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CERTAIN RESULTS OF RICCI SOLITONS ON (LCS) MANIFOLDS[‡]

MUNDALAMANE MANJAPPA PRAVEENA, MALLANNARA SIDDALINGAPPA SIDDESHA *, AND CHANNABASAPPA SHANTHAPPA BAGEWADI

Abstract. In the present paper, we study Ricci solitons of (LCS)-manifolds when quasi-conformal and pseudo projective curvature tensors of (LCS)-manifolds are irrotational and flat. It is revealed that the results obtained by the above methods and using semi-symmetry and Eisenhart problems are the same.

Keywords: (LCS)-manifolds, Ricci soliton, Einstein manifold.

2010 Mathematics Subject Classification: 53B30, 53C15.

1. Introduction

Riemannian geometry gives the study of Riemannian manifolds and Riemannian manifold is equipped with symmetric bilinear and positive definite metric. The Lorentzian manifold is a special case of pseudo-Riemannian manifold which is generalized Riemannian manifold and need not have positive metric tensor.

[‡]This paper is dedicated to late Prof. K. S. Amur.

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The study of Lorentzian concircular structure manifold shortly (LCS)-manifold initiated through Shaikh [15] and Baishya [16] in 2003 and 2005 generalize the idea of LP-Sasakian manifolds inaugurated through Matsumoto (1989) [8], Mihai and Rosca (1992) [9].

Ricci flow was initiated by Hamilton in 1982 and he observed that it's an attractive mathematical model for analyzing the fabrication of the manifold. This is a rule that defaces the metric of a Riemannian manifold in an approach similar to the dissemination of heat. This process is known as the geometrization conjecture of Thurston smoothing out irregularities in the metric. It gives an understanding of the geometry and topology of the manifold. Ricci soliton is self-similar to the Ricci flow(it is extension of Einstein metric) and it is denoted on (M, g) by

$$(L_\vartheta g)(U, W) + 2S(U, W) + 2\Upsilon g(U, W) = 0 \tag{1}$$

ϑ is a vector field created by $\{\phi_t\}_{t\in R}$ one parameter group of transformations, L_ϑ means the Lie derivative along ϑ, Υ is scalar. The Ricci soliton is shrinking: $\Upsilon < 0$, steady: $\Upsilon = 0$ and expanding: $\Upsilon > 0$.

During the current two decades, many mathematicians have investigated Ricci solitons of contact and Kähler manifolds [[13], [14]]. In particular, Praveena et. al. investigated [11, 12] a study on Ricci solitons in generalized complex space form. Hui et. al. [6, 4] and Blaga [3] have studied some classes of Ricci solitons in (LCS)-manifolds. The scholars Bagewadi et. al. [2] studied geometry of Ricci solitons in (LCS)-manifolds. Prompted by the earlier investigations in this article we investigate Ricci Solitons of (LCS)-manifolds when quasi-conformal and pseudo-projective curvature tensors in these manifolds are irrotational and flat. We also study compare our results with Ricci solitons of Eisenhart problem and semi symmetric.

2. PRELIMINARIES

A Lorentzian manifold M together with unit timelike concircular vector field ξ $(g(\xi, \xi) = -1)$, its associated 1-form η $(g(X, \xi) = \eta(X))$ and a $(1, 1)$ tensor field $\phi(take\ \ \phi U = \frac{1}{\alpha}\nabla_U \xi)$ is said to be a Lorentzian concircular structure manifold (briefly, (LCS)-manifold). Especially, if we take $\alpha = 1$ then we can obtain the LP-Sasakian structure of Matsumoto in (LCS)-manifold for $(n > 2)$. Moreover in (LCS)-manifold the following relations hold: [15, 16].

$$\phi = I + \eta \otimes \xi, \ \ \eta(\xi) = -1,$$

$$\phi\xi = 0, \ \ \eta \cdot \phi = 0, \ \ g(X, \phi Y) = g(\phi X, Y),$$

$$(\nabla_X \eta)(Y) = \alpha[g(X,Y) + \eta(X)\eta(Y)], \ \ \alpha \neq 0,$$

$$\nabla_X \xi = \alpha[X + \eta(X)\xi] \tag{2}$$

$$\nabla_X \alpha = X\alpha = d\alpha(X) = \rho\eta(X), \ \rho = -\xi\alpha = -\xi \cdot \nabla\alpha \tag{3}$$

$$g(\phi U, \phi V) = g(U,V) + \eta(U)\eta(V), \ \ g(U,\xi) = \eta(U),$$

$$R(U,V)\xi = (\alpha^2 - \rho)[\eta(V)U - \eta(U)V], \tag{4}$$

$$R(\xi,U)\xi = (\alpha^2 - \rho)[\eta(U)\xi + U], \tag{5}$$

$$R(\xi,U)V = (\alpha^2 - \rho)[g(U,V)\xi - \eta(V)U],$$

for $U, V \in T(M)$.

3. RICCI SOLITONS OF IRROTATIONAL QUASI-CONFORMAL CURVATURE TENSORS

Yano and Sawaki in 1968 [17] defined and studied a quasi-conformal curvature tensor field $\bar{Q}$ on M of dimension n which includes conformal, concircular and M-projective curvature tensors as specific cases. It is given by

$$\begin{aligned} \bar{Q}(V,U)W &= aR(V,U)W + b[S(U,W)V - S(V,W)U + g(U,W)QV - g(V,W)QU] \\ &\quad -\frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(U,W)V - g(V,W)U], \end{aligned} \tag{6}$$

where $S(V,W) = g(QV,W)$.
Using (2) in $(L_\xi g)(U,W)$ we produce

$$(L_\xi g)(U,W) = 2\alpha[g(U,W) - \eta(U)\eta(W)]. \tag{7}$$

((ξ, Υ, g) is a Ricci soliton in (LCS) manifold.)
Again using (7) and (2) we have

$$S(U,W) = -[(\alpha + \Upsilon)g(U,W) + \alpha\eta(U)\eta(W)]. \tag{8}$$

The preceding equating yields that

$$QU = -[(\alpha + \Upsilon)U + \alpha\eta(U)\xi], \tag{9}$$

$$\text{i.e.,} \qquad S(U,\xi) = -\Upsilon\eta(U), \tag{10}$$

$$r = -\Upsilon n - \alpha(n-1). \tag{11}$$

Put $W = \xi$ in (6) and using (4), (8) we have

$$\bar{Q}(V,U)\xi = A[\eta(U)V - \eta(V)U], \tag{12}$$

where $A = a(\alpha^2 - \rho) - b(2\Upsilon + \alpha) - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)$. The rotation (curl) of quasi-conformal curvature tensor $\bar{Q}$ on a Riemannian manifold is given by

$$\begin{aligned} Rot\ \bar{Q} = Curl\ \bar{Q} = \quad & (\nabla_X\bar{Q})(V,U,W) + (\nabla_V\bar{Q})(X,U,W) \\ & +(\nabla_U\bar{Q})(X,V,W) - (\nabla_W\bar{Q})(V,U,X). \end{aligned} \tag{13}$$

Under second Bianchi identity

$$(\nabla_X\bar{Q})(V,U,W) + (\nabla_V\bar{Q})(X,U,W) + (\nabla_U\bar{Q})(X,V,W) = 0. \tag{14}$$

Using (14) in reduces to

$$curl\ \bar{Q} = -(\nabla_W\bar{Q})(V,U,X).$$

If $\bar{Q}$ is irrotational then $curl\ \bar{Q} = 0$ and we should have

$$(\nabla_W\bar{Q})(V,U,X) = 0$$

$$\implies \nabla_W\{\bar{Q}(V,U)X\} = \bar{Q}(\nabla_W V,U)X + \bar{Q}(V,\nabla_W U)X + \bar{Q}(V,U)\nabla_W X. \tag{15}$$

Put $X = \xi$ in (15) and by virtue of (2), (3) and (12) we have

$$\bar{Q}(V,U)W = A[g(U,W)V - g(V,W)U]. \tag{16}$$

Exercising the inner product of (16) with X

$$\bar{Q}(V,U,W,X) = A[g(U,W)g(V,X) - g(V,W)g(U,X)]. \tag{17}$$

On contraction of the above equation (17) over V and X and using (6) we get

$$S(U,W) = \left[\frac{a(n-1)(\alpha^2-\rho) - b(n-1)(2\Upsilon+\alpha) - br}{a + b(n-2)}\right] g(U,W). \tag{18}$$

Put $U = W = \xi$ in (18) and using (10), (11) we get the value of Υ as

$$\Upsilon = -(n-1)(\alpha^2 - \rho). \tag{19}$$

We can consequently declare the following:

Theorem 1. *A Ricci soliton (g, ξ, Υ) in irrotational quasi-conformal (LCS) manifold is steady $\alpha^2 - \rho = 0$, shrinking $\alpha^2 - \rho < 0$ and expanding $\alpha^2 - \rho > 0$. i.e.,* $\alpha^2 + \xi\alpha = 0, \alpha^2 + \xi\alpha > 0, \alpha^2 + \xi\alpha < 0$.

Theorem 2. *If an (LCS)-manifold is quasi-conformally flat then it is η-Einstein provided* $b \neq 0$.

Proof. Suppose (LCS) is quasi-conformally flat then (6) becomes

$$aR(V,U)W + b[S(U,W)V - S(V,W)U + g(U,W)QV - g(V,W)QU]$$
$$- \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[g(U,W)V - g(V,W)U] = 0.$$

Put $V = W = \xi$ and using (5), (8) in preceding equation then we get

$$a[\eta(U)\xi + U](\alpha^2 - \rho) + b[(n-1)(\alpha^2 - \rho)\eta(U)\xi + (n-1)(\alpha^2 - \rho)U$$
$$+ \eta(U)Q\xi + QU] - \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[\eta(U)\xi + U] = 0.$$

Taking the inner product by X

$$a(\alpha^2 - \rho)[\eta(U)\eta(X) + g(U,X)] + b[(n-1)(\alpha^2 - \rho)\eta(U)\eta(X)$$
$$+ (n-1)(\alpha^2 - \rho)g(U,X) + \eta(U)S(X,\xi) + S(U,X)]$$
$$- \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[\eta(U)\eta(X) + g(U,X)] = 0.$$

$$\Longrightarrow a(\alpha^2 - \rho)[\eta(U)\eta(X) + g(U,X)] + b[(n-1)(\alpha^2 - \rho)\eta(U)\eta(X)$$
$$+ (n-1)(\alpha^2 - \rho)g(U,X) + (n-1)(\alpha^2 - \rho)\eta(Y)\eta(W) + S(U,X)]$$
$$- \frac{r}{n}\left(\frac{a}{n-1} + 2b\right)[\eta(U)\eta(X) + g(U,X)] = 0.$$
$$\Longrightarrow bS(U,X) = \left[\frac{r}{n}\left(\frac{a}{n-1} + 2b\right) - (\alpha^2 - \rho)(a + 2b(n-1))\right]\eta(U)\eta(X)$$
$$+ \left[\frac{r}{n}\left(\frac{a}{n-1} + 2b\right) - b(n-1)(\alpha^2 - \rho)\right]g(U,X). \tag{20}$$

$\therefore$ (LCS) manifold is η-Einstein provided $b \neq 0$. ∎

Let (g, ξ, Υ) be Ricci soliton then

$$(L_\xi g)(U, X) + 2S(U, X) + 2\Upsilon g(U, X) = 0.$$

$$\implies \alpha[\eta(U)\eta(X) + g(U, X)] + S(U, X) + \Upsilon g(U, X) = 0.$$

Replacing $U = X = \xi$ in preceding equation we get

$$S(\xi, \xi) + \Upsilon g(\xi, \xi) = 0 \implies \Upsilon = S(\xi, \xi) \implies b\Upsilon = bS(\xi, \xi).$$

Setting $U = X = \xi$ in (20) and equate to above we get

$$\begin{aligned} b\Upsilon = & \left[\frac{r}{n}\left(\frac{a}{n-1} + 2b\right) - (\alpha^2 - \rho)(a + 2b(n-1))\right] \qquad (21) \\ & + \left[\frac{r}{n}\left(\frac{a}{n-1} + 2b\right) - b(n-1)(\alpha^2 - \rho)\right](-1) \\ = & -(\alpha^2 - \rho)[a + 2b(n-1) - b(n-1)] \\ = & -(\alpha^2 - \rho)[a + b(n-1)]. \qquad (22) \end{aligned}$$

Hence Υ exists if $b = 0$. So we declare the following:

Theorem 3. *The Ricci soliton (g, ξ, Υ) in quasi-conformally flat (LCS)-manifold exists if $b \neq 0$.*

Remark 1. *(i) If $a = 1, b = -\frac{1}{n-2}$ then $\bar{Q}$ decreases to conformal curvature tensor. In this case $\Upsilon = -(\alpha^2 - \rho)$.*
(ii) If $a = 1, b = -\frac{1}{2(n-1)}$ then $\bar{Q}$ decreases to M-projective curvature tensor. In this case $\Upsilon = (n-1)(\alpha^2 - \rho)$.

4. RICCI SOLITONS IN IRROTATIONAL PSEUDO PROJECTIVE (LCS)-MANIFOLDS

Prasad in 2002 [10] defined and studied a pseudo projective curvature tensor field $\bar{P}$ on M of dimension n which includes projective curvature tensor as specific case. It is given by

$$\begin{aligned} \bar{P}(V, U)W = & \ aR(V, U)W + b[S(U, W)V - S(V, W)U] \\ & -\tfrac{r}{n}\left(\tfrac{a}{n-1} + b\right)[g(U, W)V - g(V, W)U]. \qquad (23) \end{aligned}$$

Put $W = \xi$ in (23) and using (4), (8) we have

$$\bar{P}(V, U)\xi = \theta[\eta(V)U - \eta(V)U], \qquad (24)$$

where $\theta = a(\alpha^2 - \rho) - \Upsilon b - \frac{r}{n}\left(\frac{a}{n-1} + b\right)$.

The rotation (curl) of pseudo projective curvature tensor $\bar{P}$ on a Riemannian manifold is given by

$$\begin{aligned} Rot\ \bar{P} = \quad & (\nabla_X \bar{P})(V,U,W) + (\nabla_V \bar{P})(X,U,W) \\ & +(\nabla_U \bar{P})(X,V,W) - (\nabla_W \bar{P})(V,U,X). \end{aligned} \tag{25}$$

Under second Bianchi identity we get

$$(\nabla_X \bar{P})(V,U,W) + (\nabla_V \bar{P})(X,U,W) + (\nabla_U \bar{P})(X,V,W) = 0, \tag{26}$$

using above in (25), it becomes

$$curl\ \bar{P} = -(\nabla_W \bar{P})(V,U,X).$$

If $\bar{P}$ is irrotational then $curl\ \bar{P} = 0$ and we obtain

$$\begin{aligned} (\nabla_W \bar{P})(V,U,X) =& 0. \\ \implies \nabla_W\{\bar{P}(V,U)X\} = \bar{P}(\nabla_W V,U)X + & \bar{P}(V,\nabla_W U)X + \bar{P}(V,U)\nabla_W X. \end{aligned} \tag{27}$$

Put $X = \xi$ in (27) and by virtue of (2), (3) and (24) we have

$$\bar{P}(V,U)W = \theta[g(U,W)V - (V,W)U]. \tag{28}$$

Taking inner product of (28) with W

$$\bar{P}(V,U,W,X) = \theta[g(U,W)g(V,X) - g(V,W)g(U,X)]. \tag{29}$$

On contraction of equation (29) over V and X, and using (6) we gain

$$S(U,W) = \left[\frac{(a(\alpha^2 - \rho) - b\Upsilon)(n-1)}{a + b(n-1)}\right] g(U,W). \tag{30}$$

Put $U = W = \xi$ in (30) and using (10), (11) we gain the value of Υ

$$\Upsilon = -(n-1)(\alpha^2 - \rho) = -(n-1)(\alpha^2 + \xi\alpha). \tag{31}$$

We can consequently say the following:

Theorem 4. *A Ricci soliton in irrotational pseudo projective (LCS)-manifold is steady, shrinking and expanding accordingly if*

$$\alpha^2 + \xi\alpha = 0, \alpha^2 + \xi\alpha > 0, \alpha^2 + \xi\alpha < 0$$

.

Theorem 5. *A pseudo projectively flat (LCS)-manifold is η-Einstein provided $b \neq 0$.*

Proof. Suppose (LCS) is pseudo projectively flat then (23) can write

$$aR(V,U)W + b[S(U,W)V - S(V,W)U - \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(U,W)V - g(V,W)U] = 0.$$

Put $V = \xi$, using and using (5), (8) in preceding equation then we gain

$$aR(\xi,U)W + b[S(U,W)\xi - S(\xi,W)U - \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(U,W)\xi - g(\xi,W)U] = 0.$$

$$\text{i.e., } a(\alpha^2-\rho)[g(U,W)\xi - \eta(W)U] + b[S(U,W)\xi - (n-1)(\alpha^2-\rho)\eta(W)U]$$
$$- \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(U,W)\xi - \eta(W)\eta(U)] = 0.$$

Taking the inner product ξ to precede equation then we obtained

$$\begin{aligned} &a(\alpha^2-\rho)[-g(U,W) - \eta(W)\eta(U)] + b[-S(U,W) - (n-1) \\ &\cdot(\alpha^2-\rho)\eta(W)\eta(U)] + \frac{r}{n}\left(\frac{a}{n-1}+b\right)[g(U,W) + \eta(W)\eta(U)] = 0. \\ \Longrightarrow bS(U,W) = &\left[\frac{r}{n}\left(\frac{a}{n-1}+b\right) - (\alpha^2-\rho)(a+(n-1)b)\right]\eta(U)\eta(W) \\ &+ \left[\frac{r}{n}\left(\frac{a}{n-1}+b\right) - a(\alpha^2-\rho)\right]g(U,W). \end{aligned} \tag{32}$$

Thus (LCS)-manifold is η-Einstein. ∎

Next let (ξ, Υ, g) be Ricci soliton then

$$(L_\xi g)(U,W) + 2S(U,W) + 2\Upsilon g(U,W) = 0.$$
$$\Longrightarrow \alpha[\eta(U)\eta(W) + g(U,W)] + S(U,W) + \Upsilon g(U,W) = 0.$$

Put $U = W = \xi$, then the above reduces to

$$S(\xi,\xi) - \Upsilon = 0 \quad \text{i.e.,} \quad \Upsilon = S(\xi,\xi). \tag{33}$$

From (32) and (33), $\Upsilon b = bS(\xi,\xi)$

$$\begin{aligned} \Upsilon b = &\left[\frac{r}{n}\left(\frac{a}{n-1}+b\right) - (\alpha^2-\rho)(a+(n-1)b)\right] \\ &+ \left[\frac{r}{n}\left(\frac{a}{n-1}+b\right) - a(\alpha^2-\rho)\right] \\ = &-(n-1)b(\alpha^2-\rho). \end{aligned}$$

Suppose $b \neq 0$ then

$$\Upsilon = -(n-1)(\alpha^2 - \rho). \tag{34}$$

Thus we have the following theorem:

Theorem 6. *A Ricci soliton (g, ξ, Υ) in pseudo projectively flat (LCS)-manifold exists if $b \neq 0$ and $\Upsilon = -(n-1)(\alpha^2 - \rho)$.*

Bagewadi et. al. [2], proved the following result:

Theorem 7. *If a Ricci soliton in (LCS)-manifold satisfying $R(\xi, X).\tilde{M}$ then $\Upsilon = -(n-1)(\alpha^2 - \rho)$.*

- *It is shrinking if characteristic vector field ξ is orthogonal to $\nabla\alpha$.*
- *It is shrinking of the angle between characteristic vector field ξ and the gradient vector field $\nabla\alpha$ is acute.*
- *It is shrinking if $\alpha^2 > k \mid \nabla\alpha \mid$, expanding if $\alpha^2 < k \mid \nabla\alpha \mid$, and steady if $\alpha^2 = k \mid \nabla\alpha \mid$.*

The value Υ in (19), (22), (31) and (34) is same as Υ in above theorem. Hence we conclude the following result:

Theorem 8. *If a Ricci soliton in (LCS)-manifold satisfies conditions such as irrotational quasi-conformal, quasi-conformally flat, irrotational pseudo projective and pseudo projective flat then $\Upsilon = -(n-1)(\alpha^2 - \rho)$. Further*

- *It is shrinking if characteristic vector field ξ is orthogonal to $\nabla\alpha$.*
- *It is shrinking of the angle between characteristic vector field ξ and the gradient vector field $\nabla\alpha$ is acute.*
- *It is shrinking if $\alpha^2 > k \mid \nabla\alpha \mid$, expanding if $\alpha^2 < k \mid \nabla\alpha \mid$, and steady if $\alpha^2 = k \mid \nabla\alpha \mid$.*

Shaikh et.al. [5], proved the following result i.e., Ricci solitons using the Eisenhart problem in (LCS)-manifolds.

Theorem 9. *Suppose that in (LCS)-manifold the $(0, 2)$ type tensor field $L_\vartheta g + 2S$ is parallel, where ϑ is a given vector field, then (g, ϑ) yields Ricci soliton and it is given by $\Upsilon = -(n-1)(\alpha^2 - \rho)$.*

Situated on the earlier all results we resolve that the value of $\Upsilon = -(n-1)(\alpha^2 - \rho)$ is same as Theorem (8) and Theorem (9).

5. CONCLUSION

The condition obtained for Ricci solitons of (LCS) manifold all the four methods: semi-symmetry, irrotational, flatness and Eisenhart problem is same i.e. $\Upsilon = -(n-1)(\alpha^2 - \rho)$. Hence the geometry of (LCS) manifold is same i.e. $\Upsilon = -(n-1)(\alpha^2 - \rho)$ in all these cases: semi-symmetry, irrotational, flatness and Eisenhart problem.

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ON f-BIHARMONIC AND BI-f-HARMONIC FRENET LEGENDRE CURVES

ŞERİFE NUR BOZDAĞ * AND FEYZA ESRA ERDOĞAN

ABSTRACT. This paper is devoted to study the f-harmonic, f-biharmonic, bi-f-harmonic, biminimal and f-biminimal Frenet Legendre curves in three dimensional normal almost paracontact metric manifolds and determine the necessary and sufficient conditions for these properties. Besides these, some characterizations for such curves have been defined in particular cases of a three dimensional normal almost paracontact metric manifold and some nonexistence theorems have been obtained.

Keywords: Frenet curves, Legendre curves, Normal almost paracontact metric manifolds, f-Harmonic curves, f-Biharmonic curves, Bi-f-Harmonic curves, f-Biminimal curves.

2010 Mathematics Subject Classification: 53D15, 53C25, 53C43, 58E20.

1. INTRODUCTION

The theory of curves is one of the most important topic in differential geometry and up to date from the past to the present. In the theory of curves there are many special types such as Frenet curves; slant curves, Legendre curves and these are studied in many different manifolds. In particular, Legendre curves have an important role in geometry and topology of almost contact manifolds. Among the papers on Legendre curves studied on contact manifolds in the literature, the most basic ones can be listed as [3, 19].

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On the other hand, studies on Frenet Legendre curves are newer. These studies, which are a source of motivation for us, can be briefly listed as [27, 23]. In this study, different from previous studies which are focused on curvature and torsion, we handled the maps, which briefly mentioned below, in terms of different cases of α, β and δ.

Harmonic maps which were defined by Sampson and Eells, in [8] have a wide field of study due to their wide applications such as physics, mathematics and engineering.

Besides, in [14], Jiang obtained biharmonic maps between the Riemannian manifolds by generalizing harmonic maps.

f-harmonic maps have a physical meaning as the solution of inhomogeneous Heisenberg spin systems and continuous spin systems, [4]. For this reason, the maps in question are of interest not only for mathematicians but also for physicists. f-harmonic maps between Riemannian manifolds were introduced by Lichnerowicz in 1970 and then examined by Eells and Lemaire in [9].

On the other hand, the strong relationship between f-harmonic and harmonic maps is summarized by Perktaş et.al. as follows, in [25]. The first one, extending bienergy functional to bi-f-energy functional and obtaining a new type of harmonic map called bi-f-harmonic map. The second one extending the f-energy functional to the f-bienergy functional and obtain another type of harmonic map called f-biharmonic map as critical points of f-bienergy functional, [30, 22].

f-biharmonic maps, which are the generalization of biharmonic maps, are defined by Lu, in [18]. Lu defined also f-biharmonic maps between Riemannian manifolds, in [6]. However, Ou gave complete classification of f-biharmonic curves in three dimensional Euclidean space and characterization of f-biharmonic curves in n-dimensional space forms, [21]. In addition, recent studies can be summarized as; [12, 1, 13, 16].

Moreover, bi-f-harmonic maps as a generalization of biharmonic and f-harmonic maps introduced by Ouakkas et. al., in [22]. In addition, Roth defined a non-f-harmonic, f-biharmonic map as a proper f-biharmonic map, [26]. It should be emphasized that there is no relationship between f-biharmonic and bi-f-harmonic maps.

Biminimal immersions and biminimal curves in a Riemannian manifold were defined by Loubeau and Montaldo, [17].

Finally, f-biminimal immersions were defined by Karaca and Özgür, in [11]. They considered f-biminimal curves in a Riemannian manifolds.

Based on these studies in this paper, first we give basic notions which will be needed in other sections. In section 3.1, we show that there is no f-harmonic Frenet Legendre curve in three dimensional normal almost paracontact metric manifold. In section 3.2, we get f-biharmonicity condition of a Frenet Legendre curve in three dimensional normal almost paracontact metric manifold and determine this condition in different cases such as β-para-Sasakian, α-para-Kenmotsu and paracosymplectic manifolds. In section 3.3, we obtain bi-f-harmonicity condition of a Frenet Legendre curve in three dimensional normal almost paracontact metric manifold and also discuss this condition in various manifolds. In section 3.4, we obtain biminimality condition of a Frenet Legendre curve in three dimensional normal almost paracontact metric manifold. Finally in section 3.5, we get f-biminimality conditions of Frenet Legendre curves in three dimensional normal almost paracontact metric manifold.

2. Preliminaries

This section, includes some definitions and propositions that will be required throughout the paper.

Definition 2.1. *Let (N,g) and $(\bar{N},\bar{g})$ be Riemannian manifolds, then a harmonic map $\phi:(N,g)\to(\bar{N},\bar{g})$ is defined as the critical point of the energy functional*

$$E(\phi)=\frac{1}{2}\int_N |d\phi|^2 dv_g,$$

where v_g is the volume element of (N,g). Then by using Euler-Lagrange equation $\tau(\phi)$ of the energy functional $E(\phi)$, where it is the tension field of map ϕ, a map called as harmonic if

$$\tau(\phi):=trace\nabla d\phi=0. \tag{2.1}$$

Here ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{N}}$ of $\bar{N}$ and the pull-back connection ∇^{ϕ}, [11].

Biharmonic maps, which can be considered as a natural generalization of harmonic maps, are defined as below.

Definition 2.2. *A map $\phi:(N,g)\to(\bar{N},\bar{g})$ is defined as a biharmonic map if it is a critical point, for all variations, of the bienergy functional*

$$E_2(\phi)=\frac{1}{2}\int_N |\tau(\phi)|^2 dv_g.$$

Then the Euler-Lagrange equation $\tau_2(\phi)$, for the bienergy functional $E_2(\phi)$, where $\tau_2(\phi)$ is the bitension field of map ϕ equals to

$$\tau_2(\phi) = trace(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{\nabla})\tau(\phi) - trace(R^{\bar{N}}(d\phi, \tau(\phi))d\phi) = 0, \tag{2.2}$$

if ϕ is a biharmonic map. Here $R^{\bar{N}}$, the curvature tensor field of $\bar{N}$, is defined as

$$R^{\bar{N}}(X,Y)Z = \nabla^{\bar{N}}_X\nabla^{\bar{N}}_Y Z - \nabla^{\bar{N}}_Y\nabla^{\bar{N}}_X Z - \nabla^{\bar{N}}_{[X,Y]}Z,$$

for any $X, Y, Z \in \Gamma(T\bar{N})$ and ∇^{ϕ} is the pull-back connection, [11].

