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RULED SURFACES WITH THE BASE RECTIFYING CURVES IN EUCLIDEAN 3-SPACE

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Abstract. There are many studies about rectifying curves. In this present study, we examine the ruled surfaces that have rectifying curves as base curves. We say that co-centrode curves defined by Chen and Dillen are the parameter curves for the special case $u = 1$ on the ruled surface with a base rectifying curve. Also, we answer the question when does the parameter curves of the surface are geodesic.

Keywords: Rectifying curve, Ruled surface, Modified darboux vector

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1. Introduction

The curves are the fundamental structure of differential geometry. In this study, we examine rectifying curves which are one of the subfamilies of the curves in Euclidean 3-space. A regular curve $\alpha(s)$ is called a rectifying curve, if its position vector always lies its rectifying plane. So, the position vector of a rectifying curve satisfies the equation

$$\alpha(s) = \lambda(s) T(s) + \mu(s) B(s)$$

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for differentiable functions λ and μ according to arc length parameter s. The notion of rectifying curves is introduced by B.Y. Chen in [1]. Also B.Y. Chen and Dillen show that there exists a relationship between the rectifying curves and the centrodes [2].

In the differential geometry of a regular curve, the curvature functions κ and τ of a regular curve play an important role to determine what is the type of the curve. One of the most interesting characteristics of rectifying curves is that the ratio of their torsion and curvature is a non-constant linear function of the arc length parameter s.

There are many studies about rectifying curves. K. Ilarslan et.al in [4, 5] introduce the rectifying curves in the Minkowski $3-$space. Also E. Özbey et.al study rectifying curves in dual Lorentzian space and they show that rectifying dual Lorentzian curves can be stated by the aid of dual unit spherical curves in [7]. In recent years, the rectifying curves from various viewpoints have been studied in Pseudo-Galilean space and three-dimensional sphere in [6, 8].

In this paper, we define the ruled surface whose the base curve is a rectifying curve by using modified Darboux vector field in Euclidean $3-$space. So, we examine the relationship between rectifying curves and ruled surfaces. In [2], Chen and Dillen introduce co-centrode curves. Accordingly, we say that co-centrode curves are the parameter curve for the special case $u = 1$ on this ruled surface. Also, we give the hypothesis that the curve whose the base curve for the given surface is a rectifying curve. Finally, we investigate the connection between the rectifying curve and the parameter curves of the surface which are the geodesic. We study the whole theory for the any orthonormal frame and also examine for special cases.

2. Preliminaries

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary curve in three dimensional Euclidean space. A moving orthonormal frame is defined as $\{N_1, N_2, N_3\}$ in the $\mathbb{E}^3$ along to curve α. Derivative of the frame is given by

$$\begin{bmatrix} N_1'(s) \\ N_2'(s) \\ N_3'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1(s) & \kappa_2(s) \\ -\kappa_1(s) & 0 & \kappa_3(s) \\ -\kappa_2(s) & -\kappa_3(s) & 0 \end{bmatrix} \begin{bmatrix} N_1(s) \\ N_2(s) \\ N_3(s) \end{bmatrix} \tag{2.1}$$

where $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ are the curvatures of the curve α. This any orthonormal frame encompasses some other frames. So, this frame is substantially important in terms of generality. For example, if we take $N_1 = T, N_2 = N, N_3 = B, \kappa_1 = \kappa, \kappa_2 = 0$ and $\kappa_3 = \tau$, above orthonormal frame coincides with the Serret Frenet frame. Also, if we take

$N_1 = T, N_2 = N_1, N_3 = N_2, \kappa_1 = k_1, \kappa_2 = 0$ and $\kappa_3 = k_3$, we have Bishop frame. Similarly, if we take $N_1 = T, N_2 = Y, N_3 = Z, \kappa_1 = k_g, \kappa_2 = k_n$ and $\kappa_3 = \tau_r$, orthonormal frame coincides with the Darboux frame on a curve. Using the equations $N_1 = N, N_2 = C, N_3 = W, \kappa_1 = f, \kappa_2 = 0$ and $\kappa_3 = g$, we get the alternative moving frame defined by Uzunoglu et.al in [9].

