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Editor-in-Chief

Bayram Sahin
Department of Mathematics, Faculty of Science, Ege University, Izmir, Turkey
journalofmapsinmathematics@gmail.com

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EDITORIAL

BAYRAM ŞAHIN

Dear readers and authors, welcome to the new volume of International Journal of Maps in Mathematics. In the first issue of this volume, besides presenting 5 important papers to you, we also announce our new member in the editorial board of the journal. Dr. İbrahim Şentürk, who has been with us since the establishment of the journal, will now serve as the new area editor for the Mathematical Logic section of our Journal.

Dr. İbrahim Şentürk received his PhD in Mathematics, in 2018. His research interests include algebraic logical structures, decision systems, lattice theory, multi-valued logic and fuzzy logic. He is currently an Associative Professor at the Department of Mathematics at Ege University, İzmir, Türkiye.

We welcome and congratulate Dr. İbrahim Şentürk.

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THE SPECIAL CURVES OF FIBONACCI AND LUCAS CURVES

EDANUR ERGÜL * AND SALIM YÜCE

Abstract. In this paper, we introduce the contrapedal, radial, inverse, conchoid and strophoid curves of Fibonacci and Lucas curves which are defined by Horadam and Shannon, [18]. Moreover, the graphs of these special curves are drawn by using Mathematica.

Keywords: Fibonacci curve, Lucas curve, Contrapedal curve, Radial curve, Inverse curve, Conchoid curve, Strophoid curve

2010 Mathematics Subject Classification: 53A04, 11B39.

1. Introduction

The plane curves in the Euclidean plane are one of the most essential subjects in differential geometry. Thanks to a growing interest in this subject, it is demonstrated that any plane curve brings about other plane curves through several constructions. Some of these are contrapedal, radial, inverse, conchoid and strophoid curves. Contrapedal curves are employed in many areas such as mathematics (see [16]) and physics (see [20]). Radial curve was studied by Robert Tucker in 1864, [25]. Geometrical inversion is originated from Jakob Steiner in 1824. In 1825, Adolphe Quetelet followed closely him by giving some examples. Apparently, it independently discovered by Giusto Bellavitis in 1836, by Stubbs and Ingram in 1842-3, and by Lord Kelvin who employed it in his electrical researches in 1845, [25]. Inverse curve has a important role in mathematics (see [6]). Conchoid is a plane curve invented

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* Corresponding author

Edanur Ergül; edanur.ergul@marmara.edu.tr; https://orcid.org/0000-0002-8598-5544

Salim Yüce; sayuce@yildiz.edu.tr; https://orcid.org/0000-0002-8296-6495

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by the Greek mathematician Nicomedes, who applied it to the problems of duplication the cube. The conchoid has been used by later mathematicians, notably Sir Isaac Newton, in the construction of various cubic curves, [23]. Conchoids make a significant contribution in many applications as optics (see [2]), astronomy (see [9]), engineering in medicine and biology (see [8], [12]), mechanical in fluid processing (see [21]), physics (see [22]), electromagnetic research (see [26]), etc. The *Conchoid of Nicomedes*, which is the conchoid of a line, and the *Limaçon of Pascal*, which is the conchoid of a circle, are the two most famous conchoids, [17]. Strophoid curve initially appears in work by the English mathematician Isaac Barrow, who was Isaac Newton's teacher, in 1670. However, the curve actually is described in his letters by Evangelista Torricelli before Barrow's work around 1645. In 1846, the strophoid, whose meaning is a "belt with a twist", was named by Montucci, [4]. J. Booth called it the logocyclic curve in his article in the 19th century, [3]. For further information about contrapedal, radial, inverse, conchoid, and strophoid curve, we recommend the reader to go through [7], [11], and [25].

The famous book called the Liber Abaci of Italian mathematician Leonardo de Pisa who is known as Fibonacci also posed a problem concerning the progeny of a single pair of rabbits which is the foundation of the Fibonacci sequence, [5]. During the time Fibonacci wrote Liber Abaci, Fibonacci numbers were not recognized as something special. The sequence was given the current name "Fibonacci numbers" by French mathematician Edouard Lucas who later created his own sequence based on the pattern set by Fibonacci. Lucas numbers are very similar to Fibonacci numbers in that they form a sequence of numbers and also closely related to Fibonacci numbers, [15].

In 1988, Horadam and Shannon defined Fibonacci and Lucas curves on Euclidean plane, (see [18]). Moreover, there are many articles about three dimensional Fibonacci curve, (see [13], [19]). In addition, Akyiğit, Erişir and Tosun studied on the evolute, parallel and pedal of Fibonacci and Lucas curves in 2015, (see [1]). In 2017, Özvatan and Pashaev had a study on generalized Fibonacci sequences and Binet-Fibonacci curves, (see [14]). They constructed Binet-Fibonacci curve in complex plane by extending Binet's formula to arbitrary real numbers. In this article, we are interested in investigation of the contrapedal, radial, inverse, conchoid and strophoid curves of Fibonacci and Lucas curves and obtaining the figures of these special curves.

1.1. **Fibonacci and Lucas Numbers.** This subsection gives a brief overview of Fibonacci and Lucas numbers. More detailed information about them can be found in [10] and [24].

1.1.1. *Fibonacci Numbers.*

Definition 1.1. *The nth Fibonacci number F_n is defined by*

$$F_n = F_{n-1} + F_{n-2}$$

with initial conditions

$$F_1 = F_2 = 1,$$

where $n \geq 3$. In this case, Fibonacci numbers are given by

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots, F_n, \ldots$$

The ratio of consecutive Fibonacci numbers gives us a new sequence:

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \ldots, \frac{F_{n+1}}{F_n}, \ldots$$

Lemma 1.1. *The ratio of two consecutive Fibonacci numbers approaches $\frac{1+\sqrt{5}}{2}$ as $n \to \infty$. More precisely,*

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}.$$

Definition 1.2. *The positive root $\frac{1+\sqrt{5}}{2} = 1.618...$ of the equation $x^2 - x - 1 = 0$ is called golden ratio.*

Theorem 1.1. *Let α and β be the solutions of the quadratic equation $x^2 - x - 1 = 0$; so $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then, the relation that gives us the nth term of Fibonacci sequence is given by*

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} = \frac{\alpha^n - \beta^n}{\sqrt{5}},$$

where $n \geq 1$.

Corollary 1.1. *Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then,*

1. $\alpha\beta = -1$ ***2.*** $\alpha + \beta = 1$ ***3.*** $\alpha - \beta = \sqrt{5}$

4. $\alpha^2 + 1 = \sqrt{5}\alpha$ ***5.*** $\alpha = 2 - \beta^2$ ***6.*** $\alpha^2 + \beta^2 = 3$

1.1.2. *Lucas Numbers.*

Definition 1.3. *The nth Lucas number L_n is defined by*

$$L_n = L_{n-1} + L_{n-2}$$

with initial conditions

$$L_1 = 1, \quad L_2 = 3,$$

where $n \geq 3$. In this case, Lucas numbers are given by

$$1, 3, 4, 7, 11, 18, 29, 47, \ldots, L_n, \ldots$$

Lemma 1.2. *The ratio of two consecutive Lucas numbers approaches $\frac{1+\sqrt{5}}{2}$ as $n \to \infty$. That is,*

$$\lim_{n\to\infty} \frac{L_{n+1}}{L_n} = \frac{1+\sqrt{5}}{2}.$$

Theorem 1.2. *Let α and β be the solutions of the quadratic equation $x^2 - x - 1 = 0$; so $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$. Then, the relation that gives us the nth term of Lucas sequence is given by*

$$L_n = \alpha^n + \beta^n,$$

where $n \geq 0$.

1.2. **Fibonacci and Lucas Curves.**

Definition 1.4. *Let $I \subseteq \mathbb{R}$ be an open interval of $\mathbb{R}$. Then, Fibonacci curve is defined by*

$$f : I \to \mathbb{R}^2$$
$$\theta \mapsto f(\theta) = (x(\theta), y(\theta)),$$

where

$$x(\theta) = \frac{\alpha^\theta - \alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}} \tag{1.1}$$

and

$$y(\theta) = \frac{-\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}} \tag{1.2}$$

including $\alpha = \frac{1+\sqrt{5}}{2}$, [18].

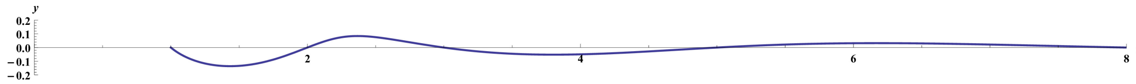


FIGURE 1. Fibonacci curve

In the interval $I=(2,6)$, the graph of Fibonacci curve can be seen in Figure 1. By taking derivative of the equations (1.1) and (1.2) with respect to θ, we obtain that

$$\frac{dx}{d\theta}=x'(\theta)=\frac{\alpha^{-\theta}\left[\alpha^{2\theta}s+s\cos(\theta\pi)+\pi\sin(\theta\pi)\right]}{\sqrt{5}} \tag{1.3}$$

and

$$\frac{dy}{d\theta}=y'(\theta)=\frac{\alpha^{-\theta}\left[-\pi\cos(\theta\pi)+s\sin(\theta\pi)\right]}{\sqrt{5}}, \tag{1.4}$$

where $\alpha=\frac{1+\sqrt{5}}{2}$ and $s=\log\left(\frac{1+\sqrt{5}}{2}\right)$. After taking derivative of the equations (1.3) and (1.4) with respect to θ, we obtain

$$\frac{d^2x}{d\theta^2}=x''(\theta)=\frac{\alpha^{-\theta}\left[(\pi^2-s^2)\cos(\theta\pi)+\alpha^{2\theta}s^2-2\pi s\sin(\theta\pi)\right]}{\sqrt{5}} \tag{1.5}$$

and

$$\frac{d^2y}{d\theta^2}=y''(\theta)=\frac{\alpha^{-\theta}\left[2\pi s\cos(\theta\pi)+(\pi^2-s^2)\sin(\theta\pi)\right]}{\sqrt{5}}, \tag{1.6}$$

[18], [1].

Definition 1.5. *Let $I\subseteq\mathbb{R}$ be an open interval of $\mathbb{R}$. Then, Lucas curve is defined by*

$$l: I\to\mathbb{R}^2$$

$$\theta\mapsto l(\theta)=(x(\theta),y(\theta)),$$

where

$$x(\theta)=\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi) \tag{1.7}$$

and

$$y(\theta)=\alpha^{-\theta}\sin(\theta\pi) \tag{1.8}$$

including $\alpha=\frac{1+\sqrt{5}}{2}$, [18].

In the interval $I=(1,5)$, the graph of Lucas curve can be seen in Figure 2.

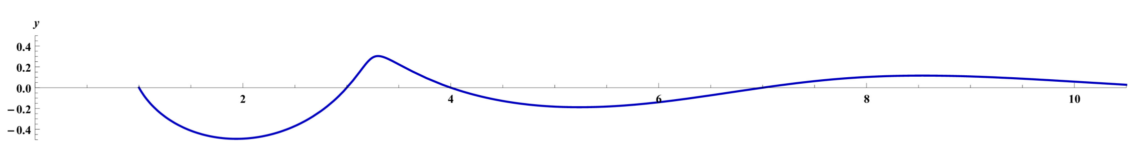


FIGURE 2. Lucas curve

By taking derivative of the equations (1.7) and (1.8) with respect to θ, we obtain that

$$\frac{dx}{d\theta} = x'(\theta) = \alpha^{-\theta}\big[s\alpha^{2\theta} - s\cos(\theta\pi) - \pi\sin(\theta\pi)\big] \tag{1.9}$$

and

$$\frac{dy}{d\theta} = y'(\theta) = \alpha^{-\theta}\big[\pi\cos(\theta\pi) - s\sin(\theta\pi)\big], \tag{1.10}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ and $s = \log\left(\frac{1+\sqrt{5}}{2}\right)$. After taking derivative of the equations (1.9) and (1.10) with respect to θ, we obtain

$$\frac{d^2x}{d\theta^2} = x''(\theta) = \alpha^{-\theta}\left[\alpha^{2\theta}s^2 + (s^2-\pi^2)\cos(\theta\pi) + 2\pi s\sin(\theta\pi)\right] \tag{1.11}$$

and

$$\frac{d^2y}{d\theta^2} = y''(\theta) = \alpha^{-\theta}\big[-2\pi s\cos(\theta\pi) + (s^2-\pi^2)\sin(\theta\pi)\big], \tag{1.12}$$

[18], [1].

2. The Special Curves of Fibonacci Curve

In this section, we will present the special plane curves of Fibonacci curve by using equations (1.3), (1.4), (1.5) and (1.6).

2.1. **The Contrapedal Curve of Fibonacci Curve.** The parametric equation of contrapedal curve [1] of Fibonacci curve $f(\theta)$ with respect to point $P = (p_1, p_2)$ on the plane is that

$$Cp_f(\theta) = (A(\theta), B(\theta)), \tag{2.13}$$

where

$$A(\theta) = p_1 + \frac{\alpha^{-\theta}\left(s\alpha^{2\theta} + s\cos(\pi\theta) + \pi\sin(\pi\theta)\right)\left(\sqrt{5}s\left(\alpha^{4\theta}-1\right) - 5sp_1\alpha^{3\theta} + \alpha^{\theta}v_\theta\right)}{5\left(s^2\left(\alpha^{4\theta}+1\right) + 2s\alpha^{2\theta}(\pi\sin(\pi\theta) + s\cos(\pi\theta)) + \pi^2\right)}$$

and

$$B(\theta) = p_2 - \frac{\alpha^{-\theta}(\pi\cos(\pi\theta) - s\sin(\pi\theta))\left(\sqrt{5}s\left(\alpha^{4\theta}-1\right) - 5sp_1\alpha^{3\theta} + \alpha^{\theta}v_\theta\right)}{5\left(s^2\left(\alpha^{4\theta}+1\right) + 2s\alpha^{2\theta}(\pi\sin(\pi\theta) + s\cos(\pi\theta)) + \pi^2\right)}$$

including

$$v_\theta = \Big((\sqrt{5}\pi\alpha^{\theta} - 5sp_2 - 5\pi p_1)\sin(\theta\pi) + 5(\pi p_2 - sp_1)\cos(\theta\pi)\Big).$$

In Figure 3, Fibonacci curve which is represented by blue curve and the contrapedal curves $Cp_f(\theta)$ of Fibonacci curve $f(\theta)$ with respect to points $(0,6)$, $(3,4)$ $(2,2)$, and $(-1,-2)$ is

[1] Let $\alpha(t) = (x(t), y(t))$ be a regular plane curve and P be a fixed point on $\mathbb{R}^2$. The locus of bases of perpendicular lines from $P = (p_1, p_2)$ to a variable normal line to α is ***contrapedal curve*** and the equation of contrapedal curve of α is that $Cp_\alpha(t) = (f(t), g(t))$ where $f(t) = p_1 + \frac{(x(t)-p_1)x'(t)+(y(t)-p_2)y'(t)}{x'(t)^2+y'(t)^2}x'(t)$ and $g(t) = p_2 + \frac{(x(t)-p_1)x'(t)+(y(t)-p_2)y'(t)}{x'(t)^2+y'(t)^2}y'(t)$, [7].

plotted, from top to down respectively. As seen in the figure, in the interval where Fibonacci curve is injective, whether the contrapedal curve of Fibonacci curve is injective or not depends on given point P.

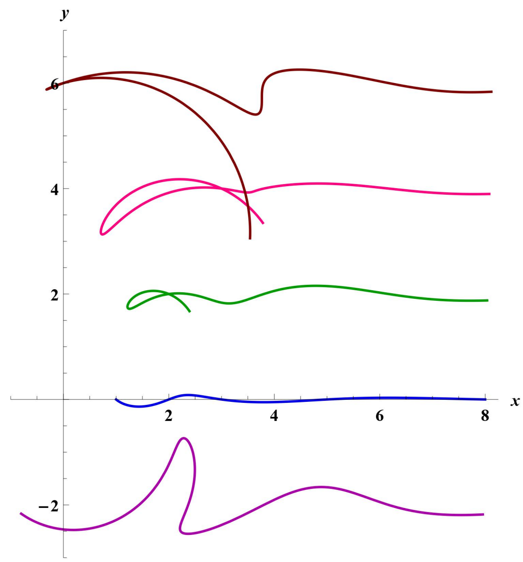


FIGURE 3. Fibonacci curve and its contrapedal curves

2.2. **The Radial Curve of Fibonacci Curve.** The parametric equation of radial curve[2] of Fibonacci curve $f(\theta)$ with respect to point $P=(p_1,p_2)$ on the plane is that

$$R_f(\theta)=\left(R_1(\theta),R_2(\theta)\right), \tag{2.14}$$

where

$$R_1(\theta)=p_1+\frac{\alpha^{-\theta}(\pi\cos(\theta\pi)-s\sin(\theta\pi))\left(s^2\left(\alpha^{4\theta}+1\right)+2s\alpha^{2\theta}z_\theta+\pi^2\right)}{\sqrt{5}\left(s\alpha^{2\theta}\left(\left(\pi^2-2s^2\right)\sin(\theta\pi)+3\pi s\cos(\theta\pi)\right)+\pi\left(s^2+\pi^2\right)\right)}$$

and

$$R_2(\theta)=p_2+\frac{\alpha^{-\theta}\left(s\alpha^{2\theta}+s\cos(\theta\pi)+\pi\sin(\theta\pi)\right)\left(s^2\left(\alpha^{4\theta}+1\right)+2s\alpha^{2\theta}z_\theta+\pi^2\right)}{\sqrt{5}\left(s\alpha^{2\theta}\left(\left(\pi^2-2s^2\right)\sin(\theta\pi)+3\pi s\cos(\theta\pi)\right)+\pi\left(s^2+\pi^2\right)\right)}$$

including

$$z_\theta=\pi\sin(\theta\pi)+s\cos(\theta\pi).$$

From the equation (2.14), we can see that point P plays a role in just the translation of the created shape. In Figure 4, Fibonacci curve which is represented by blue curve and, from

[2]Let $\alpha(t)=(x(t),y(t))$ be a regular plane curve on $\mathbb{R}^2$. Suppose that lines are drawn from a fixed point $P=(p_1,p_2)\in\mathbb{R}^2$ such that these lines are equal and parallel to the radii of curvature of $\alpha(t)$. The locus of the end points is ***radial curve*** and the equation of radial curve is that $R_\alpha(t)=(f(t),g(t))$ where $f(t)=p_1-\frac{y'(t)\left((x'(t))^2+(y'(t))^2\right)}{x'(t)y''(t)-x''(t)y'(t)}$ and $g(t)=p_2+\frac{x'(t)\left((x'(t))^2+(y'(t))^2\right)}{x'(t)y''(t)-x''(t)y'(t)}$, [11].

left to right respectively, the $R_f(\theta)$ radial curves with respect to $(3,1)$ and $(6,1)$ points are plotted by restricting $x-$axis to $(-1,8)$ interval and $y-$axis to $(-1,3)$ interval. The figure indicates that the radial curve of Fibonacci curve is not injective.

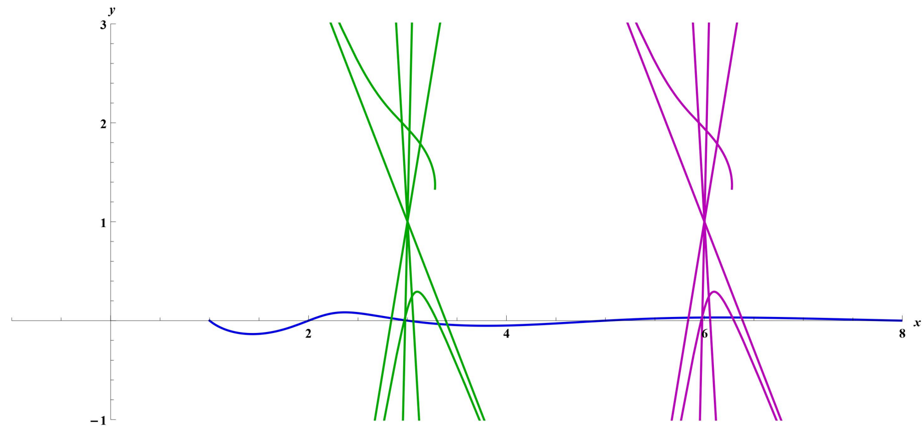


FIGURE 4. Fibonacci curve and its radial curves

2.3. **The Inverse Curve of Fibonacci Curve.** The parametric equation of inverse curve[3] of Fibonacci curve $f(\theta)$ with respect to point $R=(r_1,r_2)$ and value k is that

$$In_f(\theta) = (I_1(\theta), I_2(\theta)), \tag{2.15}$$

where

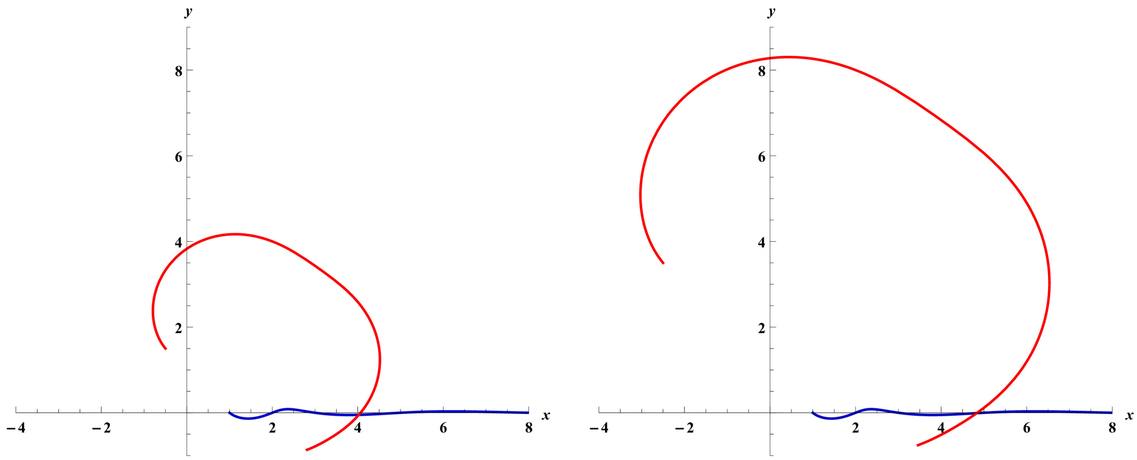
$$I_1(\theta) = r_1 + k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}$$

and

$$I_2(\theta) = r_2 - k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}.$$

The equation (2.15) demonstrates that if the point R is kept constant, the value $k>0$ has a role in changing the size of the shape. The more we increase the value k, the more the figure enlarges by preserving its basic form. In contrast, the more we decrease the value k, the more the size of the shape is dwindled by preserving its basic form. That is, the value k is the radial ratio. In Figure 5, Fibonacci curve which is represented by blue curve and its inverse curves $In_f(\theta)$ with $k=5$ and $k=9$ with respect to the point $(2,-1)$ are plotted.

[3]Let $\alpha(t)=(x(t),y(t))$ be a regular plane curve and $R=(r_1,r_2)$ be a fixed point on $\mathbb{R}^2$. Suppose that a line L is drawn through R by intersecting α at P, and let Q be a point on L so that $|RP|.|RQ|=k$, a constant. Then, P and Q are inverse points, and the locus of Q is an ***inverse*** of α with respect to R. k may be negative, in which case P and Q lie on opposite sides of R. The parametric equation of inverse curve of α is that $In_\alpha(t)=(f(t),g(t))$ where $f(t)=r_1+k\frac{x(t)-r_1}{(x(t)-r_1)^2+(y(t)-r_2)^2}$ and $g(t)=r_2+k\frac{y(t)-r_2}{(x(t)-r_1)^2+(y(t)-r_2)^2}$, [11].



(A) when $R = (2, -1)$ and $k = 5$ (B) when $R = (2, -1)$ and $k = 9$

FIGURE 5. Fibonacci curve and its inverse curves

Moreover, if one keeps the point R constant and gets the negative of the value k, then the shape is rotated around the point R at a rotation of 180°.

