International Journal of Maps in Mathematics

*Volume **5**, Issue **1**, 2022, Pages:**78-100***

ISSN: 2636-7467 (Online)

www.journalmim.com

CONFORMAL SLANT RIEMANNIAN MAPS

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Abstract. Conformal slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds are introduced. We give a non-trivial example of proper conformal slant Riemannian maps, obtain conditions for certain distributions to be integrable and find totally geodesicity conditions for leaves of distributions. We adjust the notion of pluriharmonicity by considering distributions on the total manifold of a conformal slant Riemannian map, and get conditions for such maps to be horizontally homothetic maps.

Keywords: Kaehler manifold, Slant immersion, Slant submersion, Slant Riemannian map

2010 Mathematics Subject Classification: 53C43.

1. Introduction

The concept of Riemannian submersion was introduced by Gray [13] and O'Neill [19]. Then, this notion was widely studied [10] and new kinds of Riemannian submersions such as invariant, anti-invariant and slant submersion were introduced [26]. Let F be a Riemannian submersion (respectively, horizontally conformal submersion, $m > n$) from $(M^m, g_{_M}, J)$ an almost Hermitian manifold to $(N^n, g_{_N})$ a Riemannian manifold. If the angle $\theta(U)$ between the space $(kerF_{*p})$ and JU is a constant for any non-zero vector field $U \in \Gamma(kerF_{*p})$; $p \in M$, i.e., it is independent from the choice of the tangent vector field U in $(kerF_{*p})$ and choice of the point $p \in M$, then we say that F is a slant submersion (respectively, conformal slant submersion) [5, 14, 22].

Received: 2021.04.11 *Revised: 2021.11.25* *Accepted: 2021.12.05*

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The notions of isometric immersions and Riemannian submersions are generalized by Riemannian maps between Riemannian manifolds [10, 11, 13, 19]. Let $F : (M_1, g_1) \longrightarrow (M_2, g_2)$ be a smooth map between Riemannian manifolds such that $0 < rankF < min\{dim(M_1), dim(M_2)\}$. So, the tangent bundle TM_1 of M_1 has the sequent decomposition:

$$TM_1 = kerF_* \oplus (kerF_*)^\perp.$$

Because of $rankF < min\{dim(M_1), dim(M_2)\}$, we always have $(rangeF_*)^\perp$. Consequently, the tangent bundle TM_2 of M_2 has the sequent decomposition:

$$TM_2 = (rangeF_*) \oplus (rangeF_*)^\perp.$$

Hence, a smooth map $F : (M_1^m, g_1) \longrightarrow (M_2^m, g_2)$ is called Riemannian map at $p_1 \in M_1$ if the horizontal restriction $F^h_{*p_1} : (kerF_{*p_1})^\perp \longrightarrow (rangeF_*)$ is a linear isometry. Therefore a Riemannian map provides the equation

$$g_1(E, G) = g_2(F_*(E), F_*(G)) \tag{1.1}$$

for $E, G \in \Gamma((kerF_*)^\perp)$. Isometric immersions and Riemannian submersions are particular Riemannian maps with $kerF_* = \{0\}$ and $(rangeF_*)^\perp = \{0\}$, respectively, [11]. As an another generalization of Riemannian submersions defined and studied independently horizontally conformal submersions [12, 15]. By following these studies and B. Ṣahin's papers including anti-invariant Riemannian, semi-invariant, slant submersions (see also [20]) and conformal anti-invariant [3], conformal slant [7], conformal semi-invariant [4] and conformal semi-slant submersions [2] have appeared in the literature. At the same time, the notion of slant submanifolds was introduced by Chen [9]. Inspiring from this notion, as a general map of Hermitian, anti-invariant and slant submersions, slant Riemannian maps were given in [24, 25] as follows; let F be a Riemannian map from an almost Hermitian manifold $(M, g_{_M}, J)$ to a Riemannian manifold $(N, g_{_N})$. If the angle $\theta(U)$ is a constant between JU and the space $kerF_*$ for any non-zero vector field $U \in \Gamma(kerF_*)$; i.e., it is independent from the choice of the tangent U in $kerF_*$ and choice of the point $p \in M$, then we say that F is a slant Riemannian map [24, 25]. On the other hand, we say that $F : (M^m, g_M) \longrightarrow (N^n, g_N)$ is a conformal Riemannian map at $p \in M$ if $0 < rankF_{*p} \leq min\{m, n\}$ and F_{*p} maps the horizontal space $(kerF_{*p})^\perp)$ conformally onto $range(F_{*p})$, i.e., there exist a number $\lambda^2(p) \neq 0$ such that

$$g_N(F_{*p}(E), F_{*p}(G)) = \lambda^2(p) g_M(E, G)$$

for $E, G \in \Gamma((kerF_{*p})^{\perp})$. Also F is said to be conformal Riemannian if F is conformal Riemannian at each $p \in M$ [21]. Conformal Riemannian maps have many application areas, some of them are computer vision [16], geometric modelling [29] and medical imaging [30]. In a previous paper, the second author and Akyol have studied conformal slant Riemannian maps from a Riemannian manifold to a Kaehler manifold and they have studied the geometry determined by the existence of these maps [5].

In this paper, we present conformal slant Riemannian maps from almost Hermitian manifolds to Riemannian manifolds, investigate geometric properties of the base manifold and the total manifold by the existence of such maps and give examples. We also obtain certain geodesicity conditions for conformal slant Riemannian maps. Moreover, we obtain several conditions for conformal slant Riemannian maps to be horizontally homothetic maps by using the adapted version of the notion of pluri-harmonic maps.

2. Preliminaries

In this section, some definitions and useful results which will be used at this paper for conformal slant Riemannian maps are given. Let (M, g_M) and (N, g_N) be Riemannian manifolds and suppose that $F : M \longrightarrow N$ is a smooth map between them. The second fundamental form of F is given by

$$(\nabla F_*)(X,Y) = \overset{N}{\nabla}{}^F_X F_*(Y) - F_*(\overset{M}{\nabla}_X Y) \tag{2.2}$$

for $X, Y \in \Gamma(TM)$. We know that (∇F_*) is symmetric [17]. Here, $\overset{N}{\nabla}{}^F$ is pull-back connection of $\overset{N}{\nabla}$ on N along F.

