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# A NOTE ON $CSI-$SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

SUSHIL KUMAR *, SUMEET KUMAR , AND RAJ KUMAR SRIVASTAVA

Abstract. In this paper, our main objective is to study the notion of Clairaut semi-invariant submersions ($CSI-$ submersions, in short) from Cosymplectic manifolds onto Riemannian manifolds. We investigate some fundamental results pertaining to the geometry of such submersions. We also obtain totally geodesicness conditions for the distributions. Moreover, we provide a non-trivial example of such Riemannian submersion.

**Keywords**: Riemannian submersions, Clairaut semi-invariant submersions, Almost contact metric manifolds.

**2010 Mathematics Subject Classification**: 53C15, 53B20.

## 1. Introduction

Firstly, O' Neill [17] and Gray [9] separately studied the concept of Riemannian submersions between Riemannian manifolds in the 1960s. Using the notion of Riemannian submersions between almost complex manifolds, Watson [34] studied almost Hermitian submersions. Further, the concept of anti-invariant submersion was first defined by Sahin [23] from almost Hermitian manifolds onto Riemannian manifolds. Later, he introduced semi-invariant submersion [25] from almost Hermitian manifolds onto Riemannian manifolds as a generalization of holomorphic submersions and anti-invariant submersion. Further, different kinds of Riemannian submersions on different structures have been studied, such as: slant submersions

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[24], semi-slant submersions [18], conformal semi-slant submersion ([13],[21]), hemi-slant Riemannian submersions [31], conformal hemi-slant submersion [12], quasi-bi-slant submersions [19] (see also [14], [20], [22], [26], [28], [29]) etc.

Presently, the Riemannian submersions have abundant applications in pure mathematics and physics, for example, Kaluza-Klein theory [7], Yang-Mills theory [8], Supergravity and superstring theories [11] etc. C. Altafini [2] commenced using the notion of Riemannian submersions for the modeling and control of redundant robotic chain and proved that Riemannian submersion gives a close relationship between inverse kinematic in robotics and the pull back vectors.

In the theory of surfaces created by rotating the curves, we note that, for any geodesic $c(c : I_1 \subset R \to \mathcal{N}_1$ on $\mathcal{N}_1)$ on the rotating surface $\mathcal{N}_1$, the product $r \sin\Theta$ is constant along geodesic $c$, where $\Theta(s)$ is the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$, called Clairaut's theorem [5]. It means that it is independent of $s$. In 1972, Bishop [5] applied this idea to the Riemannian submersions and introduced the concept of Clairaut submersion. Afterwards, Clairaut submersions have been studied in Spacelike spaces, Timelike and Lorentzian spaces ([16], [32], [33]) and its applications in Static spacetimes [1]. Later on this notion has been generalized in [3] and [16]. Kumar et al., in [15], introduce the notion of Clairaut semi-invariant Riemannian map and Gupta and Singh in [10] initiate the notion of Clairaut semi-invariant submersion from Kähler manifold and investigate some interesting geometric properties of these submersions.

In the present paper, our focus is on investigating the notion of the $CSI-$submersions from Cosymplectic manifolds onto Riemannian manifolds. The paper is organized as follows: In the second section, we gather main notions and formulae for other sections. In the third section, we give the definition of the $CSI-$submersions from Cosymplectic manifolds onto Riemannian manifolds. We investigate differential geometric properties of such submersions. In the last section, we illustrate a non-trivial example of the $CSI-$submersions from Cosymplectic manifolds onto Riemannian manifolds.

## 2. Preliminaries

A $(2n+1)$-dimensional smooth manifold $\mathcal{N}_1$ is said to have an almost contact structure [26] if there exist on $\mathcal{N}_1$, a tensor field $\phi$ of type $(1,1)$, a vector field $\xi$ and 1-form $\eta$ such that

$$\phi^2 = -I + \eta \otimes \xi, \ \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g_1(\xi,\xi)=\eta(\xi)=1. \tag{2.2}$$

If there exists a Riemannian metric $g_1$ on an almost contact manifold $\mathcal{N}_1$ satisfying:

$$\begin{aligned} g_1(\phi Z_1,\phi Z_2) &= g_1(Z_1,Z_2)-\eta(Z_1)\eta(Z_2), \qquad (2.3)\\ g_1(Z_1,\phi Z_2) &= -g_1(\phi Z_1,Z_2), \end{aligned}$$

$$g_1(Z_1,\xi)=\eta(Z_1), \tag{2.4}$$

where $Z_1,Z_2$ are any vector fields on $\mathcal{N}_1$. Then $\mathcal{N}_1$ is called almost contact metric manifold [6] with almost contact structure $(\phi,\xi,\eta)$ and is represented by $(\mathcal{N}_1,\phi,\xi,\eta,g_1)$.

An almost contact structure $(\phi,\xi,\eta)$ is said to be normal if the almost complex structure $J$ on the product manifold $\mathcal{N}_1\times R$ is given by

$$J(Z_1,\mathcal{F}\frac{d}{dt})=(\phi Z_1-\mathcal{F}\xi,\eta(Z_1)\frac{d}{dt}), \tag{2.5}$$

where $J^2=-I$ and $\mathcal{F}$ is a differentiable function on $\mathcal{N}_1\times R$ that has no torsion, i.e., $J$ is integrable. The form for normality in terms of $\phi$, $\xi$ and $\eta$ is given by $[\phi,\phi]+2d\eta\otimes\xi=0$ on $\mathcal{N}_1$, where $[\phi,\phi]$ is the Nijenhuis tensor of $\phi$ . Further, the fundamental 2-form $\Phi$ is defined by $\Phi(Z_1,Z_2)=g_1(Z_1,\phi Z_2)$.

A manifold $\mathcal{N}_1$ with the structure $(\phi,\xi,\eta,g_1)$ is said to be Cosymplectic [26] if

$$(\nabla_{Z_1}\phi)Z_2=0 \tag{2.6}$$

for any vector fields $Z_1,Z_2$ on $\mathcal{N}_1$, where $\nabla$ stands for the Riemannian connection of the metric $g_1$ on $\mathcal{N}_1$. For a Cosymplectic manifold, we have

$$\nabla_{Z_1}\xi=0 \tag{2.7}$$

for any vector field $Z_1$ on $\mathcal{N}_1$.

O'Neill's tensors [17] $\mathcal{T}$ and $\mathcal{A}$ are given by

$$\mathcal{A}_{X_1}X_2=\mathcal{H}\nabla_{\mathcal{H}X_1}\mathcal{V}X_2+\mathcal{V}\nabla_{\mathcal{H}X_1}\mathcal{H}X_2, \tag{2.8}$$

$$\mathcal{T}_{X_1}X_2=\mathcal{H}\nabla_{\mathcal{V}X_1}\mathcal{V}X_2+\mathcal{V}\nabla_{\mathcal{V}X_1}\mathcal{H}X_2 \tag{2.9}$$

for any $X_1,X_2$ on $\mathcal{N}_1$. For vertical vector fields $Y_1,Y_2$, the tensor field $\mathcal{T}$ has the symmetry property, that is,

$$\mathcal{T}_{Y_1}Y_2=\mathcal{T}_{Y_2}Y_1, \tag{2.10}$$

while for horizontal vector fields $X_1,X_2$, the tensor field $\mathcal{A}$ has alternation property, that is

$$\mathcal{A}_{X_1}X_2=-\mathcal{A}_{X_2}X_1. \tag{2.11}$$

From the equations (2.8) and (2.9), we have

$$\nabla_{Y_1} Y_2 = \mathcal{T}_{Y_1} Y_2 + \mathcal{V}\nabla_{Y_1} Y_2, \tag{2.12}$$

$$\nabla_{Y_1} Z_1 = \mathcal{T}_{Y_1} Z_1 + \mathcal{H}\nabla_{Y_1} Z_1, \tag{2.13}$$

$$\nabla_{Z_1} Y_1 = \mathcal{A}_{Z_1} Y_1 + \mathcal{V}\nabla_{Z_1} Y_1, \tag{2.14}$$

$$\nabla_{Z_1} Z_2 = \mathcal{H}\nabla_{Z_1} Z_2 + \mathcal{A}_{Z_1} Z_2 \tag{2.15}$$

for all $Y_1, Y_2 \in \Gamma(\ker F_*)$ and $Z_1, Z_2 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{Y_1} Z_1 = \mathcal{A}_{Z_1} Y_1$, if $Z_1$ is basic. It can be easily seen that $\mathcal{T}$ acts on the fibers as the second fundamental form, while $\mathcal{A}$ acts on the horizontal distribution and measures the obstruction to the integrability of this distribution.

The Riemannian submersion $F$ between two Riemannian manifolds is totally geodesic if

$$(\nabla F_*)(U_1, U_2) = 0 \ \ \forall \ \ U_1, U_2 \in \Gamma(T\mathcal{N}_1).$$

Totally umbilical Riemannian submersion is a Riemannian submersion with totally umbilical fibers ([4], [5]) if

$$\mathcal{T}_{Z_1} Z_2 = g_1(Z_1, Z_2) H \tag{2.16}$$

for all $Z_1, Z_2 \in \Gamma(\ker F_*)$, where $H$ denotes the mean curvature vector field of fibers.

Let $F : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ be a Riemannian submersion between Riemannian manifolds. The differential map $F_*$ of $F$ can be viewed as a section of the bundle $Hom(T\mathcal{N}_1, F^{-1}T\mathcal{N}_2) \to \mathcal{N}_1$, where $F^{-1}T\mathcal{N}_2$ is the pullback bundle whose fibers at $q \in \mathcal{N}_1$ is $(F^{-1}T\mathcal{N}_2)_q = T_F(q)\mathcal{N}_2, q \in \mathcal{N}_1$. The bundle $Hom(T\mathcal{N}_1, F^{-1}T\mathcal{N}_2)$ has a connection $\nabla$ induced from the Levi-Civita connection $\nabla^{\mathcal{N}_1}$ and the pullback connection $\nabla^F$. Then the second fundamental form of $F$ is given by

$$(\nabla F_*)(V_1, V_2) = \nabla^F_{V_1} F_*(V_2) - F_*(\nabla^{\mathcal{N}_1}_{V_1} V_2) \tag{2.17}$$

for the vector fields $V_1, V_2 \in \Gamma(T\mathcal{N}_1)$.

## 3. The $CSI-$submersions from Cosymplectic manifolds

In the theory of Riemannian submersions, Bishop [5] initiated the concept of Clairaut submersion as: a submersion $F : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ is called a Clairaut submersion if there exist a function $r : \mathcal{N}_1 \to R^+$ in such a way that any geodesic that makes an angle $\Theta$ with a horizontal subspace, $r \sin \Theta$ is constant.

On the other side, Sahin [27] generalized the concept of Clairaut submersion and initiated the study of Clairaut Riemannian maps and investigated its geometric properties.

**Theorem 3.1.** [5] *Let $F : (\mathcal{N}_1, g_1) \to (\mathcal{N}_2, g_2)$ be a Riemannian submersion with connected fibers. Then, $F$ is a Clairaut Riemannian submersion with $r = e^h$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla h$, where $\nabla h$ is the gradient of the function $h$ with respect to $g_1$.*

**Definition 3.1.** [26] *Let $F$ be a Riemannian submersion from an almost contact metric manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. Then we say that $F$ is a semi-invariant submersion if there is a distribution $\mathfrak{D}_1 \subseteq \ker F_*$ such that*

$$\ker F_* = \mathfrak{D}_1 \oplus \mathfrak{D}_2, \quad \phi(\mathfrak{D}_1) = \mathfrak{D}_1, \phi(\mathfrak{D}_2) \subseteq (\ker F_*)^\perp,$$

*where $\mathfrak{D}_1$ and $\mathfrak{D}_2$ mutually orthogonal distributions in $(\ker F_*)$.*

Let $\mu$ denotes the complementary orthogonal subbundle to $\phi(\mathfrak{D}_2)$ in $(\ker F_*)^\perp$. Then we have

$$(\ker F_*)^\perp = \phi(\mathfrak{D}_2) \oplus \mu.$$

Obviously $\mu$ is an invariant subbundle of $(\ker F_*)^\perp$ with respect to the contact structure $\phi$.

We say that a semi-invariant submersion $F : \mathcal{N}_1 \to \mathcal{N}_2$ admits a vertical Reeb vector field $\xi$ if it is tangent to $(\ker F_*)$ and it admits horizontal Reeb vector field $\xi$ it is normal to $(\ker F_*)$. One can easily observe that $\mu$ contains the Reeb vector field in case of the Riemannian submersion admits horizontal Reeb vector field.

We now define the notion of $CSI-$ submersion in contact manifolds as follows:

**Definition 3.2.** *A semi-invariant submersions from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ is called $CSI-$ submersion if it satisfies the condition of Clairaut Riemannian submersion.*

For any vector field $W_1 \in \Gamma(\ker F_*)$, we put

$$W_1 = PW_1 + QW_1, \tag{3.18}$$

where $P$ and $Q$ are projection morphisms [4] of $\ker F_*$ onto $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.

For $U_1 \in (\ker F_*)$, we get

$$\phi U_1 = \psi U_1 + \omega U_1, \tag{3.19}$$

where $\psi U_1 \in \Gamma(\mathfrak{D}_1)$ and $\omega U_1 \in \Gamma(\phi\mathfrak{D}_2)$. Also for $V_2 \in \Gamma(\ker F_*)^\perp$, we get

$$\phi V_2 = BV_2 + CV_2, \tag{3.20}$$

where $BV_2 \in \Gamma(\mathfrak{D}_2)$ and $CV_2 \in \Gamma(\mu)$.

**Definition 3.3.** [30] *Let $F$ be a $CSI-$ submersion from an almost contact metric manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. If $\mu = \{0\}$ or $\mu =< \xi >$, i.e., $(\ker F_*)^\perp = \phi(\mathfrak{D}_2)$ or $(\ker F_*)^\perp = \phi(\mathfrak{D}_2) \oplus < \xi >$ respectively, then we call $\phi$ a Lagrangian Riemannian submersion. In this case, for any horizontal vector field $Z_1$, we have*

$$BZ_1 = \phi Z_1 \text{ and } CZ_1 = 0. \tag{3.21}$$

**Lemma 3.1.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then, we get*

$$\mathcal{V}\nabla_{W_1}\psi W_2 + \mathcal{T}_{W_1}\omega W_2 = B\mathcal{T}_{W_1}W_2 + \psi\mathcal{V}\nabla_{W_1}W_2, \tag{3.22}$$

$$\mathcal{T}_{W_1}\psi W_2 + \mathcal{H}\nabla_{W_1}\omega W_2 = C\mathcal{T}_{W_1}W_2 + \omega\mathcal{V}\nabla_{W_1}W_2, \tag{3.23}$$

$$\mathcal{V}\nabla_{U_1}BU_2 + \mathcal{A}_{U_1}CU_2 = B\mathcal{H}\nabla_{U_1}U_2 + \psi\mathcal{A}_{U_1}U_2, \tag{3.24}$$

$$\mathcal{A}_{U_1}BU_2 + \mathcal{H}\nabla_{U_1}CU_2 = C\mathcal{H}\nabla_{U_1}U_2 + \omega\mathcal{A}_{U_1}U_2, \tag{3.25}$$

$$\mathcal{V}\nabla_{W_1}BU_1 + \mathcal{T}_{W_1}CU_1 = \psi\mathcal{T}_{W_1}U_1 + B\mathcal{H}\nabla_{W_1}U_1, \tag{3.26}$$

$$\mathcal{T}_{W_1}BU_1 + \mathcal{H}\nabla_{W_1}CU_1 = \omega\mathcal{T}_{W_1}U_1 + C\mathcal{H}\nabla_{W_1}U_1, \tag{3.27}$$

$$\mathcal{V}\nabla_{U_1}\psi W_1 + \mathcal{A}_{U_1}\omega W_1 = B\mathcal{A}_{U_1}W_1 + \psi\mathcal{V}\nabla_{U_1}W_1, \tag{3.28}$$

$$\mathcal{A}_{U_1}\psi W_1 + \mathcal{H}\nabla_{U_1}\omega W_1 = C\mathcal{A}_{U_1}W_1 + \omega\mathcal{V}\nabla_{U_1}W_1, \tag{3.29}$$

*where $W_1, W_2 \in \Gamma(\ker F_*)$ and $U_1, U_2 \in \Gamma(\ker F_*)^\perp$.*

**Proof.** Using (2.12)−(2.15),(3.19) and (3.20), we get Lemma 3.1.

**Corollary 3.1.** *Let $F$ be a Lagrangian submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then we get*

$$\mathcal{V}\nabla_{V_1}\psi V_2 + \mathcal{T}_{V_1}\omega V_2 = B\mathcal{T}_{V_1}V_2 + \psi\mathcal{V}\nabla_{V_1}V_2, \mathcal{T}_{V_1}\psi V_2 + \mathcal{H}\nabla_{V_1}\omega V_2 = \omega\mathcal{V}\nabla_{V_1}V_2,$$

$$\mathcal{V}\nabla_{Y_1}BY_2 = B\mathcal{H}\nabla_{Y_1}Y_2 + \psi\mathcal{A}_{Y_1}Y_2, \mathcal{A}_{Y_1}BY_2 = \omega\mathcal{A}_{Y_1}Y_2,$$

$$\mathcal{V}\nabla_{V_1}BY_1 = \psi\mathcal{T}_{V_1}Y_1 + B\mathcal{H}\nabla_{V_1}Y_1, \mathcal{T}_{V_1}BY_1 = \omega\mathcal{T}_{V_1}Y_1,$$

$$\mathcal{V}\nabla_{Y_1}\psi V_1 + \mathcal{A}_{Y_1}\omega V_1 = B\mathcal{A}_{Y_1}V_1 + \psi\mathcal{V}\nabla_{Y_1}V_1, \mathcal{A}_{Y_1}\psi V_1 + \mathcal{H}\nabla_{Y_1}\omega V_1 = \omega\mathcal{V}\nabla_{Y_1}V_1,$$

*where $V_1, V_2 \in \Gamma(\ker F_*)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.*

**Lemma 3.2.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting vertical or horizontal Reeb vector field. Then we have*

$$\mathcal{T}_{Z_1}\xi = 0, \mathcal{A}_{Z_2}\xi = 0 \tag{3.30}$$

*for $Z_1 \in \Gamma(\ker F_*)^\perp$ and $Z_2 \in \Gamma(\ker F_*)^\perp$.*

**Proof.** Using (2.12)−(2.15) and (2.7), we get Lemma 3.2.

**Lemma 3.3.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. If $\gamma : I_2 \subset R \to \mathcal{N}_1$ is a regular curve and $Z_1(t)$ and $Z_2(t)$ are the vertical and horizontal components of the tangent vector field $\dot\gamma = E$ of $\gamma(t)$, respectively, then $\gamma$ is a geodesic if and only if along $\gamma$ the following equations hold:*

$$\mathcal{V}\nabla_{\dot\gamma}BZ_2 + \mathcal{V}\nabla_{\dot\gamma}\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1 = 0, \tag{3.31}$$

$$\mathcal{H}\nabla_{\dot\gamma}CZ_2 + \mathcal{H}\nabla_{\dot\gamma}\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + (\mathcal{A}_{Z_2} + \mathcal{T}_{Z_1})\psi Z_1 = 0. \tag{3.32}$$

**Proof.** Let $\gamma : I_2 \to \mathcal{N}_1$ be a regular curve on $\mathcal{N}_1$. Since $\dot\gamma(t) = Z_1(t) + Z_2(t)$, where $Z_1(t)$ and $Z_2(t)$ are the vertical and horizontal components of $\dot\gamma(t)$. Using (2.6),(2.12)−(2.15), (3.19) and (3.20), we have

$$\begin{aligned}
\phi\nabla_{\dot\gamma}\dot\gamma &= \nabla_{\dot\gamma}\phi\dot\gamma \\
&= \nabla_{Z_1}\psi Z_1 + \nabla_{Z_1}\omega Z_1 + \nabla_{Z_2}\psi Z_1 + \nabla_{Z_2}\omega Z_1 + \\
&\quad \nabla_{Z_1}BZ_2 + \nabla_{Z_1}CZ_2 + \nabla_{Z_2}BZ_2 + \nabla_{Z_2}CZ_2, \\
&= \mathcal{V}\nabla_{\dot\gamma}BZ_2 + \mathcal{V}\nabla_{\dot\gamma}\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1 + \\
&\quad \mathcal{H}\nabla_{\dot\gamma}CZ_2 + \mathcal{H}\nabla_{\dot\gamma}\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + (\mathcal{A}_{Z_2} + \mathcal{T}_{Z_1})\psi Z_1.
\end{aligned}$$

From above, vertical and horizontal components are:

$$\mathcal{V}\phi\nabla_{\dot\gamma}\dot\gamma = \mathcal{V}\nabla_{\dot\gamma}BZ_2 + \mathcal{V}\nabla_{\dot\gamma}\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1,$$

$$\mathcal{H}\phi\nabla_{\dot\gamma}\dot\gamma = \mathcal{H}\nabla_{\dot\gamma}CZ_2 + \mathcal{H}\nabla_{\dot\gamma}\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2 + (\mathcal{A}_{Z_2} + \mathcal{T}_{Z_1})\psi Z_1.$$

Thus $\gamma$ is a geodesic on $\mathcal{N}_1$ if and only if $\mathcal{V}\phi\nabla_{\dot\gamma}\dot\gamma = 0$ and $\mathcal{H}\phi\nabla_{\dot\gamma}\dot\gamma = 0$.

**Theorem 3.2.** *Let $F$ be a Clairaut semi-invariant submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. Then $F$ is a $CSI-$ submersion with $r = e^h$ if and only if*

$$\begin{aligned} g_1(\nabla h, Z_2)||Z_1||^2 &= g_1(\mathcal{V}\nabla_{\dot{\gamma}} BZ_2, \psi Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) + \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}} CZ_2, \omega Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1), \end{aligned}$$

*where $\gamma : I_2 \to \mathcal{N}_1$ is a geodesic on $\mathcal{N}_1$, $Z_1(t)$ and $Z_2(t)$ are vertical and horizontal components of $\dot{\gamma}(t)$, respectively.*

**Proof.** Let $\gamma : I_2 \to \mathcal{N}_1$ be a geodesic on $\mathcal{N}_1$ with $Z_1(t) = \mathcal{V}\dot{\gamma}(t)$ and $Z_2(t) = \mathcal{H}\dot{\gamma}(t)$. Let $\Theta(t)$ denotes the angle in $[0, \pi]$ between $\dot{\gamma}(t)$ and $Z_2(t)$. Assuming $\upsilon = ||\dot{\gamma}(t)||,^2$ then we get

$$g_1(Z_1(t), Z_1(t)) = \upsilon \sin^2 \Theta(t), \tag{3.33}$$

$$g_1(Z_2(t), Z_2(t)) = \upsilon \cos^2 \Theta(t). \tag{3.34}$$

Now, differentiating (3.33), we get

$$\begin{aligned} \frac{d}{dt} g_1(Z_1(t), Z_1(t)) &= 2\upsilon \sin \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt}, \\ g_1(\nabla_{\dot{\gamma}} Z_1(t), Z_1(t)) &= \upsilon \cos \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt}. \end{aligned}$$

Using equations (2.3) and (2.6) in above equation, we get

$$g_1(\nabla_{\dot{\gamma}} \phi Z_1(t), \phi Z_1(t)) = \upsilon \sin \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt}. \tag{3.35}$$

Now we obtain

$$\begin{aligned} g_1(\nabla_{\dot{\gamma}} \phi Z_1, \phi Z_1) &= g_1(\mathcal{V}\nabla_{\dot{\gamma}} \psi Z_1, \psi Z_1) + g_1(\mathcal{H}\nabla_{\dot{\gamma}} \omega Z_1, \omega Z_1) + \\ &\quad g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\psi Z_1, \omega Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})\omega Z_1, \psi Z_1). \end{aligned} \tag{3.36}$$

Using equations (3.31) and (3.32) in (3.37), we have

$$\begin{aligned} g_1(\nabla_{\dot{\gamma}} \phi Z_1, \phi Z_1) &= -g_1(\mathcal{V}\nabla_{\dot{\gamma}} BZ_2, \psi Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) - \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}} CZ_2, \omega Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1). \end{aligned} \tag{3.37}$$

From (3.35) and (3.38), we have

$$\begin{aligned} \upsilon \cos \Theta(t) \cos \Theta(t) \frac{d\Theta}{dt} &= -g_1(\mathcal{V}\nabla_{\dot{\gamma}} BZ_2, \psi Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) - \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}} CZ_2, \omega Z_1) - g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1). \end{aligned} \tag{3.38}$$

Moreover, $\pi$ is a $CSI-$ Riemannian submersion with $r = e^h$ if and only if $\frac{d}{dt}(e^{h\circ\gamma}\sin\Theta) = 0$, i.e., $e^{h\circ\gamma}(\cos\Theta\frac{d\Theta}{dt} + \sin\Theta\frac{dh}{dt}) = 0$. By multiplying this with non-zero factor $\upsilon\sin\Theta$, we have

$$\begin{aligned} -\upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= \upsilon\sin^2\Theta\frac{dh}{dt}, \\ \upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= -g_1(Z_1, Z_1)\frac{dh}{dt}, \\ \upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= -g_1(\nabla h, \dot{\gamma})||Z_1||^2, \\ \upsilon\cos\Theta\sin\Theta\frac{d\Theta}{dt} &= -g_1(\nabla h, Z_2)||Y_1||^2. \end{aligned} \tag{3.39}$$

Thus, from equations (3.39) and (3.39), we have

$$\begin{aligned} g_1(\nabla h, Z_2)||Z_1||^2 &= g_1(\mathcal{V}\nabla_{\dot{\gamma}}BZ_2, \psi Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})CZ_2, \psi Z_1) + \\ &\quad g_1(\mathcal{H}\nabla_{\dot{\gamma}}CZ_2, \omega Z_1) + g_1((\mathcal{T}_{Z_1} + \mathcal{A}_{Z_2})BZ_2, \omega Z_1). \end{aligned}$$

Hence Theorem 3.2 is proved.

