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ON f-BIHARMONIC AND BI-f-HARMONIC FRENET LEGENDRE CURVES

ŞERİFE NUR BOZDAĞ * AND FEYZA ESRA ERDOĞAN

ABSTRACT. This paper is devoted to study the f-harmonic, f-biharmonic, bi-f-harmonic, biminimal and f-biminimal Frenet Legendre curves in three dimensional normal almost paracontact metric manifolds and determine the necessary and sufficient conditions for these properties. Besides these, some characterizations for such curves have been defined in particular cases of a three dimensional normal almost paracontact metric manifold and some nonexistence theorems have been obtained.

Keywords: Frenet curves, Legendre curves, Normal almost paracontact metric manifolds, f-Harmonic curves, f-Biharmonic curves, Bi-f-Harmonic curves, f-Biminimal curves.

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1. Introduction

The theory of curves is one of the most important topic in differential geometry and up to date from the past to the present. In the theory of curves there are many special types such as Frenet curves; slant curves, Legendre curves and these are studied in many different manifolds. In particular, Legendre curves have an important role in geometry and topology of almost contact manifolds. Among the papers on Legendre curves studied on contact manifolds in the literature, the most basic ones can be listed as [3, 19].

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* Corresponding author

Şerife Nur Bozdağ; serife.nur.yalcin@ege.edu.tr; https://orcid.org/0000-0002-9651-7834

Feyza Esra Erdoğan; feyza.esra.erdogan@ege.edu.tr; https://orcid.org/0000-0003-0568-7510

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On the other hand, studies on Frenet Legendre curves are newer. These studies, which are a source of motivation for us, can be briefly listed as [27, 23]. In this study, different from previous studies which are focused on curvature and torsion, we handled the maps, which briefly mentioned below, in terms of different cases of α, β and δ.

Harmonic maps which were defined by Sampson and Eells, in [8] have a wide field of study due to their wide applications such as physics, mathematics and engineering.

Besides, in [14], Jiang obtained biharmonic maps between the Riemannian manifolds by generalizing harmonic maps.

f-harmonic maps have a physical meaning as the solution of inhomogeneous Heisenberg spin systems and continuous spin systems, [4]. For this reason, the maps in question are of interest not only for mathematicians but also for physicists. f-harmonic maps between Riemannian manifolds were introduced by Lichnerowicz in 1970 and then examined by Eells and Lemaire in [9].

On the other hand, the strong relationship between f-harmonic and harmonic maps is summarized by Perktaş et.al. as follows, in [25]. The first one, extending bienergy functional to bi-f-energy functional and obtaining a new type of harmonic map called bi-f-harmonic map. The second one extending the f-energy functional to the f-bienergy functional and obtain another type of harmonic map called f-biharmonic map as critical points of f-bienergy functional, [30, 22].

f-biharmonic maps, which are the generalization of biharmonic maps, are defined by Lu, in [18]. Lu defined also f-biharmonic maps between Riemannian manifolds, in [6]. However, Ou gave complete classification of f-biharmonic curves in three dimensional Euclidean space and characterization of f-biharmonic curves in n-dimensional space forms, [21]. In addition, recent studies can be summarized as; [12, 1, 13, 16].

Moreover, bi-f-harmonic maps as a generalization of biharmonic and f-harmonic maps introduced by Ouakkas et. al., in [22]. In addition, Roth defined a non-f-harmonic, f-biharmonic map as a proper f-biharmonic map, [26]. It should be emphasized that there is no relationship between f-biharmonic and bi-f-harmonic maps.

Biminimal immersions and biminimal curves in a Riemannian manifold were defined by Loubeau and Montaldo, [17].

Finally, f-biminimal immersions were defined by Karaca and Özgür, in [11]. They considered f-biminimal curves in a Riemannian manifolds.

Based on these studies in this paper, first we give basic notions which will be needed in other sections. In section 3.1, we show that there is no f-harmonic Frenet Legendre curve in three dimensional normal almost paracontact metric manifold. In section 3.2, we get f-biharmonicity condition of a Frenet Legendre curve in three dimensional normal almost paracontact metric manifold and determine this condition in different cases such as β-para-Sasakian, α-para-Kenmotsu and paracosymplectic manifolds. In section 3.3, we obtain bi-f-harmonicity condition of a Frenet Legendre curve in three dimensional normal almost paracontact metric manifold and also discuss this condition in various manifolds. In section 3.4, we obtain biminimality condition of a Frenet Legendre curve in three dimensional normal almost paracontact metric manifold. Finally in section 3.5, we get f-biminimality conditions of Frenet Legendre curves in three dimensional normal almost paracontact metric manifold.

2. Preliminaries

This section, includes some definitions and propositions that will be required throughout the paper.

Definition 2.1. *Let (N, g) and $(\bar{N}, \bar{g})$ be Riemannian manifolds, then a harmonic map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is defined as the critical point of the energy functional*

$$E(\phi) = \frac{1}{2} \int_N |d\phi|^2 dv_g,$$

where v_g is the volume element of (N, g). Then by using Euler-Lagrange equation $\tau(\phi)$ of the energy functional $E(\phi)$, where it is the tension field of map ϕ, a map called as harmonic if

$$\tau(\phi) := trace \nabla d\phi = 0. \tag{2.1}$$

Here ∇ is the connection induced from the Levi-Civita connection $\nabla^{\bar{N}}$ of $\bar{N}$ and the pull-back connection ∇^{ϕ}, [11].

Biharmonic maps, which can be considered as a natural generalization of harmonic maps, are defined as below.

Definition 2.2. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is defined as a biharmonic map if it is a critical point, for all variations, of the bienergy functional*

$$E_2(\phi) = \frac{1}{2} \int_N |\tau(\phi)|^2 dv_g.$$

Then the Euler-Lagrange equation $\tau_2(\phi)$, for the bienergy functional $E_2(\phi)$, where $\tau_2(\phi)$ is the bitension field of map ϕ equals to

$$\tau_2(\phi) = trace(\nabla^{\phi}\nabla^{\phi} - \nabla^{\phi}_{\nabla})\tau(\phi) - trace(R^{\bar{N}}(d\phi, \tau(\phi))d\phi) = 0, \tag{2.2}$$

if ϕ is a biharmonic map. Here $R^{\bar{N}}$, the curvature tensor field of $\bar{N}$, is defined as

$$R^{\bar{N}}(X,Y)Z = \nabla^{\bar{N}}_X \nabla^{\bar{N}}_Y Z - \nabla^{\bar{N}}_Y \nabla^{\bar{N}}_X Z - \nabla^{\bar{N}}_{[X,Y]} Z,$$

for any $X, Y, Z \in \Gamma(T\bar{N})$ and ∇^{ϕ} is the pull-back connection, [11].

One can easily see that harmonic maps are always biharmonic. Biharmonic maps which are not harmonic are called proper biharmonic maps, [24].

Definition 2.3. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is said to be an f-harmonic if it is critical point of f-energy functional,*

$$E_f(\phi) = \frac{1}{2}\int_N f|d\phi|^2 dv_g,$$

where $f \in C^{\infty}(N, \mathbb{R})$ is a positive smooth function. Then the f-harmonic map equation obtained by using Euler-Lagrange equation as follows;

$$\tau_f(\phi) = f\tau(\phi) + d\phi(gradf) = 0, \tag{2.3}$$

where $\tau_f(\phi)$ is the f-tension field of the map ϕ.
f-harmonic maps are generalizations of harmonic maps, [2, 7].