One can easily see that harmonic maps are always biharmonic. Biharmonic maps which are not harmonic are called proper biharmonic maps, [24].

Definition 2.3. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is said to be an f-harmonic if it is critical point of f-energy functional,*

$$E_f(\phi) = \frac{1}{2}\int_N f|d\phi|^2 dv_g,$$

where $f \in C^{\infty}(N, \mathbb{R})$ is a positive smooth function. Then the f-harmonic map equation obtained by using Euler-Lagrange equation as follows;

$$\tau_f(\phi) = f\tau(\phi) + d\phi(gradf) = 0, \tag{2.3}$$

where $\tau_f(\phi)$ is the f-tension field of the map ϕ.
f-harmonic maps are generalizations of harmonic maps, [2, 7].

Definition 2.4. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is said to be an f-biharmonic if it is critical point of the f-bienergy functional*

$$E_{2,f}(\phi) = \frac{1}{2}\int_N f|\tau(\phi)|^2 dv_g.$$

The Euler-Lagrange equation for the f-biharmonic map is given by

$$\tau_{2,f}(\phi) = f\tau_2(\phi) + \Delta f\tau(\phi) + 2\nabla^{\phi}_{gradf}\tau(\phi) = 0, \tag{2.4}$$

where $\tau_{2,f}(\phi)$ is the f-bitension field of the map ϕ.
A f-biharmonic map turns into a biharmonic map if f is a constant, [6].

Definition 2.5. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is said to be a bi-f-harmonic if it is critical point of the bi-f-energy functional*

$$E_{f,2}(\phi) = \frac{1}{2}\int_N |\tau_f(\phi)|^2 dv_g.$$

The Euler-Lagrange equation for the bi-f-harmonic map is given by

$$\tau_{f,2}(\phi) = trace\big((\nabla^{\phi} f(\nabla^{\phi}\tau_f(\phi)) - f\nabla^{\phi}_{\nabla^N}\tau_f(\phi) + fR^{\bar{N}}(\tau_f(\phi), d\phi)d\phi\big) = 0, \tag{2.5}$$

where $\tau_{f,2}(\phi)$ is the bi-f-tension field of the map ϕ, [22].

Definition 2.6. *An immersion $\phi : (N, g) \to (\bar{N}, \bar{g})$ is called biminimal if it is critical point of the bienergy functional $E_2(\phi)$ for variations normal to the image $\phi(N) \subset \bar{N}$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that ϕ is a critical point of the λ-bienergy functional,*

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi).$$

The Euler-Lagrange equation for a λ- biminimal immersion is

$$[\tau_{2,\lambda}(\phi)]^{\perp} = [\tau_2(\phi)]^{\perp} - \lambda[\tau(\phi)]^{\perp} = 0, \tag{2.6}$$

for some value of $\lambda \in \mathbb{R}$, where $[.]^{\perp}$ denotes the normal component of $[.]$. An immersion is called free biminimal if it is biminimal for $\lambda = 0$, [11, 17].

Definition 2.7. *An immersion $\phi : (N, g) \to (\bar{N}, \bar{g})$ is called f-biminimal if it is a critical point of the f-bienergy functional $E_{2,f}(\phi)$ for variations normal to the image $\phi(N) \subset \bar{N}$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that ϕ is a critical point of the λ-f-bienergy functional,*

$$E_{2,\lambda,f}(\phi) = E_{2,f}(\phi) + \lambda E_f(\phi).$$

Using the Euler-Lagrange equations for f-harmonic and f-biharmonic maps, an immersion is f-biminimal if

$$[\tau_{2,\lambda,f}(\phi)]^{\perp} = [\tau_{2,f}(\phi)]^{\perp} - \lambda[\tau_f(\phi)]^{\perp} = 0, \tag{2.7}$$

for some value of $\lambda \in \mathbb{R}$. An immersion is called free f-biminimal if it is f-biminimal for $\lambda = 0$. If f is a constant then the immersion is biminimal, [11].

Definition 2.8. *A differentiable manifold N^{2n+1} is called almost paracontact metric manifold if it admits a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g satisfying the following conditions:*

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.8}$$

where $X, Y \in TN$ and I is the identity endomorphism on vector fields. g is called compatible metric and any compatible metric is necessarily of signature $(n+1, n)$. In an almost paracontact metric manifold N, $\eta \circ \varphi = 0$ and $rank(\varphi) = 2n$. From (2.8), $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$, for any $X, Y \in TN$. The fundamental 2-form of N is defined by $\Phi(X, Y) = g(X, \varphi Y)$. An almost paracontact metric manifold $(N, \varphi, \xi, \eta, g)$ is said to be normal if $\mathscr{N}(X, Y) - 2d\eta(X, Y)\xi = 0$, where $\mathscr{N}$ is the Nijenhuis torsion tensor of φ, [15, 29].

Proposition 2.1. [27] *For a three dimensional almost paracontact metric manifold N, the following conditions are mutually equivalent:*

i- *N is normal,*

ii- *there exist α, β functions on N such that*

$$(\nabla_X \varphi) Y = \alpha \left(g(\varphi X, Y) \xi - \eta(Y) \varphi X \right) + \beta \left(g(X, Y)\xi - \eta(Y) X \right), \tag{2.9}$$

iii- *there exist α, β functions on N such that*

$$\nabla_X \xi = \alpha \left(X - \eta(X)\xi \right) + \beta \varphi X. \tag{2.10}$$

Moreover, the functions α, β realizing (2.9) as well as (2.10) are given by

$$2\alpha = trace\{X \to \nabla_X \xi\}, \qquad 2\beta = trace\{X \to \varphi \nabla_X \xi\}.$$

For a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$, the curvature tensor field equation becomes

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2} + 2\left(\alpha^2+\beta^2\right)\right)\left(g(Y,Z)X - g(X,Z)Y\right) \\ &+ g(X,Z)\left(\frac{r}{2} + 3\left(\alpha^2+\beta^2\right)\right)\eta(Y)\xi \\ &- \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)\eta(Y)\eta(Z)X \\ &- g(Y,Z)\left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)\eta(X)\xi \\ &+ \left(\frac{r}{2} + 3(\alpha^2+\beta^2)\right)\eta(X)\eta(Z)Y, \end{aligned} \tag{2.11}$$

where $X, Y, Z \in TN$ and r is the scalar curvature, [24].

Definition 2.9. *A three dimensional normal almost paracontact metric manifold is called;*

. *β-para-Sasakian if $\alpha = 0$, $\beta \neq 0$ and β is constant,*

. *para-Sasakian if $\alpha = 0$, $\beta = -1$,*

. *quasi-para-Sasakian if $\alpha = 0$ and $\beta \neq 0$,*

. *α-para-Kenmotsu if $\alpha \neq 0$, $\beta = 0$ and α is constant,*

. *paracosymplectic if* $\alpha = \beta = 0$, [29].

Definition 2.10. *Let* $(N, \varphi, \xi, \eta, g)$ *be a three dimensional normal almost paracontact metric manifold where* $\alpha, \beta = constant$. *The structural function of the immersed curve* $\gamma : I \subset \mathbb{R} \to (N, g)$ *is the map* $c_\gamma : I \to \mathbb{R}$ *given by*

$$c_\gamma(s) = g(T(s), \xi) = \eta(T(s)),$$

where $T = \gamma'$. *Then the curve* γ *called as Legendre curve if* $c_\gamma = \eta(T(s)) = 0$, [5].

With the help of these definitions, we get f-tension field, f-bitension field, bi-f-tension field, the biminimality and f-biminimality conditions of a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold as in following sections.

3. FRENET LEGENDRE CURVES

Let $\gamma : I \longrightarrow N$ be a curve in a three dimensional pseudo-Riemannian manifold N such that $g(\gamma', \gamma') = \varepsilon_1$ where $\varepsilon_1 = \pm 1$ and $\nabla_{\gamma'}\gamma'$ denotes the covariant differentiation along γ. Then γ is a Frenet curve with $\{T, N, B\}$ Frenet Frame if one of the following three cases hold:

(1) γ is of osculating order 1, $\nabla_{\gamma'}\gamma' = 0$ (geodesics),

(2) γ is of osculating order 2, there exist two ortonormal vector fields T, N and a positive function κ along γ such that

$$\nabla_{\gamma'}T = \kappa\varepsilon_2 N, \quad \nabla_{\gamma'}N = -\kappa\varepsilon_1 T,$$

(3) γ is of osculating order 3, there exist three ortonormal vector fields T, N, B and two positive function κ and τ along γ such that

$$\nabla_{\gamma'}T = \kappa\varepsilon_2 N, \quad \nabla_{\gamma'}N = -\kappa\varepsilon_1 T + \tau\varepsilon_3 B, \quad \nabla_{\gamma'}B = -\tau\varepsilon_2 N,$$

where $T = \gamma', g(N, N) = \varepsilon_2 = \pm 1, g(B, B) = \varepsilon_3 = \pm 1$, κ is the curvature and τ is the torsion function, [27].

Note that in this paper, we study with $\gamma : I \subset \mathbb{R} \longrightarrow N$ non-null curve parametrized by arc length on a pseudo-Riemannian manifold N which is a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$. In this case, from Definition 2.1 and Definition 2.2, tension and bitension fields reduces to

$$\tau(\gamma) = \nabla_T T \tag{3.12}$$

and

$$\tau_2(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T = 0 \tag{3.13}$$

[20].

Now, let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in N and $\{T, \varphi T, \xi\}$ are ortonormal vector fields along γ where $\gamma' = T$. By differentiating $g(T, \xi) = 0$ along γ, it is obvious that $g(\nabla_T T, \xi) = -\varepsilon_1 \alpha$. Then $\nabla_T T$ obtained as below

$$\nabla_T T = -\varepsilon_1 \alpha \xi - \varepsilon_1 \delta \varphi T, \tag{3.14}$$

where δ is a function defined by $\delta = g(\nabla_T T, \varphi T)$, [27].

Let investigate the necessary and sufficient conditions of a Frenet Legendre curve to be f-harmonic, f-biharmonic, bi-f-harmonic, biminimal and f-biminimal in a three dimensional normal almost paracontact metric manifold in terms of different cases of α, β and δ.

It should be noted that; throughout our paper, for the sake of shortness, only N will be called instead of a three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$.

4. f-Harmonic Frenet Legendre Curves

In this subsection, we investigated the f-harmonicity condition of a Frenet Legendre curve in N.

Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in N. Then with the help of Definition 2.3 and equation (3.12), f-harmonicity condition obtained as below;

$$\tau_f(\gamma) = f\tau(\gamma) + d\gamma(gradf) = f\nabla_T T + f' T = 0. \tag{4.15}$$

Based on this result, we can express the following theorem:

Theorem 4.1. *There is no f-harmonic Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$.*

Proof. The f-harmonicity condition for this kind of curves obtained by substituting equation (3.14), in equation (4.15) as below;

$$\begin{aligned} \tau_f(\gamma) &= f\nabla_T T + f' T \\ &= f(-\varepsilon_1 \alpha \xi - \varepsilon_1 \delta \varphi T) + f' T \\ &= f' T - (\varepsilon_1 \alpha f)\xi - (\varepsilon_1 \delta f)\varphi T = 0. \end{aligned} \tag{4.16}$$

From equation (4.16); it is easy to see that $f^{'}=0$ namely, f is a constant function. This is a contradiction with the definition of f-harmonic curves.

5. f-Biharmonic Frenet Legendre Curves

In this section, we obtain the f-biharmonicity condition of a Frenet Legendre curve in N. In addition, we make detailed examinations for α-para-Kenmotsu, β-para-Sasakian and paracosymplectic manifolds.

First, let determine the f-biharmonicity condition for this kind of curves. By using tension and bitension field equations, f-bitension field $\tau_{2,f}(\gamma)$ obtained as below, [21];

$$\begin{aligned}\tau_{2,f}(\gamma) &= f\tau_2(\gamma)+(\Delta f)\tau(\gamma)+2\nabla^{\gamma}_{gradf}\tau(\gamma)\\ &= f(\nabla^3_T T-R(T,\nabla_T T)T)+f^{''}\nabla_T T+2f^{'}\nabla^2_T T=0. \end{aligned}\tag{5.17}$$

Then by differentiating $\nabla_T T=-\varepsilon_1\alpha\xi-\varepsilon_1\delta\varphi T$ with respect to T, we obtain $\nabla^2_T T$ and $\nabla^3_T T$ as below;

$$\nabla^2_T T=(\delta^2-\varepsilon_1\alpha^2)T-\varepsilon_1(\alpha\beta+\delta^{'})\varphi T-\delta\beta\xi \tag{5.18}$$

and

$$\nabla^3_T T=3\delta\delta^{'}T+(\alpha^2\delta-\delta\beta^2-\varepsilon_1\delta^3-\varepsilon_1\delta^{''})\varphi T+(\alpha^3-\alpha\beta^2-\varepsilon_1\alpha\delta^2-2\beta\delta^{'})\xi. \tag{5.19}$$

After that by substutiting $\nabla_T T$ into the curvature tensor field formula (2.11) we find,

$$R(T,\nabla_T T)T=-\alpha(\alpha^2+\beta^2)\xi+\delta(\frac{r}{2}+2(\alpha^2+\beta^2))\varphi T. \tag{5.20}$$

Finally, we determined the f-biharmonicity condition as below:

$$\begin{aligned}\tau_{2,f}(\gamma) &= f(\nabla^3_T T-R(T,\nabla_T T)T)+f^{''}\nabla_T T+2f^{'}\nabla^2_T T\\ &= (3\delta\delta^{'}f+2(\delta^2-\varepsilon_1\alpha^2)f^{'})T\\ &+ ((-\alpha^2\delta-3\beta^2\delta-\varepsilon_1\delta^3-\varepsilon_1\delta^{''}-\frac{r}{2}\delta)f-2\varepsilon_1(\alpha\beta+\delta^{'})f^{'}-\varepsilon_1\delta f^{''})\varphi T\\ &+ ((2\alpha^3-\varepsilon_1\alpha\delta^2-2\beta\delta^{'})f-2\delta\beta f^{'}-\varepsilon_1\alpha f^{''})\xi\\ &= 0. \end{aligned}\tag{5.21}$$

With the help of this result, we can state the following theorems:

Theorem 5.1. *Let $\gamma: I\longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional normal almost paracontact metric manifold N where α,β are constants.*

Then γ is an f-biharmonic Frenet Legendre curve iff the following equations hold:

$$\begin{cases} 3\delta\delta' f + 2(\delta^2 - \varepsilon_1\alpha^2)f' = 0, \\ (\alpha^2\delta + 3\beta^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' + \dfrac{r}{2}\delta)f + 2\varepsilon_1(\alpha\beta + \delta')f' + \varepsilon_1\delta f'' = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta')f - 2\delta\beta f' - \varepsilon_1\alpha f'' = 0. \end{cases} \tag{5.22}$$

Theorem 5.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional normal almost paracontact metric manifold N where α, β are constants. Then γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{4}} + c$$

and

$$r = -2\big[\alpha^2 + 3\beta^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta''}{\delta} + 3\frac{\varepsilon_1\delta'(\alpha\beta + \delta')}{\varepsilon_1\alpha^2 - \delta^2} + \frac{6(\delta')^2\alpha^2 + 6\delta\delta''\alpha^2 - 6\varepsilon_1\delta^3\delta'' + 15\varepsilon_1(\delta\delta')^2}{4(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where $2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta' - 2\delta\beta A - \varepsilon_1\alpha(A' + A^2) = 0$ *for* $A = \frac{3\delta\delta'}{2(\varepsilon_1\alpha^2 - \delta^2)}$ *and* $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Now, we give the interpretations of **Theorem 5.1**.

Case I : Assume that δ is not equal to a constant.

Case I-1: If N is a three dimensional β-para-Sasakian manifold and $\delta \neq constant$ then we have following equations from (5.22);

$$\begin{cases} 3\delta\delta' f + 2\delta^2 f' = 0, \\ (3\beta^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' + \frac{r}{2}\delta)f + 2\varepsilon_1\delta' f' + \varepsilon_1\delta f'' = 0, \\ \beta\delta' f + \delta\beta f' = 0. \end{cases} \tag{5.23}$$

Hence we obtain the following theorem;

Theorem 5.3. *There is no f-biharmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta \neq constant$.*

Proof. By solving the first and third equations of (5.23) together, it is easy to see that there is a contradiction between them.

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq \textit{constant}$ then we have following equations from (5.22);

$$\begin{cases} 3\delta\delta^{'} f + 2(\delta^2 - \varepsilon_1\alpha^2)f^{'} = 0, \\ (\alpha^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''} + \frac{r}{2}\delta)f + 2\varepsilon_1\delta^{'} f^{'} + \varepsilon_1\delta f^{''} = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2)f - \varepsilon_1\alpha f^{''} = 0. \end{cases}$$

So we have the following corollary;

Corollary 5.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional α-para-Kenmotsu manifold N with δ is not equal to a constant. Then γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{4}} + c$$

and

$$r = -2\big[\alpha^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta^{''}}{\delta} + 3\frac{\varepsilon_1\delta^{'}\delta^{'}}{\varepsilon_1\alpha^2 - \delta^2} + \frac{6(\delta^{'})^2\alpha^2 + 6\delta\delta^{''}\alpha^2 - 6\varepsilon_1\delta^3\delta^{''} + 15\varepsilon_1(\delta\delta^{'})^2}{4(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where $2\alpha^3 - \varepsilon_1\alpha\delta^2 - \varepsilon_1\alpha(A^{'} + A^2) = 0$ *for* $A = \frac{3\delta\delta^{'}}{2(\varepsilon_1\alpha^2 - \delta^2)}$ *and* $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Case I-3: If N is a three dimensional paracosymplectic manifold and $\delta \neq \textit{constant}$ then we have following equations from (5.22);

$$\begin{cases} 3\delta\delta^{'} f + 2\delta^2 f^{'} = 0, \\ (\varepsilon_1\delta^3 + \varepsilon_1\delta^{''} + \frac{r}{2}\delta)f + 2\varepsilon_1\delta^{'} f^{'} + \varepsilon_1\delta f^{''} = 0. \end{cases}$$

Therefore, we obtain the following corollary.

Corollary 5.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional paracosymplectic manifold N. Then for $\delta \neq constant$; γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the scalar curvature r equal to:*

$$f = \delta^{-\frac{3}{2}} + c$$

and

$$r = -2\varepsilon_1\big[\delta^2 + \delta^{-1}\delta^{''} - 3\delta^{-2}(\delta^{'})^2 + \frac{15}{4}\delta^{-2}\delta^{'} - \frac{3}{2}\delta^{''}\delta^{-1}\big].$$

Case II : Assume that $\delta=$ constant $\neq 0$. Then we investigate the following subcases:

Case II-1: If N is a three dimensional β-para-Sasakian manifold and $\delta=constant\neq 0$ then we have following equations from (5.22);

$$\begin{cases}\delta^2 f^{'}=0,\\ (3\beta^2+\varepsilon_1\delta^2+\frac{r}{2})f+\varepsilon_1 f^{''}=0,\\ \beta f^{'}=0.\end{cases} \tag{5.24}$$

Hence we obtain the following theorem;

Theorem 5.4. *There is no proper f-biharmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold with $\delta=constant\neq 0$.*

Proof. For $\delta=constant\neq 0$, from the first equation of (5.24) we obtain that $f^{'}=0$, this situation contradicts the definition of the f-biharmonic curve.

Case II-2: If N be a three dimensional α-para-Kenmotsu manifold and $\delta=constant\neq 0$ then we have following equations from (5.22);

$$\begin{cases}(\delta^2-\varepsilon_1\alpha^2)f^{'}=0,\\ (\alpha^2+\varepsilon_1\delta^2+\dfrac{r}{2})f+\varepsilon_1 f^{''}=0,\\ (2\alpha^2-\varepsilon_1\delta^2)f-\varepsilon_1 f^{''}=0.\end{cases}$$

So, we have;

Corollary 5.3. *Let $\gamma: I\longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional α-para-Kenmotsu manifold N. Then γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the constant scalar curvature r are given by*

$$f=c_1e^{\alpha s}+c_2e^{-\alpha s}$$

and

$$r=-6\alpha^2,$$

where $f\in C^\infty(N,\mathbb{R})$ is a positive smooth function dependent on s arc length parameter, $\delta=|\alpha|$ and $\varepsilon_1=1$.

Case II-3: Let N be a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then we have followings from (5.22);

$$\begin{cases} \delta^2 f' = 0, \\ (\varepsilon_1 \delta^2 + \frac{r}{2}) f + \varepsilon_1 f'' = 0. \end{cases} \tag{5.25}$$

Hence we have the following nonexistence theorem;

Theorem 5.5. *There is no proper f-biharmonic Frenet Legendre curve in a three dimensional paracosymplectic manifold where $\delta = constant \neq 0$.*

6. Bi-f-Harmonic Frenet Legendre Curves

In this subsection, we handle bi-f-harmonic Frenet Legendre curves in N. Also we obtained bi-f-harmonicity conditions for α-para-Kenmotsu, β-para-Sasakian and paracosymplectic manifolds.

First let determine the bi-f-harmonicity condition in a three dimensional normal almost paracontact metric manifold. By substutiting equations (3.14), (5.18), (5.19) and (5.20) into the bi-f-tension field formula, $\tau_{f,2}(\gamma)$ obtained as below, [25];

$$\begin{aligned} \tau_{f,2}(\gamma) &= trace(\nabla^{\gamma} f(\nabla^{\gamma}\tau_f(\gamma)) - f\nabla^{\gamma}_{\nabla^N}\tau_f(\gamma) + fR(\tau_f(\gamma), d\gamma)d\gamma) \\ &= (ff'')'T + (3ff'' + 2(f')^2)\nabla_T T + 4ff'\nabla^2_T T + f^2\nabla^3_T T + f^2 R(\nabla_T T, T)T \\ &= [(ff'')' + 4ff'(\delta^2 - \varepsilon_1\alpha^2) + 3f^2\delta\delta']T \\ &+ [-3\varepsilon_1\alpha ff'' - 2\varepsilon_1\alpha(f')^2 - 4ff'\delta\beta + f^2(2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta')]\varphi T \\ &+ [-3\varepsilon_1\delta ff'' - 2\varepsilon_1\delta(f')^2 - 4\varepsilon_1 ff'(\alpha\beta + \delta') \\ &\quad + f^2(-\frac{r}{2}\delta - \alpha^2\delta - 3\beta^2\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta''))]\xi \\ &= 0, \end{aligned} \tag{6.26}$$

which implies the following.

Theorem 6.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$.*

Then γ is a bi-f-harmonic curve iff the following equations hold:

$$\begin{cases} (ff^{''})^{'} + 4(\delta^2 - \varepsilon_1\alpha^2)ff^{'} + 3\delta\delta^{'}f^2 = 0, \\ \\ 3\varepsilon_1\alpha ff^{''} + 2\varepsilon_1\alpha(f^{'})^2 + 4\delta\beta ff^{'} - (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta^{'})f^2 = 0, \\ \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4(\alpha\beta + \delta^{'})ff^{'} + (\frac{r}{2}\varepsilon_1\delta + \alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \delta^{''})f^2 = 0. \end{cases} \tag{6.27}$$

Now, we give the interpretations of **Theorem 6.1**.

Case I : Assume that δ is not equal to constant. Then we investigate the following subcases:

Case I-1: If N a three dimensional β-para-Sasakian manifold and $\delta \neq$ *constant* then we have following equations from (6.27);

$$\begin{cases} (ff^{''})^{'} + 4\delta^2 ff^{'} + 3\delta\delta^{'}f^2 = 0, \\ \\ 2\delta f^{'} + \delta^{'}f = 0, \\ \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4\delta^{'}ff^{'} + (\frac{r}{2}\varepsilon_1\delta + 3\beta^2\varepsilon_1\delta + \delta^3 + \delta^{''})f^2 = 0. \end{cases}$$

Then we obtain the following corollary.

Corollary 6.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional β-para-Sasakian manifold N. Then γ is a bi-f-harmonic curve where the function f and the constant scalar curvature r are given by;*

$$f = \delta^{-\frac{1}{2}} + c$$

and

$$r = 3\varepsilon_1\delta^{-2}(\delta^{'})^2 + \varepsilon_1\delta^{-1}\delta^{''} - \frac{9}{2}\varepsilon_1\delta^{-2}\delta^{'} - 2\varepsilon_1\delta^2 - 6\beta^2,$$

for where $\delta \neq$ constant is the solution of $-9(\delta^{'})^3 + 10\delta\delta^{'}\delta^{''} - 2\delta^2\delta^{'''} + 4\delta^4\delta^{'} = 0$ differential equation.

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq$ *constant* then from (6.27), we obtain following equations;

$$\begin{cases} (ff^{''})^{'} + 4(\delta^2 - \varepsilon_1\alpha^2)ff^{'} + 3\delta\delta^{'}f^2 = 0, \\ \\ 3ff^{''} + 2(f^{'})^2 + f^2(\delta^2 - 2\alpha^2\varepsilon_1) = 0, \\ \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4\delta^{'}ff^{'} + (\frac{r}{2}\varepsilon_1\delta + \alpha^2\delta\varepsilon_1 + \delta^3 + \delta^{''})f^2 = 0. \end{cases} \tag{6.28}$$

So, we have the following corollary.

Corollary 6.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional α-para-Kenmotsu manifold N where $\delta \neq constant$. Then γ is a bi-f-harmonic curve iff f is a solution of the non-linear differential equations given in (6.28).*

Case I-3: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then from (6.27), we obtain the following equations;

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}\delta^2 + 3\delta\delta^{'}f^2 = 0, \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4\delta^{'}ff^{'} + (\frac{r}{2}\varepsilon_1\delta + \delta^3 + \delta^{''})f^2 = 0. \end{cases} \tag{6.29}$$

Hence we obtain following corollary.

Corollary 6.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional paracosymplectic manifold N where $\delta \neq constant$. Then γ is a bi-f-harmonic curve iff f is a solution of the non-linear differential equations given in (6.29).*

Case I-4: If N $ff^{''} = 0$ and $\delta \neq constant$ then via equation (6.27), we obtain following equations;

$$\begin{cases} 4f^{'}(\delta^2 - \varepsilon_1\alpha^2) + 3f\delta\delta^{'} = 0, \\ 2\varepsilon_1\alpha(f^{'})^2 + 4ff^{'}\delta\beta - f^2(2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta^{'}) = 0, \\ 2\delta(f^{'})^2 + 4ff^{'}(\alpha\beta + \delta^{'}) + f^2(\frac{r}{2}\delta\varepsilon_1 + \alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \delta^{''}) = 0. \end{cases} \tag{6.30}$$

We have the following corollary.