**Corollary 3.2.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ to a Riemannian manifold $(\mathcal{N}_2, g_2)$ admitting horizontal Reeb vector field. Then we get*

$$g_1(\nabla h, \xi) = 0.$$

**Theorem 3.3.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ with $r = e^h$, then at least one of the following statement is true:*

$(i)$ $h$ is constant on $\phi(\mathfrak{D}_2)$,

$(ii)$ the fibers are one-dimensional,

$(iii)$ $\overset{F}{\nabla}_{\phi X_1} F_*(W_1) = -W_1(h)F_*(\phi X_1)$, for all $X_1 \in \Gamma(\mathfrak{D}_2), W_1 \in \Gamma(\mu)$ and $\xi \neq W_1$.

**Proof.** Let $F$ be $CSI-$ submersion from a Cosymplectic manifold onto a Riemannian manifold. For $Y_1, Y_2 \in \Gamma(\mathfrak{D}_2)$, using (2.16) and Theorem 3.1, we get

$$\mathcal{T}_{Y_1}Y_2 = -g_1(Y_1, Y_2)gradh. \tag{3.40}$$

Taking the inner product in (3.40) with $\phi X_1$, we get

$$g_1(\mathcal{T}_{Y_1}Y_2, \phi X_1) = -g_1(Y_1, Y_2)g_1(gradh, \phi X_1) \tag{3.41}$$

for all $X_1 \in \Gamma(\mathfrak{D}_2)$.

From $(2.3), (2.6)$ and $(3.41)$, we obtain

$$g_1(\nabla_{Y_1}\phi Y_2, X_1) = g_1(Y_1, Y_2)g_1(gradh, \phi X_1).$$

By using (2.3) and (2.16) in above equation, we have

$$g_1(Y_1, X_1)g_1(gradh, \phi Y_2) = g_1(Y_1, Y_2)g_1(gradh, \phi X_1). \tag{3.42}$$

Taking $X_1 = Y_2$ and interchanging the role of $Y_1$ and $Y_2$, we get

$$g_1(Y_2, Y_2)g_1(gradh, \phi Y_1) = g_1(Y_1, Y_2)g_1(gradh, \phi Y_2). \tag{3.43}$$

Using (3.42) with $X_1 = Y_1$ in (3.43), we have

$$g_1(gradh, \phi Y_1) = \frac{(g_1(Y_1, Y_2))^2}{||Y_1||^2||Y_2||^2} g_1(gradh, \phi Y_1). \tag{3.44}$$

If $gradh \in \Gamma(\phi(\mathfrak{D}_2))$, then (3.44) and the equality condition of Schwarz inequality implies that either $h$ is constant on $\phi(\mathfrak{D}_2)$ or the fibers are 1-dimensional. This implies the proof of $(i)$ and $(ii)$.

Now, from (2.15) and (2.16), we get

$$g_1(\nabla_{Y_1} X_1, W_1) = -g_1(Y_1, X_1)g_1(gradh, W_1), \tag{3.45}$$

for all $W_1 \in \Gamma(\mu)$ and $\xi \neq W_1$. Using (2.3), (2.6) and (3.45), we have

$$g_1(\nabla_{Y_1} \phi X_1, \phi W_1) = -g_1(Y_1, X_1)g_1(gradh, W_1),$$

which implies

$$g_1(\nabla_{\phi X_1} Y_1, \phi W_1) = -g_1(Y_1, X_1)g_1(gradh, W_1). \tag{3.46}$$

By using (2.14) and (3.46), we have

$$g_1(\mathcal{H}\nabla_{\phi X_1} W_1, \phi Y_1) = -g_1(\phi Y_1, \phi X_1)g_1( gradh, W_1).$$

Also for Riemannian submersion $F$, we have

$$g_2(F_*(\nabla^{\mathcal{N}_1}_{\phi X_1} W_1), F_*(\phi Y_1)) = -g_2(F_*(\phi Y_1), F_*(\phi X_1))g_1(gradh, W_1). \tag{3.47}$$

Again, using (2.17) and (3.47), we get

$$g_2(\overset{F}{\nabla}_{\phi X_1} F_*(W_1), F_*(\phi Y_1)) = -g_2(F_*(\phi Y_1), F_*(\phi X_1))g_1(gradh, W_1),$$

which implies.

$$\overset{F}{\nabla}_{\phi X_1} F_*(W_1) = -W_1(h)F_*(\phi X_1). \tag{3.48}$$

If $gradh \in \Gamma(\mu)\backslash\{\xi\}$, then (3.48) implies $(iii)$.

**Corollary 3.3.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ with $r = e^h$ and $\dim(D_2) > 1$. Then the fibers of $F$ are totally geodesic if and only if $\overset{F}{\nabla}_{\phi X_1} F_*(W_1) = 0\ \forall X_1 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\mu)$.*

**Lemma 3.4.** *Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$ with $r = e^h$ and $\dim(D_2) > 1$. Then $\overset{F}{\nabla}_{W_1} F_*(\phi Y_1) = W_1(h) F_*(\phi Y_1)$ for $Y_1 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\ker F_*)^{\perp} \backslash \{\xi\}$.*

**Proof.** Let $F$ be a $CSI-$ submersion from a Cosymplectic manifold onto a Riemannian manifold. From Theorem 3.1, fibers are totally umbilical with mean curvature vector field $H = -gradh$, then we get

$$\begin{aligned} -g_1(\nabla_{Y_1} W_1, Y_2) &= g_1(\nabla_{Y_1} Y_2, W_1), \\ -g_1(\nabla_{Y_1} W_1, Y_2) &= -g_1(Y_1, Y_2) g_1(gradh, W_1) \end{aligned}$$

for $Y_1, Y_2 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\ker F_*)^{\perp} \backslash \{\xi\}$.

Using (2.3) and (2.6) in above equation, we get

$$g_1(\nabla_{W_1} \phi Y_1, \phi Y_2) = g_1(\phi Y_1, \phi Y_2) g_1(gradh, W_1). \tag{3.49}$$

Since $F$ is $CSI-$ submersion and using (3.49), we have

$$g_2(F_*(\nabla^F_{W_1} \phi Y_1), F_*(\phi Y_2)) = g_2(F_*(\phi Y_1), F_*(\phi Y_2)) g_1(gradh, W_1). \tag{3.50}$$

From (2.17) in (3.50), we obtain

$$g_2(\overset{F}{\nabla}_{W_1} F_*(\phi Y_1), F_*(\phi Y_2)) = g_2(F_*(\phi Y_1), F_*(\phi Y_2)) g_1(gradh, W_1), \tag{3.51}$$

which implies $\overset{F}{\nabla}_{W_1} F_*(\phi Y_1) = W_1(h) F_*(\phi Y_1)$ for $Y_1 \in \Gamma(\mathfrak{D}_2)$ and $W_1 \in \Gamma(\ker F_*)^{\perp} \backslash \{\xi\}$.

**Theorem 3.4.** *Let $F$ be a $CSI-$ submersion with $r = e^h$ from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution $\mathfrak{D}_1$ cannot defined a totally geodesic foliation on $\mathcal{N}_1$.*

**Proof.** For $Y_1, Y_2 \in \Gamma(\mathfrak{D}_1)$ and $U_1 \in \Gamma(\mathfrak{D}_2)$, using $(2.3), (2.6), (2.13)$ and $(2.16)$, we get

$$\begin{aligned} g_1(\nabla_{Y_1} Y_2, U_1) &= g_1(\nabla_{Y_1} \phi Y_2, \phi U_1), \\ &= g_1(\mathcal{T}_{Y_1} \phi Y_2, \phi U_1), \\ &= -g_1(Y_1, \phi Y_2) g_1(gradh, \phi U_1). \end{aligned}$$

Thus, one can easily obtain the assertion from above equation and the fact that $gradh \in \phi(\mathfrak{D}_2)$.

**Theorem 3.5.** *The $CSI-$ submersion $F$ with $r = e^h$ from a Cosymplectic manifold $(\mathcal{N}_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold $(\mathcal{N}_2, g_2)$. Then the fibers of $F$ are totally geodesic or the anti-invariant distribution $\mathfrak{D}_2$ is one-dimensional.*

**Proof.** The result is quite obvious when we take the fibers of $F$ are totally geodesic. For second one, since $F$ is a $CSI-$ submersion, then either $\dim(\mathfrak{D}_2) = 1$ or $\dim(\mathcal{D}_2) > 1$. If $\dim(\mathcal{D}_2) > 1$, then we can choose $U_1, U_2 \in \Gamma(\mathfrak{D}_2)$ such that $\{U_1, U_2\}$ is orthonormal. From $(2.13), (3.19)$ and $(3.20)$, we get

$$\begin{aligned} \mathcal{T}_{U_1}\phi U_2 + \mathcal{H}\nabla_{U_1}\phi U_2 &= \nabla_{U_1}\phi U_2, \\ \mathcal{T}_{U_1}\phi U_2 + \mathcal{H}\nabla_{U_1}\phi U_2 &= B\mathcal{T}_{U_1}U_2 + C\mathcal{T}_{U_1}U_2 + \psi\mathcal{V}\nabla_{U_1}U_2 + \omega\mathcal{V}\nabla_{U_1}U_2. \end{aligned}$$

Taking the inner product above equation with $U_1$, we obtain

$$g_1(\mathcal{T}_{U_1}\phi U_2, U_1) = g_1(B\mathcal{T}_{U_1}U_2, U_1) + g_1(\psi\mathcal{V}\nabla_{U_1}U_2, U_1). \tag{3.52}$$

From $(2.3), (2.6)$ and $(2.13)$, we have

$$g_1(\mathcal{T}_{U_1}U_1, \phi U_2) = -g_1(\mathcal{T}_{U_1}\phi U_2, U_1) = g_1(\mathcal{T}_{U_1}U_2, \phi U_1). \tag{3.53}$$

Now, using (2.16) and (3.53), we get

$$g_1(\mathcal{T}_{U_1}U_1, \phi U_2) = -g_1(gradh, \phi U_2). \tag{3.54}$$

From equations (2.16) and (3.54), we obtain

$$-g_1(gradh, \phi U_2) = g_1(\mathcal{T}_{U_1}U_1, \phi U_2) = -g_1(\mathcal{T}_{U_1}\phi U_2, U_1) = g_1(\mathcal{T}_{U_1}U_2, \phi U_1). \tag{3.55}$$

From above equation, we get

$$\begin{aligned} g_1(gradh, \phi U_2) &= -g_1(\mathcal{T}_{U_1}U_2, \phi U_1), \\ g_1(gradh, \phi U_2) &= g_1(U_1, U_2)g_1(gradh, \phi U_1), \\ g_1(gradh, \phi U_2) &= 0. \end{aligned}$$

Thus, we get $gradh \perp \phi(\mathfrak{D}_2)$.

Therefore, the dimension of $\mathfrak{D}_2$ must be one.

## 4. Example

**Example 4.1.** *Taking an Euclidean space $\mathcal{N}_1$, given by $\mathcal{N}_1 = \{(x_1, x_2, y_{1,}y_2, z) \in R^5 : (x_1, x_2, y_{1,}y_2) \neq (0,0,0,0)$ and $z \neq 0\}$. We define the Riemannian metric $g_1$ on $\mathcal{N}_1$ defined as $g_1 = e^{2z}dx_1^2 + e^{2z}dx_2^2 + e^{2z}dy_1^2 + e^{2z}dy_2^2 + dz^2$ and the Cosymplectic structure $\phi$ on $\mathcal{N}_1$ defined as $\phi(x_1, x_2, y_{1,}y_2, z) = (y_{1,}y_2, -x_1, -x_2, z)$.*

Let $\mathcal{N}_2 = \{(v_1, v_2) \in R^2\}$ be a Riemannian manifold with Riemannian metric $g_2$, given by $g_2 = e^{2z}dv_1^2 + dv_2^2$. Define a map $F : R^5 \to R^2$ by

$$F(x_1, x_2, y_{1,}y_2, z) = (\frac{x_2 - y_2}{\sqrt{2}}, z).$$

Then, we have

$$\ker F_* =< X_1 = e_1, X_2 = e_2 + e_4, X_3 = e_3 >,$$

$$\mathfrak{D}_1 =< X_1 = e_1, X_3 = e_3 >, \mathfrak{D}_2 =< X_2 = e_2 + e_4 >,$$

$$(\ker F_*)^{\perp} =< H_1 = e_2 - e_4, H_2 = e_5 >,$$

where $\{e_1 = e^{-z}\frac{\partial}{\partial x_1}, e_2 = e^{-z}\frac{\partial}{\partial x_2}, e_3 = e^{-z}\frac{\partial}{\partial y_1}, e_4 = e^{-z}\frac{\partial}{\partial y_2}, e_7 = \frac{\partial}{\partial z}\}$, $\{e_1^* = \frac{\partial}{\partial v_1}, e_2^* = \frac{\partial}{\partial v_2}\}$ are bases on $T_p\mathcal{N}_1$ and $T_{F(p)}\mathcal{N}_2$, respectively, for all $p \in \mathcal{N}_1$. By direct computations, we can see that $F_*(H_1) = \sqrt{2}e^{-z}e_1^*, F_*(H_2) = e_2^*$, and $g_1(H_i, H_j) = g_2(F_*H_i, F_*H_j)$ for all $H_i, H_j \in \Gamma(\ker F_*)^{\perp}$, $i, j = 1, 2$. Thus, $F$ is submersion. Moreover, it is easy to see that $\phi X_1 = -X_3, \phi X_2 = -H_1$ and $\phi X_3 = X_1$. Therefore $F$ is a $CSI-$ submersion.

Now, using the Cosymplectic structure, we see that

$$\begin{aligned} [e_1, e_1] &= [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = [e_5, e_5] = 0, \\ [e_1, e_2] &= 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1, \\ [e_2, e_3] &= 0, [e_2, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_4] = 0, \\ [e_3, e_5] &= e_3, [e_4, e_5] = e_4. \end{aligned} \tag{4.56}$$

The Levi-Civita connection $\nabla$ of the metric $g_1$ is given by the Koszul′s formula which is

$$\begin{aligned} &2g_1(\nabla_X Y, Z) \\ = \; &Xg_1(Y, Z) + Yg_1(Z, X) - Zg_1(X, Y) + g_1([X, Y], Z) - g_1([Y, Z], X) + g_1([Z, X], Y). \end{aligned} \tag{4.57}$$

Using equations (4.56) and (4.57), we obtain

$$\begin{aligned}\nabla_{e_1}e_1 &= \nabla_{e_2}e_2=\nabla_{e_3}e_3=\nabla_{e_4}e_4=-\frac{\partial}{\partial z}, \\ \nabla_{e_1}e_2 &= \nabla_{e_1}e_3=\nabla_{e_1}e_4=\nabla_{e_2}e_1=\nabla_{e_2}e_3=\nabla_{e_2}e_4=0, \\ \nabla_{e_3}e_1 &= \nabla_{e_3}e_2=\nabla_{e_3}e_4=\nabla_{e_4}e_1=\nabla_{e_4}e_2=\nabla_{e_4}e_3=0. \\ \nabla_{e_1}e_5 &= e_1,\nabla_{e_2}e_5=e_2,\nabla_{e_3}e_5=e_3,\nabla_{e_4}e_5=e_4,\nabla_{e_5}e_5=0.\end{aligned} \tag{4.58}$$

Therefore, we have

$$\begin{aligned}\nabla_{X_1}X_1 &= \nabla_{e_1}e_1=-\frac{\partial}{\partial z},\nabla_{X_2}X_2=\nabla_{e_2+e_4}e_2+e_4=-2\frac{\partial}{\partial z}, \\ \nabla_{X_3}X_3 &= \nabla_{e_3}e_3=-2\frac{\partial}{\partial z},\nabla_{X_1}X_2=\nabla_{e_1}e_2=\nabla_{X_1}X_3=\nabla_{e_1}e_3=0, \\ \nabla_{X_2}X_3 &= \nabla_{e_2}e_3=0,\nabla_{X_2}X_1=\nabla_{e_2}e_1=0,\nabla_{X_3}X_1=\nabla_{e_3}e_1=0, \\ \nabla_{X_3}X_2 &= \nabla_{e_3}e_2+e_4=0.\end{aligned} \tag{4.59}$$

Thus, we have

$$\mathcal{T}_V V=\mathcal{T}_{\lambda_1X_1+\lambda_2X_2+\lambda_3X_3}\lambda_1V_1+\lambda_2V_2+\lambda_3V_3,\lambda_1,\lambda_2,\lambda_3\in R,$$

$$\begin{aligned}\mathcal{T}_V V &= \lambda_1^2\mathcal{T}_{X_1}X_1+\lambda_2^2\mathcal{T}_{X_2}X_2+\lambda_3^2\mathcal{T}_{X_3}X_3+ \\ &\quad \lambda_1\lambda_2\mathcal{T}_{X_1}X_2+\lambda_1\lambda_3\mathcal{T}_{X_1}X_3+\lambda_2\lambda_3\mathcal{T}_{X_2}X_3+ \\ &\quad \lambda_1\lambda_2\mathcal{T}_{X_2}X_1+\lambda_1\lambda_3\mathcal{T}_{X_3}X_1+\lambda_2\lambda_3\mathcal{T}_{X_3}X_2.\end{aligned} \tag{4.60}$$

Using equations (2.12) and (4.59), we obtain

$$\begin{aligned}\mathcal{T}_{X_1}X_1 &= -\frac{\partial}{\partial z},\mathcal{T}_{X_2}X_2=-2\frac{\partial}{\partial z},\mathcal{T}_{X_3}X_3=-\frac{\partial}{\partial z}, \\ \mathcal{T}_{X_1}X_2 &= 0,\mathcal{T}_{X_1}X_3=0,\mathcal{T}_{X_2}X_3=0,\mathcal{T}_{X_2}X_1=0, \\ \mathcal{T}_{X_2}X_3 &= 0,\mathcal{T}_{X_3}X_1=0.\end{aligned} \tag{4.61}$$

Now using equations (4.60) and (4.61), we get

$$\mathcal{T}_V V=-(\lambda_1^2+2\lambda_2^2+\lambda_3^2)\frac{\partial}{\partial z}. \tag{4.62}$$

Since $X=\lambda_1X_1+\lambda_2X_2+\lambda_3X_3$, so $g_1(\lambda_1V_1+\lambda_2V_2+\lambda_3V_3,\lambda_1V_1+\lambda_2V_2+\lambda_3V_3)=\lambda_1^2+2\lambda_2^2+\lambda_3^2$. For a smooth function $h$ on $R^5$, the $\nabla h$ w. r. t. the metric $g_1$ is given by $\nabla h=e^{-2z}\frac{\partial h}{\partial x_1}\frac{\partial}{\partial x_1}+e^{-2z}\frac{\partial h}{\partial x_2}\frac{\partial}{\partial x_2}+e^{-2z}\frac{\partial h}{\partial y_1}\frac{\partial}{\partial y_1}+e^{-2z}\frac{\partial h}{\partial y_2}\frac{\partial}{\partial y_2}+\frac{\partial h}{\partial z}\frac{\partial}{\partial z}$. Hence $\nabla h=\frac{\partial}{\partial z}$ for the function $h=z$. Then one can easily find that $\mathcal{T}_V V=-g_1(V,V)\nabla h$, thus by Theorem 3.1, the map $F$ is a $CSI-$ submersion from Cosymplectic manifold onto Riemannian manifold.

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# A STUDY ON THE SEMI-CONFORMAL DEFORMATION OF BERGER-TYPE METRIC

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Abstract. The main purpose of this article is to introduce a new class of metric on an anti-paraKähler manifold $(M^{2m}, \varphi, g)$. First we investigate the Levi-Civita connection of this metric. Secondly, we study some properties of Riemannian curvature tensors. Finally, we characterizes some class of harmonic maps.

**Keywords**: Riemannian manifold, Semi-conformal deformation of Berger-type metric, Scalar curvature, Harmonic map.

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## 1. Introduction

Let $(M^m, g)$ be an $m$-dimensional Riemannian manifold and $\Im^1_0(M)$ the set of all vector fields on $M$. We denote by $\Gamma^k_{ij}$ the Christoffel symbols of $g$ and by $\nabla$ the Levi-Civita connection of $g$, this connection is characterized by the Koszul formula

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) + g(Z,[X,Y]) \\ &\quad + g(Y,[Z,X]) - g(X,[Y,Z]). \end{aligned} \tag{1.1}$$

for all $X, Y, Z \in \Im^1_0(M)$.

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The Riemannian curvature tensor $R$, the Ricci tensor $Ricci$ and the Ricci curvature $Ric$ of $(M^m, g)$ are defined respectively by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.2}$$

$$Ricci(X) = \sum_{i=1}^{m} R(X, E_i)E_i, \tag{1.3}$$

$$Ric(X,Y) = \sum_{i=1}^{m} g(R(X, E_i)E_i, Y) = g(Ricci(X), Y), \tag{1.4}$$

for all vector fields $X, Y, Z \in \Im_0^1(M)$, where $(E_1, \cdots, E_m)$ be a local orthonormal frame on $M$.

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the tension field of $\phi$ is defined by

$$\tau(\phi) = trace_g \nabla d\phi. \tag{1.5}$$

The energy functional of $\phi$ is defined by

$$E(\phi, D) = \frac{1}{2} \int_D |d\phi|^2 \, v_g, \tag{1.6}$$

such that $D$ is any compact of $M$, where $v_g$ is the volume element on $(M^m, g)$.

A map $\phi$ is called harmonic if it is a critical point of the energy functional $E$. Equivalently, $\phi$ is harmonic if it satisfies the associated Euler-Lagrange equations given by the following formula:

$$\tau(\phi) = 0. \tag{1.7}$$

For more detail on harmonic maps, see [7, 6, 8]. In recent years, this theme has been widely developed even on the tangent bundle and on the cotangent bundle has been done by many authors [2, 3, 4, 5, 13, 14, 15]. These and more general mappings of Riemannian and affine connected spaces are explored in monograph [9].

In the present paper, we first introduce a new class of metric on an anti-paraKähler manifold, namely the semi-conformal deformation of Berger-type metric. Then we calculate Levi-Civita connection of this metric ( Theorem 2.1). Secondly, we investigate all forms of curvature tensors (the Riemannian curvature, the sectional curvature ,the Ricci curvature and the scalar curvature) see ( Theorem 3.1, Theorem 3.2, Theorem 3.3, Theorem 3.4 and Theorem 3.5). In the last section we study the harmonicity with respect to the semi-conformal deformation of Berger-type metric which is an interesting research task, as we studied on some class of harmonic maps ( Proposition 4.1, Theorem 4.1, Proposition 4.3, Theorem 4.3, Proposition 4.4 and Theorem 4.4).

## 2. Semi-conformal deformation of Berger-type metric

Let $M$ be a $2m$-dimensional Riemannian manifold with a Riemannian metric $g$. An almost paracomplex manifold is an almost product manifold $(M^{2m}, \varphi)$, $\varphi^2 = id$, $\varphi \neq \pm id$ such that the two eigenbundles $T^+M$ and $T^-M$ associated to the two eigenvalues $+1$ and $-1$ of $\varphi$, respectively, have the same rank.

The integrability of an almost paracomplex structure is equivalent to the vanishing of the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y] + [X,Y].$$

An anti-paraHermitian metric (B-metric)[10] with respect to the almost paracomplex structure $\varphi$ is a Riemannian metric $g$ such that

$$g(\varphi X, \varphi Y) = g(X,Y), \tag{2.8}$$

i.e. is a (pure metric)

$$g(\varphi X, Y) = g(X, \varphi Y), \tag{2.9}$$

for any vector fields $X, Y$ on $M$.

If $(M^{2m}, \varphi)$ is an almost paracomplex manifold with an anti-paraHermitian metric $g$, we say that the triple $(M^{2m}, \varphi, g)$ is an almost anti-paraHermitian manifold (an almost B-manifold)[10]. If $\varphi$ is integrable, we say that $(M^{2m}, \varphi, g)$ is an anti-paraKähler manifold (B-manifold)[10].