Let F be a Riemannian map from a Riemannian manifold (M^m, g_M) to a Riemannian manifold (N^n, g_N). We characterize $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{A}_X Y = h\overset{M}{\nabla}_{hX} vY + v\overset{M}{\nabla}_{hX} hY, \tag{2.3}$$

$$\mathcal{T}_X Y = h\overset{M}{\nabla}_{vX} vY + v\overset{M}{\nabla}_{vX} hY, \tag{2.4}$$

for $X, Y \in \Gamma(TM)$, where $\overset{M}{\nabla}$ is the Levi-Civita connection of g_M. Actually, we could see that these are O'Neill's tensor fields for Riemannian submersions [19]. $\mathcal{T}_X$ and $\mathcal{A}_X$ are skew-symmetric operators and reversing the vertical and the horizontal distributions on $(\Gamma(TM), g)$ for any $X \in \Gamma(TM)$. Also, it can be seen easily that $\mathcal{T}$ is vertical, $\mathcal{T}_X = \mathcal{T}_{vX}$, and $\mathcal{A}$ is horizontal, $\mathcal{A}_X = \mathcal{A}_{hX}$. We should know that $\mathcal{T}$ is symmetric on the vertical distribution

[10, 19]. Following these, from (2.3) and (2.4) we have

$$\overset{M}{\nabla}_U V = \mathcal{T}_U V + \hat{\nabla}_U V, \tag{2.5}$$

$$\overset{M}{\nabla}_U E = h\overset{M}{\nabla}_U E + \mathcal{T}_U E, \tag{2.6}$$

$$\overset{M}{\nabla}_E V = \mathcal{A}_E V + v\overset{M}{\nabla}_E V, \tag{2.7}$$

$$\overset{M}{\nabla}_E G = h\overset{M}{\nabla}_E G + \mathcal{A}_E G \tag{2.8}$$

for $E, G \in \Gamma((ker\, F_*)^{\perp})$ and $U, V \in \Gamma(kerF_*)$, where $\hat{\nabla}_U V = v\overset{M}{\nabla}_U V$ [10].

A vector field on M is called a projectable vector field if it is related to a vector field on N . Thus, we say a vector field is basic on M if it is both a horizontal and a projectable vector field. From now on, when we mention a horizontal vector field, we always consider a basic vector field [8].

On the other hand, let F be a conformal Riemannian map between Riemannian manifolds $(M^m, g_{_M})$ and $(N^n, g_{_N})$. Then, we have

$$\begin{aligned} (\nabla F_*)(E,G)\mid_{rangeF_*} &= E(\ln\lambda)F_*(G) + G(\ln\lambda)F_*(E) \\ &- g_M(E,G)F_*(grad(\ln\lambda)), \end{aligned} \tag{2.9}$$

where $E, G \in \Gamma((kerF_*)^{\perp})$ [6, 21]. Therefore from (2.9), we obtain $\overset{N}{\nabla}{}^F_E F_*(G)$ as

$$\begin{aligned} \overset{N}{\nabla}{}^F_E F_*(G) &= F_*(h\overset{M}{\nabla}_E G) + E(\ln\lambda)F_*(G) + G(\ln\lambda)F_*(E) \\ &- g_M(E,G)F_*(grad(\ln\lambda)) + (\nabla F_*)^{\perp}(E,G) \end{aligned} \tag{2.10}$$

where $(\nabla F_*)^{\perp}(E,G)$ is the component of $(\nabla F_*)(E,G)$ on $(rangeF_*)^{\perp}$ for $E, G \in \Gamma((kerF_*)^{\perp})$ [27, 28].

Finally, we recall the following notion. A map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is a pluriharmonic map if F provides the following equation

$$(\nabla F_*)(X,Y) + (\nabla F_*)(JX,JY) = 0 \tag{2.11}$$

for $X, Y \in \Gamma(TM)$ [18].

3. Conformal Slant Riemannian maps

In this section we are going to introduce conformal slant Riemannian maps as a generalization of slant Riemannian maps and conformal slant submersions, present examples and examine the geometry of source manifolds, target manifolds and maps themselves. We present the sequent definition.

Definition 3.1. *Let $F : (M, g_M, J_M) \longrightarrow (N, g_N)$ be a conformal Riemannian map from an almost Hermitian manifold (M, g_M, J_M) to a Riemannian manifold (N, g_N) . If for any non-zero vector $X \in \Gamma(kerF_*)$ at a point $p \in M$; the angle $\theta(X)$ between the space $kerF_*$ and $J_M X$ is a constant, i.e. it is independent of the choice of the tangent vector $X \in \Gamma(kerF_*)$ and choice of the point $p \in M$, then we say that F is a conformal slant Riemannian map. In this situation, the angle θ is called the slant angle of the conformal slant Riemannian map.*

We say that a conformal slant Riemannian map is proper if F is not a conformal invariant and a conformal anti-invariant Riemannian map. The sequent example is for a proper conformal slant Riemannian map.

Example 3.1. *Let $F : (R^4, g_4, J) \longrightarrow (R^4, g_4)$ be a map from a Kaehlerian manifold (R^4, g_4, J) to a Riemannian manifold (R^4, g_4) defined by*

$$(e^{x_2} \sin x_4, e^{x_2} \cos x_4, -e^{x_2} \sin x_4, -e^{x_2} \cos x_4).$$

Then, F is a conformal Riemannian map with $\lambda = e^{x_2}\sqrt{2}$ and $rankF = 2$. One can easily see that F is a proper conformal slant Riemannian map with the slant angle $\theta = \alpha$ via $J_\alpha = \cos\alpha(-c, -d, a, b) + \sin\alpha(-b, a, d, -c)$, $0 < \alpha \leq \frac{\pi}{2}$.

Let F be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then for $V \in \Gamma(kerF_*)$, we write

$$JV = \phi V + \omega V, \tag{3.12}$$

where $\phi V \in \Gamma(kerF_*)$ and $\omega V \in \Gamma((kerF_*)^\perp)$. Also for $X \in \Gamma((kerF_*)^\perp)$, we write

$$JX = BX + CX, \tag{3.13}$$

where $BX \in \Gamma(kerF_*)$ and $CX \in \Gamma((kerF_*)^\perp)$. We have covariant derivatives of ϕ and ω:

$$(\overset{M}{\nabla}_U \omega)V = h\overset{M}{\nabla}_U \omega V - \omega\hat{\nabla}_U V, \tag{3.14}$$

$$(\overset{M}{\nabla}_U \phi)V = \hat{\nabla}_U \phi V - \phi\hat{\nabla}_U V \tag{3.15}$$

for any $U, V \in \Gamma(kerF_*)$.

We give the following result by using equations (2.5), (2.6), (3.12), (3.13) and covariant derivatives of ϕ and ω.

Lemma 3.1. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is a conformal slant Riemannian map, we get*

$$h\overset{M}{\nabla}_U \omega V - \omega \hat{\nabla}_U V = CT_U V - T_U \phi V,$$

$$\hat{\nabla}_U \phi V - \phi \hat{\nabla}_U V = BT_U V - T_U \omega V$$

for any $U, V \in \Gamma(kerF_*)$.

Now, we present the following characterization for conformal slant Riemannian maps.

Theorem 3.1. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then F is a conformal slant Riemannian map if and only if there exists a constant $\lambda \in [-1, 0]$ such that*

$$\phi^2 U = \lambda U$$

for $U \in \Gamma(kerF_*)$. If F is a conformal slant Riemannian map, then $\lambda = -\cos^2\theta$.

