4 \in \mathbb{R}$. Moreover, the left-invariant Riemannian metric (3.2) is an expanding Yamabe soliton.

Proof. In view of (3.2) and (3.8), we have

$$\begin{cases} (\mathcal{L}_V g)(E_1, E_1) = 2(f_3 + e^{-z}\partial_x f_1),\ (\mathcal{L}_V g)(E_1, E_2) = e^{-z}\partial_x f_2 + e^z \partial_y f_1, \\ (\mathcal{L}_V g)(E_1, E_3) = e^{-z}\partial_x f_3 - f_1 + \partial_z f_1,\ (\mathcal{L}_V g)(E_2, E_2) = -2(f_3 - e^z\partial_y f_2), \\ (\mathcal{L}_V g)(E_2, E_3) = e^z \partial_y f_3 + f_2 + \partial_z f_2,\ (\mathcal{L}_V g)(E_3, E_3) = 2\partial_z f_3. \end{cases} \tag{3.9}$$

Thus, by using (3.2), (3.7), and (3.9) in (1.1), we find that (Sol_3, g) is a Yamabe soliton if and only if the following system of equations holds:

$$f_3 + e^{-z}\partial_x f_1 = \nu + 2, \tag{3.10}$$

$$e^{-z}\partial_x f_2 + e^z \partial_y f_1 = 0, \tag{3.11}$$

$$e^{-z}\partial_x f_3 - f_1 + \partial_z f_1 = 0, \tag{3.12}$$

$$-f_3 + e^z \partial_y f_2 = \nu + 2, \tag{3.13}$$

$$e^z \partial_y f_3 + f_2 + \partial_z f_2 = 0, \tag{3.14}$$

$$\partial_z f_3 = \nu + 2. \tag{3.15}$$

From (3.15), we get

$$f_3 = (\nu + 2)z + F(x, y), \tag{3.16}$$

where $F = F(x, y)$ is a real-valued smooth function on $\mathbb{R}^2$.

From (3.10), we obtain

$$\partial_x f_1 = e^z\big((\nu + 2) - (\nu + 2)z - F\big). \tag{3.17}$$

Differentiating (3.17) with respect to z, we get

$$\partial_z \partial_x f_1 = -e^z\big((\nu + 2)z + F\big). \tag{3.18}$$

Differentiating (3.12) with respect to x, and using (3.16), (3.17), and (3.18) therein we find $\nu = -2$ and

$$\partial_x^2 F = 0. \tag{3.19}$$

Further, (3.13) gives

$$\partial_y f_2 = e^{-z}\big((\nu + 2) + (\nu + 2)z + F\big). \tag{3.20}$$

Differentiating (3.20) with respect to z, we obtain

$$\partial_z \partial_y f_2 = -e^{-z}\big((\nu + 2)z + F\big). \tag{3.21}$$

Next, differentiating (3.14) with respect to y and therein using (3.16), (3.20), and (3.21), we find $\nu = -2$ and

$$\partial_y^2 F = 0. \tag{3.22}$$

From (3.16), (3.19), and (3.22), we derive that

$$f_3 = F(x, y) = \delta_1 x + \delta_2 y + \delta_3 xy + \delta_4, \tag{3.23}$$

where $\delta_i \in \mathbb{R}$.

From (3.10) and (3.23), we get

$$f_1 = -e^z\Big(\frac{\delta_1 x^2}{2} + \delta_2 xy + \frac{\delta_3 yx^2}{2} + \delta_4 x\Big) + T(y, z), \tag{3.24}$$

where T is a smooth function.

From (3.13) and (3.23), we obtain

$$f_2 = e^{-z}\Big(\delta_1 xy + \frac{\delta_2 y^2}{2} + \frac{\delta_3 xy^2}{2} + \delta_4 y\Big) + I(x, z), \tag{3.25}$$

where I is a smooth function.

Using (3.24) and (3.25) in (3.11), we get

$$\Big(e^z \partial_y T + e^{-2z}\Big(\delta_1 y + \frac{\delta_3 y^2}{2}\Big)\Big) + \Big(e^{-z}\partial_x I - e^{2z}\Big(\delta_2 x + \frac{\delta_3 x^2}{2}\Big)\Big) = 0. \tag{3.26}$$

Since (3.26) holds for all values of z, therefore, it implies that

$$e^z \partial_y T + e^{-2z}\Big(\delta_1 y + \frac{\delta_3 y^2}{2}\Big) = 0, \quad e^{-z}\partial_x I - e^{2z}\Big(\delta_2 x + \frac{\delta_3 x^2}{2}\Big) = 0. \tag{3.27}$$

By integration (3.27) gives

$$T = -e^{-3z}\big(\frac{\delta_1 y^2}{2} + \frac{\delta_3 y^3}{6}\big) + \bar{T}(z), \quad I = e^{3z}\big(\frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}\big) + \bar{I}(z), \tag{3.28}$$

where $\bar{T}$ and $\bar{I}$ are smooth functions. So,

$$\begin{cases} f_1 = -e^z\Big(\frac{\delta_1 x^2}{2} + \delta_2 xy + \frac{\delta_3 yx^2}{2} + \delta_4 x\Big) - e^{-3z}\Big(\frac{\delta_1 y^2}{2} + \frac{\delta_3 y^3}{6}\Big) + \bar{T}(z), \\ f_2 = e^{-z}\Big(\delta_1 xy + \frac{\delta_2 y^2}{2} + \frac{\delta_3 xy^2}{2} + \delta_4 y\Big) + e^{3z}\big(\frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}\big) + \bar{I}(z). \end{cases} \tag{3.29}$$

Now putting the values of f_1 and f_3 in (3.12), we obtain

$$e^{-z}\big(\delta_1 + \delta_3 y\big) + 4e^{-3z}\Big(\frac{\delta_1 y^2}{2} + \frac{\delta_3 y^3}{6}\Big) - \bar{T}(z) + \bar{T}'(z) = 0. \tag{3.30}$$

Since (3.30) holds for all z, therefore, it gives that $\delta_1 = \delta_3 = 0$, and

$$\bar{T} = \alpha_1 e^z,$$

where $\alpha_1 \in \mathbb{R}$.

Putting the values of f_2 and f_3 in (3.14), we find

$$e^z\big(\delta_2+\delta_3 x\big)+4e^{3z}\Big(\frac{\delta_2 x^2}{2}+\frac{\delta_3 x^3}{6}\Big)+\bar{I}(z)+\bar{I}'(z)=0. \tag{3.31}$$

Since (3.31) holds for all z, therefore, it implies that $\delta_2=\delta_3=0$, and

$$\bar{I}=\alpha_2 e^{-z},$$

where $\alpha_2\in\mathbb{R}$.

Hence, the solution of the system of equations (3.10)∼(3.15) is given by

$$f_1=(\alpha_1-\delta_4 x)e^z,\quad f_2=(\alpha_2+\delta_4 y)e^{-z},\quad f_3=\delta_4, \tag{3.32}$$

where α_1, α_2, $\delta_4\in\mathbb{R}$.

Hence, the Riemannian three-dimensional Lie group Sol_3 admits an expanding Yamabe soliton for appropriate vector fields given by (3.32). □

Theorem 3.2. *The Sol_3 space with a left-invariant Riemannian metric given by (3.2) satisfies a gradient Yamabe soliton equation*

$$\mathcal{L}_{\operatorname{grad}F}\,g=2(-2-r)g,$$

where the potential function F is constant.

Proof. Let $V=\operatorname{grad}F$ be an arbitrary gradient vector field on (Sol_3,g) with potential function F. Then V is given by

$$\operatorname{grad}F=e^{-2z}\partial_x F\,\partial_x+e^{2z}\,\partial_y F\,\partial_y+\partial_z F\,\partial_z.$$

From (3.32), we see that (Sol_3,g) is a gradient Yamabe soliton if and only if the potential function F satisfies the following systems:

$$\partial_x F = e^{2z}(\alpha_1-\delta_4 x), \tag{3.33}$$

$$\partial_y F = e^{-2z}(\alpha_2+\delta_4 y), \tag{3.34}$$

$$\partial_z F = \delta_4. \tag{3.35}$$

Differentiating (3.33) with respect to z and (3.35) with respect to x, and equating them we obtain

$$2e^{2z}(\alpha_1-\delta_4 x)=0, \tag{3.36}$$

which gives $\alpha_1 = 0$ and $\delta_4 = 0$. Further, taking the derivative of (3.34) with respect to z and (3.35) with respect to y and equating them we get

$$2e^{-2z}(\alpha_2 + \delta_4 y) = 0, \tag{3.37}$$

which gives $\alpha_2 = 0$ and $\delta_4 = 0$. So, $F =$ constant. Hence the result. □

4. Yamabe and gradient Yamabe solitons on Sol_3 space with Lorentzian metric

In this section, we examine the existence of Yamabe and gradient Yamabe solitons on a three-dimensional solvable Lie group with the Lorentzian metric.

We consider Sol_3 space with a left-invariant Lorentzian metric

$$g = e^{2z}dx^2 - e^{-2z}dy^2 + dz^2, \tag{4.38}$$

with a left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}, \tag{4.39}$$

where $(x, y, z) \in \mathbb{R}^3$.

The non-vanishing Lie brackets are

$$[E_1, E_3] = E_1,\ [E_2, E_3] = -E_2. \tag{4.40}$$

Using (4.38), (4.39), and (4.40) the Levi-Civita connection is given by

$$(\nabla_{E_i} E_j) = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & -E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.41}$$

where $i,\ j = 1, 2, 3$.

The non-vanishing components of Riemann curvature tensor and Ricci tensor are

$$\begin{cases} R(E_1, E_j)E_j = -E_1, \text{for } j = 2, 3,\ R(E_1, E_2)E_1 = -E_2 = R(E_2, E_3)E_3, \\ R(E_1, E_3)E_1 = E_3 = -R(E_2, E_3)E_2,\ S(E_3, E_3) = -2. \end{cases} \tag{4.42}$$

The scalar curvature r of the Lorentzian Sol_3 Lie group is

$$r = \sum_{i=1}^{3} g(E_i, E_i)S(E_i, E_i) = -2. \tag{4.43}$$

Now, we have

Theorem 4.1. *The Sol_3 space with a left-invariant Lorentzian metric given by (4.38) satisfies a Yamabe soliton equation*

$$\mathcal{L}_V g = 2(-2-r)g,$$

with

$$V = (\beta_1 - \gamma_4 x)\partial_x + (\beta_2 + \gamma_4 y)\partial_y + \gamma_4 \partial_z,$$

where β_1, β_2, $\gamma_4 \in \mathbb{R}$. Moreover, the left-invariant Lorentzian metric (4.38) is an expanding Yamabe soliton.

Proof. In view of (3.8) and (4.38), we have

$$\begin{cases} (\mathcal{L}_V g)(E_1, E_1) = 2(f_3 + e^{-z}\partial_x f_1), \ (\mathcal{L}_V g)(E_1, E_2) = -e^{-z}\partial_x f_2 + e^{z}\partial_y f_1, \\ (\mathcal{L}_V g)(E_1, E_3) = e^{-z}\partial_x f_3 - f_1 + \partial_z f_1, \ (\mathcal{L}_V g)(E_2, E_2) = 2(f_3 - e^{z}\partial_y f_2), \\ (\mathcal{L}_V g)(E_2, E_3) = e^{z}\partial_y f_3 - f_2 - \partial_z f_2, \ (\mathcal{L}_V g)(E_3, E_3) = 2\partial_z f_3. \end{cases} \tag{4.44}$$

Thus, by using (4.38), (4.43), and (4.44) in (1.1), we find that (Sol_3, g) is a Yamabe soliton if and only if the following system of equations holds,

$$f_3 + e^{-z}\partial_x f_1 = \nu + 2, \tag{4.45}$$

$$-e^{-z}\partial_x f_2 + e^{z}\partial_y f_1 = 0, \tag{4.46}$$

$$e^{-z}\partial_x f_3 - f_1 + \partial_z f_1 = 0, \tag{4.47}$$

$$f_3 - e^{z}\partial_y f_2 = -\nu - 2, \tag{4.48}$$

$$e^{z}\partial_y f_3 - f_2 - \partial_z f_2 = 0, \tag{4.49}$$

$$\partial_z f_3 = \nu + 2. \tag{4.50}$$

From (4.50), we get

$$f_3 = (\nu + 2)z + H(x, y), \tag{4.51}$$

where $H = H(x, y)$ is a real-valued smooth function on $\mathbb{R}^2$.

Using (4.51) in (4.48), we obtain

$$\partial_y f_2 = e^{-z}\big((\nu + 2) + (\nu + 2)z + H\big). \tag{4.52}$$

Differentiating (4.52) with respect to z, we get

$$\partial_z \partial_y f_2 = -e^{-z}\big((\nu + 2)z + H\big). \tag{4.53}$$

Further, diferentiating (4.49) with respect to y and using (4.51)∼(4.53), we find that $\nu = -2$ and

$$\partial_y^2 H = 0. \tag{4.54}$$

On the other hand, using (4.51) in (4.45), we obtain

$$\partial_x f_1 = e^z\big((\nu+2) - (\nu+2)z - H\big). \tag{4.55}$$

Differentiating (4.55) with respect to z, we get

$$\partial_z\partial_x f_1 = -e^z\big((\nu+2)z + H\big). \tag{4.56}$$

Further, differentiating (4.47) with respect to x and using (4.51), (4.55), and (4.56), we find that $\nu = -2$ and

$$\partial_x^2 H = 0. \tag{4.57}$$

From (4.51), (4.54), and (4.57), we derive that

$$f_3 = H(x,y) = \gamma_1 x + \gamma_2 y + \gamma_3 xy + \gamma_4, \tag{4.58}$$

where $\gamma_i \in \mathbb{R}$.

From (4.45) and (4.58), we get

$$f_1 = -e^z\Big(\frac{\gamma_1 x^2}{2} + \gamma_2 xy + \frac{\gamma_3 yx^2}{2} + \gamma_4 x\Big) + K(y,z), \tag{4.59}$$

where K is a smooth function.

From (4.48) and (4.58), we obtain

$$f_2 = e^{-z}\Big(\gamma_1 xy + \frac{\gamma_2 y^2}{2} + \frac{\gamma_3 xy^2}{2} + \gamma_4 y\Big) + L(x,z), \tag{4.60}$$

where L is a smooth function.

Using (4.59) and (4.60) in (4.46), we get

$$\Big(e^z\partial_y K - e^{-2z}\Big(\gamma_1 y + \frac{\gamma_3 y^2}{2}\Big)\Big) - \Big(e^{-z}\partial_x L + e^{2z}\Big(\gamma_2 x + \frac{\gamma_3 x^2}{2}\Big)\Big) = 0. \tag{4.61}$$

Since (4.61) holds for all values of z, therefore, it implies that

$$e^z\partial_y K - e^{-2z}\big(\gamma_1 y + \frac{\gamma_3 y^2}{2}\big) = 0, \quad e^{-z}\partial_x L + e^{2z}\big(\gamma_2 x + \frac{\gamma_3 x^2}{2}\big) = 0. \tag{4.62}$$

By integration (4.62) gives

$$K = e^{-3z}\big(\frac{\gamma_1 y^2}{2} + \frac{\gamma_3 y^3}{6}\big) + \bar{K}(z), \quad L = -e^{3z}\big(\frac{\gamma_2 x^2}{2} + \frac{\gamma_3 x^3}{6}\big) + \bar{L}(z), \tag{4.63}$$

where $\bar{K}$ and $\bar{L}$ are smooth functions. So, from (4.59) and (4.60), we get

$$\begin{cases} f_1 = -e^z\Big(\frac{\gamma_1 x^2}{2} + \gamma_2 xy + \frac{\gamma_3 yx^2}{2} + \gamma_4 x\Big) + e^{-3z}\Big(\frac{\gamma_1 y^2}{2} + \frac{\gamma_3 y^3}{6}\Big) + \bar{K}(z), \\ f_2 = e^{-z}\Big(\gamma_1 xy + \frac{\gamma_2 y^2}{2} + \frac{\gamma_3 xy^2}{2} + \gamma_4 y\Big) - e^{3z}\big(\frac{\gamma_2 x^2}{2} + \frac{\gamma_3 x^3}{6}\big) + \bar{L}(z). \end{cases} \tag{4.64}$$

Now, putting the values of f_1 and f_3 in (4.47), we obtain

$$e^{-z}\big(\gamma_1 + \gamma_3 y\big) - 4e^{-3z}\Big(\frac{\gamma_1 y^2}{2} + \frac{\gamma_3 y^3}{6}\Big) - \bar{K}(z) + \bar{K}'(z) = 0. \tag{4.65}$$

Since (4.65) holds for all z, therefore, it gives that $\gamma_1 = \gamma_3 = 0$, and

$$\bar{K} = \beta_1 e^z,$$

where $\beta_1 \in \mathbb{R}$.

Putting the values of f_2 and f_3 in (4.49), we find

$$e^{z}\big(\gamma_2 + \gamma_3 x\big) + 4e^{3z}\Big(\frac{\gamma_2 x^2}{2} + \frac{\gamma_3 x^3}{6}\Big) - \bar{L}(z) - \bar{L}'(z) = 0. \tag{4.66}$$

Since (4.66) holds for all z, therefore, it implies that $\gamma_2 = \gamma_3 = 0$, and

$$\bar{L} = \beta_2 e^{-z},$$

where $\beta_2 \in \mathbb{R}$.

Hence, the solution of the system of equations (4.45)∼(4.50) is given by

$$f_1 = (\beta_1 - \gamma_4 x)e^z, \quad f_2 = (\beta_2 + \gamma_4 y)e^{-z}, \quad f_3 = \gamma_4, \tag{4.67}$$

where β_1, β_2, $\gamma_4 \in \mathbb{R}$.

Hence Lorentzian three-dimensional Lie group Sol_3 admits an expanding Yamabe soliton for appropriate vector fields given by (4.67). □

Theorem 4.2. *The Sol_3 space with a left-invariant Lorentzian metric given by (4.38) satisfies a gradient Yamabe soliton equation*

$$\mathcal{L}_{\operatorname{grad} F}\, g = 2(-2 - r)g,$$

where the potential function F is constant.

Proof. Let $V = \operatorname{grad} F$ be an arbitrary gradient vector field on (Sol_3, g) with potential function F. Then V is given by

$$\operatorname{grad} F = e^{-2z}\partial_x F\,\partial_x + e^{2z}\,\partial_y F\,\partial_y + \partial_z F\,\partial_z.$$

From (4.67), we see that (Sol_3, g) is a gradient Yamabe soliton if and only if the potential function F satisfies the following systems:

$$\partial_x F = e^{2z}(\beta_1 - \gamma_4 x), \tag{4.68}$$

$$\partial_y F = e^{-2z}(\beta_2 + \gamma_4 y), \tag{4.69}$$

$$\partial_z F = \gamma_4. \tag{4.70}$$

Differentiating (4.68) with respect to z and (4.70) with respect to x, and equating them we obtain

$$2e^{2z}(\beta_1 - \gamma_4 x) = 0, \tag{4.71}$$

which gives $\beta_1 = 0$ and $\gamma_4 = 0$. Further, taking the derivative of (4.69) with respect to z and (4.70) with respect to y and equating them we get

$$2e^{-2z}(\beta_2 + \gamma_4 y) = 0, \tag{4.72}$$

which gives $\beta_2 = 0$ and $\gamma_4 = 0$. So, $F =$ constant. Hence the result. □

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GENERALIZED SOLITONIC CHARACTERISTICS IN TRANS PARA SASAKIAN MANIFOLDS

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In the current research, we quantify the almost generalized Ricci soliton (AGRS) on the trans-para-Sasakian manifold (TPS-manifold) as well as the gradient almost generalized Ricci soliton (GAGRS). Trans-para Sasakian manifolds that meet certain criteria are also required to be Einstein manifolds. It is demonstrated that the almost generalized Ricci soliton equation is also satisfied by some manifolds, notably β-para-Kenmotsu manifolds, α-para-Sasakian. The fact that a compact trans-para-Sasakian admits both a convex Einstein potential with non-negative scalar curvature and a gradient almost generalized Ricci soliton with Hodge-de Rham potential has also been covered. Finally, we furnished an example which illustrates our finding.

Keywords: Almost generalized Ricci solitons, gradient almost generalized Ricci soliton, Trans-para Sasakian manifold, Einstein manifold.

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1. Introduction

The most important geometrical tool to explain the geometric structures in Riemannian geometry (semi-Riemannian) over the last two decades has been the theory of geometric flows. Since they arise as potential models of discontinuities, the study of discontinuities

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(singularities) of the flows involves a special class of solutions where the metric changes via dilations and diffeomorphisms. They are often called soliton solutions. In 1982, R. S. Hamilton [10] developed the idea of Ricci flow such that

$$\frac{\partial g}{\partial t} = -2S_{ric}(g). \tag{1.1}$$

On a Riemannian manifold (M, g), a Ricci soliton struture (g, V, λ) can be expressed by

$$S_{ric} + \frac{1}{2}\mathfrak{L}_\theta g + \Lambda g = 0, \tag{1.2}$$

here $\mathfrak{L}_\theta$ is the Lie derivative along the vector field θ, Λ is a scalar, and S_{ric} is the Ricci tensor. Ricci soliton is defined as $\Lambda < 0, \Lambda = 0$, and $\Lambda > 0$, respectively. It can also be described as expanding, stable, or shrinking.

Equation (1.2) takes on the form of a gradient Ricci soliton if the vector field $\theta = grad(\psi)$, where ψ is potential function on manifold.

$$Hess\psi = S_{ric} + \Lambda g. \tag{1.3}$$

Pigola et al. [21] argue that if we consider $\Lambda \in C^\infty(M)$, sometimes referred to as a soliton function, so we could assert that (M, g) is *almost generalized Ricci solitons* (AGRS).

Plenty of mathematicians are drawn to this idea. Therefore, how self-similar solutions are categorized to Ricci flows has received a lot of attention in recent years.This problem has significant practical implications in fields such as thermodynamics, control theory, optics, mechanics, phase space of dynamical systems, and many other departments of pure mathematics.

Ricci solitons are significant because they are both logical generalizations of Einstein metrics. A few generalizations, for example quasi-Einstein manifolds [4], generalized quasi-Einstein manifolds [5] and gradient Ricci solitons [3], are crucial in the solutions of some manifolds have their local structure derived from Ricci flows.

Overarching in reference [19], Nurowski and Randall initially defined Ricci soliton as a kind of over determined framework for equations.

$$\frac{1}{2}\mathfrak{L}_\theta g - bS_{ric} - \Lambda g + a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp = 0, \tag{1.4}$$

where $\mathcal{U}^\sharp$ denotes the canonical 1-from and a, b are real constants .

If $\mathcal{U} = \nabla\psi$, where $\psi \in C^\infty(M)$, (M, g) is referred to as a gradient almost generalized Ricci soliton (GAGRS) in that case. As a result, (1.4) becomes

$$\nabla^2\psi - 2bS_{ric} - 2\Lambda g + 2a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp = 0. \tag{1.5}$$

However, Kaneyuki and Konzai started researching an almost para-contact structure on semi-Riemannian manifolds [12]. Zamkovoy has done extensive research on para contact metric manifolds [35]. Furthermore, trans-para-Sasakian manifold geometry was given by Zamkovoy in 2019 [37]. Siddiqi also has investigated lightlike hypersurfaces [27] and null hypersurfaces of trans-para-Sasakian manifold [26].