Definition 2.4. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is said to be an f-biharmonic if it is critical point of the f-bienergy functional*

$$E_{2,f}(\phi) = \frac{1}{2}\int_N f|\tau(\phi)|^2 dv_g.$$

The Euler-Lagrange equation for the f-biharmonic map is given by

$$\tau_{2,f}(\phi) = f\tau_2(\phi) + \Delta f\tau(\phi) + 2\nabla^{\phi}_{gradf}\tau(\phi) = 0, \tag{2.4}$$

where $\tau_{2,f}(\phi)$ is the f-bitension field of the map ϕ.
A f-biharmonic map turns into a biharmonic map if f is a constant, [6].

Definition 2.5. *A map $\phi : (N, g) \to (\bar{N}, \bar{g})$ is said to be a bi-f-harmonic if it is critical point of the bi-f-energy functional*

$$E_{f,2}(\phi) = \frac{1}{2}\int_N |\tau_f(\phi)|^2 dv_g.$$

The Euler-Lagrange equation for the bi-f-harmonic map is given by

$$\tau_{f,2}(\phi) = trace\big((\nabla^{\phi} f(\nabla^{\phi}\tau_f(\phi)) - f\nabla^{\phi}_{\nabla^N}\tau_f(\phi) + fR^{\bar{N}}(\tau_f(\phi), d\phi)d\phi\big) = 0, \qquad (2.5)$$

where $\tau_{f,2}(\phi)$ is the bi-f-tension field of the map ϕ, [22].

Definition 2.6. *An immersion $\phi : (N, g) \to (\bar{N}, \bar{g})$ is called biminimal if it is critical point of the bienergy functional $E_2(\phi)$ for variations normal to the image $\phi(N) \subset \bar{N}$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that ϕ is a critical point of the λ-bienergy functional,*

$$E_{2,\lambda}(\phi) = E_2(\phi) + \lambda E(\phi).$$

The Euler-Lagrange equation for a λ- biminimal immersion is

$$[\tau_{2,\lambda}(\phi)]^{\perp} = [\tau_2(\phi)]^{\perp} - \lambda[\tau(\phi)]^{\perp} = 0, \qquad (2.6)$$

for some value of $\lambda \in \mathbb{R}$, where $[.]^{\perp}$ denotes the normal component of $[.]$. An immersion is called free biminimal if it is biminimal for $\lambda = 0$, [11, 17].

Definition 2.7. *An immersion $\phi : (N, g) \to (\bar{N}, \bar{g})$ is called f-biminimal if it is a critical point of the f-bienergy functional $E_{2,f}(\phi)$ for variations normal to the image $\phi(N) \subset \bar{N}$, with fixed energy. Equivalently, there exists a constant $\lambda \in \mathbb{R}$ such that ϕ is a critical point of the λ-f-bienergy functional,*

$$E_{2,\lambda,f}(\phi) = E_{2,f}(\phi) + \lambda E_f(\phi).$$

Using the Euler-Lagrange equations for f-harmonic and f-biharmonic maps, an immersion is f-biminimal if

$$[\tau_{2,\lambda,f}(\phi)]^{\perp} = [\tau_{2,f}(\phi)]^{\perp} - \lambda[\tau_f(\phi)]^{\perp} = 0, \qquad (2.7)$$

for some value of $\lambda \in \mathbb{R}$. An immersion is called free f-biminimal if it is f-biminimal for $\lambda = 0$. If f is a constant then the immersion is biminimal, [11].

Definition 2.8. *A differentiable manifold N^{2n+1} is called almost paracontact metric manifold if it admits a tensor field φ of type $(1,1)$, a vector field ξ, a 1-form η and a pseudo-Riemannian metric g satisfying the following conditions:*

$$\varphi^2 = I - \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi\xi = 0, \quad g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \qquad (2.8)$$

where $X, Y \in TN$ and I is the identity endomorphism on vector fields. g is called compatible metric and any compatible metric is necessarily of signature $(n+1, n)$. In an almost paracontact metric manifold N, $\eta \circ \varphi = 0$ and $rank(\varphi) = 2n$. From (2.8), $g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$, for any $X, Y \in TN$. The fundamental 2-form of N is defined by $\Phi(X, Y) = g(X, \varphi Y)$. An almost paracontact metric manifold $(N, \varphi, \xi, \eta, g)$ is said to be normal if $\mathscr{N}(X, Y) - 2d\eta(X, Y)\xi = 0$, where $\mathscr{N}$ is the Nijenhuis torsion tensor of φ, [15, 29].

Proposition 2.1. [27] *For a three dimensional almost paracontact metric manifold N, the following conditions are mutually equivalent:*

i- *N is normal,*

ii- *there exist α, β functions on N such that*

$$(\nabla_X \varphi) Y = \alpha \left(g(\varphi X, Y)\, \xi - \eta(Y)\varphi X \right) + \beta \left(g(X, Y)\xi - \eta(Y)X \right), \tag{2.9}$$

iii- *there exist α, β functions on N such that*

$$\nabla_X \xi = \alpha \left(X - \eta(X)\xi \right) + \beta \varphi X. \tag{2.10}$$

Moreover, the functions α, β realizing (2.9) as well as (2.10) are given by

$$2\alpha = trace\{X \to \nabla_X \xi\}, \qquad 2\beta = trace\{X \to \varphi \nabla_X \xi\}.$$

For a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$, the curvature tensor field equation becomes

$$\begin{aligned} R(X,Y)Z &= \left(\frac{r}{2} + 2\left(\alpha^2 + \beta^2\right)\right)\left(g(Y,Z)X - g(X,Z)Y\right) \\ &+ g(X,Z)\left(\frac{r}{2} + 3\left(\alpha^2 + \beta^2\right)\right)\eta(Y)\xi \\ &- \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(Y)\eta(Z)X \\ &- g(Y,Z)\left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(X)\xi \\ &+ \left(\frac{r}{2} + 3(\alpha^2 + \beta^2)\right)\eta(X)\eta(Z)Y, \end{aligned} \tag{2.11}$$

where $X, Y, Z \in TN$ and r is the scalar curvature, [24].

Definition 2.9. *A three dimensional normal almost paracontact metric manifold is called;*

. *β-para-Sasakian if $\alpha = 0$, $\beta \neq 0$ and β is constant,*

. *para-Sasakian if $\alpha = 0$, $\beta = -1$,*

. *quasi-para-Sasakian if $\alpha = 0$ and $\beta \neq 0$,*

. *α-para-Kenmotsu if $\alpha \neq 0$, $\beta = 0$ and α is constant,*

. *paracosymplectic if* $\alpha = \beta = 0$, [29].