The purity conditions for a $(0,q)$-tensor field $S$ with respect to the almost paracomplex structure $\varphi$ given by

$$S(\varphi X_1, X_2, \ldots, X_q) = S(X_1, \varphi X_2, \ldots, X_q) = \ldots = S(X_1, X_2, \ldots, \varphi X_q),$$

for any vector fields $X_1, X_2, \ldots, X_q$ on $M$ [10].

It is well known that if $(M^{2m}, \varphi, g)$ is a anti-paraKähler manifold, the Riemannian curvature tensor is pure [10], and we have

$$R(\varphi Y, Z) = R(Y, \varphi Z) = R(Y,Z)\varphi = \varphi R(Y,Z), \tag{2.10}$$

for all vector fields $Y, Z$ on $M$.

**Definition 2.1.** *Let $(M^{2m}, \varphi, g)$ be an almost anti-paraHermitian manifold. We define semi-conformal deformation of Berger-type metric of $g$ on $M$ noted ${}^{SB}g$ by*

$$ {}^{SB}g(X,Y) = g(X,Y) + \delta^2 g(X,\varphi\xi)g(Y,\varphi\xi), $$

*for all $X, Y \in \Im^1_0(M)$ and $\xi \in \Im^1_0(M)$ such that $g(\xi,\xi) = 1$, where $\delta$ is some constant. (there are other works on the deformation of Berger-type metric on the tangent bundle and on the cotangent bundle see for example,* [1, 11, 12]*).*

In the following, we consider $g(\nabla_X(\varphi\xi), Y) = g(\nabla_Y(\varphi\xi), X)$, where $\nabla$ denote the Levi-Civita connection of $(M^{2m}, \varphi, g)$.

Note that we have,

$$ \begin{cases} g(\varphi\xi,\varphi\xi) = 1, \\ g(\nabla_X(\varphi\xi),\varphi\xi) = 0, \end{cases} \tag{2.11} $$

for all vector field $X \in \Im^1_0(M)$.

**Lemma 2.1.** *Let $(M^{2m}, \varphi, g)$ be an anti-paraKähler manifold, then we have*

$$ \begin{aligned} X\,{}^{SB}g(Y,Z) &= {}^{SB}g(\nabla_X Y, Z) + {}^{SB}g(Y, \nabla_X Z) + \delta^2 g(Z,\varphi\xi)g(Y,\nabla_X(\varphi\xi)) \\ &\quad + \delta^2 g(Y,\varphi\xi)g(Z,\nabla_X(\varphi\xi)), \end{aligned} \tag{2.12} $$

*for all vector fields $X, Y, Z \in \Im^1_0(M)$.*

**Theorem 2.1.** *Let $(M^{2m}, \varphi, g)$ be an anti-paraKähler manifold. If ${}^{SB}\nabla$ denote the Levi-Civita connection of $(M^{2m}, {}^{SB}g)$, then we have the following*

$$ {}^{SB}\nabla_X Y = \nabla_X Y + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)\varphi\xi \tag{2.13} $$

*for all vector fields $X, Y \in \Im^1_0(M)$.*

**Proof.** From Kozul formula (1.1), we have

$$ \begin{aligned} 2\,{}^{SB}g({}^{SB}\nabla_X Y, Z) &= X\,{}^{SB}g(Y,Z) + Y\,{}^{SB}g(Z,X) - Z\,{}^{SB}g(X,Y) + {}^{SB}g(Z,[X,Y]) \\ &\quad + {}^{SB}g(Y,[Z,X]) - {}^{SB}g(X,[Y,Z]). \end{aligned} $$

Using (2.12), we get

$$\begin{aligned}
2^{SB}g(^{SB}\nabla_X Y, Z) &= {}^{SB}g(\nabla_X Y, Z) + {}^{SB}g(Y, \nabla_X Z) + \delta^2 g(Z, \varphi\xi)g(Y, \nabla_X(\varphi\xi)) \\
&\quad +\delta^2 g(Y, \varphi\xi)g(Z, \nabla_X(\varphi\xi)) + {}^{SB}g(\nabla_Y Z, X) + {}^{SB}g(Z, \nabla_Y X) \\
&\quad +\delta^2 g(X, \varphi\xi)g(Z, \nabla_Y(\varphi\xi)) + \delta^2 g(Z, \varphi\xi)g(X, \nabla_Y(\varphi\xi)) \\
&\quad -{}^{SB}g(\nabla_Z X, Y) - {}^{SB}g(X, \nabla_Z Y) - \delta^2 g(Y, \varphi\xi)g(X, \nabla_Z(\varphi\xi)) \\
&\quad -\delta^2 g(X, \varphi\xi)g(Y, \nabla_Z(\varphi\xi)) + {}^{SB}g(Z, \nabla_X Y) - {}^{SB}g(Z, \nabla_Y X) \\
&\quad +{}^{SB}g(Y, \nabla_Z X) - {}^{SB}g(Y, \nabla_X Z) - {}^{SB}g(X, \nabla_Y Z) - {}^{SB}g(X, \nabla_Z Y) \\
&= 2^{SB}g\big(\nabla_X Y, Z) + 2\delta^2 g(\nabla_X(\varphi\xi), Y)g(\varphi\xi, Z) \\
&= 2^{SB}g\big(\nabla_X Y, Z) + \frac{2\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)G(\varphi\xi, Z).
\end{aligned}$$

Hence, we get

$$^{SB}\nabla_X Y = \nabla_X Y + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)\varphi\xi.$$

Using (2.11) and (2.13), we obtain the following

$$^{SB}\nabla_X(\varphi\xi) = \nabla_X(\varphi\xi), \tag{2.14}$$

for all vector field $X \in \Im_0^1(M)$.

## 3. Curvatures of semi-conformal deformation of Berger-type metric

**Theorem 3.1.** *Let $(M^{2m}, \varphi, g)$ be an anti-paraKähler manifold. If $^{SB}R$ denote the Riemannian curvature tensor of $(M^{2m}, {}^{SB}g)$, then we have the following*

$$\begin{aligned}
^{SB}R(X,Y)Z &= R(X,Y)Z + \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi, Z)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\nabla_X(\varphi\xi) \\
&\quad - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Z)\nabla_Y(\varphi\xi),
\end{aligned} \tag{3.15}$$

*for all vector fields $X, Y, Z \in \Im_0^1(M)$, where $R$ denote the curvature tensor of $(M^{2m}, \varphi, g)$.*

**Proof.** For all $X, Y, Z \in \Im_0^1(M)$,

$$^{SB}R(X,Y)Z = {}^{SB}\nabla_X {}^{SB}\nabla_Y Z - {}^{SB}\nabla_Y {}^{SB}\nabla_X Z - {}^{SB}\nabla_{[X,Y]} Z.$$

By virtue of (2.13) and (2.14), we obtain

$$\begin{aligned}
{}^{SB}\nabla_X {}^{SB}\nabla_Y Z &= {}^{SB}\nabla_X\Big(\nabla_Y Z + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\varphi\xi\Big) \\
&= {}^{SB}\nabla_X(\nabla_Y Z) + \frac{\delta^2}{1+\delta^2} X\big(g(\nabla_Y(\varphi\xi), Z)\big)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\big)^{SB}\nabla_X(\varphi\xi) \\
&= \nabla_X(\nabla_Y Z) + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), (\nabla_Y Z))\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_X\nabla_Y(\varphi\xi), Z)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), \nabla_X Z)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\nabla_X(\varphi\xi).
\end{aligned}$$

In fact, by substituting $X$ by $Y$ into the ${}^{SB}\nabla_X {}^{SB}\nabla_Y Z$, we get,

$$\begin{aligned}
{}^{SB}\nabla_Y {}^{SB}\nabla_X Z &= \nabla_Y(\nabla_X Z) + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), (\nabla_X Z))\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y\nabla_X(\varphi\xi), Z)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), \nabla_Y Z)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Z)\nabla_Y(\varphi\xi).
\end{aligned}$$

We also find

$$ {}^{SB}\nabla_{[X,Y]}Z = \nabla_{[X,Y]}Z + \frac{\delta^2}{1+\delta^2} g(\nabla_{[X,Y]}(\varphi\xi), Z)\varphi\xi. $$

Hence, we have

$$\begin{aligned}
{}^{SB}R(X,Y)Z &= R(X,Y)Z + \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi, Z)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Z)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Z)\nabla_Y(\varphi\xi).
\end{aligned}$$

**Theorem 3.2.** *Let $(M^{2m}, \varphi, g)$ be an anti-paraKähler manifold. If $K$ (resp., ${}^{SB}K$) denote the sectional curvature of $(M^{2m}, \varphi, g)$ (resp., $(M^{2m}, {}^{SB}g)$), then we have the following*

$$\begin{aligned}
{}^{SB}K(X,Y) &= \frac{1}{1+\delta^2\Big(g(X,\varphi\xi)^2 + g(Y,\varphi\xi)^2\Big)}\Big(K(X,Y) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi), Y)^2 \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi), Y) g(\nabla_X(\varphi\xi), X)\Big), \qquad (3.16)
\end{aligned}$$

*for any $X, Y \in \Im^1_0(M)$ two vector fields orthonormal with respect to $g$.*

**Proof.** For $x \in M$, $V, W \in \Im_0^1(M)$ and such that $V_x$ and $W_x$ are linearly independent, the sectional curvature of the plane spanned by $V_x$ and $W_x$ is given by

$$^{SB}K(V,W) = \frac{^{SB}g(^{SB}R(V,W)W,V)}{^{SB}g(V,V)^{SB}g(W,W) - {}^{SB}g(V,W)^2}.$$

First we calculate,

$$^{SB}g(^{SB}R(X,Y)Y,X) = g(^{SB}R(X,Y)Y,X) + \delta^2 g(^{SB}R(X,Y)Y,\varphi\xi)g(X,\varphi\xi).$$

From (2.11) and (3.15) with direct computation we get,

$$\begin{aligned} ^{SB}g(^{SB}R(X,Y)Y,X) &= g(R(X,Y)Y,X) + \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi,Y)g(\varphi\xi,X) \\ &+ \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi),X) \\ &- \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi),Y)g(\nabla_Y(\varphi\xi),X) \\ &+ \delta^2 g(X,\varphi\xi)\Big(g(R(X,Y)Y,\varphi\xi) \\ &+ \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi,Y)g(\varphi\xi,\varphi\xi) \\ &+ \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi),\varphi\xi) \\ &- \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi),Y)g(\nabla_Y(\varphi\xi),\varphi\xi)\Big). \end{aligned}$$

By simple calculation, we find

$$\begin{aligned} ^{SB}g(^{SB}R(X,Y)Y,X) &= g(R(X,Y)Y,X) + \frac{\delta^2}{1+\delta^2} g(R(X,Y)\varphi\xi,Y)g(X,\varphi\xi) \\ &+ \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi)X) \\ &- \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi),Y)^2 - \delta^2 g(R(X,Y)\varphi\xi,Y)g(X,\varphi\xi) \\ &+ \frac{\delta^4}{1+\delta^2} g(R(X,Y)\varphi\xi,Y)g(X,\varphi\xi) \\ &= K(X,Y) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi),Y)^2 \\ &+ \frac{\delta^2}{1+\delta^2} g(\nabla_Y(\varphi\xi),Y)g(\nabla_X(\varphi\xi),X). \end{aligned} \tag{3.17}$$

On the other hand, we have

$$^{SB}g(X,X)^{SB}g(Y,Y) - {}^{SB}g(X,Y)^2 = 1 + \delta^2\Big(g(X,\varphi\xi)^2 + g(Y,\varphi\xi)^2\Big). \tag{3.18}$$

From (3.17) and (4.33), we get the formula (3.16).

**Corollary 3.1.** *If $\nabla\xi = 0$, the sectional curvature ${}^{SB}K$ of $(M^{2m}, {}^{SB}g)$ is given by*

$$ {}^{SB}K(X,Y) = \frac{K(X,Y)}{1+\delta^2\Big(g(X,\varphi\xi)^2 + g(Y,\varphi\xi)^2\Big)} $$

*for any $X, Y$ two vector fields orthonormal with respect to $g$.*

**Remark 3.1.** *Let $\{E_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, \varphi, g)$, such that $E_1 = \varphi\xi$, we define the orthonormal vector fields*

$$ \widetilde{E}_1 = \frac{1}{\sqrt{1+\delta^2}} E_1, \widetilde{E}_i = E_i,\ i = \overline{2,2m}, \tag{3.19} $$

*then $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ is a local orthonormal frame on $(M^{2m}, {}^{SB}g)$.*

**Theorem 3.3.** *Let $(M^{2m}, \varphi, g)$ be an anti-paraKähler manifold. If Ricci (resp. ${}^{SB}Ricci$) denote the Ricci tensor of $(M^{2m}, \varphi, g)$ (resp., $(M^{2m}, {}^{SB}g)$), then we have the following*

$$ \begin{aligned} {}^{SB}Ricci(X) &= Ricci(X) - \frac{\delta^2}{1+\delta^2} R(X,\xi)\xi - \frac{\delta^2}{1+\delta^2} Ric(X,\varphi\xi)\varphi\xi \\ &\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2}\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi), \end{aligned} \tag{3.20} $$

*for all vector field $X \in \Im^1_0(M)$.*

**Proof.** Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{SB}g)$ defined by (3.19). By the definition of Ricci tensor, we have

$$ \begin{aligned} {}^{SB}Ricci(X) &= \sum_{i=1}^{2m} {}^{SB}R(X,\widetilde{E}_i)\widetilde{E}_i \\ &= \frac{1}{1+\delta^2} {}^{SB}R(X,\varphi\xi)\varphi\xi + \sum_{i=2}^{2m} {}^{SB}R(X,E_i)E_i. \end{aligned} $$

From (2.10), (2.11) and (3.15) with direct computation we get,

$$\begin{aligned}
{}^{SB}Ricci(X) &= \frac{1}{1+\delta^2}\Big(R(X,\varphi\xi)\varphi\xi + \frac{\delta^2}{1+\delta^2} g(R(X,\varphi\xi)\varphi\xi,\varphi\xi)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_{\varphi\xi}(\varphi\xi),\varphi\xi)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi),\varphi\xi)\nabla_{\varphi\xi}(\varphi\xi)\Big) \\
&\quad + \sum_{i=2}^{m}\Big(R(X,E_i)E_i + \frac{\delta^2}{1+\delta^2} g(R(X,E_i)\varphi\xi,E_i)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} g(\nabla_{E_i}(\varphi\xi),E_i)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2} g(\nabla_X(\varphi\xi),E_i)\nabla_{E_i}(\varphi\xi)\Big) \\
&= \frac{1}{1+\delta^2} R(X,\xi)\xi + Ricci(X) - R(X,\varphi\xi)\varphi\xi - \frac{\delta^2}{1+\delta^2} Ric(X,\varphi\xi)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2}\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi) \\
&= Ricci(X) - \frac{\delta^2}{1+\delta^2} R(X,\xi)\xi - \frac{\delta^2}{1+\delta^2} Ric(X,\varphi\xi)\varphi\xi \\
&\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi)\nabla_X(\varphi\xi) - \frac{\delta^2}{1+\delta^2}\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi).
\end{aligned}$$

**Theorem 3.4.** *Let $(M^{2m},\varphi,g)$ be an anti-paraKähler manifold. If Ric (resp. ${}^{SB}Ric$) denote the Ricci curvature of $(M^{2m},\varphi,g)$ (resp., $(M^{2m},{}^{SB}g)$), then we have*

$$\begin{aligned}
{}^{SB}Ric(X,Y) &= Ric(X,Y) - \frac{\delta^2}{1+\delta^2} g(R(X,\xi)\xi,Y) - \frac{\delta^2}{1+\delta^2} g(\nabla_X\xi,\nabla_Y\xi) \\
&\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi) g(\nabla_X(\varphi\xi),Y), \qquad (3.21)
\end{aligned}$$

*for any vector field $X \in \Im_0^1(M)$.*

**Proof.** Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{SB}g)$ defined by (3.19). By the definition of Ricci tensor, we have

$$\begin{aligned}
{}^{SB}Ric(X,Y) &= {}^{SB}g({}^{SB}Ricci(X),Y) \\
&= g({}^{SB}Ricci(X),Y) + \delta^2 g({}^{SB}Ricci(X),\varphi\xi) g(Y,\varphi\xi).
\end{aligned}$$

From the formula (3.20) and direct computation we get,

$$\begin{aligned}
{}^{SB}Ric(X,Y) &= g(Ricci(X),Y) - \frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y) \\
&\quad -\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(\varphi\xi,Y) + \frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y) \\
&\quad -\frac{\delta^2}{1+\delta^2}g(\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi),Y) \\
&\quad +\delta^2 g(Y,\varphi\xi)\Big(g(Ricci(X),\varphi\xi) - \frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,\varphi\xi) \\
&\quad -\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(\varphi\xi,\varphi\xi) + \frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),\varphi\xi) \\
&\quad -\frac{\delta^2}{1+\delta^2}g(\nabla_{\nabla_X(\varphi\xi)}(\varphi\xi),\varphi\xi)\Big) \\
&= Ric(X,Y) - \frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y) - \frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(Y,\varphi\xi) \\
&\quad +\frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y) - \frac{\delta^2}{1+\delta^2}g(\nabla_Y\xi,\nabla_X\xi) \\
&\quad +\frac{\delta^2}{1+\delta^2}Ric(X,\varphi\xi)g(Y,\varphi\xi) \\
&= Ric(X,Y) - \frac{\delta^2}{1+\delta^2}g(R(X,\xi)\xi,Y) + \frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_X(\varphi\xi),Y) \\
&\quad -\frac{\delta^2}{1+\delta^2}g(\nabla_X\xi,\nabla_Y\xi).
\end{aligned}$$

**Theorem 3.5.** *Let $(M^{2m},\varphi,g)$ be an anti-paraKähler manifold. If $\sigma$ (resp., ${}^{SB}\sigma$) denote the scalar curvature of $(M^{2m},\varphi,g)$ (resp., $(M^{2m},{}^{SB}g)$), then we have the following*

$$ {}^{SB}\sigma = \sigma - \frac{2\delta^2}{1+\delta^2}Ric(\xi,\xi) + \frac{\delta^2}{1+\delta^2}(div(\varphi\xi))^2 - \frac{\delta^2}{1+\delta^2}trace_g g(\nabla\xi,\nabla\xi). \quad (3.22)$$

**Proof.** Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},{}^{SB}g)$ defined by (3.19). We have

$$\begin{aligned}
{}^{SB}\sigma &= \sum_{i=1}^{2m} {}^{SB}Ric(\widetilde{E}_i,\widetilde{E}_i) \\
&= \frac{1}{1+\delta^2}{}^{SB}Ric(\varphi\xi,\varphi\xi) + \sum_{i=2}^{2m} {}^{SB}Ric(E_i,E_i).
\end{aligned}$$

From the formula (3.21) and direct computation we get,

$$\begin{aligned}
{}^{SB}\sigma &= \frac{1}{1+\delta^2}\Big(Ric(\varphi\xi,\varphi\xi) - \frac{\delta^2}{1+\delta^2}g(R(\varphi\xi,\xi)\xi,\varphi\xi) \\
&\quad + \frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_{\varphi\xi}(\varphi\xi),\varphi\xi) - \frac{\delta^2}{1+\delta^2}g(\nabla_{\varphi\xi}\xi,\nabla_{\varphi\xi}\xi)\Big) \\
&\quad + \sum_{i=2}^{2m}\Big(Ric(E_i,E_i) - \frac{\delta^2}{1+\delta^2}g(R(E_i,\xi)\xi,E_i) \\
&\quad + \frac{\delta^2}{1+\delta^2}div(\varphi\xi)g(\nabla_{E_i}(\varphi\xi),E_i) - \frac{\delta^2}{1+\delta^2}g(\nabla_{E_i}\xi,\nabla_{E_i}\xi)\Big) \\
&= \frac{1}{1+\delta^2}Ric(\xi,\xi) - \sigma - Ric(\xi,\xi) - \frac{\delta^2}{1+\delta^2}Ric(\xi,\xi) + \frac{\delta^2}{1+\delta^2}(div(\varphi\xi))^2 \\
&\quad - \frac{\delta^2}{1+\delta^2}trace_g g(\nabla\xi,\nabla\xi) \\
&= \sigma - \frac{2\delta^2}{1+\delta^2}Ric(\xi,\xi) + \frac{\delta^2}{1+\delta^2}(div(\varphi\xi))^2 - \frac{\delta^2}{1+\delta^2}trace_g g(\nabla\xi,\nabla\xi).
\end{aligned}$$

## 4. Harmonicity of semi-conformal deformation of Berger-type metric

### 4.1. **The harmonicity of the Identity map.**

We study the both cases $Id:(M^{2m},\varphi,g)\to(M^{2m},{}^{SB}g)$ or $Id:(M^{2m},{}^{SB}g)\to(M^{2m},\varphi,g)$.

**Proposition 4.1.** *The tension field $\tau(Id)$ of $Id:(M^{2m},\varphi,g)\to(M^{2m},{}^{SB}g)$ is given by*

$$\tau(Id) = \frac{\delta^2}{1+\delta^2}div(\varphi\xi)\varphi\xi. \tag{4.23}$$

**Proof.** Let $\{e_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m},\varphi,g)$, the tension field $\tau(Id)$ of $Id:(M^{2m},\varphi,g)\to(M^{2m},{}^{SB}g)$ is give by.

$$\begin{aligned}
\tau(Id) &= \sum_{i=1}^{2m}\big({}^{SB}\nabla^{Id}_{e_i}dId(e_i) - dId(\nabla_{e_i}e_i)\big) \\
&= \sum_{i=1}^{2m}\big({}^{SB}\nabla_{dId(e_i)}dId(e_i) - \nabla_{e_i}e_i\big) \\
&= \sum_{i=1}^{2m}\big({}^{SB}\nabla_{e_i}e_i - \nabla_{e_i}e_i\big),
\end{aligned}$$

by virtue of theorem 2.1, we have

$$\begin{aligned}
\tau(Id) &= \sum_{i=1}^{2m}\Big(\nabla_{e_i}e_i + \frac{\delta^2}{1+\delta^2}g(\nabla_{e_i}(\varphi\xi),e_i)\varphi\xi - \nabla_{e_i}e_i\Big) \\
&= \frac{\delta^2}{1+\delta^2}\sum_{i=1}^{2m}g(\nabla_{e_i}(\varphi\xi),e_i)\varphi\xi \\
&= \frac{\delta^2}{1+\delta^2}div(\varphi\xi)\varphi\xi.
\end{aligned}$$

From the proposition 4.1 we find the following theorem.

**Theorem 4.1.** *$Id : (M^{2m}, \varphi, g) \to (M^{2m}, {}^{SB}g)$ is harmonic if and only if*

$$div(\varphi\xi) = 0. \tag{4.24}$$

**Proposition 4.2.** *The tension field of $Id : (M^{2m}, {}^{SB}g) \to (M^{2m}, \varphi, g)$ is given by*

$$\tau(Id) = -\frac{\delta^2}{1+\delta^2} div(\varphi\xi)\varphi\xi. \tag{4.25}$$

**Proof.** Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{SB}g)$ defined by (3.19).

$$\begin{aligned}
\tau(Id) &= \sum_{i=1}^{2m} \big(\nabla^{Id}_{\widetilde{E}_i} dId(\widetilde{E}_i) - dId({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i)\big) \\
&= \sum_{i=1}^{2m} \big(\nabla_{dId(\widetilde{E}_i)} dId(\widetilde{E}_i) - {}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i\big) \\
&= \sum_{i=1}^{2m} \big(\nabla_{\widetilde{E}_i}\widetilde{E}_i - {}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i\big),
\end{aligned}$$

by virtue of theorem 2.1, we get

$$\begin{aligned}
\tau(Id) &= \sum_{i=1}^{2m} \Big(\nabla_{\widetilde{E}_i}\widetilde{E}_i - \nabla_{\widetilde{E}_i}\widetilde{E}_i - \frac{\delta^2}{1+\delta^2} g(\nabla_{\widetilde{E}_i}(\varphi\xi), \widetilde{E}_i)\varphi\xi\Big) \\
&= -\frac{\delta^2}{1+\delta^2} \sum_{i=1}^{2m} g(\nabla_{\widetilde{E}_i}(\varphi\xi), \widetilde{E}_i)\varphi\xi \\
&= -\frac{\delta^2}{(1+\delta^2)^2} g(\nabla_{\varphi\xi}(\varphi\xi), \varphi\xi)\varphi\xi - \frac{\delta^2}{1+\delta^2} \sum_{i=2}^{2m} g(\nabla_{E_i}(\varphi\xi), E_i)\varphi\xi \\
&= -\frac{\delta^2}{1+\delta^2} div(\varphi\xi)\varphi\xi.
\end{aligned}$$

From the proposition 4.2, we obtain the next theorem.