Structures that are an almost contact manifold M are known as trans-Sasakian structures [20], if $M \times \mathbb{R}$, the product manifolds, are members of class W_4 [9]. Marrero and Chinea are fully characterized trans-Sasakian structures of type (α, β) in [16].

The trans-para-Sasakian manifolds are seen by Zamkovoy in [37] as an analogy of the trans-Sasakian manifolds. A trans-para-Sasakian structure of type (α, β), where α and β are smooth functions, is called a trans-para-Sasakian manifold [28]. The manifolds of type (α, β) that are trans-para-Sasakian are the para-Sasakian manifolds in the case of $\alpha = 1$, the para-Kenmostu manifolds in the case of $\beta = 1$ [37], and the para-cosympletic manifolds $(\alpha = \beta = 0)$ [13].

During last two decades, many geometers exclusively studied the Ricci solitons and an extension [24] of Ricci solitons namely, η-Ricci solitons on different manifolds such as Riemannian manifold [22], Kenmotsu manifold [18], K-contact manifolds and (k, μ)-contact manifolds [29] and trans-Sasakian manifolds [31]. Following Siddiqi [25], who also discussed generalized Ricci soliton. Mekki and Cherif studied another generic concept known as generalized Ricci soliton on Sasakian manifolds [17]. In this research note, we studied the almost generalized Ricci soliton and almost gradient generalized Ricci soliton in trans-para-Sasakian manifolds as a result of the aforementioned sources and comments.

2. Preliminaries

If a $(2n+1)$-dimensional smooth manifold Θ admits a vector field ζ, a 1-form γ, and a tensor field Φ of type $(1, 1)$, and a pseudo-Riemannian metric g then it has an almost paracontact structure (Φ, ζ, γ, g) such that [2]

$$\Phi^2 p = p - \gamma(p)\zeta, \quad \Phi(\zeta) = 0, \quad \gamma \circ \Phi = 0, \quad \gamma(\zeta) = 1. \tag{2.6}$$

The definition of almost paracontact structure immediately leads to the rank $2n$ of the endomorphism Φ.

$$g(\Phi p, \Phi q) = -g(p, q) + \gamma(p)\gamma(q), \tag{2.7}$$

then g is said to be compatible with signature $(n+1, n)$ and Θ has an almost paracontact metric structure.

Observe that when $q = \zeta$ is set, $\gamma(p) = g(p, \zeta)$. Moreover, a compatible metric admits any almost paracontact structure. If

$$g(p, \Phi q) = d\gamma(p, q),$$

where $d\gamma(p, q) = \frac{1}{2}(p\gamma(q) - q\gamma(p) - \gamma([p, q]))$, then γ is a paracontact form and the almost paracontact metric manifold $(\Theta, \Phi, \gamma, \zeta, g)$ is defined as a paracontact metric manifold.

An almost paracomplex structure on the product $\Theta^{(2n+1)} \times \mathbb{R}$ easily arises from a paracontact structure on a $\Theta^{(2n+1)}$. The provided paracontact metric manifold is called para-Sasakian if this almost paracomplex structure is integrable. Comparably, a paracontact metric manifold is a para-Sasakian if and only if (see [36]).

$$(\nabla_p \Phi)q = -g(p, q)\zeta + \gamma(q)p, \tag{2.8}$$

the manifold $(\Theta, \Phi, \zeta, \gamma, g)$ of dimension $(2n+1)$ is said to be trans-para-Sasakian manifolds (TPS-manifolds) if and only if

$$(\nabla_p \Phi)Y = \alpha(-g(p, q)\zeta + \gamma(q)p) + \beta(g(p, \Phi q)\zeta + \gamma(q)\Phi p), \tag{2.9}$$

from (2.9), we also have

$$\nabla_p \zeta = -\alpha \Phi p - \beta(p - \gamma(p)\zeta). \tag{2.10}$$

The gradient of a smooth function ψ on Θ is defined as follows

$$g(grad\psi, p) = p(\psi). \tag{2.11}$$

The definition of ψ's *Hessian* is

$$(Hess\Psi)(p, q) = g(\nabla_p grad\Psi, q), \tag{2.12}$$

where $p, q \in \Gamma(T\Theta)$.

We defined $p \in \Gamma(T\Theta)$. $\mathcal{U}^\sharp \in \Gamma(\bar{T}\Theta)$ by

$$\mathcal{U}^\sharp(q) = g(p, q). \tag{2.13}$$

The AGRS equation in Riemannian manifold Θ is given by [19]

$$\mathfrak{L}_\theta g = -2a\mathcal{U}^\sharp \odot \mathcal{U}^\sharp + 2bS_{ric} + 2\Lambda g, \tag{2.14}$$

where $p \in \Gamma(T\Theta)$ and the Lie-derivative is defined as

$$(\mathfrak{L}_\theta g)(q, t) = g(\nabla_q \theta, t) + g(\nabla_t \theta, q) \tag{2.15}$$

where $q, t \in \Gamma(T\Theta)$. Equation (1.4), furthermore, is refers to an expansion of

(1) If $a = b = \Lambda = 0$, then Killing's equation.

(2) If $a = b = 0$, then equation for homotheties.

(3) If $a = 0, b = -1$,then Ricci soliton.

(4) If $a = 1, b = \frac{-1}{n-2}$, then Einstein-Weyl geometry.

(5) If $a = 1, b = \frac{-1}{n-2}, \lambda = 0$, then we have metric projective structures with skew-symmetric Ricci tensor in projective class.

(6) If $a = 1, b = \frac{1}{2}$, then we have Vacuum near-horzion geometry equation (for more details see [7], [8], [11], [14]).

A generalization of Einstein manifolds [5] is given by equation (1.4). Observe that the gradient AGRS equation is provided by: if $p = grad\psi$, where $\psi, \Lambda \in C^\infty(\Theta)$

$$Hess\psi + adf \odot df = bS_{ric} + \Lambda g. \tag{2.16}$$

3. Gradient almost generalized Ricci soliton on trans para Sasakian manifolds

The following relations hold in a $(2n+1)$-dimensional TPS manifold Θ [37]:

$$\Re(p, q)\zeta = -(\alpha^2 + \beta^2)[\gamma(q)p - \gamma(p)q] - 2\alpha\beta[\gamma(q)\Phi p - \gamma(p)\Phi q] \tag{3.17}$$
$$+[(q\alpha)\Phi p - (p\alpha)\Phi q + (q\beta)\Phi^2 p - (p\beta)\Phi^2 q].$$

$$S_{ric}(p, \zeta) = [(-2n(\alpha^2 + \beta^2) - (\zeta\beta)]\gamma(p) + ((\Phi p)\alpha) + (n-2)(p\beta), \tag{3.18}$$

$$Q\zeta = -2n(\alpha^2 + \beta^2) - (\zeta\beta))\zeta + \Phi(grad\alpha) - (n-2)(grad\beta), \tag{3.19}$$

where Q is the Ricci operator provided by $S_{ric}(p,q) = g(Qp,q)$, and $\Re$ is the curvature tensor.

Furthermore, we have a TPS manifold

$$\Phi(grad\alpha) = -(2n-1)(grad\beta), \tag{3.20}$$

$$2\alpha\beta - (\zeta\alpha) = 0. \tag{3.21}$$

Lemma [15] follows from combining (3.17) and (3.21) for constants α and β.

Lemma 3.1. [15] *Let $(\Theta^{(2n+1)}, \Phi, \gamma, \zeta, g)$ be a TPS-manifold. Then we have*

$$\Re(p,q)\zeta = -(\alpha^2+\beta^2)[\gamma(q)p - \gamma(p)q], \tag{3.22}$$

$$\Re(\zeta,q)t = -(\alpha^2+\beta^2)[g(q,t)\zeta - \gamma(t)q], \tag{3.23}$$

$$S_{ric}(p,\zeta) = -2n(\alpha^2+\beta^2)\gamma(p), \tag{3.24}$$

$$(\nabla_p\gamma)q = \alpha g(p,\Phi q) - \beta(g(p,q) - \gamma(p)\gamma(q)), \tag{3.25}$$

$$Q\zeta = -[2n(\alpha^2+\beta^2)]\zeta, \tag{3.26}$$

where for all $p,q,t \in T(\Theta)$.

Example 3.1. *Let (x,y,z) be the Cartesian coordinates in $\mathbb{R}^3$. Assume a 3-dimensional manifold $\Theta = \{(x,y,z) \in \mathbb{R}^3 | z \neq 0\}$. Let the linearly independent vector fields $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are linearly independent at each point of Θ defined as*

$$\mathcal{E}_1 = e^z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad \mathcal{E}_2 = e^z\frac{\partial}{\partial y}, \quad \mathcal{E}_3 = \frac{\partial}{\partial z}.$$

Let g be the pseudo-Riemannian metric defined by

$$g(\mathcal{E}_1,\mathcal{E}_1) = -g(\mathcal{E}_2,\mathcal{E}_2) = g(\mathcal{E}_3,\mathcal{E}_3) = 1, \quad g(\mathcal{E}_1,\mathcal{E}_2) = g(\mathcal{E}_2,\mathcal{E}_3) = g(\mathcal{E}_3,\mathcal{E}_1) = 0.$$

Moreover, the 1-form γ is given by $\zeta = \mathcal{E}_3$ and $\gamma(p) = g(p,\mathcal{E}_3)$. Let Φ be the (1,1) tensor field defined by

$$\Phi(\mathcal{E}_1) = \mathcal{E}_2, \ \Phi(\mathcal{E}_2) = \mathcal{E}_1, \ \Phi(\mathcal{E}_3) = 0,$$

for any vector field p on Θ. Using the linearity of Φ and g, we then obtain $\gamma(\mathcal{E}_3) = 1$, $\phi^2 p = p - \gamma(p)\zeta$, with $\zeta = \mathcal{E}_3$.

Moreover, for all vector fields p and q on Θ, we have

$$g(\Phi p, \Phi q) = -g(p, q) + \gamma(p)\gamma(q).$$

Therefore, in $\mathbb{R}^3$, the structure (Φ, ζ, γ, g) defines a paracontact structure for $\mathcal{E}_3 = \zeta$ [36]. Let $\Re$ be the curvature tensor of g and ∇ be the Levi-Civita connection with respect to metric g. Next, we have

$$[\mathcal{E}_1, \mathcal{E}_2] = ye^z\mathcal{E}_2 - e^{2z}\mathcal{E}_3[\mathcal{E}_1, \mathcal{E}_3] = -\mathcal{E}_3 \qquad [\mathcal{E}_2, \mathcal{E}_3] = -\mathcal{E}_2.$$

Now, we have Koszul's formula

$$2g(\nabla_p q, t) = pg(q, t) + qg(t, p) - tg(p, q) - g(p, [q, t])$$

$$-g(q, [p, t]) + g(t, [p, q]).$$

Therefore, in light of above formula, we turn up

$$\nabla_{\mathcal{E}_1}\mathcal{E}_1 = \mathcal{E}_3, \quad \nabla_{\mathcal{E}_1}\mathcal{E}_2 = -\frac{1}{2}e^{2z}\mathcal{E}_3, \quad \nabla_{\mathcal{E}_1}\mathcal{E}_3 = -\mathcal{E}_1 - \frac{1}{2}e^{2z}\mathcal{E}_2, \tag{3.27}$$

$$\nabla_{\mathcal{E}_2}\mathcal{E}_2 = -ye^z\mathcal{E}_1 - \mathcal{E}_3, \quad \nabla_{\mathcal{E}_2}\mathcal{E}_1 = -ye^z\mathcal{E}_2 + \frac{1}{2}e^{2z}\mathcal{E}_3,$$

$$\nabla_{\mathcal{E}_3}\mathcal{E}_1 = -\frac{1}{2}e^{2z}\mathcal{E}_2, \quad \nabla_{\mathcal{E}_3}\mathcal{E}_2 = -\frac{1}{2}e^{2z}\mathcal{E}_1, \quad \nabla_{\mathcal{E}_3}\mathcal{E}_3 = 0.$$

The fact that (Φ, ζ, γ, g) is a TPS-structure on Θ is evident from the above. Thus, $\Theta^3(\Phi, \zeta, \gamma, g)$, with $\beta = 1$ and $\alpha = \frac{1}{2}e^{2z} \neq 0$, is a TPS- manifold.

Theorem 3.1. *If $\Theta^{(2n+1)}$ be a TPS-manifolds, and satisfies the $AGRS$ (1.4) with restriction $a[\lambda - 2nb(\alpha^2 + \beta^2)] \neq -1$. Then ψ is a constant function. In addition, if $b \neq 0$, then Θ is an Einstein.*

Lemma 3.1 gives us the following observations:

Corollary 3.1. *If $\Theta^{(2n+1)}$ be a TPS-manifolds, and satisfies the $AGRS$ $Hess\psi + S_{ric} = \Lambda g$, then ψ is a constant function and Θ is an Einstein.*

Corollary 3.2. *In a TPS-manifolds Θ, there is no non-constant smooth function ψ, such that $Hess\psi = \Lambda g$, for some constant Λ.*

We must first show the following lemmas in order to proceed with the proof of the Theorem (3.1).

Lemma 3.2. *Let Θ be a TPS-manifold. Then we have*

$$(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) = -(\alpha^2+\beta^2)g(p,q) + g(\nabla_\zeta\nabla_\zeta p, q) + qg(\nabla_\zeta p, \zeta), \tag{3.28}$$

where $p, q \in \Gamma(T\Theta)$ and q is orthogonal to ζ.

Proof. Based on the Lie-derivative property, we may observe that

$$(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) = \zeta((\mathfrak{L}_p g)(q,\zeta)) - (\mathfrak{L}_p g)(\mathfrak{L}_\zeta q, \zeta) - (\mathfrak{L}_p g)(q, \mathfrak{L}_\zeta \zeta). \tag{3.29}$$

Since $\mathfrak{L}_\zeta q = [\zeta, q]$, $\mathfrak{L}_\zeta \zeta = [\zeta,\zeta]$, by adopting (2.16) and (4.51), we have

$$\begin{aligned}(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) &= \zeta g(\nabla_q p, \zeta) + \zeta g(\nabla_\zeta p, q) - g(\nabla_{[\zeta,q]} p, \zeta) \\ &\quad - g(\nabla_\zeta p, [\zeta, q]) \\ &= g(\nabla_\zeta\nabla_q p, \zeta) + g(\nabla_q p, \nabla_\zeta \zeta) + g(\nabla_\zeta\nabla_\zeta p, q) \\ &\quad + g(\nabla_\zeta p, \nabla_\zeta q) - g(\nabla_\zeta p, \nabla_\zeta q) - g(\nabla_{[\zeta,q]} p, \zeta) + g(\nabla_\zeta p, \nabla_q \zeta).\end{aligned} \tag{3.30}$$

By (1.4), we turn up $\nabla_\zeta \zeta = \Phi\zeta = 0$, therefore we gain

$$\begin{aligned}(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) &= g(\nabla_\zeta\nabla_q p, \zeta) + g(\nabla_\zeta\nabla_\zeta p, q) - g(\nabla_{[\zeta,q]} p, \zeta) \\ &\quad + qg(\nabla_\zeta p, \zeta) - g(\nabla_q\nabla_\zeta p, \zeta).\end{aligned} \tag{3.31}$$

Utilizing (4.51) and (3.29), we turn up

$$(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) = g(\Re(\zeta,q)p, \zeta) + g(\nabla_\zeta\nabla_\zeta p, q) + qg(\nabla_\zeta p, \zeta). \tag{3.32}$$

When $g(q,\zeta) = 0$ is taken from (4.51), we find

$$g(\Re(\zeta,q)p,\zeta) = g(R(q,\zeta)\zeta, p) = (\alpha^2+\beta^2)g(p,q). \tag{3.33}$$

(3.29) and (4.52) provide the Lemma. ☐

We now have another helpful Lemma.

Lemma 3.3. *If Θ be a Riemannian manifold, and let $\psi \in C^\infty(\Theta)$. Then we have*

$$(\mathfrak{L}_\zeta(df \odot df))(q,\zeta) = q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\xi(\psi)), \tag{3.34}$$

where $\zeta, q \in \Gamma(T\Theta)$.

Proof. We compute

$$(\mathfrak{L}_\zeta(df \odot df))(q,\zeta) = \zeta(q(\psi)\zeta(\psi) - [\zeta,q](\psi)\zeta(\psi) - q(\psi)[\zeta,\zeta](\psi)$$

$$= \zeta(q(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi)) - [\zeta,q](\psi)\zeta(\psi).$$

Since $[\zeta,q](\psi) = \zeta(q(\psi)) - q(\zeta(\psi))$, we gain

$$(\mathfrak{L}_\zeta(df \odot df))(q,\zeta) = [\zeta,q](\psi)\zeta(\psi) + q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi)) - [\zeta,q](\psi)\zeta(\psi)$$

$$= q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi)).$$

□

Lemma 3.4. *If Θ^{2n+1} be a TPS-manifold and satisfies the AGRS equation (2.16). Then we have*

$$\nabla_\zeta \; grad\psi = [\Lambda - 2nb(\alpha^2+\beta^2)]\zeta - a\zeta(\psi)grad\psi. \tag{3.35}$$

Proof. Let $q \in \Gamma(T\Theta)$, adopting the definition of Ricci curvature S_{ric} (1.4) , and the curvature restriction (4.51), we gain

$$\begin{aligned} S_{ric}(p,q) &= g(\Re(\zeta,\mathcal{E}_i)\mathcal{E}_i,q) \\ &= g(\Re(\mathcal{E}_i,q)\xi,\mathcal{E}_i) \\ &= -(\alpha^2+\beta^2)[\gamma(q)g(\mathcal{E}_i,\mathcal{E}_i) - \gamma(\mathcal{E}_i)g(p,\mathcal{E}_i) \\ &= (\alpha^2+\beta^2)[(2n+1)\gamma(q) - \gamma(q)] \\ &= -2n(\alpha^2+\beta^2)\gamma(q) \\ &= -2n(\alpha^2+\beta^2)g(\zeta,q), \end{aligned}$$

where $\{\mathcal{E}_1,\mathcal{E}_2,\cdots,\mathcal{E}_i\}$, and $1 \le i \le n$ is an orthonormal frame of Θ, indicates that

$$\Lambda g(\zeta,q) + bS_{ric}(\zeta,q) = \Lambda g(\zeta,q) - 2nb(\alpha^2+\beta^2)g(\zeta,q) \tag{3.36}$$

$$= [\Lambda - 2nb(\alpha^2+\beta^2)]g(\zeta,q).$$

In light of (1.4) and (3.35), we turn up

$$(Hess\psi)(\zeta,q) = -a\zeta(\psi)(q)(\psi) + [\Lambda - 2nb(\alpha^2+\beta^2)]g(\zeta,q) \tag{3.37}$$

$$= -a\zeta(\psi)g(grad\psi,q) + [\Lambda - 2nb(\alpha^2+\beta^2)]g(\zeta,q).$$

Accordingly, Lemma is inferred from both equation (3.35) and *Hessian* Definition (1.5). □

We can now establish Theorem 3.1 with the aid of Lemma 3.2, Lemma 3.3, and Lemma 3.4.

Proof. (Proof of Theorem 3.1) Consider $q \in \Gamma(T\Theta)$, such that $g(\zeta, q) = 0$. Lemma 3.1 gives us that, given $X = grad\ \psi$,

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) + g(\nabla_\zeta \nabla_\zeta\ grad\psi, q) + qg(\nabla_\zeta\ grad\psi, \zeta). \tag{3.38}$$

Using equation (3.37) and Lemma 3.1, we obtain

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) + [\Lambda + b(n-1)(\alpha^2+\beta^2)]g(\nabla_\zeta, q) - ag(\nabla_\zeta(\zeta(\psi)grad\ \psi), q)$$
$$+[\Lambda + b(n-1)(\alpha^2+\beta^2)]qg(\zeta,\zeta) - aq(\zeta(\psi)^2). \tag{3.39}$$

Since $\nabla_\zeta \zeta = 0$ and $g(\zeta,\zeta) = 1$, in view of equation (3.38), we get

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) - a\zeta(\zeta(\psi))q(\psi) - a\zeta(\psi)g(\nabla_\zeta\ grad\psi, q) \tag{3.40}$$
$$-2a\zeta(\psi)q(\zeta(\psi)).$$

Given $g(\xi, Y) = 0$ and Lemma 3.1 and equation (3.39), we have

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) - a\zeta(\zeta(\psi))q(\psi) + a^2\zeta(\psi)^2 q(\psi) \tag{3.41}$$
$$-2a\zeta(\psi)q(\zeta(\psi)).$$

Observe that $\mathfrak{L}_\zeta g = 0$, a Killing vector filed, follows from (1.4) and (1.5). This suggests that $\mathfrak{L}_\zeta S = 0$, which is what the Lie derivative to the GRS equation (2.16) delivers.

$$q(\psi) - a\zeta(\zeta(\psi))q(\psi) + a^2\zeta(\psi)^2 q(\psi) - 2a\zeta(\psi)q(\zeta(\psi)) \tag{3.42}$$
$$= -2aq(\zeta(\psi))\zeta(\psi) - 2aq(\psi)\zeta(\zeta(\psi)),$$

is equivalent to

$$q(\psi)[1 + a\zeta(\zeta(\psi)) + a^2\zeta(\psi)^2] = 0. \tag{3.43}$$

Lemma 3.1 states that we have

$$a\zeta(\zeta(\psi)) = a\zeta g(\zeta, grad\ \psi) \tag{3.44}$$
$$= ag(\zeta, \nabla_\zeta\ grad\psi)$$
$$= a[\Lambda - 2nb(\alpha^2+\beta^2)] - a^2\zeta(\psi)^2.$$

In view of equations (3.42) and (3.43), we gain

$$q(\psi)[\Lambda - 2nb(\alpha^2+\beta^2)] = 0. \tag{3.45}$$

$$[\Lambda - 2nb(\alpha^2+\beta^2)] \neq -1,$$

which indicates that $grad\psi$ is parallel to ζ, and so $q(\psi) = 0$. Since $D = ker\gamma$ is not integrable anywhere, $grad\ \psi = 0$, indicating that ψ is a constant function. □

Now, the following scenarios exist for specific values of α and β:

Case 1.: For $\alpha = 0$ or $(\beta = 1)$, we can state:

Corollary 3.3. *If $\Theta^{(2n+1)}$ be a β-para Kenmotsu (or para Kenmotsu) manifold and satisfies the AGRS (1.5) with condition $a[\Lambda - 2nb\beta^2)] \neq -1$, then ψ is a constant function. In addition, if $b \neq 0$, then $\Theta^{(2n+1)}$ is Einstein .*

Case 2.: For $\beta = 0$, or $(\alpha = 1)$ we can state:

Corollary 3.4. *If $\Theta^{(2n+1)}$ be a α-para Sasakian (or para Sasakian) manifold and satisfies the AGRS (1.5) with condition $a[\Lambda - 2nb\alpha^2)] \neq -1$, then ψ is a constant function. Moreover, if $b \neq 0$, then $\Theta^{(2n+1)}$ is Einstein.*

4. Almost generalized Ricci solitons on compact trans para Sasakian manifolds

de Rham-Hodge's classical theorem states that harmonic forms can express the cohomology of an oriented closed Riemannian manifold. For an orientated compact Riemannian manifold with boundary, the analogous one still holds by imposing certain boundary requirements, including relative and absolute ones. However, these examples come from fully Riemannian manifolds. The following are some helpful definitions.