Proof. For $U \in \Gamma(kerF_*)$ we have $\cos\theta = \frac{\|\phi U\|}{\|JU\|}$. Since M is a Kaehler manifold, we get

$$g_M(\phi^2 U, U) = -g_M(\phi U, \phi U) = -\cos^2\theta g_M(U, U).$$

Hence, we have $\phi^2 U = \lambda U$. Conversely, suppose that $\phi^2 U = \lambda U$ for $\forall U \in \Gamma(kerF_*)$ with $\lambda \in [-1, 0]$. Hence, we obtain

$$\cos\theta = \frac{g_M(JU, \phi U)}{\|JU\|\|\phi U\|} = -\lambda \frac{\|JU\|}{\|\phi U\|}. \tag{3.16}$$

Using $\cos\theta = \frac{\|\phi U\|}{\|JU\|}$ in (3.16) we get $\lambda = -\cos^2\theta$.

From (3.12) and Theorem 3.1. we have the next result.

Theorem 3.2. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N) with the slant angle θ. Then, we have*

$$g_M(\phi U, \phi V) = \cos^2\theta g_M(U, V) \tag{3.17}$$

$$g_M(\omega U, \omega V) = \sin^2\theta g_M(U, V) \tag{3.18}$$

for any $U, V \in \Gamma(kerF_*)$.

Let F be a conformal slant Riemannian map from an almost Hermitian manifold (M, g_M, J) onto a Riemannian manifold (N, g_N) with the slant angle θ; then we say that ω is parallel with respect to $\overset{M}{\nabla}$ on $kerF_*$ if its covariant derivative according to $\overset{M}{\nabla}$ vanishes, i.e.

$$(\overset{M}{\nabla}_U \omega)V = 0 \tag{3.19}$$

for $U, V \in \Gamma(kerF_*)$.

Theorem 3.3. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If ω is parallel according to $\overset{M}{\nabla}$ on $kerF_*$, then we have*

$$T_{\phi U}\phi U = -cos^2\theta T_U U \tag{3.20}$$

for $U \in \Gamma(kerF_*)$.

Proof. If ω is parallel according to $\overset{M}{\nabla}$ on $kerF_*$, we obtain using (3.14) and Lemma 3.1. for $U, V \in \Gamma(kerF_*)$

$$CT_U V = T_U \phi V. \tag{3.21}$$

Now, changing roles of U and V in (3.21) we get

$$CT_V U = T_V \phi U. \tag{3.22}$$

Because vertical vector field T is symmetric, from (3.21) and (3.22) we get

$$T_U \phi V = T_V \phi U. \tag{3.23}$$

Since $\phi^2 V = \lambda V$ and for $V = \phi U$ in (3.23) we obtain

$$-\cos^2\theta T_U U = T_{\phi U}\phi U,$$

which gives the assertion.

Theorem 3.4. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, two of the below assertions imply the third assertion,*

i- *The horizontal distribution $(kerF_*)^\perp$ is integrable,*

ii- $X(\ln\lambda)g_M(Y, \omega\phi U) = Y(\ln\lambda)g_M(X, \omega\phi U)$,

iii- $g_N(F_*(Ch\overset{M}{\nabla}_X \omega U + \omega A_X \omega U), F_*(Y)) + g_N(\overset{N}{\nabla}{}^F_X F_*(\omega\phi U), F_*(Y))$
$= g_N(F_*(Ch\overset{M}{\nabla}_Y \omega U + \omega A_Y \omega U), F_*(X)) + g_N(\overset{N}{\nabla}{}^F_Y F_*(\omega\phi U), F_*(X))$

for $X, Y \in \Gamma((kerF_*)^\perp)$ *and* $U \in \Gamma(kerF_*)$.

Proof. Now, for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$, using (2.8) and (3.12), we obtain

$$\begin{aligned} g_M([X,Y],U) &= g_M(\overset{M}{\nabla}_X J\phi U, Y) + g_M(JA_X\omega U + Jh\overset{M}{\nabla}_X\omega U, Y) \\ &- g_M(\overset{M}{\nabla}_Y J\phi U, X) - g_M(JA_Y\omega U + Jh\overset{M}{\nabla}_Y\omega U, X). \end{aligned}$$

Since F is a conformal map, from Theorem 3.1., (2.8) and (3.13) we get

$$\begin{aligned} g_M([X,Y],U) &= cos^2\theta g_M([X,Y],U) + \frac{1}{\lambda^2}\{g_N(F_*(h\overset{M}{\nabla}_X\omega\phi U), F_*(Y)) \\ &+ g_N(F_*(\omega A_X\omega U), F_*(Y)) + g_N(F_*(Ch\overset{M}{\nabla}_X\omega U), F_*(Y)) \\ &- g_N(F_*(h\overset{M}{\nabla}_Y\omega\phi U), F_*(X)) - g_N(F_*(\omega A_Y\omega U), F_*(X)) \\ &- g_N(F_*(Ch\overset{M}{\nabla}_Y\omega U), F_*(X))\}. \end{aligned}$$

Now, from (2.2) and (2.9) we have

$$\begin{aligned} sin^2\theta g_M([X,Y],U) &= \frac{1}{\lambda^2}\{g_N(F_*(Ch\overset{M}{\nabla}_X\omega U + \omega A_X\omega U), F_*(Y)) \\ &- g_N(F_*(Ch\overset{M}{\nabla}_Y\omega U + \omega A_Y\omega U), F_*(X)) \\ &+ g_N(F_*(\overset{N}{\nabla}{}^F_X F_*(\omega\phi U), F_*(Y)) \\ &- g_N(F_*(\overset{N}{\nabla}{}^F_Y F_*(\omega\phi U), F_*(X)) \\ &- X(\ln\lambda) g_N(F_*(\omega\phi U), F_*(Y)) \\ &- \omega\phi U(\ln\lambda) g_N(F_*(X), F_*(Y)) \\ &+ g_M(X, \omega\phi U) g_N(F_*(grad(\ln\lambda)), F_*(Y)) \\ &- g_N((\nabla F_*)^\perp(X, \omega\phi U), F_*(Y)) \\ &+ Y(\ln\lambda) g_N(F_*(\omega\phi U), F_*(X)) \\ &+ \omega\phi U(\ln\lambda) g_N(F_*(Y), F_*(X)) \\ &- g_M(Y, \omega\phi U) g_N(F_*(grad(\ln\lambda)), F_*(X)) \\ &+ g_N((\nabla F_*)^\perp(Y, \omega\phi U), F_*(X))\}. \end{aligned}$$

Using conformality of F we obtain

$$\begin{aligned} sin^2\theta g_M([X,Y],U) &= \frac{1}{\lambda^2}\{g_N(F_*(Ch\overset{M}{\nabla}_X\omega U+\omega A_X\omega U),F_*(Y)) \\ &- g_N(F_*(Ch\overset{M}{\nabla}_Y\omega U+\omega A_Y\omega U),F_*(X)) \\ &+ g_N(F_*(\overset{N}{\nabla}{}^F_X F_*(\omega\phi U),F_*(Y)) \\ &- g_N(F_*(\overset{N}{\nabla}{}^F_Y F_*(\omega\phi U),F_*(X))\} \\ &+ 2Y(\ln\lambda)g_M(X,\omega\phi U)-2X(\ln\lambda)g_M(Y,\omega\phi U). \end{aligned}$$

The proof is completed from the above equation.

Now we will examine the geometry of leaves of the vertical distribution.