Definition 2.10. *Let* $(N, \varphi, \xi, \eta, g)$ *be a three dimensional normal almost paracontact metric manifold where* $\alpha, \beta = constant$. *The structural function of the immersed curve* $\gamma : I \subset \mathbb{R} \to (N, g)$ *is the map* $c_\gamma : I \to \mathbb{R}$ *given by*

$$c_\gamma(s) = g(T(s), \xi) = \eta(T(s)),$$

where $T = \gamma'$. *Then the curve* γ *called as Legendre curve if* $c_\gamma = \eta(T(s)) = 0$, [5].

With the help of these definitions, we get f-tension field, f-bitension field, bi-f-tension field, the biminimality and f-biminimality conditions of a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold as in following sections.

3. FRENET LEGENDRE CURVES

Let $\gamma : I \longrightarrow N$ be a curve in a three dimensional pseudo-Riemannian manifold N such that $g(\gamma', \gamma') = \varepsilon_1$ where $\varepsilon_1 = \pm 1$ and $\nabla_{\gamma'}\gamma'$ denotes the covariant differentiation along γ. Then γ is a Frenet curve with $\{T, N, B\}$ Frenet Frame if one of the following three cases hold:

(1) γ is of osculating order 1, $\nabla_{\gamma'}\gamma' = 0$ (geodesics),

(2) γ is of osculating order 2, there exist two ortonormal vector fields T, N and a positive function κ along γ such that

$$\nabla_{\gamma'}T = \kappa\varepsilon_2 N, \quad \nabla_{\gamma'}N = -\kappa\varepsilon_1 T,$$

(3) γ is of osculating order 3, there exist three ortonormal vector fields T, N, B and two positive function κ and τ along γ such that

$$\nabla_{\gamma'}T = \kappa\varepsilon_2 N, \quad \nabla_{\gamma'}N = -\kappa\varepsilon_1 T + \tau\varepsilon_3 B, \quad \nabla_{\gamma'}B = -\tau\varepsilon_2 N,$$

where $T = \gamma', g(N, N) = \varepsilon_2 = \pm 1, g(B, B) = \varepsilon_3 = \pm 1$, κ is the curvature and τ is the torsion function, [27].

Note that in this paper, we study with $\gamma : I \subset \mathbb{R} \longrightarrow N$ non-null curve parametrized by arc length on a pseudo-Riemannian manifold N which is a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$. In this case, from Definition 2.1 and Definition 2.2, tension and bitension fields reduces to

$$\tau(\gamma) = \nabla_T T \tag{3.12}$$

and

$$\tau_2(\gamma) = \nabla_T^3 T - R(T, \nabla_T T)T = 0 \tag{3.13}$$

[20].

Now, let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in N and $\{T, \varphi T, \xi\}$ are ortonormal vector fields along γ where $\gamma' = T$. By differentiating $g(T, \xi) = 0$ along γ, it is obvious that $g(\nabla_T T, \xi) = -\varepsilon_1 \alpha$. Then $\nabla_T T$ obtained as below

$$\nabla_T T = -\varepsilon_1 \alpha \xi - \varepsilon_1 \delta \varphi T, \tag{3.14}$$

where δ is a function defined by $\delta = g(\nabla_T T, \varphi T)$, [27].

Let investigate the necessary and sufficient conditions of a Frenet Legendre curve to be f-harmonic, f-biharmonic, bi-f-harmonic, biminimal and f-biminimal in a three dimensional normal almost paracontact metric manifold in terms of different cases of α, β and δ.

It should be noted that; throughout our paper, for the sake of shortness, only N will be called instead of a three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$.

4. f-Harmonic Frenet Legendre Curves

In this subsection, we investigated the f-harmonicity condition of a Frenet Legendre curve in N.

Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in N. Then with the help of Definition 2.3 and equation (3.12), f-harmonicity condition obtained as below;

$$\tau_f(\gamma) = f\tau(\gamma) + d\gamma(gradf) = f\nabla_T T + f' T = 0. \tag{4.15}$$

Based on this result, we can express the following theorem:

Theorem 4.1. *There is no f-harmonic Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$.*

Proof. The f-harmonicity condition for this kind of curves obtained by substituting equation (3.14), in equation (4.15) as below;

$$\begin{aligned} \tau_f(\gamma) &= f\nabla_T T + f' T \\ &= f(-\varepsilon_1 \alpha \xi - \varepsilon_1 \delta \varphi T) + f' T \\ &= f' T - (\varepsilon_1 \alpha f)\xi - (\varepsilon_1 \delta f)\varphi T = 0. \end{aligned} \tag{4.16}$$

From equation (4.16); it is easy to see that $f^{'}=0$ namely, f is a constant function. This is a contradiction with the definition of f-harmonic curves.

5. f-Biharmonic Frenet Legendre Curves

In this section, we obtain the f-biharmonicity condition of a Frenet Legendre curve in N. In addition, we make detailed examinations for α-para-Kenmotsu, β-para-Sasakian and paracosymplectic manifolds.

First, let determine the f-biharmonicity condition for this kind of curves. By using tension and bitension field equations, f-bitension field $\tau_{2,f}(\gamma)$ obtained as below, [21];

$$\begin{aligned} \tau_{2,f}(\gamma) &= f\tau_2(\gamma)+(\Delta f)\tau(\gamma)+2\nabla^{\gamma}_{gradf}\tau(\gamma) \\ &= f(\nabla^3_T T-R(T,\nabla_T T)T)+f^{''}\nabla_T T+2f^{'}\nabla^2_T T=0. \end{aligned} \tag{5.17}$$

Then by differentiating $\nabla_T T=-\varepsilon_1\alpha\xi-\varepsilon_1\delta\varphi T$ with respect to T, we obtain $\nabla^2_T T$ and $\nabla^3_T T$ as below;

$$\nabla^2_T T=(\delta^2-\varepsilon_1\alpha^2)T-\varepsilon_1(\alpha\beta+\delta^{'})\varphi T-\delta\beta\xi \tag{5.18}$$

and

$$\nabla^3_T T=3\delta\delta^{'}T+(\alpha^2\delta-\delta\beta^2-\varepsilon_1\delta^3-\varepsilon_1\delta^{''})\varphi T+(\alpha^3-\alpha\beta^2-\varepsilon_1\alpha\delta^2-2\beta\delta^{'})\xi. \tag{5.19}$$

After that by substutiting $\nabla_T T$ into the curvature tensor field formula (2.11) we find,

$$R(T,\nabla_T T)T=-\alpha(\alpha^2+\beta^2)\xi+\delta(\frac{r}{2}+2(\alpha^2+\beta^2))\varphi T. \tag{5.20}$$

Finally, we determined the f-biharmonicity condition as below:

$$\begin{aligned} \tau_{2,f}(\gamma) &= f(\nabla^3_T T-R(T,\nabla_T T)T)+f^{''}\nabla_T T+2f^{'}\nabla^2_T T \\ &= (3\delta\delta^{'}f+2(\delta^2-\varepsilon_1\alpha^2)f^{'})T \\ &+ ((-\alpha^2\delta-3\beta^2\delta-\varepsilon_1\delta^3-\varepsilon_1\delta^{''}-\frac{r}{2}\delta)f-2\varepsilon_1(\alpha\beta+\delta^{'})f^{'}-\varepsilon_1\delta f^{''})\varphi T \\ &+ ((2\alpha^3-\varepsilon_1\alpha\delta^2-2\beta\delta^{'})f-2\delta\beta f^{'}-\varepsilon_1\alpha f^{''})\xi \\ &= 0. \end{aligned} \tag{5.21}$$