Definition 4.1. [33] *A C^2-function $\omega : \Theta \longrightarrow \mathbb{R}$ is considered to be harmonic if $\Delta\omega = 0$. The function ω is named subharmonic (resp. superharmonic) if $\Delta \geq 0$ (resp. $\Delta\omega \leq 0$), where Δ is the Laplacian operator in Θ.*

Definition 4.2. [35] *A function $\omega : \Theta \longrightarrow \mathbb{R}$ is called convex if the following inequality holds*

$$\omega \circ \delta(T) \leq (1 - T)\omega \circ \delta(0) + T\omega \circ \delta(1), \qquad \forall T \in [0, 1],$$

for any geodesic $\delta : [0, 1] \longrightarrow \Theta$. Therefore in this case ω is differentiable, then ω is convex if and only if ω satisfies

$$g(\nabla\omega, p) \leq \omega(e^x\nabla\omega) - \omega(x), \qquad \forall p \in T_x\Theta.$$

Let $p \in \chi(\Theta)$ and Θ be compact orientable TPS- manifolds. Then, according to the [1] *Hodge-de Rham* decomposition theorem, p can be stated as

$$p = \nabla \hbar + q, \tag{4.46}$$

where $\hbar \in C^{\infty}(\Theta)$ and $div(q) = 0$. The *Hodge-de Rham* potential is the name given to the function h [22].

Let (g, p, λ) be a compact AGRS on compact TPS-manifold Θ, we turn up

$$div(p) + 2n(2n+1)b(\alpha^2 + \beta^2) = n\Lambda - tr(a\mathcal{U}^{\sharp} \otimes \mathcal{U}^{\sharp}). \tag{4.47}$$

$div(X) = \Delta\hbar$ is implied by the Hodge-de Rham decomposition, so, using equation(4.47), we obtain

$$2n(2n+1)b(\alpha^2 + \beta^2) = n\Lambda - \Delta\hbar - tr(a\mathcal{U}^{\sharp} \otimes \mathcal{U}^{\sharp}). \tag{4.48}$$

Since Θ is $GAGRS$ with potential function, we obtain

$$2n(2n+1)b(\alpha^2 + \beta^2) = n\Lambda - \Delta f - tr(a\mathcal{U}^{\sharp} \otimes \mathcal{U}^{\sharp}). \tag{4.49}$$

Now, on equating (4.48) and (4.49), we turn

$$\Delta(f - \hbar) = 0.$$

Consequently, although Θ is compact, f is a harmonic function in Θ. Thus, for some constant c, $f = \hbar + c$. As so, we possess the following outcome.

Theorem 4.1. *If (g, p, Λ) is a compact $GAGRS$. If TPS- manifold Θ is also a $GAGRS$ with potential function f, then, up to constant, f equals to the Hodge-de Rham potential.*

Theorem 4.2. *Let $(\Theta, \zeta, \gamma, \Phi, g)$ be a complete TPS-manifold satisfying*

$$\frac{1}{2}\mathfrak{L}_p g - bS_{ric} \geq \Lambda g - a\mathcal{U}^{\sharp} \otimes \mathcal{U}^{\sharp}, \tag{4.50}$$

where $\mathcal{U}^{\sharp}$ is a canonical 1-from associated with p, a, b, and Λ are smooth functions, and p is a smooth vector field. If one of the following requirements is fulfilled and $\|p\|$ is bounded, then the TPS-manifold Θ is compact:

(1) $\Lambda \geq 0$ *and* $a > 0$, $c > 0$,

(2) $\Lambda > c > 0$ *and* $a \geq 0$,

for a constant $c > 0$.

Proof. If $\pi \in \Theta$ be a fixed point and $\delta : (0, \infty] \longrightarrow \Theta$ be a geodesic such that $\delta(0) = p$. Then along δ we compute

$$\mathfrak{L}_p g(\delta_1, \delta_1) = 2g(\nabla_{\delta_1} p, \delta_1) = 2\frac{d}{dt}[g(p, \delta_1)]. \tag{4.51}$$

Now, from (4.50) and (4.51), we have

$$-\int_0^T bS_{ric}(\delta_1, \delta_1)dt \geq \int_0^T \Lambda(\delta(t))g(\delta_1, \delta_1)dt - \int_0^T \frac{d}{dt}[g(p, \delta_1)dt - \int_0^t a(\delta(T))(\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp)(\delta_1, \delta_1)dt$$

$$= -\frac{1}{b}\left[\int_0^T \Lambda(\delta(t))dt + g(p_\pi, \delta_1(0)) - g(p_{\delta(T)}, \delta_1(T)) + \int_0^T a(\delta(T))\mathcal{U}^{\sharp^2}(\delta_1)dt\right]$$

$$\geq -\frac{1}{b}\left[\int_0^T \Lambda(\delta(t))dt + g(p_\pi, \delta_1(0)) - \left\|X_{\delta(T)}\right\| + \int_0^T a(\delta(T))\mathcal{U}^{\sharp^2}(\delta_1)dt\right].$$

Cauchy-Schwarz inequality leads to the final inequality. If either of the two conditions (1) or (2) is true, the inequality above suggests that

$$\int_0^\infty bS_{ric}(\delta_1, \delta_1)dt = \infty. \tag{4.52}$$

Hence by Ambrose's Compactness Theorem [1] implies that TPS-manifold Θ is compact.

□

5. Gradient almost generalized Ricci soliton on compact trans para Sasakian manifolds

In this segment, we discuss some results based on gradient almost generalized Ricci solitonon compact trans-para Sasakian manifold $n \geq 2$. Next, we articulate the following.

Theorem 5.1. [32] *IF $(\Theta, \Phi, \gamma, \zeta, g)$ be a compact TPS-manifold with constant scalar curvature and Θ admits a non-trivial conformal vector field p. If $\mathfrak{L}_p S_{ric} = \rho g$ for some $\rho \in C^\infty(\Theta)$, then Θ is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Hence, from Theorem (5.1) we can also state the next theorem:

Theorem 5.2. *Let $(\Theta, \Phi, \zeta, \gamma, g)$ be a compact $GAGRS$ with Einstein potential f. If ∇f is non-trivial conformal vector field, then TPS-manifold Θ is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Proof. Let (Θ, g) be a $GAGRS$. Then from (1.4) we deduce

$$\nabla^2 f - bS_{ric} = \Lambda g - a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp.$$

For each $\psi \in C^\infty(\Theta)$, $\nabla^2 f - \psi g$, if ∇f is a conformal vector field. The equation above now takes the form

$$bS_{ric} = (\psi - \Lambda)g - a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp. \tag{5.53}$$

As a result, S_{ric} is only dependent on Θ points. Schur's lemma thus implies that R is constant. Once more, using $p = \nabla f$, we get

$$a\mathfrak{L}_p S_{ric} = (\psi - \Lambda)\mathfrak{L}_p g - a\mathfrak{L}_p(\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp) \tag{5.54}$$

$$a\mathfrak{L}_p S_{ric} = (\psi - \Lambda)\psi g - a[q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi))]. \tag{5.55}$$

This completes the proof. □

In [32] Yano already proved a following results.

Theorem 5.3. [32] *A compact manifold Θ with constant scalar curvature admits a non-trivial conformal vector field p such that $\mathfrak{L}_p g = 2\psi g, \quad \psi \neq 0$, then*

$$\int_\Theta \psi dV = 0. \tag{5.56}$$

Therefore in light of Theorem 5.3 we can state.

Theorem 5.4. *Let $(\Theta, \Phi, \zeta, \gamma, g)$ be a compact $GAGRS$ with Einstein potential f and $(\alpha^2 + \beta^2) \leq 0$. If ∇f is conformal vector field then TPS manifold Θ is shrinking or steady $GAGRS$.*

Proof. Taking the trace in (5.53)

$$2n(2n+1)b(\alpha^2 + \beta^2) = (2n+1)(\psi - \Lambda) - a\,|\zeta|^2 \tag{5.57}$$

which implies

$$\int_\Theta 2nb(\alpha^2 + \beta^2) + \frac{a}{(2n+1)}\,|\zeta|^2 = \int_\Theta (\psi - \Lambda). \tag{5.58}$$

If p is conformal vector field and the scalar curvature of Θ is constant $2n(2n+1)(\alpha^2 + \beta^2)$, then applying Theorem (5.3) we get

$$2n(2n+1)(\alpha^2 + \beta^2)\int_\Theta \left[b + \frac{a}{2n(2n+1)(\alpha^2 + \beta^2)}\,|\zeta|^2\right] = -(2n+1)\int_\Theta \Lambda. \tag{5.59}$$

Now, if $\Lambda < 0$, then above equation reduced

$$2n(2n+1)(\alpha^2 + \beta^2)\int_\Theta \left[b + \frac{a}{2n(2n+1)(\alpha^2 + \beta^2)}\,|\zeta|^2\right] < 0. \tag{5.60}$$

If M is a compact TPS-manifold, then Theorem (5.2) implies that Θ is isometric to $\mathbb{S}^n$. Because scalar curvature is preserved via isometry so $2n(2n+1)(\alpha^2+\beta^2) > 0$. Hence the above equation entails that

$$Vol(M) < \frac{1}{2n(2n+1)} \int_\Theta \left[2n(2n+1)b + \frac{a}{(\alpha^2+\beta^2)} |\zeta|^2 \right]. \tag{5.61}$$

□

Lemma 5.1. [5] *If $(\Theta, \Phi, \zeta, \gamma, g)$ be a GAGRS with Einstein potential f. Then we have*

$$\Delta f = 2n(2n+1)b(\alpha^2+\beta)^2) + (2n+1)\Lambda - a\,|\zeta|^2. \tag{5.62}$$

Currently, function f convexity suggests that it is harmonic, or that $\Delta f = 0$, [32]. Therefore, (5.62) implies

$$2n(2n+1)b(\alpha^2+\beta)^2) + (2n+1)\Lambda - a\,|\zeta|^2 = 0. \tag{5.63}$$

$$\Lambda = \frac{a\,|\zeta|^2}{(2n+1)} - 2nb(\alpha^2+\beta^2). \tag{5.64}$$

Therefore, this leads the following result:

Theorem 5.5. *If f is a convex harmonic function on TPS-manifold $(\Theta, \Phi, \zeta, \gamma, g)$ and has non negative scalar curvature, then admitting a GAGRS with Einstein potential f is expanding, stable, or shrinking according as*

(1) $\frac{a|\zeta|^2}{(2n+1)} > 2nb(\alpha^2+\beta^2)$,

(2) $\frac{a|\zeta|^2}{(2n+1)} = 2nb(\alpha^2+\beta^2)$ *and*

(3) $\frac{a|\zeta|^2}{(2n+1)} < 2nb(\alpha^2+\beta^2)$, *respectively.*

Moreover, Lemma 5.1 entails the following:

Corollary 5.1. *If $(\Theta, \Phi, \zeta, \gamma, g)$ be a TPS-manifold admitting a GAGRS with Einstein potential f, then the Poisson equation satisfied by f becomes*

$$\Delta f = 2n(2n+1)b(\alpha^2+\beta)^2) + (2n+1)\Lambda - a\,|\zeta|^2. \tag{5.65}$$

Example 5.1. *Let $(\Theta, \Phi, \zeta, \gamma, g)$ be the 3-dimensional TPS-manifold considered in example 3.1.*

Let ∇ be a Levi-Civita connection. From (3.27), we obtain the following components of Riemannina curvature tensor and Ricci tensor:

$$\Re(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_2=\left(-\frac{3}{4}e^{4z}+1\right)\mathcal{E}_1,\quad \Re(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_3=-\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_1,\quad \Re(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2=-e^{3z}y\mathcal{E}_1, \tag{5.66}$$

$$\Re(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_3=-\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_2,\quad \Re(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_2=-\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_3,$$

$$\Re(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_1=-e^{2z}\mathcal{E}_1-\left(\frac{1}{2}e^{4z}+1\right)\mathcal{E}_2+e^{3z}y\mathcal{E}_3,\quad \Re(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1=0,$$

$$\Re(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_1=e^{3z}ye_2+\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_3,\quad \Re(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=-e^{3z}y\mathcal{E}_1.$$

$$S_{ric}(\mathcal{E}_1,\mathcal{E}_1)=-\frac{3}{4}e^{4z}-2,\quad S_{ric}(\mathcal{E}_2,\mathcal{E}_2)=-\frac{1}{2}e^{4z}+2\mathcal{E}_2,\quad S_{ric}(\mathcal{E}_3,\mathcal{E}_3,)=-\frac{1}{2}e^{4z}-2. \tag{5.67}$$

From (2.14), we have

$$bS_{ric}(\mathcal{E}_i,\mathcal{E}_i)=-(\beta+\Lambda)g(\mathcal{E}_i,\mathcal{E}_i)+(a-\beta)\delta^i_j,\quad \{i=1,2,3\} \tag{5.68}$$

Now, we find the following cases corresponding to the different values of a and b in equation (2.14):

Case(1). For Killing vector field i.e., $a=b=0$, from (5.68) we find $\Lambda=-2\beta$, which is shrinking.

Case(2). In case of Ricci soliton $a=0,b=-1$, from (5.68), $\Lambda=-\left(\frac{3}{4}e^{4z}+2\right)-\beta$. Therefore, the data (g,ζ,Λ,a,b) is an AGRS on TPS-manifold $(\Theta,\Phi,\zeta,\gamma,g)$, is steady and shrinking according as $\frac{3}{4}e^{4z}+2<-\beta$, $\frac{3}{4}e^{4z}+2=\beta$, respectively.

Case(3). For Einstein-Weyl geometry case $a=1,b=\frac{-1}{n-2)}$, from (5.68), $\Lambda=(2\beta+1)-\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$. Now, the data (g,ζ,Λ,a,b) is an AGRS on TPS-manifold $(\Theta,\Phi,\zeta,\gamma,g)$ is steady, shrinking or expanding according as $(2\beta+1)=\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$, $(2\beta+1)<\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$ or $(2\beta+1)>\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$, respectively.

Case(4). For the geometry of Vacuum near horizon equation $a=1,b=\frac{1}{2}$, from (5.68), $\Lambda=(2\beta-2)-\left(\frac{1}{4}e^{4z}\right)$. The data (g,ζ,Λ,a,b) is an AGRS on TPS-manifold $(\Theta,\Phi,\zeta,\gamma,g)$, is steady, shrinking or expanding according as $(2\beta-2)=\left(\frac{1}{4}e^{4z}\right)$, $(2\beta-2)<\left(\frac{1}{4}e^{4z}\right)$ or $(2\beta-2)>\left(\frac{1}{4}e^{4z}\right)$, respectively.

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POINTWISE HEMI-SLANT RIEMANNIAN MAPS INTO ALMOST HERMITIAN MANIFOLDS AND CASORATI INEQUALITIES

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In the present paper, we introduce a new class of Riemannian maps which are called *pointwise hemi-slant Riemannian maps* from Riemannian manifolds to almost Hermitian manifolds as a natural generalization of hemi-slant submanifolds, hemi-slant submersions and hemi-slant Riemannian maps in a very natural way. We mention some examples, present a characterization and obtain the geometry of foliations in terms of the distributions which are involved in the definition of such maps. We also find necessary and sufficient conditions for pointwise hemi-slant Riemannian maps to be totally geodesic. Finally, we obtain Casorati curvatures for pointwise hemi-slant Riemannian maps in complex space form.

Keywords: Kaehler manifold, Riemannian map, pointwise hemi-slant submanifold, hemi-slant function, pointwise hemi-slant Riemannian map.

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1. Introduction

A considerable flaw in Riemannian geometry is to compare some geometric properties of suitable types of maps between Riemannian manifolds. Such suitable maps between Riemannian manifolds are isometric immersions and Riemannian submersions. Many geometers

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have been studied these maps between manifolds in ([1, 2, 9, 10, 12, 13, 14, 16, 18, 19, 25, 27, 32, 34, 33, 36, 35, 40, 37, 45, 48, 49]).

As a natural generalization of isometric immersions and Riemannian submersions, Fischer [17] defined the concept of Riemannian maps between Riemannian manifolds as follows: Let (M, g) and $(N, \bar{g})$ be Riemannian manifolds and Ψ is a smooth map between them. Then the tangent bundle of M has the following decompostion

$$TM = ker\Psi_* \oplus (ker\Psi_*)^{\perp},$$

where $ker\Psi_*$ denotes the kernel space of Ψ_* and $(ker\Psi_*)^{\perp}$ is the orthogonal complementary space to $ker\Psi_*$. In a similar way, the tangent bundle of N has the following decompostion

$$TN = (range\Psi_*) \oplus (range\Psi_*)^{\perp}$$

where $range\Psi_*$ denotes the range of Ψ_* and $range\Psi_*)^{\perp}$ is the orthogonal complementary space to $range\Psi_*$. Now, if the horizontal restriction $\Psi^h_{*p_1} : (ker\Psi_*^{\perp}) \longrightarrow (range\Psi_{*p_1})$ is a linear isometry between the inner product spaces $((ker\Psi_{*p_1})^{\perp}, g(p_1) \mid_{(ker\Psi_{*p_1})^{\perp}})$ and $(range\Psi_{*p_1}, \bar{g}(p_2) \mid_{(range\Psi_*)_{p_1}}), p_2 = \Psi(p_1)$ then a smooth map $\Psi : (M, g) \longrightarrow (N, \bar{g})$ is called Riemannian map at $p_1 \in M$. One can see that Riemannian submersions and isometric immersions are particular Riemannian maps with $(range\Psi_*)^{\perp} = 0$ and $ker\Psi_* = 0$, respectively.

Inspried by Fischer's article, B. Şahin introduced anti invariant Riemannian maps, holomorphic Riemannian maps and semi-invariant Riemannian maps to almost Hermitian manifolds and studied the geomerty of total spaces and base spaces ([39, 41]). This notion has opened a new original and effective area in the theory of Riemannian maps. Since then many geometers have studied Riemannian maps in different kinds of structures in [3, 4, 5, 20, 29, 28, 31, 38, 44, 43]. Recent developments in the theory of Riemannian map can be found in the books [30, 42].

On the other hand, in [11], Casorati introduced Casorati curvature which is a very natural concept of regular surfaces in the three-dimensional Euclidean space. One can see some optimal inequalities involving Casorati curvatures in ([7, 6, 15, 22, 24, 46, 47, 23, 51, 52]).

Hemi-slant submanifolds were introduced by Carriazo (Bi-slant immersions. in: Proc. ICRAMS 2000, Kharagpur, India, 2000, 88–97.) and Şahin (Annales Polonici Mathematici 95 (2009), 207-226) as a generalization of slant submanifolds. Hemi-slant submersions were introduced by Taştan, Şahin and Yanan (Mediterr. J. Math. 13, 2171–2184 (2016)) as a natural generalization of slant submersions. On the other hand, hemi-slant Riemannian maps were defined by Şahin (Mediterr. J. Math. 14, 10 (2017)) as a natural generalization of hemi-slant submanifolds and hemi-slant submersions. In 2022, Gündüzalp and Akyol defined pointwise slant Riemannian maps as a generalization of pointwise slant submanifolds [14] and pointwise slant submersions [25] in a natural way in [21]. They obtained simple characterizations and geometrical properties of pointwise slant Riemannian maps. As far as we know, no author has studied pointwise hemi-slant Riemannian maps so far. In the present paper, we are motivated to fill a gap in the literature by giving the notion of pointwise hemi-slant Riemannian maps, in which the base space consist of an anti-invariant and a slant distribution, as a special case of slant submanifold, hemi-slant submanifold, pointwise slant submanifold, slant submersions, hemi-slant submersions and hemi-slant Riemannian map and investigate the geometry of these maps.

The paper is organized as follows. Section 2 includes the main properties of the Riemannian maps, the tensors introduced by B. O'Neill and the second fundamental form of a map. Section 3 contains the definition of pointwise hemi-slant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds, many examples and investigate the geometry of foliations which are arisen from the definition of a pointwise hemi-slant Riemannian map and obtain decomposition theorems by using these maps. We also find necessary and sufficient conditions for pointwise hemi-slant Riemannian maps to be totally geodesic. Finally, we obtain Casorati curvatures for pointwise hemi-slant Riemannian maps in complex space form.

2. Preliminaries

Let (M_1, g_{M_1}, J_1) be an almost Hermitian manifold. This means that M_1 admits a tensor field J_1 of type $(1,1)$ on M_1 such that

$$J_1^2 = -I, \;\; g_{M_1}(J_1\xi_1, J_1\xi_2) = g_{M_1}(\xi_1, \xi_2), \;\; \xi_1, \xi_2 \in \Gamma(TM_1). \tag{2.1}$$

An almost Hermitian manifold M_1 is called Kaehler manifold [50] if

$$(\nabla_{\xi_1} J_1)\xi_2 = 0, \quad \xi_1, \xi_2 \in \Gamma(TM_1), \tag{2.2}$$

where ∇ denotes the Riemannian connection of the metric g_{M_1} on M_1.

Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be Riemannian manifolds and Ψ is a differentiable map between them. Then the differential Ψ_* of Ψ can be viewed a section of the bundle $Hom(TM_1, \Psi^{-1}TM_2) \to M_1$, where $\Psi^{-1}TM_2$ is the pullback bundle which has fibres $(\Psi^{-1}TM_2)_q = T_{\Psi(q)}M_2$, $q \in M_1$. $Hom(TM_1, \Psi^{-1}TM_2)$ has a connection ∇ induced from the Levi-Civita connection ∇^{M_1} and the pullback connection. The second fundamental form of Ψ is given by [8]

$$(\nabla\Psi_*)(\xi_1,\xi_2) = \nabla^{\Psi}_{\xi_1}\Psi_*\xi_2 - \Psi_*(\nabla^{M_1}_{\xi_1}\xi_2) \tag{2.3}$$

for $\xi_1,\xi_2 \in \Gamma(TM_1)$, where ∇^{Ψ} is the pullback connection. On the other hand, it is shown in ([39]) that $(\nabla\Psi_*)(\xi_1,\xi_2)$ has no components in $Im\Psi_*$, provided that $\xi_1,\xi_2 \in \Gamma((ker\Psi_*)^{\perp})$. More exactly,

$$(\nabla\Psi_*)(\xi_1,\xi_2) \in \Gamma((Im\Psi_*)^{\perp}), \ \ \forall \xi_1,\xi_2 \in \Gamma((ker\Psi_*)^{\perp}), \tag{2.4}$$

here $(Im\Psi_*)^{\perp}$ is the subbundle of $\Psi^{-1}(TM_2)$ with fibre $\Gamma(\Psi_*(T_qM_1)^{\perp})$, $q \in M_1$.
Let Ψ be a Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Riemannian manifold (M_2, g_{M_2}). Then $\forall \xi_1,\xi_2,Y_3 \in \Gamma((ker\Psi_*)^{\perp})$, we have

$$g_{M_2}((\nabla\Psi_*)(\xi_1,\xi_2), \Psi_*(Y_3)) = 0. \tag{2.5}$$

O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ are defined by, respectively,

$$\mathcal{T}_{\xi_1}\xi_2 = h\nabla_{v\xi_1}v\xi_2 + v\nabla_{v\xi_1}h\xi_2 \tag{2.6}$$

and

$$\mathcal{A}_{\xi_1}\xi_2 = v\nabla_{h\xi_1}h\xi_2 + h\nabla_{h\xi_1}v\xi_2 \tag{2.7}$$

for every $\xi_1,\xi_2 \in \Gamma(TM_1)$, where ∇ is the Levi-Civita connection of g_{M_1}. Here h and v are the projections on horizontal and vertical distributions, respectively. It is known that the tensor fields $\mathcal{T}$ is symetric and $\mathcal{A}$ is anti-symetric tensors. By using (2.6) and (2.7), we obtain

$$\nabla_{\eta_1}\eta_2 = \mathcal{T}_{\eta_1}\eta_2 + \hat{\nabla}_{\eta_1}\eta_2; \tag{2.8}$$

$$\nabla_{\eta_1}\xi_1 = \mathcal{T}_{\eta_1}\xi_1 + h\nabla_{\eta_1}\xi_1; \tag{2.9}$$

$$\nabla_{\xi_1}\eta_1 = \mathcal{A}_{\xi_1}\eta_1 + v\nabla_{\xi_1}\eta_1; \tag{2.10}$$

$$\nabla_{\xi_1}\xi_2 = \mathcal{A}_{\xi_1}\xi_2 + h\nabla_{\xi_1}\xi_2, \tag{2.11}$$

for any $\xi_1,\xi_2 \in \Gamma((ker\Psi_*)^{\perp})$, $\eta_1,\eta_2 \in \Gamma(ker\Psi_*)$, here $\hat{\nabla}_{\eta_1}\eta_2 = v\nabla_{\eta_1}\eta_2$.