Theorem 3.5. *Let $F:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M,g_M,J) to a Riemannian manifold (N,g_N). Then, the vertical distribution $kerF_*$ defines a totally geodesic foliation on M if and only if*

$$g_N((\nabla F_*)(U,JX),F_*(\omega V))=g_N((\nabla F_*)(U,X),F_*(\omega\phi V))$$

for $X\in\Gamma((kerF_*)^{\perp})$ and $U,V\in\Gamma(kerF_*)$.

Proof. Because of M is a Kaehler manifold and from Theorem 3.1., (3.12) and (3.13), we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_U V,X) &= -cos^2\theta g_M(\overset{M}{\nabla}_U X,V)-g_M(\overset{M}{\nabla}_U X,\omega\phi V) \\ &- g_M(\overset{M}{\nabla}_U BX,\omega V)-g_M(\overset{M}{\nabla}_U CX,\omega V). \end{aligned}$$

Hence we have

$$\begin{aligned} sin^2\theta g_M(\overset{M}{\nabla}_U V,X) &= -g_M(h\overset{M}{\nabla}_U X,\omega\phi V)-g_M(T_U BX,\omega V) \\ &- g_M(h\overset{M}{\nabla}_U CX,\omega V). \end{aligned}$$

Now, from (2.2) we get

$$\begin{aligned} sin^2\theta g_M(\overset{M}{\nabla}_U V,X) &= \frac{1}{\lambda^2}\{-g_N(F_*(h\overset{M}{\nabla}_U X),F_*(\omega\phi V)) \\ &- g_N(F_*(T_U BX),F_*(\omega V)) \\ &- g_N(F_*(h\overset{M}{\nabla}_U CX),F_*(\omega V))\} \\ &= \frac{1}{\lambda^2}\{g_N((\nabla F_*)(U,JX),F_*(\omega V)) \\ &- g_N((\nabla F_*)(U,X),F_*(\omega\phi V))\}. \end{aligned}$$

This completes the proof.

Now, we examine the geometry of the horizontal distribution.

Theorem 3.6. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, two of the below assertions imply the third assertion,*

i- *the horizontal distribution $(kerF_*)^\perp$ defines a totally geodesic foliation on M,*

ii- *F is a horizontally homothetic map,*

iii- $g_M(A_XY, U) = \frac{1}{\lambda^2} g_N(\overset{N}{\nabla}{}^F_X F_*(Y), F_*(\omega\phi U + C\omega U))$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$.

Proof. Now, from (2.8), (3.17) and (3.18) we have

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= g_M(JA_XY + Jh\overset{M}{\nabla}_X Y, \phi U) \\ &+ g_M(JA_XY + Jh\overset{M}{\nabla}_X Y, \omega U) \\ &= cos^2\theta g_M(A_XY, U) - g_M(h\overset{M}{\nabla}_X Y, J\phi U) \\ &+ sin^2\theta g_M(A_XY, U) - g_M(h\overset{M}{\nabla}_X Y, J\omega U) \\ &= g_M(A_XY, U) - g_M(h\overset{M}{\nabla}_X Y, \omega\phi U) - g_M(h\overset{M}{\nabla}_X Y, C\phi U) \end{aligned}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$. From (2.2) and (2.9), we obtain

$$\begin{aligned} g_M(\overset{M}{\nabla}_X Y, U) &= g_M(A_XY, U) - \frac{1}{\lambda^2} g_N(\overset{N}{\nabla}{}^F_X F_*(Y), F_*(\omega\phi U + C\omega U)) \\ &+ X(\ln\lambda) g_M(Y, \omega\phi U) + Y(\ln\lambda) g_M(X, \omega\phi U) \\ &- \omega\phi U(\ln\lambda) g_M(X, Y) + X(\ln\lambda) g_M(Y, C\omega U) \\ &+ Y(\ln\lambda) g_M(X, C\omega U) - C\omega U(\ln\lambda) g_M(X, Y). \end{aligned} \tag{3.24}$$

If the horizontal distribution $(kerF_*)^\perp$ defines a totally geodesic foliation on M for $X, Y \in \Gamma((kerF_*)^\perp)$, $U \in \Gamma(kerF_*)$ and $g_M(A_XY, U) = \frac{1}{\lambda^2} g_N(\overset{N}{\nabla}{}^F_X F_*(Y), F_*(\omega\phi U + C\omega U))$, we show that the map F is a horizontally homothetic map. If (i) and (iii) are satisfied, then we have

$$\begin{aligned} 0 &= X(\ln\lambda) g_M(Y, \omega\phi U) + Y(\ln\lambda) g_M(X, \omega\phi U) \\ &- \omega\phi U(\ln\lambda) g_M(X, Y) + X(\ln\lambda) g_M(Y, C\omega U) \\ &+ Y(\ln\lambda) g_M(X, C\omega U) - C\omega U(\ln\lambda) g_M(X, Y) \end{aligned} \tag{3.25}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$ and $U \in \Gamma(kerF_*)$. Suppose that $X = \omega\phi U$, $Y = C\omega U$ in equation (3.25), we have

$$C\omega U(\ln\lambda)g_M(\omega\phi U, \omega\phi U) + \omega\phi U(\ln\lambda)g_M(C\omega U, C\omega U) = 0. \quad (3.26)$$

If $C\omega U(\ln\lambda) = 0$ from (3.26) we get $\omega\phi U(\ln\lambda)g_M(C\omega U, C\omega U) = 0$ for $C\omega U \in \Gamma(C(kerF_*)^\perp)$. Therefore λ is a constant on $\Gamma(\omega(kerF_*))$. At the same time, if $\omega\phi U(\ln\lambda) = 0$ we derive $C\omega U(\ln\lambda)g_M(\omega\phi U, \omega\phi U) = 0$ from (3.26) for $\omega\phi U \in \Gamma(\omega(kerF_*))$. Thus λ is a constant on $\Gamma(C(kerF_*)^\perp)$. So, F is a horizontally homothetic map. The rest of the proof is clear.

Now we are going to slightly modify the notion of pluriharmonic map and use this new notion to obtain certain conditions for conformal slant Riemannian maps to be horizontally homothetic map. We say that a conformal slant Riemannian map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is $kerF_*-$ (respectively, $(kerF_*)^\perp$, $\omega(kerF_*)$, μ) pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(U, V) + (\nabla F_*)(JU, JV) = 0$$

for $U, V \in \Gamma(kerF_*)$ (respectively, $(kerF_*)^\perp$, $\omega(kerF_*)$, μ) [27, 28].

Theorem 3.7. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If F is a $kerF_*-$pluriharmonic map, then one of the below assertions imply the second assertion,*

i- *F is a horizontally homothetic map,*

ii- $F_*(A_{\omega U}\phi V + A_{\omega V}\phi U) = F_*(h\overset{M}{\nabla}_U \omega\phi V + \omega T_U \omega V + Ch\overset{M}{\nabla}_U \omega V)$ *and* $(\nabla F_*)^\perp(\omega U, \omega V) = 0$

for $U, V \in \Gamma(kerF_*)$.