**Theorem 4.2.** *$Id : (M^{2m}, {}^{SB}g) \to (M^{2m}, \varphi, g)$ is harmonic if and only if*

$$div(\varphi\xi) = 0. \tag{4.26}$$

**Example 4.1.** *Let $(M^2 = ]0, +\infty[\times]0, \pi[, \varphi, g)$ be an anti-paraKähler manifold, such that $(\varphi, g)$ in polar coordinate defined by*

$$g = dr^2 + r^2 d\theta^2,$$

*and*

$$\varphi\frac{\partial}{\partial r} = \sin(2\theta)\frac{\partial}{\partial r} + \frac{1}{r}\cos(2\theta)\frac{\partial}{\partial\theta}, \quad \varphi\frac{\partial}{\partial\theta} = r\cos(2\theta)\frac{\partial}{\partial r} - \sin(2\theta)\frac{\partial}{\partial\theta}.$$

*Let* $\xi = \cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$. *By a simple calculation, we have*

$$\begin{aligned} |\xi| &= 1, \\ \varphi\xi &= \sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\frac{\partial}{\partial\theta}, \\ \nabla(\varphi\xi) &= 0, \\ div(\varphi\xi) &= 0. \end{aligned}$$

*So, thus* $Id : (M^2, \varphi, g) \to (M^2, {}^{SB}g)$ *is harmonic, where*

$$^{SB}g = (1 + \delta^2\sin^2\theta)dr^2 + (r^2 + \delta^2\cos^2\theta)d\theta^2 + r\sin(2\theta)drd\theta.$$

### 4.2. Harmonicity of the map $\phi : (M^{2m}, {}^{SB}g) \longrightarrow (N^n, h)$.

**Proposition 4.3.** *The tension field of the map* $\phi : (M^{2m}, {}^{SB}g) \longrightarrow (N^n, h)$ *is given by*

$$^{SB}\tau(\phi) = \tau(\phi) - \frac{\delta^2}{1+\delta^2}\nabla d\phi(\varphi\xi, \varphi\xi) - \frac{\delta^2}{1+\delta^2}div(\varphi\xi)d\phi(\varphi\xi), \tag{4.27}$$

*where* $\tau(\phi)$ *is the tension field of* $\phi : (M^{2m}, \varphi, g) \longrightarrow (N^n, h)$.

**Proof.** Let $\{\widetilde{E}_i\}_{i=\overline{1,2m}}$ be a local orthonormal frame on $(M^{2m}, {}^{SB}g)$ defined by (3.19), we compute the tension field ${}^{SB}\tau(\phi)$ of the map $\phi : (M^{2m}, {}^{SB}g) \longrightarrow (N^n, h)$.

$$^{SB}\tau(\phi) = \sum_{i=1}^{2m}\nabla^N_{d\phi(\widetilde{E}_i)}d\phi(\widetilde{E}_i) - \sum_{i=1}^{m}d\phi({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i). \tag{4.28}$$

By direct calculations we obtain

$$\begin{aligned} \sum_{i=1}^{m}\nabla^N_{d\phi(\widetilde{E}_i)}d\phi(\widetilde{E}_i) &= \nabla^N_{d\phi(\widetilde{E}_1)}d\phi(\widetilde{E}_1) + \sum_{i=2}^{2m}\nabla^N_{d\phi(\widetilde{E}_i)}d\phi(\widetilde{E}_i) \\ &= \frac{1}{1+\delta^2}\nabla^N_{d\phi(\varphi\xi)}d\phi(\varphi\xi) + \sum_{i=2}^{2m}\nabla^N_{d\phi(E_i)}d\phi(E_i) \\ &= -\frac{\delta^2}{1+\delta^2}\nabla^N_{d\phi(\varphi\xi)}d\phi(\varphi\xi) + \sum_{i=1}^{2m}\nabla^N_{d\phi(E_i)}d\phi(E_i) \end{aligned} \tag{4.29}$$

and by similar calculations we obtain

$$\begin{aligned}
\sum_{i=1}^{2m} d\phi({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i) &= d\phi({}^{SB}\nabla_{\widetilde{E}_1}\widetilde{E}_1) + \sum_{i=2}^{2m} d\phi({}^{SB}\nabla_{\widetilde{E}_i}\widetilde{E}_i) \\
&= \frac{1}{1+\delta^2} d\phi(\nabla_{\varphi\xi}\varphi\xi) + \sum_{i=2}^{2m} d\phi(\nabla_{E_i}E_i) \\
&\quad + \frac{\delta^2}{1+\delta^2}\sum_{i=2}^{2m} g(\nabla_{E_i}(\varphi\xi), E_i) d\phi(\varphi\xi) \\
&= -\frac{\delta^2}{1+\delta^2} d\phi(\nabla_{\varphi\xi}\varphi\xi) + \sum_{i=1}^{2m} d\phi(\nabla_{E_i}E_i) \\
&\quad + \frac{\delta^2}{1+\delta^2} div(\varphi\xi) d\phi(\varphi\xi). \qquad (4.30)
\end{aligned}$$

In fact, by adding (4.29) and (4.30) in (4.28), we get

$$ {}^{SB}\tau(\phi) = \tau(\phi) - \frac{\delta^2}{1+\delta^2}\nabla d\phi(\varphi\xi,\varphi\xi) - \frac{\delta^2}{1+\delta^2} div(\varphi\xi) d\phi(\varphi\xi),$$

where,

$$\nabla d\phi(\varphi\xi,\varphi\xi) = \nabla^N_{d\phi(\varphi\xi)} d\phi(\varphi\xi) - d\phi(\nabla_{\varphi\xi}\varphi\xi).$$

From the proposition 4.3 we obtain the following theorem.

**Theorem 4.3.** *The map $\phi : (M^{2m}, {}^{SB}g \longrightarrow (N^n, h)$ is harmonic if and only if*

$$\tau(\phi) = \frac{\delta^2}{1+\delta^2}\nabla d\phi(\varphi\xi,\varphi\xi) + \frac{\delta^2}{1+\delta^2} div(\varphi\xi) d\phi(\varphi\xi). \qquad (4.31)$$

4.3. **Harmonicity of the map $\phi : (M^m, g) \longrightarrow (N^{2n}, {}^{SB}h)$.**

**Proposition 4.4.** *The tension field of the map $\phi : (M^m, g) \longrightarrow (N^{2n}, {}^{SB}h)$ is given by*

$$ {}^{SB}\tau(\phi) = \tau(\phi) + \frac{\delta^2}{1+\delta^2} trace_g h(\nabla^N_{d\phi(*)}(\varphi\xi), d\phi(*))\varphi\xi, \qquad (4.32)$$

*where $\tau(\phi)$ is the tension field of $\phi : (M^m, g) \longrightarrow (N^{2n}, \varphi, h)$.*

**Proof.** Let $\{e_i\}_{i=\overline{1,m}}$ be a local orthonormal frame on $(M^m, g)$, we compute the tension field ${}^{SB}\tau(\phi)$ of the map $\phi : (M^m, g) \longrightarrow (N^{2n}, {}^{SB}h)$.

$$\begin{aligned} {}^{SB}\tau(\phi) &= \sum_{i=1}^{m} \left( {}^{SB}\nabla^N_{d\phi(e_i)} d\phi(e_i) - d\phi(\nabla_{e_i} e_i) \right) \\ &= \sum_{i=1}^{m} \left( \nabla^N_{d\phi(e_i)} d\phi(e_i) + \frac{\delta^2}{1+\delta^2} h(\nabla^N_{d\phi(e_i)}(\varphi\xi), d\phi(e_i))\varphi\xi - d\phi(\nabla_{e_i} e_i) \right) \\ &= \tau(\phi) + \frac{\delta^2}{1+\delta^2} trace_g h(\nabla^N_{d\phi(*)}(\varphi\xi), d\phi(*))\varphi\xi. \end{aligned}$$

From the proposition 4.4 we obtain the following theorem.

**Theorem 4.4.** *The map $\phi : (M^m, g) \longrightarrow (N^n, {}^{SB}h)$ is harmonic if and only if*

$$\tau(\phi) = -\frac{\delta^2}{1+\delta^2} trace_g h(\nabla^N_{d\phi(*)}(\varphi\xi), d\phi(*))\varphi\xi. \tag{4.33}$$

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# CSI-$\xi^\perp$- RIEMANNIAN SUBMERSIONS FROM LORENTZIAN PARA-KENMOTSU MANIFOLDS

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ABSTRACT. The purpose of this article is to examine the characteristics of Clairaut semi-invariant-$\xi^\perp$(CSI-$\xi^\perp$,in brief) Riemannian submersions from Lorentzian para-Kenmotsu manifolds onto Riemannian manifolds and also enrich this geometrical analysis with specific condition for a semi-invariant $\xi^\perp$-Riemannian submersion to be CSI-$\xi^\perp$-Riemannian submersion. Furthermore, we discuss some results about these submersions and present a consequent non-trivial example based on this study.

**Keywords**: Riemannian submersions, Clairaut semi-invariant submersion, Lorentzian para Kenmotsu manifolds

**2010 Mathematics Subject Classification**: 53C43, 53C15, 53C22, 53C42, 53C55.

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## 1. INTRODUCTION

Let $N_1$ be a semi-Riemannian manifold endowed with a semi-Riemannian metric $g_{N_1}$. A Lorentzian manifold is a subclass of semi-Riemannian manifold. Since the Lorentzian manifold has many applications in science and technology, especially in the theory of relativity and cosmology, therefore it attracts many researchers to do the research in this area. The different classes of Lorentzian manifolds have been studied in ([15], [16], [17], [25], [26]) and by many others.

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The concept of Riemannian submersions is studied extensively together with starting the study of Riemannian geometry. In fact, the theory of Riemannian submersion was initiated by O' Neill [27] in 1966, and it has been further studied by Gray [13], in 1967. Watson [40] popularized the knowledge of Riemannian submersions considering almost Hermitian manifolds in terms of almost Hermitian submersions. The Riemannian submersions play a vital role not only in the differential geometry but also in science and technology. It is noticed that the theory of Riemannian submersions are capable to handle many issues of the singularity theory, Yang-Mills theory, quantum theory, Kaluza-Klein theory, relativity, superstring theories, mechanics, modelling, robotics etc. (see, [4], [7], [6], [10], [11], [18], [19], [20]). For more details, we cite the books ([12], [35]) and the references therein. The Riemannian submersions motivate the researchers to define the semi-Riemannian submersions and Lorentzian submersion [12], almost Hermitian submersions [40], almost contact submersions [26], anti-invariant Riemannian submersions [34], semi-slant submersion [28], conformal anti-invariant submersions ([21], [29]), conformal semi-invariant submersion [22], conformal semi-slant submersions ([22], [30]), para-contact submersions [14], quasi bi-slant submersion ([31], [32], [33]).

In 1972, Bishop [8] presented the hypothesis and conditions of a Clairaut submersion in terms of a natural generalization of a surface of a revolution. Let $c$ is any geodesic defined as $c : I_1 \subset R \to M$, $\phi(s)$ is the angle between $c(s)$ and the meridian curve through $c(s)$, $s \in I_1$. Under these conditions, the product $rsin\phi$ is constant on the revolution surface $M$ along geodesic $c$. Hence, it is apart from $s$. Afterwards, this idea has been considered in Lorentzian spaces, timelike and spacelike spaces ([24] [37], [38]).

In 1981, Allison [3] proposed Clairaut submersions in case the total manifold is Lorentzian. In addition, it is discovered that Clairaut submersions are used for static spacetime applications. Furthermore, Clairaut submersions have been generalized in [5]. The concept of anti-invariant Riemannian submersions was initiated by Lee [23] in 2013. On the other hand, Sahin [36] introduced Clairaut Riemannian map and studied it's geometric properties in 2017.

In 2017, Akyol, Sari and Aksoy [1] introduced the notion of semi-invariant $\xi^{\perp}-$Riemannian Submersions as well as semi-slant $\xi^{\perp}-$Riemannian Submersions [2], as a generalization of anti invariant $\xi^{\perp}-$Riemannian Submersions and discussed the geometry of the total space and the base space for the existence of such submersions.

The above studies inspire us to introduce the notion of CSI-$\xi^{\perp}$-Riemannian submersions from the Lorentzian para-Kenmotsu (a subclass of semi-Riemannian) manifolds to the Riemannian manifolds and characterize its geometrical properties. Throughout the paper, we denote the Lorentzian para-Kenmotsu manifold of dimension $n$ by $(N_1, g_{N_1})$. It is noticed that Akyol et al. [1] has been studied the properties of semi-invariant $\xi^{\perp}-$Riemannian Submersions from a class of Riemannian manifold (almost contact manifold) to a Riemannian manifold but in this paper, we are going to characterize the properties of CSI-$\xi^{\perp}$- Riemannian submersions from a class of semi-Riemannian manifold to a Riemannian manifold, which is an extension of [1].

We exhibit our work as follows: Section 2 contains some basic results of Lorentzian para-Kenmotsu manifold, a non-trivial example of Lorentzian para-Kenmotsu manifold. In Section 3, we give the basic definitions related to semi-invariant $\xi^{\perp}-$ Riemannian Submersions and well-known Lemma. In section 4, we define CSI-$\xi^{\perp}$- Riemannian submersions from the Lorentzian para-Kenmotsu manifolds and discuss some geometrical properties of such submersions. The last section is concerned with a non-trivial example of Lorentzian para-Kenmotsu manifold with CSI-$\xi^{\perp}$- Riemannian submersion.

## 2. Lorentzian para-Kenmotsu manifolds

Let $N_1$ be an $n-$dimensional Lorentzian metric manifold if it is endowed with a structure $(\phi, \xi, \eta, g_{N_1})$, where $\phi$ is a $(1,1)$ tensor field, $\xi$ is a vector field, $\eta$ is a $1-$form on $N_1$ and $g_{N_1}$ is a Lorentzian metric satisfying:

$$\phi^2 = I + \eta \otimes \xi, \phi \circ \xi = 0, \eta \circ \phi = 0, \tag{2.1}$$

$$g_{N_1}(\phi W_1, \phi W_2) = g_{N_1}(W_1, W_2) + \eta(W_1)\eta(W_2), g_{N_1}(\phi W_1, W_2) = g_{N_1}(W_1, \phi W_2), \tag{2.2}$$

$$\eta(\xi) = -1, g_{N_1}(W_2, \xi) = \eta(W_2), \tag{2.3}$$

for any vector field $W_1, W_2$ on $N_1$, then it is called Lorentzian almost para-contact manifold. In the Lorentzian almost para-contact manifold following relations hold:

$$\Phi(W_1, W_2) = \Phi(W_2, W_1) = g_{N_1}(W_1, \phi W_2), \tag{2.4}$$

where $\Phi$ is symmetric $(0,2)$ tensor field and vector fields $W_1$ and $W_2$ on $N_1$.

If $\xi$ is a killing vector field, the para-contact structure is called $K-$para contact.

A Lorentzian almost para-contact manifold $N_1$ is called Lorentzian para-Kenmotsu manifold [9] if

$$(\nabla_{W_1}\phi)W_2 = -g_{N_1}(\phi W_1, W_2)\xi - \eta(W_2)\phi W_1, \tag{2.5}$$

for any vector field $W_1, W_2$ on $N_1$.

In the Lorentzian para-Kenmotsu manifold, we have

$$\nabla_{W_2}\xi = -W_2 - \eta(W_2)\xi, \tag{2.6}$$

$$(\nabla_{W_1}\eta)W_2 = -g_{N_1}(W_1, W_2) - \eta(W_1)\eta(W_2), \tag{2.7}$$

where $\nabla$ denotes the operation of covariant differentiation (Levi-Civita connection) with respect to the Lorentzian metric $g_{N_1}$.

In a Lorentzian para Kenmotsu manifold, it is clear that

$$rank\phi = n - 1.$$

**Example 2.1.** [39] *We consider* $(2n+1)$ *dimensional manifold* $R^{2n+1} = \{(x^1, x^2, ...., x^n, y^1, y^2, ....y^n, z) = (x^i, y^i, z) \in R^{2n+1}, (x^i, y^i, z \in R, i = 1, 2, ...., n)\}$. *Consider* $R^{2n+1}$ *with the following structure:*

$$\phi(X_i) = Y_i, \phi(Y_i) = X_i, \phi(\xi) = 0,$$

*which are linearly independent at each point of* $N_1$. *Let* $g_{N_1}$ *is Lorentzian metric defined by*

$$g_{N_1} = -(\eta \otimes \eta) + e^{2Z}\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i),$$

$$\phi^2 X = X + \eta(X)\xi, g_{N_1}(X, \xi) = \eta(X),$$

*for all vector fields* $X$ *on* $R^{2n+1}$.

*Then,* $(R^{2m+1}, \phi, \xi, \eta, g_{N_1})$ *is a Lorentzian para-Kenmotsu manifold. The vector fields* $X_i = e^{-Z}\frac{\partial}{\partial x^i}, Y_i = e^{-Z}\frac{\partial}{\partial y^i}$ *and* $\xi = \frac{\partial}{\partial z}$ *form a* $\phi-$*basis for Lorentzian para-Kenmotsu manifold* $R^{2n+1}$, *where* $i = 1, 2, ....., n$.

## 3. Semi-invariant $\xi^{\perp}-$ Riemannian submersions

An essential background of Riemannian submersions ($F : N_1 \to N_2$) and definition of semi-invariant $\xi^{\perp}-$ Riemannian submersions are given at this section. It is well-known that the fundamental tensors $\mathcal{T}$ and $\mathcal{A}$, define by O'Neill's [27] by

$$\mathcal{A}_{Z_1}U_1 = \mathcal{H}\nabla_{\mathcal{H}Z_1}\mathcal{V}U_1 + \mathcal{V}\nabla_{\mathcal{H}Z_1}\mathcal{H}U_1, \tag{3.8}$$

$$\mathcal{T}_{Z_1}U_1 = \mathcal{H}\nabla_{\mathcal{V}Z_1}\mathcal{V}U_1 + \mathcal{V}\nabla_{\mathcal{V}Z_1}\mathcal{H}U_1 \tag{3.9}$$

for any vector fields $Z_1, U_1$ on $N_1$, where $\nabla$ is the Levi-Civita connection of $g_{N_1}$.
From equations (3.8) and (3.9), we have

$$\nabla_{Y_1} Z_2 = \mathcal{T}_{Y_1} Z_2 + \mathcal{V}\nabla_{Y_1} Z_2, \tag{3.10}$$

$$\nabla_{Y_1} U_1 = \mathcal{T}_{Y_1} U_1 + \mathcal{H}\nabla_{Y_1} U_1, \tag{3.11}$$

$$\nabla_{U_1} Y_1 = \mathcal{A}_{U_1} Y_1 + \mathcal{V}\nabla_{U_1} Y_1, \tag{3.12}$$

$$\nabla_{U_1} W_2 = \mathcal{H}\nabla_{U_1} W_2 + \mathcal{A}_{U_1} W_2 \tag{3.13}$$

for all $Y_1, Z_2 \in \Gamma(\ker F_*)$ and $U_1, W_2 \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_{Y_1} U_1 = \mathcal{A}_{U_1} Y_1$, if $U_1$ is basic. It is easy to notice that $\mathcal{A}$ performs on the horizontal distribution and estimates the interference to the integrability of this distribution and $\mathcal{T}$ performs on the fibers as the second fundamental form. .

Here $F$ between two Riemannian manifolds called totally geodesic if

$$(\nabla F_*)(U_1, W_2) = 0, \text{ for all } U_1, W_2 \in \Gamma(TN_1) \tag{3.14}$$

and $F$ is called totally umbilical if [6]

$$\mathcal{T}_{Y_1} Y_2 = g_{N_1}(Y_1, Y_2) H \tag{3.15}$$

for all $Y_1, Y_2 \in \Gamma(\ker F_*)$, where $H$ represents the mean curvature vector field of fibers.

The the second fundamental form of $F$ is given by

$$(\nabla F_*)(W_1, W_2) = \nabla^F_{W_1} F_*(W_2) - F_*(\nabla^{N_1}_{W_1} W_2) \tag{3.16}$$

for vector field $W_1, W_2 \in \Gamma(TN_1)$, where $\nabla^F$ denotes the pullback connection [6] and it is easy to see that the second fundamental form is symmetric.

**Lemma 3.1.** [6] *Let $(N_1, g_{N_1})$ and $(N_2, g_{N_2})$ are two Riemannian manifolds. If $F : N_1 \to N_2$ Riemannian submersion between Riemannian manifolds, then for any horizontal vector fields $Y_1, Y_2$ and vertical vector fields $Z_1, Z_2$, we have*

*(a)* $(\nabla F_*)(Y_1, Y_2) = 0$,
*(b)* $(\nabla F_*)(Z_1, Z_2) = -F_*(\mathcal{T}_{Z_1} Z_2) = -F_*(\nabla^{N_1}_{Z_1} Z_2)$,
*(c)* $(\nabla F_*)(Y_1, Z_1) = -F_*(\nabla^{N_1}_{Y_1} Z_1) = -F_*(\mathcal{A}_{Y_1} Z_1)$.

**Definition 3.1.** [35] *Let $(N_1, g_{N_1})$ be an almost Hermitian manifold and $(N_2, g_{N_2})$ be a Riemannian manifold. Then we say that $F$ is a semi-invariant Riemannian submersion if there is a distribution $D_1 \subseteq \ker F_*$ such that*

$$\ker F_* = D_1 \oplus D_2, J(D_1) = D_1, J(D_2) \subseteq (\ker F_*)^{\perp}.$$

We can write

$$(\ker F_*)^{\perp} = J(D_2) \oplus \mu,$$

where, $\mu$ is an invariant subbundle of $(\ker F_*)^{\perp}$.

**Definition 3.2.** [23] *Let $F : (N_1, \phi, \xi, \eta, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion in such a manner that $\xi$ is normal to $(\ker F_*)$ and $(\ker F_*)$ is anti-invariant with respect to $\phi$. Then $F$ is called an anti-invariant $\xi^{\perp}$-Riemannian submersion.*

**Definition 3.3.** [1] *Let $F : (N_1, \phi, \xi, \eta, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion from an almost para-contact metric manifold onto a Riemannian manifold. $F$ is called a semi-invariant $\xi^{\perp}$-Riemannian submersion if $D_1 \subset (\ker F_*)$ is such that*

$$(\ker F_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subset (\ker F_*)^{\perp}.$$

## 4. CSI-$\xi^{\perp}$-Riemannian submersions from a Lorentzian para-Kenmotsu manifolds

In this section, we define and study CSI-$\xi^{\perp}$-Riemannian submersion from Lorentzian para-Kenmotsu manifolds onto a Riemannian manifolds.

In the theory of Riemannian submersions, Bishop [8] defines the notion of Clairaut submersion:

**Definition 4.1.** *Let $\alpha$ is any geodesic on $N_1$, $r$ is a positive function on $N_1$ and $\theta(t)$ is the angle between $\dot{\alpha}$ and the horizontal space at $\alpha(t)$ for any $t$. If the function $(r \circ \alpha)\sin\theta$ is constant on $N_1$, then a Riemannian submersion $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ is called a Clairaut submersion.*

**Theorem 4.1.** [8] *Let $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion with connected fibers. Then, $F$ is a Clairaut Riemannian submersion with $r = e^f$ if each fiber is totally umbilical and has the mean curvature vector field $H = -\nabla f$ is the gradient of the function $f$ with respect to $g_{N_1}$.*

**Definition 4.2.** *A semi-invariant $\xi^{\perp}$-Riemannian submersion $F$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$ is called Csi-$\xi^{\perp}$-Riemannian submersion if it satisfies the condition of Clairaut Riemannian submersion i.e., if each fiber is totally umbilical with mean curvature vector field $H = -\nabla f$ with respect to $g_{N_1}$, then $F$ is a Clairaut Riemannian submersion with $r = e^f$.*

Now, using definition (4.1), we have

$$(\ker F_*) = D_1 \oplus D_2, \phi(D_1) = D_1, \phi(D_2) \subseteq (\ker F_*)^{\perp}.$$

Thus for any $V_1 \in (\ker F_*)$, we put

$$V_1 = PV_1 + QV_1, \tag{4.17}$$

where $PV_1 \in \Gamma(D_1)$ and $QV_1 \in \Gamma(D_2)$.

In addition, for $Y_1 \in (\ker F_*)$, we get

$$\phi Y_1 = \psi Y_1 + \omega Y_1, \tag{4.18}$$

where $\phi Y_1 \in \Gamma(D_1)$ and $\omega Y_1 \in \Gamma(\phi D_2)$.

$\Gamma(\ker F_*)^{\perp}$ is decomposed as

$$\Gamma(\ker F_*)^{\perp} = \phi(D_2) \oplus \mu.$$

Here $\mu$ is invariant and contains $\xi$.

Also for $X_2 \in \Gamma(\ker F_*)^{\perp}$, we have

$$\phi X_2 = BX_2 + CX_2, \tag{4.19}$$

where $BX_2 \in \Gamma(D_2)$ and $CX_2 \in \Gamma(\mu)$.