We denote by ∇^2 both the levi-Civita connection of (M_2, g_{M_2}) and its pullback along Ψ. Then according to [26], for any vector field ξ_1 on M_1 and any section η_1 of $(range\Psi_*)^{\perp}$, where $(range\Psi_*)^{\perp}$ is the subbundle of $\Psi^{-1}(TM_2)$ with fiber $(\Psi_*(T_qM_1))^{\perp}-$ orthogonal complement of $(\Psi_*(T_qM_1))$ for g_{M_2} over q, we have $\nabla^{\Psi\perp}_{\xi_1}\eta_1$ which is the orthogonal projection of $\nabla^2_{\xi_1}\eta_1$ on $(\Psi_*(T_qM_1))^{\perp}-$such that $\nabla^{\Psi\perp}g_{M_2} = 0$. We now define $\mathcal{S}_{\eta_1}$ as

$$\nabla^2_{\Psi_*\xi_1}\eta_1 = -\mathcal{S}_{\eta_1}\Psi_*\xi_1 + \nabla^{\Psi\perp}_{\xi_1}\eta_1 \tag{2.12}$$

where $\mathcal{S}_{\eta_1}\Psi_*\xi_1$ is tangential component of $\nabla^2_{\Psi_*\xi_1}\eta_1$. It is easy to see that $\mathcal{S}_{\eta_1}\Psi_*\xi_1$ is bilinear in η_1 and $\Psi_*\xi_1$ and $\mathcal{S}_{\eta_1}\Psi_*\xi_1$ at q depends only on U_{1q} and $\Psi_{*q}Y_{1q}$. Thus,for $\xi_1, \xi_2 \in \Gamma((ker\Psi_*^{\perp})$ and $\eta_1 \in \Gamma((range\Psi_*)^{\perp})$, we get

$$g_{M_2}(\mathcal{S}_{\eta_1}\Psi_*\xi_1, \Psi_*\xi_2) = g_{M_2}(\eta_1, (\nabla\Psi_*)(\xi_1, \xi_2)). \tag{2.13}$$

Since $(\nabla\Psi_*)$ is symmetric, it follows that $\mathcal{S}_{\eta_1}$ is a symmetric linear transformation of $range\Psi_*$.

3. Pointwise hemi-slant Riemannian maps to Kaehler manifolds

Let $\Psi : (M_1, g_{M_1}) \to (M_2, g_{M_2}, J_2)$ be a Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to an almost Hermitian manifold (M_2, g_{M_2}, J_2). If, at each given point $p \in M_2$, the Wirtinger angle $\phi(X)$ between $J_2\Psi_*(X)$ and the space $range\Psi_*$ is independent of the choice of the nonzero tangent vector $\Psi_*(X)$ in $range\Psi_*$, then we say that Ψ is a pointwise slant Riemannian map. In this case, the angle ϕ can be regarded as a function on M_2, which is called the slant function of the pointwise slant Riemannian map.

Let $\mathcal{D}$ be a differentiable distribution on M_2. Then $\mathcal{D}$ is pointwise slant if and only if there exists a function $\mu \in [-1, 0]$ such that $(\gamma Q_\phi)^2\eta = \mu\eta$ for $\eta \in \mathcal{D}$, where Q_ϕ denotes the orthogonal projection on $\mathcal{D}$. Moreover, in this case $\mu = -\cos^2\phi$.

Definition 3.1. *Let (M_1, g_{M_1}) be a Riemannian manifold and (M_2, g_{M_2}, J_2) be an almost Hermitian manifold. Then we say that a Riemannian map $\Psi : M_1 \to M_2$ is a pointwise hemi-slant Riemannian map if there exists a pair of orthogonal distributions $\mathcal{D}^\phi$ and $\mathcal{D}^\perp$ on $range\Psi_*$ such that*

(1) *The space $range\Psi_*$ admits the orthogonal direct decomposition $\mathcal{D}^\phi \oplus \mathcal{D}^\perp$.*
(2) *The distribution $\mathcal{D}^\perp$ is totally real.*
(3) *The distribution $\mathcal{D}^\phi$ is pointwise slant with slant function ϕ.*

In this case, the angle ϕ can be regarded as a function on M_2, which is called the hemi-slant function of the pointwise hemi-slant Riemannian map.

Now we say that the pointwise hemi-slant Riemannian map Ψ is proper if $\mathcal{D}^\perp \neq \{0\}$ and $\phi \neq 0, \frac{\pi}{2}$.

Then, for $\eta_1 \in \Gamma(range\Psi_*)$, we can write

$$J_2\eta_1 = \mathcal{N}_1\eta_1 + \mathcal{N}_2\eta_1, \tag{3.14}$$

here $\mathcal{N}_1\eta_1 \in \Gamma(\mathcal{D}^\phi)$ and $\mathcal{N}_2\eta_1 \in \Gamma(\mathcal{D}^\perp)$ and we can write

$$J_2\eta_1 = \gamma\eta_1 + \delta\eta_1, \tag{3.15}$$

here $\gamma\eta_1 \in \Gamma(range\Psi_*)$ and $\delta\eta_1 \in \Gamma((range\Psi_*)^\perp)$. Also, for any $Y_1 \in \Gamma((range\Psi_*)^\perp)$, we get

$$J_2Y_1 = \bar{\gamma}Y_1 + \bar{\delta}Y_1, \tag{3.16}$$

here $\bar{\gamma}Y_1 \in \Gamma(range\Psi_*)$ and $\bar{\delta}Y_1 \in \Gamma((range\Psi_*)^\perp)$.

Theorem 3.1. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to an almost Hermitian manifold (M_2, g_{M_2}, J_2) with hemi-slant function ϕ.*

$$\gamma^2\eta_1 = -(\cos^2\phi)\eta_1 \tag{3.17}$$

for any $\eta_1 \in \Gamma(\mathcal{D}^\phi)$.

Proof. Since,

$$\cos\phi = \frac{g_{M_2}(J_2\eta_1, \gamma\eta_1)}{|J_2\eta_1||\gamma\eta_1|} = -\frac{g_{M_2}(\eta_1, \gamma^2\eta_1)}{|\eta_1||\gamma\eta_1|}$$

and $\cos\phi == \frac{|\gamma\eta_1|}{|J_2\eta_1|}$, for $\eta_1 \in \Gamma(\mathcal{D}^\phi)$ we obtain

$$\cos^2\phi = -\frac{g_{M_2}(\eta_1, \gamma^2\eta_1)}{|\eta_1|^2}.$$

Hence,

$$\gamma^2\eta_1 = -(\cos^2\phi)\eta_1.$$

Also converse of Theorem 3.1, it can be directly verified. □

Moreover, for any $\eta_1, U_2 \in \Gamma(\mathcal{D}^\phi)$ we have

$$g_{M_2}(\gamma\eta_1, \gamma U_2) = \cos^2\phi g_{M_2}(\eta_1, U_2) \tag{3.18}$$

$$g_{M_2}(\delta\eta_1, \delta U_2) = \sin^2\phi g_{M_2}(\eta_1, U_2). \tag{3.19}$$

Furthermore, for $\eta_1 \in \Gamma(\mathcal{D}^\phi)$ we obtain

$$\bar{\gamma}\delta\eta_1 = -\sin^2\phi\eta_1, \ \ \bar{\delta}\delta\eta_1 = -\delta\gamma\eta_1. \tag{3.20}$$

Example 3.1. *Let $(\mathbb{R}^8, g_{\mathbb{R}^8})$ be the Euclid space. Consider $\{J_1, J_2\}$ a pair of almost complex structures on R^8 satisfying $J_1J_2 = -J_2J_1$, here*

$$J_1(a_1, ..., a_8) = (-a_3, -a_4, a_1, a_2, -a_7, -a_8, a_5, a_6)$$

and

$$J_2(a_1, ..., a_8) = (-a_2, a_1, a_4, -a_3, -a_6, a_5, a_8, -a_7).$$

For any real-valued function $\lambda : \mathbb{R}^8 \to \mathbb{R}$, we define new almost complex structure J_λ on $\mathbb{R}^8$ by $J_\lambda = (\cos\lambda)J_1 + (\sin\lambda)J_2$.

Then $\mathbb{R}^8_\lambda = (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8})$ *is an almost Hermitian manifold.*

Consider a Riemannian map $\Psi : \mathbb{R}^8 \to \mathbb{R}^8_\lambda$ *by*

$$\Psi(y_1, ..., y_8) = (y_1, y_3, y_6, y_8, \pi, e, c_1, c_2).$$

Then the map Ψ *is a proper pointwise hemi-slant Riemannian map with the hemi-slant function* λ *such that*

$$\mathcal{D}^\phi = span\{\frac{\partial}{\partial z_6}, \frac{\partial}{\partial z_8}\}, \text{ and } \mathcal{D}^\perp = span\{\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\}.$$

Also, we obtain

$$(range_*)^\perp = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_4}, \frac{\partial}{\partial z_5}, \frac{\partial}{\partial z_7}\},$$

here $z_1, ..., z_8$ *are the local coordinates on* $\mathbb{R}^8$.

Theorem 3.2. *Let* Ψ_1 *be a Riemannian submersion from a Riemannian manifold* (M_1, g_{M_1}) *onto an almost Hermitian manifold* (M_2, g_{M_2}, J_2) *and* Ψ_2 *a poitwise hemi-slant immersion from* (M_2, g_{M_2}, J_2) *to an almost Hermitian manifold* (M_3, g_{M_3}, J_2). *Then* $\Psi_2 \circ \Psi_1$ *is a pointwise hemi-slant Riemannian map.*

This theorem is obvious from ([38], Theorem 5.2), and therefore we omit its proof.

As an application of the above Theorem, we give the following example of proper pointwise hemi-slant Riemannian map.

Example 3.2. *Let* $(\mathbb{R}^8, g_{\mathbb{R}^8})$ *be the Euclid space. Consider* $\{J_1, J_2\}$ *a pair of almost complex structures on* R^8 *satisfying* $J_1J_2 = -J_2J_1$, *here*

$$J_1(a_1, ..., a_8) = (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$$

and

$$J_2(a_1, ..., a_8) = (-a_3, a_4, a_1, -a_2, -a_7, a_8, a_5, -a_6).$$

For any real-valued function $\lambda : \mathbb{R}^8 \to \mathbb{R}$, *we define new almost complex structure* J_λ *on* $\mathbb{R}^8$ *by* $J_\lambda = (\cos\lambda)J_1 + (\sin\lambda)J_2$.

Then, $\mathbb{R}^8_\lambda = (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8})$ *is an almost Hermitian manifold. Consider the map*

$$\Psi : (\mathbb{R}^8, g) \to (\mathbb{R}^8_\lambda, J_\lambda, g_{\mathbb{R}^8}), \;\; \Psi(y_1, ..., y_8) = (y_1, 0, 0, y_4, 0, 0, y_8, y_7)$$

which is the the composition of the Riemannian submersion

$$\Psi_1 : (\mathbb{R}^8, g) \to \mathbb{E}^4, \;\; \Psi_1(y_1, ..., y_8) = (y_1, y_4, y_7, y_8)$$

followed by the pointwise hemi-slant immersion

$$\Psi_2 : \mathbb{E}^4 \to (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8}),\ \ \Psi_2(u_1, ..., u_4) = (u_1, 0, 0, u_2, 0, 0, u_4, u_3).$$

It is easy to verify that Ψ *is a pointwise hemi-slant Riemannian map with the slant function* $\phi = f$ *such that*

$$\mathcal{D}^\phi = span\{\frac{\partial}{\partial z_7}, \frac{\partial}{\partial z_8}\},\ \ and\ \ \mathcal{D}^\perp = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_4}\}.$$

Also, we obtain

$$(range_*)^\perp = span\{\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_5}, \frac{\partial}{\partial z_6}\},$$

here $z_1, ..., z_8$ *are the local coordinates on* $\mathbb{R}^8_\lambda$.

First note that for $\Psi_*\xi_1 \in \mathcal{D}^\phi$ and $\Psi_*\xi_2 \in \mathcal{D}^\perp$, we get $g_{M_2}(\Psi_*\xi_1, \Psi_*\xi_2) = 0$. Then, Riemannian map Ψ implies that $g_{M_1}(\xi_1, \xi_2) = 0$. So we obtain two orthogonal distributions $\tilde{\mathcal{D}^\phi}$ and $\tilde{\mathcal{D}^\perp}$ such that

$$(ker\Psi_*)^\perp = \tilde{\mathcal{D}^\phi} \oplus \tilde{\mathcal{D}^\perp}.$$

Let Ψ be a $C^\infty-$map from a Riemannian manifold (M_1, g_{M_1}) to a Riemannian manifold (M_2, g_{M_2}). Then, the adjoint map $^*(\Psi_*)_{q_1}$ of the differential $(\Psi_*)_{q_1}$, $q_1 \in M_1$, is given by

$$g_{M_2}((\Psi_*)_{q_1}\eta_1, Y_1) = g_{M_1}(\eta_1,^* (\Psi_*)_{q_1} Y_1) \tag{3.21}$$

for any $\eta_1 \in T_{q_1}M_1$ and $Y_1 \in T_{\Psi(q_1)}M_2$. Furthermore if the map Ψ is a Riemannian map, then for $\eta_1 \in (range\Psi_*)_{\Psi(q_1)}$ and $Y_1 \in (ker(\Psi_*)_{q_1})^\perp$, we obtain

$$(\Psi_*)^*_{q_1}(\Psi_*)_{q_1}\eta_1 = \eta_1,\ \ ^*(\Psi_*)_{q_1}(\Psi_*)_{q_1}Y_1 = Y_1,$$

thus the linear map $^*(\Psi_*)_{q_1} : (range\Psi_*)_{\Psi(q_1)} \to (ker(\Psi_*)_{q_1})^\perp$ is an isomorphism. Define $\mathrm{C} =^* (\Psi_*)_{q_1}\gamma(\Psi_*)$. From Theorem 3.1, we obtain:

Corollary 3.1. *Let* Ψ *be a pointwise hemi-slant Riemannian map from a Riemannian manifold* (M_1, g_{M_1}) *to an Hermitian manifold* (M_2, g_{M_2}, J_2) *with the hemi-slant function* ϕ. *Then,* $\eta_1 \in \Gamma(\mathrm{D}^\phi)$ *we have*

$$\mathrm{C}^2\eta_1 = -\cos^2\phi\eta_1. \tag{3.22}$$

For $Y_1, Y_2, \acute{Y_2} \in (ker(\Psi_*)_{q_1})^\perp$ with $\Psi_*\acute{Y_2} = \gamma\Psi_*Y_2$, we define

$$(\nabla^\Psi_{Y_1}\delta)\Psi_*Y_2 = \bar{\delta}(\nabla\Psi_*)(Y_1, Y_2) - (\nabla\Psi_*)(Y_1, \acute{Y_2}). \tag{3.23}$$

Proposition 3.1. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. If the tensor δ is parallel,then $\eta_1, U_2 \in \Gamma(\mathcal{D}^{\phi})$ we obtain*

$$(\nabla\Psi_*)(\mathrm{C}\eta_1, \mathrm{C}U_2) = -\cos^2\phi(\nabla\Psi_*)(\eta_1, U_2). \tag{3.24}$$

Proof. Assume that δ is parallel. Then, using (3.24), for $\eta_1, U_2 \in \Gamma(\mathcal{D}^{\phi})$ we get

$$\bar{\delta}(\nabla\Psi_*)(\eta_1, U_2) = (\nabla\Psi_*)(\eta_1, \mathrm{C}U_2).$$

By replacing η_1 and U_2 , we have

$$\bar{\delta}(\nabla\Psi_*)(U_2, \eta_1) = (\nabla\Psi_*)(U_2, \mathrm{C}\eta_1).$$

Since the tensor $(\nabla\Psi_*)$ is symmetric, we obtain

$$(\nabla\Psi_*)(\eta_1, \mathrm{C}U_2) = (\nabla\Psi_*)(U_2, \mathrm{C}\eta_1).$$

Thus we have

$$(\nabla\Psi_*)(\mathrm{C}\eta_1, \mathrm{C}U_2) = (\nabla\Psi_*)(\eta_1, \mathrm{C}^2U_2) = -\cos^2\phi(\nabla\Psi_*)(\eta_1, U_2).$$

□

Theorem 3.3. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. Then, the following assertions are equivalent:*

(a) distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliation on M_2,

(b)

$$g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*((\gamma\Psi_*(U_3)))), J_2\Psi_*(U_2)) = g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2))$$

and

$$g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*(\bar{\gamma}Y_1)), J_2\Psi_*(U_2)) = g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}J_2\Psi_*(U_2), \bar{\delta}Y_1), \tag{3.25}$$

(c) Ψ satisfies (3.25) and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, U_2), \delta\gamma\Psi_*(U_3)) = g_{M_2}(\bar{\gamma}\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), \Psi_*(U_2))$$

for any $\Psi_*(\eta_1), \Psi_*(U_2) \in \Gamma(\mathcal{D}^{\perp})$, $\Psi_*(U_3) \in \Gamma(\mathcal{D}^{\phi})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$.

Proof. For any $\Psi_*(\eta_1), \Psi_*(U_2) \in \Gamma(\mathcal{D}^\perp)$ and $\Psi_*(U_3) \in \Gamma(\mathcal{D}^\phi)$, using (2.2),(3.15) and (2.12) we obtain

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = -g_{M_2}(S_{J_2\Psi_*(U_2)}\Psi_*(\eta_1), \gamma\Psi_*(U_3)) + g_{M_2}((\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \delta\Psi_*(U_3)).$$

From (2.13), we arrive at

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = -g_{M_2}((\nabla\Psi_*)(\eta_1,^* \Psi_*(\gamma\Psi_*(U_3))), J_2\Psi_*(U_2)) + g_{M_2}(\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \delta\Psi_*(U_3)). \tag{3.26}$$

On the other hand, using (2.2),(3.16) and (2.12), for $Y_1 \in \Gamma((range\Psi_*)^\perp)$ we have

$$\begin{aligned} g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), Y_1) &= -g_{M_2}(S_{J_2\Psi_*(U_2)}\Psi_*(\eta_1), \bar{\gamma}Y_1) \\ &+ g_{M_2}(\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \bar{\delta}Y_1). \end{aligned}$$

From (2.13), we get

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), Y_1) = -g_{M_2}((\nabla\Psi_*)(\eta_1,^* \Psi_*(\bar{\gamma}Y_1)), J_2\Psi_*(U_2)) + g_{M_2}(\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \bar{\delta}Y_1). \tag{3.27}$$

(3.26) and (3.27) gives $(a) \Leftrightarrow (b)$. For any $\Psi_*(\eta_1), \Psi_*(U_2) \in \Gamma(\mathcal{D}^\perp)$ and $\Psi_*(U_3) \in \Gamma(\mathcal{D}^\phi)$, from (2.2) and (3.15) we get

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = g_{M_2}(\nabla^\Psi_{\eta_1}\gamma^2\Psi_*(U_3), \Psi_*(U_2)) + g_{M_2}(\nabla^\Psi_{\eta_1}\delta\gamma\Psi_*(U_3), \Psi_*(U_2)) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2)).$$

Using (2.12)and (3.17), we obtain

$$\sin^2\phi g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = \sin 2\phi\Psi_*\eta_1(\phi)g_{M_2}(\Psi_*(U_3), \Psi_*(U_2) - g_{M_2}(S_{\delta\gamma\Psi_*(U_3)}\Psi_*(\eta_1), \Psi_*(U_2)) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2)).$$

So, from (2.13) we arrive at

$$\sin^2\phi g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = -g_{M_2}((\nabla\Psi_*)(\eta_1, U_2), \delta\gamma\Psi_*(U_3)) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2)). \tag{3.28}$$

(3.27) and (3.28) gives $(a) \Leftrightarrow (c)$. □

In a similar way we obtain:

Theorem 3.4. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. Then, the following assertions are equivalent:*
(a) distribution $\mathcal{D}^{\phi}$ defines a totally geodesic foliation on M_2,
(b)

$$g_{M_2}((\nabla\Psi_*)(U_2,{}^*\Psi_*((\gamma\Psi_*(U_3)))), J_2\Psi_*(\eta_1)) = g_{M_2}(\nabla^{\Psi\perp}_{U_2} J_2\Psi_*(\eta_1), \delta\Psi_*(U_3))$$

and

$$\begin{aligned} g_{M_2}((\nabla\Psi_*)(U_2,{}^*\Psi_*(\bar{\gamma}Y_1)), \delta\Psi_*(U_3)) &= g_{M_2}(\nabla^{\Psi\perp}_{U_2}\delta\Psi_*(U_3), \bar{\delta}Y_1) \\ &- g_{M_2}(\nabla^{\Psi\perp}_{U_2}\delta\gamma\Psi_*(U_3), Y_1), \end{aligned} \tag{3.29}$$

(c) Ψ satisfies (3.29) and

$$g_{M_2}((\nabla\Psi_*)(U_2,\eta_1), \delta\gamma\Psi_*(U_3)) = g_{M_2}(\bar{\gamma}\nabla^{\Psi\perp}_{U_2}\delta\Psi_*(U_3), \Psi_*(\eta_1))$$

for any $\Psi_*(\eta_1) \in \Gamma(\mathcal{D}^{\perp})$, $\Psi_*(U_2), \Psi_*(U_3) \in \Gamma(\mathcal{D}^{\phi})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$.

Using Theorems 3.3 and 3.4, we obtain:

Theorem 3.5. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with integrable distribution and the hemi-slant function ϕ. Then, the leaf of $(range\Psi_*)$ is a locally product Riemannian manifold $M_1^{\perp} \times M_2^{\phi}$ if and only if*

$$\begin{aligned} g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*\bar{\gamma}(Y_1)), \delta\Psi_*(U_2)) &= g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2), \bar{\delta}Y_1) \\ &- g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\gamma\Psi_*(U_2), Y_1) \end{aligned}$$

and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, U_2), \delta\Psi_*(U_3)) = g_{M_2}(\bar{\gamma}\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), \Psi_*(U_2))$$

for any $\eta_1, U_2 \in \Gamma((ker\Psi^*)^{\perp})$, $\Psi_*(U_3) \in \Gamma(\mathcal{D}^{\phi})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$, here $M_1^{\perp}$ and M_2^{ϕ} denotes the leaves of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\phi}$, respectively.