Proof. From the definition of $kerF_*-$pluriharmonic map, (2.2) and (2.10), we have

$$\begin{aligned} 0 &= F_*(\overset{M}{\nabla}_U J\phi V + J\overset{M}{\nabla}_U \omega V) - F_*(\overset{M}{\nabla}_{\phi U}\phi V) - F_*(\overset{M}{\nabla}_{\omega V}\phi U) \\ &- F_*(\overset{M}{\nabla}_{\omega U}\phi V) + (\nabla F_*)^\perp(\omega U, \omega V) + \omega U(\ln\lambda)F_*(\omega V) \\ &+ \omega V(\ln\lambda)F_*(\omega U) - g_M(\omega U, \omega V)F_*(grad(\ln\lambda)). \end{aligned}$$

Now, using (2.6), (3.20) and Theorem 3.1., we get

$$\begin{aligned} 0 &= F_*(h\overset{M}{\nabla}_U\omega\phi V + \omega T_U\omega V + Ch\overset{M}{\nabla}_U\omega V - A_{\omega U}\phi V - A_{\omega V}\phi U) \\ &+ (\nabla F_*)^{\perp}(\omega U, \omega V) + \omega U(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(\omega U) \\ &- g_M(\omega U, \omega V)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.27}$$

If (i) is provided we have from (3.27)

$$\omega U(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(\omega U) - g_M(\omega U, \omega V)F_*(grad(\ln\lambda)) = 0$$

for $U, V \in \Gamma(kerF_*)$. So one can see second assertion clearly. Now if (ii) is satisfied in (3.27) we have $F_*(A_{\omega U}\phi V + A_{\omega V}\phi U) = F_*(h\overset{M}{\nabla}_U\omega\phi V + \omega T_U\omega V + Ch\overset{M}{\nabla}_U\omega V)$ and $(\nabla F_*)^{\perp}(\omega U, \omega V) = 0$ for $U, V \in \Gamma(kerF_*)$, respectively. Thus, by (3.27) we get

$$\begin{aligned} 0 &= \omega U(\ln\lambda)F_*(\omega V) + \omega V(\ln\lambda)F_*(\omega U) \\ &- g_M(\omega U, \omega V)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.28}$$

For $\omega U \in \Gamma(\omega(kerF_*))$ from (3.28) we get $0 = \lambda^2\omega V(\ln\lambda)g_M(\omega U, \omega U)$, which implies that $\omega(kerF_*)(grad(\ln\lambda)) = 0$. At the same time, from (3.28) if we take $\omega U = \omega V$ and for $X \in \Gamma(C(kerF_*)^{\perp})$ we get

$$0 = 2\lambda^2\omega U(\ln\lambda)g_M(X, \omega U) - \lambda^2 X(\ln\lambda)g_M(\omega U, \omega U). \tag{3.29}$$

Because of λ is a constant on $\omega(kerF_*)$ we have $2\lambda^2\omega U(\ln\lambda)g_M(X, \omega U) = 0$. Thus, by (3.29) we get $\lambda^2 X(\ln\lambda)g_M(\omega U, \omega U) = 0$, which implies that $(C(kerF_*)^{\perp})\ (grad(\ln\lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln\lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

Theorem 3.8. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If F is a $(kerF_*)^{\perp}-$ pluriharmonic map, then F is a horizontally homothetic map if and only if the following conditions*

$$(\nabla F_*)^{\perp}(X, Y) + (\nabla F_*)^{\perp}(CX, CY) = 0$$

and

$$F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) = 0,$$

are satisfied for $X, Y \in \Gamma((kerF_*)^{\perp})$.

Proof. From the definition of a $(kerF_*)^\perp$-pluriharmonic map, (2.2) and (2.9), we have

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X,Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X,Y)F_*(grad(\ln\lambda)) + (\nabla F_*)^\perp(CX,CY) + CX(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(CX) - g_M(CX,CY)F_*(grad(\ln\lambda)) \\ &- F_*(\overset{M}{\nabla}_{BX}BY) - F_*(\overset{M}{\nabla}_{CY}BX) - F_*(\overset{M}{\nabla}_{CX}BY) \end{aligned}$$

or

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X,Y) + (\nabla F_*)^\perp(CX,CY) + X(\ln\lambda)F_*(Y) \\ &+ Y(\ln\lambda)F_*(X) - g_M(X,Y)F_*(grad(\ln\lambda)) + CX(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(CX) - g_M(CX,CY)F_*(grad(\ln\lambda)) \\ &- F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) \end{aligned} \tag{3.30}$$

for $X,Y \in \Gamma((kerF_*)^\perp)$. If F is a horizontally homothetic map we have from equation (3.30)

$$\begin{aligned} 0 &= X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X,Y)F_*(grad(\ln\lambda)) + CX(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(CX) - g_M(CX,CY)F_*(grad(\ln\lambda)) \end{aligned}$$

for $X,Y \in \Gamma((kerF_*)^\perp)$. Since F is a horizontally homothetic map from (3.30) we obtain $(\nabla F_*)^\perp(X,Y)+(\nabla F_*)^\perp(CX,CY) = 0$ and $F_*(T_{BX}BY+A_{CY}BX+A_{CX}BY) = 0$ for $X,Y \in \Gamma((kerF_*)^\perp)$. Now suppose that $(\nabla F_*)^\perp(X,Y) + (\nabla F_*)^\perp(CX,CY) = 0$ and $F_*(T_{BX}BY + A_{CY}BX + A_{CX}BY) = 0$ in (3.30) for $X,Y \in \Gamma((kerF_*)^\perp)$, respectively. Thus, by (3.30) we get

$$\begin{aligned} 0 &= X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X,Y)F_*(grad(\ln\lambda)) + CX(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(CX) - g_M(CX,CY)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.31}$$

For $X = CX$, $Y = CY$ and $CY \in \Gamma(C(kerF_*)^\perp)$ in (3.31), we get $0 = 2\lambda^2 CX(\ln\lambda)$ $g_M(CY,CY)$, which implies that $(C(kerF_*)^\perp)(grad(\ln\lambda)) = 0$. At the same time, from (3.31) if we take $X = Y = CX$ and $\omega U \in \Gamma(\omega(kerF_*))$, we get

$$0 = 4\lambda^2 CX(\ln\lambda)g_M(CX,\omega U) - 2\lambda^2\omega U(\ln\lambda)g_M(CX,CX). \tag{3.32}$$

Since λ is a constant on $C(kerF_*)^\perp$ we have $4\lambda^2 CX(\ln\lambda)g_M(CX,\omega U)=0$. Thus, by (3.32) we get $-2\lambda^2\omega U(\ln\lambda)g_M(CX,CX)=0$, which implies that $(\omega(kerF_*))(grad(\ln\lambda))=0$. Thus, $\mathcal{H}(grad(\ln\lambda))=0$. It can be seen from here that F is a horizontally homothetic map.

We say that a conformal slant Riemannian map F from a complex manifold (M,g_M,J) to a Riemannian manifold (N,g_N) is $\{(kerF_*)^\perp-(kerF_*)\}-$ pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X,V)+(\nabla F_*)(JX,JV)=0$$

for $X\in\Gamma((kerF_*)^\perp)$ and $V\in\Gamma(kerF_*)$ [27, 28].