Corollary 6.4. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional normal almost paracontact metric manifold N where $ff^{''} = 0$ and $\delta \neq constant$. Then γ is a bi-f-harmonic Frenet Legendre curve where the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{8}} + c$$

and

$$r = -2\big[\alpha^2 + 3\beta^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta^{''}}{\delta} + 3\frac{(\alpha\beta + \delta^{'})\delta^{'}}{\varepsilon_1\alpha^2 - \delta^2} + \frac{9\delta^2(\delta^{'})^2}{8(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where $2\varepsilon_1\alpha A^2 + 4A\delta\beta - (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta^{'}) = 0$ *for* $A = \frac{3\delta\delta^{'}}{4(\varepsilon_1\alpha^2 - \delta^2)}$ *and* $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Case I-5: If N a three dimensional β-para-Sasakian manifold, $ff^{''}=0$ and $\delta \neq constant$ then from equation (6.30), we obtain following equations;

$$\begin{cases} 4f^{'}\delta + 3\delta^{'}f = 0, \\ 2f^{'}\delta + \delta^{'}f = 0, \\ 2\varepsilon_1\delta(f^{'})^2 + 4\varepsilon_1\delta^{'}ff^{'} + f^2(\frac{r}{2}\delta + 3\beta^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''}) = 0. \end{cases} \tag{6.31}$$

We have the following nonexistence theorem.

Theorem 6.2. *There is no bi-f-harmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $ff^{''}=0$ and $\delta \neq constant$.*

Proof. When first and the second equations of (6.31) solved together, we obtain $\delta^{'}f = 0$. For $\delta \neq constant$ and $\delta^{'}f = 0$; we get that $f = 0$ which is a contradiction to the definition of bi-f-harmonic curve.

Case I-6: If N a α-para-Kenmotsu manifold, $ff^{''}=0$ and $\delta \neq constant$ then from equation (6.30) we have following equations;

$$\begin{cases} 4f^{'}(\delta^2 - \varepsilon_1\alpha^2) + 3\delta\delta^{'}f = 0, \\ 2\varepsilon_1(f^{'})^2 - f^2(2\alpha^2 - \varepsilon_1\delta^2) = 0, \\ 2\varepsilon_1\delta(f^{'})^2 + 4\varepsilon_1ff^{'}\delta^{'} + f^2(\frac{r}{2}\delta + \alpha^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''}) = 0. \end{cases}$$

Then, we have the following corollary.

Corollary 6.5. *Let N be a α-para-Kenmotsu manifold where $ff^{''}=0$, $\delta \neq constant$ and $\gamma : I \longrightarrow N$ be a Frenet Legendre curve. Then γ is a bi-f-harmonic curve where the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{8}} + c$$

and

$$r = -2\big[\alpha^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta^{''}}{\delta} + 3\frac{(\delta^{'})^2}{\varepsilon_1\alpha^2 - \delta^2} + \frac{9\delta^2(\delta^{'})^2}{8(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where δ is the solution of $3\varepsilon_1\delta^2(\delta^{'})^2 - 2(2\alpha^2 - \varepsilon_1\delta^2)(\varepsilon_1\alpha^2 - \delta^2)^2 = 0$ differential equation and and $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Case I-7: If N is a paracosymplectic manifold, $ff^{''}=0$ and $\delta \neq constant$ then from (6.30), we obtain following equations;

$$\begin{cases} 4ff^{'}\delta^2+3f^2\delta\delta^{'}=0, \\ 2\varepsilon_1\delta(f^{'})^2+4\varepsilon_1 ff^{'}\delta^{'}+f^2(\frac{r}{2}\delta+\varepsilon_1\delta^3+\varepsilon_1\delta^{''})=0. \end{cases}$$

We have the following corollary.

Corollary 6.6. *Let N be a paracosymplectic manifold where $ff^{''}=0$, $\delta \neq constant$ and $\gamma : I \longrightarrow N$ be a Frenet Legendre curve. Then γ is a bi-f-harmonic curve where the function f and the scalar curvature r are given by;*

$$f=\delta^{-\frac{3}{4}}+c$$

and

$$r=-2\varepsilon_1\delta^2-2\varepsilon_1\frac{\delta^{''}}{\delta}+\frac{6\varepsilon_1\delta^{'}}{\delta^2}-\frac{9\varepsilon_1}{4\delta^2}.$$

Case II : Assume that $\delta = constant$ is not equal to 0. Then we shall investigate the following subcases:

Case II-1: If N a three dimensional β-para-Sasakian manifold then we have following equations from (6.27);

$$\begin{cases} (ff^{''})^{'}+4ff^{'}\delta^2=0, \\ ff^{'}\beta=0, \\ 3ff^{''}+2(f^{'})^2+f^2(\frac{r}{2}\varepsilon_1+3\beta^2\varepsilon_1+\delta^2)=0. \end{cases} \tag{6.32}$$

Hence, we give the following theorem;

Theorem 6.3. *There is no proper bi-f-harmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta = constant \neq 0$.*

Proof. From (6.32), the proof is obvious.

Case II-2: If N a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}(\delta^2 - \varepsilon_1\alpha^2) = 0, \\ 3ff^{''} + 2(f^{'})^2 - f^2(2\alpha^2\varepsilon_1 - \delta^2) = 0, \\ 3ff^{''} + 2(f^{'})^2 + f^2(\frac{r}{2}\varepsilon_1 + \alpha^2\varepsilon_1 + \delta^2) = 0. \end{cases}$$

So, we have the following corollary;

Corollary 6.7. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N. Then γ is a bi-f-harmonic curve where $\delta = constant \neq 0$, the constant scalar curvature equals to $r = -6\alpha^2$ and the function f is a solution of the non-linear differential equations given as;*

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}(\delta^2 - \varepsilon_1\alpha^2) = 0, \\ 3\alpha ff^{''} + 2\alpha(f^{'})^2 - f^2(2\alpha^3\varepsilon_1 - \alpha\delta^2) = 0. \end{cases}$$

Case II-3: If N a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then we obtain the following equations from (6.27);

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}\delta^2 = 0, \\ 3ff^{''} + 2(f^{'})^2 + f^2(\frac{r}{2}\varepsilon_1 + \delta^2) = 0. \end{cases} \tag{6.33}$$

Then we have,

Corollary 6.8. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a pracosymplectic manifold N. Then γ is a bi-f-harmonic curve where $\delta = constant \neq 0$, the scalar curvature r is given by;*

$$r = -6\varepsilon_1\frac{f^{''}}{f} - 4\varepsilon_1(\frac{f^{'}}{f})^2 - 2\varepsilon_1\delta^2$$

and the function f is a solution of the non-linear differential equations given in equation (6.33).

Case II-4: If N a three dimensional normal almost paracontact metric manifold, $ff^{''} = 0$ and $\delta = constant \neq 0$ then from (6.27), we obtain that γ is a bi-f-harmonic Frenet Legendre curve if and only if

$$\begin{cases} 4ff'(\delta^2-\varepsilon_1\alpha^2)=0, \\ 2\varepsilon_1\alpha(f')^2+4ff'\delta\beta-f^2(2\alpha^3-\varepsilon_1\alpha\delta^2)=0, \\ 2\varepsilon_1\delta(f')^2+4\varepsilon_1ff'\alpha\beta+f^2(\frac{r}{2}\delta+\alpha^2\delta+3\beta^2\delta+\varepsilon_1\delta^3)=0. \end{cases} \tag{6.34}$$

Hence we give,

Corollary 6.9. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in N where $\alpha, \beta = constant$, $ff'' = 0$ and $\delta = constant \neq 0$. Then γ is a bi-f-harmonic curve iff f is a solution of non-linear differential equations given in equation (6.34).*

Case II-5: If N a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} 4ff'\delta^2=0, \\ ff'\delta\beta=0, \\ \varepsilon_1\delta(f')^2+\frac{f^2}{2}(\frac{r}{2}\delta+3\beta^2\delta+\varepsilon_1\delta^3)=0. \end{cases} \tag{6.35}$$

So, we have the following nonexistence theorem.

Theorem 6.4. *There is no proper bi-f-harmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta = constant \neq 0$.*

Case II-6: If N a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} ff'(\delta^2-\varepsilon_1\alpha^2)=0, \\ 2\varepsilon_1(f')^2-f^2(2\alpha^2-\varepsilon_1\delta^2)=0, \\ 2\varepsilon_1(f')^2+f^2(\frac{r}{2}+\alpha^2+\varepsilon_1\delta^2)=0. \end{cases} \tag{6.36}$$

Corollary 6.10. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional α-para-Kenmotsu manifold N where $\delta = constant \neq 0$. Then γ is a proper bi-f-harmonic curve iff the scalar curvature equals to $r = -6\alpha^2$ and the function f is the solution of* $2(f')^2+ff'(\varepsilon_1\delta^2-\alpha^2)-f^2(2\varepsilon_1\alpha^2-\delta^2)=0$.

Case II-7: If N a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} 4ff^{'}\delta^2 = 0, \\ \varepsilon_1\delta(f^{'})^2 + \frac{f^2}{2}(\frac{r}{2}\delta + \varepsilon_1\delta^3) = 0. \end{cases} \tag{6.37}$$

Then we give

Theorem 6.5. *There is no bi-f-harmonic Frenet Legendre curve in a three dimensional paracosymplectic manifold where $\delta = constant \neq 0$.*

7. Biminimal Frenet Legendre Curves

In this section, the conditions for a Frenet curve to be biminimal are obtained in N. Besides, detailed calculations have been made for various manifolds as in the previous sections. By using normal components of tension and bitension fields, the condition of being biminimal curve is obtained by using the formula given as below, [11, 17];

$$[\tau_{2,\lambda}(\gamma)]^{\perp} = [\tau_2(\gamma)]^{\perp} - \lambda[\tau(\gamma)]^{\perp} = 0. \tag{7.38}$$

Let determine the biminimality condition for a Frenet Legendre curve in N. First, let give the tension and bitension fields respectively;

$$\tau(\gamma) = -\varepsilon_1\alpha\xi - \varepsilon_1\delta\varphi T,$$

$$\tau_2(\gamma) = 3\delta\delta^{'}T + (-3\beta^2\delta - \alpha^2\delta - \frac{r}{2}\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta^{''})\varphi T + (-2\beta\delta^{'} + 2\alpha^3 - \alpha\varepsilon_1\delta^2)\xi.$$

Hence by using normal components of tension and bitension fields the biminimality condition is obtained as below;

$$\begin{aligned} [\tau_{2,\lambda}(\gamma)]^{\perp} &= (-3\beta^2\delta - \alpha^2\delta - \frac{r}{2}\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta^{''} + \lambda\varepsilon_1\delta)\varphi T \\ &+ (-2\beta\delta^{'} + 2\alpha^3 - \alpha\varepsilon_1\delta^2 + \lambda\varepsilon_1\alpha)\xi \\ &= 0. \end{aligned} \tag{7.39}$$

By using this condition, we can give the following theorems;

Theorem 7.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$. Then γ is a biminimal curve* *iff the following equations hold:*

$$\begin{cases} 3\beta^2\delta + \alpha^2\delta + \frac{r}{2}\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' - \lambda\varepsilon_1\delta = 0, \\ \\ -2\beta\delta' + 2\alpha^3 - \alpha\varepsilon_1\delta^2 + \lambda\varepsilon_1\alpha = 0. \end{cases} \tag{7.40}$$

Theorem 7.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$. Then γ is a biminimal curve where the scalar curvature r is given by;*

$$r = -2\varepsilon_1\frac{\delta''}{\delta} - 4\frac{\beta}{\alpha}\delta' - 6\alpha^2 - 6\beta^2,$$

where δ is the solution of the second differential equation of (7.40).

Now, we give the interpretations of **Theorem 7.1**.

Case I: Assume that δ is not constant. Then we shall investigate the following subcases.

Case I-1: If N is a three dimensional β-para-Sasakian manifold and $\delta \neq constant$ then from (7.40), we obtain following equations;

$$\begin{cases} 3\beta^2\delta + \frac{r}{2}\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' - \lambda\varepsilon_1\delta = 0, \\ \\ 2\beta\delta' = 0. \end{cases} \tag{7.41}$$

Then we obtain the following nonexistence theorem.

Theorem 7.3. *There is no biminimal Frenet Legendre curve in a β-para-Sasakian manifold where $\delta \neq constant$.*

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq constant$ then from (7.40), we obtain following equations;

$$\begin{cases} -\alpha^2\delta - \frac{r}{2}\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta'' + \lambda\varepsilon_1\delta = 0, \\ \\ 2\alpha^3 - \alpha\varepsilon_1\delta^2 + \lambda\varepsilon_1\alpha = 0. \end{cases} \tag{7.42}$$

So we give,

Theorem 7.4. *There is no biminimal Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N where $\delta \neq constant$.*

Proof. From (7.42), we find that $\delta = \sqrt{2\varepsilon_1\alpha^2 + \lambda}$ but we accept $\delta \neq constant$ where $\alpha = constant$.

Case I-3: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then from (7.40), we obtain following equation;

$$\frac{r}{2}\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' - \lambda\varepsilon_1\delta = 0.$$

Hence we have,

Corollary 7.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional paracosymplectic manifold N and $\delta \neq constant$. Then γ is a biminimal curve iff the scalar curvature r is given by;*

$$r = -2\varepsilon_1\frac{\delta''}{\delta} - 2\varepsilon_1\delta^2 - 2\lambda\varepsilon_1.$$

Case II: Assume that δ=constant is not equal to 0. Then we shall investigate the following subcases:

Case II-1: If N is a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then from (7.40), we obtain following equation;

$$3\beta^2 + \frac{r}{2} + \varepsilon_1\delta^2 - \lambda\varepsilon_1 = 0.$$

Hence, we give the following theorem.

Corollary 7.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional β-para-Sasakian manifold N and $\delta = constant \neq 0$. Then γ is a biminimal curve where the constant scalar curvature r is given by;*

$$r = 2\varepsilon_1\delta^2 - 6\beta^2 + 2\lambda\varepsilon_1.$$

Case II-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then from (7.40), we obtain we obtain following equations;

$$\begin{cases} -\alpha^2 - \frac{r}{2} - \varepsilon_1\delta^2 + \lambda\varepsilon_1 = 0, \\ \\ 2\alpha^2 - \varepsilon_1\delta^2 + \lambda\varepsilon_1 = 0. \end{cases}$$

Then we obtain the following corollary.

Corollary 7.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N and $\delta = constant \neq 0$. Then γ is a biminimal curve where the constant scalar curvature r is given by;*

$$r = -6\alpha^2.$$

Case II-3: If N is a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then from (7.40), we obtain following equation;

$$\frac{r}{2} + \varepsilon_1\delta^2 - \lambda\varepsilon_1 = 0.$$

So we have,

Corollary 7.4. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional para-cosymplectic manifold N. Then γ is a biminimal curve where the constant scalar curvature r is given by;*

$$r = -2\varepsilon_1\delta^2 + 2\lambda\varepsilon_1.$$

8. f-Biminimal Frenet Legendre Curves

Finally in this section, we give f-biminimality conditions for a Frenet curve in N and also particular cases such as: β-para-Sasakian, α-para-Kenmotsu and paracosymplectic manifolds. From the Definition 2.7, we know that the condition of being f-biminimal curve given as below, [11];

$$[\tau_{2,\lambda,f}(\gamma)]^{\perp} = [\tau_{2,f}(\gamma)]^{\perp} - \lambda[\tau_f(\gamma)]^{\perp} = 0.$$

Then using the normal components of tension and bitension fields, given by (4.16) and (5.21), f-biminimality condition is obtained as below;

$$\begin{aligned} [\tau_{2,\lambda,f}(\gamma)]^{\perp} &= \big[(-\alpha^2\delta - 3\beta^2\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta'' - \frac{r}{2}\delta + \lambda\varepsilon_1\delta)f \\ &- 2\varepsilon_1(\alpha\beta + \delta')f' - \varepsilon_1\delta f''\big]\varphi T \\ &+ ((2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta' + \lambda\varepsilon_1\alpha)f - 2\delta\beta f' - \varepsilon_1\alpha f'')\xi \\ &= 0. \end{aligned} \tag{8.43}$$

Theorem 8.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$. Then γ is an f-biminimal curve* *iff the following equations hold:*

$$\begin{cases}(\alpha^2\delta\varepsilon_1+3\beta^2\delta\varepsilon_1+\delta^3+\delta^{''}+\frac{r}{2}\varepsilon_1\delta-\lambda\delta)f+2(\alpha\beta+\delta^{'})f^{'}+\delta f^{''}=0,\\ \\ (2\alpha^3-\varepsilon_1\alpha\delta^2-2\beta\delta^{'}+\lambda\varepsilon_1\alpha)f-2\delta\beta f^{'}-\varepsilon_1\alpha f^{''}=0.\end{cases} \tag{8.44}$$

Now, we give the interpretations of **Theorem 8.1**.

Case I: Assume that δ is not constant. Then we shall investigate the following subcases:

Case I-1: If N is a three dimensional β-para-Sasakian manifold and $\delta \neq constant$ then from (8.44), we obtain following equations;

$$\begin{cases}(3\beta^2\delta\varepsilon_1+\delta^3+\delta^{''}+\frac{r}{2}\varepsilon_1\delta-\lambda\delta)f+2\delta^{'}f^{'}+\delta f^{''}=0,\\ \\ \beta(\delta f)^{'}=0.\end{cases} \tag{8.45}$$

Corollary 8.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional β-para-Sasakian manifold N where $\delta \neq constant$. Then γ is an f-biminimal curve iff the function f and the scalar curvature r equals:*

$$f=\frac{1}{\delta}+c$$

and

$$r=2\varepsilon_1(\lambda-\delta^2-\frac{\delta^{''}}{\delta}-3\beta^2\varepsilon_1)-4\varepsilon_1(\frac{\delta^{'}}{\delta})^2-2\varepsilon_1\delta(2(\delta)^{'}\delta^{''}-\delta^{''}\delta^{-2}).$$

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq constant$ then from (8.44), we obtain following equations;

$$\begin{cases}(\alpha^2\delta\varepsilon_1+\delta^3+\delta^{''}+\frac{r}{2}\varepsilon_1\delta-\lambda\delta)f+2\delta^{'}f^{'}+\delta f^{''}=0,\\ \\ (2\alpha^3-\varepsilon_1\alpha\delta^2+\lambda\varepsilon_1\alpha)f-\varepsilon_1\alpha f^{''}=0.\end{cases} \tag{8.46}$$

Corollary 8.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional α-para-Kenmotsu manifold N and $\delta \neq constant$. Then γ is an f-biminimal curve iff f is a solution of non-linear differential equations given in (8.46).*

Case I-2: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then from (8.44), we obtain following equation;

$$(\varepsilon_1\delta^3+\varepsilon_1\delta^{''}+\frac{r}{2}\delta-\lambda\varepsilon_1\delta)f+2\varepsilon_1\delta^{'}f^{'}+\varepsilon_1\delta f^{''}=0. \tag{8.47}$$

Corollary 8.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional paracosymplectic manifold N and $\delta \neq constant$. Then γ is an f-biminimal curve iff f is a solution of non-linear differential equation given in (8.47).*

Case II: Assume that $\delta = constant$ is not equal to 0. Then we shall investigate the following subcases:

$$\begin{cases} (\alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \frac{r}{2}\varepsilon_1\delta - \lambda\delta)f + 2(\alpha\beta)f' + \delta f'' = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2 + \lambda\varepsilon_1\alpha)f - 2\delta\beta f' - \varepsilon_1\alpha f'' = 0. \end{cases} \tag{8.48}$$

Case II-1: If N is a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then from (8.44), we obtain following equations;

$$\begin{cases} (3\beta^2\delta\varepsilon_1 + \delta^3 + \frac{r}{2}\varepsilon_1\delta - \lambda\delta)f + \delta f'' = 0, \\ 2\delta\beta f' = 0. \end{cases} \tag{8.49}$$

Then we obtain the following nonexistence theorem;

Theorem 8.2. *There is no proper f-biminimal Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta = constant \neq 0$.*

Proof. From the second equation of (8.49), the proof is obvious.

Case II-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then from (8.44), we obtain following equations;

$$\begin{cases} (\alpha^2\varepsilon_1 + \delta^2 + \frac{r}{2}\varepsilon_1 - \lambda)f + f'' = 0, \\ (2\alpha^2\varepsilon_1 - \delta^2 + \lambda)f - f'' = 0. \end{cases} \tag{8.50}$$

Corollary 8.4. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N and $\delta = constant \neq 0$. Then γ is an f-biminimal curve where the constant scalar curvature equals to $r = -6\alpha^2$ and the function f is a solution of the non-linear differential equations given in (8.50).*

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CERTAIN RESULTS OF RICCI SOLITON ON 3-DIMENSIONAL LORENTZIAN PARA α-SASAKIAN MANIFOLDS

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Abstract. The paper deals with the study of almost Ricci (AR) soliton and gradient almost Ricci (GAR) soliton on 3-dimensional Lorentzian para α-Sasakian manifolds (α- LPS manifolds). Finally, we also provide an example of AR soliton.

Keywords: Lorentzian Para α-Sasakian manifold, Ricci soliton , Gradient Ricci soliton, Almost Ricci soliton, Gradient almost Ricci soliton.

2010 Mathematics Subject Classification: 53B30, 53C15, 53C25.

1. Introduction

As a generalization of an Einstein metric [6], Ricci soliton first defined in 1982 by Hamilton [19]. A pseudo-Riemannian manifold (M, g_*) defines a Ricci soliton with a smooth vector field V on M such that

$$\pounds_V g_* + 2S - 2\tau_1 g_* = 0, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative along the vector field V and S is the Ricci tensor on M and τ_1 is a real scalar. Ricci soliton is said to be shrinking $\tau_1 < 0$, steady $\tau_1 = 0$ or expanding $\tau_1 > 0$, [8]. A Ricci soliton is changed into Einstein equation with V zero or killing vector field.

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The study of almost Ricci soliton was presented by Pigola et al. [23], in this manner they gave new version of the definition of Ricci soliton by adding new condition on the parameter τ_1 to be a variable function, we say that a Riemannian manifold (M, g_*) admits an almost Ricci soliton, if there exists a complete vector field V, called potential vector field and a smooth soliton function $\tau_1 : M \to \mathbb{R}$ satisfying

$$S + \frac{1}{2}\pounds_V g_* = \tau_1 g_*, \tag{1.2}$$

where S and $\pounds$ represent Ricci tensor and Lie derivative along the direction of soliton vector field V. We shall now refer to this equation as the fundamental equation of an almost Ricci soliton (M, g_*, V, τ_1). Ricci soliton will be called shrinking, steady or expanding, respectively, if $\tau_1 < 0, \tau_1 = 0$ or $\tau_1 > 0$. For remaining it will be called indefinite. When the vector field V is gradient of a smooth function $f : M \to \mathbb{R}$ the metric will be called gradient almost Ricci soliton. So, we obtain

$$S + \bar{\nabla}^2 f = \tau_1 g_*, \tag{1.3}$$

where $\bar{\nabla}^2 f$ means for the Hessian of f.

Additionally, if the vector field X_1 is trivial, or the potential f is constant, the almost Ricci soliton is said to be trivial, otherwise it is said to be non-trivial almost Ricci soliton. We observe that when $n \geq 3$ and X_1 is a killing vector field almost Ricci solitons will be Ricci solitons. So in this situtation we have an Einstein manifold. The soliton function τ_1 is not necessarily constant, certainly comparison with soliton theory will be modified. In particular the rigidity result contained in Theorem 1.3 of [23] inform that almost Ricci solitons should reveal a reasonably broad generalization of the important concept of classical soliton.

The presence of Ricci almost soliton has been affirmed by Pigola et al. [23] on some specific class of warped product manifolds. Some characterization of Ricci almost soliton on Riemannian manifolds can be found in [1, 4, 5, 7, 18, 26]. It is important to note that if the potential vector field V of the Ricci almost soliton (M, g_*, V, τ_1) is Killing then the soliton becomes trivial, provided the dimension of $M > 2$. Additionally, if V is conformal then M is isometric to Euclidean sphere S^n. Thus the Ricci almost soliton is a generalization of Einstein metric as well as Ricci soliton.

In [15], authors studied Ricci solitons and gradient Ricci solitons geometric properties on 3-dimensional normal almost contact metric manifolds. In [16] authors studied compact Ricci soliton. In [17] author studied K-contact and Sasakian manifolds whose metric is gradient almost Ricci solitons. Conditions of K-contact and Sasakian manifolds are more stronger than almost normal contact metric manifolds in the sense of the 1-form of almost normal contact metric manifolds are not contact form. Ricci soliton as well as gradient Ricci soliton have been studied by many authors such as [2, 13, 14].

Sharma [24] obtained results on Ricci almost solitons in K-contact geometry, also in author [17] studied Ricci almost solitons and gradient Ricci almost solitons in (k, μ)-contact geometry and Majhi [22] on 3-dimensional f-Kenmotsu manifolds also De and Mandal [12] studied for structure (k, μ)-Paracontact geometry. Motivated by above studies in this paper, we are interested to study almost Ricci solitons and gradient Ricci almost solitons with Lorentzian para α-Sasakian manifolds.

We are studying the following sections: Section 2 contains important definitions and some preliminary results of Lorentzian para α-Sasakian (α- LPS) manifolds needed for the study. In section 3, we deal second order parallel symmetric tensors α- LPS manifolds. In section 4, we obtain result for almost Ricci (AR) soliton in 3-dimensional α-LPS manifolds. In the Section 5, we deduce theorem for such manifolds with gradient almost Ricci (GAR) solitons. Finally, we give an example of 3-dimensional (α- LPS)manifolds with almost Ricci soliton.

2. α- LPS manifolds

A differentiable manifold M of $(2n+1)$ dimensional is said to be an α- LPS manifolds, if it cosist a tensor field J of type $(1,1)$, a characteristic vector field ζ_1, a 1-form η_* and g_* as Lorentzian metric satisfy (see [10, 21]) :

$$J^2 X_1 = X_1 + \eta_*(X_1)\zeta_1, \tag{2.4}$$

$$\eta_*(\zeta_1) = -1, \ \eta_*(X_1) = g_*(X_1, \zeta_1), \tag{2.5}$$

$$J\zeta_1 = 0, \ \eta_* \circ J = 0, \tag{2.6}$$

$$g_*(JX_1, JY_1) = g_*(X_1, Y_1) + \eta_*(X_1)\eta_*(Y_1). \tag{2.7}$$

Definition 2.1. *A differentiable manifold M with an almost contact Lorentzian metric structure $(J, \zeta_1, \eta_*, g_*)$ is said to be an α-LS manifold if*

$$(\bar{\nabla}_{X_1} J)Y_1 = \alpha\{g_*(X_1, Y_1)\zeta_1 + \eta_*(Y_1)X_1\}, \tag{2.8}$$

where α is a constant function on M.

An almost contact metric structure is called a LPS manifold (or simply Lorentzian para-Sasakian manifold) if, (for details see [27, 11, 9])

$$(\bar{\nabla}_{X_1} J)Y_1 = g_*(X_1, Y_1)\zeta_1 + \eta_*(Y_1)X_1 + 2\eta_*(X_1)\eta_*(Y_1)\zeta_1, \tag{2.9}$$

where $\bar{\nabla}$ is the Levi-Civita connection with respect to g_*. Using above equation, one can obtain

$$\bar{\nabla}_{X_1}\zeta_1 = JX_1, \quad (\bar{\nabla}_{X_1}\eta_*)Y_1 = g_*(X_1, JY_1). \tag{2.10}$$

Definition 2.2. *A differentiable manifold M with an almost contact Lorentzian metric structure $(J, \zeta_1, \eta_*, g_*)$ is called an α-LPS manifold if*

$$(\bar{\nabla}_{X_1} J)Y_1 = \alpha\{g_*(X_1, Y_1)\zeta_1 + \eta_*(Y_1)X_1 + 2\eta_*(X_1)\eta_*(Y_1)\zeta_1\}, \tag{2.11}$$

where α is a smooth function on M.