**Lemma 4.1.** *Let $F$ be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$. Then, we get*

$$\mathcal{V}\nabla_{Y_1}\psi Z_1 + \mathcal{T}_{Y_1}\omega Z_1 = B\mathcal{T}_{Y_1}Z_1 + \psi\mathcal{V}\nabla_{Y_1}Z_1, \tag{4.20}$$

$$\mathcal{T}_{Y_1}\psi Z_1 + \mathcal{H}\nabla_{Y_1}\omega Z_1 + g_{N_1}(\psi Y_1, Z_1)\xi = C\mathcal{T}_{Y_1}Z_1 + \omega\mathcal{V}\nabla_{Y_1}Z_1, \tag{4.21}$$

$$\mathcal{V}\nabla_{V_1}BW_1 + \mathcal{A}_{V_1}CW_1 + \eta(W_1)BV_1 = B\mathcal{H}\nabla_{V_1}W_1 + \psi\mathcal{A}_{V_1}W_1, \tag{4.22}$$

$$\mathcal{A}_{V_1}BW_1 + \mathcal{H}\nabla_{V_1}CW_1 + \eta(W_1)CV_1 + g_{N_1}(CV_1, W_1)\xi = C\mathcal{H}\nabla_{V_1}W_1 + \omega\mathcal{A}_{V_1}W_1, \tag{4.23}$$

$$\mathcal{V}\nabla_{Y_1}BV_1 + \mathcal{T}_{Y_1}CV_1 + \eta(V_1)\psi Y_1 = \psi\mathcal{T}_{Y_1}V_1 + B\mathcal{H}\nabla_{Y_1}V_1, \tag{4.24}$$

$$\mathcal{T}_{Y_1}BV_1 + \mathcal{H}\nabla_{Y_1}CV_1 + \eta(V_1)\omega Y_1 + g_{N_1}(\omega Y_1, V_1)\xi = \omega\mathcal{T}_{Y_1}V_1 + C\mathcal{H}\nabla_{Y_1}V_1, \tag{4.25}$$

$$\mathcal{V}\nabla_{V_1}\psi Y_1 + \mathcal{A}_{V_1}\omega Y_1 = B\mathcal{A}_{V_1}Y_1 + \psi\mathcal{V}\nabla_{V_1}Y_1, \tag{4.26}$$

$$\mathcal{A}_{V_1}\psi Y_1 + \mathcal{H}\nabla_{V_1}\omega Y_1 + g_{N_1}(BV_1, Y_1)\xi = C\mathcal{A}_{V_1}Y_1 + \omega\mathcal{V}\nabla_{V_1}Y_1, \tag{4.27}$$

*where* $Y_1, Z_1 \in \Gamma(\ker F_*)$ *and* $V_1, W_1 \in \Gamma(\ker F_*)^{\perp}$.

**Proof.** Using equations (2.5), (3.10)-(3.13), (4.18) and (4.19), we get Lemma 4.1.

**Lemma 4.2.** *Let* $F$ *be a semi-invariant* $\xi^{\perp}$*-Riemannian submersion from a Lorentzian para-Kenmotsu manifold* $(N_1, \phi, \xi, \eta, g_{N_1})$ *onto a Riemannian manifold* $(N_2, g_{N_2})$*. If* $\alpha : I_2 \subset R \to N_1$ *is a regular curve and* $Z_1(t)$ *and* $U_1(t)$ *are the vertical and horizontal components of the tangent vector field* $\dot{\alpha} = E$ *of* $\alpha(t)$*, respectively, then* $\alpha$ *is a geodesic if and only if along* $\alpha$ *the following equations hold:*

$$\mathcal{V}\nabla_{\dot{\alpha}}\psi Z_1 + \mathcal{V}\nabla_{\dot{\alpha}}BU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})CU_1 + \eta(U_1)(\psi Z_1 + BZ_1) = 0,$$

$$\mathcal{H}\nabla_{\dot{\alpha}}\omega Z_1 + \mathcal{H}\nabla_{\dot{\alpha}}CU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})BU_1 + g_{N_1}(\phi\dot{\alpha}, \dot{\alpha})\xi + \eta(U_1)(\omega Z_1 + CZ_1) = 0.$$

**Proof.** Let $\alpha : I_2 \to N_1$ is a regular curve on $N_1$ and $\dot{\alpha}(t)$ is the tangent vector field. If $Z_1(t)$ and $U_1(t)$ are the vertical and horizontal parts of the tangent vector field, respectively. Then $\dot{\alpha}(t) = Z_1(t) + U_1(t)$. From equations (2.5), (3.10)-(3.13), (4.18) and (4.19), we get

$$\begin{aligned}
\phi\nabla_{\dot{\alpha}}\dot{\alpha} &= \nabla_{\dot{\alpha}}\phi\dot{\alpha} - (\nabla_{\dot{\alpha}}\phi)\dot{\alpha},\\
&= \nabla_{Z_1}\psi Z_1 + \nabla_{Z_1}\omega Z_1 + \nabla_{Z_1}BU_1 + \nabla_{Z_1}CU_1 + \nabla_{U_1}\psi Z_1 + \nabla_{U_1}\omega Z_1\\
&+ \nabla_{U_1}BU_1 + \nabla_{U_1}CU_1 + g_{N_1}(\phi\dot{\alpha}, \dot{\alpha})\xi + \eta(U_1)(\psi Z_1 + BZ_1) + \eta(U_1)(\omega Z_1 + CZ_1),\\
&= \mathcal{T}_{Z_1}\psi Z_1 + \mathcal{V}\nabla_{Z_1}\psi Z_1 + \mathcal{T}_{Z_1}\omega Z_1 + \mathcal{H}\nabla_{Z_1}\omega Z_1 + \mathcal{T}_{Z_1}BU_1 + \mathcal{V}\nabla_{Z_1}BU_1\\
&+ \mathcal{T}_{Z_1}CU_1 + \mathcal{H}\nabla_{Z_1}CU_1 + \mathcal{A}_{U_1}\psi Z_1 + \mathcal{V}\nabla_{U_1}\psi Z_1 + \mathcal{H}\nabla_{U_1}\omega Z_1 + \mathcal{A}_{U_1}\omega Z_1\\
&+ \mathcal{A}_{U_1}BU_1 + \mathcal{V}\nabla_{U_1}BU_1 + \mathcal{H}\nabla_{U_1}CU_1 + \mathcal{A}_{U_1}CU_1 + g_{N_1}(\phi\dot{\alpha}, \dot{\alpha})\xi\\
&+ \eta(U_1)(\psi Z_1 + BZ_1) + \eta(U_1)(\omega Z_1 + CZ_1).
\end{aligned}$$

Taking the vertical and horizontal components in above equation, we have

$$\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha} = \mathcal{V}\nabla_{\dot{\alpha}}\psi Z_1 + \mathcal{V}\nabla_{\dot{\alpha}}BU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\omega Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})CU_1 + \eta(U_1)(\psi Z_1 + BZ_1),$$

$$\begin{aligned}
\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha} &= \mathcal{H}\nabla_{\dot{\alpha}}\omega Z_1 + \mathcal{H}\nabla_{\dot{\alpha}}CU_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})\psi Z_1 + (\mathcal{T}_{Z_1} + \mathcal{A}_{U_1})BU_1\\
&+ g_{N_1}(\phi\dot{\alpha}, \dot{\alpha})\xi + \eta(U_1)(\omega Z_1 + CZ_1),
\end{aligned}$$

Hence, $\alpha$ is a geodesic on $N_1$ if and only if $\mathcal{V}\phi\nabla_{\dot{\alpha}}\dot{\alpha}$ and $\mathcal{H}\phi\nabla_{\dot{\alpha}}\dot{\alpha}$ both are vanish, which gives our result.

**Theorem 4.2.** *Let $F$ be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$. Then $F$ is a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if*

$$(g_{N_1}(\nabla f, Z_1) - \eta(Z_1)|)||V_1||^2 = -g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}} BZ_1, \psi V_1) - g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}} CZ_1, \omega V_1) - g_{N_1}((\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1, \psi V_1) - g_{N_1}((\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \omega V_1)$$

*where $\alpha : I_2 \to N_1$ is a geodesic on $N_1$ and $V_1, Z_1$ are vertical and horizontal components of $\dot{\alpha}(t)$.*

**Proof.** Let $\alpha : I_2 \to N_1$ be a geodesic on $N_1$ with $V_1(t) = \mathcal{V}\dot{\alpha}(t)$ and $Z_1(t) = \mathcal{H}\dot{\alpha}(t)$. Let $\theta(t)$ denote the angle in $[0, \pi]$ between $\dot{\alpha}(t)$ and $Z_1(t)$. Assuming $\nu = ||\dot{\alpha}(t)||^2$ then we get

$$g_{N_1}(V_1(t), V_1(t)) = \nu \sin^2 \theta(t), \tag{4.28}$$

$$g_{N_1}(Z_1(t), Z_1(t)) = \nu \cos^2 \theta(t). \tag{4.29}$$

Now, differentiating (4.28), we get

$$\frac{d}{dt} g_{N_1}(V_1(t), V_1(t)) = 2\nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt}.$$

Using equation (2.2), we get

$$g_{N_1}(\phi\nabla_{\dot{\alpha}} V_1, \phi V_1) = \nu \cos\theta(t) \sin\theta(t) \frac{d\theta}{dt}. \tag{4.30}$$

Now, using equation (2.5), we get

$$\nabla_{\dot{\alpha}} \phi V_1 = \phi \nabla_{\dot{\alpha}} V_1 + g_{N_1}(\phi\dot{\alpha}, V_1)\xi,$$

$$\begin{aligned} g_{N_1}(\phi\nabla_{\dot{\alpha}} V_1, \phi V_1) &= g_{N_1}(\nabla_{\dot{\alpha}} \phi V_1, \phi V_1), \\ &= g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}} \psi V_1, \psi V_1) + g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}} \omega V_1, \omega V_1) + g_{N_1}((\mathcal{A}_{Z_1} + \mathcal{T}_{V_1})\psi V_1, \omega V_1) \\ &+ g_{N_1}((\mathcal{A}_{Z_1} + \mathcal{T}_{V_1})\omega V_1, \psi V_1). \end{aligned}$$

Using Lemma 4.2 in above equation, we get

$$\begin{aligned} g_{N_1}(\phi\nabla_{\dot{\alpha}} V_1, \phi V_1) = &-g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}} BZ_1, \psi V_1) - g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}} CZ_1, \omega V_1) \\ &- g_{N_1}((\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1, \psi V_1) - g_{N_1}((\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \omega V_1) \\ &- \eta(Z_1) g_{N_1}(V_1, V_1). \end{aligned} \tag{4.31}$$

From equations (4.30) and (4.31), we have

$$\nu\cos\theta(t)\sin\theta(t)\frac{d\theta}{dt} = -g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}}BZ_1,\psi V_1) - g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}}CZ_1,\omega V_1)$$
$$- g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})CZ_1,\psi V_1) - g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})BZ_1,\omega V_1)$$
$$- \eta(Z_1)g_{N_1}(V_1,V_1). \tag{4.32}$$

Moreover, $F$ is a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ if and only if

$$\frac{d}{dt}(e^{f\circ\alpha}\sin\theta) = 0$$
$$e^{f\circ\alpha}(\cos\theta\frac{d\theta}{dt} + \sin\theta\frac{df}{dt}) = 0. \tag{4.33}$$

Multiplying with non-zero factor $\upsilon\sin\theta$ on both sides, we have

$$\begin{aligned} -\upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= \upsilon\sin^2\theta\frac{df}{dt}, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_{N_1}(V_1,V_1)\frac{df}{dt}, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_{N_1}(\nabla f,\dot{\alpha})||V_1||^2, \\ \upsilon\cos\theta\sin\theta\frac{d\theta}{dt} &= -g_{N_1}(\nabla f,Z_1)||V_1||^2. \end{aligned} \tag{4.34}$$

Thus, from equations (4.32) and (4.34), we have

$$(g_{N_1}(\nabla f,Z_1) - \eta(Z_1))||V_1||^2 = g_{N_1}(\mathcal{V}\nabla_{\dot{\alpha}}BZ_1,\psi V_1) + g_{N_1}(\mathcal{H}\nabla_{\dot{\alpha}}CZ_1,\omega V_1)$$
$$+ g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})CZ_1,\psi V_1) + g_{N_1}((\mathcal{T}_{V_1}+\mathcal{A}_{Z_1})BZ_1,\omega V_1).$$

Hence the theorem 4.2 is proved.

**Corollary 4.1.** *Let $F$ be a semi-invariant $\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1,\phi,\xi,\eta,g_{N_1})$ onto a Riemannian manifold $(N_2,g_{N_2})$. Then, we get*

$$g_{N_1}(\nabla f,\xi) = -1.$$

**Theorem 4.3.** *Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1,\phi,\xi,\eta,g_{N_1})$ onto a Riemannian manifold $(N_2,g_{N_2})$ with $r = e^f$. Then, we get*

$$\mathcal{A}_{\phi V_1}\phi Y_1 = Y_1(f)V_1 \tag{4.35}$$

*for $Y_1 \in \Gamma(\mu)$ and $V_1 \in \Gamma(D_2)$, such that $\phi Y_1$ is basic.*

**Proof.** Let $F$ be Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold onto a Riemannian manifold. For $X_1, X_2 \in \Gamma(D_2)$, using equation (3.15) and Theorem 4.1, we get

$$\mathcal{T}_{X_1} X_2 = -g_{N_1}(X_1, X_2) gradf. \tag{4.36}$$

Now, we take inner product with $\phi V_1$ in equation (4.36),

$$g_{N_1}(\mathcal{T}_{X_1} X_2, \phi V_1) = -g_{N_1}(X_1, X_2) g_{N_1}(gradf, \phi V_1), \tag{4.37}$$

for all $V_1 \in \Gamma(D_2)$.

From equations (2.2) and (2.5), we obtain

$$g_{N_1}(\nabla_{X_1} \phi X_2, V_1) = -g_{N_1}(X_1, X_2) g_{N_1}(gradf, \phi V_1).$$

Here $\nabla$ is metric connection, so we can use equations (3.15) and (4.37) in above equation and get

$$g_{N_1}(X_1, V_1) g_{N_1}(gradf, \phi X_2) = -g_{N_1}(X_1, X_2) g_{N_1}(gradf, \phi V_1). \tag{4.38}$$

Now, we take $V_1 = X_2$ and obtain the following equatin by interchanging the role of $X_1$ and $X_2$,

$$g_{N_1}(X_2, X_2) g_{N_1}(gradf, \phi X_1) = g_{N_1}(X_1, X_2) g_{N_1}(gradf, \phi X_2). \tag{4.39}$$

Using equation (4.39) with $V_1 = X_1$ in (4.38), we have

$$g_{N_1}(gradf, \phi X_1) = \frac{(g_{N_1}(X_1, X_2))^2}{||X_1||^2 ||X_2||^2} g_{N_1}(gradf, \phi X_1). \tag{4.40}$$

If $gradf \in \Gamma(\phi(D_2))$, then equation (4.40) and the condition of equality in the Schwarz inequality implies that either $f$ is constant on $\phi(D_2)$ or the fibers are one dimensional.

On the other hand, using equation (2.5), we get

$$g_{N_1}(\phi \nabla_{X_1} V_1, \phi Y_1) = g_{N_1}(\nabla_{X_1} \phi V_1, \phi Y_1)$$

for $Y_1 \in \Gamma(\mu)$ and $Y_1 \neq \xi$. Now, using equation (2.2), we obtain

$$g_{N_1}(\nabla_{X_1} \phi V_1, \phi Y_1) = g_{N_1}(\nabla_{X_1} V_1, Y_1).$$

Using equations (2.2) and (2.5) in above equation, we get

$$g_{N_1}(\nabla_{X_1} \phi V_1, \phi Y_1) = -g_{N_1}(X_1, V_1) g_{N_1}(gradf, Y_1).$$

Since $\phi Y_1$ is basic and using the fact that $\mathcal{H}\nabla_{X_1}\phi V_1 = \mathcal{A}_{\phi V_1}X_1$, we get

$$\begin{aligned} g_{N_1}(\nabla_{X_1}\phi V_1, \phi Y_1) &= -g_{N_1}(X_1, V_1)g_{N_1}(gradf, Y_1), \\ g_{N_1}(\mathcal{A}_{\phi V_1}X_1, \phi Y_1) &= -g_{N_1}(X_1, V_1)g_{N_1}(gradf, Y_1), \\ g_{N_1}(\mathcal{A}_{\phi V_1}\phi Y_1, X_1) &= g_{N_1}(X_1, V_1)g_{N_1}(gradf, Y_1) \\ g_{N_1}(\mathcal{A}_{\phi V_1}\phi Y_1, X_1) &= g_{N_1}(X_1, V_1)g_{N_1}(\nabla f, Y_1). \end{aligned} \tag{4.41}$$

Since $\mathcal{A}_{\phi V_1}\phi Y_1$ and $V_1$ are vertical and $\nabla f$ is horizontal, we obtain equation (4.35).

**Theorem 4.4.** *Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$ with $r = e^f$ and* $\dim(D_2) > 1$. *Then, for all $Y_1 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker F_*)^{\perp}$,*

$$\overset{F}{\nabla}_{V_1}F_*(\phi Y_1) = V_1(f)F_*(\phi Y_1).$$

**Proof.** Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion from a Lorentzian para-Kenmotsu manifold onto a Riemannian manifold. Since each fiber is totally umbilical with mean curvature vector field $H = -gradf$, then from theorem (4.1), we have

$$\begin{aligned} -g_{N_1}(\nabla_{Y_1}V_1, Y_2) &= g_{N_1}(\nabla_{Y_1}Y_2, V_1), \\ -g_{N_1}(\nabla_{Y_1}V_1, Y_2) &= -g_{N_1}(Y_1, Y_2)g_{N_1}(gradf, V_1), \end{aligned}$$

for all $Y_1, Y_2 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker F_*)^{\perp}$.

Using equations (2.2),(2.5) and (3.15) in above equation, we get

$$g_{N_1}(\nabla_{V_1}\phi Y_1, \phi Y_2) = g_{N_1}(\phi Y_1, \phi Y_2)g_{N_1}(gradf, V_1). \tag{4.42}$$

Since $F$ is the semi-invariant Riemannian submersion and using equation (4.42), we have

$$g_{N_2}(F_*(\nabla_{V_1}\phi Y_1), F_*(\phi Y_2)) = g_{N_2}(F_*(\phi Y_1), F_*(\phi Y_2))g_{N_1}(gradf, V_1). \tag{4.43}$$

From (3.16) in (4.43), we obtain

$$g_{N_2}(\overset{F}{\nabla}_{V_1}F_*(\phi Y_1), F_*(\phi Y_2)) = g_{N_2}(F_*(\phi Y_1), F_*(\phi Y_2))g_{N_1}(gradf, V_1), \tag{4.44}$$

which implies $\overset{F}{\nabla}_{V_1}F_*(\phi Y_1) = V_1(f)F_*(\phi Y_1)$, for all $Y_1 \in \Gamma(D_2)$ and $V_1 \in \Gamma(\ker F_*)^{\perp}$, hence the proof.

**Theorem 4.5.** *Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$. If $\mathcal{T}$ is not equal to zero identically, then the invariant distribution $D_1$ cannot defined a totally geodesic foliation on $N_1$.*

**Proof.** For $W_1, W_2 \in \Gamma(D_1)$ and $X_1 \in \Gamma(D_2)$, using equations $(2.2), (2.5), (3.11)$ and $(3.15)$, we get

$$\begin{aligned} g_{N_1}(\nabla_{W_1} W_2, X_1) &= g_{N_1}(\nabla_{W_1} \phi W_2, \phi X_1), \\ &= g_{N_1}(\mathcal{T}_{W_1} \phi W_2, \phi X_1), \\ &= -g_{N_1}(W_1, \phi W_2) g_{N_1}(gradf, \phi X_1). \end{aligned}$$

Thus, the assertion can be seen from above equation and the fact that $gradf \in \phi(D_2)$.

**Theorem 4.6.** *Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$ with $r = e^h$. Then, $D_2$ is not totally geodesic foliation on $N_1$.*

**Proof.** For $Z_1, Z_2 \in \Gamma(D_2)$ and $\xi \in \Gamma(\ker \pi_*)^{\perp}$, using (2.6), we get

$$g_{N_1}(\nabla_{Z_1} Z_2, \xi) = -g_{N_1}(\nabla_{Z_1} \xi, Z_2) = g_{N_1}(Z_1, Z_2) \neq 0.$$

Hence $D_2$ is not totally geodesic foliation on $N_1$.

Using Theorems (4.5) and (4.6), one can give the following Theorem.

**Theorem 4.7.** *Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$ with $r = e^f$. Then, $(\ker \pi_*)$ is not totally geodesic foliation on $N_1$.*

**Theorem 4.8.** *Let $F$ be a Csi-$\xi^{\perp}$-Riemannian submersion with $r = e^f$ from a Lorentzian para-Kenmotsu manifold $(N_1, \phi, \xi, \eta, g_{N_1})$ onto a Riemannian manifold $(N_2, g_{N_2})$. Then, the anti-invariant distribution $D_2$ one-dimensional.*

**Proof.** Since $F$ is a Clairaut proper semi-invariant submersion, then either $\dim(D_2) = 1$ or $\dim(D_2) > 1$. If $\dim(D_2) > 1$, then we can choose $Z_1, Z_2 \in \Gamma(D_2)$ such that $\{Z_1, Z_2\}$ is orthonormal. From equations $(2.5), (3.10), (4.18)$ and $(4.19)$, we get

$$\begin{aligned} \mathcal{T}_{Z_1} \phi Z_2 + \mathcal{H} \nabla_{Z_1} \phi Z_2 &= \nabla_{Z_1} \phi Z_2, \\ \mathcal{T}_{Z_1} \phi Z_2 + \mathcal{H} \nabla_{Z_1} \phi Z_2 &= B \mathcal{T}_{Z_1} Z_2 + C \mathcal{T}_{Z_1} Z_2 + \psi \mathcal{V} \nabla_{Z_1} Z_2 + \omega \mathcal{V} \nabla_{Z_1} Z_2. \end{aligned}$$

Now, we take inner product with $Z_1$ in above equation and obtain

$$g_{N_1}(\mathcal{T}_{Z_1}\phi Z_2, Z_1) = g_{N_1}(B\mathcal{T}_{Z_1}Z_2, Z_1) + g_{N_1}(\psi\mathcal{V}\nabla_{Z_1}Z_2, Z_1). \tag{4.45}$$

From equation (2.5), (3.10) and (3.15), we have

$$g_{N_1}(\mathcal{T}_{Z_1}Z_1, \phi Z_2) = -g_{N_1}(\mathcal{T}_{Z_1}\phi Z_2, Z_1) = -g_{N_1}(gradf, \phi Z_2) = g_{N_1}(\mathcal{T}_{Z_1}Z_2, \phi Z_1). \tag{4.46}$$

From above equation, we obtain

$$\begin{aligned} g_{N_1}(gradf, \phi Z_2) &= g_{N_1}(\mathcal{T}_{Z_1}Z_2, \phi Z_1), \\ g_{N_1}(gradf, \phi Z_2) &= g_{N_1}(Z_1, Z_2)g_{N_1}(gradf, \phi Z_1), \\ g_{N_1}(gradf, \phi Z_2) &= 0. \end{aligned}$$

So, we get

$$gradf \perp \phi(D_2).$$

Therefore, the dimension of $D_2$ must be one.

## 5. Example

Let $N_1$ be a 5-dimensional space given by the following:

$$\mathbb{R}^5 = \{(x_1, x_2, y_1, y_2, z) \in R^5 | (x_1, x_2, y_1, y_2) \neq (0,0,0,0) \text{ and } z \neq 0\}.$$

Let $\eta$ be a 1-form defined by $\eta = dz$. The vector field $\xi$ is given by $\frac{\partial}{\partial z}$ and its Lorentzian metric $g_{N_1}$ and tensor field $\phi$ are given by

$$g_{N_1} = e^{2z}(dx_1)^2 + e^{2z}(dx_2)^2 + e^{2z}(dy_1)^2 + e^{2z}(dy_2)^2 - (dz)^2,$$

$$\phi = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

This gives a Lorentzian para-Kenmotsu structure $(\phi, \xi, \eta, g_{N_1})$ on $N_1$.

A $\phi$-basis for this structure can be given by $\{e_1 = e^{-z}\frac{\partial}{\partial x_1}, e_2 = e^{-z}\frac{\partial}{\partial x_2}, e_3 = e^{-z}\frac{\partial}{\partial y_1}, e_4 = e^{-z}\frac{\partial}{\partial y_2}, e_5 = \xi = \frac{\partial}{\partial z}\}$.

Let $N_2$ be $\{(u_1, u_2) \in R^2 | u_2 = z \neq 0\}$. We choose the Riemannian metric $g_{N_2} = e^{2z}(du_1)^2 + (du_2)^2$ on $N_2$.