Theorem 3.9. *Let $F:(M,g_M,J)\longrightarrow(N,g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M,g_M,J) to a Riemannian manifold (N,g_N). If F is a $\{(kerF_*)^\perp-(kerF_*)\}-$pluriharmonic map, then two of the below assertions imply the third assertion,*

- i- *F is a horizontally homothetic map,*
- ii- $F_*(T_{BX}\omega U+A_{\omega U}BX+A_{CX}\phi U+h\overset{M}{\nabla}_X\omega\phi U)=F_*(\omega A_X\omega U+Ch\overset{M}{\nabla}_X\omega U)$ *and* $(\nabla F_*)^\perp(CX,\omega U)=0$,
- iii- *The vertical distribution $kerF_*$ is parallel along the horizontal distribution $(kerF_*)^\perp$ on M,*

for $X\in\Gamma((kerF_*)^\perp)$ and $U\in\Gamma(kerF_*)$.

Proof. From the definition of $\{(kerF_*)^\perp-(kerF_*)\}-$pluriharmonic map we get

$$0=(\nabla F_*)(X,U)+(\nabla F_*)(JX,JU)$$

for $X\in\Gamma((kerF_*)^\perp)$ and $U\in\Gamma(kerF_*)$. Using symmetry property of second fundamental form of a map by (2.2), (3.12) and (3.13) we get

$$\begin{aligned}0 &= -F_*(\overset{M}{\nabla}_X U)+(\nabla F_*)(BX,\phi U)+(\nabla F_*)(\omega U,BX)\\ &+ (\nabla F_*)(CX,\phi U)+(\nabla F_*)(CX,\omega U).\end{aligned}$$

From (2.7), (2.8) and (2.10) we get

$$\begin{aligned}0 &= F_*(\overset{M}{\nabla}_X J\phi U)+F_*(JA_X\omega U+Jh\overset{M}{\nabla}_X\omega U)-F_*(T_{BX}\phi U)\\ &- F_*(A_{\omega U}BX)-F_*(A_{CX}\phi U)+(\nabla F_*)^\perp(CX,\omega U)\\ &+ CX(\ln\lambda)F_*(\omega U)+\omega U(\ln\lambda)F_*(CX)\\ &- g_M(CX,\omega U)F_*(grad(\ln\lambda)).\end{aligned}$$

Now, from Theorem 3.1. , we have

$$\begin{aligned} cos^2\theta F_*(\overset{M}{\nabla}_X U) &= F_*(h\overset{M}{\nabla}_X \omega\phi U + \omega A_X \omega U + Ch\overset{M}{\nabla}_X \omega U) \\ &- F_*(T_{BX}\phi U + A_{\omega U}BX + A_{CX}\phi U) \\ &+ (\nabla F_*)^{\perp}(CX, \omega U) \\ &+ CX(\ln\lambda)F_*(\omega U) + \omega U(\ln\lambda)F_*(CX) \\ &- g_M(CX, \omega U)F_*(grad(\ln\lambda)) \end{aligned} \tag{3.33}$$

for $X \in \Gamma((kerF_*)^{\perp})$ and $U \in \Gamma(kerF_*)$. If (i) and (ii) are satisfied in (3.33) we have

$$0 = CX(\ln\lambda)F_*(\omega U) + \omega U(\ln\lambda)F_*(CX) - g_M(CX, \omega U)F_*(grad(\ln\lambda)),$$

$$(\nabla F_*)^{\perp}(CX, \omega U) = 0$$

and

$$F_*(T_{BX}\omega U + A_{\omega U}BX + A_{CX}\phi U + h\overset{M}{\nabla}_X \omega\phi U) = F_*(\omega A_X \omega U + Ch\overset{M}{\nabla}_X \omega U),$$

respectively. Then we get $F_*(\overset{M}{\nabla}_X U) = 0$. Therefore the vertical distribution $kerF_*$ is parallel along the horizontal distribution $(kerF_*)^{\perp}$ on M for $X \in \Gamma((kerF_*)^{\perp})$ and $U \in \Gamma(kerF_*)$. Suppose that (i) and (iii) are satisfied in (3.33), one can see clearly that (ii) is satisfies. Assume that (ii) and (iii) are satisfied in (3.33) we get

$$\begin{aligned} 0 &= CX(\ln\lambda)F_*(\omega U) + \omega U(\ln\lambda)F_*(CX) \\ &- g_M(CX, \omega U)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.34}$$

For $CX \in \Gamma(C(kerF_*)^{\perp})$ in (3.34) we get $0 = \lambda^2 \omega U(\ln\lambda)g_M(CX, CX)$, which implies that $(\omega(kerF_*))(grad(\ln\lambda)) = 0$. At the same time, from (3.34) for $\omega U \in \Gamma(\omega(kerF_*))$ we get $0 = \lambda^2 CX(\ln\lambda)g_M(\omega U, \omega U)$, which implies that $(C(kerF_*)^{\perp})$ $(grad(\ln\lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln\lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

Theorem 3.10. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If F is a $\omega(kerF_*)-$ pluriharmonic map, then F is a horizontally homothetic map if and only if the following conditions*

$$(\nabla F_*)^{\perp}(Z, Y) + (\nabla F_*)^{\perp}(CZ, CY) = 0$$

and

$$F_*(T_{BZ}BY + A_{CZ}BY + A_{CY}BZ) = 0$$

are satisfied for $Z, Y \in \Gamma(\omega(kerF_*))$.

Proof. From the definition of $\omega(kerF_*)-$ pluriharmonic map we have

$$0 = (\nabla F_*)(Z, Y) + (\nabla F_*)(JZ, JY)$$

for $Z, Y \in \Gamma(\omega(kerF_*))$. From (2.2), (2.9) and (3.13) we get

$$\begin{aligned} 0 &= (\nabla F_*)^{\perp}(Z, Y) + Z(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(Z) \\ &- g_M(Z, Y)F_*(grad(\ln\lambda)) - F_*(\overset{M}{\nabla}_{BZ}BY) - F_*(\overset{M}{\nabla}_{CZ}BY) \\ &- F_*(\overset{M}{\nabla}_{CY}BZ) + (\nabla F_*)^{\perp}(CY, CZ) + CZ(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(CZ) - g_M(CZ, CY)F_*(grad(\ln\lambda)). \end{aligned}$$

Using (2.5) and (2.7) we get

$$\begin{aligned} 0 &= (\nabla F_*)^{\perp}(Z, Y) + (\nabla F_*)^{\perp}(CY, CZ) + Z(\ln\lambda)F_*(Y) \\ &+ Y(\ln\lambda)F_*(Z) - g_M(Z, Y)F_*(grad(\ln\lambda)) + CZ(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(CZ) - g_M(CZ, CY)F_*(grad(\ln\lambda)) \\ &- F_*(T_{BZ}BY) - F_*(A_{CZ}BY) - F_*(A_{CY}BZ). \end{aligned} \tag{3.35}$$