Remark- Note that if $\alpha = 1$, then LPS manifold is the special case of α-LPS manifold. For an α-LPS manifold following relations are holds [3]:

$$\bar{\nabla}_{X_1}\zeta_1 = \alpha JX_1, \tag{2.12}$$

$$(\bar{\nabla}_{X_1}\eta_*)Y_1 = \alpha g_*(JX_1, Y_1), \tag{2.13}$$

$$\begin{aligned} R(X_1, Y_1)\zeta_1 &= \alpha^2\{\eta_*(Y_1)X_1 - \eta_*(X_1)Y_1\} \\ &\quad +\{(X_1\alpha)JY_1 - (Y_1\alpha)JX_1\}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} R(\zeta_1, Y_1)\zeta_1 &= \alpha^2\{Y_1 + \eta_*(Y_1)\zeta_1\} \\ &\quad +(\zeta_1\alpha)JY_1, \end{aligned} \tag{2.15}$$

$$R(\zeta_1, \zeta_1)\zeta_1 = 0, \tag{2.16}$$

$$\begin{aligned} R(\zeta_1, Y_1)X_1 &= \alpha^2\{g_*(X_1, Y_1)\zeta_1 - \eta_*(X_1)Y_1\} \\ &\quad -(X_1\alpha)JY_1 + g_*(JX_1, Y_1)(grad\alpha), \end{aligned} \tag{2.17}$$

$$S(Y_1,\zeta_1)=2n\alpha^2\eta_*(Y_1)-\{(Y_1\alpha)w+(JY_1)\alpha\}, \tag{2.18}$$

for any vector field Y_1 on M, $w=g_*(J(e_i),e_i)$ and S defines the Ricci curvature on M.

$$S(\zeta_1,\zeta_1)=-2n\alpha^2-(\zeta_1\alpha)w, \tag{2.19}$$

and

$$\begin{aligned}\eta_*(R(X_1,Y_1)Z_1 \quad &= \quad \alpha^2\{g_*(Y_1,Z_1)\eta_*(X_1)-g_*(X_1,Z_1)\eta_*(Y_1)\}\\ &\quad -\{(X_1\alpha)g_*(JY_1,Z_1)-(Y_1\alpha)g_*(X_1J,Z_1)\}.\end{aligned} \tag{2.20}$$

In a 3-dimensional Riemannian manifold, we always have

$$\begin{aligned}R(X_1,Y_1)Z_1 \quad &= \quad g_*(Y_1,Z_1)QX_1-g_*(X_1,Z_1)QY_1\\ &\quad +S(Y_1,Z_1)X_1-S(X_1,Z_1)Y_1\\ &\quad -\frac{r}{2}[g_*(Y_1,Z_1)X_1-g_*(X_1,Z_1)Y_1].\end{aligned} \tag{2.21}$$

In a 3-dimensional α-LPS manifold, we have

$$\begin{aligned}R(X_1,Y_1)Z_1 \quad &= \quad [\frac{r}{2}-\alpha^2][g_*(Y_1,Z_1)X_1-g_*(X_1,Z_1)Y_1]\\ &\quad +[\frac{r}{2}-3\alpha^2][g_*(Y_1,Z_1)\eta_*(X_1)\zeta_1\\ &\quad -g_*(X_1,Z_1)\eta_*(Y_1)\zeta_1+\eta_*(Y_1)\eta_*(Z_1)X_1\\ &\quad -\eta_*(X_1)\eta_*(Z_1)Y_1],\end{aligned} \tag{2.22}$$

and

$$\begin{aligned}S(X_1,Z_1) \quad &= \quad [\frac{r}{2}-\alpha^2]g_*(X_1,Z_1)\\ &\quad +[\frac{r}{2}-3\alpha^2]\eta_*(X_1)\eta_*(Y_1).\end{aligned} \tag{2.23}$$

Putting $Z_1=\zeta_1$ in (2.17), we have

$$R(X_1, Y_1)\zeta_1 = \eta_*(Y_1)QX_1 - \eta_*(X_1)QY_1 + S(Y_1, \zeta_1)X_1 - S(X_1, \zeta_1)Y_1 - \frac{r}{2}[\eta_*(Y_1)X_1 - \eta_*(X_1)Y_1], \tag{2.24}$$

and

$$S(X_1, \zeta_1) = 2\alpha^2\eta_*(X_1). \tag{2.25}$$

where Q is the Ricci operator define by $S(X_1, Y_1) = g_*(QX_1, Y_1)$.

Definition 2.3. *An α-LPS manifold M is called an Einstein like if its Ricci tensor S satisfies*

$$S(X_1, Y_1) = ag_*(X_1, Y_1) + bg_*(JX_1, Y_1) + c\eta_*(X_1)\eta_*(Y_1), \tag{2.26}$$

$X_1, Y_1 \in (M)$ for some real constants a, b and c.

3. Second order parallel symmetric tensors in an α-LPS manifold

Fix h a symmetric tensor field of $(0, 2)$-type which we suppose to be parallel with respect to $\bar\nabla$ that is $\bar\nabla h = 0$. Applying the Ricci identity [25]

$$\bar\nabla^2 h(X_1, Y_1; Z_1, W_1) - \bar\nabla^2 h(X_1, Y_1; W_1, Z_1) = 0, \tag{3.27}$$

we obtain the relation

$$h(R(X_1, Y_1)Z_1, W_1) + h(Z_1, R(X_1, Y_1)W_1) = 0. \tag{3.28}$$

Replacing $Z_1 = W_1 = \zeta_1$ in (3.2) and by using (2.11) and by the symmetry of h, we have

$$\alpha^2[\eta_*(Y_1)h(X_1, \zeta_1) - \eta_*(X_1)h(Y_1, \zeta_1)] + (X_1\alpha)h(JY_1, \zeta_1) - (Y_1\alpha)h(JX_1, \zeta_1) = 0. \tag{3.29}$$

Putting $X_1 = \zeta_1$ in (3.3) and by virtue of (2.2) and (2.3), we obtain

$$\alpha^2[\eta_*(Y_1)h(\zeta_1, \zeta_1) + h(Y_1, \zeta_1)] + (\zeta_1\alpha)h(JY_1, \zeta_1) = 0. \tag{3.30}$$

Replacing $Y_1 = JY_1$ in (3.4), we have

$$(\zeta_1\alpha)[\eta_*(Y_1)h(\zeta_1,\zeta_1)+h(Y_1,\zeta_1)]+\alpha^2h(JY_1,\zeta_1)=0. \tag{3.31}$$

Solving (3.4) and (3.5), we have

$$(\alpha^4-(\zeta_1\alpha)^2)[\eta_*(Y_1)h(\zeta_1,\zeta_1)+h(Y_1,\zeta_1)]=0. \tag{3.32}$$

Since $\alpha^4-(\zeta_1\alpha)^2\neq 0$, it results

$$h(Y_1,\zeta_1)=-\eta_*(Y_1)h(\zeta_1,\zeta_1), \tag{3.33}$$

from (3.7), we obtain

$$h(Y_1,\zeta_1)+g_*(Y_1,\zeta_1)h(\zeta_1,\zeta_1)=0. \tag{3.34}$$

Putting $Y_1=\bar{\nabla}_{X_1}Y_1$ in (3.7), we have

$$h(\bar{\nabla}_{X_1}Y_1,\zeta_1)+g_*(\bar{\nabla}_{X_1}Y_1,\zeta_1)h(\zeta_1,\zeta_1)=0. \tag{3.35}$$

Covariantly differentiating (3.7) with respect to X_1, we obtain

$$\begin{aligned} & (\bar{\nabla}_{X_1}h)(Y_1,\zeta_1)+h(\bar{\nabla}_{X_1}Y_1,\zeta_1)+h(Y_1,\bar{\nabla}_{X_1}\zeta_1) \\ = & -[g_*(\bar{\nabla}_{X_1}Y_1,\zeta_1)+g_*(Y_1,\bar{\nabla}_{X_1}\zeta_1)]h(\zeta_1,\zeta_1) \\ & -\eta_*(Y_1)[(\bar{\nabla}_{X_1}h)(\zeta_1,\zeta_1)+2h(\bar{\nabla}_{X_1}\zeta_1,\zeta_1)] \\ = & \ 0. \end{aligned} \tag{3.36}$$

Applying the parallel condition $\bar{\nabla}h=0$, $\eta_*(\bar{\nabla}_{X_1}\zeta_1)=0$ and using (2.9) and (3.6) in (3.9), we infer

$$\alpha[h(Y_1,JX_1)+g_*(Y_1,JX_1)h(\zeta_1,\zeta_1)]=0. \tag{3.37}$$

Replacing $X_1=JX_1$ in (3.11) and on simplification, we get

$$\alpha[h(X_1,Y_1)+g_*(X_1,Y_1)h(\zeta_1,\zeta_1)]=0, \tag{3.38}$$

since α is non-zero smooth function in an α-LPS manifold and this implies that

$$h(X_1,Y_1)=-g_*(X_1,Y_1)h(\zeta_1,\zeta_1), \tag{3.39}$$

which is together with the standard fact that the parallelism of h implies that $h(\zeta_1, \zeta_1)$ is a constant, via (3.6). Now using the above conditions, we can write the following:

Theorem 3.1. *A second order covariant symmetric parallel tensor in an α-LPS manifold is a constant multiple of the metric tensor.*

4. AR solitons on 3-dimensional α-LPS manifolds

This section deal with the characterization of AR solitons on 3-dimensional α-LPS manifolds. Consider the potential vector field V be pointwise collinear, $V = b\zeta_1$, where b is a function on M. Then from (1.1) we have

$$g_*(\bar{\nabla}_{X_1} b\zeta_1, Y_1) + g_*(\bar{\nabla}_{Y_1} b\zeta_1, X_1) + 2S(X_1, Y_1) = 2\tau_1 g_*(X_1, Y_1). \tag{4.40}$$

By virtue of (2.9) and (4.1), we have

$$\begin{aligned} & 2b\alpha g_*(JX_1, Y_1) + (X_1 b)\eta_*(Y_1) \\ & +(Y_1 b)\eta_*(X_1) + 2S(X_1, Y_1) \\ = \ & 2\tau_1 g_*(X_1, Y_1). \end{aligned} \tag{4.41}$$

Substituting $Y_1 = \zeta_1$ in (4.2) and using (2.21), we get

$$-(X_1 b) + (\zeta_1 b)\eta_*(X_1) + 4\alpha^2\eta_*(X_1) = 2\tau_1\eta_*(X_1). \tag{4.42}$$

Taking $X_1 = \zeta_1$ in (4.3), we infer

$$\zeta_1 b = \tau_1 - 2\alpha^2. \tag{4.43}$$

Substituting the value of $\zeta_1 b$ in (4.3), we have

$$db = (2\alpha^2 - \tau_1)\eta_*. \tag{4.44}$$

Operating d on (4.5) and using $d^2 = 0$, we obtain

$$0 = d^2 b = (2\alpha^2 - \tau_1)d\eta_*. \tag{4.45}$$

It follows from the above equation

$$\tau_1 = 2\alpha^2,$$

which implies $db = 0$, i.e., $b =$ constant, by virtue of $db = (2\alpha^2 - \tau_1)\eta_*$. Thus, using constancy of b in (4.2), we infer

$$\begin{aligned} S(X_1, Y_1) &= \tau_1 g_*(X_1, Y_1) - \alpha b g_*(JX_1, Y_1) \\ &\quad -2(2\alpha^2 - \tau_1)\eta_*(X_1)\eta_*(Y_1), \end{aligned} \tag{4.46}$$

which is of the form $S(X_1, Y_1) = a g_*(X_1, Y_1) + b g_*(JX_1, Y_1) + c\eta_*(X_1)\eta_*(Y_1)$. Hence, we can state the following result:

Theorem 4.1. *A 3-dimensional α-LPS manifold $(M, \zeta_1, \eta_*, g_*)$ with constant α admitting an AR soliton with pointwise collinear vector field V with the structure vector field ζ_1, is an Einstein like manifold provided $\tau_1 = 2\alpha^2 > 0$ i.e., expanding.*

Now let $V = \zeta_1$. Then (4.1) reduces to

$$(\pounds_{\zeta_1} g_*)(X_1, Y_1) + 2S(X_1, Y_1) = 2\tau_1 g_*(X_1, Y_1). \tag{4.47}$$

Now, by using (2.9) we have

$$\begin{aligned} (\pounds_{\zeta_1} g_*)(X_1, Y_1) &= g_*(\bar{\nabla}_{X_1}\zeta_1, Y_1) + g_*(\bar{\nabla}_{Y_1}\zeta_1, X_1) \\ &= 2\alpha g_*(JX_1, Y_1). \end{aligned} \tag{4.48}$$

Using (2.19), we get

$$\begin{aligned} (\pounds_{\zeta_1} g_*)(X_1, Y_1) &= -2[\left(\frac{r}{2} - \alpha^2\right) g_*(X_1, Y_1) \\ &\quad + \left(\frac{r}{2} - 3\alpha^2\right) \eta_*(X_1)\eta_*(Y_1)] \\ &\quad +2\tau_1 g_*(X_1, Y_1). \end{aligned} \tag{4.49}$$

In view of (4.9) and (4.10), we obtain

$$\begin{aligned} \alpha g_*(JX_1, Y_1) &= -[\left(\frac{r}{2} - \alpha^2\right) g_*(X_1, Y_1) \\ &\quad + \left(\frac{r}{2} - 3\alpha^2\right) \eta_*(X_1)\eta_*(Y_1)] \\ &\quad +\tau_1 g_*(X_1, Y_1). \end{aligned} \tag{4.50}$$

Taking $X_1 = Y_1 = \zeta_1$ in (4.11), we obtain

$$\tau_1 = 2\alpha^2. \tag{4.51}$$

Since α is constant. This implies $\tau_1 = 2\alpha^2$ =constant. Hence, we can establish the following result.

Theorem 4.2. *A 3-dimensional α-LPS manifold $(M, \zeta_1, \eta_*, g_*)$ admits AR soliton then it reduces to a Ricci soliton for α =constant.*

5. Gradient Almost Ricci (GAR) Solitons

In this part, we study 3-dimensional α-LPS manifolds admitting GAR soliton. For a GAR soliton, we have

$$\bar{\nabla}_{Y_1} Df = \tau_1 Y_1 - QY_1, \tag{5.52}$$

where D symbolize the gradient operator of g_*.

Now taking covariant differentiation of (5.1) along arbitrary vector field X_1, we have

$$\bar{\nabla}_{X_1}\bar{\nabla}_{Y_1} Df = d\tau_1(X_1)Y_1 + \tau_1\bar{\nabla}_{X_1}Y_1 - (\bar{\nabla}_{X_1}Q)Y_1. \tag{5.53}$$

In above equation d is exterior derivative, using this similarly we obtain

$$\bar{\nabla}_{Y_1}\bar{\nabla}_{X_1} Df = d\tau_1(Y_1)X_1 + \tau_1\bar{\nabla}_{Y_1}X_1 - (\bar{\nabla}_{Y_1}Q)X_1, \tag{5.54}$$

and

$$\bar{\nabla}_{[X_1,Y_1]} Df = \tau_1[X_1, Y_1] - Q[X_1, Y_1]. \tag{5.55}$$

In view of $(5.2), (5.3)$ and (5.4), we get

$$\begin{aligned} R(X_1, Y_1)Df &= \bar{\nabla}_{X_1}\bar{\nabla}_{Y_1} Df - \bar{\nabla}_{Y_1}\bar{\nabla}_{X_1} Df - \bar{\nabla}_{[X_1,Y_1]} Df \\ &= (\bar{\nabla}_{Y_1}Q)X_1 - (\bar{\nabla}_{X_1}Q)Y_1 - (Y_1\tau_1)X_1 + (X_1\tau_1)Y_1. \end{aligned} \tag{5.56}$$

From (2.19), we have

$$QX_1 = [\frac{r}{2} - \alpha^2]X_1 + [\frac{r}{2} - 3\alpha^2]\eta_*(X_1)\zeta_1. \tag{5.57}$$

Taking covariant differentiation of (5.6) along arbitrary vector field X_1 and using (2.9), we have

$$\begin{aligned}(\bar{\nabla}_{X_1}Q)Y_1 &= \left(\frac{X_1 r}{2}\right)[Y_1+\eta_*(Y_1)\zeta_1] \\ &+\alpha\left(\frac{r}{2}-3\alpha^2\right)[g_*(JX_1,Y_1)+\eta_*(Y_1)JX_1]. \end{aligned} \tag{5.58}$$

Similarly, we have

$$\begin{aligned}(\bar{\nabla}_{Y_1}Q)X_1 &= \left(\frac{Y_1 r}{2}\right)[X_1+\eta_*(X_1)\zeta_1] \\ &+\alpha\left(\frac{r}{2}-3\alpha^2\right)[g_*(JY_1,X_1)+\eta_*(X_1)JY_1]. \end{aligned} \tag{5.59}$$

Using (5.7) and (5.8) in (5.5), we have

$$\begin{aligned}R(X_1,Y_1)Df &= \left(\frac{Y_1 r}{2}\right)[X_1+\eta_*(X_1)\zeta_1]+\alpha\left(\frac{r}{2}-3\alpha^2\right)\eta_*(X_1)JY_1 \\ &-\left(\frac{X_1 r}{2}\right)[Y_1+\eta_*(Y_1)\zeta_1]-\alpha\left(\frac{r}{2}-3\alpha^2\right)\eta_*(Y_1)JX_1 \\ &-(Y_1\tau_1)X_1+(X_1\tau_1)Y_1. \end{aligned} \tag{5.60}$$

Taking an inner product with ζ_1 in above equation, then we obtain

$$g_*(R(X_1,Y_1)Df,\zeta_1) = -(Y_1\tau_1)\eta_*(X_1)+(X_1\tau_1)\eta_*(Y_1). \tag{5.61}$$

Taking $Y_1=\zeta_1$, then we infer

$$g_*(R(X_1,\zeta_1)Df,\zeta_1) = -(\zeta_1\tau_1)\eta_*(X_1)-(X_1\tau_1). \tag{5.62}$$

Also from (2.18), it follows that

$$g_*(R(X_1,\zeta_1)Df,\zeta_1) = \alpha^2[(\zeta_1 f)\eta_*(X_1)-(X_1 f)]. \tag{5.63}$$

Using (5.9) in (5.10), we get

$$\alpha^2[(\zeta_1 f)\eta_*(X_1)-(X_1 f)] = -(\zeta_1\tau_1)\eta_*(X_1)-(X_1\tau_1). \tag{5.64}$$

Assuming that f is constant. Then it follows from (5.11) that

$$d\tau_1 + (\zeta_1\tau_1)\eta_* = 0. \tag{5.65}$$

Applying d both sides of (5.14), we obtain

$$\zeta_1\tau_1 = 0. \tag{5.66}$$

By virtue of (5.14) and (5.15), we get

$$d\tau_1 = 0. \tag{5.67}$$

This implies τ_1 is constant. Hence, we can establish the following result:

Theorem 5.1. *A 3-dimensional α-LPS manifold $(M, \zeta_1, \eta_*, g_*)$ admits a GAR soliton then it reduces to a Ricci soliton provided f is constant.*

6. Example

We consider the 3-dimensional manifold $M = \{(x, y, t) \in R^3 : t \neq 0\}$, where (x, y, t) are the standard coordinates in R^3. We choose the vector fields

$$\tilde{E}_1 = e_*^t \frac{\partial}{\partial y}, \tilde{E}_2 = e_*^t (\frac{\partial}{\partial x} + \frac{\partial}{\partial y}) \text{ and } \tilde{E}_3 = e_*^t \frac{\partial}{\partial t},$$

which are linearly independent at each point of M. Let g_* be the Lorentzian metric defined by

$$g_*(\tilde{E}_1, \tilde{E}_2) = g_*(\tilde{E}_2, \tilde{E}_3) = g_*(\tilde{E}_3, \tilde{E}_1) = 0,$$

$$g_*(\tilde{E}_1, \tilde{E}_1) = g_*(\tilde{E}_2, \tilde{E}_2) = 1, \; g_*(\tilde{E}_3, \tilde{E}_3) = -1.$$

Let η_* be the 1- form defined by $\eta_*(Z_1) = g_*(Z_1, \tilde{E}_3)$ for any vector field Z_1 on M. We define the $(1,1)$ tensor field J as $J(\tilde{E}_1) = -\tilde{E}_1, J(\tilde{E}_2) = -\tilde{E}_2$ and $J(\tilde{E}_3) = 0$. Then using the linearity of J and g_*, we have

$$\eta_*(\tilde{E}_3) = -1, J^2 Z_1 = Z_1 + \eta_*(Z_1)\tilde{E}_3,$$

$$g_*(JZ_1, JW_1) = g_*(Z_1, W_1) + \eta_*(Z_1)\eta_*(W_1),$$

for any vector fields Z_1, W_1 on M. Thus for $\tilde{E}_3 = \zeta_1$, the structure $(J, \zeta_1, \eta_*, g_*)$ defines an almost contact metric structure on M.

Let $\bar{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric g_*. Then, we have

$$[\tilde{E}_1, \tilde{E}_2] = 0, \quad [\tilde{E}_1, \tilde{E}_3] = -e_*^t \tilde{E}_1 \text{ and } \quad [\tilde{E}_2, \tilde{E}_3] = -e_*^t \tilde{E}_2.$$

Koszul's formula is defined by

$$\begin{aligned} 2g_*(\bar{\nabla}_{X_1} Y_1, Z_1) \quad &= \quad X_1 g_*(Y_1, Z_1) + Y_1 g_*(Z_1, X_1) - Z_1 g_*(X_1, Y_1) \\ &\quad -g_*(X_1, [Y_1, Z_1]) - g_*(Y_1, [X_1, Z_1]) + g_*(Z_1, [X_1, Y_1]). \end{aligned}$$

Using Koszul's formula, we can easily calculate

$$\bar{\nabla}_{\tilde{E}_1} \tilde{E}_3 = -e_*^t \tilde{E}_1, \quad \bar{\nabla}_{\tilde{E}_1} \tilde{E}_2 = 0, \quad \bar{\nabla}_{\tilde{E}_1} \tilde{E}_1 = -e_*^t \tilde{E}_3,$$

$$\bar{\nabla}_{\tilde{E}_2} \tilde{E}_3 = -e_*^t \tilde{E}_2, \quad \bar{\nabla}_{\tilde{E}_2} \tilde{E}_2 = -e_*^t \tilde{E}_3, \quad \bar{\nabla}_{\tilde{E}_2} \tilde{E}_1 = 0,$$

$$\bar{\nabla}_{\tilde{E}_3} \tilde{E}_3 = 0, \quad \bar{\nabla}_{\tilde{E}_3} \tilde{E}_2 = 0, \quad \bar{\nabla}_{\tilde{E}_3 \tilde{E}_1 = 0}.$$

From the above, it follows that the manifold satisfies

$$(\bar{\nabla}_{X_1} J) Y_1 = \alpha \{ g_*(X_1, Y_1) \zeta_1 + \eta_*(Y_1) X_1 + 2\eta_*(X_1) \eta_*(Y_1) \zeta_1 \},$$

for $\tilde{E}_3 = \zeta_1$. and $\alpha = e_*^t$, $(J, \zeta_1, \eta_*, g_*)$ is a 3-dimensional α-LPS structure on M. Consequently $M^3(J, \zeta_1, \eta_*, g_*)$ is a 3-dimensional α-LPS manifold. Also, the Riemannian curvature tensor R is given by

$$R(X_1, Y_1) Z_1 = \bar{\nabla}_{X_1} \bar{\nabla}_{Y_1} Z_1 - \bar{\nabla}_{Y_1} \bar{\nabla}_{X_1} Z_1 - \bar{\nabla}_{[X_1, Y_1]} Z_1.$$

With the help of above results, we obtain

$$R(\tilde{E}_1, \tilde{E}_2) \tilde{E}_1 = -e_*^{2t} \tilde{E}_2, \; R(\tilde{E}_1, \tilde{E}_2) \tilde{E}_3 = 0, \; R(\tilde{E}_1, \tilde{E}_2) \tilde{E}_2 = -e_*^{2t} \tilde{E}_1,$$

$$R(\tilde{E}_1, \tilde{E}_3) \tilde{E}_1 = -e_*^{2t} \tilde{E}_3, \mathbf{R}(\tilde{E}_1, \tilde{E}_3) \tilde{E}_2 = 0, \mathbf{R}(\tilde{E}_1, \tilde{E}_3) \tilde{E}_3 = -e_*^{2t} \tilde{E}_3.$$

$$R(\tilde{E}_2, \tilde{E}_3) \tilde{E}_1 = 0, \mathbf{R}(\tilde{E}_2, \tilde{E}_3) \tilde{E}_2 = -e_*^{2t} \tilde{E}_3, \mathbf{R}(\tilde{E}_2, \tilde{E}_3) \tilde{E}_3 = -e_*^{2t} \tilde{E}_2.$$

Then, the Ricci tensor S is given by

$$S(\tilde{E}_1, \tilde{E}_1) = 0, S(\tilde{E}_2, \tilde{E}_2) = 0 \text{ and } \quad S(\tilde{E}_3, \tilde{E}_3) = -2e_*^{2t}.$$

from equation (1.2) and above calculation, we find $\tau_1 = 2e_*^t(1 - e_*^t)$.

Thus 3-dimensional α-LPS manifold admitting an AR soliton.

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BIWARPED PRODUCT SUBMANIFOLDS WITH A SLANT BASE FACTOR

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Abstract. We study biwarped product submanifolds with a slant base factor in locally product Riemannian manifolds. We prove an existence theorem for such submanifolds. Then we give a necessary and sufficient condition for such a manifold to be a warped product. We establish a general inequality for such submanifolds. The equality case is also considered. Moreover, we give an application of this inequality.

Keywords: Biwarped product submanifold, Slant distribution, Invariant distribution, Anti-invariant distribution, Locally product Riemannian manifold

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1. Introduction

Let (M_i, g_i) be Riemannian manifolds for $i \in \{0, 1, 2\}$ and let $f_{1,2} : M_0 \to (0, \infty)$ be smooth functions. Then the *biwarped product or twice warped product manifold* [5, 14] $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ is the product manifold $\bar{M} = M_0 \times M_1 \times M_2$ endowed with the metric

$$g = \pi_0^*(g_0) \oplus (f_1 \circ \pi_0)^2 \pi_1^*(g_1) \oplus (f_2 \circ \pi_0)^2 \pi_2^*(g_2).$$

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More precisely, for any vector fields $\bar{X}$ and $\bar{Y}$ of $\bar{M}$, we have

$$g(\bar{X},\bar{Y}) = g_0(\pi_{0_*}\bar{X},\pi_{0_*}\bar{Y}) + \sum_{i=1}^{2}(f_i\circ\pi_0)^2 g_i(\pi_{i_*}\bar{X},\pi_{i_*}\bar{Y}),$$

where $\pi_i : \bar{M} \to M_i$ is the canonical projection of $\bar{M}$ onto M_i, $\pi_i^*(g_i)$ is the pullback of g_i by π_i and the subscript π_{i*} denotes the derivative map of π_i for each i. The functions f_1 and f_2 are called *warping functions* and each manifold (M_j, g_j), $j \in \{1,2\}$ is called a *fiber* of the biwarped product $\bar{M}$. The factor (M_0, g_0) is called a *base manifold* of $\bar{M}$. As well known, the base manifold of $\bar{M}$ is totally geodesic and the fibers of $\bar{M}$ are totally umbilic in $\bar{M}$. We say that a biwarped product manifold is *trivial*, if the warping functions f_1 and f_2 are constants. Of course, biwarped product manifolds are natural generalizations of warped product manifolds [7] and special case of multiply warped product manifolds [14].