In the Euclidean space, the Darboux vector may be interpreted kinematically as the direction of the instantaneous axis of rotation in the moving trihedron. The direction of the Darboux vector is the instantaneous axis of rotation. In terms of the moving frame apparatus, the general Darboux vector field D can be expressed as

$$D = \kappa_3(s)\, N_1(s) - \kappa_2(s)\, N_2(s) + \kappa_1(s)\, N_3(s) \tag{2.2}$$

and it provides the following symmetrical properties

$$\begin{aligned} D \times N_1(s) &= N_1'(s) \\ D \times N_2(s) &= N_2'(s) \\ D \times N_3(s) &= N_3'(s) \end{aligned} \tag{2.3}$$

where $\times$ is the wedge product in Euclidean space $\mathbb{E}^3$.

Izumiya and Takeuchi define the modified Darboux vector field as follows

$$\overline{D} = \left(\frac{\tau}{\kappa}\right)(s)\, T(s) + B(s)$$

with $\kappa(s) \neq 0$ and another modified Darboux vector field is defined as $\widetilde{D} = T(s) + \left(\frac{\kappa}{\tau}\right)(s)\, B(s)$ with $\tau(s) \neq 0$ [3].

In [1], Chen proves that the curve $\alpha(s)$ is congruent to a rectifying curve if and only if the ratio $\frac{\tau}{\kappa}$ with $\kappa > 0$ is a non-constant linear according to arc length parameter s in $\mathbb{E}^3$.

3. Ruled Surfaces with The Base Rectifying Curves in Euclidean 3-Space

In this section, we examine the relationship between rectifying curves and ruled surfaces according to any orthonormal frame $\{N_1, N_2, N_3\}$. We consider this any orthonormal frame with $\kappa_2 = 0$ but note that the frame different from Frenet frame. Also, we give the hypothesis the parameter curves of the ruled surfaces with the base rectifying curve are geodesic. We can define the rectifying curve with this orthonormal frame. So, if the rate of the curvatures $\frac{\kappa_3}{\kappa_1}$ is a non-constant linear function according to arc length function s, then we can say the curve is a rectifying curve.

Theorem 3.1. *Let $\alpha(s) = \int N_1(s)\,ds$ be a unit speed curve with any orthonormal frame $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. The curve α is a rectifying curve if and only $s-$parameter curves of the surface $\phi(s,u) = \alpha(s) + u\overline{D}(s)$ are rectifying curve where $\overline{D}(s) = \left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3$ is modified Darboux vector field and $u \neq -\frac{1}{a}$.*

Proof. Let $\alpha(s) = \int N_1(s)\,ds$ be a unit speed and rectifying curve with the frame apparatus $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. If the parameter u is a constant, we obtain the $s-$parameter curves of the surface as $\beta(s) = \int N_1(s)\,ds + u\left(\left(\frac{\kappa_3}{\kappa_1}\right)(s)\,N_1(s) + N_3(s)\right)$. If we take the derivative of β according to its arc length parameter, then we have

$$\begin{aligned} \frac{d\beta}{d\overline{s}} &= \frac{d\beta}{ds}\frac{ds}{d\overline{s}}, \\ \overline{N}_1 &= \left(1 + u\left(\frac{\kappa_3}{\kappa_1}\right)'\right) N_1 \frac{ds}{d\overline{s}} \end{aligned}$$

where $\left\{\overline{N}_1, \overline{N}_2, \overline{N}_3, \overline{\kappa}_1, \overline{\kappa}_3\right\}$ is the any orthonormal frame apparatus of β. If we take the norm of both sides of above equation, we have

$$d\overline{s} = \left(1 + u\left(\frac{\kappa_3}{\kappa_1}\right)'\right) ds.$$

If we integrate the last equation, we obtain

$$\overline{s} = s + u\left(\frac{\kappa_3}{\kappa_1}\right) + c,\ c \text{ constant}, \tag{3.4}$$

and we can easily see that

$$\overline{N}_1 = N_1.$$

Similarly, if a derivative of this equation is taken with respect to s, we obtain

$$\begin{aligned} \frac{d\overline{N}_1}{d\overline{s}}\frac{d\overline{s}}{ds} &= \kappa_1 N_2, \\ \overline{\kappa}_1 \overline{N}_2 &= \kappa_1 N_2 \frac{1}{1 + u\left(\frac{\kappa_3}{\kappa_1}\right)'} \end{aligned}$$

where $\left(\frac{\kappa_3}{\kappa_1}\right)' \neq -\frac{1}{u}$. If we take the norm of last equation, we get

$$\overline{\kappa}_1 = \frac{\kappa_1}{1 + u\left(\frac{\kappa_3}{\kappa_1}\right)'}. \tag{3.5}$$