If F is a horizontally homothetic map we have from (3.35)

$$\begin{aligned} 0 &= Z(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(Z) - g_M(Z, Y)F_*(grad(\ln\lambda)) \\ &+ CZ(\ln\lambda)F_*(CY) + CY(\ln\lambda)F_*(CZ) - g_M(CZ, CY)F_*(grad(\ln\lambda)) \end{aligned}$$

for $Z, Y \in \Gamma(\omega(kerF_*))$. Since F is a horizontally homothetic map from (3.35) we obtain $(\nabla F_*)^{\perp}(Z, Y) + (\nabla F_*)^{\perp}(CZ, CY) = 0$ and $F_*(T_{BZ}BY + A_{CZ}BY + A_{CY}BZ) = 0$ for $Z, Y \in \Gamma(\omega(kerF_*))$. Suppose that

$$(\nabla F_*)^{\perp}(Z, Y) + (\nabla F_*)^{\perp}(CZ, CY) = 0$$

and $F_*(T_{BZ}BY + A_{CZ}BY + A_{CY}BZ) = 0$ in (3.35) for $Z, Y \in \Gamma(\omega(kerF_*))$. Thus, by (3.35) we get

$$\begin{aligned} 0 &= Z(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(Z) - g_M(Z, Y)F_*(grad(\ln\lambda)) \\ &+ CZ(\ln\lambda)F_*(CY) + CY(\ln\lambda)F_*(CZ) \\ &- g_M(CZ, CY)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.36}$$

We know $g_M(Y, CY) = g_M(Y, JY - BY) = g_M(Y, JY) = 0$. For $Z = Y$ and $CY \in \Gamma(C(kerF_*)^\perp)$ in (3.36) we get $0 = -\lambda^2 CY(\ln\lambda)\{g_M(Y, Y) - g_M(CY, CY)\}$ which implies that $(C(kerF_*)^\perp)(grad(\ln\lambda)) = 0$. At the same time, from (3.36) if we take $Z = Y$ and $Y \in \Gamma(\omega(kerF_*))$ we get $0 = \lambda^2 Y(\ln\lambda)\{g_M(Y, Y) - g_M(CY, CY)\}$ which implies that $(\omega(kerF_*))(grad(\ln\lambda)) = 0$. Thus $\mathcal{H}(grad(\ln\lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

We say that a conformal slant Riemannian map F from a complex manifold (M, g_M, J) to a Riemannian manifold (N, g_N) is $(\mu - \omega(kerF_*))-$pluriharmonic map if F satisfies the following equation

$$(\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) = 0$$

for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(kerF_*))$.

Theorem 3.11. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). If F is a $(\mu - \omega(kerF_*))-$pluriharmonic map, then F is a horizontally homothetic map if and only if the following conditions*

$$(\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\perp(JX, CY) = 0$$

and

$$F_*(A_{JX}BY) = 0$$

are satisfied for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(kerF_*))$.

Proof. From the definition of $(\mu - \omega(kerF_*))-$ pluriharmonic map, (2.2), (2.10) and (3.13) we have

$$\begin{aligned} 0 &= (\nabla F_*)(X, Y) + (\nabla F_*)(JX, JY) \\ 0 &= (\nabla F_*)^\perp(X, Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X, Y)F_*(grad(\ln\lambda)) + (\nabla F_*)(JX, BY) + (\nabla F_*)(JX, CY). \end{aligned}$$

Since the distributions μ and $\omega(kerF_*)$ are orthogonal to each other, we have $g_M(X, Y) = 0$. So, we obtain

$$\begin{aligned} 0 &= (\nabla F_*)^\perp(X, Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- F_*(A_{JX}BY) + (\nabla F_*)^\perp(JX, CY) + JX(\ln\lambda)F_*(CY) \\ &+ CY(\ln\lambda)F_*(JX) - g_M(JX, CY)F_*(grad(\ln\lambda)) \end{aligned} \quad (3.37)$$

for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(kerF_*))$. Suppose that F is a horizontally homothetic map. From (3.37) we have

$$\begin{aligned} 0 &= X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &+ JX(\ln\lambda)F_*(CY) + CY(\ln\lambda)F_*(JX) \\ &- g_M(JX, CY)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.38}$$

Since F is a horizontally homothetic map from (3.37) we obtain $F_*(A_{JX}BY) = 0$ and $(\nabla F_*)^\perp(X,Y) + (\nabla F_*)^\perp(JX, CY) = 0$ for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(kerF_*))$. Suppose that $F_*(A_{JX}BY) = 0$ and $(\nabla F_*)^\perp(X,Y) + (\nabla F_*)^\perp(JX, CY) = 0$ for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(kerF_*))$ in (3.37). Using conformality of F for $X \in \Gamma(\mu)$ in (3.38) we get

$$\begin{aligned} 0 &= \lambda^2\{X(\ln\lambda)g_M(Y,X) + Y(\ln\lambda)g_M(X,X) \\ &+ JX(\ln\lambda)g_M(CY,X) + CY(\ln\lambda)g_M(JX,X) \\ &- X(\ln\lambda)g_M(JX,CY)\}. \end{aligned} \tag{3.39}$$

We know $g_M(CY,X) = g_M(JY,X) = -g_M(Y,JX) = 0$, $g_M(JX,CY) = 0$ for $X \in \Gamma(\mu)$ and $Y \in \Gamma(\omega(kerF_*))$ from (3.13). Then we obtain from (3.39) $\lambda^2 Y(\ln\lambda)g_M(X,X) = 0$, which implies that $\omega(kerF_*)(grad(\ln\lambda)) = 0$. For $X \in \Gamma(\mu)$ and $JX = X$ in (3.38) we get

$$\begin{aligned} 0 &= \lambda^2\{X(\ln\lambda)g_M(Y,X) + Y(\ln\lambda)g_M(X,X) \\ &+ X(\ln\lambda)g_M(CY,X) + CY(\ln\lambda)g_M(X,X) \\ &- X(\ln\lambda)g_M(X,CY)\}. \end{aligned} \tag{3.40}$$

Since λ is a constant on $\omega(kerF_*)$ we have $Y(\ln\lambda) = 0$. We get from (3.40) $0 = \lambda^2 CY(\ln\lambda)$ $g_M(X,X)$ that implies $(C(kerF_*)^\perp)(grad(\ln\lambda)) = 0$. It means λ is a constant on $C(kerF_*)^\perp$. Lastly for $Y \in \Gamma(\omega(kerF_*))$ and $JX = X$ in (3.38) we get

$$\begin{aligned} 0 &= \lambda^2\{X(\ln\lambda)g_M(Y,Y) + Y(\ln\lambda)g_M(X,Y) \\ &+ X(\ln\lambda)g_M(CY,Y) + CY(\ln\lambda)g_M(X,Y) \\ &- Y(\ln\lambda)g_M(X,CY)\}. \end{aligned} \tag{3.41}$$

We know $g_M(CY,Y) = g_M(JY,Y) = 0$ from (3.13) for $Y \in \Gamma(\omega(kerF_*))$. Then we obtain from (3.41) $0 = \lambda^2 X(\ln\lambda)g_M(Y,Y)$, which implies that $\mu(grad(\ln\lambda)) = 0$. Thus, $\mathcal{H}(grad(\ln\lambda)) = 0$. It can be seen from here that F is a horizontally homothetic map.