With the help of this result, we can state the following theorems:

Theorem 5.1. *Let $\gamma: I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional normal almost paracontact metric manifold N where α,β are constants.*

Then γ is an f-biharmonic Frenet Legendre curve iff the following equations hold:

$$\begin{cases} 3\delta\delta' f + 2(\delta^2 - \varepsilon_1\alpha^2)f' = 0, \\ (\alpha^2\delta + 3\beta^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' + \dfrac{r}{2}\delta)f + 2\varepsilon_1(\alpha\beta + \delta')f' + \varepsilon_1\delta f'' = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta')f - 2\delta\beta f' - \varepsilon_1\alpha f'' = 0. \end{cases} \tag{5.22}$$

Theorem 5.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional normal almost paracontact metric manifold N where α, β are constants. Then γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{4}} + c$$

and

$$r = -2\big[\alpha^2 + 3\beta^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta''}{\delta} + 3\frac{\varepsilon_1\delta'(\alpha\beta + \delta')}{\varepsilon_1\alpha^2 - \delta^2} + \frac{6(\delta')^2\alpha^2 + 6\delta\delta''\alpha^2 - 6\varepsilon_1\delta^3\delta'' + 15\varepsilon_1(\delta\delta')^2}{4(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where $2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta' - 2\delta\beta A - \varepsilon_1\alpha(A' + A^2) = 0$ *for* $A = \frac{3\delta\delta'}{2(\varepsilon_1\alpha^2 - \delta^2)}$ *and* $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Now, we give the interpretations of **Theorem 5.1**.

Case I : Assume that δ is not equal to a constant.

Case I-1: If N is a three dimensional β-para-Sasakian manifold and $\delta \neq constant$ then we have following equations from (5.22);

$$\begin{cases} 3\delta\delta' f + 2\delta^2 f' = 0, \\ (3\beta^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' + \frac{r}{2}\delta)f + 2\varepsilon_1\delta' f' + \varepsilon_1\delta f'' = 0, \\ \beta\delta' f + \delta\beta f' = 0. \end{cases} \tag{5.23}$$

Hence we obtain the following theorem;

Theorem 5.3. *There is no f-biharmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta \neq constant$.*

Proof. By solving the first and third equations of (5.23) together, it is easy to see that there is a contradiction between them.

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq constant$ then we have following equations from (5.22);

$$\begin{cases} 3\delta\delta^{'} f + 2(\delta^2 - \varepsilon_1\alpha^2) f^{'} = 0, \\ (\alpha^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''} + \frac{r}{2}\delta) f + 2\varepsilon_1\delta^{'} f^{'} + \varepsilon_1\delta f^{''} = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2) f - \varepsilon_1\alpha f^{''} = 0. \end{cases}$$

So we have the following corollary;

Corollary 5.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional α-para-Kenmotsu manifold N with δ is not equal to a constant. Then γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{4}} + c$$

and

$$r = -2\big[\alpha^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta^{''}}{\delta} + 3\frac{\varepsilon_1\delta^{'}\delta^{'}}{\varepsilon_1\alpha^2 - \delta^2} + \frac{6(\delta^{'})^2\alpha^2 + 6\delta\delta^{''}\alpha^2 - 6\varepsilon_1\delta^3\delta^{''} + 15\varepsilon_1(\delta\delta^{'})^2}{4(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where $2\alpha^3 - \varepsilon_1\alpha\delta^2 - \varepsilon_1\alpha(A^{'} + A^2) = 0$ for $A = \frac{3\delta\delta^{'}}{2(\varepsilon_1\alpha^2 - \delta^2)}$ and $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Case I-3: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then we have following equations from (5.22);

$$\begin{cases} 3\delta\delta^{'} f + 2\delta^2 f^{'} = 0, \\ (\varepsilon_1\delta^3 + \varepsilon_1\delta^{''} + \frac{r}{2}\delta) f + 2\varepsilon_1\delta^{'} f^{'} + \varepsilon_1\delta f^{''} = 0. \end{cases}$$

Therefore, we obtain the following corollary.

Corollary 5.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional paracosymplectic manifold N. Then for $\delta \neq constant$; γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the scalar curvature r equal to:*

$$f = \delta^{-\frac{3}{2}} + c$$

and

$$r = -2\varepsilon_1\big[\delta^2 + \delta^{-1}\delta^{''} - 3\delta^{-2}(\delta^{'})^2 + \frac{15}{4}\delta^{-2}\delta^{'} - \frac{3}{2}\delta^{''}\delta^{-1}\big].$$

Case II : Assume that δ = constant $\neq 0$. Then we investigate the following subcases:

Case II-1: If N is a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then we have following equations from (5.22);

$$\begin{cases} \delta^2 f' = 0, \\ (3\beta^2 + \varepsilon_1 \delta^2 + \frac{r}{2})f + \varepsilon_1 f'' = 0, \\ \beta f' = 0. \end{cases} \tag{5.24}$$

Hence we obtain the following theorem;

Theorem 5.4. *There is no proper f-biharmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold with $\delta = constant \neq 0$.*

Proof. For $\delta = constant \neq 0$, from the first equation of (5.24) we obtain that $f' = 0$, this situation contradicts the definition of the f-biharmonic curve.

Case II-2: If N be a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then we have following equations from (5.22);

$$\begin{cases} (\delta^2 - \varepsilon_1 \alpha^2) f' = 0, \\ (\alpha^2 + \varepsilon_1 \delta^2 + \dfrac{r}{2})f + \varepsilon_1 f'' = 0, \\ (2\alpha^2 - \varepsilon_1 \delta^2) f - \varepsilon_1 f'' = 0. \end{cases}$$

So, we have;

Corollary 5.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional α-para-Kenmotsu manifold N. Then γ is an f-biharmonic Frenet Legendre curve if and only if the function f and the constant scalar curvature r are given by*

$$f = c_1 e^{\alpha s} + c_2 e^{-\alpha s}$$

and

$$r = -6\alpha^2,$$

where $f \in C^\infty(N, \mathbb{R})$ is a positive smooth function dependent on s arc length parameter, $\delta = |\alpha|$ and $\varepsilon_1 = 1$.

Case II-3: Let N be a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then we have followings from (5.22);

$$\begin{cases} \delta^2 f^{'} = 0, \\ (\varepsilon_1 \delta^2 + \frac{r}{2}) f + \varepsilon_1 f^{''} = 0. \end{cases} \tag{5.25}$$

Hence we have the following nonexistence theorem;

Theorem 5.5. *There is no proper f-biharmonic Frenet Legendre curve in a three dimensional paracosymplectic manifold where $\delta = constant \neq 0$.*

6. Bi-f-Harmonic Frenet Legendre Curves

In this subsection, we handle bi-f-harmonic Frenet Legendre curves in N. Also we obtained bi-f-harmonicity conditions for α-para-Kenmotsu, β-para-Sasakian and paracosymplectic manifolds.