Now, we define the map $F : (N_1, \phi, \xi, \eta, g_{N_1}) \to (N_2, g_{N_2})$ by the following:

$$F(x_1, x_2, y_1, y_2, z) = (\frac{x_2 + y_2}{\sqrt{2}}, z).$$

By direct calculations, we have

$$\begin{aligned} \ker F_* &= span\{X_1 = e_1, X_2 = (e_2 - e_4), X_3 = e_3\}, \\ D_1 &= span\{X_1 = e_1, X_3 = e_3\}, D_2 = span\{X_2 = (e_2 - e_4)\}, \\ (\ker F_*)^\perp &= span\{V_1 = (e_2 + e_4), V_2 = \xi = e_5\}. \end{aligned}$$

After some computations, we find that

$$\begin{aligned} F_*(V_1) = \sqrt{2} e^{-z} \frac{\partial}{\partial u_1}, \\ F_*(V_2) = \frac{\partial}{\partial u_2}, \\ g_{N_1}(V_i, V_j) = g_{N_2}(F_* V_i, F_* V_j) \end{aligned} \tag{5.47}$$

for all $V_i, V_j \in \Gamma(\ker F_*)^\perp, i, j = 1, 2$. Thus $F$ is semi-invariant $\xi^\perp$-Riemannian submersion.

Now, we will obtain smooth function $f$ on $\mathbb{R}^5$ which satisfy the condition $\mathcal{T}_X X = g_1(X, X)\nabla f$, for all $X \in \Gamma(\ker \pi_*)$.

Using the given Kenmotsu structure, we find

$$\begin{aligned} [e_1, e_1] &= [e_2, e_2] = [e_3, e_3] = [e_4, e_4] = [e_5, e_5] = 0, \\ [e_1, e_2] &= 0, [e_1, e_3] = 0, [e_1, e_4] = 0, [e_1, e_5] = e_1, \\ [e_2, e_3] &= 0, [e_2, e_4] = 0, [e_2, e_5] = e_2, [e_3, e_4] = 0, \\ [e_3, e_5] &= e_3, [e_4, e_5] = e_4, \end{aligned} \tag{5.48}$$

The Levi-Civita connection $\nabla$ of the metric $g_{N_1}$ is given by the Koszul′s formula which is

$$\begin{aligned} 2g_{N_1}(\nabla_X Z, W) = X g_{N_1}(Z, W) + Z g_{N_1}(W, X) - W g_{N_1}(X, Z) + g_{N_1}([X, Z], W) \\ - g_{N_1}([Z, W], X) + g_{N_1}([W, X], Z). \end{aligned} \tag{5.49}$$

Using (5.48) and (5.49), we get

$$\begin{aligned} \nabla_{e_1} e_1 &= \nabla_{e_2} e_2 = \nabla_{e_3} e_3 = \nabla_{e_4} e_4 = \frac{\partial}{\partial z}, \\ \nabla_{e_1} e_2 &= \nabla_{e_1} e_3 = \nabla_{e_1} e_4 = \nabla_{e_2} e_1 = \nabla_{e_2} e_3 = \nabla_{e_2} e_4 = 0, \\ \nabla_{e_3} e_1 &= \nabla_{e_3} e_2 = \nabla_{e_3} e_4 = \nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = 0. \end{aligned} \tag{5.50}$$

Therefore

$$\begin{aligned} \nabla_{X_1}X_1 &= \nabla_{e_1}e_1 = \frac{\partial}{\partial z}, \nabla_{X_2}X_2 = \nabla_{e_2-e_4}e_2 - e_4 = 2\frac{\partial}{\partial z} \\ \nabla_{X_3}X_3 &= \nabla_{e_3}e_3 = \frac{\partial}{\partial z}, \nabla_{X_1}X_2 = 0, \nabla_{X_1}X_3 = 0, \\ \nabla_{X_2}X_3 &= 0, \nabla_{X_2}X_1 = 0, \nabla_{X_3}X_1 = 0, \nabla_{X_3}X_2 = 0. \end{aligned} \tag{5.51}$$

Now, we have

$$\mathcal{T}_X X = \mathcal{T}_{\lambda_1 X_1+\lambda_2 X_2+\lambda_3 X_3}\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1, \lambda_2, \lambda_3 \in R.$$

$$\begin{aligned} \mathcal{T}_X X &= \lambda_1^2 \mathcal{T}_{X_1}X_1 + \lambda_2^2 \mathcal{T}_{X_2}X_2 + \lambda_3^2 \mathcal{T}_{X_3}X_3 + \\ &\quad \lambda_1\lambda_2 \mathcal{T}_{X_1}X_2 + \lambda_1\lambda_3 \mathcal{T}_{X_1}X_3 + \lambda_2\lambda_3 \mathcal{T}_{X_2}X_3 + \\ &\quad \lambda_1\lambda_2 \mathcal{T}_{X_2}X_1 + \lambda_1\lambda_3 \mathcal{T}_{X_3}X_1 + \lambda_2\lambda_3 \mathcal{T}_{X_3}X_2. \end{aligned} \tag{5.52}$$

Using (5.51), we obtain

$$\begin{aligned} \mathcal{T}_{X_1}X_1 &= \frac{\partial}{\partial z}, \mathcal{T}_{X_2}X_2 = 2\frac{\partial}{\partial z}, \mathcal{T}_{X_3}X_3 = \frac{\partial}{\partial z}, \\ \mathcal{T}_{X_1}X_2 &= 0, \mathcal{T}_{X_1}X_3 = 0, \mathcal{T}_{X_2}X_3 = 0, \mathcal{T}_{X_2}X_1 = 0, \\ \mathcal{T}_{X_3}X_1 &= 0, \mathcal{T}_{X_3}X_2 = 0. \end{aligned} \tag{5.53}$$

Next, using (5.52) and (5.53), we get

$$\mathcal{T}_X X = (\lambda_1^2 + 2\lambda_2^2 + \lambda_3^2 + \lambda_3^2)\frac{\partial}{\partial z}.$$

Since $X = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3$, so $g_{N_1}(\lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3) = \lambda_1^2 + 2\lambda_2^2 + \lambda_3^2$. For any smooth function $f$ on $R^5$, $\nabla f$ with respect to the metric $g_{N_1}$ is given by

$\nabla f = e^{-2Z}\frac{\partial f}{\partial x_1}\frac{\partial}{\partial x_1} + e^{-2Z}\frac{\partial f}{\partial x_2}\frac{\partial}{\partial x_2} + e^{-2Z}\frac{\partial f}{\partial y_1}\frac{\partial}{\partial y_1} + e^{-2Z}\frac{\partial f}{\partial y_2}\frac{\partial}{\partial y_2} + \frac{\partial f}{\partial z}\frac{\partial}{\partial z}$.

Therefore, $\nabla f = \frac{\partial}{\partial z}$ for the function $f = z$. Now, we can see that $\mathcal{T}_X X = g_{N_1}(X, X)\nabla f$ and by Theorem (4.1), it is clear that $F$ is a CSI-$\xi^{\perp}$-Riemannian submersions.

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# QUASI HEMI-SLANT CONFORMAL SUBMERSIONS FROM KENMOTSU MANIFOLD

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Abstract. We take into account the quasi hemi-slant conformal submersion from Kenmotsu manifold onto the Riemannian manifold as a generalization of anti-invariant submersions, semi-slant submersions, and hemi-slant submersions. We discussed the integrability and totally geodesicness of the different distributions. Moreover, we have obtained a condition under which the conformal hemi-slant submersions become a homothetic map.

**Keywords**: Kenmotsu manifolds, slant submersion, hemi-slant submersion, quasi-hemi slant submersion

**2010 Mathematics Subject Classification**: 53D15, 53D10.

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## 1. Introduction

The concept discussed by B. O'Neill [26] and A Gray [16] is known as Riemannian submersions. In 1976, B. Watson [42], considered the submersion between almost Hermitian manifolds with name as almost Hermitian submersions. He established that, if the whole manifold is a Kaehler manifold, then the base manifold is also a Kaehler manifold. The Riemannian submersions consist many applications in mathematics and in physics, specially in Yang-Mills theory ([8],[44]), Kaluza-Klein theory ([9],[22]). The Riemannian submersions are very interesting tools in geometry to study Riemannian manifolds having differentiable structures. B. Sahin, in ([37], [39]), respectively, presented the idea of anti-invariant Riemannian

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submersions and slant-submersion from virtually Hermitian manifold as a generalisation of Riemannian submersions.

The notion of almost contact Riemannian submersions from almost contact manifold was introduced by Chinea in [11]. He also studied the fibre space, base space and total space with differential geometric point of view. As a generalization of Riemannian submersions, Fuglede [15] and Ishihara [23] separately, studied horizontally conformal submersions. Later on, many authors investigated different kinds of Riemannian submersions like anti-invariant submersions ([5], [37]), slant submersions ([4], [39]), semi-slant submersions ([2], [19], [28]) and hemi-slant submersions ([43], [1]) between almost Hermitian manifolds as well as almost contact manifolds. R Prasad et al. ([31], [32], [33], [34]) studied Quasi-bi-slant submersion from Kenmotsu manifold onto Riemannian manifolds and they also studied Riemannian submersion from Kenmotsu manifolds with different aspect whereas Sezin [41] studied bi-slant submersions from contact manifold with taking $\xi$ as horizontal vector field.

In this paper, we study quasi hemi-slant conformal submersions from Kenmotsu manifold onto a Riemannian manifold taking 4 mutually orthogonal complementary distributions. This paper contains 4 sections. Section 2 consists some definitions of almost contact metric manifold and specially kenmotsu manifold, In section 3, we study some basic results for quasi hemi-slant conformal submersion from Kenmotsu manifold which are needed for our main sections. Section 4 contains the results of integrability and totally geodesicness of distributions.

## 2. Preliminaries

Let $M$ be a $(2n+1)$-dimensional almost contact manifold with almost contact structures $(\phi, \xi, \eta)$, where $\phi$ is a $(1,1)$ tensor field $\xi$, a vector field and $\eta$, a 1- form satisfying

$$\phi^2 = -I + \eta \otimes \xi, \ \eta(\xi) = 1, \ \eta \circ \phi = 0. \tag{2.1}$$

On an almost contact manifold, there exists a Riemannian metric $g$ which is compatible with the almost contact structure $(M, \phi, \xi, \eta)$ in the sense that

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V), \tag{2.2}$$

from which it can be observed that

$$g(U, \xi) = \eta(U), \tag{2.3}$$

for any $U,\ V \in \Gamma(TM)$ and the manifold $(M, \phi, \xi, \eta, g)$ is called an *almost contact metric manifold*. If $[\phi, \phi]$ denotes the Nijenhuis tensor of $\phi$, then the almost contact structure is normal if and only if the torsion tensor $[\phi, \phi] + 2d\eta \otimes \xi$ vanishes. An almost contact metric structure is called a *contact metric structure* if $d\eta = \Phi$, where $\Phi$ is the fundamental 2-form defined by $\Phi(U, V) = g(U, \phi V)$. Almost contact metric structure $(\phi, \xi, \eta, g)$ are said to define a *Kenmotsu structure* on $M$ if the following characterizing tensorial equation is satisfied

$$(\bar{\nabla}_U \phi)V = g(\phi U, V)\xi - \eta(V)\phi U. \tag{2.4}$$

One can deduce from the above relations that

$$\bar{\nabla}_U \xi = U - \eta(U)\xi. \tag{2.5}$$

It is also seen that

$$g(\phi U, V) = -g(U, \phi V). \tag{2.6}$$

The covariant derivative of $\phi$ is defined by

$$(\nabla_U \phi)V = \nabla_U \phi V - \phi \nabla_U V. \tag{2.7}$$

Now, we recall the notion of Riemannian submersion and horizontally conformal submersion followed by some basic results those will be useful throughout the text.

**Definition 2.1.** *Let $(M, g)$ and $(N, g')$ be two Riemannian manifolds and $F : M \to N$ be a smooth Riemannian submersion. Then $F$ is called a horizontally conformal submersion, with a positive function $\lambda$ such that*

$$g(X, Y) = \frac{1}{\lambda^2} g'(F_* X, F_* Y), \tag{2.8}$$

*for any $X, Y \in \Gamma(ker F_*)^{\perp}$. It is clear that a horizontally conformal submersion with $\lambda = 1$ is Riemannian submersions.*

Let $F : M \to N$ be a conformal submersion. A vector field $E$ on $M$ is called projectable if there exists a vector field $\bar{E}$ on $N$ such that $F_*(E_p) = \bar{E}$ for any $p \in \Gamma(TM)$.

B. O' Neill defined the tensors $\mathcal{T}$ and $\mathcal{A}$ called fundamental tensors and defined by for vector fields $E_1$ and $E_2$ on $M$ such that

$$\mathcal{A}_{E_1} E_2 = \mathcal{H}\nabla_{\mathcal{H}E_1} \mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{H}E_1} \mathcal{H}E_2 \tag{2.9}$$

$$\mathcal{T}_{E_1} E_2 = \mathcal{H}\nabla_{\mathcal{V}E_1} \mathcal{V}E_2 + \mathcal{V}\nabla_{\mathcal{V}E_1} \mathcal{H}E_2 \tag{2.10}$$

where the vertical and horizontal projections are $\mathcal{V}$ and $\mathcal{H}$ respectively. Considering the equations (2.9) and (2.10), we have

$$\nabla_{U_1} V_1 = \mathcal{T}_{U_1} V_1 + \mathcal{V} \nabla_{U_1} V_1 \tag{2.11}$$

$$\nabla_{U_1} X_1 = \mathcal{T}_{U_1} X_1 + \mathcal{H} \nabla_{U_1} X_1 \tag{2.12}$$

$$\nabla_{X_1} U_1 = \mathcal{A}_{X_1} U_1 + \mathcal{V} \nabla_{X_1} U_1 \tag{2.13}$$

$$\nabla_{X_1} Y_1 = \mathcal{H} \nabla_{X_1} Y_1 + \mathcal{A}_{X_1} Y_1 \tag{2.14}$$

for any $U_1, V_1 \in \Gamma(kerF_*)$ and $X_1, Y_1 \in \Gamma(kerF_*)^{\perp}$.

For $q \in M$, $V \in \mathcal{V}_q$ and $X \in \mathcal{H}_q$, the linear operators $\mathcal{T}_V, \mathcal{A}_X : T_pM \to T_pM$ are skew-symmetric, that is

$$g(\mathcal{A}_X E_1, E_2) = -g(E_1, \mathcal{A}_X E_2) \tag{2.15}$$

$$g(\mathcal{T}_V E_1, E_2) = -g(E_1, \mathcal{T}_V E_2) \tag{2.16}$$

for any $E_1, E_2 \in \Gamma(T_pM)$.

Let $(M, g)$ and $(N, g')$ be two Riemannian manifolds. Let $\varphi : M \to N$ be a smooth map. Then, the second fundamental form of $\varphi$ is given by

$$(\nabla \varphi_*)(X, Y) = \nabla^{\varphi}_X \varphi_* Y - \varphi_*(\nabla_X Y), \tag{2.17}$$

for all $X, Y \in \Gamma(T_pM)$, where $\nabla$ the Levi-Civita connection of the metrics $g$ and $g'$ and $\nabla^{\varphi}$ is the pullback connection. The map $\varphi$ is said to be totally geodesic map if $(\nabla \varphi_*)(U, V) = 0$ for any $U, V \in \Gamma(T_pM)$.

**Lemma 2.1.** *Let $F : M \to N$ be a horizontal conformal submersion. Then, for any horizontal vector fields $X_1, Y_1$ and vertical vector fields $U_1, V_1$*

(i) $(\nabla F_*)(X_1, Y_1) = X_1(ln\lambda)F_*(Y_1) + Y_1(ln\lambda)F_*(X_1) - g(X_1, Y_1)F_*(grad\ ln\lambda)$,

(ii) $(\nabla F_*)(U_1, V_1) = -F_*(\mathcal{T}_{U_1} V_1)$,

(iii) $(\nabla F_*)(X_1, U_1) = -F_*(\nabla_{X_1} U_1) = -F_*(\mathcal{A}_{X_1} U_1)$.

## 3. Quasi Hemi-slant Conformal Submersions

**Definition 3.1.** *A conformal submersion $F$ from almost contact metric manifold $(M, \phi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g')$ is said to be a quasi hemi-slant conformal submersion (QHSC submersions) if its vertical distribution* kerF$*$ *of $F$ admits four orthogonal complementary distributions $D_T$, $D_\theta$, $D_\perp$ and $<\xi>$ such that*

(i) $\mathrm{kerF}_* = D_T \oplus D_\theta \oplus D_\perp \oplus <\xi>$

(ii) *$D_T$ is invariant, i.e., $\phi D_T = D_T$*

(iii) *$D_\perp$ is anti-invariant, i.e., $\phi D_\perp \subseteq (\,ker F_*{}^\perp)$*

(iv) *for any non-zero vector field $X \in (D_\theta)_p, p \in M$, the angle $\theta$ between $\phi X$ and $(D_\theta)_p$ is constant and independent of the choice of point $p$ and $X$ in $(D_\theta)_p$,*

*where $<\xi>$ is 1-dimensional distribution spanned by $\xi$. Then, we say that $F$ is QHSC submersion where angle $\theta$ is called the quasi hemi-slant angle of submersion. Here we have some particular cases which are stated as :*

(i) If the distribution $D_T = 0$ then the map $F$ is a conformal hemi-slant submersion.

(ii) If the distribution $D_\theta = 0$ then the map $F$ is a conformal semi-invariant submersion.

(iii) If the distribution $D_\perp = 0$ then the map $F$ is a conformal semi-slant submersion.

Hence, it is clear that the QHSC submersions are generalized version of conformal hemi-slant submersions, conformal semi-invariant submersions and conformal semi-slant submersions.

Let $F$ be a QHSC submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g')$. Then, for any $U \in \Gamma(ker F_*)$, we have

$$U = PU + QU + RU + \eta(U)\xi \tag{3.18}$$

where $P, Q$ and $R$ are the projections morphism onto $D_T$, $D_\theta$ and $D_\perp$. Now, For any $U \in \Gamma(ker F_*)$

$$\phi U = \beta U + \delta U \tag{3.19}$$

where $\beta U \in \Gamma(ker F_*)$ and $\delta U \in \Gamma((ker F_*)^\perp)$. From equations (3.18), (3.19) and definition 3.1, we have

$$\begin{aligned}\phi U &= \phi(PU) + \phi(QU) + \phi(RU)\\ &= \beta(PU) + \delta(PU) + \beta(QU) + \delta(QU) + \beta(RU) + \delta(RU)\end{aligned}$$

We obtain $\delta \bar{P} U = 0$ and $\beta \bar{R} U = 0$, we have

$$\phi U = \beta(PU) + \beta(QU) + \delta(QU) + \delta(RU).$$

Hence, we have the decomposition as :

$$\mathrm{kerF_*}^{\perp} = \delta D_\theta \oplus \delta D_\perp \oplus \mu, \tag{3.20}$$

where $\mu$ is the orthogonal complementary distribution to $\delta D_\theta \oplus \delta D_\perp$ in $((\mathrm{kerF_*})^{\perp})$ and $\mu$ is invariant with respect to $\phi$. Now, for any $X \in (\Gamma(kerF_*)^{\perp})$, we have

$$\phi X = BX + CX \tag{3.21}$$

where $BX \in \Gamma(kerF_*)$ and $CX \in \Gamma(\mu)$.

**Lemma 3.1.** *Let $(M, \phi, \xi, \eta, g)$ be a Kenmotsu manifold and $(N, g')$ be a Riemannian manifold. If $F : M \to N$ is a QHSC submersion, then we have*

$$\delta BX + C^2X = X,\ \beta BX + BCX = 0$$

$$\beta^2U + B\delta U = U - \eta(U)\xi,\ \delta\beta U + C\delta U = 0$$

*for $U \in \Gamma(\mathrm{kerF_*})$ and $X \in \Gamma((\,kerF_*)^{\perp})$.*

**Proof.** On using equations (2.1), (3.19) and (3.21), we get the desired results.

**Lemma 3.2.** [31] *Let $F$ be a QHSC submersion from an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ onto a Riemannian manifold $(N, g')$, then we have*

(i) $\beta^2U = -\cos^2\theta\, U$

(ii) $g(\beta U, \beta V) = \cos^2\theta\, g(U, V)$

(iii) $g(\delta U, \delta V) = \sin^2\theta\, g(U, V)$,

$U, V \in \Gamma(D_\theta)$.

**Proof.** The proof of above Lemma is similar to the proof of the Theorem (3.5) of [35].

**Lemma 3.3.** *Let $(M, \phi, \xi, \eta, g)$ be a Kenmotsu manifold and $(N, g')$ be a Riemannian manifold. If $F : M \to N$ is a QHSC submersion, then we have*

$$\mathcal{A}_X BY + \mathcal{H}\nabla_X CY = \beta\mathcal{H}\nabla_X Y + B\mathcal{A}_X Y - g(\phi X, Y)\xi \tag{3.22}$$

$$\mathcal{V}\nabla_X BY + \mathcal{A}_X CY = \delta\mathcal{H}\nabla_X Y + C\mathcal{A}_X Y. \tag{3.23}$$

$$\mathcal{V}\nabla_X \beta V + \mathcal{A}_X \delta V = B\mathcal{A}_X V + \beta\mathcal{V}\nabla_X V + g(BX, V)\xi - \eta(V)BX \tag{3.24}$$

$$\mathcal{A}_X \beta V + \mathcal{H}\nabla_X \delta V = C\mathcal{A}_X V + \delta\mathcal{V}\nabla_X V + \eta(V)CX. \tag{3.25}$$

$$\mathcal{V}\nabla_V BX + \mathcal{T}_V CX = \beta\mathcal{T}_V CX + B\mathcal{H}\nabla_V X + g(\delta V, X)\xi \tag{3.26}$$

$$\mathcal{T}_V BX + \mathcal{H}\nabla_V CX = \delta\mathcal{T}_V X + C\mathcal{H}\nabla_V X. \tag{3.27}$$

$$\mathcal{V}\nabla_U \beta V + \mathcal{T}_U \delta V + \eta(V)\beta U = B\mathcal{T}_U V + \beta\mathcal{V}\nabla_U V + g(\phi U, V)\xi \tag{3.28}$$

$$\mathcal{T}_U \beta V + \mathcal{H}\nabla_U \delta V + \eta(V)\delta U = C\mathcal{T}_U V + \delta\mathcal{V}\nabla_U V. \tag{3.29}$$

*for* $U, V \in \Gamma(\ker F_*)$ *and* $X, Y \in \Gamma((\mathrm{kerF}_*)^{\perp})$.

Now we define the following :

$$(\nabla_U \beta)V = \mathcal{V}\nabla_U \beta V - \beta\mathcal{V}\nabla_U V \tag{3.30}$$

$$(\nabla_U \delta)V = \mathcal{H}\nabla_U \delta V - \delta\mathcal{V}\nabla_U V \tag{3.31}$$

$$(\nabla_X B)Y = \mathcal{V}\nabla_X BY - B\mathcal{H}\nabla_X Y \tag{3.32}$$

$$(\nabla_X C)Y = \mathcal{H}\nabla_X CY - C\mathcal{H}\nabla_X Y \tag{3.33}$$

for $U, V \in \Gamma(\ker F_*)$ and $X, Y \in \Gamma((\ker F_*)^{\perp})$.

**Lemma 3.4.** *Let $(M, \phi, \xi, \eta, g)$ be Kenmotsu manifold and $(N, g')$ be a Riemannian manifold. If $F : M \to N$ is a QHSC submersion, then we have*

$$(\nabla_U \beta)V = B\mathcal{T}_U V - \mathcal{T}_U \delta V + g(\phi U, V)\xi - \eta(V)\beta U$$

$$(\nabla_U \delta)V = C\mathcal{T}_U V - \mathcal{T}_U \beta V - \eta(V)\delta U$$

$$(\nabla_X B)Y = \beta\mathcal{A}_X Y - \mathcal{A}_X CY + g(\phi X, Y)\xi - \eta(Y)BX$$

$$(\nabla_X C)Y = \delta\mathcal{A}_X Y - \mathcal{A}_X BY - \eta(Y)CX,$$

*for* $U, V \in \Gamma(\mathrm{kerF}_*)$ *and* $X, Y \in \Gamma((\,kerF_*{}^{\perp}))$.

**Proof.** On using equations (2.7), (2.11)- (2.14), equations (3.19)-(3.21) and equations (3.30)-(3.32), we get the proof of the lemma.

If the tenors $\beta$ and $\delta$ are parallel with respect to the connection $\nabla$ of $M$, then we have

$$B\mathcal{T}_U V = \mathcal{T}_U \delta V - g(\phi U, V)\xi + \eta(V)\beta U$$

$$C\mathcal{T}_U V = \mathcal{T}_U \delta V + \eta(V)\delta U$$

for $X, Y \in \Gamma(TM)$.