Theorem 3.6. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. Then,*

Ψ is totally geodesic if and only if the following conditions are satisfied:

(a)

$$g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*\bar{\gamma}(Y_1)),\delta\Psi_*(U_2)) = g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2),\bar{\delta}Y_1) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\gamma\Psi_*(U_2),Y_1)$$

for any $\eta_1, U_2 \in \Gamma((ker\Psi^*)^\perp)$ and $Y_1 \in \Gamma((range\Psi_*)^\perp)$,

(b)

$$\begin{aligned}\sin 2\phi\eta_1(\phi)g_{M_2}(\Psi_*(U_2),\Psi_*(U_3)) &= g_{M_2}((\nabla\Psi_*)(\eta_1,U_3),\delta\gamma\Psi_*(U_2)) \\ &- g_{M_2}(S_{\delta\Psi_*(U_2)}\eta_1,\gamma\Psi_*(U_3) \\ &+ g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2),\delta\Psi_*(U_3)) \\ &- \sin^2\phi g_{M_1}(h\nabla_{\eta_1}U_2,U_3)\end{aligned}$$

for any $\eta_1, U_2, U_3 \in \Gamma((ker\Psi_*)^\perp)$,

(c) the distribution $ker\Psi_*$ is totally geodesic,

(d) the distribution $(ker\Psi_*)^\perp$ is integrable.

Proof. For any $\eta_1, U_2, U_3 \in \Gamma((ker\Psi_*)^\perp)$, from (2.2),(2.3),(2.11) and (3.15) we have

$$\begin{aligned}g_{M_2}((\nabla\Psi_*)(\eta_1,U_2),\Psi_*(U_3)) &= -g_{M_2}(\nabla^{\Psi}_{\eta_1}\gamma^2\Psi_*(U_2),\Psi_*(U_3)) \\ &- g_{M_2}(\nabla^{\Psi}_{\eta_1}\delta\gamma\Psi_*(U_2),\Psi_*(U_3)) \\ &+ g_{M_2}(\nabla^{\Psi}_{\eta_1}\delta\Psi_*(U_2),\gamma\Psi_*(U_3)) \\ &+ g_{M_2}(\nabla^{\Psi}_{\eta_1}\delta\Psi_*(U_2),\delta\Psi_*(U_3)) - g_{M_1}(h\nabla_{\eta_1}U_2,U_3).\end{aligned}$$

Then, using (2.12),(2.13) and (3.17) we obtain

$$\begin{aligned}\sin^2\phi g_{M_2}((\nabla\Psi_*)(\eta_1,U_2),\Psi_*(U_3)) &= -\sin 2\phi\eta_1(\phi)g_{M_2}(\Psi_*(U_2),\Psi_*(U_3)) \\ &+ g_{M_2}((\nabla\Psi_*)(\eta_1,U_3),\delta\gamma\Psi_*(U_2)) \\ &- g_{M_2}(S_{\delta\Psi_*(U_2)}\eta_1,\gamma\Psi_*(U_3) \\ &+ g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2),\delta\Psi_*(U_3)) \\ &- \sin^2\phi g_{M_1}(h\nabla_{\eta_1}U_2,U_3). \qquad (3.30)\end{aligned}$$

On the other hand, for $\eta_1, U_2 \in \Gamma((ker\Psi_*)^\perp)$ and $V_1, V_2 \in \Gamma(ker\Psi_*)$, using (2.3), (2.8) and (2.11) we get

$$g_{M_2}((\nabla\Psi_*)(V_1,V_2),\Psi_*(\eta_1)) = -g_{M_1}(\mathrm{T}_{V_1}V_2,\eta_1) \qquad (3.31)$$

and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, V_1), \Psi_*(U_2)) = -g_{M_1}(\mathrm{A}_{\eta_1}U_2, V_1). \tag{3.32}$$

Now, by using (3.30),(3.31), (3.32) and Theorem 3.3, the proof is completed. □

4. Casorati Inequalities along hemi-slant Riemannian maps to complex space forms

Lemma 4.1. [46] *Let* $W = \{(y_1, y_2, ..., y_m) \in \mathbb{R}^m : y_1 + y_2 + ... + y_m = z\}$ *be a hyperplane of* $\mathbb{R}^m$*, and* $g : \mathbb{R}^m \to \mathbb{R}$ *a quadratic form given by*

$$g(y_1, y_2, ..., y_m) = c\Sigma_{k=1}^{m-1}(y_k)^2 + d(y_m)^2 - 2\Sigma_{1\leq k<s\leq m}y_k y_s,\ c > 0,\ d > 0.$$

Then the constrained extremum problem $min_{(y_1,y_2,...,y_m)\in W}\, g$ *has the following solution:*

$$y_1 = y_2 = ... = y_{m-1} = \frac{z}{c+1},\ y_m = \frac{z}{d+1} = \frac{z(m-1)}{(c+1)d} = (c-m+2)\frac{z}{c+1},$$

provided that $d = \frac{m-1}{c-m+2}$.

Let (M_2, g_{M_2}, J_2) be a Kaehler manifold. The Riemannian-Christoffel curvature tensor of a complex space form $M_2(\nu)$ of constant holomorphic sectional curvature ν satisfies

$$\begin{aligned} R_{\mathcal{B}_2}(Y_1, Y_2, \mathcal{Y}_3, \mathcal{Y}_4) = \frac{\nu}{4}\{&g_{\mathcal{B}_2}(Y_1, \mathcal{Y}_4)g_{\mathcal{B}_2}(Y_2, \mathcal{Y}_3) - g_{\mathcal{B}_2}(Y_1, \mathcal{Y}_3)g_{\mathcal{B}_2}(Y_2, \mathcal{Y}_4) \\ &+ g_{\mathcal{B}_2}(Y_1, J_2\mathcal{Y}_3)g_{\mathcal{B}_2}(J_2Y_2, \mathcal{Y}_4) - g_{\mathcal{B}_2}(Y_2, J_2\mathcal{Y}_3)g_{\mathcal{B}_2}(J_2Y_1, \mathcal{Y}_4) \\ &+ 2g_{\mathcal{B}_2}(Y_1, J_2Y_2)g_{\mathcal{B}_2}(J_2\mathcal{Y}_3, \mathcal{Y}_4)\} \end{aligned} \tag{4.33}$$

for all vector fields $Y_1, Y_2, Y_3, Y_4 \in \Gamma(TM_2)$ ([50]).

Let Ψ be a Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Riemannian manifold (M_2, g_{M_2}). Let R_{M_1} and R_{M_2} be the curvature tensor fields of ∇^{M_1} and ∇^{M_2}, respectively. Then, for all $Y_1, Y_2, Y_3, Y_4 \in \Gamma((ker\Psi_*)^\perp$, we obtain the Gauss formula given by ([40])

$$\begin{aligned} g_{M_2}(R_{\mathcal{B}_2}(\Psi_*Y_1, \Psi_*Y_2)\Psi_*Y_3, \Psi_*Y_4) = {}& g_{M_1}(R_{\mathcal{B}_1}(Y_1, Y_2)Y_3, Y_4) \\ &+ g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1, Y_3), (\nabla\Psi_*)(Y_2, Y_4)) \\ &- g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1, Y_4), (\nabla\Psi_*)(Y_2, Y_3)). \end{aligned} \tag{4.34}$$

Now, we suppose that Ψ is a pointwise hemi-slant Riemannian map from a Riemannian manifold $(M_1^{b_1}, g_{M_1})$ to the complex space form $(M_2^{b_2}(\nu), J_2, g_{M_2})$ such that $3 \leq p = rank\Psi <$

$min\{b_1, b_2\}$. Using (4.33) and (4.34) , for all $Y_1, Y_2, Y_3, Y_4 \in \Gamma((ker\Psi_*)^\perp$, we obtain

$$\begin{aligned} g_{M_1}(R_{\mathcal{B}_1}(Y_1, Y_2)Y_3, Y_4) = &\frac{\nu}{4}\{g_{\mathcal{B}_2}(Y_1, Y_4)g_{\mathcal{B}_2}(Y_2, Y_3) - g_{\mathcal{B}_2}(Y_1, Y_3)g_{\mathcal{B}_2}(Y_2, Y_4) \\ &+ g_{\mathcal{B}_2}(\Psi_*Y_1, J_2\Psi_*Y_3)g_{\mathcal{B}_2}(J_2\Psi_*Y_2, \Psi_*Y_4) \\ &- g_{\mathcal{B}_2}(\Psi_*Y_2, J_2\Psi_*Y_3)g_{\mathcal{B}_2}(J_2\Psi_*Y_1, \Psi_*Y_4) \\ &+ 2g_{\mathcal{B}_2}(\Psi_*Y_1, J_2\Psi_*Y_2)g_{\mathcal{B}_2}(J_2\Psi_*Y_3, \Psi_*Y_4)\} \\ &- g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1, Y_3), (\nabla\Psi_*)(Y_2, Y_4)) \\ &+ g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1, Y_4), (\nabla\Psi_*)(Y_2, Y_3)). \end{aligned} \tag{4.35}$$

Let $q \in M_1$ and consider

$\{\Psi_*E_1, \Psi_*E_2 = \sec\phi\gamma\Psi_*E_1, ..., \Psi_*E_{2n-1}, \Psi_*E_{2n} = \sec\phi\gamma\Psi_*E_{2n-1}, \Psi_*E_{2n+1}, ..., \Psi_*E_p\}$ and $\{E_{p+1}, E_{p+2}, ..., E_{b_2}\}$ two orthonormal bases of $(ker\Psi_*)^\perp$ and $(range\Psi_*)^\perp$, respectively. Then, it follows that the dimension of $range\Psi_*$ is p. We defined the scalar curvature $\tau^{(ker\Psi_*)^\perp}$ on the horizontal space $(ker\Psi_{*q})^\perp$ by

$$\tau^{(ker\Psi_*)^\perp} = \Sigma_{k,s=1}^p g_{M_1}(R_{M_1}(E_k, E_s)E_s, E_k) \tag{4.36}$$

and the normalized scalar curvature $\kappa^{(ker\Psi_*)^\perp}$ of $(ker\Psi_{*q})^\perp$ as

$$\kappa^{(ker\Psi_*)^\perp} = \frac{2\tau^{(ker\Psi_*)^\perp}}{p(p-1)}. \tag{4.37}$$

Then, we can write

$$\psi_{ks}^\beta = g_{\mathcal{B}_2}((\nabla\Psi_*)(E_k, E_s), E_\beta), \ k, s = 1, ..., p, \ \beta = p+1, ..., b_2, \tag{4.38}$$

$$\|\psi\|^2 = \Sigma_{k,s=1}^p g_{\mathcal{B}_2}((\nabla\Psi_*)(E_k, E_s), (\nabla\Psi_*)(E_k, E_s)) \tag{4.39}$$

$$trace\psi = \Sigma_{k=1}^p (\nabla\Psi_*)(E_k, E_k), \ \|trace\psi\|^2 = g_{\mathcal{B}_2}(trace\psi, trace\psi). \tag{4.40}$$

The squared norm of ψ, the second fundamental form of the horizontal space $(ker\Psi_*)^\perp$ over the manifold $(M_2^{b_2}, J_2, g_{M_2})$, is denoted by $\mathcal{C}$ and is called the Casorati curvature of the horizontal space $(ker\Psi_*)^\perp$. Thus, we obtain

$$\mathcal{C} = \frac{1}{p}\|\psi\|^2 = \frac{1}{p}\Sigma_{\beta=p+1}^{b_2}\Sigma_{k,s=1}^p(\psi_{ks}^\beta)^2. \tag{4.41}$$

Now, assume that $\mathrm{L}^{(ker\Psi_*)^\perp}$ is a $t-$dimensional subspace $(ker\Psi_*)_q^\perp$, $2 \le t$ and let $\{E_1, E_2, ..., E_t\}$ be an orthonormal basis of $\mathrm{L}^{(ker\Psi_*)^\perp}$. Then the Casorati curvature $\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}$ of $\mathrm{L}^{(ker\Psi_*)^\perp}$ defined as

$$\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) = \frac{1}{t}\|T\|^2 = \frac{1}{t}\Sigma_{\beta=p+1}^{b_2}\Sigma_{k,s=1}^t(T_{ks}^\beta)^2.$$

The normalized $\sigma^{(ker\Psi_*)^\perp}-$ Casorati curvatures $\sigma_{\mathcal{C}}^{(ker\Psi_*)_\perp}(p-1)$ and $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1)$ of $(ker\Psi_*)_q)^\perp$ are given by

$[\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1)]_q = \frac{1}{2}\mathcal{C}_q^{(ker\Psi_*)^\perp} + \frac{p+1}{2p} inf\{\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) : \mathrm{L}^{(ker\Psi_*)^\perp}$ a hyperplane of $(ker\Psi_*)_q)^\perp\}$, and

$[\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1)]_q = 2\mathcal{C}_q^{(ker\Psi_*)^\perp} - \frac{2p-1}{2p} inf\{\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) : \mathrm{L}^{(ker\Psi_*)^\perp}$ a hyperplane of $(ker\Psi_*)_q^\perp\}$.

Using (4.35), (4.36) and (4.41) we arrive at

$$\frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi = 2\tau^{(ker\Psi_*)^\perp}(q) + p\mathcal{C}^{(ker\Psi_*)^\perp} - \|trace\psi\|^2, \tag{4.42}$$

here $\tau^{(ker\Psi_*)^\perp}$ is the scalar curvature of $(ker\Psi_*)^\perp$.

Now we define a function $\mathcal{Q}^{(ker\Psi_*)^\perp}$ associated with the following quadratic polynomial with respect to the components of ψ :

$$\begin{aligned}\mathcal{Q}^{(ker\Psi_*)^\perp} &= \frac{1}{2}[(p^2-p)\mathcal{C}^{(ker\Psi_*)^\perp} + (p^2-1)\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ &- 2\tau^{(ker\Psi_*)^\perp} + \frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi.\end{aligned}$$

Without loos of generality, by supposing that the hyperplane $\mathrm{L}^{(ker\Psi_*)^\perp}$ is spanned by $\{E_1,...,E_{p-1}\}$, using (4.42) one can produce

$$\begin{aligned}\mathcal{Q}^{(ker\Psi_*)^\perp} &= \Sigma_{\beta=p+1}^{b_2}\Sigma_{k=1}^{p-1}[p(\psi_{kk}^\beta)^2 + (p+1)(\psi_{kp}^\beta)^2] \\ &+ \Sigma_{\beta=p+1}^{b_2}[2(p+1)\Sigma_{1=k<s}^{p-1}(\psi_{ks}^\beta)^2 \\ &- 2\Sigma_{1=k<s}^{p}\psi_{kk}^\beta\psi_{ss}^\beta + \frac{p-1}{2}(\psi_{pp}^\beta)^2] \\ &\geq \Sigma_{\beta=p+1}^{b_2}[\Sigma_{k=1}^{p-1}p(\psi_{kk}^\beta)^2 + \frac{p-1}{2}(\psi_{pp}^\beta)^2 \\ &- 2\Sigma_{1=k<s}^{p}\psi_{kk}^\beta\psi_{ss}^\beta].\end{aligned} \tag{4.43}$$

For $\beta = p+1,...,b_2$, let us consider the quadratic form $g_\beta : R^{b_2} \to R$ defined by

$$g_\beta(\psi_{11}^\beta,...,\psi_{pp}^\beta) = \Sigma_{k=1}^{p-1}p(\psi_{kk}^\beta)^2 + \frac{p-1}{2}(\psi_{pp}^\beta)^2 - 2\Sigma_{k<s=1}^{p}\psi_{kk}^\beta\psi_{ss}^\beta, \tag{4.44}$$

and the constrained extremum problem, $min g_\beta$, subject to

$$\Phi^\beta : \psi_{11}^\beta + ... + \psi_{pp}^\beta = z^\beta,$$

here z^β is a real constant. From Lemma 4.1, we obtain $c = p, \ d = \frac{p-1}{2}$.

Thus, by Lemma 4.1 we get the critical point $(\psi_{11}^\beta,...,\psi_{pp}^\beta)$, given by

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1\,p-1}^\beta = \frac{z^\beta}{p+1},\ \psi_{pp}^\beta = \frac{2z^\beta}{p+1},$$

is a global minimum point. Also, $g_\beta(\psi^\beta_{11}, ..., \psi^\beta_{pp}) = 0$. Moreover we obtain

$$\mathcal{Q}^{(ker\Psi_*)^\perp} \geq 0, \tag{4.45}$$

which implies

$$\begin{aligned} 2\tau^{(ker\Psi_*)^\perp} &\leq \frac{1}{2}[(p^2-p)\mathcal{C}^{(ker\Psi_*)^\perp} + (p^2-1)\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ &+ \frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi \end{aligned} \tag{4.46}$$

and using (4.46) we obtain

$$\begin{aligned} \kappa^{(ker\Psi_*)^\perp} &\leq [\frac{1}{2}\mathcal{C}^{(ker\Psi_*)^\perp} + \frac{p+1}{2p}\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ &+ \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi \end{aligned} \tag{4.47}$$

for all hyperplane $\mathrm{L}^{(ker\Psi_*)^\perp}$ of $(ker\Psi_*)^\perp$.

Similarly, we can write

$$\begin{aligned} \mathcal{Z}^{(ker\Psi_*)^\perp} = 2(p^2-p)\mathcal{C}^{(ker\Psi_*)^\perp} - \frac{1}{2}(2p^2-3p+1)\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) \\ - 2\tau^{(ker\Psi_*)^\perp} + \frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi, \end{aligned}$$

here hyperplane $\mathrm{L}^{(ker\Psi_*)^\perp}$ is a hyperplane of $(ker\Psi_*)^\perp$. From here,

$$\mathcal{Z}^{(ker\Psi_*)^\perp} \geq 0, \tag{4.48}$$

which implies

$$\begin{aligned} \kappa^{(ker\Psi_*)^\perp} \leq 2\mathcal{C}^{(ker\Psi_*)^\perp} - \frac{2p-1}{2p}\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ + \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi. \end{aligned} \tag{4.49}$$

Now, taking the infimum in (4.47) and the supremum in (4.49) over all hyperplanes $\mathrm{L}^{(ker\Psi_*)^\perp}$ of $(ker\Psi_*)^\perp$ and analyzing the equality case in (4.45) and (4.48), respectively, we get:

Theorem 4.1. *Let Ψ be a pointwise hemi-slant Riemannian map a Riemannian manifold $(M_1^{b_1}, g_{M_1})$ to a complex space form $(M_2^{b_2}(\nu), J_2, g_{M_2})$ with hemi-slant function ϕ, $3 \leq p = rank\Psi < min\{b_1, b_2\}$. Then the normalized $\sigma-$ Casorati curvatures $\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ and $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ on $(ker\Psi_*)^\perp_q$ satisfy*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi, \tag{4.50}$$

$$(ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi. \tag{4.51}$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ *if and only if with respect to suitable orthonormal basis* $\{E_1, ..., E_p\}$ *on* $(ker\Psi_*)_q^\perp$ *and* $\{E_{p+1}, ..., E_{b_2}\}$ *on* $((range\Psi_*)_q)^\perp$*, the components of* ψ *satisfy*

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1p-1}^\beta = \frac{1}{2}\psi_{pp}^\beta,\ \ \beta \in \{p+1, p+2, ..., b_2\},$$

$$\psi_{ks}^\beta = 0,\ \ k, s \in \{1, , ..., p\}(k \neq s),\ \ \beta \in \{p+1, p+2, ..., b_2\}.$$

Using the Theorem 4.1, we obtain the following results.

Corollary 4.1. *Let* Ψ *be a pointwise hemi-slant Riemannian map a Riemannian manifold* $(M_1^{b_1}, g_{M_1})$ *to a complex space form* $(M_2^{b_2}(\nu), J_2, g_{M_2})$ *with hemi-slant function* $\phi = \frac{\pi}{2}$, $3 \leq p = rank\Psi < min\{b_1, b_2\}$*. Then the normalized* $\sigma-$ *Casorati curvatures* $\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *and* $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *on* $(ker\Psi_*)_q^\perp$ *satisfy*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4}$$

$$(ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4}.$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ *if and only if with respect to suitable orthonormal basis* $\{E_1, ..., E_p\}$ *on* $(ker\Psi_*)_q^\perp$ *and* $\{E_{p+1}, ..., E_{b_2}\}$ *on* $((range\Psi_*)_q)^\perp$*, the components of* ψ *satisfy*

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1p-1}^\beta = \frac{1}{2}\psi_{pp}^\beta,\ \ \beta \in \{p+1, p+2, ..., b_2\},$$

$$\psi_{ks}^\beta = 0,\ \ k, s \in \{1, , ..., p\}(k \neq s),\ \ \beta \in \{p+1, p+2, ..., b_2\}.$$

Corollary 4.2. *Let* Ψ *be a pointwise hemi-slant Riemannian map a Riemannian manifold* $(M_1^{b_1}, g_{M_1})$ *to a complex space form* $(M_2^{b_2}(\nu), J_2, g_{M_2})$ *with hemi-slant function* $\phi = 0$, $3 \leq p = rank\Psi < min\{b_1, b_2\}$*. Then the normalized* $\sigma-$ *Casorati curvatures* $\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *and* $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *on* $(ker\Psi_*)_q^\perp$ *satisfy*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{2}(\frac{1}{2} + \frac{3n}{p(p-1)})$$

$$(ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{2}(\frac{1}{2} + \frac{3n}{p(p-1)}).$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ if and only if with respect to suitable orthonormal basis $\{E_1, ..., E_p\}$ on $(ker\Psi_*)_q^{\perp}$ and $\{E_{p+1}, ..., E_{b_2}\}$ on $((range\Psi_*)_q)^{\perp}$, the components of ψ satisfy

$$\psi_{11}^{\beta} = \psi_{22}^{\beta} = ... = \psi_{p-1p-1}^{\beta} = \frac{1}{2}\psi_{pp}^{\beta}, \ \ \beta \in \{p+1, p+2, ..., b_2\},$$

$$\psi_{ks}^{\beta} = 0, \ \ k, s \in \{1, , ..., p\}(k \neq s), \ \ \beta \in \{p+1, p+2, ..., b_2\}.$$

Corollary 4.3. *Let Ψ be a pointwise hemi-slant Riemannian map a Riemannian manifold $(M_1^{b_1}, g_{M_1})$ to the complex Euclidean space $\mathbb{C}^{\frac{b_2}{2}}$ with hemi-slant function ϕ, $3 \leq p = rank\Psi < min\{b_1, b_2\}$. Then we get*

$$(i)\ \kappa^{(ker\Psi_*)^{\perp}} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^{\perp}}(p-1), \ (ii)\ \kappa^{(ker\Psi_*)^{\perp}} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^{\perp}}(p-1).$$

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TRANSLATION FRAMED SURFACES GENERATED BY NON-NULL FRAMED CURVES IN MINKOWSKI 3-SPACE $\mathbb{E}_1^3$

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In this paper, first we obtain the conditions for the existence and uniqueness of non-null framed curves as well as non-null framed surfaces in Minkowski 3-space. Further, we study the timelike and spacelike translation framed surfaces generated by non-null framed curves and obtain the basic invariants of such surfaces in $\mathbb{E}_1^3$. We also find the curvatures of timelike and spacelike translation framed surfaces generated by non-null framed curves. Finally, we classify the translation framed surfaces generated by non-null framed curves lying in mutually perpendicular coordinate planes of $\mathbb{E}_1^3$ with $\mu_K \equiv 0$ and $\mu_H \equiv 0$.

Keywords: Framed curve, Framed surface, Translation framed surface, Curvature and invariants of a translation framed surface.