Let $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product manifold with the Levi-Civita connection $\bar{\nabla}$ and ∇^i denote the Levi-Civita connection of M_i for $i \in \{0,1,2\}$. By usual convenience, we denote the set of lifts of vector fields on M_i by $\mathcal{L}(M_i)$ and use the same notation for a vector field and for its lifts. On the other hand, since the map π_0 is an isometry and π_1 and π_2 are (positive) homotheties, they preserve the Levi-Civita connections. Thus there is no confusion using the same notation for a connection on M_i and for its pullback via π_i. Then, the covariant derivative formulas [23] for a biwarped product manifold are given by

$$\bar{\nabla}_U V = \nabla^0_U V \tag{1.1}$$

$$\bar{\nabla}_V X = \bar{\nabla}_X V = V(\ln f_i)X \tag{1.2}$$

$$\bar{\nabla}_X Z = \begin{cases} 0 & if \quad i \neq j, \\ \nabla^i_X Z - g(X,Z)\nabla^0(\ln f_i) & if \quad i = j, \end{cases} \tag{1.3}$$

where $U, V \in \mathcal{L}(M_0)$, $X \in \mathcal{L}(M_i)$ and $Z \in \mathcal{L}(M_j)$.

The theory of warped product submanifolds has been become a popular research area since Chen [8] studied the warped product CR-submanifolds in Kaehler manifolds. Actually, several classes of warped product submanifolds appeared in the last eighteen years. Also, warped product submanifolds have been studied for different kinds of structures. Most of the studies related to the theory of warped product submanifolds can be found in Chen's book [10]. Recently, Taştan studied biwarped product submanifolds of a Kaehler manifold $(\bar{M}, J, g)$ of the form $M^T \times_f M^\perp \times_\sigma M^\theta$, where M^T is a holomorphic, $M^\perp$ is a totally real and M^θ is a pointwise slant submanifold of $\bar{M}$ [20]. Afterwards, biwarped product submanifolds have been studying by many geometers for different kinds of structures (see, [2, 21, 22]).

In this paper, we study biwarped product submanifolds with a slant base factor in locally product Riemannian manifolds. More precisely, we consider biwarped product submanifolds of the form $M^\theta \times_f M^T \times_\sigma M^\perp$, where M^θ is a slant, $M^\perp$ is an anti-invariant and M^T is an invariant submanifold of the locally product Riemannian manifold. After giving a non-trivial example and some auxiliary results, we prove an existence theorem for such submanifolds. Then, we investigate the behavior of the second fundamental form of such a submanifold and as a result, we get a condition for this kind of submanifold to be a warped product. Finally, we obtain an inequality for the squared norm of the second fundamental form in terms of the warping functions for such submanifolds. The equality case is also considered. Moreover, we give an application of this inequality for certain types of locally product Riemannian manifolds.

Remark 1.1. *Biwarped product submanifolds of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ in locally product Riemannian manifolds were also studied in* [22]*. However, expect the first four equations of Lemma 5.1, our results are completely different from the results of* [22]*. Besides, biwarped product submanifolds of the form $M^\perp \times_f M^T \times_\sigma M^\theta$ in locally product Riemannian manifolds were studied in* [2]*, where M^θ is a proper pointwise slant submanifold of the locally product Riemannian manifold. But, the geometry of $M^\theta \times_f M^T \times_\sigma M^\perp$ and the geometry of $M^\perp \times_f M^T \times_\sigma M^\theta$ are quite different.*

2. Preliminaries

We first recall the fundamental definitions and notions needed for further study. In fact, we will give the notions for submanifolds of Riemannian manifolds in subsection 2.1. In subsection 2.2, we recall the definition of a locally product Riemannian manifold.

2.1. **Riemannian submanifolds.** Let M be a Riemannian manifold isometrically immersed in a Riemannian manifold $(\bar{M}, g)$ and $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to the metric g. Also, let ∇ and $\nabla^\perp$ be the Levi-Civita connection and normal connection of M, respectively. Then the Gauss and Weingarten formulas [24] are given respectively by

$$\bar{\nabla}_V W = \nabla_V W + h(V, W) \qquad \textit{and} \qquad \bar{\nabla}_V Z = -A_Z V + \nabla^\perp_V Z. \tag{2.4}$$

Here V, W are the tangent vector fields to M and Z is normal to M. In addition, h is the second fundamental form and A_Z is the Weingarten operator of M associated with Z. Then, we have

$$g(h(V, W), Z) = g(A_Z V, W). \tag{2.5}$$

For a submanifold M of a Riemannian manifold $\bar{M}$, the equation of Gauss is given by

$$\bar{R}(U, V, Z, W) = R(U, V, Z, W) + g(h(U, Z), h(V, W)) - g(h(U, W), h(V, Z)) \tag{2.6}$$

for any $U, V, Z, W \in \Gamma(TM)$, where $\bar{R}$ and R are the curvature tensors on $\bar{M}$ and M respectively. The mean curvature vector H for an orthonormal frame $\{e_1, \ldots, e_m\}$ of tangent space T_pM, $p \in M$ on M is defined by

$$H = \frac{1}{m} trace(h) = \frac{1}{m} \sum_{i=1}^{m} h(e_i, e_i), \tag{2.7}$$

where $m = dimM$. Also, we set

$$h_{ij}^r = g(h(e_i, e_j), e_r) \quad and \quad \| h \|^2 = \sum_{i,j=1}^{m} g(h(e_i, e_j), h(e_i, e_j)). \tag{2.8}$$

Moreover, the *sectional curvature* [24] of a plane section spanned by e_i and e_j, denoted by K_{ij}, is

$$K_{ij} = R(e_i, e_j, e_j, e_i). \tag{2.9}$$

The *scalar curvature* [9] of M of is given by

$$\tau(TM) = \sum_{1 \leq i \neq j \leq m} K_{ij}. \tag{2.10}$$

Let G_r be a r-plane section on TM and $\{e_1, \ldots, e_r\}$ any orthonormal basis of G_r. Then the scalar curvature $\tau(G_r)$ of G_r is given by

$$\tau(G_r) = \sum_{1 \leq i \neq j \leq r} K_{ij}. \tag{2.11}$$

For a smooth function f on M, the *Laplacian* of f is defined by

$$\Delta f = \sum_{i=1}^{m} \{(\nabla_{e_i} e_i) f - e_i(e_i(f))\} = -\sum_{i=1}^{m} g(\nabla_{e_i} \nabla f, e_i), \tag{2.12}$$

where ∇f is the gradient of f [9].

2.2. **Locally product Riemannian manifolds.** Let $\bar{M}$ be a Riemannian manifold. Suppose $\bar{M}$ is endowed with a tensor field

$$\mathcal{F}^2 = I, \qquad (\mathcal{F} \neq \mp I), \tag{2.13}$$

of type $(1,1)$. Here, I is the identity endomorphism on $T\bar{M}$. Then, $(\bar{M}, g, \mathcal{F})$ called an *almost product manifold* and $\mathcal{F}$ is called an almost product structure. Also, we assume that g and $\mathcal{F}$ satisfy

$$g(\mathcal{F}\bar{X}, \mathcal{F}\bar{Y}) = g(\bar{X}, \bar{Y}), \tag{2.14}$$

for all vector fields $\bar{X}, \bar{Y}$ tangent to M. Then, it is known that $(\bar{M}, g, \mathcal{F})$ is an *almost product Riemannian manifold*. Let $\bar{\nabla}$ be the Levi-Civita connection of $(\bar{M}, g, \mathcal{F})$. If we have

$$\bar{\nabla}\mathcal{F} \equiv 0, \tag{2.15}$$

then $(\bar{M}, g, \mathcal{F})$ is a *locally product Riemannian manifold*, (briefly, *l.p.R. manifold*).

Let $M_1(c_1)$ (resp. $M_2(c_2)$) be a real space form and have sectional curvature c_1 (resp. c_2). Then, the Riemannian curvature tensor $\bar{R}$ of l.p.R. manifold $\bar{M} = M_1 \times M_2$ has the form

$$\begin{aligned} \bar{R}(U,V)Z &= \frac{1}{4}(c_1 + c_2)\Big\{ g(V,Z)U - g(U,Z)V + g(\mathcal{F}V,Z)\mathcal{F}U - g(\mathcal{F}U,Z)\mathcal{F}V \Big\} \\ &= \frac{1}{4}(c_1 - c_2)\Big\{ g(V,Z)\mathcal{F}U - g(U,Z)\mathcal{F}V + g(\mathcal{F}V,Z)U - g(\mathcal{F}U,Z)V \Big\}, \end{aligned} \tag{2.16}$$

for all $U, V, Z \in \Gamma(T\bar{M})$ [24].

3. Skew semi-invariant submanifolds of order 1 in locally product Riemannian manifolds

We first recall the definition of the skew semi-invariant submanifolds of order 1 of a locally product Riemannian manifold and get some useful results for the further study.

Let $(\bar{M}, g, \mathcal{F})$ be a l.p.R. manifold and let M be a submanifold of $\bar{M}$. If for $X \in \mathcal{D}_p$, the angle θ between $\mathcal{F}X$ and $\mathcal{D}_p$ is constant, i.e., it is independent of $p \in M$ and $X \in \mathcal{D}_p$, then $\mathcal{D}$ is called a *slant distribution* on M. θ is said the *slant angle* of the slant distribution $\mathcal{D}$. Thus, the invariant and anti-invariant distributions with respect to $\mathcal{F}$ are slant distributions with slant angle $\theta = 0$ and $\theta = \pi/2$, respectively. If the tangent bundle TM of M is slant [12, 15] then the submanifold M of $\bar{M}$ is called a *slant submanifold*. A slant submanifold that is neither invariant nor anti-invariant is called a *proper* slant submanifold.

Let M be a slant submanifold with slant angle θ of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$, for any $V \in \Gamma(TM)$, we write

$$\mathcal{F}V = PV + NV. \tag{3.17}$$

Here PV is the tangential part of $\mathcal{F}V$ and NV is the normal part of $\mathcal{F}V$. Then, for any $U, V \in \Gamma(TM)$ we have [15]

$$P^2V = \cos^2\theta V, \tag{3.18}$$

$$g(PU, PV) = \cos^2\theta g(U, V) \quad and \quad g(NU, NV) = \sin^2\theta g(U, V). \tag{3.19}$$

A submanifold M of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ is said a *skew semi-invariant submanifold of order 1* (briefly, s.s-i.) [18] if the tangent bundle TM of M has the form

$$TM = \mathcal{D}_\perp \oplus \mathcal{D}_T \oplus \mathcal{D}_\theta,$$

where $\mathcal{D}_\theta$ is slant distribution with slant angle θ, $\mathcal{D}_T$ is an invariant distribution, i.e., $\mathcal{F}\mathcal{D}_T \subseteq \mathcal{D}_T$, $\mathcal{D}_\perp$ is an anti-invariant distribution, i.e. $\mathcal{F}\mathcal{D}_\perp \subseteq T^\perp M$. In that case, the normal bundle $T^\perp M$ of M can be decomposed as

$$T^\perp M = N(\mathcal{D}_\theta) \oplus \mathcal{F}(\mathcal{D}_\perp) \oplus \bar{\mathcal{D}}_T, \tag{3.20}$$

where $\bar{\mathcal{D}}_T$ is the orthogonal complementary distribution of $N(\mathcal{D}_\theta) \oplus \mathcal{F}(\mathcal{D}_\perp)$ in $T^\perp M$ and it is an invariant subbundle of $T^\perp M$ with respect to $\mathcal{F}$.

Remark 3.1. *The class of s.s-i. submanifolds of order 1 of locally product Riemannian manifolds is a special subclass of skew semi-invariant submanifolds* [12] *and a natural generalization of invariant, anti-invariant* [1], *semi-invariant* [6], *slant* [15], *semi-slant* [13] *and hemi-slant submanifolds* [19] *of locally product Riemannian manifolds.*

Lemma 3.1. [18] *Let M be a proper s.s-i. submanifold of order 1 of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. Then,*

$$g(\nabla_Z W, U) = -\csc^2\theta\Big\{g(A_{NPW}Z, U) + g(A_{NW}Z, \mathcal{F}U)\Big\}, \tag{3.21}$$

$$g(\nabla_Z W, X) = \sec^2\theta\Big\{g(A_{\mathcal{F}X}Z, PW) + g(A_{NPW}Z, X)\Big\}, \tag{3.22}$$

$$g(\nabla_U V, Z) = \csc^2\theta\Big\{g(A_{NPZ}U, V) + g(A_{NZ}U, \mathcal{F}V)\Big\}, \tag{3.23}$$

$$g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V), \tag{3.24}$$

$$g(\nabla_X Y, Z) = -\sec^2\theta\Big\{g(A_{\mathcal{F}Y}X, PZ) + g(A_{NPZ}X, Y)\Big\}, \tag{3.25}$$

$$g(\nabla_X Y, V) = -g(A_{\mathcal{F}Y}X, \mathcal{F}V), \tag{3.26}$$

$$g(\nabla_X Z, V) = -\csc^2\theta\Big\{g(A_{NPZ}X, V) + g(A_{NZ}X, \mathcal{F}V)\Big\}, \tag{3.27}$$

$$g(\nabla_Z X, V) = -g(A_{\mathcal{F}X}Z, \mathcal{F}V), \tag{3.28}$$

$$g(\nabla_U X, Z) = -\sec^2\theta\Big\{g(A_{\mathcal{F}X}U, PZ) + g(A_{NPZ}U, X)\Big\} \tag{3.29}$$

for $Z, W \in \Gamma(\mathcal{D}_\theta)$, $U, V \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$.

Theorem 3.1. *Let M be a proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then the slant distribution $\mathcal{D}_\theta$ is totally geodesic iff the following equations hold*

$$g(A_{NPW}Z, V) = -g(A_{NW}Z, \mathcal{F}V), \tag{3.30}$$

$$g(A_{\mathcal{F}X}Z, PW) = -g(A_{NPW}Z, X), \tag{3.31}$$

for $Z, W \in \Gamma(\mathcal{D}_\theta)$, $V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$.

Proof. The distribution $\mathcal{D}_\theta$ is totally geodesic iff $g(\nabla_Z W, X) = 0$ and $g(\nabla_Z W, V) = 0$ for all $Z, W \in \Gamma(\mathcal{D}_\theta)$, $X \in \Gamma(\mathcal{D}_\perp)$ and $V \in \Gamma(\mathcal{D}_T)$. Thus, the assertions (3.30) and (3.31) follow from (3.21) and (3.22), respectively.

Theorem 3.2. *Let M be a proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then the invariant distribution $\mathcal{D}_T$ is integrable iff the following equations hold*

$$g(A_{\mathcal{F}X}U, \mathcal{F}V) = g(A_{\mathcal{F}X}V, \mathcal{F}U), \tag{3.32}$$

$$g(A_{NPZ}U, V) + g(A_{NZ}U, \mathcal{F}V) = g(A_{NPZ}V, U) + g(A_{NZ}V, \mathcal{F}U), \tag{3.33}$$

for $U, V \in \Gamma(\mathcal{D}_T)$, $X \in \Gamma(\mathcal{D}_\perp)$ and $Z \in \Gamma(\mathcal{D}_\theta)$.

Proof. The distribution $\mathcal{D}_T$ is integrable iff $g([U,V],X)=0$ and $g([U,V],Z)=0$ for all $Z\in\Gamma(\mathcal{D}_\theta)$, $X\in\Gamma(\mathcal{D}_\perp)$ and $U,V\in\Gamma(\mathcal{D}_T)$. Thus, the assertions (3.32) and (3.33) follow from (3.23) and (3.24), respectively.

Theorem 3.3. *Let M be a proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M},g,\mathcal{F})$. Then the anti-invariant distribution $\mathcal{D}_\perp$ is integrable iff the following equations hold*

$$g(A_{\mathcal{F}X}Y,\mathcal{F}V)=g(A_{\mathcal{F}Y}X,\mathcal{F}V), \tag{3.34}$$

$$g(A_{\mathcal{F}Y}X,PZ)=g(A_{\mathcal{F}X}Y,PZ), \tag{3.35}$$

for $X,Y\in\Gamma(\mathcal{D}_\perp)$, $V\in\Gamma(\mathcal{D}_T)$ and $Z\in\Gamma(\mathcal{D}_\theta)$.

Proof. The distribution $\mathcal{D}_\perp$ is integrable iff $g([X,Y],Z)=0$ and $g([X,Y],V)=0$ for all $Z\in\Gamma(\mathcal{D}_\theta)$, $X,Y\in\Gamma(\mathcal{D}_\perp)$ and $V\in\Gamma(\mathcal{D}_T)$. Thus, the assertions (3.34) and (3.35) follow from (3.25) and (3.26), respectively.

4. Biwarped Product Submanifolds in Locally Product Riemannian Manifolds

We first check that the existence of biwarped product submanifolds of the form, $M^T\times_f M^\perp\times_\sigma M^\theta$, $M^\perp\times_f M^\theta\times_\sigma M^T$ and $M_\theta\times_f M^T\times_\sigma M^\perp$, where $M^\perp$ is an anti-invariant, M^θ ia a proper slant and M^T is an invariant submanifold of a l.p.R. manifold $(\bar{M},g,\mathcal{F})$.

M. Atçeken and B. Ṣahin independently proved that there do not exist (non-trivial) warped product semi-invariant submanifolds of the form $M^T\times_f M^\perp$ in a l.p.R. manifold $(\bar{M},g,\mathcal{F})$, such that M^T is an invariant submanifold and $M^\perp$ is an anti-invariant submanifold of $(\bar{M},g,\mathcal{F})$ in [4, Theorem 3.1] and [16, Theorem 3.1], respectively. Again, M. Atçeken and B. Ṣahin independently proved that there do not exist (non-trivial) warped product semi-slant submanifolds of the form $M^T\times_f M^\theta$ in a l.p.R. manifold $\bar{M}$, such that M^T is an invariant submanifold and M^θ is a proper slant submanifold of $\bar{M}$ in [3, Theorem 3.3] and [17, Theorem 3.1], respectively. Thus, we obtain the following result.

Corollary 4.1. *There do not exist (non-trivial) biwarped product submanifolds of the form $M^T\times_f M^\perp\times_\sigma M^\theta$ of a l.p.R. manifold $(\bar{M},g,\mathcal{F})$ such that M^T is an invariant, $M^\perp$ is an anti-invariant and M^θ is a proper slant submanifold of $\bar{M}$.*

On the other hand, it was proved that there do not exist (non-trivial) warped product submanifolds of the form $M^\perp\times_f M^\theta$ in a l.p.R. manifold $\bar{M}$ such that $M^\perp$ is an anti-invariant submanifold and M^θ is a proper slant submanifold of $\bar{M}$ in [3, Theorem 3.4]. Thus, we deduce the following result.

Corollary 4.2. *There do not exist (non-trivial) biwarped product submanifolds of the form $M^\perp \times_f M^\theta \times_\sigma M^T$ of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$ such that $M^\perp$ is an anti-invariant, M^θ is a proper slant submanifold and M^T is an invariant submanifold of $\bar{M}$.*

Now, we consider (non-trivial) biwarped product submanifolds in the form $M^\theta \times_f M^T \times_\sigma M^\perp$ in a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$ such that M^T is an invariant, $M^\perp$ is an anti-invariant and M^θ is a proper slant submanifold of $\bar{M}$. Firstly, we present an example of such a submanifold.

Example 4.1. *Consider the 8-dimensional Euclidean space $\mathbb{R}^8$ with standard metric g and almost product structure $\mathcal{F}$ given by*

$$\mathcal{F}\partial_1 = \partial_1, \qquad \mathcal{F}\partial_2 = \partial_2, \qquad \mathcal{F}\partial_3 = -\partial_3, \qquad \mathcal{F}\partial_4 = -\partial_4,$$

$$\mathcal{F}\partial_5 = \partial_6, \qquad \mathcal{F}\partial_6 = \partial_5, \qquad \mathcal{F}\partial_7 = \partial_8, \qquad \mathcal{F}\partial_8 = \partial_7,$$

where $\partial_k = \frac{\partial}{\partial x_k}$, $k \in \{1, \dots, 8\}$ and $(x_1, x_2, \dots, x_8)$ are natural coordinates of $\mathbb{R}^8$. Upon a straightforward calculation, we see that $(\mathbb{R}^8, \mathcal{F}, g)$ is a l.p.R. manifold. Let M be a submanifold of $(\mathbb{R}^8, \mathcal{F}, g)$ given by

$$x_1 = t \sin u, \qquad x_2 = t \cos u, \qquad x_3 = \frac{t}{\sqrt{2}} \cos v, \qquad x_4 = \frac{t}{\sqrt{2}} \sin v$$

$$x_5 = 2t \sin x, \qquad x_6 = 0, \qquad x_7 = 2t \cos x, \qquad x_8 = 0,$$

where $u, v \in (0, \frac{\pi}{2})$ and $t > 0$. Then, the local frame of TM is given by

$$\begin{aligned} Z &= \sin u \partial_1 + \cos u \partial_2 + \frac{1}{\sqrt{2}} \cos v \partial_3 + \frac{1}{\sqrt{2}} \sin v \partial_4 + 2 \sin x \partial_5 + 2 \cos x \partial_7, \\ U &= t \cos u \partial_1 - t \sin u \partial_2, \\ V &= -\frac{t}{\sqrt{2}} \sin v \partial_3 + \frac{t}{\sqrt{2}} \cos v \partial_4, \\ X &= 2t \cos x \partial_5 - 2t \sin x \partial_7. \end{aligned}$$

After some calculation, we see that $\mathcal{D}_\theta = span\{Z\}$ is a proper slant distribution with slant angle $\theta = \cos^{-1}(\frac{1}{11})$ and $\mathcal{D}_T = span\{U, V\}$ is an invariant distribution and $\mathcal{D}_\perp = span\{X\}$ is an anti-invariant distribution. Moreover, $\mathcal{D}_\theta$ is totally geodesic and both $\mathcal{D}_T$ and $\mathcal{D}_\perp$ are *integrable distributions. If we denote the integral manifolds of $\mathcal{D}_\theta$, $\mathcal{D}_T$ and $\mathcal{D}_\perp$ by M^θ, M^T and $M^\perp$, respectively, then the induced metric tensor of M is*

$$\begin{aligned} ds^2 &= \frac{11}{2}dt^2 + t^2(du^2 + \frac{1}{2}dv^2) + 4t^2dx^2 \\ &= g_{M^\theta} + t^2 g_{M^T} + (2t)^2 g_{M^\perp} . \end{aligned}$$

Thus, $M = M^\theta \times_f M^T \times_\sigma M^\perp$ is a (non-trivial) biwarped product proper s.s-i. submanifold of order 1 of $(\mathbb{R}^8, \mathcal{F}, g)$ with warping functions $f = t$ and $\sigma = 2t$.

5. Biwarped product proper skew semi-invariant submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$

First, we give a characterization for biwarped product proper s.s-i. submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$, where M^θ is a proper slant submanifold, M^T is an invariant and $M^\perp$ is an anti invariant submanifold of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. After that we investigate the behavior of the second fundamental form of such submanifolds and as a result, we give a condition for these submanifolds to be locally warped product. Firstly, we recall the following fact given in [11] to prove our theorem.

Remark 5.1. *(*[11, Remark 2.1]*) Suppose that the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \ldots \oplus \mathcal{D}_k$ of non-trivial distributions such that each $\mathcal{D}_j$ is spherical and its complement in TM is autoparallel for $j \in \{1, 2, \ldots, k\}$. Then the manifold M is locally isometric to a multiply warped product $M_0 \times_{f_1} M_2 \times_{f_2} \times \ldots \times_{f_k} M_k$.*

Now, we give one of the main theorems of this paper.

Theorem 5.1. *Let M be a $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic proper s.s-i. submanifold of order 1 of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. Then M is a locally biwarped product submanifold of type $M^\theta \times_f M^T \times_\sigma M^\perp$ iff we have*

$$A_{NPZ}X = \cos^2\theta Z(\lambda)X, \tag{5.36}$$

$$A_{NZ}V + A_{NPZ}\mathcal{F}V = -\sin^2\theta Z(\mu)\mathcal{F}V \tag{5.37}$$

for smooth functions λ and μ satisfying $X(\lambda) = V(\lambda) = 0$ and $X(\mu) = V(\mu) = 0$ and

$$g(A_{\mathcal{F}X}Z, PW) = -g(A_{NPW}Z, X), \tag{5.38}$$

$$g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0, \tag{5.39}$$

$$g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0, \tag{5.40}$$

$$g(A_{\mathcal{F}X}Z,\mathcal{F}U)=0, \tag{5.41}$$

$$g(A_{\mathcal{F}X}U,PZ)=-g(A_{NPZ}U,X), \tag{5.42}$$

for $Z,W\in\Gamma(\mathcal{D}_\theta),\ U,V\in\Gamma(\mathcal{D}_T),\ X,Y\in\Gamma(\mathcal{D}_\perp)$.