Firstly, we start to make R become the origin. So, $(I_1', I_2') = (I_1, I_2) - (r_1, r_2) = (I_1 - r_1, I_2 - r_2)$ then we get that

$$I_1' = k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2},$$
$$I_2' = -k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}.$$

We know that to rotate a point 180° counterclockwise about the origin, we need to multiply the $x-$ and $y-$coordinates by -1 i.e. $(x, y) \to (-x, -y)$. Therefore, we get that

$$I_1'' = -k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2},$$
$$I_2'' = k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}.$$

Finally, we make the point R center again. So, $(I_1''', I_2''') = (I_1'', I_2'') + (r_1, r_2) = (I_1' + r_1, I_2' + r_2)$ then we get that

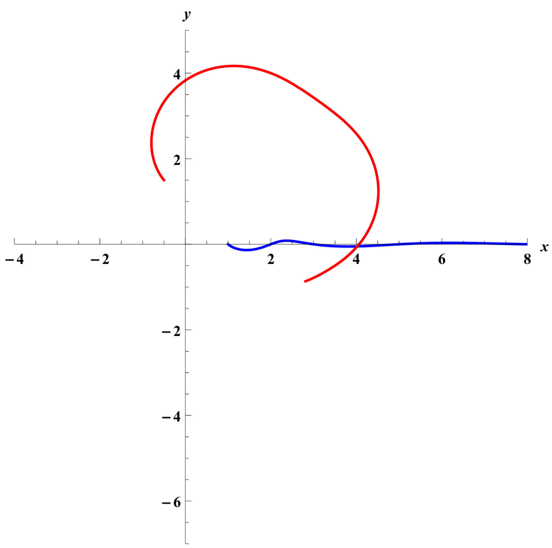
$$\begin{aligned} I_1''' &= r_1 - k\frac{\sqrt{5}\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}, \\ I_2''' &= r_2 + k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}. \end{aligned} \tag{2.16}$$

In addition, if we write $-k$ instead of k in the equation (2.15), then we obtain that

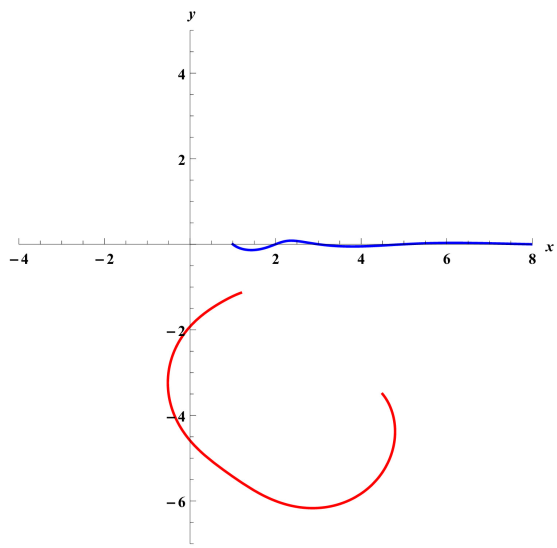
$$\begin{aligned} I_1(\theta) &= r_1 - k\frac{\sqrt{5}\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)}{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}, \\ I_2(\theta) &= r_2 + k\frac{\sqrt{5}\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)}{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}. \end{aligned} \tag{2.17}$$

Consequently, from the equations (2.16) and (2.17), we see that the statement is true.

In Figure 6, Fibonacci curve which is represented by blue curve and its inverse curves $In_f(\theta)$ with $k=5$ and $k=-5$ with respect to the point $(2,-1)$ are plotted.



(A) when $R=(2,-1)$ and $k=5$



(B) when $R=(2,-1)$ and $k=-5$

FIGURE 6. Fibonacci curve and its inverse curve with negative value k

2.4. **The Conchoid Curve of Fibonacci Curve.** The parametric equation of conchoid curve[4] of Fibonacci curve $f(\theta)$ with respect to point $R=(r_1,r_2)$ and value k is that

$$C_f(\theta) = \left(c_1(\theta), c_2(\theta)\right), \tag{2.18}$$

where

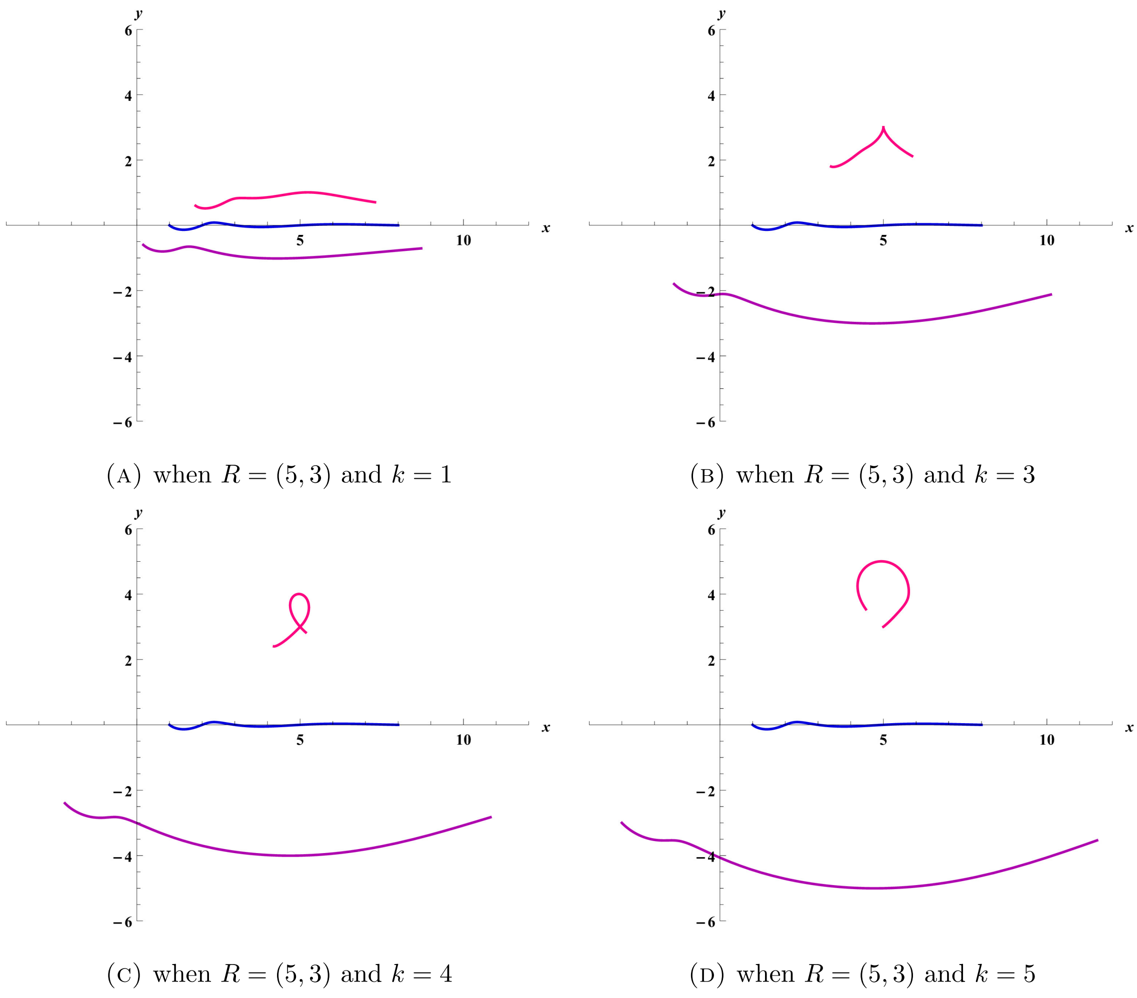
$$c_1(\theta) = \frac{\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}} \pm k\frac{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)}{\sqrt{\left(\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)+\sqrt{5}r_2\right)^2}}$$

[4]Let $\alpha(t)=(x(t),y(t))$ be a regular plane curve and $R=(r_1,r_2)$ be fixed point on $\mathbb{R}^2$. Suppose that a line L is drawn through R by intersecting α at Q. The locus of points P_1 and P_2 on L such that $|P_1Q|=|QP_2|=k$, a constant is the ***conchoid curve*** of α with respect to $R=(r_1,r_2)$. The parametric equation of conchoid curve of α is $C_\alpha(t)=(f(t),g(t))$ where $f(t)=x(t)\pm k\frac{x(t)-r_1}{\sqrt{(x(t)-r_1)^2+(y(t)-r_2)^2}}$ and $g(t)=y(t)\pm k\frac{y(t)-r_2}{\sqrt{(x(t)-r_1)^2+(y(t)-r_2)^2}}$, [11].

and

$$c_2(\theta) = \frac{-\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}} \mp k \frac{\left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)}{\sqrt{\left(\alpha^{\theta} - \alpha^{-\theta}\cos(\theta\pi) - \sqrt{5}r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) + \sqrt{5}r_2\right)^2}}.$$

In Figure 7, Fibonacci curve and its conchoid curves $C_f(\theta)$ with respect to different values k and the point $(5,3)$ are plotted. The blue, purple and pink curves in the figure, respectively, represent Fibonacci curve, the locus of P_1 and the locus of P_2. As it is seen in this figure, if we fix the point R, whether its conchoid curve is injective or not depends on the value k in the interval which Fibonacci curve is injective.



(A) when $R=(5,3)$ and $k=1$ (B) when $R=(5,3)$ and $k=3$

(C) when $R=(5,3)$ and $k=4$ (D) when $R=(5,3)$ and $k=5$

Figure 7. Fibonacci curve and its conchoid curves

2.5. **The Strophoid Curve of Fibonacci Curve.** The parametric equation of strophoid curve[5] of Fibonacci curve $f(\theta)$ with respect to points $R=(r_1,r_2)$ and $A=(a_1,a_2)$ is that

$$S_f(\theta)=\left(s_1(\theta),s_2(\theta)\right), \tag{2.19}$$

where

$$s_1(\theta)=\frac{\alpha^\theta-\alpha^{-\theta}\cos(\theta\pi)}{\sqrt{5}}\pm\frac{\left(\alpha^\theta-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)\omega_\theta}{\sqrt{\left(\alpha^\theta-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(-\alpha^{-\theta}\sin(\theta\pi)-\sqrt{5}r_2\right)^2}}$$

and

$$s_2(\theta)=-\frac{\alpha^{-\theta}\sin(\theta\pi)}{\sqrt{5}}\pm\frac{\left(-\alpha^{-\theta}\sin(\theta\pi)-\sqrt{5}r_2\right)\omega_\theta}{\sqrt{\left(\alpha^\theta-\alpha^{-\theta}\cos(\theta\pi)-\sqrt{5}r_1\right)^2+\left(-\alpha^{-\theta}\sin(\theta\pi)-\sqrt{5}r_2\right)^2}}$$

including

$$\omega_\theta=\frac{1}{\sqrt{5}}\sqrt{\left(\sqrt{5}a_1-\alpha^\theta+\alpha^{-\theta}\cos(\theta\pi)\right)^2+\left(\sqrt{5}a_2+\alpha^{-\theta}\sin(\theta\pi)\right)^2}.$$

In Figure 8, Fibonacci curve and its strophoid curves $S_f(\theta)$ with respect to $R=(4,1)$ and $A=(-1,-1)$ are plotted. The blue, purple and pink curves, respectively, in the figure represent Fibonacci curve, the locus of P_1 and the locus of P_2.

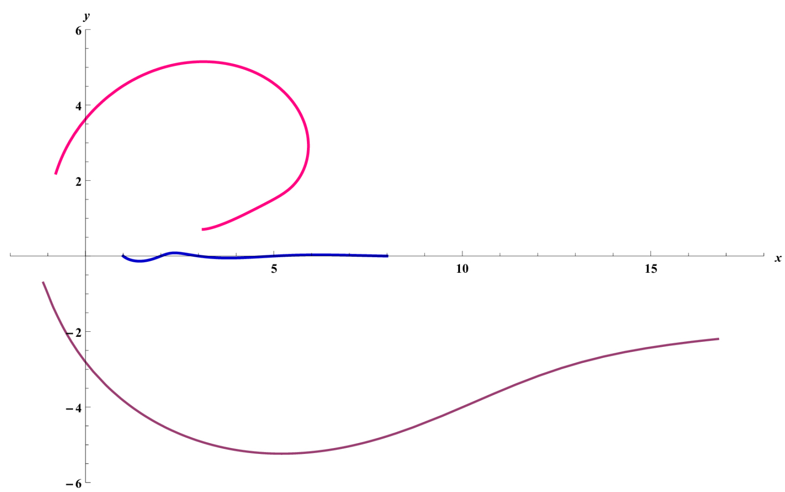


FIGURE 8. Fibonacci curve and its strophoid curve when $R=(4,1)$ and $A=(-1,-1)$

[5]Let $\alpha(t)=(x(t),y(t))$ be a regular plane curve and $R=(r_1,r_2)$ and $A=(a_1,a_2)$ be two fixed points on $\mathbb{R}^2$. Here, the point R is called the pole point. The locus of points P_1 and P_2 on a line L through R and intersecting α at a point Q such that $|P_2Q|=|QP_1|=|QA|$ is the ***strophoid curve*** of α with respect to R and A. The parametric equation of strophoid curve of α is $S_\alpha(t)=(f(t),g(t))$ where $f(t)=x(t)\pm\frac{1}{\sqrt{1+m^2}}\left[(a_1-x(t))^2+(a_2-y(t))^2\right]^{1/2}$ and $g(t)=y(t)\pm\frac{m}{\sqrt{1+m^2}}\left[(a_1-x(t))^2+(a_2-y(t))^2\right]^{1/2}$ included $m=\frac{y(t)-r_2}{x(t)-r_1}$, [11].

3. The Special Curves of Lucas Curves

In this section, we will find the equations of special plane curves of Lucas curve by using equations (1.9), (1.10), (1.11) and (1.12) and give their graphs.

3.1. **The Contrapedal Curve of Lucas Curve.**

The parametric equation of contrapedal curve of Lucas curve $l(\theta)$ with respect to point $P=(p_1,p_2)$ on the plane is that

$$Cp_l(\theta)=(A(\theta),B(\theta))\,, \tag{3.20}$$

where

$$A(\theta)=p_1-\frac{\alpha^{-\theta}\left(s\alpha^{2\theta}-s\cos(\theta\pi)-\pi\sin(\theta\pi)\right)\left(s\left(1-\alpha^{4\theta}+p_1\alpha^{3\theta}\right)+\alpha^{\theta}v_\theta\right)}{s^2\left(\alpha^{4\theta}+1\right)+\pi^2-2s\alpha^{2\theta}(\pi\sin(\theta\pi)+s\cos(\theta\pi))}$$

and

$$B(\theta)=p_2-\frac{\alpha^{-\theta}(\pi\cos(\theta\pi)-s\sin(\theta\pi))\left(s\left(1-\alpha^{4\theta}+p_1\alpha^{3\theta}\right)+\alpha^{\theta}v_\theta\right)}{s^2\left(\alpha^{4\theta}+1\right)+\pi^2-2s\alpha^{2\theta}(\pi\sin(\theta\pi)+s\cos(\theta\pi))}$$

including

$$v_\theta=\pi\alpha^{\theta}\sin(\theta\pi)+(\pi p_2-sp_1)\cos(\theta\pi)-(\pi p_1+sp_2)\sin(\theta\pi).$$

In Figure 9, Lucas curve which is represented by blue curve and its contrapedal curves $Cp_l(\theta)$ with respect to $(4,3)$ and $(1,-3)$ are plotted, from top to down respectively. As it can be seen in the figure, whether the contrapedal curve of Lucas curve is injective depends on point P in the interval where Lucas curve is injective.

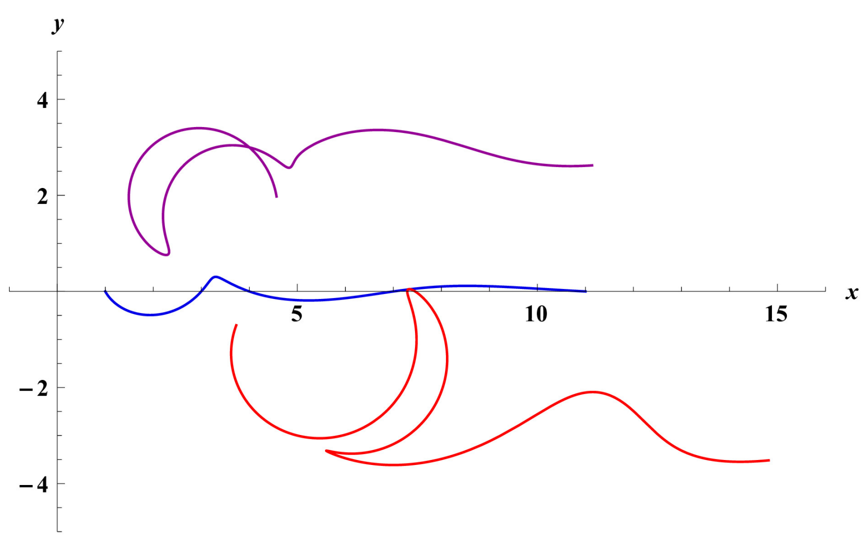


Figure 9. Lucas curve and its contrapedal curves

3.2. **The Radial Curve of Lucas Curve.** The parametric equation of radial curve of Lucas curve $l(\theta)$ with respect to point $P = (p_1, p_2)$ is that

$$R_l(\theta) = (R_1(\theta), R_2(\theta)), \tag{3.21}$$

where

$$R_1(\theta) = p_1 - \frac{\alpha^{-\theta}(\pi\cos(\theta\pi) - s\sin(\theta\pi))\left(s^2\left(\alpha^{4\theta}+1\right) - 2s\alpha^{2\theta}z_\theta + \pi^2\right)}{\pi\left(s^2+\pi^2\right) - s\alpha^{2\theta}\left(\left(\pi^2-2s^2\right)\sin(\theta\pi) + 3\pi s\cos(\theta\pi)\right)}$$

and

$$R_2(\theta) = p_2 + \frac{\alpha^{-\theta}\left(s\alpha^{2\theta} - s\cos(\theta\pi) - \pi\sin(\theta\pi)\right)\left(s^2\left(\alpha^{4\theta}+1\right) - 2s\alpha^{2\theta}z_\theta + \pi^2\right)}{\pi(s^2+\pi^2) - s\alpha^{2\theta}\left(3\pi s\cos(\theta\pi) + \left(\pi^2-2s^2\right)\sin(\theta\pi)\right)}$$

including

$$z_\theta = \pi\sin(\theta\pi) + s\cos(\theta\pi).$$

It can be understood from the equation (3.21) that point P plays a role in the translation of the shape created by radial curve. In Figure 10, Lucas curve which is represented by blue curve and its radial curves $R_l(\theta)$, from left to right respectively, at $(-1, 2)$ and $(6, 2)$ points have been plotted by restricting $x-$axis to $(-5, 11)$ interval and $y-$axis to $(-10, 10)$ interval. The figure indicates that the radial curve of Lucas curve is not injective.

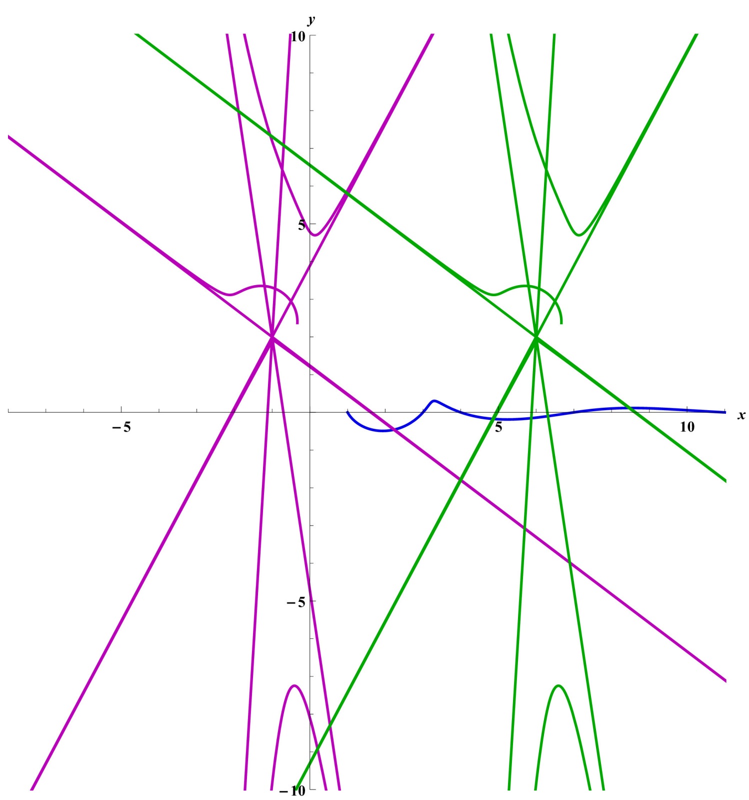


FIGURE 10. Lucas curve and its radial curves

3.3. **The Inverse Curve of Lucas Curve.** The parametric equation of inverse curve of Lucas curve $l(\theta)$ with respect to point $R = (r_1, r_2)$ and value k is that

$$In_l(\theta) = (I_1(\theta), I_2(\theta)), \tag{3.22}$$

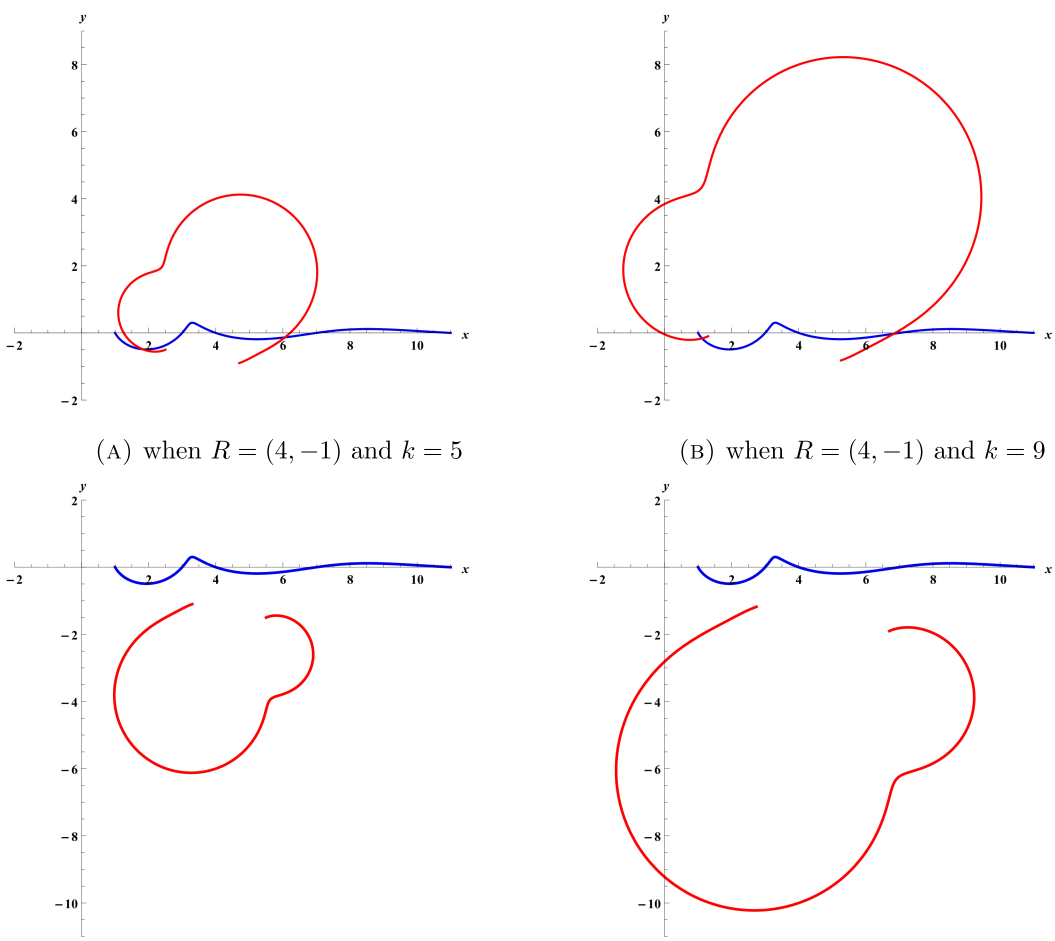
where

$$I_1(\theta) = r_1 + k\frac{\alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) - r_1}{\left(\alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) - r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) - r_2\right)^2}$$

and

$$I_2(\theta) = r_2 + k\frac{\alpha^{-\theta}\sin(\theta\pi) - r_2}{\left(\alpha^\theta + \alpha^{-\theta}\cos(\theta\pi) - r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) - r_2\right)^2}.$$

Results obtained by investigating the special cases of value k for the inverse curve of Fibonacci curve are also valid for the inverse curve of Lucas curve. In Figure 11, Lucas curve which is represented by blue curve and its inverse curves $In_l(\theta)$ for $k = -5$, $k = 5$, $k = -9$, and $k = 9$ with respect to the point $(4, -1)$ are plotted.