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1. Introduction

A translation surface is a special case of Darboux surface which is the union of 'equivalent' curves ('equivalent' in the sense that, the curves are images of one another by some isometries of the space), also known as generating curves of the surface. A Darboux surface is defined as the movement of curves by rigid motions of the space. Therefore, it can be parametrized as $X(u, v) = A(v).\alpha(u) + \beta(v)$, where α, β are two space curves and A is an orthogonal matrix. When the orthogonal matrix A is identity matrix the surface is called a translation

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surface. Thus, a generalized type of a translation surface is given by

$$X(u,v) = \alpha(u) + \beta(v).$$

Translation surface which is known as the double curve in differential geometry is base for roofing structures. The construction and design of free form glass roofing structures are generally created with the help of curved (formed) glass panes or planar triangular glass facets. Recently, classification of translation surfaces under some conditions on curvatures has been studied in Euclidean as well as Minkowski space ([1],[10],[11],[15]).

A framed curve in Minkowski 3-space is a curve with an assigned frame which moves along the curve. In [7], Honda and Takahashi defined the curvature functions of the framed curve in $\mathbb{E}^3$, similar to a regular curve. By using curvature functions, they obtained the existence and the uniqueness theorem for the framed curves. The curvature functions of a framed curve are used to investigate the curve along with its singularities. On the other hand, a framed surface is defined to be a surface with an assigned moving frame which is used to analyze properties and singularities of the surface. In [4], by using the moving frames in $\mathbb{E}^3$, the basic invariants and the curvatures of framed surfaces are introduced by Fukunaga and Takahashi. They studied the properties of framed surfaces using the basic invariants of the surfaces and gave some examples.

In [5], Fukunaga and Takahashi reviewed the theories for framed surfaces, framed curves and one-parameter families of framed curves in $\mathbb{E}^3$. They showed that up to congruence, the surface along with the moving frame can be determined by the basic invariants of the framed surface and the curvature of a one parameter family of framed curves. In [6], the authors studied the translation surfaces with assigned moving frame and discussed the various singularities that arise on such surfaces with help of the notion of framed curves and surfaces.

In this paper, we study the non-null translation framed surfaces generated by non-null framed curves in $\mathbb{E}^3_1$. The paper is arranged as follows. There are some basic results in section 2. In section 3, we study non-null framed curves in $\mathbb{E}^3_1$ and obtain the conditions for the existence and uniqueness of non-null framed curves. In section 4, first we study non-null framed surfaces in $\mathbb{E}^3_1$ and find their curvatures and existence and uniqueness conditions. Further, we study the timelike and spacelike translation framed surfaces generated by non-null framed curves and obtain the basic invariants of such surfaces in $\mathbb{E}^3_1$. We also find the curvatures of timelike and spacelike translation framed surfaces generated by non-null framed curves.

Finally, we classify the translation framed surfaces generated by non-null framed curves lie in the coordinate planes of $\mathbb{E}_1^3$ with $\mu_K \equiv 0$ and $\mu_H \equiv 0$.

2. Preliminaries

The Minkowski 3-space, denoted by $\mathbb{E}_1^3$, is a three dimensional real vector space $\mathbb{R}^3$ endowed with the metric tensor $\langle .,. \rangle = -dx^2 + dy^2 + dz^2$. The (Lorentzian) scalar and cross product are defined by:

$$\begin{cases} \langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3, \\ x \times y = (-x_2y_3 + x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1), \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ belong to $\mathbb{E}_1^3$. This space is also known as Lorentz-Minkowski space. A vector $x \in \mathbb{E}_1^3$ is said to be spacelike when $\langle x, x \rangle > 0$ or $x = 0$, timelike when $\langle x, x \rangle < 0$ and lightlike(null) when $\langle x, x \rangle = 0$, $x \neq 0$. A curve in $\mathbb{E}_1^3$ is called spacelike, timelike or lightlike when the velocity vector of the curve is spacelike, timelike or lightlike, respectively. The norm of a vector $x \in \mathbb{E}_1^3$ is defined as $\|x\| = \sqrt{|\langle x, x \rangle|}$. The hyperbolic and Lorentzian unit spheres are defined as

$$H_0^2 = \{x \in \mathbb{E}_1^3 \| \langle x, x \rangle = -1\}$$

and

$$S_1^2 = \{x \in \mathbb{E}_1^3 \| \langle x, x \rangle = 1\},$$

respectively. Let $\gamma = \gamma(s) : I \to \mathbb{E}_1^3$ be an arbitrary curve. The curve γ is said to be an unit speed curve (or parameterized by the arc-length parameter s) if $\langle \gamma'(s), \gamma'(s) \rangle = \pm 1$ for any $s \in I$.

For a spacelike curve $\gamma : I \to \mathbb{E}_1^3$ parametrized with arclength parameter s, let $\{t, n, b\}$ be the moving Frenet frame along the curve, where $t(s) = \gamma'(s)$ is the unit tangent vector, n is the unit normal vector defined as the unit vector in the direction $t'(s)$ such that $t'(s) = \kappa(s)\, n(s)$, where $\kappa(s)$ is the curvature of the curve and $b(s) = t(s) \times n(s)$. The second curvature (torsion) of the curve is given by $\tau = \epsilon \langle b', n \rangle$, where $\epsilon = \langle n, n \rangle$.The Frenet-Serret equations of the spacelike curve are given as

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} = \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t,t\rangle = 1$, $\langle n,n\rangle = \epsilon$, $\langle b,b\rangle = -\epsilon$, $\langle t,b\rangle = \langle t,n\rangle = \langle n,b\rangle = 0$. If $\epsilon = 1$, $\gamma(s)$ is a spacelike curve with the spacelike principal normal n and the timelike binormal b, while if ϵ = -1 then γ is a spacelike curve with the timelike principal normal n and the spacelike binormal b. For a timelike curve γ, we define Frenet frame in similar way except for the torsion is given by $\tau = -\langle b', n\rangle$. The Frenet-Serret equations are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} = \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t,t\rangle$ = -1, $\langle n,n\rangle = 1$, $\langle b,b\rangle = 1$, $\langle t,n\rangle = \langle t,b\rangle = \langle n,b\rangle = 0$.

A surface in $\mathbb{E}^3_1$ is said to be a spacelike, timelike or lightlike if the metric on the surface is positive definite, indefinite or degenerate, respectively. The type of a surface can also be expressed in terms of the causal character of the normal vector of the surface by the following lemma.

Lemma 2.1. [8] *A surface in Minkowski 3-space is spacelike, timelike or lightlike if and only if at every point of the surface there exists a normal which is timelike, spacelike or lightlike, respectively.*

Definition 2.1. [14] *Let v and w be two spacelike vectors. Then, there exists a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = \|v\|\|w\| \cos\theta$.*

Definition 2.2. [14] *Let v be a spacelike vector and w be a timelike vector in $\mathbb{E}^3_1$. Then, there exists a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = \|v\|\|w\| \sinh\theta$.*

Definition 2.3. [12] *Let v and w be two timelike vectors in the same time cone of $\mathbb{E}^3_1$, i.e. $\langle v,w\rangle < 0$. Then, there exists a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = -\|v\|\|w\| \cosh\theta$.*

Lagrange's Identity: For any vectors $\eta, \xi \in \mathbb{E}^3_1$, we have $\langle \eta\times\xi, \eta\times\xi\rangle = -\langle\eta,\eta\rangle\langle\xi,\xi\rangle + \langle\eta,\xi\rangle^2$.

3. Framed curves in Minkowski 3-space

In this section we define the Frenet type formula for the framed curves and give existence and uniqueness theorem of the framed curves in $\mathbb{E}^3_1$.

Definition 3.1. [9] *Let $\gamma : I \to \mathbb{E}^3_1$ be a curve in $\mathbb{E}^3_1$. Then the map $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}^3_1 \times \Theta$ is called a spacelike framed curve if*

$$\langle \gamma'(t), \vartheta_1(t)\rangle = 0, \ \langle \gamma'(t), \vartheta_2(t)\rangle = 0, \forall t \in I,$$

such that $\rho(t) = \vartheta_1(t) \times \vartheta_2(t)$ is an arbitrary spacelike vector field, where

$\Theta = \{(u, v) \in S_1^2 \times H_0^2 | \langle u, v\rangle = 0\}$ *or* $\Theta = \{(u, v) \in H_0^2 \times S_1^2 | \langle u, v\rangle = 0\}$.

Definition 3.2. [9] *Let $\gamma : I \to \mathbb{E}_1^3$ be an arbitrary curve in $\mathbb{E}_1^3$. Then the map $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ is called a timelike framed curve if*

$$\langle \gamma'(t), \vartheta_1(t)\rangle = 0,\ \langle \gamma'(t), \vartheta_2(t)\rangle = 0, \forall t \in I,$$

such that $\rho(t) = \vartheta_1(t) \times \vartheta_2(t)$ is a timelike vector field, where

$\Theta = \{(u, v) \in S_1^2 \times S_1^2 | \langle u, v\rangle = 0\}$.

Definition 3.3. *Let $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar\gamma, \bar\vartheta_1, \bar\vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ are framed curves. We say that γ and $\bar\gamma$ have the same causal character of the moving frame if the vector triplets $\{\vartheta_1, \vartheta_2, \rho\}$ and $\{\bar\vartheta_1, \bar\vartheta_2, \bar\rho\}$ have the same causal characters, respectively.*

3.1. **Frenet-Serret type formula for framed curves.** Let $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ be an spacelike framed curve and $\rho(t) = \vartheta_1(t) \times \vartheta_2(t)$. The Frenet-Serret type formula is given by

$$\begin{bmatrix} \vartheta_1' \\ \vartheta_2' \\ \rho' \end{bmatrix} = \begin{bmatrix} 0 & -\delta\kappa_1 & \kappa_2 \\ -\delta\kappa_1 & 0 & \kappa_3 \\ -\delta\kappa_2 & \delta\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \rho \end{bmatrix}, \tag{3.2}$$

where $\delta = \langle \vartheta_1, \vartheta_1\rangle = -\langle \vartheta_2, \vartheta_2\rangle$. $\kappa_1 = \langle \vartheta_1', \vartheta_2\rangle, \kappa_2 = \langle \vartheta_1', \rho\rangle, \kappa_3 = \langle \vartheta_2', \rho\rangle$. Moreover we can find a smooth function $\tau(t)$ such that $\gamma'(t) = \tau(t)\rho(t)$. We call the functions $(\tau(t), \kappa_1(t), \kappa_2(t), \kappa_3(t))$ the curvature of the framed curve.

Similarly, the Frenet-Serret type formula for a timelike framed curve $(\gamma, \vartheta_1, \vartheta_2)$ can be given by

$$\begin{bmatrix} \vartheta_1' \\ \vartheta_2' \\ \rho' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \rho \end{bmatrix}, \tag{3.3}$$

where $\kappa_1 = \langle \vartheta_1', \vartheta_2\rangle, \kappa_2 = \langle \vartheta_1', \rho\rangle, \kappa_3 = \langle \vartheta_2', \rho\rangle$.

3.2. **Existence and uniqueness of the framed curves in $\mathbb{E}_1^3$.**

Theorem 3.1. *Let $(\tau(t), \kappa_1(t), \kappa_2(t), \kappa_3(t)) : I \to \mathbb{R}^4$ be a smooth map. Then there exist framed curves $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ with three different causality whose curvatures are $(\tau(t), \kappa_1(t), \kappa_2(t), \kappa_3(t))$.*

Proof. Let $t_0 \in I$ and let $\{e_1, e_2, e_3\}$ be an pseudo orthonormal basis for $\mathbb{E}^3_1$. First we suppose that e_3 is a timelike vector and the basis is positively oriented. We need to solve the following ODE system

$$\vartheta_1' = \kappa_1\vartheta_2 + \kappa_2\rho, \tag{3.4}$$
$$\vartheta_2' = -\kappa_1\vartheta_1 + \kappa_3\rho,$$
$$\rho' = \kappa_2\vartheta_1 + \kappa_3\vartheta_2,$$

with initial conditions, $\vartheta_1(t_0) = e_1, \vartheta_2(t_0) = e_2, \rho(t_0) = e_3$. Then by existence and uniqueness of the solution of a system of ODE, we get $\{\vartheta_1, \vartheta_2, \rho\}$ to be the unique solution and define

$$\gamma(t) = \int_{t_0}^{t} \tau(s)\rho(s)ds. \tag{3.5}$$

Then we have to prove that the framed curve $(\gamma(t), \vartheta_1(t), \vartheta_2(t))$ is a timelike curve with curvature functions $(\tau, \kappa_1, \kappa_2, \kappa_3)$. We first show that the moving frame $\{\vartheta_1(t), \vartheta_2(t), \rho(t)\}$ is an pseudo orthonormal basis of $\mathbb{E}^3_1$ with the same causal properties as of the initial basis $\{e_1, e_2, e_3\}$. Consider the ODE system,

$$\langle\vartheta_1, \vartheta_1\rangle' = 2\kappa_1\langle\vartheta_1, \vartheta_2\rangle + 2\kappa_2\langle\rho, \vartheta_1\rangle,$$
$$\langle\vartheta_2, \vartheta_2\rangle' = -2\kappa_1\langle\vartheta_1, \vartheta_2\rangle + 2\kappa_3\langle\rho, \vartheta_2\rangle,$$
$$\langle\rho, \rho\rangle' = 2\kappa_2\langle\vartheta_1, \rho\rangle + 2\kappa_3\langle\rho, \vartheta_2\rangle,$$
$$\langle\vartheta_1, \vartheta_2\rangle' = \kappa_1\langle\vartheta_2, \vartheta_2\rangle + \kappa_2\langle\rho, \vartheta_2\rangle - \kappa_1\langle\vartheta_1, \vartheta_1\rangle + \kappa_3\langle\rho, \vartheta_1\rangle,$$
$$\langle\vartheta_1, \rho\rangle' = \kappa_1\langle\vartheta_2, \rho\rangle + \kappa_2\langle\rho, \rho\rangle + \kappa_2\langle\vartheta_1, \vartheta_1\rangle + \kappa_3\langle\vartheta_2, \vartheta_1\rangle,$$
$$\langle\vartheta_2, \rho\rangle' = -\kappa_1\langle\vartheta_1, \rho\rangle + \kappa_3\langle\rho, \rho\rangle + \kappa_2\langle\vartheta_1, \vartheta_2\rangle + \kappa_3\langle\vartheta_2, \vartheta_2\rangle,$$

with initial conditions $\langle\vartheta_1, \vartheta_1\rangle = 1, \langle\vartheta_2, \vartheta_2\rangle = 1, \langle\rho, \rho\rangle = -1, \langle\vartheta_1, \vartheta_2\rangle = 0, \langle\vartheta_1, \rho\rangle = 0, \langle\vartheta_2, \rho\rangle = 0$. On the other hand, the constant functions $f_1(t) = 1, f_2(t) = 1, f_3(t) = -1, f_4(t) = 0, f_5(t) = 0, f_6(t) = 0$ satisfy the same ODE system and initial conditions, so by uniqueness of the solution,

$$-\langle\rho, \rho\rangle = \langle\vartheta_1, \vartheta_1\rangle = \langle\vartheta_2, \vartheta_2\rangle = 1,\ \langle\vartheta_1, \vartheta_2\rangle = \langle\vartheta_1, \rho\rangle = \langle\vartheta_2, \rho\rangle = 0.$$

This implies that $\{\vartheta_1, \vartheta_2, \rho\}$ is a pseudo orthonormal basis of $\mathbb{E}^3_1$. From (3.4), $\gamma'(t) = \tau(t)\rho(t)$, and hence $\langle\gamma', \gamma'\rangle = \tau^2\langle\rho, \rho\rangle = -\tau^2 < 0$, considering $\tau \neq 0$, this implies that γ is a timelike framed curve with curvatures $(\tau, \kappa_1, \kappa_2, \kappa_3)$.

Similarly, we can show that $(\gamma, \vartheta_1, \vartheta_2)$ is a spacelike framed curve with the spacelike vector ϑ_1 if e_2 is timelike, and is a spacelike framed curve with the timelike vector ϑ_1 if e_1 is timelike. □

Proposition 3.1. [2] *For any vectors $a, b \in \mathbb{E}_1^3$ and an isometry $M \in SO_1(3)$, we have*

$$\langle a, b\rangle = \langle Ma, Mb\rangle, \tag{3.6}$$
$$a \times b = Ma \times Mb.$$

Definition 3.4. [9] *Let $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2) : I \to \mathbb{E}_1^3 \times \Theta$ be framed curves of same causal character. We say that $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2)$ are congruent as framed curves through a Lorentzian motion if there exists a matrix $M \in SO_1(3)$ and a constant vector $c \in \mathbb{E}_1^3$ such that*

$$\bar{\gamma}(t) = M(\gamma(t)) + c, \tag{3.7}$$
$$\bar{\vartheta}_i(t) = M(\vartheta_i(t)),$$

for all $t \in I$, where the matrix M satisfies $M^T G M = G$, $Det(M) = 1, G = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Lemma 3.1. [9] *Let the framed curves $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2)$ be congruent. Then their curvatures coincide, i.e. the curvatures $(\tau, \kappa_1, \kappa_2, \kappa_3)$ are invariant under a Lorentzian motion.*

Theorem 3.2. *Let $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2) : I \to \mathbb{E}_1^3 \times \Theta$ be framed curves that have the same causal character of the moving frames. If they have the same corresponding curvatures then they are congruent as framed curves through a Lorentzian motion.*

Proof. Let $t_0 \in I$ and consider the isometry $A \in SO_1(3)$ such that $\bar{\vartheta}_i(t_0) = A\vartheta_i(t_0)$, $\bar{\rho}(t_0) = A\rho(t_0)$. If $c = \bar{\gamma}(t_0) - A \circ \gamma(t_0)$, define the rigid motion $Mx = Ax + c$. We know that by above lemma 3.6, that the framed curve $(M \circ \gamma, A\vartheta_1, A\vartheta_2)$ satisfies the same ODE system as $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2)$. As the initial conditions coincide, then by uniqueness of ODE system,

$$\bar{\gamma}(t) = M \circ \gamma(t),$$
$$\bar{\vartheta}_i(t) = A\vartheta_i(t), \ i = 1, 2,$$

which completes the proof. □

4. Translation Framed surfaces in $\mathbb{E}^3_1$

Definition 4.1. *A smooth map* $(\sigma, \xi, \eta) : \Omega \subset \mathbb{E}^2 \to \mathbb{E}^3_1 \times \Theta$ *is said to be a spacelike framed surface if the following conditions hold*

$$\sigma_s(s,t).\xi(s,t) = 0,\ \sigma_t(s,t).\xi(s,t) = 0, \forall (s,t) \in \Omega, \tag{4.8}$$

where $\Theta = \{(u,v) \in H^2_0 \times S^2_1 | u.v = 0\}$.
Also, we say that the map $(\sigma, \xi, \eta) : \Omega \subset \mathbb{E}^2 \to \mathbb{E}^3_1 \times \Theta$ *is a timelike framed surface if condition* (4.8) *holds with* $\Theta = \{(u,v) \in S^2_1 \times S^2_1 | u.v = 0\}$ *or* $\Theta = \{(u,v) \in S^2_1 \times H^2_0 | u.v = 0\}$.
For a framed surface (σ, ξ, η), *the map* $(\xi, \eta) : \Omega \to \Theta$, *is a moving frame while* $\sigma : \Omega \to \mathbb{E}^3_1$ *is called the framed base surface.*

4.1. **Basic invariants of a framed surface.** Let's define $\zeta(s,t) = \xi(s,t) \times \eta(s,t)$, then with respect to the moving frame $\{\xi(s,t), \eta(s,t), \zeta(s,t)\}$ along $\sigma(s,t)$, the basic invariants are defined as follows

Case(i):- For the spacelike surface, ξ is a timelike vector and η, ζ are spacelike vectors. Then

$$\begin{bmatrix} \sigma_s \\ \sigma_t \end{bmatrix} = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix}, \tag{4.9}$$

where $c_1 = \sigma_s.\eta$, $c_2 = \sigma_t.\eta$, $d_1 = \sigma_s.\zeta$, $d_2 = \sigma_t.\zeta$.

$$\begin{bmatrix} \xi_s \\ \eta_s \\ \zeta_s \end{bmatrix} = \begin{bmatrix} 0 & l_1 & m_1 \\ l_1 & 0 & n_1 \\ m_1 & -n_1 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{4.10}$$

$$\begin{bmatrix} \xi_t \\ \eta_t \\ \zeta_t \end{bmatrix} = \begin{bmatrix} 0 & l_2 & m_2 \\ l_2 & 0 & n_2 \\ m_2 & -n_2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix},$$

where $l_1 = \xi_s.\eta$, $m_1 = \xi_s.\zeta$, $n_1 = \eta_s.\zeta$ and $l_2 = \xi_t.\eta$, $m_2 = \xi_t.\zeta$, $n_2 = \eta_t.\zeta$.
We call the smooth functions $c_i, d_i, l_i, m_i, n_i : \Omega \to R$, i = 1, 2 the basic invariants of the framed surface. Let the above matrices be denoted by $\Lambda, \Delta_1, \Delta_2$, respectively, as follows

$$\Lambda = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & l_1 & m_1 \\ l_1 & 0 & n_1 \\ m_1 & -n_1 & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0 & l_2 & m_2 \\ l_2 & 0 & n_2 \\ m_2 & -n_2 & 0 \end{bmatrix}.$$

Then, using the integrability condition $\sigma_{st} = \sigma_{ts}$ and $\Delta_{2,s} - \Delta_{1,t} = \Delta_1\Delta_2 - \Delta_2\Delta_1$, the basic invariants satisfy the following conditions:

$$\begin{cases} c_{1,t} - d_1 g_2 = c_{2,s} - d_2 n_1, \\ d_{1,t} - c_2 g_1 = d_{2,s} - c_1 n_2, \\ c_1 e_2 + d_1 f_2 = c_2 e_1 + d_2 m_1. \end{cases} \tag{4.11}$$

$$\begin{cases} l_{1,t} - m_1 n_2 = l_{2,s} - m_2 n_1, \\ m_{1,t} - l_2 n_1 = m_{2,s} - l_1 n_2, \\ n_{1,t} + l_1 m_2 = n_{2,s} + l_2 m_1. \end{cases} \tag{4.12}$$

Case(ii):- For the timelike surface, ξ is a spacelike vector and one of the vectors η or ζ is a timelike vector and other is spacelike. So let $\langle \eta, \eta \rangle = \delta = -\langle \zeta, \zeta \rangle$, where $\delta = \pm 1$, accordingly. Then

$$\begin{bmatrix} \sigma_s \\ \sigma_t \end{bmatrix} = \delta \begin{bmatrix} c_1 & -d_1 \\ c_2 & -d_2 \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix}, \tag{4.13}$$

where $c_1 = \sigma_s.\eta$, $c_2 = \sigma_t.\eta$, $d_1 = \sigma_s.\zeta$, $d_2 = \sigma_t.\zeta$.

$$\begin{bmatrix} \xi_s \\ \eta_s \\ \zeta_s \end{bmatrix} = \delta \begin{bmatrix} 0 & l_1 & -m_1 \\ -\delta l_1 & 0 & -n_1 \\ -\delta m_1 & -n_1 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{4.14}$$

$$\begin{bmatrix} \xi_t \\ \eta_t \\ \zeta_t \end{bmatrix} = \delta \begin{bmatrix} 0 & l_2 & -m_2 \\ -\delta l_2 & 0 & -n_2 \\ -\delta m_2 & -n_2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix},$$

where $l_1 = \xi_s.\eta$, $m_1 = \xi_s.\zeta$, $n_1 = \eta_s.\zeta$ and $l_2 = \xi_t.\eta$, $m_2 = \xi_t.\zeta$, $n_2 = \eta_t.\zeta$.