## 4. Integrability and totally geodesicness of distributions

Now, we start the discussion of the integrability of distributions and firstly we finding out the integrability of slant distribution as follows:

**Theorem 4.1.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then $D_\theta$ is integrable if and only if*

$$\begin{aligned} & g'((\nabla F_*)(V_1, \beta P\alpha), F_*(\delta V_2)) + g'((\nabla F_*)(V_2, \beta P\alpha), F_*(\delta V_1)) \\ & = \lambda^2 g(\mathcal{V}\nabla_{V_1}\beta\mathcal{P}\alpha - \mathcal{T}_{V_1}\delta\mathcal{R}\alpha, \beta V_2) \\ & + \lambda^2 g(\mathcal{V}\nabla_{V_2}\beta\mathcal{P}\alpha - \mathcal{T}_{V_2}\delta\mathcal{R}\alpha, \beta V_1) \\ & + g(\mathcal{H}\nabla_{V_2}\delta\mathcal{R}\alpha, \delta V_1) - g(\mathcal{H}\nabla_{V_1}\delta\mathcal{R}\alpha, \delta V_2) \end{aligned}$$

*for any $V_1, V_2 \in \Gamma(D_\theta)$ and $\alpha \in \Gamma(D_T \oplus D_\perp \oplus < \xi >)$.*

**Proof.** For any $V_1, V_2 \in \Gamma(D_\theta)$ and $\alpha \in \Gamma(D_T \oplus D_\perp \oplus < \xi >)$ with using (2.2).(2.7) and (2.4), we get

$$g([V_1, V_2], \alpha) = g(\nabla_{V_2}\phi\alpha, \phi V_1) - g(\nabla_{V_1}\phi\alpha, \phi V_2).$$

Taking equation (3.18), we have

$$\begin{aligned} g([V_1, V_2], \alpha) = & \; g(\nabla_{V_2}\beta P\alpha, \beta V_1) + g(\nabla_{V_2}\delta R\alpha, \phi V_1) \\ & - g(\nabla_{V_1}\beta P\alpha, \phi V_2) - g(\nabla_{V_1}\delta R\alpha, \phi V_2). \end{aligned}$$

From (2.11) and (2.12), we can write

$$\begin{aligned} g([V_1, V_2], \alpha) = & \; g(\mathcal{T}_{V_1}\beta P\alpha - \mathcal{H}\nabla_{V_1}\delta R\alpha, \delta V_2) \\ & + g(\mathcal{V}\nabla_{V_1}\beta P\alpha - \mathcal{T}_{V_1}\delta R\alpha, \beta V_2) \\ & + g(\mathcal{T}_{V_2}\beta P\alpha - \mathcal{H}\nabla_{V_2}\delta R\alpha, \delta V_1) \\ & + g(\mathcal{V}\nabla_{V_2}\beta P\alpha - \mathcal{T}_{V_2}\delta R\alpha, \beta V_1). \end{aligned}$$

Considering equation (2.17), we may write

$$\begin{aligned} g([V_1, V_2], \alpha) = & \; g(\mathcal{V}\nabla_{V_2}\beta P\alpha - \mathcal{T}_{V_2}\delta R\alpha, \beta V_1) \\ & + g(\mathcal{V}\nabla_{V_1}\beta P\alpha - \mathcal{T}_{V_1}\delta R\alpha, \beta V_2) \\ & - g(\mathcal{H}\nabla_{V_1}\delta R\alpha, \delta V_2) + g(\mathcal{H}\nabla_{V_2}\delta R\alpha, \delta V_1) \\ & - \frac{1}{\lambda^2} g'((\nabla F_*)(V_!, \beta P\alpha), F_*(\delta V_2)) \\ & - \frac{1}{\lambda^2} g'((\nabla F_*)(V_2, \beta P\alpha), F_*(\delta V_1)) \end{aligned}$$

from which we get the desired result.

**Theorem 4.2.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then invariant distribution $D_T$ is integrable if and only if*

$$P(\mathcal{V}\nabla_{U_1}\beta Q\alpha + \mathcal{T}_{U_1}\delta\alpha) = 0 \tag{4.34}$$

*for $U_1 \in \Gamma(D_T)$ and $\alpha \in \Gamma(D_\theta \oplus D_\perp \oplus <\xi>)$.*

**Proof.** On using (2.2), (2.4) and (3.18), we have

$$g(\nabla_{U_1}U_2, \alpha) = -g(\nabla_{U_1}(\phi Q\alpha + \phi R\alpha), \phi U_2) - \eta(\alpha)g(\phi U_1, \phi U_2),$$

for $U_1 \in \Gamma(D_T)$ and $\alpha \in \Gamma(D_\theta \oplus D_\perp \oplus <\xi>)$. Since $\delta(Q\alpha + R\alpha) = \delta\alpha$ and from (2.11), (2.12), we can write

$$\begin{aligned} g(\nabla_{U_1}U_2, \alpha) = &-g(\mathcal{V}\nabla_{U_1}\beta Q\alpha, \phi U_2) - g(\mathcal{T}_{U_1}\delta\alpha, \phi U_2) \\ &- \eta(\alpha)g(\phi U_1, \phi U_2) \end{aligned}$$

Change the role of $U_1$ and $U_2$, we have

$$\begin{aligned} g([U_1, U_2], \alpha) = &-g(\mathcal{V}\nabla_{U_1}\beta Q\alpha + \mathcal{T}_{U_1}\delta\alpha, \phi U_2) \\ &+ g(\mathcal{V}\nabla_{U_2}\beta Q\alpha + \mathcal{T}_{U_2}\delta\alpha, \phi U_1). \end{aligned}$$

We obtain the proof of the theorem from above equation.

**Theorem 4.3.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then anti-invariant distribution $D_\perp$ is integrable if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}[g'(\nabla_{Z_2}F_*\delta Q\alpha, F_*(\delta Z_1)) - g'(\nabla_{Z_1}F_*\delta Q\alpha, F_*(\delta Z_2))] \\ &= g(grad(\ln\lambda), Z_1)g(\delta Q\alpha, \delta Z_2) \\ &\ - g(grad(\ln\lambda), Z_2)g(\delta Q\alpha, \delta Z_1) \\ &\ - g(\mathcal{T}_{Z_2}\beta\alpha, \delta Z_1) + g(\mathcal{T}_{Z_1}\beta\alpha, \delta Z_2) \end{aligned} \tag{4.35}$$

*for $Z_1, Z_2 \in \Gamma(D_\perp)$ and $\alpha \in \Gamma(D_T \oplus D_\theta \oplus <\xi>)$.*

**Proof.** From (2.2), (2.3), (2.4) and (3.18), we have

$$g(\nabla_{Z_1}Z_2, \alpha) = -\eta(\alpha)g(Z_1, \phi Z_2) - g(\nabla_{Z_1}(\beta P\alpha + \beta Q\alpha + \delta R\alpha), \phi Z_2).$$

Since $\beta(P\alpha + Q\alpha) = \beta\alpha$, we can write

$$g(\nabla_{Z_1}Z_2, \alpha) = -\eta(\alpha)g(Z_1, \phi Z_2) - g(\nabla_{Z_1}\beta\alpha + \nabla_{Z_1}\delta Q\alpha, \delta Z_2).$$

Now, change the roles of $Z_1$ and $Z_2$, we can write

$$g([Z_1, Z_2], \alpha) = g(\nabla_{Z_2}\beta\alpha + \nabla_{Z_1}\delta Q\alpha, \delta Z_1) - g(\nabla_{Z_1}\beta\alpha + \nabla_{Z_2}\delta Q\alpha, \delta Z_2).$$

Considering equations (2.11) and (2.12), we get

$$\begin{aligned} g([Z_1, Z_2], \alpha) &= g(\nabla_{Z_2}\beta\alpha, \delta Z_1) + g(\mathcal{H}\nabla_{Z_2}\delta Q\alpha, \delta Z_2) \\ &\quad - g(\mathcal{T}_{Z_1}\beta\alpha, \delta Z_2) + g(\mathcal{H}\nabla_{Z_1}\delta Q\alpha, \delta Z_2). \end{aligned}$$

From (2.8), (2.17) and lemma 2.1, we have

$$\begin{aligned} g([Z_1, Z_2], \alpha) = \frac{1}{\lambda^2}&[g'(\nabla_{Z_2} F_*\delta Q\alpha, F_*(\delta Z_1)) - g'(\nabla_{Z_1} F_*\delta Q\alpha, F_*(\delta Z_2))] \\ &+ g(\mathcal{T}_{Z_2}\beta\alpha, \delta Z_1) - g(\mathcal{T}_{Z_1}\beta\alpha, \delta Z_2) \\ &+ g(grad(\ln\lambda), Z_2)g(\delta Q\alpha, \delta Z_1) \\ &- g(grad(\ln\lambda), Z_1)g(\delta Q\alpha, \delta Z_2) \end{aligned}$$

which completes the proof of the theorem.

Now, we will discussed the totally geodesicness of fibers of the distributions. Firstly, we will start with the totally geodesicness of the invariant distribution $D_T$.

**Theorem 4.4.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then $D_T$ is not totally geodesic.*

**Proof.** On considering $U, V \in \Gamma(D_T)$ and since $V$ and $\xi$ are orthogonal, we have

$$g(\nabla_U V, \xi) = -g(V, \nabla_U \xi)$$

Taking account the fact of equation (2.5), we have

$$g(\nabla_U V, \xi) = -g(U, V).$$

For $U, V \in \Gamma(D_T), -g(U, V) \neq 0$, that is $g(\nabla_U V, \xi) \neq 0$. Hence, the distribution is not totally geodesic.

**Theorem 4.5.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then $(D_T \oplus \xi)$ defines totally geodesic foliation on $M$ if and only if*

(i) $g(\mathcal{V}\nabla_{U_1}\phi U_2, \beta\alpha) = \frac{1}{\lambda^2}[g'(\nabla F_*)(U_1, \phi U_2), F_*(\delta\alpha)]$

(ii) $g(\mathcal{V}\nabla_{U_1}\phi U_2, BX) = \frac{1}{\lambda^2}[g'((\nabla F_*)(U_1, \phi U_2), F*(CX))]$

*for* $U_1, U_2 \in \Gamma(D_T \oplus <\xi>), X \in \Gamma((kerF_*)^{\perp})$ *and* $\alpha \in \Gamma(D_\theta \oplus D_\perp)$.

**Proof.** On using (2.2), (2.4) and (2.7), we get

$$g(\nabla_{U_1}U_2, \alpha) = g(\nabla_{U_1}\phi U_2, \phi\alpha),$$

for any $U_1, U_2 \in \Gamma(D_T \oplus <\xi>)$ and $\alpha \in \Gamma(D_\theta \oplus D_\perp)$. Now, from (2.11) and decomposition (3.19), we can write

$$g(\nabla_{U_1}U_2, \alpha) = g(\nabla_{U_1}\phi U_2, \delta\alpha) + g(\mathcal{V}\nabla_{U_1}\phi U_2, \beta\alpha).$$

Considering (2.8) and (2.17), we may have

$$g(\nabla_{U_1}U_2, \alpha) = -\frac{1}{\lambda^2}g'((\nabla F_*)(U_1, \phi U_2), F_*(\delta\alpha)) + g(\mathcal{V}\nabla_{U_1}\phi U_2, \beta\alpha) \tag{4.36}$$

On the other hand, for $U_1, U_2 \in \Gamma(D_T)$ and $X \in \Gamma((kerF_*)^{\perp})$ with using (2.2), (2.4), (2.7) and decomposition (3.21), we get

$$g(\nabla_{U_1}U_2, X) = g(\nabla_{U_1}\phi U_2, BX) + g(\nabla_{U_1}\phi U_2, CX).$$

Considering equation (2.11), we may write

$$g(\nabla_{U_1}U_2, X) = g(\mathcal{V}\nabla_{U_1}\phi U_2, BX) + g(\mathcal{T}_{U_1}\phi U_2, CX).$$

From (2.17) and (2.17), we have

$$g(\nabla_{U_1}U_2, X) = g(\mathcal{V}\nabla_{U_1}\phi U_2, BX) + \frac{1}{\lambda^2}g'((\nabla F_*)(U_1, \phi U_2), F_*(CX)). \tag{4.37}$$

From equations (4.36) and (4.37), we get $(i)$ and $(ii)$ part of theorem 4.5.

**Theorem 4.6.** *Let* $F : (M, \phi, \xi, \eta, g) \to (N, g')$ *be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold* $N$. *Then* $D_\theta$ *is not totally geodesic on* $M$.

**Proof.** On considering $Z, W \in \Gamma(D_\theta)$ and since $W$ and $\xi$ are orthogonal, we have

$$g(\nabla_Z W, \xi) = -g(W, \nabla_Z \xi)$$

Taking account the fact of equation (2.5), we have

$$g(\nabla_Z W, \xi) = -g(Z, W).$$

For $Z, W \in \Gamma(D_\theta), -g(Z, W) \neq 0$, that is $g(\nabla_Z W, \xi) \neq 0$. Hence, the distribution is not totally geodesic.

**Theorem 4.7.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then $\Gamma(D_\theta \oplus <\xi>)$ defines totally geodesic foliation if and only if*

(i) $$\lambda^2[g(\mathcal{H}\nabla_{V_1}\delta QV_2, \phi R\alpha) - \cos^2\theta g(\nabla_{V_1}QV_2, \alpha) = g'((\nabla F_*)(V_1, \alpha), F_*(\delta\beta QV_2)) - g'((\nabla F_*)(V_1, \phi P\alpha), F_*(\delta QV_2)) - \eta(\beta QV_2)g(\phi V_1, \alpha)]$$

(ii) $$\lambda^2[g(\mathcal{H}\nabla_{V_1}\delta\beta QV_2, X) - g(\mathcal{H}\nabla_{V_1}\delta QV_2, CX) - \eta(\beta QV_2)g(V_1, BX)] = g'(\nabla F_*)(V_1, BX), F_*(\delta QV_2) - g'((\nabla F_*)(V_1, QV_2), F_*(X))$$

*for any $V_1, V_2 \in \Gamma(D_\theta \oplus <\xi>)$, $X \in \Gamma((kerF_*)^\perp)$ and $\alpha \in \Gamma(D_T \oplus D_\perp)$.*

**Proof.** From equations (2.2), (2.4), (3.18) and decomposition (3.19), we get

$$g(\nabla_{V_1}V_2, \alpha) = g(\nabla_{V_1}\beta QV_2, \phi\alpha) + g(\nabla_{V_1}\delta QV_2, \phi\alpha)$$

for any $V_1, V_2 \in \Gamma(D_\theta \oplus <\xi>)$ and $\alpha \in \Gamma(D_T \oplus D_\perp)$. Again on using (2.4) and (2.7), we can write

$$g(\nabla_{V_1}V_2, \alpha) = g(\nabla_{V_1}\delta QV_2, \phi P\alpha + \phi R\alpha) - g(\nabla_{V_1}\phi\beta QV_2, \alpha) - \eta(\beta QV_2)g(\phi V_1, \alpha)$$

Considering lemma 3.2, equation (2.12) and skew symmetry property of $\mathcal{T}$, we have

$$\begin{aligned} g(\nabla_{V_1}V_2, \alpha) = &-\cos^2\theta g(\nabla_{V_1}QV_2, \alpha) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, \phi R\alpha) \\ &+ g(\mathcal{T}_{V_1}\alpha, \delta\beta QV_2) - g(\mathcal{T}_{V_1}\phi P\alpha, \delta QV_2) \\ &- \eta(\beta QV_2)g(\phi V_1, \alpha) \end{aligned}$$

Finally, from equations (2.8) and (2.17), we yield

$$\begin{aligned} g(\nabla_{V_1}V_2, \alpha) = &-\cos^2\theta g(\nabla_{V_1}QV_2, \alpha) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, \phi R\alpha) \\ &- \frac{1}{\lambda^2}g'((\nabla F_*)(V_1, \alpha), F_*(\delta\beta QV_2)) \\ &+ \frac{1}{\lambda^2}g'((\nabla F_*)(V_1, \phi P\alpha), F_*(\delta QV_2)) \\ &- \eta(\beta QV_2)g(\phi V_1, \alpha). \end{aligned} \tag{4.38}$$

In similar way, for any $V_1, V_2 \in \Gamma(D_\theta)$ and $X \in \Gamma((kerF_*)^\perp)$ with using (2.2), (2.4), (2.7) and (3.19), we get

$$g(\nabla_{V_1}V_2, X) = g(\nabla_{V_1}\beta QV_2, \phi X) - g(\nabla_{V_1}\delta QV_2, \phi X).$$

From equation (2.2), (2.4), (2.7) and (3.19), (2.11) we can write

$$\begin{aligned} g(\nabla_{V_1} V_2, X) = & -g(\nabla_{V_1}\beta^2 QV_2, X) - g(\nabla_{V_1}\delta\beta QV_2, X) \\ & + g(\mathcal{T}_{V_1}\delta QV_2, BX) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, CX) \\ & + \eta(\beta QV_2) g(V_1, BX) \end{aligned}$$

At last, considering equation (2.8), (2.17), (2.12), and lemma 3.2, we have

$$\begin{aligned} g(\nabla_{V_1} V_2, X) = & \frac{1}{\lambda^2} g'((\nabla F_*)(V_1, BX), F_*(\delta QV_2)) \\ & - \cos^2\theta \frac{1}{\lambda^2} g'((\nabla F_*)(V_1, QV_2), F_*(X)) \\ & - g(\mathcal{H}\nabla_{V_1}\delta\beta QV_2, X) + g(\mathcal{H}\nabla_{V_1}\delta QV_2, CX) \\ & + \eta(\beta QV_2) g(V_1, BX). \end{aligned} \tag{4.39}$$

Finally, from equation (4.38) and (4.39), we get the results $(i)$ and $(ii)$ of theorem 4.7. This completes the proof of theorem.

**Theorem 4.8.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then $D_\perp$ is not defined totally geodesic foliation on $M$.*

**Proof.** On considering $Z, W \in \Gamma(D_\perp)$ and since $W$ and $\xi$ are orthogonal, we have

$$g(\nabla_Z W, \xi) = -g(W, \nabla_Z \xi)$$

Taking account the fact of equation (2.5), we have

$$g(\nabla_Z W, \xi) = -g(Z, W).$$

For $Z, W \in \Gamma(D_\perp), -g(Z, W) \neq 0$, that is $g(\nabla_Z W, \xi) \neq 0$. Hence, the distribution is not totally geodesic.

**Theorem 4.9.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then $D_\perp \oplus <\xi>$ defines totally geodesic foliation if and only if*

(i) $\frac{1}{\lambda^2} g'((\nabla F_*)(Z_1, \beta\alpha), F_*(\phi Z_2)) = g(\mathcal{H}\nabla_{Z_1}\phi Z_2, \delta Q\alpha)$

(ii) $\frac{1}{\lambda^2} g'((\nabla F_*)(Z_1, BX), F_*(\phi Z_2)) = g(\mathcal{H}\nabla_{Z_1} CX, \phi Z_2)$

*for any $Z_1, Z_2 \in \Gamma(D_\perp \oplus <\xi>), X \in \Gamma((ker F_*)^\perp)$ and $\alpha \in \Gamma(D_T \oplus D_\theta)$.*

**Proof.** On using equations (2.2), (2.4), (2.7), we can write

$$g(\nabla_{Z_1} Z_2, \alpha) = g(\nabla_{Z_1} \phi Z_2, \phi\alpha),$$

for any $Z_1, Z_2 \in \Gamma(D_\perp \oplus < \xi >)$ and $\alpha \in \Gamma(D_T \oplus D_\theta)$. On using the fact that $\beta P\alpha + \beta Q\alpha = \beta\alpha$ with equations (3.18), (2.11), we get

$$g(\nabla_{Z_1} Z_2, \alpha) = g(\mathcal{T}_{Z_1} \phi Z_2, \beta\alpha) + g(\mathcal{H}\nabla_{Z_1} \phi Z_2, \delta Q\alpha).$$

Considering equation (2.8) and (2.17) and use anti-symmetric property of $\mathcal{T}$, we have

$$g(\nabla_{Z_1} Z_2, \alpha) = \frac{1}{\lambda^2} g'((\nabla F_*)(Z_1, \beta\alpha), F_*(\phi Z_2)) + g(\mathcal{H}\nabla_{Z_1} \phi Z_2, \delta Q\alpha). \tag{4.40}$$

On the other hand, for any $Z_1, Z_2 \in \Gamma(D_\perp)$ and $X \in \Gamma((kerF_*)^\perp)$ with using equations (2.2), (2.4), (2.7) and (3.21), we have

$$g(\nabla_{Z_1} Z_2, X) = -g(\nabla_{Z_1} BX, \phi Z_2) - g(\nabla_{Z_1} CX, \phi Z_2).$$

Considering equations (2.8), (2.11), (2.12) and (2.17), we can write

$$g(\nabla_{Z_1} Z_2, X) = \frac{1}{\lambda^2} g'((\nabla F_*)(Z_1, BX), F_*(\phi Z_2)) - g(\mathcal{H}\nabla_{Z_1} CX, \phi Z_2). \tag{4.41}$$

From equations (4.40) and (4.41), the proof of the theorem is complete.

**Theorem 4.10.** *Let $F : (M, \phi, \xi, \eta, g) \to (N, g')$ be QHSC submersion from a Kenmotsu manifold onto a Riemannian manifold $N$. Then the vertical distribution $(kerF_*)$ defines totally geodesic foliation if and only if*

$$\begin{aligned}
&\cos^2\theta \frac{1}{\lambda^2} g'((\nabla F_*)(Y_1, QY_2), F_*(X)) + \frac{1}{\lambda^2} g'((\nabla F_*)(Y_1, \beta PY_2), F_*(CX)) \\
&= g(\mathcal{V}\nabla_{Y_1} \beta PY_2 + \mathcal{T}_{Y_1} \delta QY_2 + \mathcal{T}_{Y_1} \delta RY_2, BX) \\
&\quad + g(\mathcal{H}\nabla_{Y_1} \delta QY_2 + \mathcal{H}\nabla_{Y_1} \delta RY_2, CX) - g(\mathcal{H}\nabla_{Y_1} \delta\beta QY_2, X),
\end{aligned}$$

*for any $Y_1, Y_2 \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$.*

**Proof.** On using (2.2), (2.4) and (2.7) with decomposition (3.18), we have

$$g(\nabla_{Y_1} Y_2, X) = g(\nabla_{Y_1} \beta PY_2 + \beta QY_2 + \delta QY_2 + \delta RY_2, \phi X),$$

for any $Y_1, Y_2 \in \Gamma(kerF_*)$ and $X \in \Gamma((kerF_*)^\perp)$. From equations (2.11), (2.12) and (3.21), we yield

$$\begin{aligned} g(\nabla_{Y_1} Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{T}_{Y_1}\beta PY_2 + \mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &+ g(\nabla_{Y_1}\beta QY_2, \phi X). \end{aligned}$$

Taking with equations (2.4), (2.7) and (3.18), we may have

$$\begin{aligned} g(\nabla_{Y_1} Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{T}_{Y_1}\beta PY_2 + \mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &- g(\nabla_{Y_1}\beta^2 QY_2, X) - g(\nabla_{Y_1}\delta\beta QY_2, X). \end{aligned}$$

Consider lemma 3.2 with equations (2.8) and (2.17), we get

$$\begin{aligned} g(\nabla_{Y_1} Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &+ \cos^2\theta g(\nabla_{Y_1} QY_2, X) - g(\nabla_{Y_1}\delta\beta QY_2, X) \\ &- \frac{1}{\lambda^2} g'((\nabla F_*)(Y_1, \beta PY_2), F_*(CX)). \end{aligned}$$

Again using (2.8) and (2.17), we finally have

$$\begin{aligned} g(\nabla_{Y_1} Y_2, X) &= g(\mathcal{V}\nabla_{Y_1}\beta PY_2 + \mathcal{T}_{Y_1}\delta QY_2 + \mathcal{T}_{Y_1}\delta RY_2, BX) \\ &+ g(\mathcal{H}\nabla_{Y_1}\delta QY_2 + \mathcal{H}\nabla_{Y_1}\delta RY_2, CX) \\ &+ \cos^2\theta \frac{1}{\lambda^2} g'((\nabla F_*)(Y_1, QY_2), F_* X) - g(\nabla_{Y_1}\delta\beta QY_2, X) \\ &- \frac{1}{\lambda^2} g'((\nabla F_*)(Y_1, \beta PY_2), F_*(CX)). \end{aligned}$$

This completes the proof of the theorem.

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# ON $QTAG$-MODULES CONTAINING PROPER $h$-PURITY

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ABSTRACT. There are numerous problems of determining the $QTAG$-modules in which every $h$-pure submodule is isotype or the $QTAG$-modules in which every submodule is isotype. Our global aim here is to find in this direction a new problem by generalizing the $h$-purity in $QTAG$-modules, and thereby to establish some characterizations of the $QTAG$-modules in which every $\sigma$-pure submodule is $\lambda$-pure submodule for arbitrary ordinals $\sigma$ and $\lambda$.