(A) when $R = (4, -1)$ and $k = 5$

(B) when $R = (4, -1)$ and $k = 9$

(C) when $R = (4, -1)$ and $k = -5$

(D) when $R = (4, -1)$ and $k = -9$

FIGURE 11. Lucas curve and its inverse curves

3.4. **The Conchoid Curve of Lucas Curve.** The parametric equation of conchoid curve of Lucas curve $l(\theta)$ with respect to point $R = (r_1, r_2)$ and value k is that

$$C_l(\theta) = (c_1(\theta), c_2(\theta)), \tag{3.23}$$

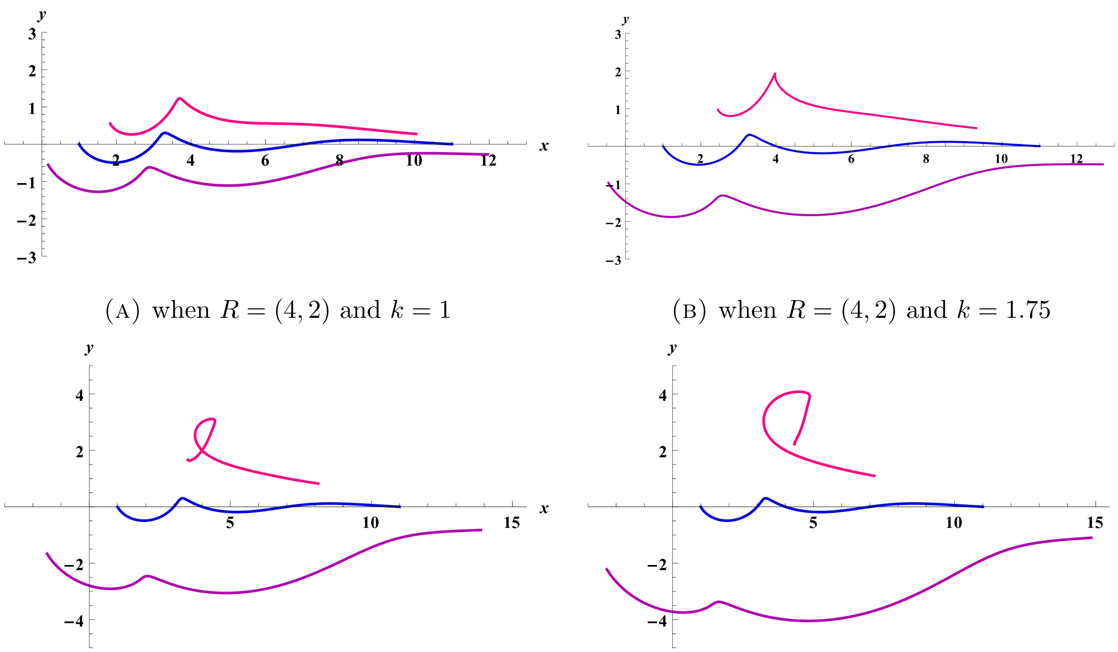
where

$$c_1(\theta) = \alpha^{\theta} + \alpha^{-\theta}\cos(\theta\pi) \pm k \frac{\alpha^{\theta} + \alpha^{-\theta}\cos(\theta\pi) - r_1}{\sqrt{\left(\alpha^{\theta} + \alpha^{-\theta}\cos(\theta\pi) - r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) - r_2\right)^2}}$$

and

$$c_2(\theta) = \alpha^{-\theta}\sin(\theta\pi) \pm k \frac{\alpha^{-\theta}\sin(\theta\pi) - r_2}{\sqrt{\left(\alpha^{\theta} + \alpha^{-\theta}\cos(\theta\pi) - r_1\right)^2 + \left(\alpha^{-\theta}\sin(\theta\pi) - r_2\right)^2}}.$$

In Figure 12, Lucas curve and its conchoid curves $C_l(\theta)$ with respect to different values k and the point $(4, 2)$ are plotted. The blue, purple and pink curves represent Lucas curve, the locus of P_1 and the locus of P_2, respectively, in the figure. As it is seen in this figure, whether its conchoid curve is injective depends on value k in the interval where Lucas curve is injective.



(A) when $R = (4, 2)$ and $k = 1$ (B) when $R = (4, 2)$ and $k = 1.75$

(C) when $R = (4, 2)$ and $k = 3$ (D) when $R = (4, 2)$ and $k = 4$

FIGURE 12. Lucas curve and its conchoid curves

3.5. **The Strophoid Curve of Lucas Curve.** The parametric equation of strophoid curve of Lucas curve $l(\theta)$ with respect to points $R=(r_1,r_2)$ and $A=(a_1,a_2)$ is that

$$S_l(\theta)=\left(s_1(\theta),s_2(\theta)\right), \tag{3.24}$$

where

$$s_1(\theta)=\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)\pm\frac{\left(\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)\omega_\theta}{\sqrt{\left(\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)^2}}$$

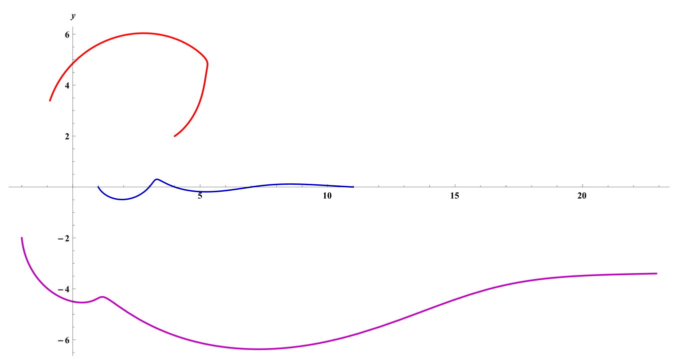
and

$$s_2(\theta)=\alpha^{-\theta}\sin(\theta\pi)\pm\frac{\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)\omega_\theta}{\sqrt{\left(\alpha^{\theta}+\alpha^{-\theta}\cos(\theta\pi)-r_1\right)^2+\left(\alpha^{-\theta}\sin(\theta\pi)-r_2\right)^2}}$$

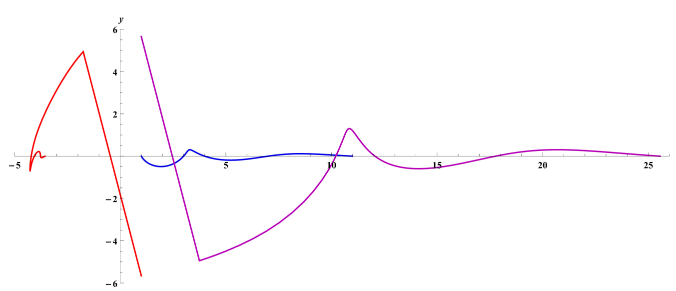
including

$$\omega_\theta=\sqrt{\left(a_1-\alpha^{\theta}-\alpha^{-\theta}\cos(\theta\pi)\right)^2+\left(a_2-\alpha^{-\theta}\sin(\theta\pi)\right)^2}.$$

In Figure 13, Lucas curve and its strophoid curves $S_l(\theta)$ with respect to different points R and A are plotted. The blue, purple and red curves represent Lucas curve, the locus of P_1 and the locus of P_2, respectively, in the figure. As it is seen in this figure, whether its strophoid curve has a critical point depends on A and R in the interval where Lucas curve has not any critical point.



(A) when $R=(4,2)$ and $A=(-1,3)$



(B) when $R=(1,0)$ and $A=(-3,-4)$

FIGURE 13. Lucas curve and its strophoid curves

4. Conclusion

In this study, firstly the notions of contrapedal, radial, inverse, conchoid and strophoid curves of the Fibonacci and Lucas curves have been investigated. Afterwards, their graphs which have been plotted by using Mathematica are examined in the interval $I = (2, 6)$ for Fibonacci curve and in the interval $I = (1, 5)$ for Lucas curve.

We have obtained some results from the notions and figures which is acquired.

- As illustrated in Figure 3 and Figure 9, if their contrapedal curves are injective or not depends on given point P in the intervals where Fibonacci and Lucas curves are injective.
- From equations (2.14) and (3.21), it is clear that the point P has a role in the translation of the figure which is created. Figure 4 and Figure 10 illustrate that their radial curves are not injective.
- The equations (2.15) and (3.22) reveals that if one fixes the point R, the value $k > 0$ has a role in changing the size of inverse curves which belongs to Fibonacci and Lucas curves. As the value k increases, the size of the shape enlarges by preserving the main form. Conversely, as the value k decreases, the size becomes smaller by preserving the main form. Moreover, if one keeps the point R constant and gets the negative of the value k, then the shape is rotated around the point R at a rotation of $180°$.
- From Figure 7 and Figure 12, it can be seen that in the interval where Fibonacci and Lucas curves are injective, if one fixes the point R, the value k is an important factor in the injectivity of their conchoid curves.
- It can be observed from Figure 13 that in the interval where Lucas curve has not any critical point whether its strophoid curve has at least one critical point or not depends on the given points R and A.

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Department of Mathematics, Faculty of Arts and Sciences,, Marmara University, Göztepe Campus,, Kadiköy, Istanbul, Turkey

Department of Mathematics, Faculty of Arts and Sciences,, Yildiz Technical University, Davutpasa Campus,, Esenler, Istanbul, Turkey

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SLANT SUBMANIFOLDS OF ALMOST POLY-NORDEN RIEMANNIAN MANIFOLDS

VİLDAN AYHAN AND SELCEN YÜKSEL PERKTAŞ *

ABSTRACT. In the present paper, we study slant submanifolds of an almost poly-Norden Riemannian manifold and give examples. Also we investigate conditions for the normality of the induced structure provided by the almost poly-Norden structure of the ambient manifold.

Keywords: Almost Poly-Norden Manifold, Slant Submanifold, Induced Structure, Normality.

2010 Mathematics Subject Classification: 53B20, 53B25, 53C15.

1. Introduction

In Riemannian (as well as semi-Riemannian) manifolds, different geometric structures such as almost complex structures, almost product structures, almost contact structures, almost paracontact structures etc. allow rich differential and geometric features to emerge while investigating geometry of submanifolds.

A solution of the equation $x^2 - x - 1 = 0$, the number $\phi = \frac{1+\sqrt{5}}{2} = 1.618...$, is known as the Golden ratio and it is also considered to be the order relation that gives the best harmony and proportions in art and architecture since ancient times. As a generalization of the Golden ratio, Spinadel introduced metallic means family or metallic proportions in [24].

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* Corresponding author

Vildan Ayhan; tayhan@gmail.com; https://orcid.org/0000-0002-4499-8509

Selcen Yüksel Perktaş; sperktas@adiyaman.edu.tr; https://orcid.org/0000-0002-8848-0621

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Members of metallic means family, namely (p,q) metallic numbers, are the positive solutions of the equation $x^2-px-q=0$ and denoted by

$$\sigma_{p,q}=\frac{p+\sqrt{p^2+4q}}{2}, \tag{1.1}$$

where p and q are positive integer numbers. The well-known members of the metallic means family are the Golden mean, the Silver mean, the Bronze mean; the Copper mean etc. These means constitute a bridge between mathematics, physics and art.

In recent years, inspired by the Golden mean and the metallic mean, the Golden structure and the metallic structure on Riemannian manifolds were introduced in [10] and [18], respectively. Golden Riemannian manifolds, considered an important subclass of metallic Riemannian manifolds and their submanifolds, have extensively been studied by many geometers (see [13, 11, 15, 16, 17]).

In 2006, by a different approach, Kalia [21] introduced a new Bronze mean and studied Bronze Fibonacci and Lucas numbers. The author revealed the relationship between the convergents of continued fractions of the power of Bronze means and the Bronze Fibonacci and Lucas numbers. Note that, unlike the Bronze mean contained by the metallic means family defined in [24], that new Bronze mean given by Kalia [21] can not be expressed with $\sigma_{p,q},$ for positive integers p and q.

Considering the differentiable structure that may occur on a semi-Riemannian manifold depending on the Bronze mean given by [21] and the study on a Riemannian manifold with the Golden structure [10], a new type of manifold equipped with the Bronze structure was introduced by Şahin [26] and the author named it an almost poly-Norden manifold. After then, Perktaş [28] studied submanifolds of almost poly-Norden Riemannian manifolds and examined fundamental geometric features of such submanifolds with the induced structure provided by the almost poly-Norden structure of the ambient manifold.

Slant submanifolds were first defined by Chen (see, [8], [9]) in complex manifolds. Later, submanifolds of this type have begun to be widely studied on different manifolds. For slant submanifolds in almost contact metric manifolds, in Sasakian manifolds, in para-Hermitian manifolds and in almost product manifolds we refer to [2, 3, 4, 7, 6, 22, 25]. Invariant, anti-invariant, semi-invariant, slant, semi-slant, hemi-slant and bi-slant submanifolds of a metallic Riemannian manifold were studied in [5, 19, 20]. Some types of lightlike submanifolds of a Golden semi-Riemannian manifold and metallic semi-Riemannian manifold were introduced in [1, 12, 14, 23, 27].

In the present paper, we study slant submanifolds of an almost poly-Norden Riemannian manifold and give examples. Also we investigate conditions for the normality of the induced structure provided by the almost poly-Norden structure of the ambient manifold.

2. Preliminaries

The Bronze mean introduced by Kalia [21] is the positive solution of $x^2-mx+1=0$,which is defined by

$$B_m = \frac{m+\sqrt{m^2-4}}{2}. \tag{2.2}$$

For detailed reading on the relations between Bronze Fibonacci numbers, Bronze Lucas numbers and family of sequences given by the recurrences, we refer to [21].

Inspired by the Bronze mean given by (2.2), Şahin [26], defined a structure on a differentiable manifold, precisely the Bronze structure. A differentiable manifold $\hat{M}$ with a $(1,1)$-tensor field $\hat{\Phi}$ satisfying

$$\hat{\Phi}^2 = m\hat{\Phi} - I, \tag{2.3}$$

where I is the identity operator on the set of cross sections of tangent bundle $T\hat{M}$ denoted by $\Gamma(T\hat{M})$, is called an almost poly-Norden manifold equipped with a poly-Norden structure $\hat{\Phi}$. Also, an almost poly-Norden manifold $(\hat{M},\hat{\Phi})$ having a semi-Riemannian metric $\hat{g}$ which is $\hat{\Phi}$-compatible, i.e.,

$$g(\hat{\Phi}X,\hat{\Phi}Y) = mg(\hat{\Phi}X,Y) - g(X,Y), \tag{2.4}$$

equivalent to

$$g(\hat{\Phi}X,Y) = g(X,\hat{\Phi}Y), \tag{2.5}$$

for any $X,Y \in \Gamma(T\hat{M})$, is called an almost poly-Norden semi-Riemannian manifold [26]. Every complex structure $\hat{F}$ allows to reduce two poly-Norden structures to a semi-Riemannian manifold given by [26]:

$$\hat{\Phi}_1 = \frac{m}{2}I + \frac{\sqrt{4-m^2}}{2}\hat{F}, \quad \hat{\Phi}_2 = \frac{m}{2}I - \frac{\sqrt{4-m^2}}{2}\hat{F}, \quad -2<m<2.$$

Conversely, every poly-Norden structure $\hat{\Phi}$ give rise to define two almost complex structures in the followings [26]:

$$\hat{F}_1 = -\frac{m}{\sqrt{4-m^2}}I + \frac{2}{\sqrt{4-m^2}}\hat{\Phi}, \quad \hat{F}_2 = \frac{m}{\sqrt{4-m^2}}I - \frac{2}{\sqrt{4-m^2}}\hat{\Phi}, \quad -2<m<2.$$

A poly-Norden semi-Riemannian manifold is an almost poly-Norden semi-Riemannian manifold with a parallel poly-Norden structure $\hat{\Phi}$ with respect to Levi-Civita connection $\hat{\nabla}$ on the manifold. The integrability of $\hat{\Phi}$ is defined by vanishing of the its Nijenhuis tensor field

$N_{\hat{\Phi}}(X,Y) := [\hat{\Phi}X, \hat{\Phi}Y] - \hat{\Phi}[\hat{\Phi}X, Y] - \hat{\Phi}[X, \hat{\Phi}Y] + \hat{\Phi}^2[X,Y]$, for any $X, Y \in \Gamma(T\hat{M})$. Note that $N_{\hat{\Phi}} = 0$ is equivalent to $\hat{\nabla}\hat{\Phi} = 0$, where $\hat{\nabla}$ is the Levi-Civita connection on $\hat{M}$. It was shown that in case of m is being zero every Norden manifold becomes an almost poly-Norden manifold [26].

Throughout the paper we will consider $m \neq 0$.

3. Submanifolds of almost poly-Norden Riemannian manifolds

Let $(\hat{M}, \hat{\Phi}, g)$ be an $(n+k)$-dimensional almost poly-Norden Riemannian manifold and M be an n-dimensional isometrically immersed submanifold of $\hat{M}$. For any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$, we put

$$\hat{\Phi}X = fX + wX, \tag{3.6}$$

$$\hat{\Phi}U = BU + CU, \tag{3.7}$$

where fX (resp., wX) is the tangential (resp., normal) part of $\hat{\Phi}X$ and BU (resp., CU) is the tangential (resp., normal) part of $\hat{\Phi}U$.

From (2.5) and (3.7) one can easily see that

$$g(fX,Y) = g(X,fY), \quad \forall X,Y \in \Gamma(TM), \tag{3.8}$$

$$g(CU,V) = g(U,CV), \quad \forall U,V \in \Gamma(TM^{\perp}). \tag{3.9}$$

Also, the maps w and B are related by $g(wX,U) = g(X,BU)$, for any $X \in \Gamma(TM)$ and $U \in \Gamma(TM^{\perp})$.

Denoting by $\hat{\nabla}$ and ∇, the Levi-Civita connections on M and $\hat{M}$, respectively, then Gauss and Weingarten formulas are given as follows:

$$\hat{\nabla}_X Y = \nabla_X Y + \sum_{\beta=1}^{k} h_\beta(X,Y)N_\beta, \tag{3.10}$$

$$\hat{\nabla}_X N_\beta = -A_{N_\beta} X + \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)N_\gamma, \tag{3.11}$$

for any $X, Y \in \Gamma(TM)$ and an orthonormal basis $\{N_1, ..., N_k\}$ of $TM^{\perp}$, where $\beta, \gamma \in \{1, ..., k\}$. Here, $h(X,Y) = \sum\limits_{\beta=1}^{k} h_\beta(X,Y)N_\beta$ and A_{N_β} is the shape operator in the direction of N_β defined by $g(A_{N_\beta} X, Y) = h_\beta(X,Y)$. Also, $\sigma_{\beta\gamma}$ $(1 \leq \beta, \gamma \leq k)$ denotes the 1-forms on the submanifold M which satisfy $\hat{\nabla}^{\perp}_X N_\beta = \sum\limits_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)N_\gamma$. Note that by taking the covariant derivative of $g(N_\beta, N_\gamma) = \delta_{\beta\gamma}$ on M, one gets $\sigma_{\beta\gamma} = -\sigma_{\gamma\beta}$.

For any $X \in \Gamma(TM)$, $\hat{\Phi}X$ and $\hat{\Phi}N_\beta$ $(1 \leq \beta \leq k)$ can be written respectively in the following forms:

$$\hat{\Phi}X = fX + \sum_{\beta=1}^{k} \upsilon_\beta(X)\, N_\beta, \tag{3.12}$$

$$\hat{\Phi}N_\beta = \zeta_\beta + \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\, N_\gamma, \tag{3.13}$$

where f is a tensor field of type $(1,1)$ on M which transforms tangent vector field X on M to the tangential component of $\hat{\Phi}X$, υ_β are 1-forms and $\theta_{\beta\gamma}$ are differentiable real valued functions on M providing a $k \times k$ matrix denoted by $(\theta_{\beta\gamma})_{k\times k}$.

Since $g(\hat{\Phi}X, N_\beta) = g(X, \hat{\Phi}N_\beta)$ and $g(\hat{\Phi}N_\beta, N_\gamma) = g(N_\beta, \hat{\Phi}N_\gamma)$, by using (2.4) and (3.8) we have

Lemma 3.1. [28] *In a submanifold M of an almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$, we have*

$$\upsilon_\beta(X) = g(\hat{\Phi}X, N_\beta) = g(X, \zeta_\beta), \tag{3.14}$$

$$g(fX, fY) = mg(X, fY) - g(X,Y) + \sum_{\beta,\gamma=1}^{k} \upsilon_\beta(X)\upsilon_\gamma(Y), \tag{3.15}$$

$$\theta_{\beta\gamma} = \theta_{\gamma\beta}, \tag{3.16}$$

for any $X, Y \in \Gamma(TM)$ and $1 \leq \beta, \gamma \leq k$.

Proposition 3.1. [28] *Let M be an n-dimensional isometrically immersed submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. Then the structure $(f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ on M induced by the almost-poly Norden structure of $\hat{M}$ satisfies*

$$(\nabla_X f)\, Y = \sum_{\beta=1}^{k} \left\{ g(wY, N_\beta) A_{N_\beta} X + h_\beta(X,Y) BN_\beta \right\}, \tag{3.17}$$

$$
\begin{aligned}
f^2X &= mfX - X - \sum_{\beta=1}^{k} \upsilon_\beta(X)\zeta_\beta, && (3.18)\\
\upsilon_\beta(fX) &= m\upsilon_\beta(X) - \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\upsilon_\gamma(X), && (3.19)\\
\upsilon_\gamma(\zeta_\beta) &= m\theta_{\beta\gamma} - \delta_{\beta\gamma} - \sum_{\lambda=1}^{k} \theta_{\beta\lambda}\theta_{\lambda\gamma}, && (3.20)\\
f\,\zeta_\beta &= m\zeta_\beta - \sum_{\gamma=1}^{k} \theta_{\beta\gamma}\zeta_\gamma, && (3.21)
\end{aligned}
$$

for any $X \in \Gamma(TM)$. Moreover, in case of $\hat{M}$ is being a poly-Norden semi-Riemannian manifold, we have

$$f\,A_{N_\beta}X + \nabla_X \zeta_\beta - \sum_{\gamma=1}^{k} \theta_{\beta\gamma} A_{N_\gamma} X - \sum_{\gamma=1}^{k} \sigma_{\beta\gamma}(X)\zeta_\gamma = 0. \tag{3.22}$$

4. Slant Submanifolds

Let M be a submanifold of an almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. By using the Cauchy-Schwartz inequality, namely,

$$g(\hat{\Phi}X, fX) \leq \left\|\hat{\Phi}X\right\| \|fX\|, \quad \forall X \in \Gamma(TM),$$

we can state that there exists a function $\theta : T_xM \to \left[0, \frac{\pi}{2}\right]$ satisfying

$$\left|g(\hat{\Phi}X, fX)\right| = \cos\theta(X) \left\|\hat{\Phi}X\right\| \|fX\|,$$

for any $X \in \Gamma(TM)$. Here $\theta(X)$ is called the Wirtinger angle of X.

Now we define the slant submanifolds of an almost poly-Norden Riemannian manifold similar to the definition given in [8]:

Definition 4.1. *Let M be a submanifold of an almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. If for any $X \in \Gamma(TM)$ the angle $\theta(X)$ between $\hat{\Phi}X$ and T_xM does not depend on $X_x \in T_xM$, then M is called a slant submanifold of $(\hat{M}, \hat{\Phi}, g)$.*

In this case, θ is called the slant angle of M. Furthermore, we have

$$\cos\theta = \frac{g(\hat{\Phi}X, fX)}{\left\|\hat{\Phi}X\right\| \|fX\|} = \frac{\|fX\|}{\left\|\hat{\Phi}X\right\|}, \tag{4.23}$$

for any $X \in \Gamma(TM)$ and $\hat{\Phi}X \neq 0$. The invariant and anti-invariant submanifolds of an almost poly-Norden Riemannian manifold are slant submanifolds with the slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively.

Proposition 4.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. If M is a slant submanifold with the slant angle θ, then we have*

$$g(fX, fY) = \cos^2\theta\{mg(\hat{\Phi}X, Y) - g(X, Y)\}, \tag{4.24}$$

$$g(wX, wY) = \sin^2\theta\{mg(\hat{\Phi}X, Y) - g(X, Y)\}, \tag{4.25}$$

for any $X, Y \in \Gamma(TM)$.