In particular, if we assume that the vector field η is timelike, then the basic invariants are given by

$$\Lambda = \begin{bmatrix} -c_1 & d_1 \\ -c_2 & d_2 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & -l_1 & m_1 \\ -l_1 & 0 & n_1 \\ -m_1 & n_1 & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0 & -l_2 & m_2 \\ -l_2 & 0 & n_2 \\ -m_2 & n_2 & 0 \end{bmatrix}.$$

Again using the integrability conditions, the basic invariants satisfy the following conditions:

$$\begin{cases} c_{1,t} - d_1 n_2 = c_{2,s} - d_2 n_1, \\ d_{1,t} + c_2 n_1 = d_{2,s} + c_1 n_2, \\ c_1 l_2 - b_1 m_2 = c_2 l_1 - d_2 m_1. \end{cases} \tag{4.15}$$

$$\begin{cases} l_{1,t} - m_1 n_2 = l_{2,s} - m_2 n_1, \\ m_{1,t} + l_2 n_1 = m_{2,s} + l_1 n_2, \\ n_{1,t} - l_1 m_2 = n_{2,s} - l_2 m_1. \end{cases} \tag{4.16}$$

4.2. **Existence and Uniqueness of framed surfaces in $\mathbb{E}^3_1$.**

Theorem 4.1. *For arbitrary given smooth functions $c_i, d_i, l_i, m_i, n_i : \Omega \to \mathbb{R}, i = 1, 2,$ defined on a simply connected domain Ω, satisfying the integrability conditions* (4.11) *and* (4.12) *(respectively,* (4.15) *and* (4.16)*), there exists a spacelike (respectively, timelike) framed surface $(\sigma, \xi, \eta) : \Omega \to \mathbb{E}^3_1 \times \Theta$ such that c_i, d_i, l_i, m_i, n_i are the basic invariants of the surface.*

Proof. By the integrability condition (4.12) (respectively, (4.16)), there exists a pseudo orthonormal frame $\{\xi, \eta, \zeta\}$ such that it satisfy ODE system (4.10) (respectively, (4.14)). Further, by the integrability condition (4.11) and (4.15), there exists a smooth map $\sigma : \Omega \to \mathbb{E}^3_1$ which satisfies the condition (4.9) and (4.13). Thus, we get a spacelike (respectively, timelike) framed surface (σ, ξ, η) with basic invariants $(\Lambda, \Delta_1, \Delta_2)$. □

Theorem 4.2. *Let (σ, ξ, η) and $(\bar{\sigma}, \bar{\xi}, \bar{\eta}) : \Omega \to \mathbb{E}^3_1 \times \Theta$ be framed surfaces of same causal character with basic invariants $(\Lambda, \Delta_1, \Delta_2)$ and $(\bar{\Lambda}, \bar{\Delta}_1, \bar{\Delta}_2)$, respectively. Then (σ, ξ, η) and $(\bar{\sigma}, \bar{\xi}, \bar{\eta})$ are congruent as framed surfaces if and only if the basic invariants coincide.*

Proof. Let $(s_0, t_0) \in U_0$ and consider the isometry $A \in O_1(3)$, such that $\bar{\xi}(s_0, t_0) = A \circ \xi(s_0, t_0), \bar{\eta}(s_0, t_0) = A \circ \eta(s_0, t_0)$ and $\bar{\zeta}(s_0, t_0) = A \circ \zeta(s_0, t_0)$. If $c = \bar{\sigma}(s_0, t_0) - A \circ \sigma(s_0, t_0)$, define the rigid motion $Mx = Ax + c$. Using the proposition 3.1, we see that the framed surface $(M \circ \sigma, A \circ \xi, A \circ \eta)$ and $(\bar{\sigma}, \bar{\xi}, \bar{\eta})$ both satisfy the same linear system of differential equations (4.13) and (4.14), i.e., basic invariants coincide. Now since initial conditions are same, by uniqueness theorem of system of ordinary differential equations, we find that $M \circ \sigma = \bar{\sigma}, A \circ \xi = \bar{\xi}, A \circ \eta = \bar{\eta}, A \circ \zeta = \bar{\zeta}$. Conversely, If (σ, ξ, η) and $(\bar{\sigma}, \bar{\xi}, \bar{\eta})$ are congruent then $M \circ \sigma = \bar{\sigma}, A \circ \xi = \bar{\xi}, A \circ \eta = \bar{\eta}, A \circ \zeta = \bar{\zeta}$, then again using proposition 3.1, we find that both framed surfaces have common basic invariants. □

4.3. **Curvatures of a Framed surface in $\mathbb{E}_1^3$.** We define curvatures of a framed surface $(\sigma, \xi, \eta) : \Omega \to \mathbb{E}_1^3 \times \Theta$ using the moving frame $\{\xi, \eta, \zeta = \xi \times \eta\}$ instead of $\{\sigma_s, \sigma_t, \xi\}$ as at singular points it may not be well defined. So first we obtain the matrix associated with the Weingarten map $W : TM \to TM$ with respect to the frame $\{\xi, \eta, \zeta = \xi \times \eta\}$ and then define the curvatures as determinant and trace of the map, where $TM = span\{\eta, \zeta\}$. Thus,

$$W(\eta) = -\eta\xi, \; W(\zeta) = -\zeta\xi, \tag{4.17}$$

where $\eta\xi$ and $\zeta\xi$ are the derivatives of the unit normal ξ with respect to the vector fields η and ζ, respectively. By using equation (4.9), we get

$$\begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} d_2 & -d_1 \\ -c_2 & c_1 \end{bmatrix} \begin{bmatrix} \sigma_s \\ \sigma_t \end{bmatrix},$$

where $\lambda = \det \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$.

$$\begin{aligned} W(\eta) &= -\eta\xi = -\frac{1}{\lambda}(d_2\sigma_s - d_1\sigma_t)\xi = -\frac{1}{\lambda}(d_2\xi_s - d_1\xi_t), \\ W(\zeta) &= -\zeta\xi = -\frac{1}{\lambda}(-c_2\sigma_s + c_1\sigma_t)\xi = -\frac{1}{\lambda}(-c_2\xi_s + c_1\xi_t). \end{aligned}$$

Also using (4.10), we get

$$\begin{aligned} W(\eta) &= -\frac{1}{\lambda}((d_2 l_1 - d_1 l_2)\eta + (m_1 d_2 - m_2 d_1)\zeta), \\ W(\zeta) &= -\frac{1}{\lambda}((c_1 l_2 - c_2 l_1)\eta + (c_1 m_2 - c_2 m_1)\zeta). \end{aligned}$$

Thus, we get the Weingarten matrix as follows

$$W = -\frac{1}{\lambda} \begin{bmatrix} l_1 d_2 - l_2 d_1 & c_1 l_2 - c_2 l_1 \\ m_1 d_2 - m_2 d_1 & c_1 m_2 - c_2 m_1 \end{bmatrix}.$$

Now, we define $\mu_K = \lambda.\text{det}W$ and $\mu_H = \lambda.\frac{1}{2}\text{trace}(W)$. By direct calculation we obtain

$$\lambda = det \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}, \; \mu_K = det \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}, \tag{4.18}$$

$$\mu_H = -\frac{1}{2}\{det \begin{bmatrix} c_1 & m_1 \\ c_2 & m_2 \end{bmatrix} - det \begin{bmatrix} d_1 & l_1 \\ d_2 & l_2 \end{bmatrix}\}. \tag{4.19}$$

Where $\kappa_f = (\lambda, \mu_K, \mu_H)$ is the curvature of a spacelike framed surface. Similarly we find the curvature of a timelike framed surface as follows

$$\lambda = -det \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}, \ \mu_K = -det \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}, \tag{4.20}$$

$$\mu_H = -\frac{\delta}{2}\{det \begin{bmatrix} c_1 & -m_1 \\ c_2 & m_2 \end{bmatrix} - det \begin{bmatrix} d_1 & -l_1 \\ d_2 & l_2 \end{bmatrix}\}, \tag{4.21}$$

where $\delta = \langle \eta, \eta \rangle$.

4.4. **Translation framed surface generated by framed curves in $\mathbb{E}^3_1$.** Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be framed curves with the curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$ in $\mathbb{E}^3_1$. Let $\sigma : I \times \bar{I} :\to \mathbb{E}^3_1$ be the translation surface parametrized as $\sigma(s,t) = \gamma(s) + \bar{\gamma}(t)$.

Proposition 4.1. [5] *Let $(\sigma, \nu_1^s, \nu_2^s) : \Omega \to \mathbb{E}^3_1 \times \Theta$ be a one parameter family of curves with respect to s and $(\sigma, \nu_1^t, \nu_2^t) : \Omega \to \mathbb{E}^3_1 \times \Theta$ be a one parameter family of curves with respect to t. If $\rho^s = \nu_1^s \times \nu_2^s$ and $\rho^t = \nu_1^t \times \nu_2^t$ are linearly independent for each $(s,t) \in \Omega$, then (σ, ξ, η) is a framed surface for some smooth mapping $(\xi, \eta) : \Omega \to \Theta$.*

For a translation surface $\sigma(s,t) = \gamma(s) + \bar{\gamma}(t)$ defined as above, we have (σ, ν_1, ν_2) and $(\sigma, \bar{\nu}_1, \bar{\nu}_2)$ as one parameter family of curves on the translation surface with respect to s and t, respectively. We consider a smooth map $(\xi, \eta) : \Omega \to \Theta$ defined by $\xi(s,t) = \frac{\rho(s) \times \bar{\rho}(t)}{\|\rho(s) \times \bar{\rho}(t)\|}$ and $\eta(s,t) = \rho(s)$, where $\rho = \nu_1 \times \nu_2$ and $\bar{\rho} = \bar{\nu}_1 \times \bar{\nu}_2$ such that the map $(\sigma, \xi, \eta) : \Omega \to E^3_1 \times \Theta$ is a framed surface and σ is a framed base surface by the Proposition 4.1. Considering the above construction we have the following corollary.

Corollary 4.1. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be framed curves in Minkowski 3-space such that $\rho(s)$ and $\bar{\rho}(t)$ are linearly independent for all $(s,t) \in I \times \bar{I}$, then $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}^3_1 \times \Theta$, defined by $\sigma(s,t) = \gamma(s) + \bar{\gamma}(t)$, $\xi(s,t) = \frac{\rho(s) \times \bar{\rho}(t)}{\|\rho(s) \times \bar{\rho}(t)\|}$ and $\eta(s,t) = \rho(s)$, is a translation framed surface.*

Theorem 4.3. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be timelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively in $\mathbb{E}^3_1$. Then the basic invariants of the timelike translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}^3_1 \times \Theta$ are obtained* *as follows*

$$
\begin{aligned}
c_1(s,t) &= -\tau(s),\\
d_1(s,t) &= 0,\\
c_2(s,t) &= \bar{\tau}(t)\rho(s).\bar{\rho}(t),\\
d_2(s,t) &= \bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2-1},\\
l_1(s,t) &= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t)-\kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
m_1(s,t) &= \frac{\rho(s).\bar{\rho}(t)}{(\rho(s).\bar{\rho}(t))^2-1}(\kappa_2(s)\nu_2(s).\bar{\rho}(t)-\kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
n_1(s,t) &= \frac{-1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t)+\kappa_3(s)\nu_2(s).\bar{\rho}(t)),\\
l_2(s,t) &= 0,\\
m_2(s,t) &= \frac{-1}{(\rho(s).\bar{\rho}(t))^2-1}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s)-\bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)),\\
n_2(s,t) &= 0,
\end{aligned}
$$

where . denotes semi-Euclidean or Lorentzian scalar product.

Proof. Since the framed curves $(\gamma(s),\nu_1(s),\nu_2(s))$ and $(\bar{\gamma}(s),\bar{\nu}_1(s),\bar{\nu}_2(s))$ are timelike, by construction $\eta(s,t)=\rho(s)$ is a timelike vector field which belongs to the tangent space of the surface σ, therefore it is a timelike surface and furthermore ξ and ζ are spacelike vector. Now by using the Lagrange's identity $\langle u\times v, u\times v\rangle = -\langle u,u\rangle\langle v,v\rangle+\langle u,v\rangle^2$ and vector triple product $(u\times v)\times w = \langle v,w\rangle u-\langle u,w\rangle v$ for Minkowski space, we have

$$\|\rho(s)\times\bar{\rho}(t)\| = \sqrt{\epsilon(-\langle\rho(s),\rho(s)\rangle\langle\bar{\rho}(t),\bar{\rho}(t)\rangle+\langle\rho(s),\bar{\rho}(t)\rangle^2)} = \sqrt{\epsilon(-1+\langle\rho(s),\bar{\rho}(t)\rangle^2)},$$

where $\epsilon=\langle\xi,\xi\rangle=1$. Since by definition (of angle), $\langle\rho(s),\bar{\rho}(t)\rangle=-\cosh\theta$, therefore $-1+\langle\rho(s),\bar{\rho}(t)\rangle^2 = -1+\cosh^2\theta=\sinh^2\theta\geq 0$. Also since ρ and $\bar{\rho}$ are linearly independent, $\rho.\bar{\rho}\neq 1$ therefore $\sinh^2\theta>0$. Thus, we have

$$
\begin{aligned}
c_1(s,t) &= \sigma_s(s,t).\eta(s,t) = \tau(s)\rho(s).\rho(s) = -\tau(s),\\
d_1(s,t) &= \sigma_s(s,t).\zeta(s,t) = \tau(s)\eta(s,t).\zeta(s,t) = 0,\\
c_2(s,t) &= \sigma_t(s,t).\eta(s,t) = \bar{\tau}(t)\bar{\eta}(t).\eta(s) = \bar{\tau}(t)\rho(s).\bar{\rho}(t),
\end{aligned}
$$

$$\begin{aligned}
d_2(s,t) &= \sigma_t(s,t).\zeta(s,t) = \bar{\tau}(t)\bar{\rho}(t).(\xi(s,t)\times\rho(s))\\
&= \bar{\tau}(t)(\rho(s)\times\bar{\rho}(t)).\frac{\rho(s)\times\bar{\rho}(t)}{\|\rho(s)\times\bar{\rho}(t)\|}\\
&= \bar{\tau}(t)\|\rho(s)\times\bar{\rho}(t)\| = \bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2-1},
\end{aligned}$$

Now, by using equation (3.3), we have $\rho_s(s) = \kappa_2(s)\nu_2(s) - \kappa_3(s)\nu_1(s)$. Thus

$$\begin{aligned}
l_1(s,t) &= \xi_s(s,t).\eta(s,t) = \frac{1}{\|\rho(s)\times\bar{\rho}(t)\|}((\rho(s)\times\rho_s(s)).\bar{\rho}(t))\\
&= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\rho(s)\times(-\kappa_2(s)\nu_1(s)-\kappa_3(s)\nu_2(s)).\bar{\rho}(t))\\
&= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t)-\kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
m_1(s,t) &= \xi_s(s,t).\zeta(s,t) = \frac{1}{\|\rho(s)\times\bar{\rho}(t)\|^2}(\rho_s(s)\times\bar{\rho}(t)).(\rho(s)\times\bar{\rho}(t)\times\rho(s))\\
&= \frac{1}{(\rho(s).\bar{\rho}(t))^2-1}(\rho_s(s)\times\bar{\rho}(t)).((\rho(s).\bar{\rho}(t))\rho(s)+\bar{\rho}(t))\\
&= \frac{1}{(\rho(s).\bar{\rho}(t))^2-1}(\rho(s).\bar{\rho}(t))(\rho(s)\times\rho_s(s)).\bar{\rho}(t)\\
&= \frac{\rho(s).\bar{\rho}(t)}{(\rho(s).\bar{\rho}(t))^2-1}(\kappa_2(s)\nu_2(s).\bar{\rho}(t)-\kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
n_1(s,t) &= \eta_s(s,t).\zeta(s,t) = \frac{1}{\|\rho(s)\times\bar{\rho}(t)\|}\rho_s(s).((\rho(s)\times\bar{\rho}(t))\times\rho(s))\\
&= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}\rho_s(s).((\rho(s).\bar{\rho}(t))\rho(s)+\bar{\rho}(t))\\
&= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\rho_s(s).\bar{\rho}(t))\\
&= \frac{-1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t)+\kappa_3(s)\nu_2(s).\bar{\rho}(t)).\\
l_2(s,t) &= \xi_t(s,t).\eta(s,t) = \frac{1}{\|\rho(s)\times\bar{\rho}(t)\|}(\rho(s)\times\bar{\rho}_t(t).\rho(s)) = 0,\\
m_2(s,t) &= \xi_t(s,t).\zeta(s,t) = \frac{1}{(\rho(s).\bar{\rho}(t))^2-1}(\rho(s)\times\bar{\rho}_t(t)).(\rho(s)\times\bar{\rho}(t)\times\rho(s))\\
&= \frac{1}{(\rho(s).\bar{\rho}(t))^2-1}(\bar{\rho}_t(t)\times\bar{\rho}(t)).\rho(s)\\
&= \frac{1}{(\rho(s).\bar{\rho}(t))^2-1}(-\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s)+\bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)),\\
n_2(s,t) &= \eta_t(s,t).\zeta(s,t) = \rho_t(s,t).\zeta(s,t) = 0.\zeta(s,t) = 0.
\end{aligned}$$

□

Corollary 4.2. *The curvature $\kappa_f = (\lambda, \mu_K, \mu_H)$ of the timelike translation framed surface in Theorem 4.4 is given as follows*

$$\lambda = \tau(s)\bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1},$$
$$\mu_K = \frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{((\rho(s).\bar{\rho}(t))^2 - 1)^{3/2}},$$
$$\mu_H = \frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) + \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2((\rho(s).\bar{\rho}(t))^2 - 1)}.$$

Proof. Using (4.20) and (4.21) and Theorem 4.4, we have

$$\lambda(s,t) = \tau(s)\bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1},$$
$$\begin{aligned}\mu_K(s,t) &= -l_1(s,t)m_2(s,t)\\ &= \frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{((\rho(s).\bar{\rho}(t))^2 - 1)^{3/2}}.\end{aligned}$$

$\delta = \langle \eta, \eta \rangle = -1$, so

$$\begin{aligned}\mu_H(s,t) =& \frac{1}{2}\{c_1 m_2 + c_2 m_1 - d_2 l_1\}\\ =& \frac{1}{2}\{\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(\rho(s).\bar{\rho}(t))^2 - 1}\\ &+ \frac{\bar{\tau}(t)(\rho(s).\bar{\rho}(t))^2}{(\rho(s).\bar{\rho}(t))^2 - 1}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\\ &- \bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}\frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\}\\ =& \frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) + \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2((\rho(s).\bar{\rho}(t))^2 - 1)}.\end{aligned}$$

□

Proposition 4.2. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be timelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the xz-plane and $\bar{\gamma}$ is contained in the xy-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$ obtained by the above curves, $\mu_K \equiv 0$ if and only if σ is a generalized cylinder.*

Proof. Let the curve γ be contained in the xz-plane and $\bar{\gamma}$ be contained in the xy-plane. Then we take $\nu_1(s) = (0, 1, 0)$, $\bar{\nu}_1(t) = (0, 0, 1)$ and $\rho(s) = (\rho_1(s), 0, \rho_3(s))$, $\bar{\rho}(t) = (\bar{\rho}_1(t), \bar{\rho}_2(t), 0)$ for some real smooth functions ρ_1, ρ_3, $\bar{\rho}_1$ and $\bar{\rho}_2$, which further gives $\nu_2(s) = \nu_1(s) \times \rho(s) =$

$(-\rho_3(s), 0, -\rho_1(s))$ and $\bar{\nu}_2(t) = \bar{\nu}_1(t) \times \bar{\rho}(t) = (-\bar{\rho}_2(t), -\bar{\rho}_1(t), 0)$. Also since ν_1 and $\bar{\nu}_1$ are fixed vectors, $\nu_1' = 0$ and $\bar{\nu}_1' = 0$ therefore from (3.3), $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Hence

$$\mu_K(s,t) = \frac{\kappa_3(s)\bar{\kappa}_3(t)\rho_3(s)\bar{\rho}_2(t)}{(\rho_1^2(s)\bar{\rho}_1^2(t) - 1)^{3/2}}.$$

Thus $\mu_K \equiv 0$ if and only if one of the functions $\bar{\kappa}_3, \kappa_3, \rho_3, \bar{\rho}_2$ is identically zero on an open interval in I or $\bar{I}$. So, if $\kappa_3 = 0$ or $\rho_3 = 0$ then γ is a part of a timelike straight line, while $\bar{\kappa}_3 = 0$ or $\bar{\rho}_2 = 0$ implies $\bar{\gamma}$ is a part of a timelike straight line. In either case σ is a generalized cylinder. □

Proposition 4.3. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be timelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the xz-plane and $\bar{\gamma}$ is contained in the xy-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$ generated by the framed curves, $\mu_H \equiv 0$ if and only if σ is a point or is a part of the following surface*

$$\sigma(s,t) = \Big(\frac{1}{c}\log\Big|\frac{\cosh(cu(s))}{\sinh(cv(t))}\Big| + B, v(t), u(s)\Big),$$

where B, c are some constants.

Proof. Using the similar constructions $\{\nu_1, \nu_2, \rho\}$ and $\{\bar{\nu}_1, \bar{\nu}_2, \bar{\rho}\}$ as in Proposition 4.2, we get $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Hence

$$\mu_H(s,t) = \frac{-\tau(s)\bar{\kappa}_3(t)\rho_3(s) - \bar{\tau}(t)\kappa_3(s)\bar{\rho}_2(t)}{2(\rho_1^2(s)\bar{\rho}_1^2(t) - 1)}.$$

Now $\mu_H \equiv 0$ if and only if $\tau(s)\bar{\kappa}_3(t)\rho_3(s) + \bar{\tau}(t)\kappa_3(s)\bar{\rho}_2(t) = 0$, or

$$\frac{\tau(s)\rho_3(s)}{\kappa_3(s)} = -\frac{\bar{\tau}(t)\bar{\rho}_2(t)}{\bar{\kappa}_3(t)} = C(constant). \tag{4.22}$$

By definition $\kappa_3(s) = \nu_2'(s).\rho(s) = \rho_{3,s}\rho_1 - \rho_{1,s}\rho_3$ and $\bar{\kappa}_3(t) = \bar{\rho}_{2,t}\bar{\rho}_1 - \bar{\rho}_{1,t}\bar{\rho}_2$, substituting into (4.22) we get,

$$C(\rho_{3,s}\rho_1 - \rho_{1,s}\rho_3) = \tau(s)\rho_3(s),$$

$$C(\bar{\rho}_{2,t}\bar{\rho}_1 - \bar{\rho}_{1,t}\bar{\rho}_2) = -\bar{\tau}(t)\bar{\rho}_2(t).$$

In the case $C = 0$, we have $\tau = 0$ or $\rho_3 = 0$ and $\bar{\tau} = 0$ or $\bar{\rho}_2 = 0$. If $\rho_3 = 0$ and $\bar{\rho}_2 = 0$ then $\kappa_3 = 0 = \bar{\kappa}_3$ which contradicts to the equation (4.22). Thus $\tau = 0$ and $\bar{\tau} = 0$ which implies that σ is a point.