First let determine the bi-f-harmonicity condition in a three dimensional normal almost paracontact metric manifold. By substutiting equations (3.14), (5.18), (5.19) and (5.20) into the bi-f-tension field formula, $\tau_{f,2}(\gamma)$ obtained as below, [25];

$$\begin{aligned}
\tau_{f,2}(\gamma) &= trace(\nabla^{\gamma} f(\nabla^{\gamma} \tau_f(\gamma)) - f \nabla^{\gamma}_{\nabla^N} \tau_f(\gamma) + f R(\tau_f(\gamma), d\gamma) d\gamma) \\
&= (f f^{''})^{'} T + (3 f f^{''} + 2(f^{'})^2) \nabla_T T + 4 f f^{'} \nabla_T^2 T + f^2 \nabla_T^3 T + f^2 R(\nabla_T T, T) T \\
&= [(f f^{''})^{'} + 4 f f^{'} (\delta^2 - \varepsilon_1 \alpha^2) + 3 f^2 \delta \delta^{'}] T \\
&+ [-3\varepsilon_1 \alpha f f^{''} - 2\varepsilon_1 \alpha (f^{'})^2 - 4 f f^{'} \delta \beta + f^2 (2\alpha^3 - \varepsilon_1 \alpha \delta^2 - 2\beta \delta^{'})] \varphi T \\
&+ [-3\varepsilon_1 \delta f f^{''} - 2\varepsilon_1 \delta (f^{'})^2 - 4\varepsilon_1 f f^{'} (\alpha\beta + \delta^{'}) \\
&\quad + f^2 (-\frac{r}{2}\delta - \alpha^2 \delta - 3\beta^2 \delta - \varepsilon_1 \delta^3 - \varepsilon_1 \delta^{''})] \xi \\
&= 0,
\end{aligned} \tag{6.26}$$

which implies the following.

Theorem 6.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$.*

Then γ is a bi-f-harmonic curve iff the following equations hold:

$$\begin{cases} (ff^{''})^{'} + 4(\delta^2 - \varepsilon_1\alpha^2)ff^{'} + 3\delta\delta^{'}f^2 = 0, \\ 3\varepsilon_1\alpha ff^{''} + 2\varepsilon_1\alpha(f^{'})^2 + 4\delta\beta ff^{'} - (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta^{'})f^2 = 0, \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4(\alpha\beta + \delta^{'})ff^{'} + (\frac{r}{2}\varepsilon_1\delta + \alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \delta^{''})f^2 = 0. \end{cases} \tag{6.27}$$

Now, we give the interpretations of **Theorem 6.1**.

Case I : Assume that δ is not equal to constant. Then we investigate the following subcases:

Case I-1: If N a three dimensional β-para-Sasakian manifold and $\delta \neq$ *constant* then we have following equations from (6.27);

$$\begin{cases} (ff^{''})^{'} + 4\delta^2 ff^{'} + 3\delta\delta^{'}f^2 = 0, \\ 2\delta f^{'} + \delta^{'}f = 0, \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4\delta^{'}ff^{'} + (\frac{r}{2}\varepsilon_1\delta + 3\beta^2\varepsilon_1\delta + \delta^3 + \delta^{''})f^2 = 0. \end{cases}$$

Then we obtain the following corollary.

Corollary 6.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional β-para-Sasakian manifold N. Then γ is a bi-f-harmonic curve where the function f and the constant scalar curvature r are given by;*

$$f = \delta^{-\frac{1}{2}} + c$$

and

$$r = 3\varepsilon_1\delta^{-2}(\delta^{'})^2 + \varepsilon_1\delta^{-1}\delta^{''} - \frac{9}{2}\varepsilon_1\delta^{-2}\delta^{'} - 2\varepsilon_1\delta^2 - 6\beta^2,$$

for where $\delta \neq$ constant is the solution of $-9(\delta^{'})^3 + 10\delta\delta^{'}\delta^{''} - 2\delta^2\delta^{'''} + 4\delta^4\delta^{'} = 0$ differential equation.

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq$ *constant* then from (6.27), we obtain following equations;

$$\begin{cases} (ff^{''})^{'} + 4(\delta^2 - \varepsilon_1\alpha^2)ff^{'} + 3\delta\delta^{'}f^2 = 0, \\ 3ff^{''} + 2(f^{'})^2 + f^2(\delta^2 - 2\alpha^2\varepsilon_1) = 0, \\ 3\delta ff^{''} + 2\delta(f^{'})^2 + 4\delta^{'}ff^{'} + (\frac{r}{2}\varepsilon_1\delta + \alpha^2\delta\varepsilon_1 + \delta^3 + \delta^{''})f^2 = 0. \end{cases} \tag{6.28}$$

So, we have the following corollary.

Corollary 6.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional α-para-Kenmotsu manifold N where $\delta \neq constant$. Then γ is a bi-f-harmonic curve iff f is a solution of the non-linear differential equations given in (6.28).*

Case I-3: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then from (6.27), we obtain the following equations;

$$\begin{cases} (ff'')' + 4ff'\delta^2 + 3\delta\delta' f^2 = 0, \\ 3\delta ff'' + 2\delta(f')^2 + 4\delta' ff' + (\frac{r}{2}\varepsilon_1\delta + \delta^3 + \delta'')f^2 = 0. \end{cases} \tag{6.29}$$

Hence we obtain following corollary.

Corollary 6.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional paracosymplectic manifold N where $\delta \neq constant$. Then γ is a bi-f-harmonic curve iff f is a solution of the non-linear differential equations given in (6.29).*

Case I-4: If N $ff'' = 0$ and $\delta \neq constant$ then via equation (6.27), we obtain following equations;

$$\begin{cases} 4f'(\delta^2 - \varepsilon_1\alpha^2) + 3f\delta\delta' = 0, \\ 2\varepsilon_1\alpha(f')^2 + 4ff'\delta\beta - f^2(2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta') = 0, \\ 2\delta(f')^2 + 4ff'(\alpha\beta + \delta') + f^2(\frac{r}{2}\delta\varepsilon_1 + \alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \delta'') = 0. \end{cases} \tag{6.30}$$

We have the following corollary.

Corollary 6.4. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in a three dimensional normal almost paracontact metric manifold N where $ff'' = 0$ and $\delta \neq constant$. Then γ is a bi-f-harmonic Frenet Legendre curve where the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{8}} + c$$

and

$$r = -2\big[\alpha^2 + 3\beta^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta''}{\delta} + 3\frac{(\alpha\beta + \delta')\delta'}{\varepsilon_1\alpha^2 - \delta^2} + \frac{9\delta^2(\delta')^2}{8(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where $2\varepsilon_1\alpha A^2 + 4A\delta\beta - (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta') = 0$ *for* $A = \frac{3\delta\delta'}{4(\varepsilon_1\alpha^2 - \delta^2)}$ *and* $\varepsilon_1\alpha^2 - \delta^2 \neq 0.$

Case I-5: If N a three dimensional β-para-Sasakian manifold, $ff^{''}=0$ and $\delta \neq constant$ then from equation (6.30), we obtain following equations;

$$\begin{cases} 4f^{'}\delta + 3\delta^{'}f = 0, \\ 2f^{'}\delta + \delta^{'}f = 0, \\ 2\varepsilon_1\delta(f^{'})^2 + 4\varepsilon_1\delta^{'}ff^{'} + f^2(\frac{r}{2}\delta + 3\beta^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''}) = 0. \end{cases} \tag{6.31}$$

We have the following nonexistence theorem.