Proof. Since M is a slant submanifold with the slant angle θ, then by putting $X+Y$ instead of X in (4.23) we get

$$\cos^2\theta g(\hat{\Phi}X,\hat{\Phi}Y)=g(fX,fY). \tag{4.26}$$

From (2.4) and the last equation we obtain (4.24).

On the other hand, using (3.6) we write

$$g(\hat{\Phi}X,\hat{\Phi}Y)=g(fX,fY)+g(wX,wY),$$

which implies

$$g(wX,wY)=(1-\cos^2\theta)\{mg(\hat{\Phi}X,Y)-g(X,Y)\}$$

via (4.24) and (2.4). Hence we obtain (4.25).

Theorem 4.1. *A submanifold M of an almost poly-Norden Riemannian manifold $(\hat{M},\hat{\Phi},g)$ is slant if and only if there exists a constant $\lambda\in[0,1]$ such that*

$$f^{\,2}=\lambda\,(mf-I)\,. \tag{4.27}$$

Proof. Since M is a slant submanifold, from (3.8) and (4.24) we write

$$\begin{aligned} g(f^{\,2}X,Y) &= g(fX,fY)=\cos^2\theta\{mg(\hat{\Phi}X,Y)-g(X,Y)\} \\ &= \cos^2\theta g(mfX-X,Y), \end{aligned}$$

for any $X,Y\in\Gamma(TM)$, which implies

$$f^{\,2}X=\cos^2\theta\,(mf-I)\,(X).$$

For $\lambda=\cos^2\theta$ gives (4.27).

Conversely, assume that there exists a constant $\lambda\in[0,1]$ which satisfies (4.27). Then, for any $X\in\Gamma(TM)$ with $fX\neq 0$, we have

$$\begin{aligned} \cos\theta &= \frac{g(\hat{\Phi}X,fX)}{\left\|\hat{\Phi}X\right\|\|fX\|}=\frac{g(X,f^2X)}{\left\|\hat{\Phi}X\right\|\|fX\|} \\ &= \lambda\frac{mg(\hat{\Phi}X,X)-g(X,X)}{\left\|\hat{\Phi}X\right\|\|fX\|}. \end{aligned}$$

By using (2.4) in the last equation we get $\cos\theta=\lambda\frac{\|\hat{\Phi}X\|}{\|fX\|}$, which shows that $\cos^2\theta=\lambda=$ *constant* and hence, M is a slant submanifold. This completes the proof.

Proposition 4.2. *If M is a slant submanifold with the slant angle θ of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$, then we have*

$$(\nabla_X f^2)Y = m\cos^2\theta(\nabla_X f)Y, \tag{4.28}$$

for any $X, Y \in \Gamma(TM)$.

Proof. From (4.27), for all $X, Y \in \Gamma(TM)$, we write

$$\nabla_X f^2 Y = \cos^2\theta(m\nabla_X fY - \nabla_X Y)$$

and

$$f^2(\nabla_X Y) = \cos^2\theta(mf\,\nabla_X Y - \nabla_X Y),$$

which completes the proof.

Hence, from (3.17) and (4.28) we give

Proposition 4.3. *Let M be an n-dimensional slant submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. Then, for any $X, Y \in \Gamma(TM)$, we have*

$$\left(\nabla_X f^2\right)Y = m\cos^2\theta\sum_{\beta=1}^{k}\left\{\upsilon_\beta(Y)A_{N_\beta}X + h_\beta(X,Y)\zeta_\beta\right\}.$$

Proposition 4.4. *If M is a slant submanifold with the slant angle θ $(\theta \neq \frac{\pi}{2})$ of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$, then we have*

$$f^2 = \cot^2\theta\sum_{\beta=1}^{k}\nu_\beta \otimes \zeta_\beta.$$

Proof. It follows from (3.18) and (4.27).

Example 4.1. *Let R^4 be the 4-dimensional real number space with a coordinate system (x, y, z, t). We define*

$$\begin{array}{rccl} \hat{\Phi}: & R^4 & \rightarrow & R^4 \\ & (x,y,z,t) & \rightarrow & \hat{\Phi}(x,y,z,t) = \left(B_m x, B_m y, (m-B_m)\,z, (m-B_m)\,t\right), \end{array}$$

where $B_m = \frac{m+\sqrt{m^2-4}}{2}$. Then $(R^4, \hat{\Phi})$ is an almost poly-Norden manifold [26]*. If we consider usual scalar product $\langle .,. \rangle$ on R^4, then we see that it is $\hat{\Phi}$-compatible and $(R^4, \hat{\Phi}, \langle .,. \rangle)$ is an almost poly-Norden Riemannian manifold. Now assume that M is a submanifold of $(R^4, \hat{\Phi}, \langle .,. \rangle)$ defined by the immersion*

$$\Omega(u_1, u_2) = (u_1 + u_2, u_1 - u_2, \sqrt{2}u_2, \sqrt{2}u_1).$$

In this case, TM is generated by

$$X = (1, 1, 0, \sqrt{2}), \quad Y = (1, -1, \sqrt{2}, 0).$$

One can see that

$$\hat{\Phi}X = \left(B_m, B_m, 0, \sqrt{2}(m - B_m)\right), \hat{\Phi}Y = \left(B_m, -B_m, \sqrt{2}(m - B_m), 0\right),$$

and

$$\begin{aligned}\left\langle \hat{\Phi}X, X \right\rangle &= 2(B_m + (m - B_m)) = 2m = \left\langle \hat{\Phi}Y, Y \right\rangle, \\ \|X\| &= \|Y\| = 2, \quad \left\|\hat{\Phi}X\right\| = \left\|\hat{\Phi}Y\right\| = \sqrt{2\left(m^2 - 2\right)},\end{aligned}$$

which imply that M is a slant submanifold of $\left(R^4, \hat{\Phi}, \langle ., . \rangle\right)$ with the slant angle

$$\theta = \cos^{-1}\left(\frac{m}{\sqrt{2\left(m^2 - 2\right)}}\right), \quad -\sqrt{2} < m < \sqrt{2}.$$

Example 4.2. *Consider the almost poly-Norden structure given by*

$$\hat{\Phi}(x_i, y_j, t) = \left(B_m x_i, \bar{B}_m y_j, B_m t\right), \quad 1 \leq i, j \leq 4,$$

and the scalar product $\langle ., . \rangle$ on R^9. Then $(R^9, \hat{\Phi}, \langle ., . \rangle)$ is an almost poly-Norden Riemannian manifold. Now let M be a submanifold of $(R^9, \hat{\Phi}, \langle ., . \rangle)$ by

$$\begin{aligned}\Psi(u, v, w, z) &= (\bar{B}_m u \cos\theta, \bar{B}_m v \cos\theta, \bar{B}_m w \cos\theta, \bar{B}_m z \cos\theta, \\ &\quad B_m u \sin\theta, B_m v \sin\theta, B_m w \sin\theta, B_m z \sin\theta, 0).\end{aligned}$$

In this case the tangent bundle of the submanifold is generated by

$$\begin{aligned}E_1 &= (\bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0, 0, 0, 0), \\ E_2 &= (0, \bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0, 0, 0), \\ E_3 &= (0, 0, \bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0, 0), \\ E_4 &= (0, 0, 0, \bar{B}_m \cos\theta, 0, 0, 0, B_m \sin\theta, 0).\end{aligned}$$

Then we calculate

$$\begin{aligned}\hat{\Phi}E_1 &= (\cos\theta, 0, 0, 0, \sin\theta, 0, 0, 0, 0), \\ \hat{\Phi}E_2 &= (0, \cos\theta, 0, 0, 0, \sin\theta, 0, 0, 0), \\ \hat{\Phi}E_3 &= (0, 0, \cos\theta, 0, 0, 0, \sin\theta, 0, 0), \\ \hat{\Phi}E_4 &= (0, 0, 0, \cos\theta, 0, 0, 0, \sin\theta, 0).\end{aligned}$$

and

$$\begin{aligned}\left\langle \hat{\Phi}E_k, E_k \right\rangle &= \bar{B}_m \cos^2\theta + B_m \sin^2\theta, \\ \|E_k\| &= \sqrt{\bar{B}_m^2 \cos^2\theta + B_m^2 \sin^2\theta}, \\ \left\| \hat{\Phi}E_k \right\| &= 1,\end{aligned}$$

where $1 \leq k \leq 4$, *which imply that*

$$\frac{\left\langle \hat{\Phi}E_k, E_k \right\rangle}{\left\| \hat{\Phi}E_k \right\| \|E_k\|} = \frac{\bar{B}_m \cos^2\theta + B_m \sin^2\theta}{\sqrt{\bar{B}_m^2 \cos^2\theta + B_m^2 \sin^2\theta}}.$$

Hence M *is a* 4*-dimensional slant submanifold of* $(R^9, \hat{\Phi}, \langle .,. \rangle)$ *with the slant angle* t *given by*

$$\cos t = \frac{\bar{B}_m \cos^2\theta + B_m \sin^2\theta}{\sqrt{\bar{B}_m^2 \cos^2\theta + B_m^2 \sin^2\theta}}.$$

5. Nijenhuis Tensor Field and Normality of the Structure

Let M be an n-dimensional isometrically immersed submanifold of an $(n+k)$-dimensional almost poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, \hat{g})$. We consider the structure

$$\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$$

on M induced by the almost poly-Norden structure of $\hat{M}$ which satisfies the properties given by Proposition 3.1.

Definition 5.1. *Let* M *be an* n*-dimensional submanifold of an* $(n+k)$*-dimensional almost poly-Norden Riemannian manifold* $(\hat{M}, \hat{\Phi}, g)$. *The structure* Π *is called normal if the Nijenhuis torsion tensor field of* f *satisfies*

$$N_f = 2\sum_{\beta=1}^{k} d\upsilon_\beta \otimes \zeta_\beta.$$

Lemma 5.1. *If* M *is an* n*-dimensional submanifold of an* $(n+k)$*-dimensional poly-Norden Riemannian manifold* $(\hat{M}, \hat{\Phi}, g)$ *and* $\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ *is the induced structure on* M, *then we have*

$$N_f(X,Y) = \sum_{\beta=1}^{k} \left\{ g(X,\zeta_\beta)B_\beta Y - g(Y,\zeta_\beta)B_\beta X - g(B_\beta X, Y)\zeta_\beta \right\}, \qquad (5.29)$$

$$2d\upsilon_\beta(X,Y) = -g(B_\beta X,Y) + \sum_{\gamma=1}^{k} \left\{\sigma_{\beta\gamma}(X)g(Y,\zeta_\gamma) - \sigma_{\beta\gamma}(Y)g(X,\zeta_\gamma)\right\}, \tag{5.30}$$

where $A_\beta = A_{N_\beta}$ *and* $B_\beta = fA_\beta - A_\beta f$, $1 \leq \beta \leq k$.

Proof. Since the Nijenhuis torsion tensor field of f is given by

$$N_f(X,Y) = (\nabla_{fX} f)\,Y - (\nabla_{fY} f)\,X - f\left[(\nabla_X f)\,Y - (\nabla_Y f)\,X\right],$$

then by using (3.17) we have

$$N_f(X,Y) = \sum_{\beta=1}^{k} \left\{ \begin{array}{c} g(wY,N_\beta)A_{N_\beta}fX + g(A_{N_\beta}fX,Y)\zeta_\beta - g(wX,N_\beta)A_{N_\beta}fY \\ -g(X,A_{N_\beta}fY)\zeta_\beta - fg(wY,N_\beta)A_{N_\beta}X + fg(wX,N_\beta)A_{N_\beta}Y \end{array} \right\},$$

which implies

$$N_f(X,Y) = \sum_{\beta=1}^{k} \left\{ \begin{array}{c} g(A_{N_\beta}fX - fA_{N_\beta}X,Y)\zeta_\beta \\ -g(X,\zeta_\beta)(A_{N_\beta}f - fA_{N_\beta})Y + g(Y,\zeta_\beta)(A_{N_\beta}f - fA_{N_\beta})X \end{array} \right\},$$

and we obtain (5.29).

From the definition of $d\upsilon_\beta$, it is well-known that

$$2d\upsilon_\beta(X,Y) = g(\nabla_X\zeta_\beta,Y) - g(X,\nabla_Y\zeta_\beta),$$

for any $X,Y \in \Gamma(TM)$. By using (3.22) we get

$$\begin{aligned} 2d\upsilon_\beta(X,Y) &= -g(B_\beta X,Y) \\ &\quad + \sum_{\gamma=1}^{k} \left\{g(A_{N_\gamma}X,Y) - g(X,A_{N_\gamma}Y)\right\}\theta_{\beta\gamma} \\ &\quad + \sum_{\gamma=1}^{k} \left\{g(Y,\zeta_\gamma)\sigma_{\beta\gamma}(X) - g(X,\zeta_\gamma)\sigma_{\beta\gamma}(Y)\right\}, \end{aligned}$$

which gives (5.30).

From (5.29) and (5.30), we obtain

Theorem 5.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M},\hat{\Phi},g)$ with the induced structure $\Pi = (f,g,\upsilon_\beta,\zeta_\beta,(\theta_{\beta\gamma})_{k\times k})$. Then we have*

$$\begin{aligned} N_f(X,Y) - 2\sum_{\beta=1}^{k} d\upsilon_\beta(X,Y)\zeta_\beta &= \sum_{\beta=1}^{k} \left\{g(X,\zeta_\beta)B_\beta Y - g(Y,\zeta_\beta)B_\beta X\right\} \\ &\quad - \sum_{\beta=1}^{k}\sum_{\gamma=1}^{k} \{g(Y,\zeta_\gamma)\sigma_{\beta\gamma}(X) \\ &\qquad - g(X,\zeta_\gamma)\sigma_{\beta\gamma}(Y)\zeta_\beta\}, \end{aligned} \tag{5.31}$$

for any $X, Y \in \Gamma(TM)$.

Since $\sigma_{\beta\gamma}$ are the components of the normal connection $\hat{\nabla}^{\perp}$ and $B_{\beta} = fA_{\beta} - A_{\beta}f$, from (5.31) we have

Corollary 5.1. *Let M be an n-dimensional submanifold of an $(n+k)$-dimensional poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. Then the induced structure $\Pi = (f, g, \upsilon_{\beta}, \zeta_{\beta}, (\theta_{\beta\gamma})_{k\times k})$ on M is normal provided that the tensor field f commutes with the Weingarten operator A_{β}, for all $\beta \in \{1, ..., k\}$ and the normal connection $\hat{\nabla}^{\perp}$ identically vanishes on the normal bundle.*

Lemma 5.2. *Let M be a non-invariant submanifold of codimension $k \geq 1$ in a poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$. If the normal connection $\hat{\nabla}^{\perp}$ vanishes on the normal bundle, then the vector fields $\zeta_1, ..., \zeta_k$ are linearly independent.*

Proof. From (3.14) and (3.20) we write

$$\upsilon_{\gamma}(\zeta_{\beta}) = m\theta_{_{\beta\gamma}} - \delta_{_{\beta\gamma}} - \sum_{\lambda=1}^{k} \theta_{\beta\lambda}\theta_{\lambda\gamma} = g\left(\zeta_{\gamma}, \zeta_{\beta}\right).$$

Assume that $\sum\limits_{i=1}^{k} c_i\zeta_i = 0$, for some real numbers $c_1, ..., c_k$. Then we have

$$0 = \sum_{i=1}^{k} c_i g(\zeta_i, \zeta_{\gamma}), \quad \gamma \in \{1, ..., k\},$$

which implies a linear equation system defined by

$$\sum_{i=1}^{k} c_i \Upsilon_{ij} = 0, \tag{5.32}$$

for any index $j \in \{1, ..., k\}$. Here, $\Upsilon_{ii} = m\theta_{_{ii}} - 1 - \sum\limits_{\lambda=1}^{k} \theta_{i\lambda}^2$ and $\Upsilon_{ij} = m\theta_{ij} - \sum\limits_{\lambda=1}^{k} \theta_{i\lambda}\theta_{\lambda j}$, for $i, j \in \{1, ..., k\}$ and $i \neq j$. The determinant of the coefficient matrix of the linear system (5.32) is the determinant of the matrix given by

$$P = m\Theta - I_k - \Theta^2, \quad \Theta = \left(\theta_{_{\beta\gamma}}\right)_{k\times k}.$$

In case of M is being a non-invariant submanifold with respect to $\hat{\Phi}$, the determinant of P cannot be zero which implies that the linear equation system (5.32) has only the trivial solution. This completes the proof.

Theorem 5.2. *Let M be a non-invariant submanifold of codimension $k \geq 1$ in a poly-Norden Riemannian manifold $(\hat{M}, \hat{\Phi}, g)$ with vanishing normal connection $\hat{\nabla}^{\perp}$ on the normal bundle.*

Then the induced structure $\Pi = (f, g, \upsilon_\beta, \zeta_\beta, (\theta_{\beta\gamma})_{k\times k})$ *on* M *is normal if and only if the induced* $(1,1)$*-tensor field* f *commutes with the Weingarten operator* A_β*, for all* $\beta \in \{1, ..., k\}$*.*

Proof. Assume that the induced structure Π is normal. Since $\hat{\nabla}^\perp = 0$ (equivalently, $\sigma_{\beta\gamma} = 0$) on the normal bundle, from (5.31) we have, for any $X, Y \in \Gamma(TM)$:

$$\sum_{\beta=1}^{k} g(X, \zeta_\beta) B_\beta Y = \sum_{\beta=1}^{k} g(Y, \zeta_\beta) B_\beta X,$$

which implies

$$\sum_{\beta=1}^{k} g(X, \zeta_\beta) g(B_\beta Y, Z) = \sum_{\beta=1}^{k} g(Y, \zeta_\beta) g(B_\beta X, Z), \tag{5.33}$$

for any $X, Y, Z \in \Gamma(TM)$. Replacing Y by Z in the last equation we write

$$\sum_{\beta=1}^{k} g(X, \zeta_\beta) g(B_\beta Z, Y) = \sum_{\beta=1}^{k} g(Z, \zeta_\beta) g(B_\beta X, Y). \tag{5.34}$$

By summing the last two equations side by side and using the skew-symmetry property of B_β, we obtain

$$\sum_{\beta=1}^{k} \{g(B_\beta X, Z)\zeta_\beta + g(Z, \zeta_\beta) B_\beta X\} = 0.$$

Interchanging X with Z in the last equation and summing these equations we get

$$\sum_{\beta=1}^{k} \{g(Z, \zeta_\beta) B_\beta X + g(X, \zeta_\beta) B_\beta Z\} = 0,$$

which gives

$$\sum_{\beta=1}^{k} \{g(Z, \zeta_\beta) g(B_\beta X, Y) + g(X, \zeta_\beta) g(B_\beta Z, Y)\} = 0. \tag{5.35}$$

From (5.33) and (5.35), we obtain

$$\sum_{\beta=1}^{k} g(Z, \zeta_\beta) g(B_\beta X, Y) = 0,$$

for any $X, Y, Z \in \Gamma(TM)$. By considering the hypothesis and using Lemma 5.2, we can observe that there exists a vector field $W \in \Gamma(TM)$ such that it is orthogonal on $Span\{\{\zeta_1, ..., \zeta_r\} \backslash \zeta_\beta\}$ and $g(W, \zeta_\beta) \neq 0$. So from the last equation we obtain that $B_\beta = 0$, for all $\beta \in \{1, ..., k\}$.

The proof of the converse part is obvious from (5.31).

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Faculty of Arts and Sciences, Department of Mathematics, Adiyaman University, Adiyaman, TURKEY

Faculty of Arts and Sciences, Department of Mathematics, Adiyaman University, Adiyaman, TURKEY

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CHEN'S BASIC INEQUALITIES FOR HYPERSURFACES OF STATISTICAL RIEMANNIAN MANIFOLDS

ESRA ERKAN * AND MEHMET GÜLBAHAR

Abstract. Some basic equalities and inequalities involving the Riemannian curvature invariants for hypersurfaces of statistical Riemannian manifolds are presented. With the help of these relations, the necessary conditions for these hypersurfaces to be total geodesic, total umbilical, or minimal have been obtained.

Keywords: Curvature, Statistical Riemannian manifold, Hypersurface.

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1. Introduction

With the J. F. Nash's embedding theorem, which concludes that every Riemannian manifold can be isometrically embedded into some Euclidean space, the question arose how to characterize a Riemannian manifold with the help of its intrinsic and extrinsic invariants. Riemann curvature invariants are utilized to solve this problem since these invariants are widely convenient tools to characterize Riemannian manifolds and the basic properties of the shape operator of a Riemannian manifold can be shown by the relations obtained on the section curvature, Ricci curvature, and scalar curvature.

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* Corresponding author

Esra Erkan; esraerkan@harran.edu.tr; https://orcid.org/0000-0003-0456-6418

Mehmet Gülbahar; mehmetgulbahar@harran.edu.tr; https://orcid.org/0000-0001-6950-7633

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During the 1990s, B.-Y. Chen established some inequalities involving the intrinsic invariants and the extrinsic invariants. Some of the important inequalities and their results are given as follows:

In [8], B.-Y. Chen proved the following relation between the sectional curvature K and the shape operator A_N for an n-dimensional submanifold M in Riemannian space form $R^m(\bar{c})$:

$$A_N > \frac{n-1}{n}(c-\bar{c})I_n, \tag{1.1}$$

where $c = \inf K \neq \bar{c}$ and I_n is the identity map. The equality case of (1.1) holds for all $p \in M$ if and only if M is totally geodesic.

In [9], B.-Y. Chen established the following inequality between the squared mean curvature and Ricci curvature for a submanifold in a real space form $R^m(\bar{c})$:

For each unit tangent vector $X \in T_pM^n$, the following inequality is satisfied

$$\|H\|^2 \geq \frac{4}{n^2}\{Ric(X) - (n-1)\bar{c}\}, \tag{1.2}$$

where $\|H\|^2$ is the squared mean curvature and $Ric(X)$ is the Ricci curvature of M^n at X.

The equality case of (1.2) holds for all unit tangent vectors at p if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.

In literature, these types of inequalities are known as Chen-like inequalities.

In addition to these facts, the theory of statistical manifolds has substantial physical and geometrical aspects. It has applications in neural networks, machine learning, artificial intelligence, and black holes [2, 7, 14, 27]. Statistical manifolds were firstly introduced by S. Amari [1] in his book. Later, the basic geometrical properties of hypersurfaces of statistical manifolds were exposed by H. Furuhata in [15, 16]. Recently, Chen-type inequalities for submanifolds of statistical manifolds have been studied by various authors in [3, 4, 5, 6, 11, 12, 13, 18, 19, 21, 22, 23, 24], etc.

The main purpose of the present paper is to establish Chen-like inequalities on hypersurfaces of statistical manifolds. Although it is clear that hypersurfaces are a special case of submanifolds and there are various studies related to Chen-like inequalities on the submanifolds of statistical manifolds in the literature, many exclusive and different results on the hypersurfaces of these manifolds have been obtained with the help of the Riemannian curvature invariants in this paper.

2. Preliminaries

Let $(\widetilde{M},\widetilde{g})$ be an $n-$dimensional Riemannian manifold equipped with a Riemannian metric $\widetilde{g}$ and $\{e_1,\ldots,e_n\}$ be any orthonormal frame field of $\Gamma(T\widetilde{M})$. The Ricci tensor $\widetilde{S}^0$ is defined by

$$\widetilde{S}^0(X,Y)=\sum_{j=1}^{n}\widetilde{g}(\widetilde{R}^0(e_j,X)Y,e_j)$$

for any $X,Y\in\Gamma(T\widetilde{M})$, where $\widetilde{R}^0$ is the Riemannian curvature tensor field of $\widetilde{M}$. The Ricci curvature $\widetilde{Ric}^0(X)$ of any vector field X is defined by

$$\widetilde{Ric}^0(X)=\widetilde{S}^0(X,X).$$

For a fixed $i\in\{1,\cdots,n\}$, we write

$$\widetilde{Ric}^0(e_i)\equiv\widetilde{S}^0(e_i,e_i)=\sum_{j=1}^{n}\widetilde{g}(\widetilde{R}^0(e_i,e_j)e_j,e_i), \tag{2.3}$$

which is equal to

$$\widetilde{Ric}^0(e_i)=\sum_{j\neq i}^{n}\widetilde{K}^0(e_i,e_j). \tag{2.4}$$

Here, $\widetilde{K}^0(e_i,e_j)$ denotes the sectional curvature of a plane section spanned by e_i and e_j for $i\neq j\in\{1,\ldots,n\}$.

