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ON TRANS-SASAKIAN 3-MANIFOLDS WITH RESPECT TO THE SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. The object of the present paper is to characterize trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection. Also, we consider Ricci solitons, η-Ricci solitons and Yamabe solitons of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. Then we give an example of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection.

Keywords: Schouten-van Kampen connection, trans-Sasakian manifolds, semi-symmetry, Ricci semi-symmetry, almost Ricci soliton, almost η-Ricci soliton, almost Yamabe soliton.

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1. Introduction

In [19], Oubina defined a new class of almost contact metric structure, which is said to be trans-Sasakian structure of type (α, β). In [7], Chinea and Gonzales introduced two subclasses of trans-Sasakian structures which contain the Kenmotsu and Sasakian structures. Trans-Sasakian structures of type $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α-Sasakian, β-Kenmotsu and cosymplectic, respectively [3, 14].

The Schouten-van Kampen connection defined as adapted to a linear connection for studying non holonomic manifolds and it is one of the most natural connections on a differentiable manifold [2, 13, 23]. Solov'ev studied hyperdistributions in Riemannian manifolds using the Schouten-van Kampen connection [24, 25, 26, 27]. Then Olszak studied the Schouten-van

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Kampen connection to almost (para) contact metric structures [18]. In recent times, Perktaş and Yildiz studied some symmetry conditions and some soliton types of quasi-Sasakian manifolds and f-Kenmotsu manifolds with respect to the Schouten-van Kampen connection [21, 22].

Let (M, g) be a Riemannian manifold. Then the metric g is called a Ricci soliton if [12]

$$L_X g + 2Ric + 2\delta g = 0, \tag{1.1}$$

where L is the Lie derivative, Ric is the Ricci tensor, X is a complete vector field and δ is a constant on M. In [8], Cho and Kimura given the notion of η-Ricci solitons. The manifold (M, g) is called an η-Ricci soliton if there exist a smooth vector field X such that the Ricci tensor satisfies

$$L_X g + 2Ric + 2\delta g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where and μ is also constant on M. Note that Ricci solitons and η-Ricci solitons are said to be shrinking, steady and expanding according as δ is negative, zero and positive, respectively.

In [12], Hamilton defined Yamabe flow to solve the Yamabe problem. The Yamabe soliton comes from the blow-up procedure along the Yamabe flow, so such solitons have been studied intensively [1, 5, 6, 10, 17].

A Yamabe soliton on a Riemannian manifold (M, g) satisfying [1]

$$\frac{1}{2}(L_X g) = (\tau - \delta)g, \tag{1.3}$$

where τ is the scalar curvature of M. Moreover, if (M, g) is of constant scalar curvature τ, then the Riemannian metric g is called a Yamabe metric.Yamabe solitons are said to be shrinking, steady and expanding according as δ is positive, zero and negative, respectively.

This paper is organized as follows: After preliminaries, we give some basic information about the Schouten-van Kampen connection and trans-Sasakian manifolds. Then we adapte the Schouten-van Kampen connection on trans-Sasakian 3-manifolds. In section 4, we consider Ricci semisymmetric trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection. In the last section, firstly we study Ricci solitons, η-Ricci solitons and Yamabe solitons of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. Then we give an example of a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is $(1,1)$-tensor field, ξ is a vector field, η is a 1-form and g is the compatible Riemannian metric such that

$$\phi^2(U) = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \tag{2.4}$$

$$g(\phi U, \phi V) = g(U,V) - \eta(U)\eta(V), \tag{2.5}$$

$$g(U, \phi V) = -g(\phi U, V), \quad g(U, \xi) = \eta(U), \tag{2.6}$$

for all $U, V \in TM$ [3]. The fundamental 2-form Φ of the manifold is defined by

$$\Phi(U,V) = g(U, \phi V). \tag{2.7}$$

This may be expressed by the condition [4]

$$(\nabla_U \phi)V = \alpha(g(U,V)\xi - \eta(V)U) + \beta(g(\phi U, V)\xi - \eta(V)\phi U), \tag{2.8}$$

for smooth functions α and β on M. Here we say that the trans-Sasakian structure is of type (α, β). From the formula (2.8) it follows that

$$\nabla_U \xi = -\alpha\phi U + \beta(U - \eta(U)\xi), \tag{2.9}$$

$$(\nabla_U \eta)V = -\alpha g(\phi U, V) + \beta g(\phi U, \phi V). \tag{2.10}$$

An explicit example of trans-Sasakian 3-manifolds was constructed in [15]. In [9], the Ricci tensor and curvature tensor for trans-Sasakian 3-manifolds were studied and their explicit formulae were given.