Proof. For any $Z\in\Gamma(\mathcal{D}_\theta)$, $U\in\Gamma(\mathcal{D}_T)$ and $X\in\Gamma(\mathcal{D}_\perp)$, using (2.4) and (3.17),

$$g(A_{NPZ}X,U)=-g(\bar\nabla_X NPZ,U)=-g(\bar\nabla_X\mathcal{F}PZ,U)+g(\bar\nabla_X P^2Z,U).$$

By using (2.13) – (2.15) and (3.18), we find

$$g(A_{NPZ}X,U)=-g(\bar\nabla_X PZ,\mathcal{F}U)+\cos^2\theta g(\bar\nabla_X Z,U).$$

Here, using (2.4), we arrive to

$$g(A_{NPZ}X,U)=-g(\nabla_X PZ,\mathcal{F}U)+\cos^2\theta g(\nabla_X Z,U).$$

So, using (1.2), we conclude that

$$g(A_{NPZ}X,U)=-PZ(\ln\sigma)g(X,\mathcal{F}U)+\cos^2\theta Z(\ln\sigma)g(X,U)=0. \tag{5.43}$$

Since M is $(\mathcal{D}_\theta,\mathcal{D}_\perp)$-mixed geodesic, for $W\in\Gamma(\mathcal{D}_\theta)$ using (2.5), we find

$$g(A_{NPZ}X,W)=g(h(X,W),NPZ)=0. \tag{5.44}$$

Next, by a similar argument, for $Y\in\Gamma(\mathcal{D}_\perp)$, using (2.4) and (3.17), we have

$$g(h(X,Y),NZ)=g(\bar\nabla_X Y,NZ)=g(\bar\nabla_X Y,\mathcal{F}Z)-g(\bar\nabla_X Y,PZ).$$

Then using (2.14),(2.15) and (1.2), we find

$$g(h(X,Y),NZ)=g(\bar\nabla_X\mathcal{F}Y,Z)+PZ(\ln\sigma)g(X,Y).$$

Hence using (2.4) and (2.5), we arrive to

$$\begin{aligned}g(h(X,Y),NZ)&=-g(A_{\mathcal{F}Y}X,Z)+PZ(\ln\sigma)g(X,Y)\\&=-g(h(X,Z),\mathcal{F}Y)+PZ(\ln\sigma)g(X,Y).\end{aligned}$$

In this equation, if we interchange Z with PZ, then we have

$$g(h(X,Y),NPZ)=-g(h(X,PZ),\mathcal{F}Y)+\cos^2\theta Z(\ln\sigma)g(X,Y).$$

Since M is $(\mathcal{D}_\theta,\mathcal{D}_\perp)$-mixed geodesic, we conclude that

$$g(A_{NPZ}X,Y)=\cos^2\theta Z(\ln\sigma)g(X,Y). \tag{5.45}$$

Moreover, we have $X(\ln\sigma) = V(\ln\sigma) = 0$, since σ depends only on the points of M^θ. So, we conclude that $\lambda = \ln\sigma$. Thus, from (5.43) – (5.45), it follows that (5.36). Now, we prove (5.37). For $Z \in \Gamma(\mathcal{D}_\theta)$, $V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, using (2.4) and (3.17), we have

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) &= g(A_{NZ}V, X) + g(A_{NPZ}\mathcal{F}V, X) \\ &= g(A_{NZ}X, V) + g(A_{NPZ}X, \mathcal{F}V) \\ &= -g(\bar{\nabla}_X NZ, V) - g(\bar{\nabla}_X NPZ, \mathcal{F}V) \\ &= -g(\bar{\nabla}_X NZ, V) - g(\bar{\nabla}_X \mathcal{F}PZ, \mathcal{F}V) \\ &\quad + g(\bar{\nabla}_X P^2 Z, \mathcal{F}V). \end{aligned}$$

Using (2.14), (2.15), (3.17) and(3.18) and, we arrive to

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) &= -g(\bar{\nabla}_X \mathcal{F}Z, V) + g(\bar{\nabla}_X PZ, V) - g(\bar{\nabla}_X PZ, V) \\ &\quad + \cos^2\theta g(\bar{\nabla}_X Z, \mathcal{F}V) + X(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\bar{\nabla}_X \mathcal{F}Z, V) + \cos^2\theta g(\bar{\nabla}_X Z, \mathcal{F}V). \end{aligned}$$

Then, using (1.2), (2.4), (2.13) – (2.15), we find

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) &= -g(\bar{\nabla}_X Z, \mathcal{F}V) + \cos^2\theta g(\nabla_X Z, \mathcal{F}V) \\ &= -g(\nabla_X Z, \mathcal{F}V) + \cos^2\theta g(\nabla_X Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla_X Z, \mathcal{F}V) \\ &= -\sin^2\theta Z(\ln\sigma) g(X, \mathcal{F}V). \end{aligned}$$

Since $g(X, \mathcal{F}V) = 0$, we conclude that

$$g(A_{NZ}V + A_{NPZ}\mathcal{F}V, X) = -\sin^2\theta Z(\ln\sigma) g(X, \mathcal{F}V) = 0. \tag{5.46}$$

Similarly, for $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $V \in \Gamma(\mathcal{D}_T)$, using (2.4) and (3.17), we have

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) &= g(A_{NZ}V, W) + g(A_{NPZ}\mathcal{F}V, W) \\ &= g(A_{NZ}W, V) + g(A_{NPZ}W, \mathcal{F}V) \\ &= -g(\bar{\nabla}_W NZ, V) - g(\bar{\nabla}_W NPZ, \mathcal{F}V) \\ &= -g(\bar{\nabla}_W NZ, V) - g(\bar{\nabla}_W \mathcal{F}PZ, \mathcal{F}V) \\ &\quad + g(\bar{\nabla}_W P^2 Z, \mathcal{F}V). \end{aligned}$$

Using (2.14), (2.15), (3.17) and (3.18), we arrive to

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) &= -g(\bar{\nabla}_W \mathcal{F}Z, V) + g(\bar{\nabla}_W PZ, V) - g(\bar{\nabla}_W PZ, V) \\ &\quad + \cos^2\theta g(\bar{\nabla}_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\bar{\nabla}_W \mathcal{F}Z, V) + \cos^2\theta g(\bar{\nabla}_W Z, \mathcal{F}V) \\ &\quad + W(\cos^2\theta) g(Z, \mathcal{F}V). \end{aligned}$$

Then, using (1.2), (2.4), (2.13) – (2.15), we find

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) &= -g(\bar{\nabla}_W Z, \mathcal{F}V) + \cos^2\theta g(\nabla_W Z, \mathcal{F}V) \\ &\quad + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\nabla_W Z, \mathcal{F}V) + \cos^2\theta g(\nabla_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla^\theta_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V). \end{aligned}$$

Since $g(\nabla^\theta_W Z, \mathcal{F}V) = 0$ and $g(Z, \mathcal{F}V) = 0$, we conclude that

$$g(A_{NZ}V + A_{NPZ}\mathcal{F}V, W) = -\sin^2\theta g(\nabla^\theta_W Z, \mathcal{F}V) + W(\cos^2\theta) g(Z, \mathcal{F}V) = 0. \tag{5.47}$$

On the other hand, for $Z \in \Gamma(\mathcal{D}_\theta)$ and $U, V \in \Gamma(\mathcal{D}_T)$, using (2.4) and (3.17), we get

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) &= g(A_{NZ}V, U) + g(A_{NPZ}\mathcal{F}V, U) \\ &= g(A_{NZ}U, V) + g(A_{NPZ}U, \mathcal{F}V) \\ &= -g(\bar{\nabla}_U NZ, V) - g(\bar{\nabla}_U NPZ, \mathcal{F}V) \\ &= -g(\bar{\nabla}_U NZ, V) - g(\bar{\nabla}_U \mathcal{F}PZ, \mathcal{F}V) \\ &\quad + g(\bar{\nabla}_U P^2 Z, \mathcal{F}V). \end{aligned}$$

Using (2.14), (2.15), (3.17) and (3.18), we arrive to

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) &= -g(\bar{\nabla}_U \mathcal{F}Z, V) + g(\bar{\nabla}_U PZ, V) - g(\bar{\nabla}_U PZ, V) \\ &\quad + \cos^2\theta g(\bar{\nabla}_U Z, \mathcal{F}V) + U(\cos^2\theta) g(Z, \mathcal{F}V) \\ &= -g(\bar{\nabla}_U \mathcal{F}Z, V) + \cos^2\theta g(\bar{\nabla}_U Z, \mathcal{F}V) \\ &\quad + U(\cos^2\theta) g(Z, \mathcal{F}V). \end{aligned}$$

Since $U[\cos^2\theta] = 0$, using (1.2), (2.4), (2.13) – (2.15), we find

$$\begin{aligned} g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) &= -g(\bar{\nabla}_U Z, \mathcal{F}V) + \cos^2\theta g(\nabla_U Z, \mathcal{F}V) \\ &= -g(\nabla_U Z, \mathcal{F}V) + \cos^2\theta g(\nabla_U Z, \mathcal{F}V) \\ &= -\sin^2\theta g(\nabla_U Z, \mathcal{F}V) \\ &= -\sin^2\theta Z(\ln f) g(U, \mathcal{F}V). \end{aligned}$$

So, we conclude that

$$g(A_{NZ}V + A_{NPZ}\mathcal{F}V, U) = -\sin^2\theta Z(\ln f) g(\mathcal{F}V, U). \tag{5.48}$$

Moreover, we have $X(\ln f) = V(\ln f) = 0$, since f depends only on the points of M^θ. So, we conclude that $\mu = \ln f$. Thus from (5.46) – (5.48), we get (5.37).

Next, we prove (5.38) – (5.42).We know M is a biwarped product proper s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then, for $Z, W \in \Gamma(\mathcal{D}_\theta)$, using (1.1), we get $\nabla_Z W = \nabla^\theta_Z W$ and for $X \in \Gamma(\mathcal{D}_\perp)$,we have

$$g(\nabla_Z W, X) = \sec^2\theta \{ g(A_{\mathcal{F}X} Z, PW) + g(A_{NPW} Z, X) \} = g(\nabla^\theta_Z W, X) = 0$$

from (3.22). Since M^θ is a proper slant submanifold, it follows that

$$g(A_{\mathcal{F}X}Z, PW) + g(A_{NPW}Z, X) = 0,$$

which gives (5.38). For $U, V \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$, using (1.3), we get $g(\nabla_U V, X) = g(\nabla^T_U V - g(U, V)\nabla(\ln f), X) = 0$. Then from (3.24) we find

$$g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0.$$

Therefore, we get (5.39). For $U \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$, using (1.3), we get $g(\nabla_X Y, U) = g(\nabla^\perp{}_X Y - g(X, Y)\nabla(\ln\sigma), U) = 0$. Then from (3.26) we find,

$$g(\nabla_X Y, U) = -g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0.$$

Hence, we conclude that (5.40). For $X \in \Gamma(\mathcal{D}_\perp)$, $Z \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_T)$, using (1.2), we write $g(\nabla_Z X, \mathcal{F}U) = g(Z(\ln\sigma)X, \mathcal{F}U) = Z(\ln\sigma)g(Z, \mathcal{F}U) = 0$. On the other hand, from (3.28) we find

$$g(\nabla_Z X, \mathcal{F}U) = -g(A_{\mathcal{F}X}Z, \mathcal{F}U) = 0.$$

Thus, we get (5.41). For $X \in \Gamma(\mathcal{D}_\perp)$, $Z \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_T)$, using (1.3), we have $g(\nabla_U X, Z) = 0$. Then, from (3.29) we find,

$$g(\nabla_U X, Z) = -\sec^2\theta\{g(A_{\mathcal{F}X}U, PZ) + g(A_{NPZ}U, X)\} = 0.$$

It follows (5.42).

Conversely, assume that M is a proper $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ such that (5.36) – (5.42) hold. From (5.38), we get (3.31). On the other hand if we write $\mathcal{F}V$ instead of V and W instead of Z in (5.37), we find $A_{NW}\mathcal{F}V + A_{NPW}V = -\sin^2\theta W(\mu)V$. If we take inner product of this equation with $Z \in \Gamma(\mathcal{D}_\theta)$, we get

$$\begin{aligned} g(A_{NW}\mathcal{F}V + A_{NPW}V, Z) &= g(A_{NW}Z, \mathcal{F}V) + g(A_{NPW}Z, V) \\ &= -\sin^2\theta W(\mu)g(V, Z) = 0. \end{aligned}$$

So, (3.30) holds. Thus from Theorem (3.1), the slant distribution $\mathcal{D}_\theta$ is totally geodesic and as a result, it is integrable. On the other hand, from (5.39), for all $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, we write $g(A_{\mathcal{F}X}V, \mathcal{F}U) = 0$. Thus, $g(A_{\mathcal{F}X}V, \mathcal{F}U) = g(A_{\mathcal{F}X}U, \mathcal{F}V)$, which is

(3.32). On the other hand, in (5.37), if we write $\mathcal{F}V$ instead of V, we find $A_{NZ}\mathcal{F}V+A_{NPZ}V = -\sin^2\theta Z(\mu)V$. If we take inner product of this equation with $U \in \Gamma(\mathcal{D}_T)$, we arrive at

$$\begin{aligned} g(A_{NZ}\mathcal{F}V + A_{NPZ}V, U) &= g(A_{NZ}\mathcal{F}V, U) + g(A_{NPZ}V, U) \\ &= -\sin^2\theta Z(\mu)g(V, U). \end{aligned} \tag{5.49}$$

Here, if we interchange U and V in (5.49), we find

$$\begin{aligned} g(A_{NZ}\mathcal{F}U + A_{NPZ}U, V) &= g(A_{NZ}\mathcal{F}U, V) + g(A_{NPZ}U, V) \\ &= -\sin^2\theta Z(\mu)g(U, V). \end{aligned} \tag{5.50}$$

From (5.49) and (5.50), we get $g(A_{NZ}U, \mathcal{F}V)+g(A_{NPZ}U, V) = g(A_{NZ}V, \mathcal{F}U)+g(A_{NPZ}V, U)$. This is (3.33). Thus, by Teorem 3.2, the invariant distribution $\mathcal{D}_T$ is integrable. On the other hand, for all $X, Y \in \Gamma(\mathcal{D}_\perp)$ and $U \in \Gamma(\mathcal{D}_T)$, we have $g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0$ from (5.40). It follows that $g(A_{\mathcal{F}Y}X, \mathcal{F}U) = g(A_{\mathcal{F}X}Y, \mathcal{F}U) = 0$. That is (3.34). Also, we get $g(\nabla_X Y, Z) = -\sec^2\theta\{g(h(Y, PZ), \mathcal{F}X) + g(A_{NPZ}X, Y)\}$ from (3.25). Since M is $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic, it follows that $g(h(Y, PZ), \mathcal{F}X) = 0$. Then, we find $g(\nabla_X Y, Z) = g(\nabla_Y X, Z)$. Thus (3.35) follows. Then by Theorem 3.3, the totally real distributions $\mathcal{D}_\perp$ is integrable. Let M^θ, M^T and $M^\perp$ be the integral manifolds of $\mathcal{D}_\theta, \mathcal{D}_T$ and $\mathcal{D}_\perp$, respectively. If we denote the second fundamental form of M^T in M by h^T, for $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$, using (2.4), (3.24) and (5.39), we have

$$g(h^T(U, V), X) = g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0. \tag{5.51}$$

For any, $U, V \in \Gamma(\mathcal{D}_T)$ and $Z \in \Gamma(\mathcal{D}_\theta)$, using (2.4) and (3.23), we get

$$g(h^T(U, V), Z) = g(\nabla_U V, Z) = \csc^2\theta g(A_{NPZ}U, V) + g(A_{NZ}U, \mathcal{F}V).$$

At this equation, if we use (5.37), we have

$$g(h^T(U, V), Z) = \csc^2\theta g(A_{NPZ}V + A_{NZ}\mathcal{F}V, U) = -Z(\mu)g(V, U).$$

After some calculation, we obtain

$$g(h^T(U, V), Z) = g(-g(U, V)\nabla\mu, Z), \tag{5.52}$$

where $\nabla\mu$ is the gradient of μ. Thus, from (5.51) and (5.52), we conclude that

$$h^T(U, V) = -g(U, V)\nabla\mu.$$

This equation says that M^T is totally umbilic in M with the mean curvature vector field $-\nabla\mu$. Now, we show that $-\nabla\mu$ is parallel. We have to satisfy $g(\nabla_U\nabla\mu, E) = 0$ for $U \in \Gamma(\mathcal{D}_T)$ and

$E \in (\mathcal{D}_T)^\perp = \mathcal{D}_\theta \oplus \mathcal{D}_\perp$. Here, we can put $E = Z + X$, where $Z \in \Gamma(\mathcal{D}_\theta)$ and $X \in \Gamma(\mathcal{D}_\perp)$. By direct computations, we obtain

$$\begin{aligned} g(\nabla_U \nabla\mu, E) &= \{U g(\nabla\mu, E) - g(\nabla\mu, \nabla_U E)\} \\ &= U(E(\mu)) - [U, E](\mu) - g(\nabla\mu, \nabla_E U) \\ &= [U, E](\mu) + E(U(\mu)) - [U, E](\mu) - g(\nabla\mu, \nabla_E U) \\ &= -g(\nabla\mu, \nabla_E U) = -g(\nabla\mu, \nabla_Z U) - g(\nabla\mu, \nabla_X U), \end{aligned}$$

since $U(\mu) = 0$. Here, for any $W \in \Gamma(\mathcal{D}_\theta)$, we have $g(\nabla_Z U, W) = -g(U, \nabla_Z W) = 0$, since M^θ is totally geodesic in M. Thus, $\nabla_Z U \in \Gamma(\mathcal{D}_T)$ or $\nabla_Z U \in \Gamma(\mathcal{D}_\perp)$. In either case, we have

$$g(\nabla\mu, \nabla_Z U) = 0. \tag{5.53}$$

On the other hand, from (3.27), we have

$$g(\nabla_X U, W) = -g(U, \nabla_X W) = -\csc^2\theta\{g(A_{NPW} X, U) + g(A_{NW} X, \mathcal{F}U)\}.$$

Here, using (5.37), we obtain

$$g(\nabla_X U, W) = g(W(\mu)U, X) = 0.$$

That is, $\nabla_X U \in \Gamma(\mathcal{D}_T)$ or $\nabla_X U \in \Gamma(\mathcal{D}_\perp)$. In either case, we get

$$g(\nabla\mu, \nabla_X U) = 0. \tag{5.54}$$

From (5.53) and (5.54), we find

$$g(\nabla_U \nabla\mu, E) = 0.$$

Thus, M^T is spherical, since it is also totally umbilic. Consequently, $\mathcal{D}_T$ is spherical.

Next, we show that $\mathcal{D}_\perp$ is spherical. Let $h^\perp$ denote the second fundamental form of $M^\perp$ in M. Then for $X, Y \in \Gamma(\mathcal{D}_\perp)$ and $U \in \Gamma(\mathcal{D}_T)$, using (2.4), (3.26) and (5.40), we have

$$g(h^\perp(X, Y), U) = g(\nabla_X Y, U) = -g(A_{\mathcal{F}Y} X, \mathcal{F}U) = 0. \tag{5.55}$$

On the other hand, for any $Z \in \Gamma(\mathcal{D}_\theta)$, using (3.25)

$$g(h^\perp(X, Y), Z) = -\sec^2\theta\{g(h(X, PZ), \mathcal{F}Y) + g(A_{NPZ} X, Y)\}.$$

Since M, $(\mathcal{D}_\theta, \mathcal{D}_\perp)$-mixed geodesic, $g(h(X, PZ), \mathcal{F}Y) = 0$. So, we have

$$g(h^\perp(X, Y), Z) = -g(A_{NPZ} X, Y).$$

Using (5.36), we obtain

$$g(h^\perp(X, Y), Z) = -Z(\lambda) g(X, Y).$$

By a direct calculation, we get

$$g(h^{\perp}(X,Y),Z) = -g(\nabla\lambda g(X,Y),Z), \tag{5.56}$$

where $\nabla\lambda$ is the gradient of λ. From (5.55) and (5.56), we obtain

$$h^{\perp}(X,Y) = -g(X,Y)\nabla\lambda.$$

So $M^{\perp}$ is totally umbilic in M and the mean curvature vector field is $-\nabla\lambda$. What's left is to show that $-\nabla\lambda$ is parallel. We have to satisfy $g(\nabla_X\nabla\lambda, E) = 0$ for $X \in \Gamma(\mathcal{D}_{\perp})$ and $E \in (\mathcal{D}_{\perp})^{\perp} = \mathcal{D}_{\theta} \oplus \mathcal{D}_T$. The proof is similar to the parallelity of $-\nabla\mu$. So we omit it. $-\nabla\lambda$ is parallel. So, $M^{\perp}$ is spherical, since it is also totally umbilic. Consequently, $\mathcal{D}_{\perp}$ is spherical.

Lastly, we prove that $(\mathcal{D}_T)^{\perp} = \mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp}$ and $(\mathcal{D}_{\perp})^{\perp} = \mathcal{D}_{\theta} \oplus \mathcal{D}_T$ are autoparallel. In fact, $\mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp}$ is autoparallel iff all for four types of covariant derivatives $\nabla_Z W, \nabla_Z X, \nabla_X Z, \nabla_X Y$ are again in $\Gamma(\mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp})$ for $Z, W \in \Gamma(\mathcal{D}_{\theta})$ and $X, Y \in \Gamma(\mathcal{D}_{\perp})$. This is equivalent to say that all four inner products $g(\nabla_Z W, U)$, $g(\nabla_Z X, U)$, $g(\nabla_X Z, U)$, $g(\nabla_X Y, U)$ vanish, where $U \in \Gamma(\mathcal{D}_T)$. Using (3.21) and (5.37), we find

$$\begin{aligned} g(\nabla_Z W, U) &= -\csc^2\theta\{g(A_{NPW}Z, U) + g(A_{NW}Z, \mathcal{F}U)\} \\ &= -\csc^2\theta g(A_{NPW}U + A_{NW}\mathcal{F}U, Z) \\ &= W(\mu)g(U, Z) = 0. \end{aligned}$$

Using (3.28) and (5.41), we find

$$g(\nabla_Z X, U) = -g(A_{\mathcal{F}X}Z, \mathcal{F}U) = 0.$$

By (3.27) and (5.37), we get

$$g(\nabla_X Z, U) = -\csc^2\theta\{g(A_{NPZ}X, U) + g(A_{NZ}X, \mathcal{F}U)\} = 0.$$

By (3.26) and (5.40), we find

$$g(\nabla_X Y, U) = -g(A_{\mathcal{F}Y}X, \mathcal{F}U) = 0.$$

Thus, $\mathcal{D}_{\theta} \oplus \mathcal{D}_{\perp}$ is autoparallel. On the other hand, $\mathcal{D}_{\theta} \oplus \mathcal{D}_T$ is autoparallel iff all four inner products $g(\nabla_Z W, X), g(\nabla_Z U, X), g(\nabla_U Z, X), g(\nabla_U V, X)$ vanish, where $Z, W \in \Gamma(\mathcal{D}_{\theta})$, $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_{\perp})$. Firstly, we have already $g(\nabla_Z U, X) = 0$ from above. Using (3.22) and (5.38), we get

$$g(\nabla_Z W, X) = \sec^2\theta\{g(A_{\mathcal{F}X}Z, PW) + g(A_{NPW}Z, X)\} = 0.$$

Using (3.24) and (5.39), we find

$$g(\nabla_U V, X) = g(A_{\mathcal{F}X}U, \mathcal{F}V) = 0.$$

And for last one, by (3.29) and (5.42), we get

$$g(\nabla_U Z, X) = -g(\nabla_U X, Z) = \sec^2\theta\{g(A_{\mathcal{F}X}U, PZ) + g(A_{NPZ}U, X)\} = 0.$$

So, $\mathcal{D}_\theta \oplus \mathcal{D}_T$ is autoparallel. Thus by Remark 5.1, M is locally biwarped product submanifold of the form $M^\theta \times_f M^T \times_\sigma M^\perp$.

Next, we investigate the behavior of the second fundamental form h of a non-trivial biwarped product s.s-i. submanifold of order 1 of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ of the form $M^\theta \times_f M^T \times_\sigma M^\perp$.

Lemma 5.1. *Let M be a biwarped product proper s.s-i. submanifold of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$. Then for h of M in $(\bar{M}, g, \mathcal{F})$, we have*

$$g(h(U,V), NW) = -W(\ln f)g(U, \mathcal{F}V) + PW(\ln f)g(U,V), \tag{5.57}$$

$$g(h(Z,U), NW) = 0, \tag{5.58}$$

$$g(h(X,U), NW) = 0, \tag{5.59}$$

$$g(h(Z,U), \mathcal{F}X) = 0, \tag{5.60}$$

$$g(h(X,U), \mathcal{F}Y) = 0, \tag{5.61}$$

$$g(h(U,V), \mathcal{F}X) = 0, \tag{5.62}$$

where $Z, W \in \Gamma(\mathcal{D}_\theta)$, $X, Y \in \Gamma(\mathcal{D}_\perp)$ and $U, V \in \Gamma(\mathcal{D}_T)$.

Proof. For $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$, using (2.4), (2.13) – (2.15) and (3.17), we have

$$\begin{aligned} g(h(U,V), NW) &= g(\bar{\nabla}_U V, NW) = -g(V, \bar{\nabla}_U NW) \\ &= -g(V, \bar{\nabla}_U \mathcal{F}W) + g(V, \bar{\nabla}_U PW) \\ &= -g(\mathcal{F}V, \bar{\nabla}_U W) + g(V, \nabla_U PW) \\ &= -g(\mathcal{F}V, \nabla_U W) + g(V, \nabla_U PW) \\ &= -W(\ln f)g(\mathcal{F}V, U) + PW(\ln f)g(U,V). \end{aligned}$$

Thus, we get (5.57). Now, using (2.4), (2.13) – (2.15) and (3.17), we get

$$\begin{aligned} g(h(Z,U),NW) &= g(\bar\nabla_Z U, NW) = -g(U, \bar\nabla_Z NW) \\ &= -g(U, \bar\nabla_Z \mathcal{F}W) + g(U, \bar\nabla_Z PW) \\ &= -g(\mathcal{F}U, \bar\nabla_Z W) + g(U, \nabla_Z PW) \\ &= g(W, \bar\nabla_Z(\mathcal{F}U)) - g(\nabla_Z U, PW) \\ &= g(W, \nabla_Z \mathcal{F}U) - g(\nabla_Z U, PW), \end{aligned}$$

for $Z, W \in \Gamma(\mathcal{D}_\theta)$ and $U \in \Gamma(\mathcal{D}_T)$. Here using (1.2), we get

$$g(h(Z,U),NW) = Z(\ln f)g(W,\mathcal{F}U) - Z(\ln f)g(U,PW) = 0$$

since $g(W,\mathcal{F}U) = g(U,PW) = 0$. So (5.58) follows. The proof of (5.59) is similar.

For $Z \in \Gamma(\mathcal{D}_\theta)$, $X \in \Gamma(\mathcal{D}_\perp)$ and $U \in \Gamma(\mathcal{D}_T)$, using (2.4), (2.13) – (2.15) and (3.17), we get

$$\begin{aligned} g(h(Z,U),\mathcal{F}X) &= g(\bar\nabla_Z U, \mathcal{F}X) = -g(U, \bar\nabla_Z \mathcal{F}X) \\ &= -g(\mathcal{F}U, \bar\nabla_Z X) = -g(\mathcal{F}U, \nabla_Z X) \\ &= -Z(\ln\sigma)g(\mathcal{F}U, X) = 0 \end{aligned}$$

since $g(\mathcal{F}U, X) = 0$. So (5.60) follows. Next, using (2.4), (2.13) – (2.15), (3.17) and (1.3) we get

$$\begin{aligned} g(h(X,U),\mathcal{F}Y) &= g(\bar\nabla_X U, \mathcal{F}Y) = -g(U, \bar\nabla_X \mathcal{F}Y) \\ &= -g(\mathcal{F}U, \bar\nabla_X Y) = -g(\mathcal{F}U, \nabla_X Y) \\ &= g(\nabla_X \mathcal{F}U, Y) = 0 \end{aligned}$$

for $U \in \Gamma(\mathcal{D}_T)$ and $X, Y \in \Gamma(\mathcal{D}_\perp)$. Thus, (5.61) follows. Lastly, using (2.4), (2.13) – (2.15), (3.17) and (1.3) we get

$$\begin{aligned} g(h(U,V),\mathcal{F}X) &= g(\bar\nabla_U V, \mathcal{F}X) = -g(V, \bar\nabla_U \mathcal{F}X) \\ &= -g(\mathcal{F}V, \bar\nabla_U X) = -g(\mathcal{F}V, \nabla_U X) = 0 \end{aligned}$$

for $U, V \in \Gamma(\mathcal{D}_T)$ and $X \in \Gamma(\mathcal{D}_\perp)$. So, we have (5.62). The other assertions can be obtained by a similar way.

The previous lemma shows partially us the behavior of the second fundamental form h of the biwarped product proper s.s-i. submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ in the normal subbundle $N(\mathcal{D}_\theta)$ and $\mathcal{F}(\mathcal{D}_\perp)$.

Remark 5.2. *The equations* (5.57), (5.58), (5.59) *and* (5.60) *also were obtained as Lemma 3.1-(ii), Lemma 3.1-(i), Lemma 3.3-(ii) and Lemma 3.3-(i), respectively in* [22].

By using (5.58) – (5.61), we immediately have the following result.

Corollary 5.1. *Let M be a biwarped-product proper s.s-i. submanifold of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$ of a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$ such that the invariant normal subbundle $\bar{\mathcal{D}}_T = \{0\}$. Then M is $(\mathcal{D}_T, \mathcal{D}_\perp)$ and $(\mathcal{D}_T, \mathcal{D}_\theta)$-mixed geodesic.*

Lastly, we give another main result of this section.