In [9], B.-Y. Chen extended the notion of Ricci curvature to k-Ricci curvature, $2\le k\le n$, in an n-dimensional Riemannian manifold. Let π_k be a k-plane section of $T_p\widetilde{M}$ and X be a unit vector field in π_k. If $k=n$ then $\pi_n=T_pM$; and if $k=2$ then π_2 is a plane section of $T_p\widetilde{M}$. Let us choose an orthonormal basis $\{e_1,\ldots,e_k\}$ of π_k such that $e_1=X$. The k-Ricci curvature of π_k at X, denoted by $\widetilde{Ric}^0_{\pi_k}(X)$, is defined by

$$\widetilde{Ric}^0_{\pi_k}(X)=\sum_{j\neq i}^{k}\widetilde{K}^0(e_1,e_j).$$

For $k=n$, the n-Ricci curvature of X is denoted by $\widetilde{Ric}^0_{T_p\widetilde{M}}(X)$.

The scalar curvature is one of the most studied classical curvature invariants. The scalar curvature $\widetilde{\tau}^0(p)$ at a point p is defined by

$$\begin{aligned}\widetilde{\tau}^0(p) &= \sum_{1\leqslant j\leqslant n}\widetilde{K}^0(e_i,e_j)\\ &= \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\widetilde{g}(\widetilde{R}^0(e_i,e_j)e_j,e_i).\end{aligned} \tag{2.5}$$

The scalar curvature $\widetilde{\tau}(\pi_k)$ of the k-plane section π_k is given by

$$\widetilde{\tau}(\pi_k) = \frac{1}{2}\sum_{i=1}^{k}\sum_{j\neq i}^{k} \widetilde{K}^0(e_i, e_j).$$

In particular, for $k = n$, the n-scalar curvature at a point p is denoted by $\widetilde{\tau}_{T_p\widetilde{M}}(p)$.

Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g})$ and N be the unit normal vector field of (M, g). Denote by the Levi-Civita connection of $(\widetilde{M}, \widetilde{g})$ by $\widetilde{\nabla}^0$. The Gauss and Weingarten formulas are, respectively, given by

$$\widetilde{\nabla}^0_X Y = \nabla^0_X Y + g(A^0_N X, Y)N \tag{2.6}$$

and

$$\widetilde{\nabla}^0_X N = -A^0_N X \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$, where ∇^0 is the induced linear connection and A^0_N is the shape operator of (M, g).

Denote the Riemannian curvature tensor of (M, g) by R^0. The equation of Gauss is given by

$$R^0(X, Y)Z = \widetilde{R}^0(X, Y)Z + g(A^0_N Y, Z)A^0_N X - g(A^0_N X, Z)A^0_N Y \tag{2.8}$$

for any $X, Y, Z \in \Gamma(TM)$.

The hypersurface (M, g) is called totally geodesic if $A^0_N = 0$, minimal if $trace A^0_N = 0$. If $A^0_N(X) = \lambda X$, where λ is a smooth function on M, then (M, g) is called totally umbilical [10].

3. Statistical Manifolds and Their Hypersurfaces

Let $(\widetilde{M}, \widetilde{g})$ be a Riemannian manifold and $\widetilde{\nabla}$ be a torsion-free connection on $(\widetilde{M}, \widetilde{g})$. The manifold is called a statistical manifold if the following relation is satisfied for any $X, Y, Z \in \Gamma(T\widetilde{M})$:

$$\widetilde{g}(\widetilde{\nabla}_Z X, Y) = Z\widetilde{g}(X, Y) - \widetilde{g}(X, \widetilde{\nabla}^*_Z Y), \tag{3.9}$$

where

$$\widetilde{\nabla}^0_X Y = \frac{1}{2}(\widetilde{\nabla}_X Y + \widetilde{\nabla}^*_X Y). \tag{3.10}$$

Here, $\widetilde{\nabla}^*$ is called the dual connection of $\widetilde{\nabla}^*$, the pair $(\widetilde{\nabla}, g)$ is called a statistical structure on $(\widetilde{M}, \widetilde{g})$. A statistical manifold with a torsion-free connection $\widetilde{\nabla}$ is usually denoted by $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ [1].

Now, let us denote the Riemannian curvature tensor fields with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ by $\widetilde{R}$ and $\widetilde{R}^*$. Then we have

$$\widetilde{g}(\widetilde{R}^*(X, Y)Z, W) = -\widetilde{g}(Z, \widetilde{R}(X, Y)W) \tag{3.11}$$

for any $X, Y, Z \in \Gamma(T\widetilde{M})$.

A statistical manifold is said to be of constant curvature c, if the equation

$$\widetilde{R}(X, Y)Z = \frac{c}{4}\{\widetilde{g}(Y, Z)X - \widetilde{g}(X, Z)Y\} \tag{3.12}$$

holds for any $X, Y, Z \in \Gamma(TM)$ [15].

Considering the eq. (3.11), we see that $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ is of constant curvature with respect to $\widetilde{\nabla}$ if and only if it is of constant curvature with respect to $\widetilde{R}^*$.

Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. The Gauss and Weingarten formulas with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are, respectively, given by

$$\begin{aligned} \widetilde{\nabla}_X Y &= \nabla_X Y + g(A_N X, Y)N, & (3.13)\\ \widetilde{\nabla}_X N &= -A^*_N X + \kappa(X)N, & (3.14)\\ \widetilde{\nabla}^*_X Y &= \nabla^*_X Y + g(A^*_N X, Y)N, & (3.15)\\ \widetilde{\nabla}^*_X N &= -A_N X - \kappa(X)N & (3.16) \end{aligned}$$

for any $X, Y \in \Gamma(TM)$. It is easy to show that the induced connection ∇^* is the dual connection of ∇. Here, κ is a 1-form, A_N and A^*_N are the shape operators with respect to $\widetilde{\nabla}$ and its dual connection $\widetilde{\nabla}^*$, respectively.

Let R and $\widetilde{R}$ denote the Riemannian curvature tensor (M, g, ∇) and $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ respectively. Then the following relation holds

$$R(X, Y)Z = \widetilde{R}(X, Y)Z - g(A^*_N X, Z)A_N Y + g(A^*_N Y, Z)A_N X \tag{3.17}$$

for any $X, Y, Z \in \Gamma(TM)$ [15].

Let $\pi = Span\{X, Y\}$ be a plane section of $\Gamma(TM)$. Then the $K-$sectional curvature is defined by [20]

$$\widetilde{K}(\pi) = \frac{1}{2}\left[\widetilde{g}(\widetilde{R}(X,Y)Y,X) + \widetilde{g}(\widetilde{R}^{*}(X,Y)Y,X)\right] - \widetilde{g}(\widetilde{R}^{0}(X,Y)Y,X). \qquad (3.18)$$

A hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ is called

i. totally geodesic with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^{*}$), if $A_N = 0$ (resp.$A_N^{*} = 0$).
ii. totally umbilical with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^{*}$), if there exists a smooth function ρ such that $A_N X = \rho X$ (resp. $A_N^{*} X = \rho X$).
iii. minimal with respect to $\widetilde{\nabla}$ (resp. $\widetilde{\nabla}^{*}$), if $traceA_N = 0$ (resp. $traceA_N^{*} = 0$).

For more details on statistical manifolds and their submanifolds, we refer to [15, 16].

4. Ricci Curvature

In this section, we shall give some relations involving Ricci curvatures of hypersurfaces immersed in $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$.

Lemma 4.1. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ and $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. For any unit tangent vector X at a point p, we have the following equalities:*

$$Ric^{0}(X) = \widetilde{Ric}^{0}_{T_pM}(X) + traceA_N^{0} g(A_N X, X) - g(A_N^{2} X, X). \qquad (4.19)$$

$$\sum_{j=2}^{n} g(R(X,e_j)e_j,X) = \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(X,e_j)e_j,X) + g(A_N X,X)traceA_N^{*} - g(A_N^{*}X, A_N X). \qquad (4.20)$$

$$\sum_{j=2}^{n} g(R(X,e_j)X,e_j) = -\sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^{*}(X,e_j)e_j,X) + g(A_N^{*}X,X)traceA_N - g(A_N^{*}X, A_N X). \qquad (4.21)$$

Proof. In view of (2.8), the proof of (4.19) is straightforward.

Now we shall prove (4.20). From (3.17), we may write

$$g(R(e_1,e_2)e_2,e_1) = \widetilde{g}(\widetilde{R}(e_1,e_2)e_2,e_1) - g(A_N^{*}e_1,e_2)g(A_N e_2,e_1) + g(A_N^{*}e_2,e_2)g(A_N e_1,e_1). \qquad (4.22)$$

Taking trace in (4.22), we get

$$\begin{aligned}
\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1) \\
&\quad + g(A_N e_1,e_1)\left(\sum_{j=2}^{n} g(A_N e_j,e_j) - g(A_N^* e_1,e_1)\right) \\
&\quad + \sum_{j=2}^{n} \left(g(A_N^* e_1,e_j)g(A_N e_1,e_j)\right) - g(A_N^* e_1,e_1) \\
&\quad - g(A_N^* e_1,e_1)g(A_N e_1,e_1),
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1) + g(A_N e_1,e_1)\mathrm{trace}A_N^* \\
&\quad - \sum_{j=2}^{n} g(A_N^* e_1,e_j)g(A_N e_1,e_j). \qquad (4.23)
\end{aligned}$$

Now, considering the fact that $\{e_1,\ldots,e_n\}$ is an orthonormal basis of T_pM, we can write

$$\begin{aligned}
A_N^* e_1 &= \lambda_1 e_1 + \cdots + \lambda_n e_n, \\
A_N e_1 &= \mu_1 e_1 + \cdots + \mu_n e_n,
\end{aligned}$$

where λ_i, μ_i are real numbers for each $i \in \{1,\ldots,n\}$. Thus, we have

$$\begin{aligned}
\sum_{j=2}^{n} g(A_N^* e_1,e_j)g(A_N e_1,e_j) &= \lambda_1\mu_1 + \cdots + \lambda_n\mu_n \\
&= g(A_N^* e_1, A_N e_1). \qquad (4.24)
\end{aligned}$$

Using (4.24) in (4.23), we get

$$\begin{aligned}
\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1) \\
&\quad + g(A_N e_1,e_1)\mathrm{trace}A_N^* - g(A_N^* e_1, A_N e_1). \qquad (4.25)
\end{aligned}$$

Putting $X = e_1$ in (4.25) we obtain (4.21).

Now we shall prove (4.21). Using (3.11) and (3.17), we have

$$\begin{aligned}
\widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1) &= \widetilde{g}(\widetilde{R}(e_1,e_j)e_1,e_j) \\
&= -g(R(e_1,e_j)e_1,e_j) - g(A_N^* e_1,e_1)g(A_N e_j,e_j) \\
&\quad + g(A_N^* e_1,e_j)g(A_N e_1,e_j). \qquad (4.26)
\end{aligned}$$

Taking trace in (4.26), we get

$$\begin{aligned}\sum_{j=2}^{n} g(R(e_1,e_j)e_1,e_j) &= \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1) + g(A_N^* e_1, e_j)\mathrm{trace}A_N \\ &\quad - g(A_N^* e_1, A_N e_1). \end{aligned} \tag{4.27}$$

Putting $X = e_1$ in (4.27), we obtain (4.21).

Now, we shall give some relations involving $K-$Ricci curvature and $K-$scalar curvature which are defined by

$$Ric^k(X) = \sum_{j\neq i}^{n} K(e_i,e_j)$$

and

$$\tau^k(p) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j\neq i}^{n} \widetilde{g}(\widetilde{R}^0(e_i,e_j)e_j,e_i).$$

Theorem 4.1. *Let (M,g) be a minimal hypersurface with respect to $\widetilde{\widetilde{\nabla}}$ and $\widetilde{\widetilde{\nabla}}^*$. Then we have*

$$Ric^k(X) + Ric^0(X) = \widetilde{Ric}^k_{T_pM}(X) + \widetilde{Ric}^0_{T_pM}(X) \tag{4.28}$$

for any unit vector $X \in T_pM$.

Proof. Under the assumption, we have from (4.19) and (4.20) that

$$\sum_{j=2}^{n} g(R(e_1,e_j)e_j,e_1) = \sum_{j=2}^{n} \widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1) - g(A_N^* e_1, A_N e_1) \tag{4.29}$$

and

$$\sum_{j=2}^{n} g(R(e_1,e_j)e_1,e_j) = -\sum_{j=2}^{n} \widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1) - g(A_N^* e_1, A_N e_1). \tag{4.30}$$

If the equations (4.29) and (4.30) are subtracted from side to side, we get

$$\begin{aligned}\sum_{j=2}^{n} [g(R(e_1,e_j)e_j,e_1) + g(R^*(e_1,e_j)e_j,e_1)] &= \sum_{j=2}^{n} \Big[\widetilde{g}(\widetilde{R}(e_1,e_j)e_j,e_1) \\ &\quad + \widetilde{g}(\widetilde{R}^*(e_1,e_j)e_j,e_1)\Big].\end{aligned}$$

In view of (3.18), we see that

$$\sum_{j=2}^{n} \left[K(e_1,e_j) + K^0(e_1,e_j)\right] = \sum_{j=2}^{n} \widetilde{K}(e_1,e_j) + \widetilde{K}^0(e_1,e_j). \tag{4.31}$$

Putting $X = e_1$ in (4.31), we obtain (4.28).

Remark 4.1. *Since*

$$A^0_N X = A_N X + A^*_N X$$

for any $X \in \Gamma(TM)$, it is clear that if $A_N X = A^*_N X = 0$, then we have $A^0_N = 0$. But the converse part of this claim is not correct in general. Considering this fact, the claim of Theorem 4.1 may not be correct when the hypersurface is minimal with respect to $\widetilde{\nabla}^0$.

Now, we recall the Chen-Ricci inequality for a Riemannian submanifold [17, 26]:

Theorem 4.2. *Let (M, g) be an $n-$dimensional submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$. Then the following statements are true.*

i. *For any unit tangent vector X, we have*

$$Ric^0(X) \leqslant \frac{1}{4} n^2 \|H\|^2 + \widetilde{Ric}^0_{T_pM}(X). \tag{4.32}$$

ii. *The equality case of (4.32) holds for all unit tangent vectors of T_pM if and only if either p is a totally geodesic point or $n = 2$ and p is a totally umbilical point.*

Theorem 4.3. *Let (M, g) be a $n > 2$ minimal hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$. Then we have*

$$Ric^k(X) \geq \widetilde{Ric}^0_{T_pM}(X) \tag{4.33}$$

for any unit tangent vector $X \in T_PM$. The equality case of (4.33) holds for all $X \in T_pM$ if and only if $A_N X = -A^*_N X$.

Proof. Using the fact that $traceA_N = traceA^*_N = 0$, we see that $H = 0$ from Remark 4.1. In view of (4.32), we get

$$Ric^0(X) \leq \widetilde{Ric}^0_{T_pM}(X). \tag{4.34}$$

Using (4.34) in (4.28), we obtain (4.33). From Theorem 4.2, the equality case of (4.33) is satisfied if and only if $A^0_N X = 0$ which shows that $A_N X = -A^*_N X$ for all $X \in T_pM$.

Now we recall the following theorem of T. Takahashi [25]:

Theorem 4.4. *The necessary condition for a submanifold of an Euclidean space to be a minimal immersion is that its Ricci curvature is negative semi-definite.*

In the following corollary, we obtain a similar claim of Theorem 4.4 for a minimal hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ on statistical manifolds with constant curvatures.

Corollary 4.1. *Let $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$ be of $K-$constant curvature with $c = 0$ and (M, g) be a minimal hypersurface with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$. Then we have*

$$Ric^k(X) \geq 0$$

for any unit tangent vector $X \in T_pM$.

Now we shall give the following lemma for later uses:

Lemma 4.2. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then the following relation is satisfied for any unit tangent vector $X \in T_pM$:*

$$\begin{aligned} Ric^k(X) &= \widetilde{Ric}_{T_pM}(X) - \frac{1}{2}g(A_NX, X)traceA_N^* \\ &+\frac{1}{2}g(A_N^*X, X)traceA_N + g(A_N^0X, X)traceA_N^0 + \left\|A_N^0X\right\|^2 . \end{aligned} \tag{4.35}$$

Proof. From (3.18), we have

$$\begin{aligned} \widetilde{Ric}^k_{T_pM}(e_i) &= \frac{1}{2}\sum_{j=2}^{n}\widetilde{g}(\widetilde{R}(e_i, e_j)e_j, e_i) + \frac{1}{2}\sum_{j=2}^{n}\widetilde{g}(\widetilde{R}^*(e_i, e_j)e_j, e_i) \\ &-\sum_{j=2}^{n}\widetilde{g}(\widetilde{R}^0(e_i, e_j)e_j, e_i). \end{aligned} \tag{4.36}$$

In view of (4.19), (4.20), and (4.21) in (4.36), we obtain

$$\begin{aligned} \widetilde{Ric}^k_{T_pM}(e_i) &= \frac{1}{2}\sum_{j=2}^{n}g(R(e_i, e_j)e_j, e_1) + \frac{1}{2}\sum_{j=2}^{n}g(R(e_1, e_j)e_j, e_1) \\ &-\sum_{j=2}^{n}g(R^0e_1, e_j)e_j, e_1) + \frac{1}{2}g(A_Ne_1, e_1)traceA_N^* \\ &-\frac{1}{2}g(A_N^*e_1, A_Ne_1) - \frac{1}{2}g(A_N^*e_1, e_1)traceA_N \\ &+\frac{1}{2}g(A_Ne_1, A_N^*e_1) - g(A_N^0e_1, e_1)traceA^0 + g(A_N^0e_1, A_N^0e_1). \end{aligned}$$

Putting $X = e_1$, the proof of (4.35) is straightforward.

From Lemma 4.2, we get the following corollary immediately:

Corollary 4.2. *Let (M, g) be a hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. Then the following inequality is satisfied for any unit tangent vector $X \in T_pM$:*

$$\begin{aligned} Ric^k(X) &\leqslant \widetilde{Ric}_{T_pM}(X) - \frac{1}{2}g(A_NX, X)traceA_N^* \\ &+\frac{1}{2}g(A_N^*X, X)traceA_N + g(A_N^0X, X)traceA_N^0. \end{aligned} \tag{4.37}$$

5. Scalar Curvature

In this section, we shall give some relations involving scalar curvatures of hypersurfaces immersed in $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$.

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. We put

$$\sigma_{ij} = g(A_N e_i, e_j) \quad \text{and} \quad \sigma^*_{ij} = g(A^*_N e_i, e_j)$$

for any $i, j \in \{1, 2, \cdots, n\}$. From (3.17), we write

$$g(R(e_i, e_j)e_j, e_i) = \widetilde{g}(\widetilde{R}(e_i, e_j)e_j, e_i) - \sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}. \tag{5.38}$$

Taking trace in (5.38), we get

$$\tau(p) = \widetilde{\tau}_{T_pM}(p) - \frac{1}{2}\sum_{i,j=1}^{n}\left(\sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}\right). \tag{5.39}$$

Let us define

$$\left|A^0_N\right| = \left(\sum_{i,j=1}^{n} g(A^0_N e_i, e_j)\right)^2. \tag{5.40}$$

In light of the above facts, we shall state the following lemma:

Lemma 5.1. *For any hypersurface of $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$, we have*

$$\begin{aligned} 2\sum_{i,j=1}^{n}\left(\sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}\right) &= 4\left[traceA^0_N\right]^2 - [traceA_N]^2 - [traceA^*_N]^2 \\ &\quad +4\left|A^0_N\right| - \|A^*_N\|^2 - \|A_N\|. \end{aligned} \tag{5.41}$$

Proof. We can write

$$\begin{aligned} 2\sum_{i,j=1}^{n}\left(\sigma^*_{ij}\sigma_{ji} + \sigma^*_{jj}\sigma_{ii}\right) &= \left(\sum_{i,j=1}^{n}\sigma_{ii} + \sigma^*_{jj}\right)^2 - \left(\sum_{i=1}^{n}\sigma_{ii}\right)^2 - \left(\sum_{i=1}^{n}\sigma^*_{jj}\right)^2 \\ &\quad + \left(\sum_{i,j=1}^{n}\sigma^*_{ij} + \sigma^*_{ji}\right)^2 - \left(\sum_{i=1}^{n}\sigma^*_{ij}\right)^2 - \left(\sum_{i=1}^{n}\sigma_{ji}\right)^2. \end{aligned} \tag{5.42}$$

On the other hand, we also have

$$\begin{aligned}\sum_{i,j=1}^{n}\sigma_{ii}+\sigma_{jj}^{*} &= \sum_{i,j=1}^{n}\left[g(A_Ne_i,e_i)+g(A_N^{*}e_j,e_j)\right]\\ &= \sum_{i=1}^{n}g(A_Ne_i,e_i)+\sum_{j=1}^{n}g(A_N^{*}e_j,e_j)\\ &= traceA_N+traceA_N^{*}\\ &= trace(A_N+A_N^{*})\\ &= 2traceA_N^{0}. \end{aligned}\tag{5.43}$$

In a similar way, we have

$$\begin{aligned}\sum_{i,j=1}^{n}\sigma_{ij}^{*}+\sigma_{ji} &= \sum_{i,j=1}^{n}\left[g(A_N^{*}e_i,e_i)+g(A_Ne_j,e_j)\right]\\ &= \sum_{i,j=1}^{n}g((A_N^{*}+A_N)e_i,e_j)\\ &= 2\sum_{i,j=1}^{n}g(A_N^{0}e_i,e_j) \end{aligned}\tag{5.44}$$

The proof is straightforward from computing the other terms on the right-hand side of (5.42) in a similar way.

From the equation (5.39) and (5.41), we get the following lemma:

Proposition 5.1. *For any hypersurface of $(\widetilde{M},\widetilde{g},\widetilde{\nabla})$, we have*

$$\begin{aligned}\tau(p) &= \widetilde{\tau}_{T_pM}(P)-2\left[traceA_N^{0}\right]^2+\frac{1}{2}\left[traceA_N\right]^2+\frac{1}{2}\left[traceA_N^{*}\right]^2\\ &\quad -2\left|A_N^{0}\right|+\frac{1}{2}\left\|A_N^{*}\right\|^2+\frac{1}{2}\left\|A_N\right\|^2. \end{aligned}\tag{5.45}$$

Theorem 5.1. *Let (M,g) be a hypersurface of $(\widetilde{M},\widetilde{g},\widetilde{\nabla})$. Then we have*

$$\tau(p)\geq\widetilde{\tau}_{T_pM}(p)-2\left[traceA_N^{0}\right]^2+\frac{1}{2}\left[traceA_N\right]^2+\frac{1}{2}\left[traceA_N^{*}\right]^2-2\left|A_N^{0}\right|\tag{5.46}$$

for any $p\in M$. The equality case of (5.46) holds for all $p\in M$ if and only if M is totally geodesic.

Proof. The proof of (5.46) is straightforward from (5.45). The equality case of (5.46) holds for all $p\in M$ if and only if we have $A_N^{*}=A_N=0$. Using the fact that $A_N^{0}=\frac{1}{2}(A_NX+A_N^{*}X)$, we obtain $A_N^{0}X=0$ for any $X\in T_pM$. This shows that M is totally geodesic. The converse part of the proof is straightforward.

Now we shall give the following lemma for later uses:

Lemma 5.2. *For any hypersurface of* $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$, *we have*

$$\tau^k(p) = \widetilde{\tau}^k(p) + (traceA_N^0)^2 + trace(A_N^0)^2. \tag{5.47}$$

Proof. Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of T_pM at a point $p \in M$. From Lemma 4.2, we write

$$\begin{aligned} Ric^k(e_i) &= \widetilde{Ric}_{T_pM}(e_i) - \frac{1}{2}g(A_Ne_i, e_i)traceA_N^* + \frac{1}{2}g(A_N^*e_i, e_i)traceA_N \\ &\quad + traceA_N^0 g(A_N^0e_i, e_i) + \left\|A_N^0e_i\right\|^2 \end{aligned} \tag{5.48}$$

for any $i \in \{1, \ldots, n\}$. Taking trace in (5.48), we get

$$\begin{aligned} \tau^k(p) &= \widetilde{\tau}^k_{T_pM}(p) + \frac{1}{2}traceA_N traceA_N^* - \frac{1}{2}traceA_N traceA_N^* \\ &\quad + (traceA_N^0)^2 + \sum_{i=1}^n g(A_N^0e_i, A_N^0e_i) \end{aligned} \tag{5.49}$$

which is equivalent to (5.47).