From [9] we know that for a trans-Sasakian 3-manifold

$$2\alpha\beta + \xi\alpha = 0, \tag{2.11}$$

$$Ric(U, \xi) = (2(\alpha^2 - \beta^2) - \xi\beta)\eta(U) - U\beta - (\phi U)\alpha, \tag{2.12}$$

$$\begin{aligned} Ric(U,V) &= (\frac{\tau}{2} + \xi\beta - (\alpha^2 - \beta^2))g(U,V) - (\frac{\tau}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(U)\eta(V) \\ &\quad -(V\beta + (\phi V)\alpha)\eta(U) - (U\beta + (\phi U)\alpha)\eta(V), \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} R(U,V)W &= (\frac{\tau}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(V,W)U-g(U,W)V) \\ &-g(V,W)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(U)\xi \\ &-\eta(U)(\phi grad\alpha-grad\beta)+(U\beta+(\phi U)\alpha)\xi] \\ &+g(U,W)[(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(V)\xi \\ &-\eta(V)(\phi grad\alpha-grad\beta)+(V\beta+(\phi V)\alpha)\xi] \qquad (2.14) \\ &-[(W\beta+(\phi W)\alpha)\eta(V)+(V\beta+(\phi V)\alpha)\eta(W) \\ &+(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(V)\eta(W)]U \\ &+[(W\beta+(\phi W)\alpha)\eta(U)+(U\beta+(\phi U)\alpha)\eta(W) \\ &+(\frac{\tau}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(U)\eta(W)]V, \end{aligned}$$

where Ric is the Ricci tensor, R is the curvature tensor and τ is the scalar curvature of the manifold M, respectively.

If α and β are constants, then equations (2.11)-(2.14) become

$$\begin{aligned} R(U,V)W &= (\frac{\tau}{2}-2(\alpha^2-\beta^2))(g(V,W)U-g(U,W)V) \\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))(g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi \qquad (2.15) \\ &\quad +\eta(V)\eta(W)U-\eta(U)\eta(W)V), \end{aligned}$$

$$\begin{aligned} Ric(U,V) &= (\frac{\tau}{2}-(\alpha^2-\beta^2))g(U,V) \qquad (2.16) \\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(U)\eta(V), \end{aligned}$$

$$Ric(U,\xi)=2(\alpha^2-\beta^2)\eta(U), \qquad (2.17)$$

$$R(U,V)\xi=(\alpha^2-\beta^2)(\eta(V)U-\eta(U)V), \qquad (2.18)$$

$$R(\xi,U)V=(\alpha^2-\beta^2)(g(U,V)\xi-\eta(V)U), \qquad (2.19)$$

$$\begin{aligned} QU &= (\frac{\tau}{2}-(\alpha^2-\beta^2))U \qquad (2.20) \\ &-(\frac{\tau}{2}-3(\alpha^2-\beta^2))\eta(U)\xi. \end{aligned}$$

From (2.11) it follows that if α and β are constants, then the manifold is either α-Sasakian or β-Kenmotsu or cosymplectic, respectively.

On the other hand we have two naturally defined distributions in the tangent bundle TM of M as follows:

$$H = \ker \eta, \qquad V = \text{span}\{\xi\}. \tag{2.21}$$

Then we have $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition allows one to define the Schouten-van Kampen connection $\tilde{\nabla}$ over an almost contact metric structure. The Schouten-van Kampen connection $\tilde{\nabla}$ on an almost contact metric manifold with respect to Levi-Civita connection ∇ is defined by [24]

$$\tilde{\nabla}_U V = \nabla_U V - \eta(V)\nabla_U \xi + (\nabla_U \eta)(V)\xi. \tag{2.22}$$

Thus with the help of the Schouten-van Kampen connection given by (2.22), many properties of some geometric objects connected with the distributions H, V can be characterized [24, 25, 26]. For example g, ξ and η are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0$, $\tilde{\nabla}g = 0$,$\tilde{\nabla}\eta = 0$. Also the torsion $\tilde{T}$ of $\tilde{\nabla}$ is defined by

$$\tilde{T}(U, V) = \eta(U)\nabla_V \xi - \eta(V)\nabla_U \xi + 2d\eta(U, V)\xi.$$

3. Trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection

Let M be a trans-Sasakian 3-manifold with α and β are constants with respect to the Schouten-van Kampen connection. Then using (2.9) and (2.10) in (2.22), we get

$$\tilde{\nabla}_U V = \nabla_U V + \alpha\{\eta(V)\phi U - g(\phi U, V)\xi\} + \beta\{g(U, V)\xi - \eta(V)U\}. \tag{3.23}$$

Let R and $\tilde{R}$ be the curvature tensors of the Levi-Civita connection ∇ and the Schouten-van Kampen connection $\tilde{\nabla}$ are given by

$$R(U, V) = [\nabla_U, \nabla_V] - \nabla_{[U,V]}, \qquad \tilde{R}(U, V) = [\tilde{\nabla}_U, \tilde{\nabla}_V] - \tilde{\nabla}_{[U,V]}.$$

Using (3.23), by direct calculations, we obtain the following formula connecting R and $\tilde{R}$ on a trans-Sasakian 3-manifold

$$\begin{aligned} \tilde{R}(U, V)W = {} & R(U, V)W \\ & + \alpha^2\{g(\phi V, W)\phi U - g(\phi U, W)\phi V + \eta(U)\eta(W)V \\ & - \eta(V)\eta(W)U - g(V, W)\eta(U)\xi + g(U, W)\eta(V)\xi\} \\ & + \beta^2\{g(V, W)U - g(U, W)V\}. \end{aligned} \tag{3.24}$$

We will also consider the Riemann curvature $(0,4)$-tensors $\tilde{R}, R$, the Ricci tensors $\tilde{Ric}, Ric$, the Ricci operators $\tilde{Q}, Q$ and the scalar curvatures $\tilde{\tau}, \tau$ of the connections $\tilde{\nabla}$ and ∇ are given by

$$\begin{aligned}\tilde{R}(U,V,W,Z) &= R(U,V,W,Z) \\ &+\alpha^2\{g(\phi V,W)g(\phi U,Z)-g(\phi U,W)g(\phi V,Z) \\ &+g(V,Z)\eta(U)\eta(W)-g(U,Z)\eta(V)\eta(W) \\ &-g(V,W)\eta(U)\eta(Z)+g(U,W)\eta(V)\eta(Z)\} \\ &+\beta^2\{g(V,W)g(U,Z)-g(U,W)g(V,Z)\},\end{aligned} \tag{3.25}$$

$$\begin{aligned}\tilde{Ric}(V,W) &= Ric(V,W) \\ &+2\beta^2 g(V,W)-2\alpha^2\eta(V)\eta(W),\end{aligned} \tag{3.26}$$

$$\tilde{Q}U = QU + 2\beta^2 U - 2\alpha^2\eta(U)\xi, \tag{3.27}$$

$$\tilde{\tau} = \tau - 2\alpha^2 + 6\beta^2, \tag{3.28}$$

respectively, where $\tilde{R}(U,V,W,Z) = g(\tilde{R}(U,V)W,Z)$ and $R(U,V,W,Z) = g(R(U,V)W,Z)$.

4. Ricci semisymetric trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection

In this section, we study Ricci semisymetric trans-Sasakian 3-manifolds with α and β are constants with respect to the Schouten-van Kampen connection.

If a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection is *Ricci semisymmetric* then we can write

$$(\tilde{R}(U,V)\cdot\tilde{Ric})(W,Y) = 0, \tag{4.29}$$

which turns to

$$\tilde{Ric}(\tilde{R}(U,V)W,Y) + \tilde{Ric}(W,\tilde{R}(U,V)Y) = 0. \tag{4.30}$$

Using (3.26) in (4.30), we obtain

$$\begin{aligned}&Ric(\tilde{R}(U,V)W,Y) - 2\alpha^2\eta(\tilde{R}(U,V)W)\eta(Y) + 2\beta^2 g(\tilde{R}(U,V)W,Y) \\ &+Ric(W,\tilde{R}(U,V)Y) - 2\alpha^2\eta(\tilde{R}(U,V)Y)\eta(W) + 2\beta^2 g(W,\tilde{R}(U,V)Y) \\ = \;&Ric(\tilde{R}(U,V)W,Y) + Ric(W,\tilde{R}(U,V)Y) = 0.\end{aligned} \tag{4.31}$$