Theorem 5.2. *Let M be a biwarped-product proper s.s-i. submanifold of order 1 in the form $M^\theta \times_f M^T \times_\sigma M^\perp$ of a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$ such that its invariant normal subbundle $\bar{\mathcal{D}}_T = \{0\}$. Then M is a locally warped product in the form $M^\theta \times M^T \times_\sigma M^\perp$ iff M is $\mathcal{D}_T$-geodesic.*

Proof. If M is a locally warped product of the form $M^\theta \times M^T \times_\sigma M^\perp$, then the warping function f is constant. By (5.57), we have

$$g(h(U,V), NW) = -W(\ln f)g(U, \mathcal{F}V) + PW(\ln f)g(U,V) = 0$$

for $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$, since $W(\ln f) = PW(\ln f) = 0$. Using this fact and (5.62), it follows that $h(U,V) = 0$. Which say us M is $\mathcal{D}_T$-geodesic.
Conversely, let M be $\mathcal{D}_T$-geodesic. Then for any $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$, we have

$$W(\ln f)g(U, \mathcal{F}V) + PW(\ln f)g(U,V) = 0 \tag{5.63}$$

from (5.57). If we put $W = PW$ in (5.63) and using (3.18), we obtain

$$PW(\ln f)g(U, \mathcal{F}V) + \cos^2\theta W(\ln f)g(U,V) = 0. \tag{5.64}$$

If we replace V by $\mathcal{F}V$ in (5.64), then (5.64) becomes

$$PW(\ln f)g(U,V) + \cos^2\theta W(\ln f)g(U, \mathcal{F}V) = 0. \tag{5.65}$$

From (5.63) and (5.65), we get

$$\sin^2\theta W(\ln f)g(U, \mathcal{F}V) = 0 \tag{5.66}$$

for any $U, V \in \Gamma(\mathcal{D}_T)$ and $W \in \Gamma(\mathcal{D}_\theta)$. Since (5.66) is true for any $U, V \in \Gamma(\mathcal{D}_T)$, it is also true for $\mathcal{F}V \in \Gamma(\mathcal{D}_T)$. So (5.66) becomes

$$\sin^2\theta W(\ln f)g(U,V) = 0. \tag{5.67}$$

Since M is proper, $\sin\theta \neq 0$, we can deduce that $W(\ln f) = 0$ from (5.67). Namely, we find f as a constant. Thus, M must be a locally warped product in the form $M^\theta \times M^T \times_\sigma M^\perp$.

6. An inequality for non-trivial biwarped product s.s-i. submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$

In this section, we shall establish an inequality for the squared norm of the second fundamental form in terms of the warping functions for biwarped product skew semi-invariant submanifolds of order 1 of the form $M^\theta \times_f M^T \times_\sigma M^\perp$, where M^θ is a proper slant, M^T is a invariant and $M^\perp$ is an anti-invariant submanifold in a l.p.R. manifold $(\bar{M}, g, \mathcal{F})$.

Let $M_0 \times_{f_1} M_1 \times_{f_2} M_2$ be a biwarped product submanifold in a Riemannian manifold $\bar{M}$. Then from [9], we write

$$\begin{aligned} K(X_0, X_i) = K_{0i} = & \ \frac{1}{f_i}((\nabla_{X_0} X_0)(f_i) - X_0(X_0(f_i))) \\ K(X_i, X_j) = K_{ij} = & \ -\frac{g(\nabla f_i, \nabla f_j)}{f_i f_j}, \quad i, j = 1, 2, \end{aligned} \tag{6.68}$$

for each unit vector X_i tangent to M_i. If we consider the local orthonormal frame $\{e_1, e_2, \dots, e_m\}$ of TM, in view of Gauss equation (2.6), we derive

$$\tau(TM) = \bar{\tau}(TM) + \sum_{r=m+1}^{\bar{m}} \sum_{1 \leq i \neq j \leq m} \Big(h_{ii}^r h_{jj}^r - (h_{ij}^r)^2 \Big), \tag{6.69}$$

where $\bar{m} - m = dim T^\perp M$.

Now we are ready to prove the general inequality. Let M be a $m = m_0 + m_1 + m_2$-dimensional biwarped product s.s-i. submanifolds of order 1 of type $M^\theta \times_f M^T \times_\sigma M^\perp$ in a locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. A canonical orthonormal basis is given by $\{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{m_0+m_1}, e_{m_0+m_1+1}, \dots, e_{m_0+m_1+m_2}, e_{m+1}, \dots, e_{\bar{m}}\}$ of $T\bar{M}$ such that $\{e_1, \dots, e_{m_0}\}$ is an orthonormal basis of TM^θ, $\{e_{m_0+1}, \dots, e_{m_0+m_1}\}$ is an orthonormal basis of TM^T, $\{e_{m_0+m_1+1}, \dots, e_{m_0+m_1+m_2}\}$ is an orthonormal basis of $TM^\perp$, $\{e_{m+1}, \dots, e_{\bar{m}}\}$ is an orthonormal basis of $T^\perp M$.

Theorem 6.1. *Let $M = M^\theta \times_f M^T \times_\sigma M^\perp$ be an m-dimensional non-trivial biwarped product s.s-i. submanifold M of order 1 of an $\bar{m}$-dimensional locally product Riemannian manifold $(\bar{M}, g, \mathcal{F})$. Then*

(i) *the second fundamental form of M satisfies*

$$\begin{aligned} \tfrac{1}{2} \parallel h \parallel^2 \geq & \ \bar{\tau}(TM) - \bar{\tau}(TM^\theta) - \bar{\tau}(TM^T) - \bar{\tau}(TM^\perp) \\ & -m_1 \frac{\Delta f}{f} - m_2 \frac{\Delta \sigma}{\sigma} + m_1 m_2 \frac{g(\nabla f, \nabla \sigma)}{f \sigma}, \end{aligned} \tag{6.70}$$

where $m_1 = dimM^T$ and $m_2 = dimM^\perp$.

(ii) *The equality case of the inequality (6.70) holds identically iff M^θ is also totally geodesic in $\bar{M}$, and both M^T and $M^\perp$ are totally umbilic in $\bar{M}$.*

Proof. Putting $U = W = e_i$ and $V = Z = e_j$ in Gauss equation (2.6), we obtain

$$\bar{R}(e_i, e_j, e_j, e_i) = R(e_i, e_j, e_j, e_i) + g(h(e_i, e_j), h(e_i, e_j)) - g(h(e_i, e_i), h(e_j, e_j)).$$

Taking summation, over $1 \leq i, j \leq m (i \neq j)$ in above equation, we obtain

$$2\bar{\tau}(TM) = 2\tau(TM) - m^2 \parallel H \parallel^2 + \parallel h \parallel^2 .$$

Then from (2.11), we derive

$$\begin{aligned}\frac{1}{2} \parallel h \parallel^2 &= \frac{m^2}{2} \parallel H \parallel^2 + \bar{\tau}(TM) - \sum_{1 \leq i < j \leq m_0} K_{ij} \\ &- \sum_{m_0+1 \leq i < j \leq m_0+m_1} K_{ij} - \sum_{m_0+m_1+1 \leq i < j \leq m_0+m_1+m_2} K_{ij} - \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} \\ &- \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} - \sum_{i=m_0+1}^{m_0+m_1} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij}.\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\frac{1}{2} \parallel h \parallel^2 &= \frac{m^2}{2} \parallel H \parallel^2 + \bar{\tau}(TM) - \tau(TM^\theta) - \tau(TM^T) - \tau(TM^\perp) \\ &- \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} - \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} - \sum_{i=m_0+1}^{m_0+m_1} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij}.\end{aligned}$$

Last three terms of first line of above equation can be obtained by using (6.69), then we get

$$\begin{aligned}\frac{1}{2} \parallel h \parallel^2 &= \frac{m^2}{2} \parallel H \parallel^2 + \bar{\tau}(TM) \\ &- \bar{\tau}(TM^\theta) - \sum_{r=m+1}^{\bar{m}} \sum_{1 \leq i \neq t \leq m_0} \left(h_{ii}^r h_{tt}^r - (h_{it}^r)^2 \right) \\ &- \bar{\tau}(TM^T) - \sum_{r=m+1}^{\bar{m}} \sum_{m_0+1 \leq j \neq l \leq m_0+m_1} \left(h_{jj}^r h_{ll}^r - (h_{jl}^r)^2 \right) \\ &- \bar{\tau}(TM^\perp) - \sum_{r=m+1}^{\bar{m}} \sum_{m_0+m_1+1 \leq a \neq b \leq m_0+m_1+m_2} \left(h_{aa}^r h_{bb}^r - (h_{ab}^r)^2 \right) \\ &- \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} - \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} - \sum_{i=m_0+1}^{m_0+m_1} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij}.\end{aligned} \tag{6.71}$$

Now, using (6.68), for a biwarped product submanifold, we find

$$\sum_{i=1}^{m_0} \sum_{j=m_0+1}^{m_0+m_1} K_{ij} = m_1 \frac{\Delta f}{f}, \qquad \sum_{i=1}^{m_0} \sum_{j=m_0+m_1+1}^{m_0+m_1+m_2} K_{ij} = m_2 \frac{\Delta \sigma}{\sigma}$$

and

$$\sum_{i=m_0+1}^{m_0+m_1}\sum_{j=m_0+m_1+1}^{m_0+m_1+m_2}K_{ij}=-m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}.$$

If we use these equations in (6.71), we obtain

$$\begin{aligned}\frac{1}{2}\parallel h\parallel^2 &= \frac{m^2}{2}\parallel H\parallel^2+\bar{\tau}(TM)-m_1\frac{\Delta f}{f}-m_2\frac{\Delta\sigma}{\sigma}+m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\ &-\bar{\tau}(TM^{\theta})-\sum_{r=m+1}^{\bar{m}}\sum_{1\leq i\neq t\leq m_0}\Big(h_{ii}^r h_{tt}^r-(h_{it}^r)^2\Big)\\ &-\bar{\tau}(TM^{T})-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\leq j\neq l\leq m_0+m_1}\Big(h_{jj}^r h_{ll}^r-(h_{jl}^r)^2\Big)\\ &-\bar{\tau}(TM^{\perp})-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\leq a\neq b\leq m_0+m_1+m_2}\Big(h_{aa}^r h_{bb}^r-(h_{ab}^r)^2\Big).\end{aligned}$$

If we arrange this equation, we arrive to

$$\begin{aligned}\frac{1}{2}\parallel h\parallel^2 &= \frac{m^2}{2}\parallel H\parallel^2+\bar{\tau}(TM)-m_1\frac{\Delta f}{f}-m_2\frac{\Delta\sigma}{\sigma}+m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\ &-\bar{\tau}(TM^{\theta})-\bar{\tau}(TM^{T})-2\bar{\tau}(TM^{\perp})\\ &\sum_{r=m+1}^{\bar{m}}\sum_{1\leq i\neq t\leq m_0}(h_{it}^r)^2+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\leq j\neq l\leq m_0+m_1}(h_{jl}^r)^2\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\leq a\neq b\leq m_0+m_1+m_2}(h_{ab}^r)^2-\sum_{r=m+1}^{\bar{m}}\sum_{1\leq i\neq t\leq m_0}(h_{ii}^r h_{tt}^r)\\ &-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\leq j\neq l\leq m_0+m_1}(h_{jj}^r h_{ll}^r)\\ &-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\leq a\neq b\leq m_0+m_1+m_2}(h_{aa}^r h_{bb}^r).\end{aligned}$$

Adding and substracting the term $\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)$ in the above equation, we find that

$$\begin{aligned}\frac{1}{2}\parallel h\parallel^2 \;=\;& \frac{m^2}{2}\parallel H\parallel^2+\bar{\tau}(TM)-m_1\frac{\Delta f}{f}-m_2\frac{\Delta\sigma}{\sigma}+m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\ &-\bar{\tau}(TM^\theta)-\bar{\tau}(TM^T)-\bar{\tau}(TM^\perp)\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\ne t\le m_0}(h_{it}^r)^2+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\le j\ne l\le m_0+m_1}(h_{jl}^r)^2\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\le a\ne b\le m_0+m_1+m_2}(h_{ab}^r)^2-\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\ne t\le m_0}(h_{ii}^rh_{tt}^r)\\ &-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\le j\ne l\le m_0+m_1}(h_{jj}^rh_{ll}^r)\\ &-\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\le a\ne b\le m_0+m_1+m_2}(h_{aa}^rh_{bb}^r)\\ &+\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)\\ &-\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2)+\ldots+(h_{mm}^r)^2\Big).\end{aligned}\tag{6.72}$$

Here, by (2.7), we have

$$\parallel H\parallel^2=\frac{1}{m^2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)+2\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\ne j\le m}(h_{ii}^rh_{jj}^r).$$

Using this equation in (6.72), we obtain

$$\begin{aligned}\frac{1}{2}\|h\|^2 \;=\;& \frac{m^2}{2}\parallel H\parallel^2+\bar{\tau}(TM)-m_1\frac{\Delta f}{f}-m_2\frac{\Delta\sigma}{\sigma}+m_1m_2\frac{g(\nabla f,\nabla\sigma)}{f\sigma}\\ &-\bar{\tau}(TM^\theta)-\bar{\tau}(TM^T)-\bar{\tau}(TM^\perp)\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\ne t\le m_0}(h_{it}^r)^2+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+1\le j\ne l\le m_0+m_1}(h_{jl}^r)^2\\ &+\sum_{r=m+1}^{\bar{m}}\sum_{m_0+m_1+1\le a\ne b\le m_0+m_1+m_2}(h_{ab}^r)^2-\frac{m^2}{2}\parallel H\parallel^2\\ &+\frac{1}{2}\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big).\end{aligned}\tag{6.73}$$

Now, the inequality (6.70) comes from (6.73). The equality sign in (6.70) holds iff

$$\begin{aligned}&\sum_{r=m+1}^{\bar{m}}\sum_{1\le i\ne t\le m}(h_{it}^r)^2=0\qquad \textit{and}\\ &\sum_{r=m+1}^{\bar{m}}\Big((h_{11}^r)^2+\ldots+(h_{mm}^r)^2\Big)=0.\end{aligned}\tag{6.74}$$

It follows that, $h_{ij}^r = g(h(e_i, e_j), e_r) = 0$ for $i, j \in 1, \dots, m$ and $r \in m+1, \dots, \bar{m}$. Which says us $h \equiv 0$. For a biwarped product submanifold of the form $M = M^\theta \times_f M^T \times_\sigma M^\perp$, we know already that M^θ is totally geodesic in M and both M^T and $M^\perp$ are totally umbilic in M. Since, the second fundamental form h of M vanishes, identically, it follows that M^θ is also totally geodesic in $\bar{M}$ and both M^T and $M^\perp$ are also totally umbilic in $\bar{M}$.

Now we give an application of the inequality (6.70).

Theorem 6.2. *Let $M = M^\theta \times_f M^T \times_\sigma M^\perp$ be an m-dimensional non-trivial biwarped product s.s-i. submanifold M of order 1 of an $\bar{m}$-dimensional locally product Riemannian manifold $(\bar{M} = M_1(c_1) \times M_2(c_2), \mathcal{F}, g)$. Then the squared norm of the second fundemental form h of M satisfies*

$$\begin{aligned} \| h \|^2 \geq\; & \frac{1}{2}(c_1 + c_2)\Big(m_0 m_1 + m_0 m_2 + m_1 m_2\Big) - 2m_1 \frac{\Delta f}{f} - 2m_2 \frac{\Delta \sigma}{\sigma} \\ & + 2m_1 m_2 \frac{g(\nabla f, \nabla \sigma)}{f\sigma}, \end{aligned} \tag{6.75}$$

where $m_0 = dim M^\theta$, $m_1 = dim M^T$, $m_2 = dim M^\perp$ and $m_0 + m_1 + m_2 = m$.

Proof. In (2.16), substituting $X = e_i$, $Y = Z = e_j$ and take inner product with e_i in the above equation, we obtain

$$\begin{aligned} \bar{R}(e_i, e_j, e_j, e_i) =\; & \frac{1}{4}(c_1 + c_2)\Big\{ g(e_j, e_j) g(e_i, e_i) - g(e_i, e_j) g(e_j, e_i) \\ & + g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) - g(\mathcal{F}e_i, e_j) g(\mathcal{F}e_j, e_i) \Big\} \\ & + \frac{1}{4}(c_1 - c_2)\Big\{ g(e_j, e_j) g(\mathcal{F}e_i, e_i) - g(e_i, e_j) g(\mathcal{F}e_j, e_i) \\ & + g(\mathcal{F}e_j, e_j) g(e_i, e_i) - g(\mathcal{F}e_i, e_j) g(e_j, e_i) \Big\}. \end{aligned}$$

Taking summation over basis vectors of TM for $1 \leq i \neq j \leq m$, we get

$$\begin{aligned} 2\bar{\tau}(TM) =\; & \frac{1}{4}(c_1 + c_2)\Big\{ \sum_{1 \leq i \neq j \leq m} g(e_j, e_j) g(e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(e_i, e_j)^2 \\ & + \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_i, e_j) g(\mathcal{F}e_j, e_i) \Big\} \\ & + \frac{1}{4}(c_1 - c_2)\Big\{ \sum_{1 \leq i \neq j \leq m} g(e_j, e_j) g(\mathcal{F}e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(e_i, e_j) g(\mathcal{F}e_j, e_i) \\ & + \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_j, e_j) g(e_i, e_i) - \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_i, e_j) g(e_j, e_i) \} \Big\}. \end{aligned}$$

Let M be an m-dimensional non-trivial biwarped product s.s-i. submanifold M of order 1 of an $\bar{m}$-dimensional locally product Riemannian manifold $\bar{M} = M_1(c_1) \times M_2(c_2)$ in the form $M^\theta \times_f M^T \times_\sigma M^\perp$. We choose the orthonormal frame fields of TM^θ and TM^T

as $\{e_1 = \sec\theta Pe_1, \ldots, e_{m_0} = \sec\theta Pe_{m_0}\}$ and $\{\mathcal{F}e_{m_0+1} = e_{m_0+1}, \ldots, \mathcal{F}e_t = e_t, \mathcal{F}e_{t+1} = -e_{t+1}, \ldots, \mathcal{F}e_{m_0+m_1} = -e_{m_0+m_1}\}$, respectively. Also, we choose the orthonormal frame fields of $TM^\perp$ as $\{e_{m_0+m_1+1}, \ldots, e_{m_0+m_1+m_2}\}$. Here, for $1 \leq i \leq m_0$, we have $g(\mathcal{F}e_i, e_i) = \cos\theta$ and for $1 \leq i \neq j \leq m_0$, we have $g(\mathcal{F}e_i, e_j) = 0$, since M^θ is a slant submanifold with slant angle θ. Also, for $m_0 + 1 \leq i \leq t$, we have $g(\mathcal{F}e_i, e_i) = 1$ and for $t + 1 \leq i \leq m_0 + m_1$, we have $g(\mathcal{F}e_i, e_i) = -1$. Moreover, for $m_0 + m_1 + 1 \leq i \leq m_0 + m_1 + m_2 = m$, we have $g(\mathcal{F}e_i, e_i) = 0$ and for $m_0 + m_1 + 1 \leq i \neq j \leq m_0 + m_1 + m_2 = m$, we have $g(\mathcal{F}e_i, e_j) = 0$, since $M^\perp$ is an anti-invariant submanifold. Thus, using these facts, we obtain the following

$$\sum_{m_0+1 \leq i \neq j \leq m_0+m_1} g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) = m_1 - 3,$$

$$\sum_{1 \leq i \neq j \leq m_0} g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) = (m_0 - 1)\cos^2\theta,$$

$$\sum_{m_0+1 \leq i \neq j \leq m_0+m_1} g(e_j, e_j) g(\mathcal{F}e_i, e_i) = 2t - m_1 - 1,$$

$$\sum_{1 \leq i \neq j \leq m_0} g(e_j, e_j) g(\mathcal{F}e_i, e_i) = (m_0 - 1)\cos\theta,$$

$$\sum_{m_0+m_1+1 \leq i \neq j \leq m} g(\mathcal{F}e_j, e_j) g(\mathcal{F}e_i, e_i) = \sum_{m_0+m_1+1 \leq i \neq j \leq m} g(e_j, e_j) g(\mathcal{F}e_i, e_i) = 0,$$

and

$$\sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_i, e_j) g(e_j, e_i) = \sum_{1 \leq i \neq j \leq m} g(\mathcal{F}e_i, e_j) g(\mathcal{F}e_j, e_i) = 0.$$

Thus, we find

$$\begin{aligned} 2\bar{\tau}(TM) \;=\;& \frac{1}{4}(c_1 + c_2)\Big\{ m(m-1) + m_1 - 3 + (m_0 - 1)\cos^2\theta \Big\} \\ &+ \frac{1}{4}(c_1 - c_2)\Big\{ 2(2t - m_1 - 1) + 2(m_0 - 1)\cos\theta \Big\}. \end{aligned} \tag{6.76}$$

Similarly for TM^θ, TM^T and $TM^\perp$, we derive

$$\begin{aligned} 2\bar{\tau}(TM^\theta) \;=\;& \frac{1}{4}(c_1 + c_2)\Big\{ m_0(m_0 - 1) + (m_0 - 1)\cos^2\theta \Big\} \\ &+ \frac{1}{4}(c_1 - c_2)\Big\{ 2(m_0 - 1)\cos\theta \Big\} \end{aligned} \tag{6.77}$$

$$\begin{aligned} 2\bar{\tau}(TM^T) \;=\;& \frac{1}{4}(c_1 + c_2)\Big\{ m_1(m_1 - 1) + m_1 - 3 \Big\} \\ &+ \frac{1}{4}(c_1 - c_2)\Big\{ 2(2t - m_1 - 1) \Big\} \end{aligned} \tag{6.78}$$

$$2\bar{\tau}(TM^\perp) = \frac{1}{4}(c_1 + c_2)\Big\{ m_2(m_2 - 1) \Big\}. \tag{6.79}$$

Thus, using (6.76) – (6.79) in (6.70), we get the inequality (6.75).

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ON THE MANNHEIM PARTNER OF A CUBIC BEZIER CURVE IN E^3

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Abstract. In this study we have examined, Mannheim partner of a cubic Bezier curve based on the control points with matrix form in E^3. Frenet vector fields and also curvatures of Mannheim partner of the cubic Bezier curve are examined based on the Frenet apparatus of the first cubic Bezier curve in E^3.

Keywords: Bézier curves, Mannheim partner, Cubic Bezier curve

2010 Mathematics Subject Classification: 53A04, 53A05.

1. Introduction and Preliminaries

French engineer Pierre Bézier, who used Bézier curves to design automobile bodies studied with them in 1962. But the study of these curves was first developed in 1959 by mathematician Paul de Casteljau using de Casteljau's algorithm, a numerically stable method to evaluate Bézier curves. A Bézier curve is frequently used in computer graphics and related fields, in vector graphics, used in animation as a tool to control motion. To guarantee smoothness, the control point at which two curves meet must be on the line between the two control points on either side. In animation applications, such as Adobe Flash and Synfig, Bézier curves are used to outline, for example, movement. Users outline the wanted path in Bézier curves, and the application creates the needed frames for the object to move along the path. For 3D animation Bézier curves are often used to define 3D paths as well as 2D

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curves for key-frame interpolation. We have been motivated by the following studies. First Bezier-curves with curvature and torsion continuity has been examined in [6]. Also in [2], [3] and [7] Bezier curves and surfaces have been given. In [4] Bézier curves are designed for Computer-Aided Geometric. Recently equivalence conditions of control points and application to planar Bezier curves have been examined. In [8] Frenet apparatus of the cubic Bézier curves has been examined in E^3. Before, the 5^{th} order Bézier curve and its, first, second, and third based on the control points of the 5^{th} order Bézier Curve in E^3 are examined too in [12]. We have already examine in cubic Bézier curves and involutes in [8] and [9], respectively. Also Bertrand mate of a cubic Bezier curve based on the control points with matrix form has been examined with Frenet apparatus in [11]. Here we will examine the Mannheim partner of a cubic Bezier curve, based on the control points with matrix representation.

The set, whose elements are Frenet vector fields and the curvatures of a curve $\alpha(t) \subset \mathbf{E}^3$, is called Frenet apparatus of the curves. Let $\alpha(t)$ be the curve, with $\eta = \|\alpha'(t)\| \neq 1$ and Frenet apparatus be $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$. Frenet vector fields are given for a non arc-length curve

$$T(t) = \frac{\alpha'(t)}{\|\alpha'(t)\|}, \qquad N(t) = B(t)\Lambda T(t), \qquad B(t) = \frac{\alpha'(t)\Lambda\alpha''(t)}{\|\alpha'(t)\Lambda\alpha''(t)\|},$$

$$\kappa(t) = \frac{\left\|\alpha'(t)\Lambda\alpha''(t)\right\|}{\|\alpha'(t)\|^3} \quad \text{and} \quad \tau(t) = \frac{\left\langle \alpha'(t)\Lambda\alpha''(t), \alpha'''(t)\right\rangle}{\|\alpha'(t)\Lambda\alpha''(t)\|^2}$$

where $\kappa(t)$ and $\tau(t)$ are curvature functions. Also Frenet formulas are well known as

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \eta\kappa & 0 \\ -\eta\kappa & 0 & \eta\tau \\ 0 & -\eta\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Generally, Béziers curve can be defined by $n+1$ control points $P_0, P_1, ..., P_n$ with the parametrization

$$\mathbf{B}(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} [P_i],$$

where $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ is known as the usual binomial coefficients. In this study we will define and work on cubic Bézier curves in E^3. For more detail see [1, 8].

Definition 1.1. *A cubic Bézier curve is a special Bézier curve and it has only four points P_0, P_1, P_2 and P_3, its parametrization is*

$$\alpha(t) = (1-t)^3 P_0 + 3t(1-t)^2 P_1 + 3t^2(1-t) P_2 + t^3 P_3$$

and matrix form of the cubic Bezier curve with control points P_0, P_1, P_2, P_3, is

$$\alpha(t) = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}.$$

Also using the derivatives of a cubic Bézier curve Frenet apparatus $\{T, N, B, \kappa, \tau\}$ have already been given as in the following theorems by using matrix representation. For more detail see in [8].

The first derivative of a cubic Bézier curve by using matrix representation is

$$\alpha'(t) = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

where $Q_0 = 3(P_1 - P_0) = (x_0, y_0, z_0)$, $Q_1 = 3(P_2 - P_1) = (x_1, y_1, z_1)$, $Q_2 = 3(P_3 - P_2) = (x_2, y_2, z_2)$ are control points.

The second derivative of a cubic Bézier curve by using matrix representation is

$$\alpha''(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} R_0 \\ R_1 \end{bmatrix}$$

where $R_0 = 6(P_2 - 2P_1 + P_0)$, $R_1 = 6(P_3 - 2P_2 + P_1)$ are control points.

The third derivative of a cubic Bézier curve is constant by using matrix representation is

$$\alpha'''(t) = [R_0 R_1]$$

with the control point $[R_0 R_1] = R_1 - R_0 = 2[Q_1 Q_2] - 2[Q_0 Q_1]$.

Frenet apparatus $\{T(t), N(t), B(t), \kappa(t), \tau(t)\}$ of a cubic Bézier curve have already been given as in the following theorems by using the matrix representation. For more detail see in [9].