So, we can easily see that

$$\overline{N}_2 = N_2.$$

Hence, we know that $\overline{N}_3 = N_3$. If we take the derivative of this equation according to s, we have

$$\overline{\kappa}_3 = \frac{\kappa_3}{1 + u\left(\dfrac{\kappa_3}{\kappa_1}\right)'}. \tag{3.6}$$

If we look at the ratio of the Eq. (3.5) and Eq. (3.6), we can say that

$$\frac{\kappa_3}{\kappa_1} = \frac{\overline{\kappa}_3}{\overline{\kappa}_1}. \tag{3.7}$$

Since α is a rectifying curve, we know that $\dfrac{\kappa_3}{\kappa_1} = as+b$ non-constant linear function for some constants a and b with $a \neq 0$ and $a \neq -\dfrac{1}{u}$. Let us write this equality in equation (3.4).

$$\begin{aligned} \overline{s} &= s + u\left(as + b\right) + c, \\ \overline{s} &= \left(1 + au\right)s + buc, \\ \overline{s} &= es + f, \end{aligned}$$

where e, f are some constants with $e \neq 0$. So, we obtain the arc length parameter of the curve α as follows

$$s = \frac{\overline{s} - f}{e}.$$

From equation (3.7), we get

$$\frac{\kappa_3}{\kappa_1} = \frac{\overline{\kappa}_3}{\overline{\kappa}_1} = a\left(\frac{\overline{s} - f}{e}\right) + b.$$

Hence, we can easily see that

$$\frac{\overline{\kappa}_3}{\overline{\kappa}_1} = \lambda\overline{s} + \mu$$

where λ and μ are some constants with $\lambda \neq 0$.

Finally, if the curve α is a rectifying curve, then s-parameter curves of the surface $\phi\left(s, u\right) = \int N_1\left(s\right)ds + u\left(\left(\dfrac{\kappa_3}{\kappa_1}\right)N_1 + N_3\right)$ are rectifying curve.

Conversely, let s-parameter curves of the surface $\beta\left(s\right) = \int N_1\left(s\right)ds + u\left(\left(\dfrac{\kappa_3}{\kappa_1}\right)N_1 + N_3\right)$ are rectifying curve. The ratio of the curvatures of the curve β is the non-constant linear function according to $\overline{s}$ for some constants λ and μ with $\lambda \neq 0$ as

$$\frac{\overline{\kappa}_3}{\overline{\kappa}_1} = \lambda\overline{s} + \mu.$$

From the equations (3.4) and (3.7), we can easily see that

$$\frac{\overline{\kappa}_3}{\overline{\kappa}_1} = \frac{\kappa_3}{\kappa_1} = \lambda\left(s + u\left(\frac{\kappa_3}{\kappa_1}\right) + c\right) + \mu.$$

If the necessary arrangements are made, we get

$$\frac{\kappa_3}{\kappa_1} = as + b$$

where a, b are some constants with $a \neq 0$. This means that α is a rectifying curve.

Corollary 3.1. *Let $\beta(s) = \int N_3(s)\, ds$ be a unit speed curve with $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. The curve β is a rectifying curve if and only if s-parameter curves of the surface $\phi(s,u) = \beta(s) + v\widetilde{D}(s)$ are rectifying curve where $\widetilde{D}(s) = N_1 + \left(\frac{\kappa_1}{\kappa_3}\right) N_3$ is modified Darboux vector field.*

Corollary 3.2. *Let $\gamma(s) = \int T(s)\, ds$ be a unit speed curve with $\{T, N, B, \kappa, \tau\}$. Then the curve γ is a rectifying curve if and only if s-parameter curves of the surface $\phi(s,u) = \gamma(s) + u\overline{D}(s)$ are rectifying curve where $\overline{D}(s) = \left(\frac{\tau}{\kappa}\right) T + B$ is modified Darboux vector field.*