Now using (3.24) in (4.31), we get

$$\begin{aligned}
&Ric(R(U,V)W,Y)+Ric(W,R(U,V)Y)+\alpha^2\{g(\phi V,W)Ric(\phi U,Y)\\
&-g(\phi U,W)Ric(\phi V,Y)+Ric(V,Y)\eta(U)\eta(W)-Ric(U,Y)\eta(V)\eta(W)\\
&+g(U,W)\eta(V)Ric(Y,\xi)-g(V,W)\eta(U)Ric(Y,\xi)+g(\phi V,Y)Ric(\phi U,W)\\
&-g(\phi U,Y)Ric(\phi V,W)+Ric(V,W)\eta(U)\eta(Y)-Ric(U,W)\eta(V)\eta(Y)\\
&+g(U,Y)\eta(V)Ric(W,\xi)-g(V,Y)\eta(U)Ric(W,\xi)\}\\
&+\beta^2\{g(V,W)Ric(Y,U)-g(U,W)Ric(Y,V)\\
&+g(V,Y)Ric(W,U)-g(U,Y)Ric(W,V)\}=0.
\end{aligned} \tag{4.32}$$

Let $\{e_i\}$, $(1\leq i\leq 3)$, be an orthonormal basis of the tangent space at any point of M. Then the sum for $1\leq i\leq 3$ of the relation (4.32) for $U=Y=e_i$ gives

$$\begin{aligned}
&Ric(R(e_i,V)W,e_i)+Ric(W,R(e_i,V)e_i)\\
&+\alpha^2\left\{Ric(V,W)-\tau\eta(V)\eta(W)\right\}\\
&+2\alpha^2\left(\alpha^2-\beta^2\right)\{3\eta(V)\eta(W)-g\left(V,W\right)\}\\
&+\beta^2\{\tau g(V,W)-3Ric(V,W)\}=0,
\end{aligned} \tag{4.33}$$

which is equal to

$$\begin{aligned}
&\lambda\left\{\tau g(V,W)-3Ric\left(V,W\right)\right\}+2\mu(\alpha^2-\beta^2)\eta(V)\eta(W)\\
&+\mu Ric(V,W)-2\mu(\alpha^2-\beta^2)g(V,W)+4\mu(\alpha^2-\beta^2)\eta(V)\eta(W)\\
&-\mu\tau\eta(V)\eta(W)\\
&+\alpha^2\left\{Ric(V,W)-\tau\eta(V)\eta(W)\right\}\\
&+2\alpha^2(\alpha^2-\beta^2)\left\{3\eta(V)\eta(W)-g(V,W)\right\}\\
&+\beta^2\{\tau g(V,W)-3Ric(V,W)\}=0,
\end{aligned} \tag{4.34}$$

where $\lambda=\frac{\tau}{2}-2(\alpha^2-\beta^2)$ and $\mu=\frac{\tau}{2}-3(\alpha^2-\beta^2)$. After some calculations we have

$$\begin{aligned}
&[-3\left(\lambda+\beta^2\right)+\left(\mu+\alpha^2\right)]Ric(V,W)\\
&+[(\lambda+\beta^2)\tau-2\left(\mu+\alpha^2\right)\left(\alpha^2-\beta^2\right)]g(V,W)\\
&+[6(\mu+\alpha^2)(\alpha^2-\beta^2)-\left(\lambda+\beta^2\right)\tau]\eta(V)\eta(W)=0,
\end{aligned}$$

i.e.,

$$Ric(V,W) = [\frac{\tau}{2} - \left(\alpha^2 - \beta^2\right)]g\left(V,W\right) + [3\left(\alpha^2 - \beta^2\right) - \frac{\tau}{2}]\eta(V)\eta(W). \tag{4.35}$$

Hence M is an η-Einstein manifold with respect to the Levi-Civita connection. Now using (4.35) in (3.26), we have

$$\tilde{R}ic(V,W) = [\frac{\tau}{2} - \alpha^2 + 3\beta^2]g(V,W) - [\frac{\tau}{2} - \alpha^2 + 3\beta^2]\eta(V)\eta(W).$$

Thus M is also an η-Einstein manifold with respect to the Schouten-van Kampen connection. Therefore we have the following:

Theorem 4.1. *Let M be a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. If M is Ricci semisymmetric with respect to the Schouten-van Kampen connection then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection and Levi-Civita connection.*

5. Soliton types on trans-Sasakian 3-manifolds with respect to the Schouten-van Kampen connection

In this section we study Ricci solitons, η-Ricci solitons and Yamabe solitons on a trans-Sasakian 3-manifold with α and β are constants with respect to the Schouten-van Kampen connection.