Theorem 6.2. *There is no bi-f-harmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $ff^{''}=0$ and $\delta \neq constant$.*

Proof. When first and the second equations of (6.31) solved together, we obtain $\delta^{'}f=0$. For $\delta \neq constant$ and $\delta^{'}f=0$; we get that $f=0$ which is a contradiction to the definition of bi-f-harmonic curve.

Case I-6: If N a α-para-Kenmotsu manifold, $ff^{''}=0$ and $\delta \neq constant$ then from equation (6.30) we have following equations;

$$\begin{cases} 4f^{'}(\delta^2 - \varepsilon_1\alpha^2) + 3\delta\delta^{'}f = 0, \\ 2\varepsilon_1(f^{'})^2 - f^2(2\alpha^2 - \varepsilon_1\delta^2) = 0, \\ 2\varepsilon_1\delta(f^{'})^2 + 4\varepsilon_1ff^{'}\delta^{'} + f^2(\frac{r}{2}\delta + \alpha^2\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''}) = 0. \end{cases}$$

Then, we have the following corollary.

Corollary 6.5. *Let N be a α-para-Kenmotsu manifold where $ff^{''}=0$, $\delta \neq constant$ and $\gamma : I \longrightarrow N$ be a Frenet Legendre curve. Then γ is a bi-f-harmonic curve where the function f and the scalar curvature r are given by;*

$$f = (\varepsilon_1\alpha^2 - \delta^2)^{-\frac{3}{8}} + c$$

and

$$r = -2\big[\alpha^2 + \varepsilon_1\delta^2 + \varepsilon_1\frac{\delta^{''}}{\delta} + 3\frac{(\delta^{'})^2}{\varepsilon_1\alpha^2 - \delta^2} + \frac{9\delta^2(\delta^{'})^2}{8(\varepsilon_1\alpha^2 - \delta^2)^2}\big],$$

where δ is the solution of $3\varepsilon_1\delta^2(\delta^{'})^2 - 2(2\alpha^2 - \varepsilon_1\delta^2)(\varepsilon_1\alpha^2 - \delta^2)^2 = 0$ differential equation and and $\varepsilon_1\alpha^2 - \delta^2 \neq 0$.

Case I-7: If N is a paracosymplectic manifold, $ff^{''}=0$ and $\delta \neq constant$ then from (6.30), we obtain following equations;

$$\begin{cases} 4ff^{'}\delta^2+3f^2\delta\delta^{'}=0, \\ \\ 2\varepsilon_1\delta(f^{'})^2+4\varepsilon_1 ff^{'}\delta^{'}+f^2(\frac{r}{2}\delta+\varepsilon_1\delta^3+\varepsilon_1\delta^{''})=0. \end{cases}$$

We have the following corollary.

Corollary 6.6. *Let N be a paracosymplectic manifold where $ff^{''}=0$, $\delta \neq constant$ and $\gamma : I \longrightarrow N$ be a Frenet Legendre curve. Then γ is a bi-f-harmonic curve where the function f and the scalar curvature r are given by;*

$$f=\delta^{-\frac{3}{4}}+c$$

and

$$r=-2\varepsilon_1\delta^2-2\varepsilon_1\frac{\delta^{''}}{\delta}+\frac{6\varepsilon_1\delta^{'}}{\delta^2}-\frac{9\varepsilon_1}{4\delta^2}.$$

Case II : Assume that $\delta = constant$ is not equal to 0. Then we shall investigate the following subcases:

Case II-1: If N a three dimensional β-para-Sasakian manifold then we have following equations from (6.27);

$$\begin{cases} (ff^{''})^{'}+4ff^{'}\delta^2=0, \\ \\ ff^{'}\beta=0, \\ \\ 3ff^{''}+2(f^{'})^2+f^2(\frac{r}{2}\varepsilon_1+3\beta^2\varepsilon_1+\delta^2)=0. \end{cases} \tag{6.32}$$

Hence, we give the following theorem;

Theorem 6.3. *There is no proper bi-f-harmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta = constant \neq 0$.*

Proof. From (6.32), the proof is obvious.

Case II-2: If N a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}(\delta^2 - \varepsilon_1\alpha^2) = 0, \\ 3ff^{''} + 2(f^{'})^2 - f^2(2\alpha^2\varepsilon_1 - \delta^2) = 0, \\ 3ff^{''} + 2(f^{'})^2 + f^2(\frac{r}{2}\varepsilon_1 + \alpha^2\varepsilon_1 + \delta^2) = 0. \end{cases}$$

So, we have the following corollary;

Corollary 6.7. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N. Then γ is a bi-f-harmonic curve where $\delta = constant \neq 0$, the constant scalar curvature equals to $r = -6\alpha^2$ and the function f is a solution of the non-linear differential equations given as;*

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}(\delta^2 - \varepsilon_1\alpha^2) = 0, \\ 3\alpha ff^{''} + 2\alpha(f^{'})^2 - f^2(2\alpha^3\varepsilon_1 - \alpha\delta^2) = 0. \end{cases}$$

Case II-3: If N a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then we obtain the following equations from (6.27);

$$\begin{cases} (ff^{''})^{'} + 4ff^{'}\delta^2 = 0, \\ 3ff^{''} + 2(f^{'})^2 + f^2(\frac{r}{2}\varepsilon_1 + \delta^2) = 0. \end{cases} \tag{6.33}$$

Then we have,

Corollary 6.8. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a pracosymplectic manifold N. Then γ is a bi-f-harmonic curve where $\delta = constant \neq 0$, the scalar curvature r is given by;*

$$r = -6\varepsilon_1\frac{f^{''}}{f} - 4\varepsilon_1(\frac{f^{'}}{f})^2 - 2\varepsilon_1\delta^2$$

and the function f is a solution of the non-linear differential equations given in equation (6.33).

Case II-4: If N a three dimensional normal almost paracontact metric manifold, $ff^{''} = 0$ and $\delta = constant \neq 0$ then from (6.27), we obtain that γ is a bi-f-harmonic Frenet Legendre curve if and only if

$$\begin{cases} 4ff'(\delta^2 - \varepsilon_1\alpha^2) = 0, \\ 2\varepsilon_1\alpha(f')^2 + 4ff'\delta\beta - f^2(2\alpha^3 - \varepsilon_1\alpha\delta^2) = 0, \\ 2\varepsilon_1\delta(f')^2 + 4\varepsilon_1 ff'\alpha\beta + f^2(\frac{r}{2}\delta + \alpha^2\delta + 3\beta^2\delta + \varepsilon_1\delta^3) = 0. \end{cases} \tag{6.34}$$

Hence we give,

Corollary 6.9. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in N where $\alpha, \beta = constant$, $ff'' = 0$ and $\delta = constant \neq 0$. Then γ is a bi-f-harmonic curve iff f is a solution of non-linear differential equations given in equation (6.34).*

Case II-5: If N a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} 4ff'\delta^2 = 0, \\ ff'\delta\beta = 0, \\ \varepsilon_1\delta(f')^2 + \frac{f^2}{2}(\frac{r}{2}\delta + 3\beta^2\delta + \varepsilon_1\delta^3) = 0. \end{cases} \tag{6.35}$$

So, we have the following nonexistence theorem.