As a result of Lemma 5.2, we obtain the following corollaries:

Corollary 5.1. *Let* (M, g) *be a hypersurface of* $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. *Then we have*

$$\tau(p) \leq \widetilde{\tau}(p) + trace(A_N^0)^2 \tag{5.50}$$

for any $p \in M$. *The equality case of (5.50) holds for all* $p \in M$ *if and only if* M *is minimal.*

Corollary 5.2. *Let* (M, g) *be a totally umbilical hypersurface of* $(\widetilde{M}, \widetilde{g}, \widetilde{\nabla})$. *Then we have*

$$\widetilde{\tau}_{T_pM}(p) < \tau_{T_pM}(p) \tag{5.51}$$

Proof. If (M, g) is a totally umbilical hypersurface, then there exists a smooth function ρ^0 on M such that we can write $A_N^0X = \rho^0X$ for any $X \in \Gamma(TM)$. Thus, we obtain from (5.47) that

$$\widetilde{\tau}_{T_pM}(p) = \tau_{T_pM}(p) + (n^2 - n)\lambda^2. \tag{5.52}$$

In view (5.52), we have (5.51).

6. Examples

Now we shall give an example satisfying some results obtained in this paper:

Example 6.1. *Let us consider a hypersurface M given by*

$$M = \{(cosx_1, sinx_1, x_2, x_3) : x_1 \in (0, 2\pi], x_2, x_3 \in \mathbb{R}\}$$

in $\mathbb{E}^4$. The natural tangent vector fields of M are given by

$$e_1 = -sinx_1\partial_1 + cosx_1\partial_2, \quad e_2 = \partial_3, \quad e_3 = \partial_4$$

and the normal vector field of M is given by

$$N = cosx_1\partial_1 + sinx_1\partial_2,$$

where $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is the natural basis of $\mathbb{E}^4$. By a straightforward computation, we easily have

$$\widetilde{\nabla}^0_{e_1} e_1 = -cosx_1\partial_1 - sinx_1\partial_2, \quad \widetilde{\nabla}^0_{e_2} e_2 = 0, \quad \widetilde{\nabla}^0_{e_3} e_3 = 0$$

and $\widetilde{\nabla}^0_{e_i} e_j = 0$ for $i \neq j \in \{1, 2, 3\}$. From (2.6), we get

$$A^0_N = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.53}$$

Now, suppose that the connections $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$ are satisfied the following relations:

$$\widetilde{\nabla}_{e_1} e_1 = -2\cos x_1\partial_1 \quad \widetilde{\nabla}_{e_2} e_2 = e_2, \quad \widetilde{\nabla}_{e_3} e_3 = e_3, \tag{6.54}$$

$$\widetilde{\nabla}^*_{e_1} e_1 = -2\sin x_1\partial_2, \quad \widetilde{\nabla}^*_{e_2} e_2 = -e_2, \quad \widetilde{\nabla}^*_{e_3} e_3 = -e_3,$$

and $\widetilde{\nabla}_{e_i} e_j = \widetilde{\nabla}^*_{e_i} e_j = 0$ for $i \neq j \in \{1, 2, 3\}$. Then we get

$$\widetilde{K}(e_1, e_2) = \widetilde{K}(e_1, e_3) = \widetilde{K}(e_2, e_3) = 0$$

and

$$\widetilde{Ric}_{T_pM}(e_1) = \widetilde{Ric}_{T_pM}(e_2) = \widetilde{Ric}_{T_pM}(e_1) = \widetilde{\tau}_{T_pM}(p) = 0. \tag{6.55}$$

In view of (3.13), (3.15) and (6.54), we have

$$A_N = \begin{bmatrix} -2\cos^2 x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^*_N = \begin{bmatrix} -2\sin^2 x_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{6.56}$$

From these facts, it is clear that

$$K(e_1, e_2) = K(e_1, e_3) = K(e_2, e_3) = 0, \tag{6.57}$$

$$Ric(e_1) = Ric(e_2) = Ric(e_3) = \tau(p) = 0. \tag{6.58}$$

Considering (6.53), (6.55), (6.56) and (6.58), we see that the hypersurface M is satisfied the claims of Theorem 4.1, Theorem 4.3, Corollary 4.2, Theorem 5.1, and Corollary 5.1.

Example 6.2. *Let us consider the following hypersurface*

$$M = \{(x_1, x_2, x_3, 0) : \forall x_1, x_2, x_3 \in \mathbb{R}\}$$

in $\mathbb{E}^4$. Then it is clear that $T_pM = Span\{e_1, e_2, e_3\}$ and $N = \partial_4$ such that $e_i = \partial_i$ for $i \in \{1, 2, 3\}$. Suppose that the connection ∇ and $\widetilde{\nabla}^*$ are satisfied

$$\widetilde{\nabla}_{e_1} e_1 = \partial_1 + \partial_4, \qquad \widetilde{\nabla}^*_{e_1} e_1 = -\partial_1 - \partial_4,$$

$$\widetilde{\nabla}_{e_1} e_1 = \partial_2 + \partial_4, \qquad \widetilde{\nabla}^*_{e_2} e_2 = -\partial_2 - \partial_4,$$

and the other component of $\widetilde{\nabla}_{e_i} e_j$ are equal to zero for $i, j \in \{1, 2, 3\}$. Then we have

$$\nabla_{e_1} e_1 = \partial_1, \qquad \nabla^*_{e_1} e_1 = -\partial_1,$$

$$\nabla_{e_2} e_2 = \partial_2, \qquad \nabla^*_{e_2} e_2 = -\partial_2,$$

and other component of $\nabla_{e_i} e_j$ are equal to zero for $i, j \in \{1, 2, 3\}$. By a straightforward computation, we obtain M is minimal with respect to $\widetilde{\nabla}$ and $\widetilde{\nabla}^*$, and $c = 0$. Also, we see that the hyperplane M satisfies of Corollary 4.1 by $Ric^k(X) = 0$ for any $X \in T_pM$ at any point $p \in M$.

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Department of Mathematics, Faculty of Science and Arts, Harran University, Sanliurfa,Turkey

Department of Mathematics, Faculty of Science and Arts, Harran University, Sanliurfa,Turkey

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SOME CHARACTERIZATIONS OF QUASI-EINSTEIN AND TWISTED PRODUCT MANIFOLDS

MOCTAR TRAORE *, HAKAN METE TAŞTAN , AND ADARA M. BLAGA

Abstract. We first consider quasi-Einstein manifolds with concircular generator vector field. Secondly, we get a result for a twisted product alternative to a result of Ponge-Reckziegel [13]. Then we study quasi-Einstein manifolds on twisted product structures. In particular, we examine the effect of the condition of quasi-Einstein on a twisted product to its factor manifolds. Also, we obtain some conditions for a twisted product satisfying the quasi-Einstein condition to be a warped or a direct product.

Keywords: Twisted product manifold, Quasi-Einstein manifold.

2010 Mathematics Subject Classification: Primary 53C20, Secondary 53C25.

1. Introduction

The concept of warped product of Riemannian manifolds [3] is a generalization of the direct product of Riemannian manifolds and plays a very important role in physics, as well as in differential geometry, especially in the theory of relativity. Indeed, the standard space-time models such as Robertson-Walker, Schwarzschild, static and Kruskal, are warped products. Also, the simplest models of neighborhoods of stars and black holes are warped products [12]. Moreover, some solutions to Einstein's field equation can be written in terms of warped products [1].

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* Corresponding author

Moctar Traore; moctar.traore@ogr.iu.edu.tr; https://orcid.org/0000-0003-2132-789X

Hakan Mete Taştan; hakmete@istanbul.edu.tr; https://orcid.org/0000-0002-0773-9305

Adara M. Blaga; adarablaga@yahoo.com; https://orcid.org/0000-0003-0237-3866

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On the other hand, there is an important notion known as Einstein manifold [2], which has a central place in both mathematics and physics. Indeed, Einstein manifolds are not only interesting themselves, but they are also related to many important topics of differential geometry such as Riemannian submersions, homogenous Riemannian spaces, Yang-Mills theory, self-dual manifolds of dimension four, holonomy groups, etc.

In this paper, we study twisted products and quasi-Einstein manifolds, which are generalizations of both of the two concepts mentioned above.

2. Preliminaries

2.1. **Twisted products.** Let M_1 and M_2 be two Riemannian manifolds endowed with the Riemannian metric tensors g_1 and g_2 and let f be a positive smooth function defined on $M_1 \times M_2$. Denote by π_1 and π_2 the canonical projections of $M_1 \times M_2$ onto M_1 and M_2, respectively. Then the *twisted product* [7] $M_1 \times_f M_2$ of (M_1, g_1) and (M_2, g_2) is the product manifold $M := M_1 \times M_2$ equipped with metric g given by

$$g = \pi_1^*(g_1) \oplus f^2 \pi_2^*(g_2), \tag{2.1}$$

where $\pi_i^*(g_i)$ is the pullback of g_i via π_i for $i \in \{1, 2\}$. Then the function f is called the *twisting function* of the twisted product $M_1 \times_f M_2 = (M, g)$. If f only depends on the points of M_1, then we get a *warped product* [3] and if f is a constant, then we get a *direct product manifold* [8].

Let $M_1 \times_f M_2$ be a twisted product manifold with the Levi-Civita connection ∇ and denote by ∇^i the Levi-Civita connection of M_i for $i \in \{1, 2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathfrak{L}(M_i)$ and use the same notation for a vector field and for its lift. On the other hand, π_1 is an isometry and π_2 is a (positive) homothety, so they preserve the Levi-Civita connection. Thus, there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas of twisted product manifold are given by the following.

Lemma 2.1. [7] *Let $M_1 \times_f M_2$ be a twisted product manifold. Then for $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, we have*

$$\nabla_X Y = \nabla^1_X Y, \tag{2.2}$$

$$\nabla_X V = \nabla_V X = X(k)V, \tag{2.3}$$

$$\nabla_U V = \nabla^2_U V + U(k)V + V(k)U - g(U,V)\nabla k, \tag{2.4}$$

where $k = \ln f$ and ∇k is the gradient of the function k.

The manifold $\{p\} \times M_2$ is called a *fiber* of the twisted product and the manifold $M_1 \times \{q\}$ is called a *base manifold* of $M_1 \times_f M_2$, where $p \in M_1$ and $q \in M_2$. The base manifold is totally geodesic and the fiber is totally umbilical in $M_1 \times_f M_2$.

As in [10], we define $h^k_1(X,Y) = XY(k) - (\nabla^1_X Y)(k)$ for all $X, Y \in \mathfrak{L}(M_1)$ and $h^k_2(U,V) = UV(k) - (\nabla^2_U V)(k)$ for all $U, V \in \mathfrak{L}(M_2)$. Then the Hessian form h^k of k on (M, g) satisfies

$$h^k(X,Y) = h^k_1(X,Y), \tag{2.5}$$

$$h^k(U,V) = h^k_2(U,V) - 2U(k)V(k) + g(U,V)g(\nabla k, \nabla k). \tag{2.6}$$

Let ${}^1R$ and ${}^2R$ be the lifts of Riemann curvature tensors of (M_1, g_1) and (M_2, g_2), respectively and let R be the Riemann curvature tensor of the twisted product $M_1 \times_f M_2$. Then, by direct computations and using (2.2)–(2.4), we have the following relations.

Lemma 2.2. *Let $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$. Then, we have*

$$R_{XY}Z = {}^1R(X,Y)Z, \tag{2.7}$$

$$R_{XY}U = 0, \tag{2.8}$$

$$R_{UV}X = UX(k)V - VX(k)U, \tag{2.9}$$

$$R_{XU}Y = \Big(h^k_1(X,Y) + X(k)Y(k)\Big)U, \tag{2.10}$$

$$R_{UX}V = -XV(k)U + \Big(X(k)\nabla k + H^k(X)\Big)g(U,V), \tag{2.11}$$

$$\begin{aligned} R_{UV}W = {}^2R(U,V)W &- \Big(h^k_2(V,W) - W(k)V(k)\Big)U \\ &+ \Big(h^k_2(U,W) - W(k)U(k)\Big)V \\ &- \Big(H^k(U) + U(k)\nabla k\Big)g(V,W) + \Big(H^k(V) + V(k)\nabla k\Big)g(U,W), \end{aligned} \tag{2.12}$$

where H^k is the Hessian tensor of k on $M_1 \times_f M_2$, i.e., $H^k(E) = \nabla_E \nabla k$ for any vector field E on $M_1 \times_f M_2$.

Next, let ${}^1\mathrm{Ric}$ and ${}^2\mathrm{Ric}$ be the lifts of Ricci curvature tensors of (M_1, g_1) and (M_2, g_2), respectively and let Ric be the Ricci curvature tensor of the twisted product $M_1 \times_f M_2$. Then, by direct computations and using (2.5)–(2.12), we have the following relations.

Lemma 2.3. *Let $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$. Then, we have*

$$\mathrm{Ric}(X,Y) = {}^1\mathrm{Ric}(X,Y) - m_2\Big(h_1^k(X,Y) + X(k)Y(k)\Big), \tag{2.13}$$

$$\mathrm{Ric}(X,U) = -(m_2 - 1)XU(k), \tag{2.14}$$

$$\begin{aligned}\mathrm{Ric}(U,V) = {}&{}^2\mathrm{Ric}(U,V) - (m_2-2)h_2^k(U,V) \\ &+ (m_2-2)U(k)V(k) - g(U,V)\Delta k,\end{aligned} \tag{2.15}$$

where Δk is the Laplacian of k on $M_1 \times_f M_2$ and $m_i = \dim(M_i)$.

2.2. **Quasi-Einstein Manifolds.** A Riemannian manifold (M^m, g), $m \geq 2$, is said to be an *Einstein manifold* [2] if its Ricci tensor Ric satisfies the condition $\mathrm{Ric} = \frac{\tau}{m} g$, where τ denotes the *scalar curvature* of M. A non-flat Riemannian manifold (M, g), $m \geq 2$, is said to be a *quasi-Einstein* manifold [6] if the Ricci tensor field of M satisfies

$$\mathrm{Ric} = \alpha g + \beta A \otimes A, \tag{2.16}$$

where α and β are scalar functions on M with $\beta \neq 0$ and A is non-zero 1-form such that $g(X, \xi) = A(X)$ for every vector field X on M, ξ being a unitary vector field which is called the generator of the manifold M. Note that if $\beta = 0$, then the manifold reduces to an Einstein manifold.

Remark 2.1. *In what follows, we shall use this notion in a slightly enlarged sense, allowing for the non-zero vector field ξ to be non-unitary. Notice also that quasi-Einstein manifolds coincide with trivial almost η-Ricci solitons* [4]*, i.e., almost η-Ricci solitons with Killing potential vector field.*

3. Main Results

Let (M^m, g), $m \geq 3$, be a quasi-Einstein manifold with associated scalar functions α and β and the generator vector field ξ. By a contraction from (2.16), we have

$$\tau = m\alpha + \beta|\xi|^2, \tag{3.17}$$

where τ is the scalar curvature of M. By taking the gradient of (3.17), we obtain

$$\nabla\tau = m\nabla\alpha + |\xi|^2\nabla\beta + \beta\nabla(|\xi|^2). \tag{3.18}$$

Now, by taking the divergence of (2.16) for any vector field X on M, we have

$$\mathrm{div}(\mathrm{Ric})(X) = g\Big(\nabla\alpha + \beta\nabla_\xi\xi + \xi(\beta)\xi + \beta\,\mathrm{div}(\xi)\xi, X\Big).$$

Using the Schur's Lemma, i.e., $d\tau = 2\,\mathrm{div}(\mathrm{Ric})$ and (3.18), we obtain

$$(m-2)\nabla\alpha = 2\beta\nabla_\xi\xi + 2\Big(\xi(\beta) + \beta\,\mathrm{div}(\xi)\Big)\xi. \tag{3.19}$$

Now, we suppose that ξ is a concircular vector field [11], i.e., $\nabla_Z\xi = aZ$ for any vector field Z on M, with a a smooth function on M. Then, we have $\mathrm{div}(\xi) = ma$ and the equation (3.19) becomes

$$(m-2)\nabla\alpha = 2\Big(\xi(\beta) + (m+1)a\beta\Big)\xi. \tag{3.20}$$

On the other hand, upon direct computations, we find

$$R(X,Y)\xi = X(a)Y - Y(a)X$$

for any vector fields X and Y on M and so, we deduce that

$$\mathrm{Ric}(\xi,\xi) = -(m-1)\xi(a). \tag{3.21}$$

But the equation (2.16) gives

$$\mathrm{Ric}(\xi,\xi) = \alpha|\xi|^2 + \beta|\xi|^4. \tag{3.22}$$

From (3.21) and (3.22), we deduce that $-(m-1)\xi(a) = |\xi|^2(\alpha + \beta|\xi|^2)$ and so

$$\alpha = -\tfrac{m-1}{|\xi|^2}\xi(a) - \beta|\xi|^2. \tag{3.23}$$

Assume now that a is constant. Using (3.23) in (3.20), we get

$$-(m-3)\Big(\beta\nabla(|\xi|^2) + |\xi|^2\nabla\beta\Big) = 2\Big(\xi(\beta) + (m+1)a\beta\Big)\xi. \tag{3.24}$$

Taking the inner product of (3.24) with ξ, we get

$$-(m-3)\beta\xi(|\xi|^2) = (m-1)|\xi|^2\xi(\beta) + 2(m+1)a\beta|\xi|^2. \tag{3.25}$$

Since $\xi(|\xi|^2) = 2a|\xi|^2$, from (3.25), we find

$$\xi(\beta) = -4a\beta.$$

Finally, using (3.18), we arrive to

$$\nabla\tau = -(m-1)\left(\beta\nabla(|\xi|^2) + |\xi|^2\nabla\beta\right) \tag{3.26}$$

and by taking the inner product of (3.26) with ξ, we find

$$\xi(\tau) = -2(m-1)a\beta|\xi|^2.$$

On the other hand, if ξ is of constant length, then $a = 0$ and $\nabla\alpha = -|\xi|^2\nabla\beta$, which is combined with (3.20) to obtain

$$-(m-2)|\xi|^2\nabla\beta = 2\xi(\beta)\xi \tag{3.27}$$

and by taking the inner product of (3.27), we get $\xi(\beta) = 0$, hence $\xi(\alpha) = 0$.

Therefore, we get the following two results.

Theorem 3.1. *Let (M^m, g), $m \geq 3$, be a quasi-Einstein manifold with associated scalar functions α and β and the generator vector field ξ such that ξ is concircular with a constant. If β is constant, then ξ is ∇-parallel or M is a Ricci-flat manifold.*

Theorem 3.2. *Let (M^m, g), $m \geq 3$, be a quasi-Einstein manifold with associated scalar functions α and β and the generator vector field ξ such that ξ is concircular. If ξ is of constant length, then ξ is ∇-parallel and the functions α and β are constant along the integral curves of ξ.*

Now we give a new characterization for twisted products.

Theorem 3.3. *Let (M, g) be a pseudo-Riemannian manifold and let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the canonical foliations on M. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ intersect perpendicularly everywhere. Then M is a locally twisted product $M_1 \times_f M_2$ with a twisting function f if and only if for any $W \in \mathfrak{L}(M_2)$, we have*

$$\mathcal{L}_W g = 0 \quad on \quad M_1 \tag{3.28}$$

and there exists a smooth function μ on M such that for any $Z \in \mathfrak{L}(M_1)$, we have

$$\mathcal{L}_Z g = 2Z(\mu)g \quad on \quad M_2, \tag{3.29}$$

where $\mathcal{L}_W$ is the Lie derivative with respect to W and M_1 (resp. M_2) is the integral manifold of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$).

Proof. Let $M_1 \times_f M_2$ be a twisted product with the metric g. Then using the Lie derivative formula for any $X, Y, Z \in \mathfrak{L}(M_1)$ and $U, V, W \in \mathfrak{L}(M_2)$, we have

$$(\mathcal{L}_W g)(X, Y) = -2g(\sigma_1(X, Y), W) \tag{3.30}$$

and

$$(\mathcal{L}_Z g)(U, V) = -2g(\sigma_2(U, V), Z), \tag{3.31}$$

where σ_1 (resp. σ_2) denotes the second fundamental form of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$), (e.g. see [5], p. 195). Hence, using (2.2), we get

$$(\mathcal{L}_W g)(X, Y) = 0$$

from (3.30) and we get (3.28). Next, using (2.4), we get

$$(\mathcal{L}_Z g)(U, V) = -2g\Big(- g(U, V)P_1\nabla(\ln f),\ Z\Big) \tag{3.32}$$

from (3.31), where $P_i : \mathfrak{L}(M_1 \times M_2) \to \mathfrak{L}(M_i)$ for $i \in \{1, 2\}$. By a direct computation, we obtain

$$(\mathcal{L}_Z g)(U, V) = 2Z(\ln f)g(U, V)$$

from (3.32). Thus, we get (3.29) for $\mu = \ln f$.

Conversely, suppose that the conditions (3.28) and (3.29) hold. Then for any $X, Y \in \mathfrak{L}(M_1)$ and $W \in \mathfrak{L}(M_2)$, using (3.28) and (3.30), we get $g(\sigma_1(X, Y), W) = 0$. It follows that $\sigma_1(X, Y) = 0$ for all $X, Y \in \mathfrak{L}(M_1)$ and so $\mathcal{F}_1$ is totally geodesic. On the other hand for any $Z \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$, using (3.29) and (3.31), we have

$$-2g(\sigma_2(U, V), Z) = 2Z(\mu)g(U, V).$$

After a straightforward computation, we get

$$g(\sigma_2(U, V), Z) = g\Big(- g(U, V)\nabla\mu,\ Z\Big).$$

It follows that $\sigma_2(U, V) = -g(U, V)P_1\nabla\mu$ for all $U, V \in \mathfrak{L}(M_2)$. Thus, $\mathcal{F}_2$ is totally umbilical with the mean curvature vector field $-P_1\nabla\mu$. Therefore, it follows from Proposition 3(b) of [13] that M is a locally twisted product $M_1 \times_f M_2$ with $f = e^{\mu}$ and M_1 (resp. M_2) is the integral manifold of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$). $\square$

Remark 3.1. *Let $\{e_1, ..., e_{m_1}, \omega_1, ..., \omega_{m_2}\}$ be an orthonormal basis of the twisted product $M_1 \times_f M_2$, where $\{e_1, ..., e_{m_1}\}$ are tangent to M_1 and $\{\omega_1, ..., \omega_{m_2}\}$ are tangent to M_2. Then by (2.1), we see that $\{e_1, ..., e_{m_1}\}$ is an orthonormal basis of (M_1, g_1) and $\{f\omega_1, ..., f\omega_{m_2}\}$ is an orthonormal basis of (M_2, g_2).*

Let Δ^1 and Δ^2 be the lifts of Laplacian operators on (M_1, g_1) and (M_2, g_2), respectively and let Δ be the Laplacian operator on the twisted product $M_1 \times_f M_2$. In view of Remark 3.1 and using (2.5) and (2.6), we get

$$\Delta k = \Delta^1 k + \frac{1}{f^2}\Delta^2 k + m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k).$$

Notice that for $m_2 \geq 2$, we have $m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k) \geq 0$. Moreover, we have

$$\Delta^1 k = \Delta k - \frac{1}{f^2}\Delta^2 k - \Big(m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k)\Big)$$

and

$$\Delta^2 k = f^2\Bigg(\Delta k - \Delta^1 k - \Big(m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k)\Big)\Bigg).$$

Also $\Delta k = 0$ if and only if

$$\Delta^1 k = -\frac{1}{f^2}\Delta^2 k - \Big(m_2 g(\nabla k, \nabla k) - 2g(P_2\nabla k, P_2\nabla k)\Big).$$

If $\Delta^2 k \geq 0$, then $\Delta^1 k \leq 0$, and by Hopf's Lemma we deduce that $k = \ln f$ is constant on both M_2 and M_1.

Therefore, we get the following result.

Proposition 3.1. *Let $M_1 \times_f M_2$ be a twisted product manifold with harmonic function $k = \ln f$ with respect to Δ and $m_2 \geq 2$. If $\Delta^2 k \geq 0$, then $\Delta^1 k \leq 0$. As a consequence, the twisted product manifold is a direct product.*

Similarly, we obtain the following.