In a trans-Sasakian 3-manifold M endowed with respect to the Schouten-van Kampen connection bearing an Ricci soliton, we can write

$$(\tilde{L}_X g + 2\tilde{R}ic + 2\delta g)(U,V) = 0. \tag{5.36}$$

Using (3.23) in (5.36), since $\tilde{\nabla} g = 0$ and $\tilde{T} \neq 0$, we have

$$(\tilde{L}_X g)(U,V) = g(\nabla_U X, V) + g(U, \nabla_V X) = (L_X g)(U,V),$$

that is,

$$g(\nabla_U X, V) + g(U, \nabla_V X) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) = 0. \tag{5.37}$$

Putting $X = \xi$ in (5.37), we obtain

$$g(\nabla_U \xi, V) + g(U, \nabla_V \xi) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) = 0. \tag{5.38}$$

Now using (2.9) in (5.38), we get

$$g(-\alpha\phi U + \beta(U - \eta(U)\xi), V) + g(U, -\alpha\phi V + \beta(V - \eta(V)\xi) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) = 0,$$

i.e.,

$$\tilde{R}ic(U,V) = -(\beta+\delta)g(U,V) + \beta\eta(U)\eta(V). \tag{5.39}$$

Thus M is an η-Einstein manifold with respect to the Schouten-van Kampen connection. Also using (3.26) in (5.39), we get

$$Ric(U,V) = -(2\beta^2+\beta+\delta)g(U,V) + (\beta+2\alpha^2)\eta(U)\eta(V).$$

Hence M is an η-Einstein manifold with respect to the Levi-Civita connection. Thus we have the following:

Theorem 5.1. *Let M be a trans-Sasakian 3-manifold bearing a Ricci soliton (ξ,δ,g) with respect to the Schouten-van Kampen connection. Then M is an η-Einstein manifold both with respect to the Schouten-van Kampen connection and Levi-Civita connection.*

Putting $V=\xi$ and using (3.26) in (5.39), we give the following:

Corollary 5.1. *A Ricci soliton (ξ,δ,g) on a trans-Sasakian 3-manifold M with respect to the Schouten-van Kampen connection is always steady.*

On the other hand, from (2.16) and (3.26), it is easy to see that a trans-Sasakian 3-manifold M is always η-Einstein with respect to the Schouten-van Kampen connection of the form $\tilde{R}ic = \gamma g + \sigma\eta\otimes\eta$, where $\gamma = -\sigma = \frac{\tau}{2} - \alpha^2 + 3\beta^2$. Then, we write

$$(\tilde{L}_\xi g + 2\tilde{R}ic + 2\delta g)(U,V) = ((2\gamma+2\delta)g - 2\sigma\eta\otimes\eta)(U,V), \tag{5.40}$$

for all $U,V\in\chi(M)$, which implies that the manifold M admits a Ricci soliton (ξ,δ,g) if $\gamma+\delta=0$ and $\sigma=0$.

Using (5.39), we can also state the following:

Corollary 5.2. *The scalar curvature of a trans-Sasakian 3-manifold M bearing a Ricci soliton (ξ,δ,g) with respect to the Schouten-van Kampen connection is $\tilde{\tau} = -3\delta - 2\beta$.*

Now we consider an η-Ricci soliton on a trans-Sasakian 3-manifold M with respect to the Schouten-van Kampen connection. Then

$$(\tilde{L}_X g + 2\tilde{R}ic + 2\delta g + 2\mu\eta\otimes\eta)(U,V) = 0, \tag{5.41}$$

that is,

$$g(\nabla_U X, V) + g(U, \nabla_V X) + 2\tilde{R}ic(U,V) + 2\delta g(U,V) + 2\mu\eta(U)\eta(V) = 0. \tag{5.42}$$