Theorem 6.4. *There is no proper bi-f-harmonic Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta = constant \neq 0$.*

Case II-6: If N a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} ff'(\delta^2 - \varepsilon_1\alpha^2) = 0, \\ 2\varepsilon_1(f')^2 - f^2(2\alpha^2 - \varepsilon_1\delta^2) = 0, \\ 2\varepsilon_1(f')^2 + f^2(\frac{r}{2} + \alpha^2 + \varepsilon_1\delta^2) = 0. \end{cases} \tag{6.36}$$

Corollary 6.10. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional α-para-Kenmotsu manifold N where $\delta = constant \neq 0$. Then γ is a proper bi-f-harmonic curve iff the scalar curvature equals to $r = -6\alpha^2$ and the function f is the solution of* $2(f')^2 + ff'(\varepsilon_1\delta^2 - \alpha^2) - f^2(2\varepsilon_1\alpha^2 - \delta^2) = 0$.

Case II-7: If N a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then we have following equations from (6.27);

$$\begin{cases} 4ff^{'}\delta^2 = 0, \\ \varepsilon_1\delta(f^{'})^2 + \frac{f^2}{2}(\frac{r}{2}\delta + \varepsilon_1\delta^3) = 0. \end{cases} \tag{6.37}$$

Then we give

Theorem 6.5. *There is no bi-f-harmonic Frenet Legendre curve in a three dimensional paracosymplectic manifold where $\delta = constant \neq 0$.*

7. Biminimal Frenet Legendre Curves

In this section, the conditions for a Frenet curve to be biminimal are obtained in N. Besides, detailed calculations have been made for various manifolds as in the previous sections. By using normal components of tension and bitension fields, the condition of being biminimal curve is obtained by using the formula given as below, [11, 17];

$$[\tau_{2,\lambda}(\gamma)]^{\perp} = [\tau_2(\gamma)]^{\perp} - \lambda[\tau(\gamma)]^{\perp} = 0. \tag{7.38}$$

Let determine the biminimality condition for a Frenet Legendre curve in N. First, let give the tension and bitension fields respectively;

$$\tau(\gamma) = -\varepsilon_1\alpha\xi - \varepsilon_1\delta\varphi T,$$

$$\tau_2(\gamma) = 3\delta\delta^{'}T + (-3\beta^2\delta - \alpha^2\delta - \frac{r}{2}\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta^{''})\varphi T + (-2\beta\delta^{'} + 2\alpha^3 - \alpha\varepsilon_1\delta^2)\xi.$$

Hence by using normal components of tension and bitension fields the biminimality condition is obtained as below;

$$\begin{aligned} [\tau_{2,\lambda}(\gamma)]^{\perp} &= (-3\beta^2\delta - \alpha^2\delta - \frac{r}{2}\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta^{''} + \lambda\varepsilon_1\delta)\varphi T \\ &+ (-2\beta\delta^{'} + 2\alpha^3 - \alpha\varepsilon_1\delta^2 + \lambda\varepsilon_1\alpha)\xi \\ &= 0. \end{aligned} \tag{7.39}$$

By using this condition, we can give the following theorems;

Theorem 7.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$. Then γ is a biminimal curve* *iff the following equations hold:*

$$\begin{cases} 3\beta^2\delta + \alpha^2\delta + \frac{r}{2}\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''} - \lambda\varepsilon_1\delta = 0, \\ \\ -2\beta\delta^{'} + 2\alpha^3 - \alpha\varepsilon_1\delta^2 + \lambda\varepsilon_1\alpha = 0. \end{cases} \tag{7.40}$$

Theorem 7.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold N where $\alpha, \beta = constant$. Then γ is a biminimal curve where the scalar curvature r is given by;*

$$r = -2\varepsilon_1\frac{\delta^{''}}{\delta} - 4\frac{\beta}{\alpha}\delta^{'} - 6\alpha^2 - 6\beta^2,$$

where δ is the solution of the second differential equation of (7.40).

Now, we give the interpretations of **Theorem 7.1**.

Case I: Assume that δ is not constant. Then we shall investigate the following subcases.

Case I-1: If N is a three dimensional β-para-Sasakian manifold and $\delta \neq constant$ then from (7.40), we obtain following equations;

$$\begin{cases} 3\beta^2\delta + \frac{r}{2}\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta^{''} - \lambda\varepsilon_1\delta = 0, \\ \\ 2\beta\delta^{'} = 0. \end{cases} \tag{7.41}$$

Then we obtain the following nonexistence theorem.

Theorem 7.3. *There is no biminimal Frenet Legendre curve in a β-para-Sasakian manifold where $\delta \neq constant$.*

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq constant$ then from (7.40), we obtain following equations;

$$\begin{cases} -\alpha^2\delta - \frac{r}{2}\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta^{''} + \lambda\varepsilon_1\delta = 0, \\ \\ 2\alpha^3 - \alpha\varepsilon_1\delta^2 + \lambda\varepsilon_1\alpha = 0. \end{cases} \tag{7.42}$$

So we give,

Theorem 7.4. *There is no biminimal Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N where $\delta \neq constant$.*

Proof. From (7.42), we find that $\delta = \sqrt{2\varepsilon_1\alpha^2 + \lambda}$ but we accept $\delta \neq constant$ where $\alpha = constant$.

Case I-3: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then from (7.40), we obtain following equation;

$$\frac{r}{2}\delta + \varepsilon_1\delta^3 + \varepsilon_1\delta'' - \lambda\varepsilon_1\delta = 0.$$

Hence we have,

Corollary 7.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional paracosymplectic manifold N and $\delta \neq$ constant. Then γ is a biminimal curve iff the scalar curvature r is given by;*

$$r = -2\varepsilon_1\frac{\delta''}{\delta} - 2\varepsilon_1\delta^2 - 2\lambda\varepsilon_1.$$

Case II: Assume that δ=constant is not equal to 0. Then we shall investigate the following subcases:

Case II-1: If N is a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then from (7.40), we obtain following equation;

$$3\beta^2 + \frac{r}{2} + \varepsilon_1\delta^2 - \lambda\varepsilon_1 = 0.$$

Hence, we give the following theorem.

Corollary 7.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional β-para-Sasakian manifold N and $\delta = constant \neq 0$. Then γ is a biminimal curve where the constant scalar curvature r is given by;*

$$r = 2\varepsilon_1\delta^2 - 6\beta^2 + 2\lambda\varepsilon_1.$$

Case II-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then from (7.40), we obtain we obtain following equations;

$$\begin{cases} -\alpha^2 - \frac{r}{2} - \varepsilon_1\delta^2 + \lambda\varepsilon_1 = 0, \\ 2\alpha^2 - \varepsilon_1\delta^2 + \lambda\varepsilon_1 = 0. \end{cases}$$

Then we obtain the following corollary.