**Keywords**: QTAG-modules, $\sigma$-pure submodules, $\lambda$-pure submodules

**2010 Mathematics Subject Classification**: Primary 16K20, Secondary 13C12, 13C13.

## 1. INTRODUCTION

The theory of abelian groups studied from time to time by many mathematicians, play a very crucial role in the theory of modules. Many authors interested in module theory have worked on generalizing the theory of abelian groups. The notion of the generalized torsion abelian groups is an important concept in the area of $TAG$-modules. It was first introduced by Singh [17] in 1976. A module $M$ over a ring $R$ is called a $TAG$-module if it satisfies the following two conditions while the rings are associated with unity.

"$(i)$ Every finitely generated submodule of any homomorphic image of $M$ is a direct sum of uniserial modules.

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$(ii)$ Given any two uniserial submodules $U_1$ and $U_2$ of a homomorphic image of $M$, for any submodule $N$ of $U_1$, any non-zero homomorphism $\phi : N \to U_2$ can be extended to a homomorphism $\psi : U_1 \to U_2$, provided the composition length $d(U_1/N) \leq d(U_2/\phi(N))$."

It was shown that the theory of these modules very closely paralleled the theory of torsion abelian groups; for this reason they were referred to as $TAG$-modules. Later on, it was shown that, for almost all applications, one of these conditions was not needed; ignoring this nearly superfluous condition, the slightly more general concept of a $QTAG$-module was initiated by the same author in [18]. Since then, many forms of this notion such as $\alpha$-modules [4, 10], $n$-layered modules [15], essentially finitely indecomposable modules [3] and semi-complete modules [6] etc. have been defined and studied by many authors. Moreover, the authors have introduced many new concepts via these types of modules. They have also investigated some of their interesting properties and characterizations of these modules. Not surprisingly, many of the developments parallel the earlier development of the structure of torsion abelian groups. The present work is a natural extension of the torsion abelian groups over to the area of $QTAG$-modules and certainly contributes to the overall knowledge of the structure of $QTAG$-modules.

## 2. Preliminaries

Throughout the text, we assume that all rings into consideration are associative with unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. By the term "uniserial module" we will mean a module $M$ over a ring $R$, whose submodules are totally ordered by inclusion, i.e., for any two submodules $N$ and $L$ of $M$, either $N \subseteq L$ or $L \subseteq N$. Likewise, we shall say $M$ is uniform if intersection of any two of its non-zero submodules is non-zero. In particular, if $M$ is a module and $u \in M$, then let $u$ denote the uniform element and let $uR$ denote the uniform (hence uniserial) module, respectively. Concerning decomposition series, we suppose that all decomposition series are unique. For any module $M$, the symbol $d(M)$ will denote its decomposition length. In addition, if $u$ is an uniform element of $M$ (i.e., $u \in M$), then $e(u)$ is called the exponent of $u$, and $e(u) = d(uR)$. As usual, for such a module $M$, we set the height of $u$ in $M$ as $H_M(u) = \sup\{d(vR/uR) : v \in M,\ u \in vR \text{ and } v \text{ uniform}\}$. For every non-negative integer $t$, $H_t(M) = \{u \in M |\ H_M(u) \geq t\}$ denotes the $t$-th copies of $M$ which can be viewed as a submodule of $M$ consisting of all elements of height at least $t$. In this way, for a module $M$, the letter $M^1$ will always denote in the sequel the submodule of $M$, containing elements of infinite height. Moreover, we denote by $Soc(M)$, the socle of $M$, i.e., the sum of all simple submodules of $M$. For any $t \geq 0$, $Soc^t(M)$ is defined inductively as follows: $Soc^0(M) = 0$ and $Soc^{t+1}(M)/Soc^t(M) = Soc(M/Soc^t(M))$.

We add some basic definitions as well from [6], which is necessary for our successful presentation. The module $M$ is named $h$-divisible if $M = M^1 = \cap_{t=0}^{\infty} H_t(M)$, or equivalently, if $H_1(M) = M$. The module $M$ is termed separable if $M^1 = 0$. A submodule $N$ of a module $M$ is said to be an $h$-pure in $M$ if for every non-negative integer $t$ the equality $N \cap H_t(M) = H_t(N)$ hold. The cardinality of the minimal generating set of $M$ is denoted by the symbol $g(M)$ that plays a significant role in our further investigation. By analogy, for all ordinals $\sigma$, one can define $f_M(\sigma)$, the $\sigma^{th}$-$Ulm$ invariant of $M$ as follows: $f_M(\sigma) = g\big(Soc(H_\sigma(M))/Soc(H_{\sigma+1}(M))\big)$.

In [5, 11], respectively, a submodule $N$ of $M$ is $L$-high, if $N \cap L = 0$ and $N$ is maximal with respect to this intersection, that is, it is not properly contained in any different submodule of $M$ having the same property.

It is well to note that various results for $TAG$-modules are also valid for $QTAG$-modules [13]. Our present work is motivated by the many significant results from the reference [14]. It is worthwhile noticing that some of the results are already investigated [7, 8] with $h$-purity. For the better understanding of the mentioned topic here one must go through the papers [9, 16]. In what follows, all notations and notions are standard and will be in agreement with those used in [1, 2]; for the specific ones, we refer the readers to [19].

## 3. Chief results

We begin by reviewing some terminology. If $\sigma$ is an ordinal, and $M$ is a $QTAG$-module, then the infinite height $H_\sigma(M)$ will be defined as $H_\sigma(M) = \cap_{\lambda<\sigma} H_\lambda(M)$ in the sense of [12], by using transfinite induction. Likewise, for any first infinite ordinal $\omega$, the submodule $M^1$ of $M$, containing elements of infinite height that hold the equality $M^1 = \cap_{t=1}^{\infty} H_t(M) = H_\omega(M)$. Clearly, $H_t(M)$ is a submodule of $M$ and the intersection $\cap_{t=1}^{\infty} H_t(M)$ form a submodule which is known as first $Ulm$ submodule.

Next, we review the following concepts from [13]. A submodule $N$ of $M$ is said to be $\sigma$-pure if, for all ordinal $\lambda$, there exists an ordinal $\sigma$ (depending on $N$) such that $H_\lambda(M) \cap N = H_\lambda(N)$. Besides, a submodule $N$ of $M$ is named isotype, if it is $\sigma$-pure for every ordinal $\sigma$. It readily follows that an isotype submodule will be $h$-pure in $M$, and hence a summand of $M$.

The theory of isotypity clearly depends on the theory of $h$-purity in $QTAG$-modules, and hence upon criteria under which a given $h$-pure submodule must necessarily be isotype (see, [7]). One important example is the determination of the $QTAG$-modules in which every $h$-pure submodule is a direct summand. Though it has been stated in a variety of forms by a number of characterizations. In this section we follow a somewhat different path and explore a new problem of determining the $QTAG$-modules in which every $\sigma$-pure submodule is $\lambda$-pure submodule for arbitrary ordinals $\sigma$ and $\lambda$.

The following elementary, but useful lemma, shed some light about the relationships between $Ulm$-invariant and $h$-purity.

**Lemma 3.1.** *Suppose $\sigma$ and $\lambda$ are ordinals such that $1 \leq \sigma < \lambda \leq \infty$ and $M$ is a $QTAG$-module with $f_M(\delta) = 0$ for $\sigma \leq \delta + 1 < \lambda$. If $N$ is a $\sigma$-pure submodule of $M$, then $N$ is also $\lambda$-pure.*

**Proof.** First observe that if $\sigma \leq \alpha < \lambda$ and $N$ is an $\alpha$-pure submodule, then $N$ is an $(\alpha+1)$-pure submodule of $M$. Next, choose $N$ is $\alpha$-pure and let $a \in N \cap H_{\alpha+1}(M)$. Then $a \in H_\alpha(N) \subset H_\sigma(N)$. Thus $a = b'$, where $d(bR/b'R) = 1$ and $b \in H_{\sigma-1}(N)$. But $a = c'$, where $d(cR/c'R) = 1$, and $c \in H_\alpha(M)$. Therefore, $b = c + x$, where $x \in Soc(H_{\sigma-1}(M))$. By hypothesis on $f_M(\delta)$, we have $Soc(H_{\sigma-1}(M)) \subset Soc(H_\alpha(M))$. Thus

$$b = c + x \in H_\alpha(M) \cap N = H_\alpha(N).$$

Therefore, $a = b' \in H_{\alpha+1}(M)$ such that $d(bR/b'R) = 1$, we are done.

Let us recall the smallest ordinal $\beta$ such that $H_\beta(M) = 0$, is said to be the length of the $QTAG$-module $M$.

Inspired and motivated by the above concept, we give a new concept of two parameters involving the $Ulm$-invariant as follows.

**Definition 3.1.** *Let $\delta$ be an ordinal and $M$ a $QTAG$-module such that $1 \leq \delta \leq H_\beta(M)$ and let $\gamma$ be any ordinal. We define $t_\delta$ and $r_\gamma$ by*

$$t_\delta = \begin{cases} \inf\{t \geq 0 : f_M(\delta - 1 + t) \neq 0\}, \text{ if } \delta - 1 \text{ exists} \\ 0, \text{ if } \delta \text{ is a limit ordinal,} \end{cases}$$

*and*

$$r_\gamma = \inf\{\alpha + 1 : \alpha + 1 < \gamma \text{ and } f_M(\alpha) \neq 0\}.$$

It is fairly to see that $t_\delta$ is a finite ordinal. This follows easily that $\delta \leq r_\gamma$ implies $\delta + t_\delta \leq \gamma$, with strict inequality holding when $\delta$ is not a limit ordinal.

Before presenting our main attainments, two preliminary technical lemmas are necessary.

**Lemma 3.2.** *Let $N$ be a submodule of a QTAG-module $M$. Then there exists a submodule $L$ of $M$ containing $N$ such that it satisfies the following conditions:*
*(i) $L$ is isotype in $M$.*
*(ii) $N$ is isotype in $L$.*
*In particular, $L$ is $\sigma$-pure in $M$ if and only if $N$ is $\sigma$-pure in $M$, where $\sigma$ is an arbitrary ordinal.*

**Proof.** In order to show that $L$ is isotype in $M$, it suffices to show that $L$ is $\sigma$-pure in $M$ for an arbitrary ordinal $\sigma$, that is to show that $L$ is $(\sigma+1)$-pure. In order to do this, among all uniform element in $L \cap H_{\sigma+1}(M)$, choose $a$ such that $a = b'$, where $d(bR/b'R) = 1$ and $b \in H_\sigma(M)$. Now $a \in N_k$ for some $k$, so that $b' \in N_k$ where $d(bR/b'R) = 1$. Thus $b \in N_{k+1}$. Therefore, $b \in L \cap H_\sigma(M) = H_\sigma(L)$ and $a = b' \in H_{\sigma+1}(L)$ such that $d(bR/b'R) = 1$, as expected.

As for the second part, we can apply the same idea. Assume that $N$ is $\sigma$-pure in $L$ and let $a \in N \cap H_{\sigma+1}(L)$. Then $a = b'$ where $d(bR/b'R) = 1$ and $b \in H_\sigma(L)$. Since $b \in L$, it follows that $nb \in N$ for some non-negative integer $n$. Therefore, $b \in N \cap H_\sigma(L) = H_\sigma(N)$. Hence, $a = b' \in H_{\sigma+1}(N)$ such that $d(bR/b'R) = 1$, as required.

Conversely, suppose that $N \cap H_\lambda(M) = H_\lambda(N)$ for all $\lambda \leq \sigma$. Let $a \in L \cap H_\lambda(M)$, it is readily checked that $na \in N$ for some non-negative integer $n$. Thus

$$na \in N \cap H_\lambda(M) = H_\lambda(N) \subset H_\lambda(L).$$

It is only a routine exercise to check that $na \in H_\lambda(L)$ and implies that $a \in H_\lambda(L)$. Thus, we conclude that $L$ is $\sigma$-pure in $M$, as asserted.

**Lemma 3.3.** *Let $\gamma$ be an ordinal and $M$ a QTAG-module such that $H_\gamma(M)$ contains a non-zero uniform element $u$ with $e(u) = \infty$. For each ordinal $\delta$ and for some $n$ let $n_\delta = -1$ if $\delta - 1$ exists and $n_\delta = 0$ otherwise. Then there exists a submodule $N_\delta$ of $M$ such that $1 \leq \delta \leq r_\gamma$ and it satisfies the following conditions:*
*(i) $N_\delta$ is $(\delta + t_\delta)$-pure in $M$.*
*(ii) $N_\alpha \subset N_\delta$ if $\alpha < \delta$.*
*(iii) $N_\delta \cap Soc(H_{\delta+t_\delta+n_\delta}(M)) = Soc(H_\gamma(M))$.*

*(iv)* $u \in N_\delta$

*(v)* $u \notin H_{\delta+t_\delta+1}(N_\delta)$

*In particular,* $N_\delta$ *is not a* $(\gamma+1)$*-pure submodule of* $M$*, and* $N_\delta$ *is not* $\gamma$*-pure in* $M$ *if* $\gamma$ *is not a limit ordinal.*

**Proof.** The proof is by induction on $\delta$. Assume that for each ordinal $\alpha < \delta$, there exists a submodule $N_\alpha$ and satisfies $(i)-(v)$. If $\delta$ is a limit ordinal, then

$$N_\delta = \cup_{\alpha<\delta} N_\alpha.$$

Certainly, if the submodule $N_\delta$ exists, then $N_\delta$ satisfies $(i)-(v)$. If $\delta-1$ exists and $t_{\delta-1} > 0$, we have $N_\delta = N_{\delta-1}$. It follows that $N_\alpha$ satisfies $(i)-(v)$, since $t_\delta = t_{\delta-1}-1$. If $\delta-1$ exists and $t_{\delta-1} = 0$, then we can construct $N_\delta$ from $N_{\delta-1}$. Since $f_M(\delta+t_\delta-1) \neq 0$, there exists an uniform element $v \in Soc(M)$ such that $H_M(v) = \delta+t_\delta-1$. Note also that $\delta+t_\delta < \gamma$, since $\delta \leq r_\gamma$. Then for any submodule $P$ of $M$ containing $v$, we have

$$Soc(H_{\delta+t_\delta-1}(M)) = Soc(H_\gamma(M)) \oplus P.$$

Thus, for $0 \neq w \in H_{\delta+t_\delta}(M)$ such that $u = w'$ and $d(wR/w'R) = 1$. This, in tern, implies that, there exists a submodule $Q$ of $M$ containing $v+w$ such that $Q = \langle N_{\delta-1}, a\rangle$, where $a = v+w$.

We first claim that $P \cap Q = 0$, if this failed, then there exist elements $b \in P$ and $c \in N_{\delta-1}$ and an integer $k$ such that $b = c + a' \neq 0$, where $d(aR/a'R) = k$. If $k = 0$, then $u = a' = -c'$ such that $d(aR/a'R) = 1$, $d(cR/c'R) = 1$ and

$$c \in N_{\delta-1} \cap H_{\delta+t_\delta-1}(M) \subset H_{\delta-1}(N_{\delta-1}).$$

Thus $u \in H_\delta(N_{\delta-1})$, which is a contradiction that satisfies $(v)$. On the other hand if $k > 0$, then $b = u' + c \in N_{\delta-1} \in P \cap N_{\delta-1}$ where $d(uR/u'R) = k-1$. But $P \cap N_{\delta-1} = 0$ because $N_{\delta-1}$ satisfies $(iii)$. This gives the desired claim that $P \cap Q = 0$.

Suppose now that $N_\delta$ is a $P$-high submodule of $M$ containing $Q$. Then $N_{\delta-1} \subset N_\delta$, which satisfies $(ii)$. In fact, the checking of $(i)$ is elementary for $N_\delta$. As for $(iii)$, using the fact that $Soc(H_\gamma(M)) \subset N_{\delta-1}$ for $N_\delta$. Observe that $N_\delta$ also satisfies $(iv)$ because $a \in N_\delta$ and $a' = u$ where $d(aR/a'R) = 1$. In order to see that $(v)$ is valid, let us suppose that $u \in H_{\delta+t_\delta+1}(N_\delta)$. Then $u = x'$ where $d(xR/x'R) = 1$ and $x \in H_{\delta+t_\delta}(N)$. Thus $a = x + y$, where $y \in N_\delta \cap Soc(H_{\delta+t_\delta-1}(M))$ and $H_M(a) = \delta+t_\delta-1$. Therefore, $y \in H_\gamma(M)$ because of $(iii)$. But $a = x + y \in H_{\delta+t_\delta}(M)$. This is a contradiction. Hence $N_\delta$ must satisfy $(v)$, as promised.

We construct now a submodule $N_1$ of $M$, imitating the method of $N_\delta$ as demonstrated in the above paragraph. Therefore to finish the induction, we choose $v \in Soc(M)$ such that $H_M(v) = t_1$ and $Soc(H_{t_1}(M)) = Soc(H_\gamma(M)) \oplus P$ with $v \in P$. Let $Q = \langle Soc(H_\gamma(M)), a\rangle$, where $a = v+w$. If $0 \neq b \in P \cap Q$, then $b = c+ta$, where $c \in Soc(H_\gamma(M))$ and $t$ is a positive integer. Bearing in mind this construction, it is apparent that $P \cap Q = 0$. Finally, we let $N_1$ be a $P$-high submodule of $M$, and a routine computations reveals that $N_1$ satisfies $(i) - (v)$. The proof is completed.

We next give an explicit definition of our main term.

**Definition 3.2.** *Let $\sigma$ and $\lambda$ be ordinals, we say a QTAG-module $M$ is $(\sigma,\ \lambda)$-module if every $\sigma$-pure submodule of $M$ is $\lambda$-pure.*

Now we have all the ingredients needed to establish the following.

**Theorem 3.1.** *Suppose $\sigma$ and $\lambda$ are ordinals with $\lambda > 0$ and $M$ is a QTAG-module. Then $M$ is a $(\sigma,\ \lambda)$-module if and only if $M$ is $h$-divisible.*

**Proof.** Foremost, assume that $M$ is $h$-divisible, that is $H_1(M) = M$. Knowing this, we yield that there is a submodule $N$ of $M$ such that $N \cap H_1(M) \subset H_\delta(N)$ for any ordinal $\delta > 0$. Hence, in particular, every $\sigma$-pure submodule of $M$ is $\lambda$-pure and we are done.

Next, we deal with the converse implication. Assume that $M$ is a $(\sigma,\ \lambda)$-module. If $H_1(M) \neq M$, then there exists an uniform element $u$ containing $H_1(M)$ such that $e(u) = \infty$. Let $N = \langle u\rangle$. Then $N$ is not $h$-pure in $M$. Henceforth, according to Lemma 3.2 , there is a submodule $L$ of $M$ such that $L$ is not $\lambda$-pure for any $\lambda > 0$. Since $L$ is isotype in $M$, we have $H_1(M) = M$, as required.

And so, we come to the following.

**Theorem 3.2.** *Suppose $\sigma$ and $\lambda$ are ordinals, $1 \leq \sigma < \lambda \leq \infty$ and $M$ is a QTAG-module. If $\sigma$ is a limit ordinal, then $M$ is a $(\sigma, \lambda)$-module if and only if the following hold:*
*$(i)$ $H_\beta(Soc(H_k(M)) < \sigma$, for some $k > 0$*
*$(ii)$ $H_\sigma(M) = U \oplus H_1(M)$, where $U$ is a direct sum of uniserial modules of exponent $k$.*

**Proof.** In virtue of Lemma 3.3 , the necessity is true. Suppose $(ii)$ is not hold, then there exists an element $x \in H_{\sigma+1}(M)$ such that $e(x) = \infty$. After this, let us assume that $(i)$

is not hold, then $\sigma = r_\sigma \leq r_{\sigma+1}$. If we replace $\delta = \sigma$ and $\gamma = \sigma + 1$ in Lemma 3.3 , we get that a $\sigma$-pure submodule $N_\sigma$ of $M$ which is not $(\sigma + 1)$-pure. Henceforth, all the conditions are satisfied for $M$ to be a $(\sigma, \lambda)$-module.

The sufficiency of $(i)$ being self-evident from Lemma 3.1 , where we replace $\sigma$ by $H_\beta(Soc(H_k(M))+1$. Let us assume that $(ii)$ is hold and let $N$ be a $\sigma$-pure submodule of $M$ with $\sigma < \alpha \leq \lambda$. Without loss of generality, we assume that $y \in N \cap H_\alpha(M)$. Then $y \in N \cap H_1(M)$, since $H_\alpha(M) = H_1(M)$. From the $\delta$-purity of $N$, we have $y \in H_\delta(N)$ for every ordinal $\delta$. Consequently, $y \in H_\alpha(N)$. Thus $N$ is $\lambda$-pure, as expected.

We continue in this way by the following.

**Theorem 3.3.** *Suppose $\sigma$ and $\lambda$ are ordinals, $1 \leq \sigma < \lambda \leq \infty$ and $M$ is a $QTAG$-module. If $\sigma - 1$ exists, then $M$ is a $(\sigma, \lambda)$-module if and only if the following hold:*
*(i) $f_M(\delta) = 0$ if $\delta$ satisfies $\sigma \leq \delta + 1 < \lambda$.*
*(ii) $H_{\sigma-1}(M) = U \oplus V \oplus H_1(M)$, where $U$ and $V$ are direct sum of uniserial modules of exponent $k$ and $k + 1$ respectively, for some $k > 0$.*

**Proof.** First assume that $M$ is a $(\sigma, \lambda)$-module such that $(i)$ is not hold. Suppose now $(ii)$ holds. Then $\sigma \leq H_\beta(Soc(H_k(M))$, for some $k$ and an ordinal $\beta$. Thus by Definition 3.1 , there exists a parameter $t_\sigma$ such that $f_{(H_{\sigma-1}(M))}(\delta) = 0$, for some $\delta < t_\sigma$. Let $x$ be an uniform element of $H_{\sigma+t_\sigma+1}(M)$ such that $e(x) = \infty$. Then by Lemma 3.3 , there exists a $(\sigma + t_\sigma)$-pure submodule $N_\sigma$ of $H_{\sigma-1}(M)$, which is not $(\sigma + t_\sigma + 1)$-pure. But this is impossible because $\sigma + t_\sigma + 1 \leq \lambda$. Utilizing the preceding point, it is straight forward to compute that

$$H_{\sigma+t_\sigma+1}(M) = H_{t_\sigma+2}(H_{\sigma-1}(M))$$

is a $QTAG$-module and besides it is direct sum of uniserial module. Let $k = t_\sigma + 1$. Then $f_M(\delta) \neq 0$ if $k-1 \leq \delta \leq k$ and the above condition on $H_{k+1}(H_{\sigma-1}(M))$ holds $(ii)$, as needed.

Concerning the sufficiency, the first condition is straight forward from Lemma 3.1. As for the second condition, let $N$ be a $\sigma$-pure submodule of $M$. Then $N$ is $(\sigma + k - 1)$-pure, in conjunction with Lemma 3.1 , since $f_M(\delta) = 0$ for $\sigma \leq \delta + 1 < \sigma + k - 1$. In fact, for every ordinal $\alpha$, we observe that $\sigma + k \leq \alpha \leq \lambda$ and choose $y \in N \cap H_\alpha(M)$. Since $H_\alpha(M) = H_{\alpha+k}(M) = H_1(M)$, we have $y \in H_\alpha(N)$. Hence $N$ is a $\lambda$-pure submodule of $M$, as required.

We now settle the example to constructing extensions of $(\sigma, \lambda)$-module, which is parallel as assertion due to Moore and Hewett [14].

**Example**: Let $M$ be a $(\sigma, \lambda)$-module with $N$ a $\gamma$-pure submodule of $M$, for $\sigma \leq \gamma < \lambda$. One can easily constructs a submodule $L$ such that $L$ is $\sigma$-pure. Applying Lemma 3.2, $L$ is $\lambda$-pure and, hence, $\delta$-pure for $\sigma \leq \gamma < \lambda$. Thus, in view of Theorem 3.3, $N$ is $\delta$-pure, as required.

## 4. Open Problems

We close the work by formulating the following problems.

**Problem 4.1.** *Suppose $M$ is a QTAG-module such that $M/H_\sigma(M)$ is a direct sum of uniserial modules and $1 \leq \sigma < \lambda \leq \infty$. Is then $M$ $(\sigma, \lambda)$-module if and only if $H_\sigma(M)$ is?*

**Problem 4.2.** *If $1 \leq \sigma < \lambda \leq \infty$ and $M$ is a $(\sigma, \lambda)$-module such that $M = \Sigma_{\lambda \in I} M_\lambda$, and $N_\lambda$ is a $\lambda$-pure submodule of $M_\lambda$, then is it true that $\Sigma_{\lambda \in I} N_\lambda$ is a $\lambda$-pure submodule of $M$?*

**Problem 4.3.** *If $\omega \leq \lambda \leq \infty$. Can $M$ is a $(\omega, \lambda)$-module if and only if $M = M_1 \oplus M_2$, where $M_1$ is an $h$-divisible and $M_2$ is a direct sum of separable modules?*

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