Tangent vector field of a cubic Bezier curve α with, $\|\alpha'\| = \eta$ has the following the matrix representation

$$T(t) = \frac{1}{\eta}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{bmatrix}$$

$$= \frac{1}{\eta}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} Q_0 \\ Q_1 \\ Q_2 \end{bmatrix}$$

$$= \frac{1}{\eta}\left(Q_0\left(t^2 - 2t + 1\right) - Q_1\left(2t^2 - 2t\right) + t^2 Q_2\right).$$

Binormal vector field of a cubic Bezier curve by using the matrix representation is

$$B(t) = \frac{6}{m}\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \frac{6}{m}\begin{bmatrix} t^2 & t & 1 \end{bmatrix}\begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$= \frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)$$

where $m = \|\alpha' \Lambda \alpha''\|$ and

$$b_{11} = \left(y_0 z_1 - y_1 z_0 - y_0 z_2 + y_2 z_0 + y_1 z_2 - y_2 z_1\right),$$

$$b_{12} = \left(x_1 z_0 - x_0 z_1 + x_0 z_2 - x_2 z_0 - x_1 z_2 + x_2 z_1\right),$$

$$b_{13} = \left(x_0 y_1 - x_1 y_0 - x_0 y_2 + x_2 y_0 + x_1 y_2 - x_2 y_1\right),$$

$$b_{21} = \left(2y_1 z_0 + y_0 z_2 - 2y_0 z_1 - y_2 z_0\right),$$

$$b_{22} = \left(2x_0 z_1 - 2x_1 z_0 - x_0 z_2 + x_2 z_0\right),$$

$$b_{23} = \left(2x_1 y_0 - 2x_0 y_1 + x_0 y_2 - x_2 y_0\right),$$

$$b_{31} = y_0 z_1 - y_1 z_0,$$

$$b_{32} = x_1 z_0 - x_0 z_1,$$

$$b_{33} = x_0 y_1 - x_1 y_0.$$

Normal vector field of a cubic Bezier curve is a $4th$ order Bezier curve and it has the matrix representation as in

$$\begin{aligned} N(t) &= \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \\ n_{41} & n_{42} & n_{43} \\ n_{51} & n_{52} & n_{53} \end{bmatrix} \\ &= \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix} \\ &= \frac{6}{\eta m}\left(N_0 t^4 + N_1 t^3 + N_2 t^2 + N_3 t + N_4\right) \end{aligned}$$

where

$$n_{11} = b_{12}d_{13} - b_{13}d_{12},$$

$$n_{21} = b_{12}d_{23} - b_{13}d_{22} + b_{22}d_{13} - b_{23}d_{12},$$

$$n_{31} = b_{12}d_{33} - b_{13}d_{32} + b_{22}d_{23} - b_{23}d_{22} + b_{32}d_{13} - b_{33}d_{12},$$

$$n_{41} = b_{22}d_{33} - b_{23}d_{32} + b_{32}d_{23} - b_{33}d_{22},$$

$$n_{51} = b_{32}d_{33} - b_{33}d_{32},$$

$$n_{12} = b_{11}d_{13} - b_{13}d_{11},$$

$$n_{22} = -b_{11}d_{23} - b_{21}d_{13} + b_{13}d_{21} + b_{23}d_{11},$$

$$n_{32} = b_{23}d_{21} + b_{33}d_{11} - b_{11}d_{33} - b_{21}d_{23} + b_{13}d_{31} - b_{31}d_{13},$$

$$n_{42} = -b_{21}d_{33} - b_{31}d_{23} + b_{23}d_{31} + b_{33}d_{21},$$

$$n_{52} = -b_{31}d_{33} + b_{33}d_{31},$$

$$n_{13} = b_{11}d_{12} - b_{12}d_{11},$$

$$n_{23} = b_{11}d_{22} - b_{12}d_{21} + b_{21}d_{12} - b_{22}d_{11},$$

$$n_{33} = b_{11}d_{32} - b_{12}d_{31} + b_{21}d_{22} - b_{22}d_{21} + b_{31}d_{12} - b_{32}d_{11},$$

$$n_{43} = b_{21}d_{32} - b_{22}d_{31} + b_{31}d_{22} - b_{32}d_{21},$$

$$n_{53} = b_{31}d_{32} - b_{32}d_{31}.$$

The first and second curvatures of a cubic Bezier curve by using the matrix representation are

$$\begin{aligned}
\kappa(t) &= \frac{6}{\eta^3}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} b_{11}^2 + b_{12}^2 + b_{13}^2 \\ 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23} \\ 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2 \\ 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33} \\ b_{31}^2 + b_{32}^2 + b_{33}^2 \end{bmatrix} \\
&= \frac{6}{\eta^3}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{bmatrix} \\
&= \frac{6}{\eta^3}\left(C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5\right)
\end{aligned}$$

where

$C_1 = b_{11}^2 + b_{12}^2 + b_{13}^2,$

$C_2 = 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23},$

$C_3 = 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2,$

$C_4 = 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33},$

$C_5 = b_{31}^2 + b_{32}^2 + b_{33}^2,$

and

$$\tau(t) = \frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}.$$

2. Mannheim partner of a cubic Bezier curve

Mannheim curve was firstly defined by A. Mannheim in 1878. A curve is called a Mannheim curve if and only if $\dfrac{\kappa}{\kappa^2+\tau^2}$ is a nonzero constant, κ is the curvature and τ is the torsion. Mannheim curve was redefined as; if the principal normal vector of first curve and binormal vector of second curve are linearly dependent, then first curve is called Mannheim curve, and the second curve is called Mannheim partner curve by Liu and Wang. As a result they called these new curves as Mannheim partner curves. For more detail see

[10]. $\alpha^*(t) = \alpha(t) + \mu(t) B^*(t)$, $N = B^*$. Hence $\alpha^*(t) = \alpha(t) + \mu(t) N(t)$. We know for a Mannheim curve α, that μ is constant.
Since $\dfrac{d\alpha^*}{dt} = \eta T + \dot{\mu}(t) N(t) + \eta\mu(-\kappa T + \tau B)$, $\dfrac{d\alpha^*}{dt} \perp B^*$ and $\dfrac{d\alpha^*}{dt} \perp N$, we get μ is constant.
Also $dtds^* = \dfrac{1}{\cos\theta}$ and $|\mu|$ is the distance between the curves α and α^*. Also we can write $\dfrac{dt}{ds^*} = \dfrac{1}{\sqrt{1+\mu\tau}}$.

Theorem 2.1. *The Mannheim partner of a cubic Bezier curve has the following matrix representation*

$$\alpha^* = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{6\mu}{\eta m} N_0 \\ \frac{6\mu}{\eta m} N_1 + P_3 + 3P_1 - 3P_2 - P_0 \\ \frac{6\mu}{\eta m} N_2 + 3P_2 - 6P_1 + 3P_0 \\ \frac{6\mu}{\eta m} N_3 + 3P_1 - 3P_0 \\ \frac{6\mu}{\eta m} N_4 + P_0 \end{bmatrix}.$$

Proof. Let $\alpha^* = \alpha(t) + \mu N$ be Mannheim partner of a cubic Bezier curve $\alpha(t)$, hence

$$\alpha^* = \begin{bmatrix} t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} + \frac{6\mu}{\eta m} \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}$$

$$\begin{aligned} &= P_2\left(3t^2 - 3t^3\right) + t^3 P_3 + P_0\left(-t^3 + 3t^2 - 3t + 1\right) + P_1\left(3t^3 - 6t^2 + 3t\right) \\ &\quad + \frac{6}{m}\frac{\mu}{\eta} N_4 + \frac{6}{m} t \frac{\mu}{\eta} N_3 + \frac{6}{m} t^2 \frac{\mu}{\eta} N_2 + \frac{6}{m} t^3 \frac{\mu}{\eta} N_1 + \frac{6}{m} t^4 \frac{\mu}{\eta} N_0 \end{aligned}$$

$$\begin{aligned} &= t^4 \frac{6\mu}{m\eta} N_0 + t^3\left(\frac{6\mu}{m\eta} N_1 + P_3 + 3P_1 - 3P_2 - P_0\right) \\ &\quad + t^2\left(\frac{6\mu}{m\eta} N_2 + 3P_2 - 6P_1 + 3P_0\right) + t\left(\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0\right) \\ &\quad + \frac{6\mu}{\eta m} N_4 + P_0. \end{aligned}$$

So we can write this as in the following matrix form

$$\alpha^* = \begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{6\mu}{\eta m} N_0 \\ \frac{6\mu}{\eta m} N_1 + P_3 + 3P_1 - 3P_2 - P_0 \\ \frac{6\mu}{\eta m} N_2 + 3P_2 - 6P_1 + 3P_0 \\ \frac{6\mu}{\eta m} N_3 + 3P_1 - 3P_0 \\ \frac{6\mu}{\eta m} N_4 + P_0 \end{bmatrix}.$$

Theorem 2.2. *The Mannheim partner of a cubic Bezier curve is a 4^{th} order Bezier curve with constant speed. It has the control points $P_0^*, P_1^*, P_2^*,$ P_3^* and P_4^* based on the control points of the cubic Bezier curve, as in the following way, where η, m are constants,*

$$\begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} P_0 + \frac{6\mu}{m\eta} N_4 \\ \frac{1}{4}P_0 + \frac{3}{4}P_1 + \frac{3\mu}{2m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ \frac{1}{2}P_1 + \frac{1}{2}P_2 + \frac{\mu}{m\eta} N_2 + \frac{3\mu}{m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ \frac{3}{4}P_2 + \frac{1}{4}P_3 + \frac{3\mu}{2m\eta} N_1 + \frac{3\mu}{m\eta} N_2 + \frac{9\mu}{2m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \\ P_3 + \frac{6\mu}{m\eta} N_0 + \frac{6\mu}{m\eta} N_1 + \frac{6\mu}{m\eta} N_2 + \frac{6\mu}{m\eta} N_3 + \frac{6\mu}{m\eta} N_4 \end{bmatrix}.$$

Proof. Let $P_0^*, P_1^*, P_2^*,$ P_3^* and P_4^* be the control points of 4th order Bezier curve which is Mannheim partner of a cubic Bezier curve, so we can write

$$\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} \frac{6\mu}{m\eta} N_0 \\ +\frac{6\mu}{m\eta} N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6\mu}{m\eta} N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0 \\ +\frac{6\mu}{m\eta} N_4 + P_0 \end{bmatrix}.$$

By using the following inverse matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

we obtain

$$\begin{bmatrix} P_0^* \\ P_1^* \\ P_2^* \\ P_3^* \\ P_4^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & \frac{1}{4} & 1 \\ 0 & 0 & \frac{1}{6} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{6\mu}{m\eta} N_0 \\ +\frac{6\mu}{m\eta} N_1 + P_3 - 3P_2 - P_0 + 3P_1 \\ +\frac{6\mu}{m\eta} N_2 + 3P_2 + 3P_0 - 6P_1 \\ +\frac{6\mu}{m\eta} N_3 + 3P_1 - 3P_0 \\ +\frac{6\mu}{m\eta} N_4 + P_0 \end{bmatrix}$$

which completes the proof.

Furthermore, the equality $\dfrac{\kappa}{\kappa^2+\tau^2}$ =constant is known as the offset property, for some non-zero constant. For some function μ, since N and B^* are linearly dependent, equation can be rewritten as $\alpha^*(t) = \alpha(t) - \mu N(t)$ where $\mu = \dfrac{-\kappa}{\kappa^2+\tau^2}$. Frenet-Serret apparatus of Mannheim partner curve α^*, based on Frenet-Serret vectors of Mannheim curve α are

$$\begin{aligned} &T^* = \cos\theta\ T - \sin\theta\ B, \\ &N^* = \sin\theta\ T + \cos\theta B, \\ &B^* = N, \\ &\mu = \frac{-\kappa}{\kappa^2+\tau^2} \end{aligned}$$

where $\theta = \sphericalangle(T, T^*)$.

Theorem 2.3. *Tangent vector field of Mannheim partner of a cubic Bezier curve based on the angle θ is*

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\cos\theta - \frac{6}{m}B_1\sin\theta \\ \frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\cos\theta - \frac{6}{m}B_2\sin\theta \\ \frac{1}{\eta}3(P_1 - P_0)\cos\theta - \frac{6}{m}B_3\sin\theta \end{bmatrix}.$$

Proof. Since $T^* = \cos\theta\ T - \sin\theta\ B$, we have

$$\begin{aligned} T^* &= \frac{1}{\eta}\left(Q_0\left(t^2 - 2t + 1\right) - Q_1\left(2t^2 - 2t\right) + t^2 Q_2\right)\cos\theta \ - \left(\frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)\right)\sin\theta \\ &= \frac{1}{\eta}\left(t^2 Q_0\cos\theta - 2t^2 Q_1\cos\theta + t^2 Q_2\cos\theta\right) - \frac{6}{m}t^2 B_1\sin\theta \\ &\quad + \frac{1}{\eta}\left(-2Q_0 t\cos\theta + 2Q_1 t\cos\theta\right) - \frac{6}{m}tB_2\sin\theta \\ &\quad + \frac{1}{\eta}Q_0\cos\theta - \frac{6}{m}B_3\sin\theta. \end{aligned}$$

Therefore, based on the control points Q_0, Q_1, Q_2, the following matrix representation can be written as

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}\left(Q_0 - 2Q_1 + Q_2\right)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}\left(-2Q_0 + 2Q_1\right)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}Q_0 \cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix}.$$

Also it can be written in the following matrix representation, based on the control points P_0, P_1, P_2, P_3

$$\begin{aligned} T^* &= \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}\left(3\left(P_1 - P_0\right) - 6\left(P_2 - P_1\right) + 3\left(P_3 - P_2\right)\right)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}\left(-6\left(P_1 - P_0\right) + 6\left(P_2 - P_1\right)\right)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}3\left(P_1 - P_0\right)\cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix} \\ &= \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}\left(6P_0 - 12P_1 + 6P_2\right)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}3\left(P_1 - P_0\right)\cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix}. \end{aligned}$$

Corollary 2.1. *Tangent vector field of Mannheim partner can be written as in the following way where η, m are constants*

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{m\eta}\left(mQ_0 \cos\theta - 6\eta B_3 \sin\theta\right) \\ -\frac{1}{m\eta}\left(3\eta B_2 \sin\theta - mQ_1 \cos\theta + 6\eta B_3 \sin\theta\right) \\ -\frac{1}{m\eta}\left(6\eta B_1 \sin\theta - mQ_2 \cos\theta + 6\eta B_2 \sin\theta + 6\eta B_3 \sin\theta\right) \end{bmatrix}.$$

Proof. As a quadratic Bezier curve, tangent vector field of Mannheim partner of a cubic Bezier curve with the control points Q_0^*, Q_1^*, Q_2^* is

$$T^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} Q_0^* \\ Q_1^* \\ Q_2^* \end{bmatrix}.$$

Hence, by using the inverse matrix the control points are

$$\begin{aligned} \begin{bmatrix} Q_0^* \\ Q_1^* \\ Q_2^* \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\eta}\left(Q_0 - 2Q_1 + Q_2\right)\cos\theta - \frac{6}{m}B_1 \sin\theta \\ \frac{1}{\eta}\left(-2Q_0 + 2Q_1\right)\cos\theta - \frac{6}{m}B_2 \sin\theta \\ \frac{1}{\eta}Q_0 \cos\theta - \frac{6}{m}B_3 \sin\theta \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{m\eta}\left(mQ_0 \cos\theta - 6\eta B_3 \sin\theta\right) \\ -\frac{1}{m\eta}\left(3\eta B_2 \sin\theta - mQ_1 \cos\theta + 6\eta B_3 \sin\theta\right) \\ -\frac{1}{m\eta}\left(6\eta B_1 \sin\theta - mQ_2 \cos\theta + 6\eta B_2 \sin\theta + 6\eta B_3 \sin\theta\right) \end{bmatrix}. \end{aligned}$$

Theorem 2.4. *Normal vector field of Mannheim partner of a cubic Bezier curve based on the angle θ is*

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}3(P_1 - P_0)\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}.$$

Proof. Since $N^* = \sin\theta\ T + \cos\theta\ B$, we have

$$\begin{aligned} N^* &= \frac{1}{\eta}\left(Q_0\left(t^2 - 2t + 1\right) - Q_1\left(2t^2 - 2t\right) + t^2 Q_2\right)\sin\theta + \frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)\cos\theta \\ &= \frac{1}{\eta}\left(t^2 Q_0\sin\theta - 2t^2 Q_1\sin\theta + t^2 Q_2\sin\theta\right) + \frac{6}{m}t^2 B_1\cos\theta \\ &+ \frac{1}{\eta}\left(-2tQ_0\sin\theta + 2tQ_1\sin\theta\right) + \frac{6}{m}tB_2\cos\theta \\ &+ \frac{1}{\eta}Q_0\sin\theta + \frac{6}{m}B_3\cos\theta. \end{aligned}$$

It can be written in the following matrix representation, based on the control points Q_0, Q_1, Q_2

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(Q_0 - 2Q_1 + Q_2)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}(-2Q_0 + 2Q_1)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}Q_0\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}.$$

Also it can be written in the following matrix representation, based on the control points P_0, P_1, P_2, P_3

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{1}{\eta}(9P_1 - 3P_0 - 9P_2 + 3P_3)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}(6P_0 - 12P_1 + 6P_2)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}3(P_1 - P_0)\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}.$$

This completes the proof.

Corollary 2.2. *Normal vector field of Mannheim partner of a cubic Bezier can be written as in the following way, where η, m are constants*

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{m\eta}(mQ_0\sin\theta + 6\eta B_3\cos\theta) \\ \frac{1}{m\eta}(mQ_1\sin\theta + 3\eta B_2\cos\theta + 6\eta B_3\cos\theta) \\ \frac{1}{m\eta}(mQ_2\sin\theta + 6\eta B_1\cos\theta + 6\eta B_2\cos\theta + 6\eta B_3\cos\theta) \end{bmatrix}.$$

Proof. As a quadratic Bezier curve normal vector field of Mannheim partner of a cubic Bezier curve with the control points N_0^*, N_1^*, N_2^* is

$$N^* = \begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} N_0^* \\ N_1^* \\ N_2^* \end{bmatrix}.$$

Hence, using the inverse matrix the control points are

$$\begin{bmatrix} N_0^* \\ N_1^* \\ N_2^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\eta}\left(Q_0 - 2Q_1 + Q_2\right)\sin\theta + \frac{6}{m}B_1\cos\theta \\ +\frac{1}{\eta}\left(-2Q_0 + 2Q_1\right)\sin\theta + \frac{6}{m}B_2\cos\theta \\ +\frac{1}{\eta}Q_0\sin\theta + \frac{6}{m}B_3\cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{m\eta}\left(mQ_0\sin\theta + 6\eta B_3\cos\theta\right) \\ \frac{1}{m\eta}\left(mQ_1\sin\theta + 3\eta B_2\cos\theta + 6\eta B_3\cos\theta\right) \\ \frac{1}{m\eta}\left(mQ_2\sin\theta + 6\eta B_1\cos\theta + 6\eta B_2\cos\theta + 6\eta B_3\cos\theta\right) \end{bmatrix}.$$

This completes the proof.

Theorem 2.5. *Binormal vector field of Mannheim partner of a cubic Bezier curve based on the angle θ are*

$$\begin{aligned} B^* &= N \\ &= \frac{6\mu}{\eta m}\left(N_0 t^4 + N_1 t^3 + N_2 t^2 + N_3 t + N_4\right). \end{aligned}$$

Theorem 2.6. *The curvature and the torsion of Mannheim partner of a cubic Bezier curve based on the angle θ are have the following equalities,*

Proof. Since

$$\kappa(t) = \frac{6}{\eta^3}\left(C_1 t^4 + C_2 t^3 + C_3 t^2 + C_4 t + C_5\right)$$

where

$$\begin{aligned} C_1 &= b_{11}^2 + b_{12}^2 + b_{13}^2, \\ C_2 &= 2b_{11}b_{21} + 2b_{12}b_{22} + 2b_{13}b_{23}, \\ C_3 &= 2b_{11}b_{31} + 2b_{12}b_{32} + 2b_{13}b_{33} + b_{21}^2 + b_{22}^2 + b_{23}^2, \\ C_4 &= 2b_{21}b_{31} + 2b_{22}b_{32} + 2b_{23}b_{33}, \\ C_5 &= b_{31}^2 + b_{32}^2 + b_{33}^2 \end{aligned}$$

and

$$\tau(t) = \frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}.$$

The curvature and the torsion have the following equalities of Mannheim partner of a cubic Bezier curve;

$$\kappa^* = -\frac{d\theta}{ds^*} = \frac{\dot{\theta}}{\cos\theta},$$

$$\begin{aligned}\tau^* &= \frac{\kappa}{\mu\tau} \\ &= \frac{\frac{6}{\eta^3}\left(C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5\right)}{\mu\left(\frac{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}{m^2}\right)} \\ &= \frac{6m^2}{\mu\eta^3}\frac{C_1t^4 + C_2t^3 + C_3t^2 + C_4t + C_5}{x_0y_1z_2 - x_0y_2z_1 - x_1y_0z_2 + x_1y_2z_0 + x_2y_0z_1 - x_2y_1z_0}.\end{aligned}$$

Theorem 2.7. *Frenet vector fields $\{T^*, N^*, B^*\}$ of Mannheim partner of any cubic Bezier curve in E^3 are*

$$T^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{(1-\mu\kappa)}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) + \frac{6\mu\tau}{m}B_1 \\ \frac{(1-\mu\kappa)}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) + \frac{6\mu\tau}{m}B_2 \\ \frac{(1-\mu\kappa)}{\eta}3\left(P_1 - P_0\right) + \frac{6\mu\tau}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

$$N^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{\mu\tau}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) - \frac{6(1-\mu\kappa)}{m}B_1 \\ \frac{\mu\tau}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) - \frac{6(1-\mu\kappa)}{m}B_2 \\ \frac{\mu\tau}{\eta}3\left(P_1 - P_0\right) - \frac{6(1-\mu\kappa)}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

$$B^* = \frac{6}{\eta m}\begin{bmatrix} t^4 \\ t^3 \\ t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} N_0 \\ N_1 \\ N_2 \\ N_3 \\ N_4 \end{bmatrix}.$$

Proof. Let a curve α^* be a Mannheim partner of α with Frenet-Serret apparatus, then

$$\begin{aligned} T^* &= \frac{(1-\mu\kappa)\, T + \mu\tau B}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}, \\ N^* &= \frac{\mu\tau T - (1-\mu\kappa)\, B}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}, \\ B^* &= N, \\ \frac{dt}{ds^*} &= \frac{1}{\eta\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}. \end{aligned}$$

Tangent vector field of Mannheim partner of a cubic Bezier curve is

$$\begin{aligned} T^* &= \frac{\frac{(1-\mu\kappa)}{\eta}\left(Q_0\left(t^2-2t+1\right) - Q_1\left(2t^2-2t\right) + t^2 Q_2\right) + \mu\tau\frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}} \\ &= \frac{\frac{(1-\mu\kappa)}{\eta}\left(Q_0 - 2tQ_0 + 2tQ_1 + t^2 Q_0 - 2t^2 Q_1 + t^2 Q_2\right) + \left(\frac{6\mu\tau}{m}B_1 t^2 + \frac{6\mu\tau}{m}B_2 t + \frac{6\mu\tau}{m}B_3\right)}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}. \end{aligned}$$

Hence its matrix representation, based on the control points Q_0, Q_1, Q_2 is

$$T^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{(1-\mu\kappa)}{\eta}\left(Q_0 - 2Q_1 + Q_2\right) + \frac{6\mu\tau}{m}B_1 \\ \frac{(1-\mu\kappa)}{\eta}\left(-2Q_0 + 2Q_1\right) + \frac{6\mu\tau}{m}B_2 \\ \frac{(1-\mu\kappa)}{\eta}\left(Q_0\right) + \frac{6\mu\tau}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}},$$

and based on the control points P_0, P_1, P_2, P_3 is

$$T^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{(1-\mu\kappa)}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) + \frac{6\mu\tau}{m}B_1 \\ \frac{(1-\mu\kappa)}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) + \frac{6\mu\tau}{m}B_2 \\ \frac{(1-\mu\kappa)}{\eta}3\left(P_1 - P_0\right) + \frac{6\mu\tau}{m}B_3 \end{bmatrix}}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}.$$

So the normal vector field of Mannheim partner of a cubic Bezier curve is

$$\begin{aligned} N^* &= \frac{\mu\tau T - (1-\mu\kappa)\, B}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}} \\ &= \frac{\frac{\mu\tau}{\eta}\left(Q_0\left(t^2-2t+1\right) - Q_1\left(2t^2-2t\right) + t^2 Q_2\right) - (1-\mu\kappa)\frac{6}{m}\left(B_1 t^2 + B_2 t + B_3\right)}{\sqrt{(1-\mu\kappa)^2 + (\mu\tau)^2}}. \end{aligned}$$

Hence its matrix representation is

$$N^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{\mu\tau}{\eta}\left(Q_0 - 2Q_1 + t^2 Q_2\right) - \frac{6\left(1-\mu\kappa\right)}{m} B_1 \\ \frac{\mu\tau}{\eta}\left(-2Q_0 + 2Q_1\right) - \frac{6\left(1-\mu\kappa\right)}{m} B_2 \\ \frac{\mu\tau}{\eta} Q_0 - \frac{6\left(1-\mu\kappa\right)}{m} B_3 \end{bmatrix}}{\sqrt{\left(1-\mu\kappa\right)^2 + \left(\mu\tau\right)^2}},$$

$$N^* = \frac{\begin{bmatrix} t^2 \\ t \\ 1 \end{bmatrix}^T \begin{bmatrix} \frac{\mu\tau}{\eta}\left(9P_1 - 3P_0 - 9P_2 + 3P_3\right) - \frac{6\left(1-\mu\kappa\right)}{m} B_1 \\ \frac{\mu\tau}{\eta}\left(6P_0 - 12P_1 + 6P_2\right) - \frac{6\left(1-\mu\kappa\right)}{m} B_2 \\ \frac{\mu\tau}{\eta} 3\left(P_1 - P_0\right) - \frac{6\left(1-\mu\kappa\right)}{m} B_3 \end{bmatrix}}{\sqrt{\left(1-\mu\kappa\right)^2 + \left(\mu\tau\right)^2}}.$$

Also, since $B^* = N$, its matrix representation is trivial.

Theorem 2.8. *The second curvature τ^* of Mannheim partner of any cubic Bezier curve is*

$$\tau^* = \frac{\sqrt{\begin{array}{c} \left(\frac{x_0 y_1 z_2 - x_0 y_2 z_1 - x_1 y_0 z_2 + x_1 y_2 z_0 + x_2 y_0 z_1 - x_2 y_1 z_0}{m^2}\right)^2 \\ -\left(\frac{6}{\eta^3} C_1 t^4 + C_2 t^3 + C_3 t^2 + C_4 t + C_5\right)^2 \end{array}}}{\sqrt{\left(1-\mu\kappa\right)^2 + \left(\mu\tau\right)^2}}.$$

Proof. Since $\frac{dB^*}{ds^*} = \frac{dB^*}{dt}\frac{dt}{ds^*} = -\tau^* N^*$ and $\langle -\tau^* N^*, -\tau^* N^* \rangle = \tau^{*2}$ we have

$$\tau^* = \frac{\sqrt{\tau^2 - \kappa^2}}{\sqrt{\left(1-\mu\kappa\right)^2 + \left(\mu\tau\right)^2}}, \quad \tau > \kappa.$$

By using $\kappa(t)$ and $\tau(t)$ of any cubic Bezier curve, we get the proof.

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Contents