Putting $X = \xi$ in (5.42), we obtain

$$\tilde{R}ic(U,V) = -\delta g(U,V) - \mu\eta(U)\eta(V). \tag{5.43}$$

Hence M is an η-Einstein manifold with respect to the Schouten-van Kampen connection. Taking $V = \xi$ in (5.43), we get $\delta + \mu = 0$. Using (3.26) in (5.43), we have

$$Ric(U,V) = [-2\beta^2 - \delta]g(U,V) + [2\alpha^2 - \mu]\eta(U)\eta(V).$$

Thus M is an η-Einstein manifold with respect to the Levi-Civita connection. Now we have the following:

Theorem 5.2. *Let M be a trans-Sasakian 3-manifold bearing an η-Ricci soliton (ξ, δ, μ, g) with respect to the Schouten-van Kampen connection. Then M is an η-Einstein manifold with respect to the Schouten-van Kampen connection and the Levi-Civita connection.*

Again let us consider equations (5.36) and (5.37). Using (3.26), we obtain

$$g(\nabla_U X, V) + g(U, \nabla_V X) + 2Ric(U,V) + 2(2\beta^2 + \delta)g(U,V) - 2\alpha^2\eta(U)\eta(V) = 0.$$

Thus we write

$$(L_X g)(U,V) + 2Ric(U,V) + 2(2\beta^2 + \delta)g(U,V) - 2\alpha^2\eta(U)\eta(V) = 0.$$

This last equation shows that if (X, δ, g) is a Ricci soliton on a trans-Sasakian 3-manifold M with respect to the Schouten-van Kampen connection, then the manifold admits an η-Ricci soliton $(X, 2\beta^2 + \delta, \alpha^2, g)$ with respect to the Levi-Civita connection. If $\alpha = 0$, then

$$(L_X g)(U,V) + 2Ric(U,V) + 2(2\beta^2 + \delta)g(U,V) = 0.$$

So we have the following:

Corollary 5.3. *Let M be a trans-Sasakian 3-manifold bearing a Ricci soliton (X, δ, g) with respect to the Schouten-van Kampen connection. Then we have: (i) If $\alpha = 0$, then M admits a Ricci soliton $(X, 2\beta^2 + \delta, g)$ with respect to the Levi-Civita connection. (ii) If $\alpha \neq 0$, then M admits an η-Ricci soliton $(X, 2\beta^2 + \delta, \alpha^2, g)$ with respect to the Levi-Civita connection.*

Example 5.1. *We consider the 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3,\ y \neq 0\}$, where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields*

$$e_1 = e^y \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = e^y \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$\begin{aligned} g(e_1, e_3) &= g(e_2, e_3) = g(e_1, e_2) = 0, \\ g(e_1, e_1) &= g(e_2, e_2) = g(e_3, e_3) = 1. \end{aligned}$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_2)$ for any $Z \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by $\phi(e_1) = e_3$, $\phi(e_2) = 0$, $\phi(e_3) = -e_1$. Then using linearity of ϕ and g we have

$$\eta(e_2) = 1, \quad \phi^2 W = -W + \eta(W)e_3,$$

$$g(\phi W, \phi Z) = g(W, Z) - \eta(W)\eta(Z),$$

for any $W, Z \in \chi(M)$. Thus for $e_2 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = -e_1, \quad [e_2, e_3] = e_3, \quad [e_1, e_3] = 0.$$

The Riemannian connection ∇ of the metric tensor g is given by the Koszul's formula which is

$$\begin{aligned} 2g(\nabla_U V, W) &= Ug(V, W) + Vg(W, U) - Wg(U, V) \\ &\quad -g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V]). \end{aligned} \tag{5.44}$$

Using (5.44), we obtain

$$\begin{aligned} \nabla_{e_1} e_1 &= e_2, \quad \nabla_{e_1} e_2 = -e_1, \quad \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 &= 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = 0, \\ \nabla_{e_3} e_1 &= 0, \quad \nabla_{e_3} e_2 = -e_3, \quad \nabla_{e_3} e_3 = e_2. \end{aligned} \tag{5.45}$$

By (5.45), we see that the manifold satisfies (2.8) for $U = e_1$, $\alpha = 0$, $\beta = -1$, and $e_2 = \xi$. Similarly, it can be shown that for $U = e_2$ and $U = e_3$ the manifold also satisfies (2.8) for $\alpha = 0$, $\beta = -1$, and $e_2 = \xi$. Hence the manifold is a trans-Sasakian manifold of type $(0, -1)$ [20]*. Now we consider the Schouten-van Kampen connection to this example. From (5.45), we have*

$$\begin{aligned} R(e_1, e_2)e_1 &= e_2, \quad R(e_1, e_2)e_2 = -e_1, \quad R(e_1, e_2)e_3 = 0, \\ R(e_1, e_3)e_1 &= e_3, \quad R(e_1, e_3)e_2 = 0, \quad R(e_1, e_3)e_3 = -e_1, \\ R(e_2, e_3)e_1 &= 0, \quad R(e_2, e_3)e_2 = e_3, \quad R(e_2, e_3)e_3 = -e_2. \end{aligned} \tag{5.46}$$