Theorem 3.12. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). F is a $\mu-$pluriharmonic map if and only if λ is a constant on $\omega(kerF_*)$.*

Proof. From the definition of $\mu-$ pluriharmonic map and (2.10), we have

$$\begin{aligned} 0 &= (\nabla F_*)^{\perp}(X,Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X) \\ &- g_M(X,Y)F_*(grad(\ln\lambda)) + (\nabla F_*)^{\perp}(JX,JY) + JX(\ln\lambda)F_*(JY) \\ &+ JY(\ln\lambda)F_*(JX) - g_M(JX,JY)F_*(grad(\ln\lambda)) \end{aligned}$$

for $X, Y \in \Gamma(\mu)$. Since $g_M(X,Y) = g_M(JX,JY)$ we obtain

$$\begin{aligned} 0 &= (\nabla F_*)^{\perp}(X,Y) + (\nabla F_*)^{\perp}(JX,JY) + X(\ln\lambda)F_*(Y) \\ &+ Y(\ln\lambda)F_*(X) + JX(\ln\lambda)F_*(JY) + JY(\ln\lambda)F_*(JX) \\ &- 2g_M(X,Y)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.42}$$

Now taking $X = Y$ in (3.42) we get

$$\begin{aligned} 0 &= (\nabla F_*)^{\perp}(X,X) + (\nabla F_*)^{\perp}(JX,JX) \\ &+ 2X(\ln\lambda)F_*(X) + 2JX(\ln\lambda)F_*(JX) \\ &- 2g_M(X,X)F_*(grad(\ln\lambda)). \end{aligned} \tag{3.43}$$

For $Z \in \Gamma(\omega(kerF_*))$ in (3.43) we get

$$\begin{aligned} 0 &= g_N((\nabla F_*)^{\perp}(X,X), F_*(Z)) + g_N((\nabla F_*)^{\perp}(JX,JX), F_*(Z)) \\ &+ 2X(\ln\lambda)g_N(F_*(X), F_*(Z)) + 2JX(\ln\lambda)g_N(F_*(JX), F_*(Z)) \\ &- 2g_M(X,X)g_N(F_*(grad(\ln\lambda)), F_*(Z)). \end{aligned}$$

Because of F is a conformal map and μ is a invariant distribution we obtain

$$\begin{aligned} 0 &= 2\lambda^2\{X(\ln\lambda)g_M(X,Z) + JX(\ln\lambda)g_M(JX,Z)\} \\ &- 2\lambda^2 g_M(X,X)g_M(grad(\ln\lambda), Z) \\ 0 &= -2\lambda^2 Z(\ln\lambda)g_M(X,X). \end{aligned} \tag{3.44}$$

From equation (3.44) we obtain $Z(\ln\lambda) = 0$, which implies that λ is a constant on $\omega(kerF_*)$ for $Z \in \Gamma(\omega(kerF_*))$. The converse is clear.

We now give necessary and sufficient conditions for a conformal slant Riemannian map to be totally geodesic map.

Theorem 3.13. *Let $F : (M, g_M, J) \longrightarrow (N, g_N)$ be a conformal slant Riemannian map from a Kaehler manifold (M, g_M, J) to a Riemannian manifold (N, g_N). Then, F is a totally geodesic map if and only if the following conditions are satisfied for $X, Y, Z \in \Gamma((kerF_*)^\perp)$ and $U, V \in \Gamma(kerF_*)$;*

i- $g_N(F_*(Ch\overset{M}{\nabla}_U \omega V) + F_*(\omega\hat{\nabla}_U \phi V + \omega T_U \omega V), F_*(X)) = 0,$

ii- *F is a horizontally homothetic map and $(\nabla F_*)^\perp(X, Y) = 0$.*

Proof. Now, from (2.2), (2.5), (3.12) and (3.13) we have

$$\begin{aligned}(\nabla F_*)(U, V) &= F_*(JT_U\phi V + J\hat{\nabla}_U \phi V)\\ &+ F_*(\omega T_U \omega V + Ch\overset{M}{\nabla}_U \omega V).\end{aligned}$$

Because T is symmetric, we get

$$\begin{aligned}(\nabla F_*)(U, V) &= \cos^2\theta F_*(T_V U) + F_*(\omega\hat{\nabla}_U \phi V)\\ &+ F_*(\omega T_U \omega V + Ch\overset{M}{\nabla}_U \omega V)\end{aligned}$$

which implies that

$$\sin^2\theta(\nabla F_*)(U, V) = F_*(\omega\hat{\nabla}_U \phi V) + F_*(\omega T_U \omega V + Ch\overset{M}{\nabla}_U \omega V) \tag{3.45}$$

for $U, V \in \Gamma(kerF_*)$. Thus, we obtain from (3.45)

$$\begin{aligned}\sin^2\theta g_N((\nabla F_*)(U, V), F_*(X)) &= g_N(F_*(\omega\hat{\nabla}_U \phi V + \omega T_U \omega V), F_*(X))\\ &+ g_N(F_*(Ch\overset{M}{\nabla}_U \omega V), F_*(X))\end{aligned} \tag{3.46}$$

for $X \in \Gamma((kerF_*)^\perp)$. (i) is satisfied in (3.46). Now, from (2.9) we get

$$\begin{aligned}(\nabla F_*)(X, Y) &= (\nabla F_*)^\perp(X, Y) + (\nabla F_*)^\top(X, Y)\\ &= (\nabla F_*)^\perp(X, Y) + X(\ln\lambda)F_*(Y) + Y(\ln\lambda)F_*(X)\\ &- g_M(X, Y)F_*(grad(\ln\lambda))\end{aligned} \tag{3.47}$$

for $X, Y \in \Gamma((kerF_*)^\perp)$. From (3.47) we have

$$\begin{aligned} g_N((\nabla F_*)(X,Y), F_*(X)) &= g_N((\nabla F_*)^\perp(X,Y), F_*(X)) \\ &+ X(\ln\lambda) g_N(F_*(Y), F_*(X)) \\ &+ Y(\ln\lambda) g_N(F_*(X), F_*(X)) \\ &- g_M(X,Y) g_N(F_*(grad(\ln\lambda)), F_*(X)) \\ &= Y(\ln\lambda) g_N(F_*(X), F_*(X)) \\ &= \lambda^2 Y(\ln\lambda) g_M(X,X) \end{aligned}$$

for $X \in \Gamma((kerF_*)^\perp)$. We have $0 = \lambda^2 Y(\ln\lambda) g_M(X,X)$ which implies $Y(\ln\lambda) = 0$. So, λ is a constant on $(kerF_*)^\perp$. F is a horizontally homothetic map and from (3.47) we get $(\nabla F_*)^\perp(X,Y) = 0$. Therefore, (ii) is satisfied. We complete the proof.

Acknowledgments. The authors would like to thank the referee for useful comments and their helpful suggestions that have improved the quality of this paper. This paper was supported by the Scientific and Technological Research Council of Turkey (TUBITAK) with number 114F339.

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