Proposition 3.2. *Let $M_1 \times_f M_2$ be a twisted product manifold with harmonic function $k = \ln f$ with respect to Δ and $m_2 \geq 2$. If $\Delta^1 k \geq 0$, then $\Delta^2 k \leq 0$. As a consequence, the twisted product manifold is a direct product.*

Next, we shall examine the condition of quasi-Einstein on a twisted product to its factor manifolds.

Theorem 3.4. *Let $M_1 \times_f M_2$ be a twisted product manifold. Then it is a quasi-Einstein manifold with associated scalar functions α and β and 1-form A if and only if the followings hold:*

(a) ${}^1\operatorname{Ric} = \alpha g_1 + \beta\tilde{A} \otimes \tilde{A} + m_2 \tilde{dk} \otimes \tilde{dk} + m_2 h_1^k$, *where* $\tilde{A} = A|_{M_1}$ *and* $\tilde{dk} = dk|_{M_1}$,

(b) ${}^2\operatorname{Ric} = f^2(\alpha + \Delta k)g_2 + (m_2 - 2)h_2^k - (m_2 - 2)\tilde{dk} \otimes \tilde{dk} + \beta f^4 \tilde{A} \otimes \tilde{A}$, *where* $\tilde{A} = A|_{M_2}$ *and* $\tilde{dk} = dk|_{M_2}$,

(c) *We have* $-(m_2 - 1)XV(k) = \beta A(X)A(V)$ *for any* $X \in \mathfrak{L}(M_1)$ *and* $V \in \mathfrak{L}(M_2)$.

Proof. On M_1, we have

$$\alpha g + \beta A \otimes A = {}^1\operatorname{Ric} - m_2 h_1^k - m_2 dk \otimes dk$$

from (2.13) and (2.16). By using (2.1) and (2.5), we obtain

$${}^1\operatorname{Ric} = \alpha g_1 + \beta\tilde{A} \otimes \tilde{A} + m_2 \tilde{dk} \otimes \tilde{dk} + m_2 h_1^k,$$

where $\tilde{A} = A|_{M_1}$ and $\tilde{dk} = dk|_{M_1}$, as desired.

Similarly, on M_2, we have

$$\alpha g + \beta A \otimes A = {}^2\mathsf{R}ic - (m_2 - 2)h_2^k + (m_2 - 2)dk \otimes dk - \Delta k g$$

from (2.15) and (2.16). By using (2.1), we obtain

$${}^2\operatorname{Ric} = f^2(\alpha + \Delta k)g_2 + (m_2 - 2)h_2^k - (m_2 - 2)\tilde{dk} \otimes \tilde{dk} + \beta f^4 \tilde{A} \otimes \tilde{A},$$

where $\tilde{A} = A|_{M_2}$ and $\tilde{dk} = dk|_{M_2}$, as desired. On the other hand, from (2.14) and (2.16), we easily get (3). The converse is just a verification. □

Theorem 3.5. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β. If the generator vector field ξ is tangent to the base manifold M_1, then the Ricci tensors of M_1 and M_2 satisfy the following equations*

$${}^1\operatorname{Ric}(X,Y) = \alpha g_1(X,Y) + m_2\Big(h_1^k(X,Y) + X(k)Y(k)\Big) + \beta g_1(X,\xi)g_1(Y,\xi), \tag{3.33}$$

$${}^2\operatorname{Ric}(U,V) = f^2 g_2(U,V)(\alpha + \Delta k) + (m_2 - 2)h_2^k(U,V) - (m_2 - 2)U(k)V(k), \tag{3.34}$$

where $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$.

Proof. For any $X, Y \in \mathfrak{L}(M_1)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(X,Y) = \alpha g_1(X,Y) + \beta g_1(X,\xi)g_1(Y,\xi).$$

By (2.13), we get (3.33).

Similarly for any $U, V \in \mathfrak{L}(M_2)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(U,V) = \alpha f^2 g_2(U,V),$$

since $g(U,\xi) = 0$. By (2.15), we get (3.34). □

Let ${}^1\tau$ and ${}^2\tau$ be the lifts of scalar curvatures of (M_1, g_1) and (M_2, g_2), respectively and let τ be the scalar curvature of the twisted product $M_1 \times_f M_2$. In view of Theorem 3.5 and Remark 3.1, we obtain the following.

Corollary 3.1. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with the associated scalar functions α and β. If the generator vector field ξ is tangent to the base manifold M_1, then, we have*

$$\tau = (m_1 + m_2)\alpha + \beta|\xi|^2,$$

$${}^1\tau = m_1\alpha + \beta|\xi|^2 + m_2\Delta^1 k + m_2 g_1(\nabla k, \nabla k), \tag{3.35}$$

$${}^2\tau = m_2 f^2(\alpha + \Delta k) + (m_2 - 2)\Delta^2 k - (m_2 - 2)f^4 g_2(\nabla k, \nabla k), \tag{3.36}$$

where Δ^i is the Laplacian operator on (M_i, g_i) for $i \in \{1, 2\}$.

Theorem 3.6. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β. If the generator vector field ξ is tangent to the fiber manifold M_2, then the Ricci tensors of M_1 and M_2 satisfy the following equations*

$${}^1\mathrm{Ric}(X,Y) = \alpha g_1(X,Y) + m_2\Big(h_1^k(X,Y) + X(k)Y(k)\Big), \tag{3.37}$$

$$\begin{aligned} {}^2\mathrm{Ric}(U,V) = {} & f^2 g_2(U,V)(\alpha + \Delta k) + (m_2 - 2)h_2^k(U,V) \\ & - (m_2 - 2)U(k)V(k) + \beta f^4 g_2(U,\xi)g_2(V,\xi), \end{aligned} \tag{3.38}$$

where $X, Y \in \mathfrak{L}(M_1)$ and $U, V \in \mathfrak{L}(M_2)$.

Proof. For any $X, Y \in \mathfrak{L}(M_1)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(X,Y) = \alpha g_1(X,Y),$$

since $g(X,\xi)=0$. By (2.13), we get (3.37).

Similarly, for any $U,V\in\mathfrak{L}(M_2)$, using (2.1) and (2.16), we have

$$\mathrm{Ric}(U,V)=\alpha f^2 g_2(U,V)+\beta f^4 g_2(U,\xi)g_2(V,\xi).$$

By using (2.15), we get (3.38). □

In view of Theorem 3.6 and Remark 3.1, we obtain the following.

Corollary 3.2. *Let $M_1\times_f M_2$ be a twisted product quasi-Einstein manifold with the associated scalar functions α and β. If the generator vector field ξ is tangent to the fiber manifold M_2, then, we have*

$$\tau=(m_1+m_2)\alpha+\beta|\xi|^2,$$

$$^1\tau=m_1\alpha+m_2\Delta^1 k+m_2 g_1(\nabla k,\nabla k), \tag{3.39}$$

$$^2\tau=m_2 f^2(\alpha+\Delta k)+(m_2-2)\Delta^2 k-(m_2-2)f^4 g_2(\nabla k,\nabla k)+\beta f^4|\xi|^2. \tag{3.40}$$

Finally, motivated by the results of [9] on warped product quasi-Einstein manifolds, we obtain the following results for twisted product quasi-Einstein manifolds.

Theorem 3.7. *Let $M_1\times_f M_2$ be a twisted product quasi-Einstein manifold with associated positive scalar functions α and β such that the generator vector field ξ tangent to M_1. If M_1 is compact and $^1\tau=0$, then the twisted product manifold is a direct product.*

Proof. We have

$$m_2\Delta^1 k=-m_1\alpha-\beta|\xi|^2-m_2 g_1(\nabla k,\nabla k)$$

from (3.35). Under the given hypothesis, it follows that $\Delta^1 k\le 0$. Namely, $\Delta^1 k$ has constant sign on M_1. By Hopf's Lemma, the function $k=\ln f$ is constant on M_1, since M_1 is compact. Therefore, the twisting function f only depends on the points of M_2. Thus, the twisted product manifold is a direct product of (M_1,g_1) and $(M_2,\tilde{g_2})$, where $\tilde{g_2}=f^2 g_2$. □

Similarly, with the help of (3.39), we obtain the following result.

Theorem 3.8. *Let $M_1\times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β such that the generator vector field ξ tangent to M_2 and $\alpha\ge 0$. If M_1 is compact and $^1\tau=0$, then the twisted product manifold is a direct product.*

Theorem 3.9. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated scalar functions α and β such that the generator vector field ξ tangent to M_1 and $\alpha + \Delta k \leq 0$. If M_2 is compact, ${}^2\tau = 0$ and $m_2 \geq 3$, then the twisted product manifold is a warped product.*

Proof. We have

$$(m_2 - 2)\Delta^2 k = -m_2 f^2(\alpha + \Delta k) + (m_2 - 2) f^4 g_2(\nabla k, \nabla k)$$

from (3.36). Under the given hypothesis, it follows that $\Delta^2 k \geq 0$. Namely, $\Delta^2 k$ has constant sign on M_2. By Hopf's Lemma, the function $k = \ln f$ is constant on M_2, since M_2 is compact. Therefore, the twisting function f only depends on the points of M_1. Thus, the twisted product manifold is a warped product of (M_1, g_1) and (M_2, g_2). □

Similarly, with the help of (3.40), we obtain the following result.

Theorem 3.10. *Let $M_1 \times_f M_2$ be a twisted product quasi-Einstein manifold with associated positive scalar functions α and β such that the generator vector field ξ tangent to M_2 and $\alpha + \Delta k \leq 0$, $\beta < 0$. If M_2 is compact, ${}^2\tau = 0$ and $m_2 \geq 3$, then the twisted product manifold is a warped product.*

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Conflicts of interests.

The authors declare that there is no conflict of interests.

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Department of Mathematics, Faculty of Science, İstanbul University, İstanbul 34134, Turkey

Department of Mathematics, Faculty of Science, İstanbul University, İstanbul 34134, Turkey

Department of Mathematics, Faculty of Mathematics and Computer Science, West University of Timişoara, Bld. V. Pârvan, 300223 Timişoara, România

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FRACTIONAL EQUIAFFINE CURVATURES OF CURVES IN 3-DIMENSIONAL AFFINE SPACE

MUHITTIN EVREN AYDIN * AND SEYMA KAYA

Abstract. In this study, we investigate the equiaffine invariants of a parametrized curve in the 3-dimensional affine space $\mathbb{R}^3$ by using a simplification of Caputo fractional derivative. We introduce the so-called fractional equiaffine arclength function for a non-degenerate parametrized curve, providing the notions of fractional equiaffine frame and curvatures. Furthermore, we give the relations between the fractional and standard equiaffine curvatures.

Keywords: Affine differential geometry; Caputo fractional derivative; Equiaffine arclength; Equiaffine curvature.

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1. Introduction

Fractional calculus extends to arbitrary orders the notions of classical derivative and integral of a function and has a remarkable historical background, which it can be found in [22]. This interesting field has applications ranging from physical phenomena ([20]), dynamical systems ([27]), viscoelasticity ([15], [24]) to medicine [8].

Recently, there have been ascending contributions to the differential geometric applications of fractional calculus. From the viewpoints of Riemannian and Finsler geometries, these

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* Corresponding author

Muhittin Evren Aydin; meaydin@firat.edu.tr https://orcid.org/0000-0001-9337-8165

Seyma Kaya; seymakayaa23@hotmail.com; https://orcid.org/0000-0002-4792-4556

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contributions can be found in [4], [5]. Also, we refer to [1], [2], [9], [10], [11], [16], [17], [19], [25], [26], [28], [29] for the contributions to the differential geometries of curves and surfaces.

We will consider a simplification of Caputo fractional derivative as follows: let $f(t)$ and $g(x)$ be smooth functions and denote by D^α Caputo fractional derivative. Then the simplification, relating to the derivative of the composite function of $f(t)$ and $g(x)$, that we will use is given by

$$(D_x^\alpha f)(g(x)) = \frac{\alpha x^{1-\alpha}}{\Gamma(2-\alpha)} \frac{df}{dt} \frac{dg}{dx}. \tag{1.1}$$

The idea of using Equation (1.1) in the study of differential geometric curves was first proposed in [26] because of the reason that Caputo fractional derivative of composite functions is given by an infinite series. The derivative of composite functions, i.e. chain rule, is an essential tool for the parametrized objects in differential geometry. To overcome this difficulty in the case of Caputo fractional derivative, we will use Equation (1.1) in our calculations as did the authors in [26].

In this study, we perform Equation (1.1) in order to investigate the equiaffine invariants of the non-degenerate parametrized curves in the 3-dimensional affine space $\mathbb{R}^3$. Our motivation of investigating the equiaffine invariants is the following.

Let $\mathbf{r}(s)$ be a regular parametrized curve in a Euclidean space $\mathbb{E}^3$ by arclength and $\times$ denote the cross product. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame along $\mathbf{r}(s)$ such that (see [21])

$$\mathbf{t} = \frac{d\mathbf{r}}{ds}, \quad \mathbf{n} = \frac{d^2\mathbf{r}/ds^2}{\|d^2\mathbf{r}/ds^2\|}, \quad \mathbf{b} = \mathbf{t} \times \mathbf{n},$$

where $\|.\|$ denotes the induced norm in $\mathbb{E}^3$ by the Euclidean scalar product.

If we use Equation (1.1) instead of the standard ordinary derivative, i.e. d/ds, then the set of Frenet vectors is again $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$. This situation changes for the equiaffine Frenet frame of a non-degenerate curve in $\mathbb{R}^3$. More explicitly, the equiaffine Frenet frame of a non-degenerate curve produced by Equation (1.1) is different than the standard equiaffine Frenet frame. This justifies why we consider the equiaffine invariants instead of Frenet invariants for the use of fractional derivative in the differential geometry of curves.

The main purpose of this study is to extend the results in [2] to 3-dimensional case where the authors ([2]) introduced the fractional equiaffine invariants of a non-degenerate curve in the affine plane $\mathbb{R}^2$. Since we use a different formula of derivative instead of the standard ordinary derivative, we will need a new equiaffine arclength function which differs by the standard one. The new equiaffine arclength function will depend on the dimension of affine space and the standard equaffine parameter of given non-degenerate curve. For this, we will provide a general formula for the fractional equiaffine arclength function of a non-degenerate curve in the $n-$dimensional affine space $\mathbb{R}^n$ ($n \geq 2$) (see Definition 4.1). Then, in 3-dimensional context, we introduce the equiaffine Frenet curvatures of fractional type (see Definition 4.3) and obtain the properties between the fractional and standard equiaffine curvatures (Theorem 4.1 and Corollaries 4.1 and 4.2). Several examples are also provided by figures.

2. Fractional tools

Denote by $\Gamma(\alpha)$ the Euler gamma function depending on the parameter $\alpha \in \mathbb{R}$, which it is defined by ([14])

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt.$$

Throughout the paper we will assume $0 < \alpha \leq 1$. The Riemann–Liouville fractional integral of order α for a function $f(x)$ is defined by ([14], [22])

$$I^\alpha_{0+} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(\xi)}{(x-\xi)^{1-\alpha}} d\xi.$$

The Riemann–Liouville fractional derivative of order α is ([14], [22])

$$(\mathcal{D}^\alpha_{0+} f)(x) = \frac{d}{dx} (I^{1-\alpha}_{0+} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x \frac{f(\xi)}{(x-\xi)^\alpha} d\xi.$$

As can be seen, the Riemann–Liouville fractional derivative uses the ordinary integral of $f(x)$ and it is a nonlocal operator, i.e. the Riemann–Liouville derivative of $f(x)$ at a point x_0 is determined by nonlocal values of $f(x)$.

The Caputo fractional derivative of order α for a function $f(x)$ is given by ([6])

$$(D^\alpha_{0+} f)(x) = I^{1-\alpha}_{0+} (\frac{df}{dx})(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x \frac{1}{(x-\xi)^\alpha} \frac{df(\xi)}{d\xi} d\xi.$$

Leibniz rule and the derivative of composite function for the Caputo fractional derivative are respectively defined by ([3])

$$(D^\alpha_{0+} fg)(x) = \sum_{i=0}^\infty \binom{\alpha}{i} \frac{d^i f}{dx^i} (D^{\alpha-i}_x g)(x) - \frac{f(0)g(0)}{\Gamma(1-\alpha)} x^{-\alpha}$$

and

$$(D^\alpha_{0+} f)(g(x)) = \sum_{i=1}^\infty \binom{\alpha}{i} \frac{x^{i-\alpha}}{\Gamma(i-\alpha+1)} \frac{d^i f(g(x))}{dx^i} + \frac{f(g(x)) - f(g(0))}{\Gamma(1-\alpha)} x^{-\alpha}. \tag{2.2}$$

Notice that the simplification (1.1) is obtained by extracting the term $i = 1$ in the infinite series in Equation (2.2).

For simplicity, we will use the following notation throughout the paper:

$$(D_{0+}^{\alpha}f)(x)=\frac{d^{\{\alpha\}}f}{dx^{\{\alpha\}}}.$$

3. Equiaffine Invariants

Let $\mathbb{R}^n$ denote the $n-$dimensional affine space ($n\geq 2$) and $\mathrm{Mat}(n,\mathbb{R})$ be the set of all square matrices of order n. We set

$$\mathrm{SL}(\mathbb{R}^n)=\{A\in \mathrm{Mat}(n,\mathbb{R}):\det(A)=1\}.$$

Then by an *equiaffine invariant* we mean an unchanged feature under the actions of $\mathrm{SL}(\mathbb{R}^n)$ and the translations of $\mathbb{R}^n$. For example, the volume is an equiaffine invariant (see e.g. [7]).

Denote by $[u_1,\dots,u_n]$ the determinant of the vectors $u_1,\dots,u_n\in\mathbb{R}^n$ where u_k represents the k.-th column. Then the value of $[u_1,\dots,u_n]$ is an equiaffine invariant because it measures the volume of parallelopipedon determined by $u_1,\dots,u_n$.

Let $t\mapsto \mathbf{r}(t)$, $t\in I\subset\mathbb{R}$, a smooth parametrized curve in $\mathbb{R}^n$. We call the curve $\mathbf{r}(t)$ *non-degenerate* if, for every $t\in I$, (see [7] and also [12], [13], [18])

$$\left[\frac{d\mathbf{r}}{dt}(t),...,\frac{d^n\mathbf{r}}{dt^n}(t)\right]\neq 0.$$

For simplicity, by a curve we will mean a non-degenerate smooth parametrized curve throughout the paper. Then the *equiaffine arclength function* is defined by

$$\sigma(t)=\int^t\left[\frac{d\mathbf{r}}{du}(u),...,\frac{d^n\mathbf{r}}{du^n}(u)\right]^{2/(n^2+n)}du.$$

We call that the curve is parametrized by *equiaffine arclength* if, for every $\sigma\in J\subset\mathbb{R}$,

$$\left[\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^n\mathbf{r}}{d\sigma^n}(\sigma)\right]=1. \tag{3.3}$$

The set $\left\{\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^n\mathbf{r}}{d\sigma^n}(\sigma)\right\}$ is called the *equiaffine Frenet frame* of $\mathbf{r}(\sigma)$. When we differentiate Equation (3.3) with respect to the parameter σ, we may observe that

$$\left[\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^{n-1}\mathbf{r}}{d\sigma^{n-1}}(\sigma),\frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)\right]=0,$$

where the following set are linearly dependent for every $\sigma\in J$:

$$\left\{\frac{d\mathbf{r}}{d\sigma}(\sigma),...,\frac{d^{n-1}\mathbf{r}}{d\sigma^{n-1}}(\sigma),\frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)\right\}.$$

Hence, this gives the existence of smooth functions $\kappa_i(\sigma)$ on J ($1\leq i\leq n-1$) such that

$$\frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma)+\sum_{i=1}^{n-1}\kappa_i(\sigma)\frac{d^i\mathbf{r}}{d\sigma^i}(\sigma)=0,$$

where

$$\kappa_i(\sigma) = (-1)^{n-i+1} \left[\frac{d\mathbf{r}}{d\sigma}(\sigma), ..., \frac{d^{i-1}\mathbf{r}}{d\sigma^{i-1}}(\sigma), \frac{d^{i+1}\mathbf{r}}{d\sigma^{i+1}}(\sigma), ..., \frac{d^{n+1}\mathbf{r}}{d\sigma^{n+1}}(\sigma) \right], \quad 1 \leq i \leq n-1.$$

The function $\kappa_i(\sigma)$ is called $i.-$th *equiaffine curvature* of the curve $\mathbf{r}(\sigma)$. The equiaffine curvatures are the equiaffine invariants in $\mathbb{R}^n$. In 3-dimensional case, that is, in the case $i \in \{1, 2\}$, we will use the notations $\kappa_1 = \kappa$ and $\kappa_2 = \tau$. In additon, the equiaffine Frenet vectors will be denoted by

$$\mathbf{T}(\sigma) = \frac{d\mathbf{r}}{d\sigma}(\sigma), \quad \mathbf{N}(\sigma) = \frac{d^2\mathbf{r}}{d\sigma^2}(\sigma), \quad \mathbf{B}(\sigma) = \frac{d^3\mathbf{r}}{d\sigma^3}(\sigma).$$

In consequence, the equiaffine equations of Frenet type are given in matrix form

$$\begin{bmatrix} \dot{\mathbf{T}}(\sigma) \\ \dot{\mathbf{N}}(\sigma) \\ \dot{\mathbf{B}}(\sigma) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa(\sigma) & -\tau(\sigma) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(\sigma) \\ \mathbf{N}(\sigma) \\ \mathbf{B}(\sigma) \end{bmatrix},$$

where $\dot{\mathbf{T}}(\sigma)$ is the derivative of $\mathbf{T}(\sigma)$ with respect to the arclength parameter σ.

4. Equiaffine invariants of fractional order

Let $\mathbf{r}(\sigma)$, $\sigma \in (a, b)$, $0 < a < b$, be a curve in $\mathbb{R}^n$, $n \geq 2$, parametrized by equiaffine arclength. Again, we consider the simplification (1.1) as

$$\frac{d^{\{\alpha\}}\mathbf{r}}{dt^{\{\alpha\}}}(\sigma(t)) = \frac{\alpha t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\mathbf{r}}{d\sigma}(\sigma(t)) \frac{d\sigma}{dt}(t). \tag{4.4}$$

Here $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$ and Equation (4.4) becomes the classical chain rule provided $\alpha = 1$.

In the following, by using Equation (4.4) we introduce an equiaffine arclength function of fractional type.

Definition 4.1. *Let* $\mathbf{r}(\sigma)$, $\sigma \in (a, b)$, $0 < a < b$, *be a curve in* $\mathbb{R}^n$ *parametrized by equiaffine arclength. The following function* $s(\sigma)$ *is called equiaffine arclength function of the curve of order* $0 < \alpha \leq 1$

$$\sigma \mapsto s(\sigma) = \left(\frac{2\alpha + n - 1}{n+1} \left(\frac{\alpha}{\Gamma(2-\alpha)} \right)^{2/(n+1)} \sigma \right)^{(n+1)/(2\alpha+n-1)}. \tag{4.5}$$

It is obvious from Equation (4.5) that $s(\sigma)$ is a smooth function of σ on (a, b) and so is $\mathbf{r}(s(\sigma))$.