Now in the case $C \neq 0$, replacing $c = \frac{1}{C}$ in the above equations we get,

$$\rho_{3,s}\rho_1 - \rho_{1,s}\rho_3 = c\tau(s)\rho_3(s), \tag{4.23}$$

$$\bar{\rho}_{2,t}\bar{\rho}_1 - \bar{\rho}_{1,t}\bar{\rho}_2 = -c\bar{\tau}(t)\bar{\rho}_2(t). \tag{4.24}$$

Since ρ is a timelike unit vector we take $\rho(s) = (\cosh(\theta(s)), 0, \sinh(\theta(s)))$, therefore $\rho_{1,s} = \theta_s \sinh(\theta)$ and $\rho_{3,s} = \theta_s \cosh(\theta)$. Using equation (4.23), we get

$$\theta_s = c\tau(s)\sinh(\theta(s)),$$
$$\int \frac{1}{\sinh(\theta)} d\theta = c\int \tau(s)ds + b,$$

which gives $e^\theta = \frac{1+Ae^{c\int \tau(s)ds}}{1-Ae^{c\int \tau(s)ds}}$, thus we get $\rho(s) = \big(\frac{1+A^2e^{2c\int \tau(s)ds}}{1-A^2e^{2c\int \tau(s)ds}}, 0, \frac{2Ae^{c\int \tau(s)ds}}{1-Ae^{2c\int \tau(s)ds}}\big)$. Now we calculate $\gamma(s) = \int \tau(s)\rho(s)ds$. Let $\gamma(s) = (\gamma_1(s), 0, \gamma_2(s))$, then we get $\gamma_1(s) = \int \tau(s)\rho_1(s)ds = -\frac{1}{c}\log\left(\frac{1-A^2e^{2c\int \tau(s)ds}}{e^{c\int \tau(s)ds}}\right)$ and $\gamma_3(s) = \int \tau(s)\rho_3(s)ds = \frac{1}{c}\log\left(\frac{1+Ae^{c\int \tau(s)ds}}{1-Ae^{c\int \tau(s)ds}}\right)$.
Let $u(s) = \frac{1}{c}\log\left(\frac{1+Ae^{c\int \tau(s)ds}}{1-Ae^{c\int \tau(s)ds}}\right)$, then γ is given by

$$\gamma(s) = \big(\frac{1}{c}\log\cosh(cu(s)) - \frac{1}{c}\log(2A), 0, u(s)\big).$$

Similarly, by equation (4.24), we obtain

$$\bar{\gamma}(t) = \big(-\frac{1}{c}\log\big|\sinh(cv(t))\big| + \frac{1}{c}\log(2\bar{A}), v(t), 0\big),$$

where $v(t) = -\frac{1}{c}\log\frac{1+\bar{A}e^{-c\int \bar{\tau}(t)dt}}{1-\bar{A}e^{-c\int \bar{\tau}(t)dt}}$. Thus

$$\begin{aligned}\sigma(s,t) &= \gamma(s) + \bar{\gamma}(t)\\ &= \Big(\frac{1}{c}\log\Big|\frac{\cosh(cu(s))}{\sinh(cv(t))}\Big| + B, v(t), u(s)\Big),\end{aligned}$$

where B is a constant. In fig. 1 we have diagram of the surface when $c = 1, B = 0$. □

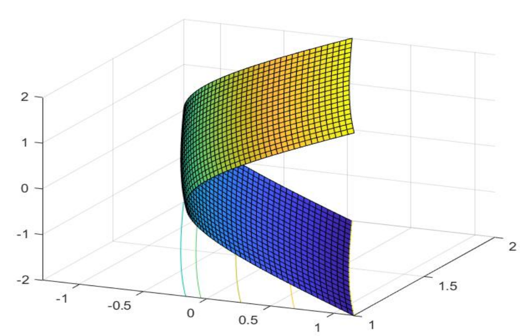

FIGURE 1.

Theorem 4.4. *Let* $(\gamma,\nu_1,\nu_2): I \to \mathbb{E}_1^3 \times \Theta$ *a spacelike and* $(\bar{\gamma},\bar{\nu}_1,\bar{\nu}_2): \bar{I} \to \mathbb{E}_1^3 \times \Theta$ *be a timelike framed curve with curvatures* $(\kappa_1,\kappa_2,\kappa_3,\tau)$ *and* $(\bar{\kappa}_1,\bar{\kappa}_2,\bar{\kappa}_3,\bar{\tau})$*, respectively in* $\mathbb{E}_1^3$*. Then the basic invariants of the timelike translation framed surface* $(\sigma,\xi,\eta): I\times\bar{I} \to \mathbb{E}_1^3\times\Theta$*, are obtained as follows*

$$
\begin{aligned}
c_1(s,t) &= \tau(s),\\
d_1(s,t) &= 0,\\
c_2(s,t) &= \bar{\tau}(t)\rho(s).\bar{\rho}(t),\\
d_2(s,t) &= \bar{\tau}(t)\sqrt{1+(\rho(s).\bar{\rho}(t))^2},\\
l_1(s,t) &= \frac{1}{\sqrt{1+(\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
m_1(s,t) &= \frac{\rho(s).\bar{\rho}(t)}{1+(\rho(s).\bar{\rho}(t))^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
n_1(s,t) &= \frac{\delta}{\sqrt{1+(\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t) - \kappa_3(s)\nu_2(s).\bar{\rho}(t)),\\
l_2(s,t) &= 0,\\
m_2(s,t) &= \frac{1}{1+(\rho(s).\bar{\rho}(t))^2}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)),\\
n_2(s,t) &= 0,
\end{aligned}
$$

where $\delta = \langle\nu_1,\nu_1\rangle = \pm 1$*.*

Proof. Since the framed curves $(\gamma(s),\nu_1(s),\nu_2(s))$ is spacelike and $(\bar{\gamma}(s),\bar{\nu}_1(s),\bar{\nu}_2(s))$ is timelike, by construction $\sigma_t(s,t) = \bar{\gamma}'(t)$ is a timelike vector field which belongs to the tangent space of the surface σ, hence σ is a timelike surface and ξ and η are spacelike vectors, ζ is a timelike vector. Thus, we have

$$\|\rho(s)\times\bar{\rho}(t)\| = \sqrt{\epsilon(\langle\rho(s),\rho(s)\rangle\langle\bar{\rho}(t),\bar{\rho}(t)\rangle + \langle\rho(s),\bar{\rho}(t)\rangle^2)} = \sqrt{\epsilon(1+\langle\rho(s),\bar{\rho}(t)\rangle^2)},$$

we have $\langle\rho(s),\bar{\rho}(t)\rangle = \sinh\theta$, $1+\langle\rho(s),\bar{\rho}(t)\rangle^2 = 1+\sinh^2\theta = \cosh^2\theta > 0$, hence $\epsilon = \langle\xi,\xi\rangle = 1$, and $\|\rho(s)\times\bar{\rho}(t)\| = \sqrt{1+\langle\rho(s),\bar{\rho}(t)\rangle^2}$, we have

$$
\begin{aligned}
c_1(s,t) &= \sigma_s(s,t).\eta(s,t) = \tau(s)\rho(s).\rho(s) = \tau(s),\\
d_1(s,t) &= \sigma_s(s,t).\zeta(s,t) = \tau(s)\eta(s,t).\zeta(s,t) = 0,\\
c_2(s,t) &= \sigma_t(s,t).\eta(s,t) = \bar{\tau}(t)\bar{\rho}(t).\rho(s) = \bar{\tau}(t)\rho(s).\bar{\rho}(t),\\
d_2(s,t) &= \sigma_t(s,t).\zeta(s,t) = \bar{\tau}(t)\bar{\rho}(t).(\xi(s,t)\times\rho(s))
\end{aligned}
$$

$$= \bar{\tau}(t)(\rho(s) \times \bar{\rho}(t)).\frac{\rho(s) \times \bar{\rho}(t)}{\|\rho(s) \times \bar{\rho}(t)\|}$$

$$= \bar{\tau}(t)\|\rho(s) \times \bar{\rho}(t)\| = \bar{\tau}(t)\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2},$$

By using (3.2), we have $\rho_s(s) = -\delta\kappa_2(s)\nu_1(s) + \delta\kappa_3(s)\nu_2(s)$, hence

$$\begin{aligned}
l_1(s,t) = \xi_s(s,t).\eta(s,t) &= \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}((\rho(s) \times \rho_s(s)).\bar{\rho}(t)) \\
&= \frac{1}{\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}}((\rho(s) \times (-\delta\kappa_2(s)\nu_1(s) + \delta\kappa_3(s)\nu_2(s)).\bar{\rho}(t)) \\
&= \frac{\delta}{\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}}(-\kappa_2(s)(-\delta\nu_2(s)) + \kappa_3(s)(-\delta\nu_1(s))).\bar{\rho}(t) \\
&= \frac{1}{\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)), \\
m_1(s,t) = \xi_s(s,t).\zeta(s,t) &= \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|^2}(\rho_s(s) \times \bar{\rho}(t)).(\rho(s) \times \bar{\rho}(t) \times \rho(s)) \\
&= \frac{1}{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}(\rho_s(s) \times \bar{\rho}(t)).((\rho(s).\bar{\rho}(t))\rho(s) + \bar{\rho}(t)) \\
&= \frac{1}{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}(\rho(s).\bar{\rho}(t))(\rho(s) \times \rho_s(s)).\bar{\rho}(t) \\
&= \frac{\rho(s).\bar{\rho}(t)}{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)), \\
n_1(s,t) = \eta_s(s,t).\zeta(s,t) &= \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}\rho_s(s).((\rho(s) \times \bar{\rho}(t)) \times \rho(s)) \\
&= \frac{1}{\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}}\rho_s(s).((\rho(s).\bar{\rho}(t))\rho(s) - \bar{\rho}(t)) \\
&= \frac{-1}{\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}}(\rho_s(s).\bar{\rho}(t)) \\
&= \frac{\delta}{\sqrt{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t) - \kappa_3(s)\nu_2(s).\bar{\rho}(t)), \\
l_2(s,t) = \xi_t(s,t).\eta(s,t) &= \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}(\rho(s) \times \bar{\rho}_t(t).\rho(s)) = 0, \\
m_2(s,t) = \xi_t(s,t).\zeta(s,t) &= \frac{1}{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}(\rho(s) \times \bar{\rho}_t(t)).(\rho(s) \times \bar{\rho}(t) \times \rho(s)) \\
&= \frac{-1}{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}(\bar{\rho}_t(t) \times \bar{\rho}(t)).\rho(s) \\
&= \frac{1}{1 + \langle\rho(s), \bar{\rho}(t)\rangle^2}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)), \\
n_2(s,t) = \eta_t(s,t).\zeta(s,t) &= \rho_t(s,t).\zeta(s,t) = 0.\zeta(s,t) = 0.
\end{aligned}$$

□

Corollary 4.3. *The curvature $\kappa_f = (\lambda, \mu_K, \mu_H)$ of the timelike translation framed surface given in Theorem 4.5 is given as follows*

$$\lambda = -\tau(s)\bar{\tau}(t)\sqrt{1 + (\rho(s).\bar{\rho}(t))^2},$$
$$\mu_K = -\frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(1 + (\rho(s).\bar{\rho}(t))^2)^{3/2}},$$
$$\mu_H = -\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) - \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2(1 + (\rho(s).\bar{\rho}(t))^2)}.$$

Proof. Using (4.20) and (4.21) and Theorem 4.5, we have

$$\lambda(s,t) = \tau(s)\bar{\tau}(t)\sqrt{1 + (\rho(s).\bar{\rho}(t))^2},$$
$$\mu_K(s,t) = -l_1(s,t)m_2(s,t)$$
$$= -\frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(1 + (\rho(s).\bar{\rho}(t))^2)^{3/2}}.$$

$\delta = \langle \eta, \eta \rangle = 1$, so

$$\mu_H(s,t) = -\frac{1}{2}\{c_1 m_2 + c_2 m_1 - d_2 l_1\}$$
$$= -\frac{1}{2}\{\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{1 + (\rho(s).\bar{\rho}(t))^2}$$
$$+ \frac{\bar{\tau}(t)(\rho(s).\bar{\rho}(t))^2}{1 + (\rho(s).\bar{\rho}(t))^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))$$
$$- \bar{\tau}(t)\sqrt{1 + (\rho(s).\bar{\rho}(t))^2}\frac{1}{\sqrt{1 + (\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\}$$
$$= -\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) - \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2(1 + (\rho(s).\bar{\rho}(t))^2)}.$$

□

Proposition 4.4. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ be an spacelike and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be a timelike framed curve with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the yz-plane and $\bar{\gamma}$ is contained in the xz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}^3_1 \times \Theta$, $\mu_K \equiv 0$ if and only if σ is a generalized cylinder.*

Proof. We take $\nu_1(s) = (1,0,0)$, $\bar{\nu}_1(t) = (0,1,0)$ and $\rho(s) = (0, \rho_2(s), \rho_3(s))$, $\bar{\rho}(t) = (\bar{\rho}_1(t), 0, \bar{\rho}_3(t))$ for some real smooth functions ρ_2, ρ_3, $\bar{\rho}_1$ and $\bar{\rho}_3$. Then we get $\nu_2(s) = \rho(s) \times \nu_1(s) = (0, \rho_3(s), \rho_2(s))$ and $\bar{\nu}_2(t) = \bar{\nu}_1(t) \times \bar{\rho}(t) = (-\bar{\rho}_3(t), 0, -\bar{\rho}_1(t))$. Since ν_1 and $\bar{\nu}_1$ are fixed vectors, $\nu_1' = 0$ and $\bar{\nu}_1' = 0$ therefore $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Now by following the similar steps to the Proposition 4.2 we get the result. □

Proposition 4.5. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ be an spacelike and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be a timelike framed curve with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the yz-plane and $\bar{\gamma}$ is contained in the xz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$, $\mu_H \equiv 0$ if and only if σ is a point or is a part of the following surface*

$$\sigma(s,t) = \Big(v(t), u(s), \frac{1}{c}\log\Big|\frac{2\csc(cu(s))}{\cosh(cv(t))}\Big|\Big),$$

where c is some constant.

Proof. Working with the same frames $\{\nu_1, \nu_2, \rho\}$ and $\{\bar{\nu}_1, \bar{\nu}_2, \bar{\rho}\}$ as defined in the Proposition 4.4, we get $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Since ρ is an spacelike unit vector and $\bar{\rho}$ is a timelike unit vector so we take $\rho(s) = (0, \cos\theta(s), \sin\theta(s))$ and $\bar{\rho} = (\cosh\theta(t), 0, \sinh\theta(t))$ and by following the similar steps to the Proposition 4.3 we obtain

$$\gamma(s) = \big(0, u(s), \frac{1}{c}\log\left(2\csc\left(cu(s)\right)\right)\big),$$
$$\bar{\gamma}(t) = \big(v(t), 0, -\frac{1}{c}\log\cosh\left(cv(t)\right)\big),$$

where $u(s) = -\frac{1}{c}\log(\tan(\frac{c}{2}\int\bar{\tau}(t)dt + b))$ and $v(t) = \frac{2}{c}\arctan(Ae^{c\int\bar{\tau}(t)dt})$. Thus

$$\begin{aligned} X(s,t) &= \gamma(s) + \bar{\gamma}(t) \\ &= \Big(v(t), u(s), \frac{1}{c}\log\Big|\frac{2\csc(cu(s))}{\cosh(cv(t))}\Big|\Big), \end{aligned}$$

where c is a constant. In fig. 2 we have diagram of the surface when $c = 1$. □

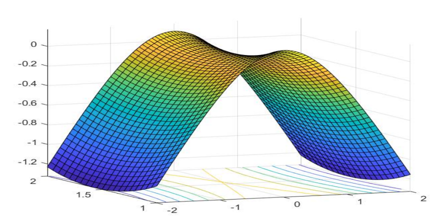

FIGURE 2.

Theorem 4.5. *Let* $(\gamma,\nu_1,\nu_2): I \to \mathbb{E}^3_1 \times \Theta$ *and* $(\bar\gamma,\bar\nu_1,\bar\nu_2): \bar I \to \mathbb{E}^3_1 \times \Theta$ *be spacelike framed curves with curvatures* $(\kappa_1,\kappa_2,\kappa_3,\tau)$ *and* $(\bar\kappa_1,\bar\kappa_2,\bar\kappa_3,\bar\tau)$*, respectively in* $\mathbb{E}^3_1$*. Then the basic invariants of the spacelike translation framed surface* $(\sigma,\xi,\eta): I \times \bar I \to \mathbb{E}^3_1 \times \Theta$*, are obtained as follows*

$$\begin{aligned}
c_1(s,t) &= \tau(s),\\
d_1(s,t) &= 0,\\
c_2(s,t) &= \bar\tau(t)\rho(s).\bar\rho(t),\\
d_2(s,t) &= -\bar\tau(t)\sqrt{1-(\rho(s).\bar\rho(t))^2},\\
l_1(s,t) &= \frac{1}{\sqrt{1-(\rho(s).\bar\rho(t))^2}}(\kappa_2(s)\nu_2(s).\bar\rho(t)-\kappa_3(s)\nu_1(s).\bar\rho(t)),\\
m_1(s,t) &= \frac{\rho(s).\bar\rho(t)}{1-(\rho(s).\bar\rho(t))^2}(\kappa_2(s)\nu_2(s).\bar\rho(t)-\kappa_3(s)\nu_1(s).\bar\rho(t)),\\
n_1(s,t) &= \frac{\delta}{\sqrt{1-(\rho(s).\bar\rho(t))^2}}(\kappa_2(s)\nu_1(s).\bar\rho(t)-\kappa_3(s)\nu_2(s).\bar\rho(t)),\\
l_2(s,t) &= 0,\\
m_2(s,t) &= \frac{1}{1-(\rho(s).\bar\rho(t))^2}(\bar\kappa_2(t)\bar\nu_2(t).\rho(s)-\bar\kappa_3(t)\bar\nu_1(t).\rho(s)),\\
n_2(s,t) &= 0,
\end{aligned}$$

where $\delta = \langle\nu_1,\nu_1\rangle = \pm 1$*.*

Proof. We can prove this theorem using similar steps as the Theorems 4.4, 4.5. □

Corollary 4.4. *The curvature* $\kappa_f = (\lambda,\mu_K,\mu_H)$ *of the spacelike translation framed surface given in Theorem 4.7 is given as follows*

$$\begin{aligned}
\lambda &= -\tau(s)\bar\tau(t)\sqrt{1-(\rho(s).\bar\rho(t))^2},\\
\mu_K &= \frac{(\kappa_2(s)\nu_2(s).\bar\rho(t)-\kappa_3(s)\nu_1(s).\bar\rho(t))(\bar\kappa_2(t)\bar\nu_2(t).\rho(s)-\bar\kappa_3(t)\bar\nu_1(t).\rho(s))}{(1-(\rho(s).\bar\rho(t))^2)^{3/2}},\\
\mu_H &= -\frac{\tau(s)(\bar\kappa_2(t)\bar\nu_2(t).\rho(s)-\bar\kappa_3(t)\bar\nu_1(t).\rho(s))-\bar\tau(t)(\kappa_2(s)\nu_2(s).\bar\rho(t)-\kappa_3(s)\nu_1(s).\bar\rho(t))}{2(1-(\rho(s).\bar\rho(t))^2)}.
\end{aligned}$$

Proof. Proof is similar to the corollaries 4.2, 4.3. □

Proposition 4.6. *Let* $(\gamma,\nu_1,\nu_2): I \to \mathbb{E}^3_1 \times \Theta$ *and* $(\bar\gamma,\bar\nu_1,\bar\nu_2): \bar I \to \mathbb{E}^3_1 \times \Theta$ *be spacelike framed curves with curvatures* $(\kappa_1,\kappa_2,\kappa_3,\tau)$ *and* $(\bar\kappa_1,\bar\kappa_2,\bar\kappa_3,\bar\tau)$*, respectively. Assume that* γ *is contained in the yz-plane and* $\bar\gamma$ *is contained in the xz-plane. Then for the translation framed surface* $(\sigma,\xi,\eta): I \times \bar I \to \mathbb{E}^3_1 \times \Theta$*,* $\mu_K \equiv 0$ *if and only if* σ *is a generalized cylinder.*

Proof. We take $\nu_1(s) = (1, 0, 0)$ and $\bar{\nu}_1(t) = (0, 1, 0)$ then there exist real smooth functions ρ_2, ρ_3, $\bar{\rho}_1$ and $\bar{\rho}_3$ such that $\rho(s) = (0, \rho_2(s), \rho_3(s))$ and $\bar{\rho}(t) = (\bar{\rho}_1(t), 0, \bar{\rho}_3(t))$. Now by definition $\nu_2(s) = \rho(s) \times \nu_1(s) = (0, \rho_3(s), \rho_2(s))$ and $\bar{\nu}_2(t) = \bar{\nu}_1(t) \times \bar{\rho}(t) = (-\bar{\rho}_3(t), 0, -\bar{\rho}_1(t))$ and since ν_1 and $\bar{\nu}_1$ are fixed vectors, $\nu_1' = 0$ and $\bar{\nu}_1' = 0$ therefore $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Now by following the similar steps to the Proposition 4.2, we get the desired result. □

Proposition 4.7. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be spacelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the xz-plane and $\bar{\gamma}$ is contained in the yz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$, $\mu_H \equiv 0$ if and only if σ is a point or is a part of the following surface*

$$\sigma(s,t) = \Big(v(t), u(s), \frac{1}{c} \log \Big| \frac{\sinh(cv(t))}{\sin(cu(s))} \Big| \Big),$$

where c is some constant.

Proof. Working with the frames $\{\nu_1, \nu_2, \rho\}$ and $\{\bar{\nu}_1, \bar{\nu}_2, \bar{\rho}\}$ as defined in the Proposition 4.6, we have $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Since ρ and $\bar{\rho}$ are spacelike unit vectors so we take $\rho(s) = (0, \cos\theta(s), \sin\theta(s))$ and $\bar{\rho}(t) = (\sinh\theta(t), 0, \cosh\theta(t))$ and by following the similar steps to the Proposition 4.3 we obtain

$$\gamma(s) = \big(0, u(s), -\frac{1}{c} \log\left(2\sin\left(cu(s)\right)\right)\big),$$
$$\bar{\gamma}(t) = \big(v(t), 0, \frac{1}{c} \log\left(2\sinh\left(cv(t)\right)\right)\big),$$

where $u(s) = -\frac{1}{c}\log(\tan(\frac{c}{2}\int \bar{\tau}(t)dt + b))$ and $v(t) = \frac{1}{c}\log(\frac{1+\bar{A}e^{c\int \bar{\tau}(t)dt}}{1-\bar{A}e^{c\int \bar{\tau}(t)dt}})$. Thus,

$$\begin{aligned} \sigma(s,t) &= \gamma(s) + \bar{\gamma}(t) \\ &= \Big(v(t), u(s), \frac{1}{c} \log \Big| \frac{\sinh(cv(t))}{\sin(cu(s))} \Big| \Big), \end{aligned}$$

where c is a constant. In fig. 3 we have diagram of the surface when $c = 1$. □

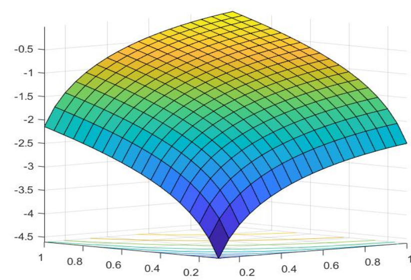

FIGURE 3.

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