Remark 3.1. *For a regular curve γ in $\mathbb{E}^3$ with $\kappa \neq 0$, the curve given by the Darboux vector $D = \tau T + \kappa B$ is called the centrode of γ and the curves $C_{\pm} = \gamma \pm D$ are called the co-centrodes of γ. Chen and Dillen show that a curve γ with non-zero constant curvature and non-constant torsion is a rectifying curve if and only if one of its co-centrodes is a rectifying curve* [2]*. If we select $u = 1$ for u constant parameter curves, then we define the u constant parameter curves correspond to the co-centrodes.*

Corollary 3.3. *Let $\sigma(s) = \int N(s)$ be a unit speed curve with $\{N, C, W, f, g\}$ defined by Uzunoğlu* [9]*. The curve σ is a rectifying curve if and only if s-parameter curves of the surface $\phi(s,u) = \sigma(s) + u\overline{D}(s)$ are rectifying curve where $\overline{D}(s) = \left(\frac{g}{f}\right) N + W$ is modified Darboux vector field.*

Theorem 3.2. *Let $\alpha(s) = \int N_1(s)\, ds$ be a unit speed curve with any orthonormal frame apparatus $\{N_1, N_2, N_3, \kappa_1, \kappa_3\}$. If α is a rectifying curve, the parameter curves of the surface $\phi(s,u) = \alpha(s) + u\overline{D}(s)$ are geodesic curve where $\overline{D}(s) = \left(\frac{\kappa_3}{\kappa_1}\right) N_1 + N_3$ is modified Darboux vector field and $u \neq -\frac{1}{a}$.*

Proof. The curve α has been always geodesic on the surface, but the parameter curves of the surface are geodesic if α is a rectifying curve. The normal vector of the surface is as follows

$$\begin{aligned}\phi_s &= \left(1+u\left(\frac{\kappa_3}{\kappa_1}\right)'\right)N_1 \text{ and } \phi_u = \left(\frac{\kappa_3}{\kappa_1}\right)N_1+N_3,\\ N_\phi &= -\left(1+u\left(\frac{\kappa_3}{\kappa_1}\right)'\right)N_2.\end{aligned}$$

Let α be a unit speed rectifying curve. Let's examine s-parameter curves of the surface

$$\phi(s,u)=\int N_1(s)\,ds+u\left(\left(\frac{\kappa_3}{\kappa_1}\right)(s)\,N_1(s)+N_3(s)\right),$$

$$\beta(s)=\int N_1(s)\,ds+u\left(\left(\frac{\kappa_3}{\kappa_1}\right)N_1+N_3\right),$$

$$\begin{aligned}\frac{d\beta}{d\bar{s}} &= \frac{d\beta}{ds}\frac{ds}{d\bar{s}}=aN_1\frac{ds}{d\bar{s}},\\ \frac{d^2\beta}{d\bar{s}^2} &= \frac{d^2\beta}{ds^2}\frac{ds^2}{d\bar{s}^2}=b\kappa_1N_2,\end{aligned}$$

where a and b are some constants.

Similar to the above thought, if we examine u-parameter curves of the surface $\phi(s,u)=\int N_1(s)\,ds+u\left(\left(\frac{\kappa_3}{\kappa_1}\right)N_1+N_3\right)$, then we have

$$\begin{aligned}\beta(s) &= \int N_1(s)\,ds+u\left(\left(\frac{\kappa_3}{\kappa_1}\right)N_1+N_3\right),\\ \frac{d^2\beta}{d\bar{u}^2} &= 0.\end{aligned}$$

So, if the curve α is a rectifying curve, then the parameter curves of the surface $\phi(s,u)=\alpha(s)+u\left(\left(\frac{\kappa_3}{\kappa_1}\right)N_1+N_3\right)$ are geodesic curve.

Corollary 3.4. *Let $\gamma=\int T(s)\,ds$ be a unit speed curve with $\{T,N,B,\kappa,\tau\}$. If γ is a rectifying curve, the parameter curves of the surface $\phi(s,u)=\gamma(s)+u\overline{D}(s)$ are geodesic curve where $\overline{D}(s)=\left(\frac{\tau}{\kappa}\right)T+B$ is modified Darboux vector field.*

Corollary 3.5. *Let $\sigma(s)=\int N(s)$ be a unit speed curve with $\{N,C,W,f,g\}$. If σ is a rectifying curve, the parameter curves of the surface $\phi(s,u)=\sigma(s)+u\overline{D}(s)$ are geodesic curve where $\overline{D}(s)=\left(\frac{g}{f}\right)N+W$ is modified Darboux vector field.*

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