Corollary 7.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N and $\delta = constant \neq 0$. Then γ is a biminimal curve where the constant scalar curvature r is given by;*

$$r = -6\alpha^2.$$

Case II-3: If N is a three dimensional paracosymplectic manifold and $\delta = constant \neq 0$ then from (7.40), we obtain following equation;

$$\frac{r}{2} + \varepsilon_1\delta^2 - \lambda\varepsilon_1 = 0.$$

So we have,

Corollary 7.4. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional para-cosymplectic manifold N. Then γ is a biminimal curve where the constant scalar curvature r is given by;*

$$r = -2\varepsilon_1\delta^2 + 2\lambda\varepsilon_1.$$

8. f-Biminimal Frenet Legendre Curves

Finally in this section, we give f-biminimality conditions for a Frenet curve in N and also particular cases such as: β-para-Sasakian, α-para-Kenmotsu and paracosymplectic manifolds. From the Definition 2.7, we know that the condition of being f-biminimal curve given as below, [11];

$$[\tau_{2,\lambda,f}(\gamma)]^{\perp} = [\tau_{2,f}(\gamma)]^{\perp} - \lambda[\tau_f(\gamma)]^{\perp} = 0.$$

Then using the normal components of tension and bitension fields, given by (4.16) and (5.21), f-biminimality condition is obtained as below;

$$\begin{aligned}[\tau_{2,\lambda,f}(\gamma)]^{\perp} &= \big[(-\alpha^2\delta - 3\beta^2\delta - \varepsilon_1\delta^3 - \varepsilon_1\delta'' - \frac{r}{2}\delta + \lambda\varepsilon_1\delta)f \\ &- 2\varepsilon_1(\alpha\beta + \delta')f' - \varepsilon_1\delta f''\big]\varphi T \\ &+ ((2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta' + \lambda\varepsilon_1\alpha)f - 2\delta\beta f' - \varepsilon_1\alpha f'')\xi \\ &= 0. \end{aligned} \tag{8.43}$$

Theorem 8.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional normal almost paracontact metric manifold where $\alpha, \beta = constant$. Then γ is an f-biminimal curve* *iff the following equations hold:*

$$\begin{cases} (\alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \delta'' + \frac{r}{2}\varepsilon_1\delta - \lambda\delta)f + 2(\alpha\beta + \delta')f' + \delta f'' = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2 - 2\beta\delta' + \lambda\varepsilon_1\alpha)f - 2\delta\beta f' - \varepsilon_1\alpha f'' = 0. \end{cases} \tag{8.44}$$

Now, we give the interpretations of **Theorem 8.1**.

Case I: Assume that δ is not constant. Then we shall investigate the following subcases:

Case I-1: If N is a three dimensional β-para-Sasakian manifold and $\delta \neq constant$ then from (8.44), we obtain following equations;

$$\begin{cases} (3\beta^2\delta\varepsilon_1 + \delta^3 + \delta'' + \frac{r}{2}\varepsilon_1\delta - \lambda\delta)f + 2\delta' f' + \delta f'' = 0, \\ \beta(\delta f)' = 0. \end{cases} \tag{8.45}$$

Corollary 8.1. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional β-para-Sasakian manifold N where $\delta \neq constant$. Then γ is an f-biminimal curve iff the function f and the scalar curvature r equals:*

$$f = \frac{1}{\delta} + c$$

and

$$r = 2\varepsilon_1(\lambda - \delta^2 - \frac{\delta''}{\delta} - 3\beta^2\varepsilon_1) - 4\varepsilon_1(\frac{\delta'}{\delta})^2 - 2\varepsilon_1\delta(2(\delta)'\delta'' - \delta''\delta^{-2}).$$

Case I-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta \neq constant$ then from (8.44), we obtain following equations;

$$\begin{cases} (\alpha^2\delta\varepsilon_1 + \delta^3 + \delta'' + \frac{r}{2}\varepsilon_1\delta - \lambda\delta)f + 2\delta' f' + \delta f'' = 0, \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2 + \lambda\varepsilon_1\alpha)f - \varepsilon_1\alpha f'' = 0. \end{cases} \tag{8.46}$$

Corollary 8.2. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional α-para-Kenmotsu manifold N and $\delta \neq constant$. Then γ is an f-biminimal curve iff f is a solution of non-linear differential equations given in (8.46).*

Case I-2: If N is a three dimensional paracosymplectic manifold and $\delta \neq constant$ then from (8.44), we obtain following equation;

$$(\varepsilon_1\delta^3 + \varepsilon_1\delta'' + \frac{r}{2}\delta - \lambda\varepsilon_1\delta)f + 2\varepsilon_1\delta' f' + \varepsilon_1\delta f'' = 0. \tag{8.47}$$

Corollary 8.3. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve parametrized by arc length in three dimensional paracosymplectic manifold N and $\delta \neq constant$. Then γ is an f-biminimal curve iff f is a solution of non-linear differential equation given in (8.47).*

Case II: Assume that $\delta = constant$ is not equal to 0. Then we shall investigate the following subcases:

$$\begin{cases} (\alpha^2\delta\varepsilon_1 + 3\beta^2\delta\varepsilon_1 + \delta^3 + \dfrac{r}{2}\varepsilon_1\delta - \lambda\delta)f + 2(\alpha\beta)f^{'} + \delta f^{''} = 0, \\ \\ (2\alpha^3 - \varepsilon_1\alpha\delta^2 + \lambda\varepsilon_1\alpha)f - 2\delta\beta f^{'} - \varepsilon_1\alpha f^{''} = 0. \end{cases} \tag{8.48}$$

Case II-1: If N is a three dimensional β-para-Sasakian manifold and $\delta = constant \neq 0$ then from (8.44), we obtain following equations;

$$\begin{cases} (3\beta^2\delta\varepsilon_1 + \delta^3 + \dfrac{r}{2}\varepsilon_1\delta - \lambda\delta)f + \delta f^{''} = 0, \\ \\ 2\delta\beta f^{'} = 0. \end{cases} \tag{8.49}$$

Then we obtain the following nonexistence theorem;

Theorem 8.2. *There is no proper f-biminimal Frenet Legendre curve in a three dimensional β-para-Sasakian manifold where $\delta = constant \neq 0$.*

Proof. From the second equation of (8.49), the proof is obvious.

Case II-2: If N is a three dimensional α-para-Kenmotsu manifold and $\delta = constant \neq 0$ then from (8.44), we obtain following equations;

$$\begin{cases} (\alpha^2\varepsilon_1 + \delta^2 + \dfrac{r}{2}\varepsilon_1 - \lambda)f + f^{''} = 0, \\ \\ (2\alpha^2\varepsilon_1 - \delta^2 + \lambda)f - f^{''} = 0. \end{cases} \tag{8.50}$$

Corollary 8.4. *Let $\gamma : I \longrightarrow N$ be a Frenet Legendre curve in a three dimensional α-para-Kenmotsu manifold N and $\delta = constant \neq 0$. Then γ is an f-biminimal curve where the constant scalar curvature equals to $r = -6\alpha^2$ and the function f is a solution of the non-linear differential equations given in (8.50).*

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Ege University, Department of Mathematics, 35100, Bornova, İzmir, TURKEY

Ege University, Department of Mathematics, 35100, Bornova, İzmir, TURKEY