Proposition 4.1. *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^n$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *Then, for every* $s \in (c,d)$,

$$\left[\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right) \right] = 1.$$

Proof. Let σ be the standard equiaffine arclength parameter of $\mathbf{r}(s)$. By Equation (4.5), we have $ds/d\sigma > 0$, yielding the existence of the inverse of the function $s(\sigma)$, namely,

$$s \mapsto \sigma(s) = \frac{n+1}{2\alpha+n-1}\left(\frac{\alpha}{\Gamma(2-\alpha)} \right)^{-2/(n+1)} s^{(2\alpha+n-1)/(n+1)}, \tag{4.6}$$

where $\sigma(s)$ is smooth on $s \in (c,d)$. Taking derivative in Equation (4.6) with respect to s,

$$\frac{d\sigma}{ds}(s) = \left(\frac{\alpha}{\Gamma(2-\alpha)} \right)^{-2/(n+1)} s^{2(\alpha-1)/(n+1)}. \tag{4.7}$$

From Equation (4.4) we have

$$\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(\sigma(s)) = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\mathbf{r}}{d\sigma}(\sigma(s)) \frac{d\sigma}{ds}(s). \tag{4.8}$$

We successively differentiate Equation (4.8) with respect to s, obtaining

$$\frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right) = (...)\frac{d\mathbf{r}}{d\sigma}(\sigma(s)) + \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \left(\frac{d\sigma}{ds}(s) \right)^2 \frac{d^2\mathbf{r}}{d\sigma^2}(\sigma(s)),$$
$$\vdots$$
$$\frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right) = (...)\frac{d\mathbf{r}}{d\sigma}(\sigma(s)) + (...)\frac{d^2\mathbf{r}}{d\sigma^2}(\sigma(s)) + ... + \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \left(\frac{d\sigma}{ds}(s) \right)^n \frac{d^n\mathbf{r}}{d\sigma^n}(\sigma(s)),$$

where since we want to find the value of the determinant determined by

$$\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right),$$

the coefficients denoted by $(\dots)$ will not effect our calculation. Noticing that $\mathbf{r}(\sigma)$ and $\sigma(s)$ are smooth, then the above derivatives exist. Hence,

$$\left[\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right) \right] =$$
$$\left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \right)^n \left(\frac{d\sigma}{ds}(s) \right)^{(n^2+n)/2} \left[\frac{d\mathbf{r}}{d\sigma}(\sigma(s)), \frac{d^2\mathbf{r}}{d\sigma^2}(\sigma(s)), ..., \frac{d^n\mathbf{r}}{d\sigma^2}(\sigma(s)) \right].$$

Because σ is the standard equiaffine arclength parameter, the value of the determinant at the right hand side is 1, yielding

$$\left[\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right), ..., \frac{d^{n-1}}{ds^{n-1}}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right) \right] = \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \right)^n \left(\frac{d\sigma}{ds}(s) \right)^{(n^2+n)/2}.$$

Considering Equation (4.7) into the above last equation, we complete the proof.

Since we are interested in the 3−dimensional case, then Equation (4.5) is now

$$s(\sigma) = \left(\frac{\alpha+1}{2} \left(\frac{\alpha}{\Gamma(2-\alpha)} \right)^{1/2} \sigma \right)^{2/(\alpha+1)}. \tag{4.9}$$

Hence,

$$\sigma(s) = \left(\frac{\alpha}{\Gamma(2-\alpha)} \right)^{-1/2} \frac{2}{\alpha+1} s^{(\alpha+1)/2}$$

and

$$\frac{d\sigma}{ds}(s) = \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \right)^{-1/2}. \tag{4.10}$$

Definition 4.2. *Let $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, be a curve in $\mathbb{R}^3$ parametrized by equiaffine arclength of order $0 < \alpha \leq 1$. Then, the set $\left\{ \mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s) \right\}$ is called equiaffine Frenet frame of $\mathbf{r}(s)$ of order α, where*

$$\mathbf{T}^{\{\alpha\}}(s) = \frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s), \quad \mathbf{N}^{\{\alpha\}}(s) = \frac{d}{ds}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right), \quad \mathbf{B}^{\{\alpha\}}(s) = \frac{d^2}{ds^2}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}(s) \right).$$

Note that when $\alpha = 1$ the set $\left\{ \mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s) \right\}$ is equivalent to the standard equiaffine Frenet frame of $\mathbf{r}(s)$, that is, $\mathbf{T}^{\{1\}} = \mathbf{T}$, $\mathbf{N}^{\{1\}} = \mathbf{N}$, $\mathbf{B}^{\{1\}} = \mathbf{B}$.

By Proposition 4.1, we have

$$\left[\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s) \right] = 1. \tag{4.11}$$

Denote by a prime the ordinary derivative with respect to the parameter s, that is, $\mathbf{N}^{\{\alpha\}}(s) = \mathbf{T}^{\{\alpha\}\prime}(s)$ and $\mathbf{B}^{\{\alpha\}}(s) = \mathbf{N}^{\{\alpha\}\prime}(s)$. Then we differentiate Equation (4.11) with respect to s, obtaining

$$\left[\mathbf{T}^{\{\alpha\}}, \mathbf{N}^{\{\alpha\}}, \mathbf{B}^{\{\alpha\}\prime} \right] = 0,$$

where it can be seen that the set $\left\{ \mathbf{T}^{\{\alpha\}}, \mathbf{N}^{\{\alpha\}}, \mathbf{B}^{\{\alpha\}\prime} \right\}$ is linearly dependent for every $s \in (c,d)$. Then there are some smooth functions on (c,d) denoted by $\kappa^{\{\alpha\}}$ and $\tau^{\{\alpha\}}$ such that

$$\kappa^{\{\alpha\}}\mathbf{T}^{\{\alpha\}} + \tau^{\{\alpha\}}\mathbf{N}^{\{\alpha\}} + \mathbf{B}^{\{\alpha\}\prime} = 0.$$

Consequently, we can give the following.

Definition 4.3. *Let $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, be a curve in $\mathbb{R}^3$ parametrized by equiaffine arclength of order $0 < \alpha \leq 1$. Then the functions $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are called the equiaffine curvatures of $\mathbf{r}(s)$ of order α, where*

$$\kappa^{\{\alpha\}}(s) = -\left[\mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}\prime}(s) \right] \tag{4.12}$$

and

$$\tau^{\{\alpha\}}(s) = \left[\mathbf{T}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}\prime}(s)\right]. \tag{4.13}$$

With this definition, we have the *equiaffine Frenet equations of order* α given in matrix form

$$\begin{bmatrix} \mathbf{T}^{\{\alpha\}\prime}(s) \\ \mathbf{N}^{\{\alpha\}\prime}(s) \\ \mathbf{B}^{\{\alpha\}\prime}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\kappa^{\{\alpha\}}(s) & -\tau^{\{\alpha\}}(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\{\alpha\}}(s) \\ \mathbf{N}^{\{\alpha\}}(s) \\ \mathbf{B}^{\{\alpha\}}(s) \end{bmatrix}.$$

We occasionally use the terms of fractional equiaffine arclength, Frenet vector and curvature instead of the equiaffine arclength, Frenet vector and curvature of order α.

Proposition 4.2. *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *Denote by* $\left\{\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)\right\}$ *and* $\{\mathbf{T}(\sigma), \mathbf{N}(\sigma), \mathbf{B}(\sigma)\}$ *the equiaffine Frenet frames of* $\mathbf{r}(s)$. *Then we have*

$$\begin{bmatrix} \mathbf{T}^{\{\alpha\}}(s) \\ \mathbf{N}^{\{\alpha\}}(s) \\ \mathbf{B}^{\{\alpha\}}(s) \end{bmatrix} = \begin{bmatrix} \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} & 0 & 0 \\ \frac{1-\alpha}{2}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} & 1 & 0 \\ \frac{\alpha^2-1}{4}\left(\frac{\alpha s^{-3-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} & \frac{1-\alpha}{2}s^{-1} & \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1/2} \end{bmatrix} \begin{bmatrix} \mathbf{T}(\sigma(s)) \\ \mathbf{N}(\sigma(s)) \\ \mathbf{B}(\sigma(s)) \end{bmatrix},$$

where σ *is the standard equiaffine arclength parameter.*

Proof. Denote by σ the standard equiaffine parameter. By Equations (4.8) and (4.10), we write

$$\mathbf{T}^{\{\alpha\}}(s) = \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \mathbf{T}(\sigma(s)) \tag{4.14}$$

where $\mathbf{T}(\sigma(s)) = \frac{d\mathbf{r}}{d\sigma}(\sigma(s))$. Differentiating Equation (4.14) with respect to s,

$$\mathbf{N}^{\{\alpha\}}(s) = \frac{1-\alpha}{2}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \mathbf{T}(\sigma(s)) + \mathbf{N}(\sigma(s)) \tag{4.15}$$

and

$$\mathbf{B}^{\{\alpha\}}(s) = \frac{\alpha^2-1}{4}\left(\frac{\alpha s^{-3-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} \mathbf{T}(\sigma(s)) + \frac{1-\alpha}{2}s^{-1}\mathbf{N}(\sigma(s)) + \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1/2} \mathbf{B}(\sigma(s)). \tag{4.16}$$

The proof is completed by expressing Equations (4.14), (4.15) and (4.16) in matrix form.

Proposition 4.2 indicates the difference between the fractional and standard equiaffine Frenet vectors. Now, we give the relations between the fractional and standard equiaffine curvatures.

Theorem 4.1. *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *The equiaffine curvatures* $\kappa^{\{\alpha\}}(s)$ *and* $\tau^{\{\alpha\}}(s)$ *of order* α *are invariants under the equiaffine transformations of* $\mathbb{R}^3$. *Furthermore, if the standard equiaffine curvatures of* $\mathbf{r}(s)$ *are denoted by* $\kappa(\sigma)$ *and* $\tau(\sigma)$, *then the following relations occur*

$$\kappa^{\{\alpha\}}(s) = \frac{(3+\alpha)(-1+\alpha)}{4}s^{-3} + \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-3/2}\kappa(\sigma(s)) - \frac{(1-\alpha)\Gamma(2-\alpha)}{2\alpha}s^{\alpha-2}\tau(\sigma(s)) \tag{4.17}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{(3+\alpha)(1-\alpha)}{4}s^{-2} + \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}\tau(\sigma(s)). \tag{4.18}$$

Proof. Since $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ are defined by determinants (see Definition 4.3), those are invariant under the equiaffine transformations of $\mathbb{R}^3$. This is the proof of first part. Differentiating (4.16) with respect to s,

$$\frac{d(\mathbf{B}^{\{\alpha\}})}{ds}(s) = p(s)\mathbf{T}(\sigma(s)) + q(s)\mathbf{N}(\sigma(s)), \tag{4.19}$$

where

$$p(s) = \frac{(3+\alpha)(1-\alpha^2)}{8}\left(\frac{\alpha s^{-5-\alpha}}{\Gamma(2-\alpha)}\right)^{1/2} - \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}\kappa(\sigma(s))$$

and

$$q(s) = \frac{(3+\alpha)(-1+\alpha)}{4}s^{-2} - \left(\frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\right)^{-1}\tau(\sigma(s)).$$

If we consider Equations (4.15), (4.16) and (4.19) in Equation (4.12), after some manipulations, we derive Equation (4.17). Analogously, Equation (4.18) is obtained by substituting equations (4.14), (4.16) and (4.19) into (4.13). This completes the proof.

As consequences, we can state the following results.

Corollary 4.1. *Let* $\mathbf{r}(s)$, $s \in (c,d)$, $0 < c < d$, *be a curve in* $\mathbb{R}^3$ *parametrized by equiaffine arclength of order* $0 < \alpha \leq 1$. *If the equiaffine curvatures of* $\mathbf{r}(s)$ *vanish identically, then*

$$\kappa^{\{\alpha\}}(s) = \frac{(3+\alpha)(-1+\alpha)}{4}s^{-3}$$

and

$$\tau^{\{\alpha\}}(s) = \frac{(3+\alpha)(1-\alpha)}{4}s^{-2}.$$

Proof. It follows by Equations (4.17) and (4.18).

Corollary 4.2. *Let $\mathbf{r}(\sigma)$, $\sigma \in (a,b)$, $0<a<b$, be a curve in $\mathbb{R}^3$ parametrized by equiaffine arclength. If the equiaffine curvatures of $\mathbf{r}(\sigma)$ of order $0<\alpha\leq 1$ vanish identically, then*

$$\kappa(\sigma) = \frac{(3+\alpha)(1-\alpha)}{(1+\alpha)^2}\sigma^{-3} \tag{4.20}$$

and

$$\tau(\sigma) = -\frac{(3+\alpha)(1-\alpha)}{(1+\alpha)^2}\sigma^{-2}. \tag{4.21}$$

Proof. If $\tau^{\{\alpha\}}(s)=0$ for every s then from Equation (4.18) we have

$$\tau(\sigma(s)) = \frac{(3+\alpha)(-1+\alpha)}{4}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right). \tag{4.22}$$

Equation (4.21) is obtained by considering Equation (4.9) in Equation (4.22). Analogously, if $\kappa^{\{\alpha\}}=0$ for every s then Equation (4.17) is now

$$\kappa(\sigma(s)) = \frac{(3+\alpha)(1-\alpha)}{4}\left(\frac{\alpha s^{-1-\alpha}}{\Gamma(2-\alpha)}\right)^{3/2} + \frac{(1-\alpha)\Gamma(2-\alpha)}{2\alpha}\left(\frac{\alpha s^{-(1+\alpha)/3}}{\Gamma(2-\alpha)}\right)^{3/2}\tau(\sigma(s)). \tag{4.23}$$

Substituting Equations (4.9) and (4.21) into Equation (4.23), we derive Equation (4.20).

5. Examples

Example 5.1. *Consider in $\mathbb{R}^3$ the following curve (see Figure 1)*

$$\mathbf{r}(\sigma) = \left(\sigma, \frac{\sigma^2}{2}, \frac{\sigma^3}{3}\right), \quad \sigma \in (a,b), \quad 0<a<b,$$

where σ is the equiaffine arclength parameter of $\mathbf{r}(\sigma)$, that is, $[\mathbf{T}(\sigma),\mathbf{N}(\sigma),\mathbf{B}(\sigma)]=1$, for every $\sigma\in(a,b)$. Because $\mathbf{B}(\sigma)=(0,0,1)$, the equiaffine curvatures $\kappa(\sigma)$ and $\tau(\sigma)$ are identically 0. By Corollary 4.1, the equiaffine curvatures of the curve of order $0<\alpha\leq 1$ are $\kappa^{\{\alpha\}}(s)=(3+\alpha)(-1+\alpha)/(4s^3)$ and $\tau^{\{\alpha\}}(s)=(3+\alpha)(1-\alpha)/(4s^2)$, where s is the equiaffine arclength parameter of order α. The graphs of the curvature functions $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ can be drawn in Figures 2 and 3 up to different values of α.

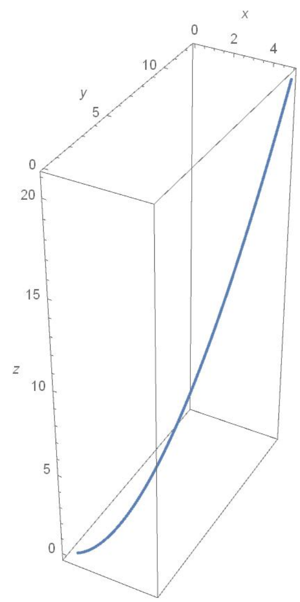

FIGURE 1. $\mathbf{r}(\sigma) = \left(\sigma, \frac{\sigma^2}{2}, \frac{\sigma^3}{3}\right)$, $\sigma \in [1/2, 5]$, with vanishing equiaffine curvatures.

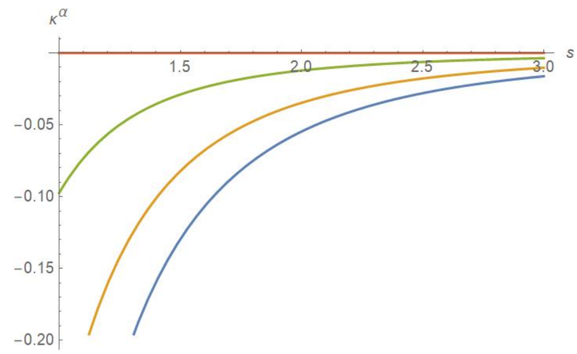

FIGURE 2. The graphs of $\kappa^{\{\alpha\}}(s) = (3+\alpha)(-1+\alpha)/(4s^3)$, $s \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

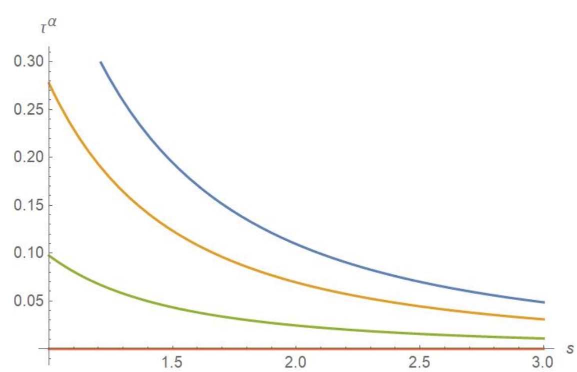

FIGURE 3. The graphs of $\tau^{\{\alpha\}}(s) = (3+\alpha)(1-\alpha)/(4s^2)$, $s \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

Example 5.2. *Let $0 < \alpha \leq 1$. We take in $\mathbb{R}^3$ the following curve (see Figure 4)*

$$\mathbf{r}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\frac{s^\alpha}{\alpha}, \frac{s^{\alpha+1}}{\alpha+1}, \frac{s^{\alpha+2}}{2(\alpha+2)}\right), \quad s \in (c,d), \quad 0 < c < d,$$

where s is the equiaffine arclength parameter of $\mathbf{r}(s)$ of order α, that is,

$$[\mathbf{T}^{\{\alpha\}}(s), \mathbf{N}^{\{\alpha\}}(s), \mathbf{B}^{\{\alpha\}}(s)] = 1,$$

for every $s \in (c,d)$. Because $\mathbf{B}^{\{\alpha\}}(s) = (0,0,1)$, the equiaffine curvatures $\kappa^{\{\alpha\}}(s)$ and $\tau^{\{\alpha\}}(s)$ of order α are identically 0. By Corollary 4.2, the standard equiaffine curvatures of $\mathbf{r}(s)$ are $\kappa(\sigma) = (3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-3}$ and $\tau(\sigma) = -(3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-2}$, where σ is the equiaffine arclength parameter (see Figures 5 and 6).

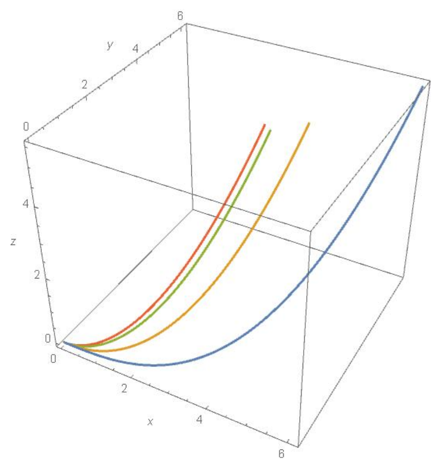

FIGURE 4. $\mathbf{r}(s) = \frac{\Gamma(2-\alpha)}{\alpha}\left(\frac{s^\alpha}{\alpha}, \frac{s^{\alpha+1}}{\alpha+1}, \frac{s^{\alpha+2}}{2(\alpha+2)}\right)$, $s \in [1/2, 5]$, with vanishing equiaffine curvatures of order $0 < \alpha \leq 1$. In blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

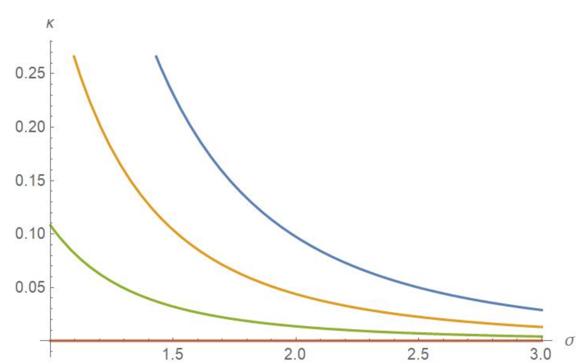

FIGURE 5. The graphs of $\kappa(\sigma) = (3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-3}$, $\sigma \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

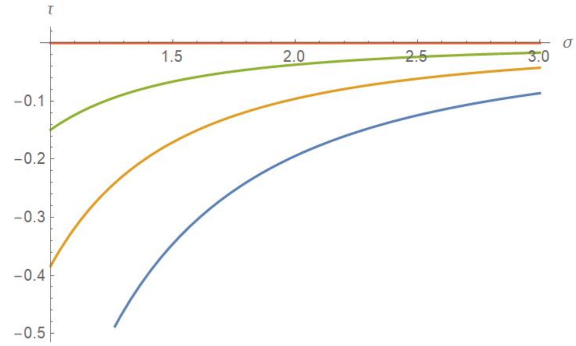

FIGURE 6. The graphs of $\tau(\sigma) = -(3+\alpha)(1-\alpha)(1+\alpha)^{-2}\sigma^{-2}$, $\sigma \in [1,3]$, in blue for $\alpha = 0.5$, in yellow for $\alpha = 0.7$, in green for $\alpha = 0.9$ and in red for $\alpha = 1$.

6. Discussions

The results of the present study may give new ideas relating to using of fractional derivative in the differential geometry of curves. For example, when emposing some natural conditions on curvatures, the classification of curves is a central problem. Or, the extension of results in 3-dimenisonal case to higher dimensions is an important problem again. Hence, the following two problems can be posed:

(1) The first one is the problem of finding parametric equations of curves when their fractional curvatures $\kappa^{\{\alpha\}}$ and $\tau^{\{\alpha\}}$ are constant. Indeed, solving this problem is equivalent to solve the following vector differential equation

$$\kappa_0^{\{\alpha\}}\mathbf{T}^{\{\alpha\}} + \tau_0^{\{\alpha\}}\mathbf{N}^{\{\alpha\}} + \mathbf{B}^{\{\alpha\}\prime} = 0, \tag{6.24}$$

where $\kappa_0^{\{\alpha\}}$ and $\tau_0^{\{\alpha\}}$ are some constants. As an example, we will find the equation of a curve that satisfies $\kappa_0^{\{\alpha\}}(s) = 0 = \tau_0^{\{\alpha\}}(s)$, for every s. Then Equation (6.24) is now $\mathbf{B}^{\{\alpha\}\prime} = 0$, or equivalently,

$$\frac{d^3}{ds^3}\left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}\right) = 0.$$

Integrating,

$$\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} = \mathbf{a} + \mathbf{b}s + \mathbf{c}\frac{s^2}{2}, \tag{6.25}$$

where $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. Since $\left[\mathbf{T}^{\{\alpha\}}, \mathbf{N}^{\{\alpha\}}, \mathbf{B}^{\{\alpha\}}\right] = 1$, we may choose that $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,1,0)$ and $\mathbf{c} = (0,0,1)$. Now if we consider Equation (4.4) into Equation (6.25) then we have

$$\frac{d\mathbf{r}}{ds} = \frac{\Gamma(2-\alpha)}{\alpha}\left(s^{-1+\alpha}, s^{\alpha}, s^{1+\alpha}\right).$$

After integrating the above last equation, up to a translation of $\mathbb{R}^3$, we find the parametrization of the curve that we are looking for. Consequently, the general solution of the posed problem can be obtained by following the similar steps.

(2) The second idea is to find the relations in higher dimensions between the fractional and standard equiaffine curvatures, that is, the analogous ones of equations (4.17) and (4.18). In particular, the main purpose of this problem is to express the relations into one equation between the fractional and standard equiaffine curvatures. For this, given a curve $\mathbf{r}(s)$ in $\mathbb{R}^n$ parametrized by equiaffine arclength of order α then the $i.-$th equiaffine curvature of order α can be defined by

$$\kappa_i^{\{\alpha\}} = (-1)^{n-i+1} \left[\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}}, ..., \frac{d^{i-2}}{ds^{i-2}} \left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} \right), \frac{d^{i}}{ds^{i}} \left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} \right), ..., \frac{d^{n}}{ds^{n}} \left(\frac{d^{\{\alpha\}}\mathbf{r}}{ds^{\{\alpha\}}} \right) \right],$$

where $i \in \{1, ..., n-1\}$. The problem proposes to establish a unique relation between $\kappa_i^{\{\alpha\}}$ and κ_i that holds for some $i \in \{1, ..., n-1\}$.

7. Conclusions

The simplification of Caputo fractional derivative given by Equation (1.1) effects the study of curves in terms of their equiaffine invariants in two ways. Given a curve $\mathbf{r}(s)$, then the first effect is obtaining a different equiaffine Frenet frame of $\mathbf{r}(s)$ from the standard one (Proposition 4.2). This situation is not valid for the Euclidean setting. The second effect can be seen on the fractional equiaffine curvatures (see Equations (4.17) and (4.18)) where the value of the terms containing the arclength s take a large value around an initial time and converges to zero for $s \to \infty$. See also Figures 2 and 3. This intention of the fractional equiaffine curvatures refers to the memory effect of fractional derivative which is decreasing for a long period of time ([23]).

As can be observed in the figures of Section 4, as α goes to 1 the geometric notions defined by using the derivative formula (1) approach to the standard ones. This implies that the idea proposed in the present study is consistent with the classical theory.

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Department of Mathematics, Faculty of Science, Firat University, Elazig, 23200, Turkey

Department of Mathematics, Faculty of Science, Firat University, Elazig, 23200, Turkey

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