Again using (3.23) and (5.45), we obtain

$$\begin{aligned}
\tilde{\nabla}_{e_1}e_1 &= (\beta+1)e_2, \qquad \tilde{\nabla}_{e_1}e_2 = -(\beta+1)e_1 + \alpha e_3, \\
\tilde{\nabla}_{e_1}e_3 &= -\alpha e_2, \quad \tilde{\nabla}_{e_2}e_1 = 0, \quad \tilde{\nabla}_{e_2}e_2 = 0, \\
\tilde{\nabla}_{e_2}e_3 &= 0, \quad \tilde{\nabla}_{e_3}e_1 = \alpha e_2, \qquad (5.47) \\
\tilde{\nabla}_{e_3}e_2 &= -(\beta+1)e_3 - \alpha e_1, \quad \tilde{\nabla}_{e_3}e_3 = (\beta+1)e_2.
\end{aligned}$$

Considering (5.47), we can see that $\tilde{\nabla}_{e_i}\xi = 0$, $(1 \leq i \leq 3)$, *for* $\xi = e_2$ *and* $\alpha = 0$, $\beta = -1$. *Hence* M *is a trans-Sasakian* 3*-manifold of type* $(0,-1)$ *with respect to the Schouten-van Kampen connection. Thus from (5.47), we get*

$$\begin{aligned}
\tilde{R}(e_1,e_2)e_1 &= (1+\alpha^2-\beta^2)e_2, \qquad \tilde{R}(e_1,e_2)e_2 = -(1+\alpha^2-\beta^2)e_1, \\
\tilde{R}(e_1,e_2)e_3 &= 0, \qquad \tilde{R}(e_1,e_3)e_1 = (1-\alpha^2-\beta^2)e_3, \\
\tilde{R}(e_1,e_3)e_2 &= 0, \qquad \tilde{R}(e_1,e_3)e_3 = (-1-\alpha^2+\beta^2)e_1, \qquad (5.48) \\
\tilde{R}(e_2,e_3)e_1 &= 0, \qquad \tilde{R}(e_2,e_3)e_2 = (1+\alpha^2-\beta^2)e_3, \\
\tilde{R}(e_2,e_3)e_3 &= (-1+\alpha^2+\beta^2)e_2.
\end{aligned}$$

Now using (5.48), we see that the non-zero components of the Ricci tensor $\tilde{R}ic$ *with respect to the Schouten-van Kampen connection as follows:*

$$\tilde{R}ic(e_1,e_1) = -2+2\beta^2, \quad \tilde{R}ic(e_2,e_2) = -2-2\alpha^2+2\beta^2, \quad \tilde{R}ic(e_3,e_3) = -2+2\beta^2.$$

For any $U,V \in \chi(M)$, *we write*

$$U = a_1e_1 + a_2e_2 + a_3e_3,$$
$$V = b_1e_1 + b_2e_2 + b_3e_3.$$

Thus we have

$$\begin{aligned}
(\tilde{L}_\xi g)(X,Y) + 2\tilde{S}(X,Y) + 2\delta g(X,Y) + 2\mu\eta(X)\eta(Y) &= (-2+2\beta^2+\delta)a_1b_1 \\
&\quad +(-2-2\alpha^2+2\beta^2+\delta+\mu)a_2b_2 \\
&\quad +(-2+2\beta^2+\delta)a_3b_3.
\end{aligned}$$

If $\delta = 2-2\beta^2$ *and* $\mu = 2\alpha^2$, *then* M *admits an* η*-Ricci soliton* (ξ,δ,μ,g) *with respect to the Schouten-van Kampen connection.*

Finally we study Yamabe solitons on a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. Assume that (M, X, δ, g) is a Yamabe soliton on a trans-Sasakian 3-manifold with respect to the Schouten-van Kampen connection. From (1.3), we can write

$$\frac{1}{2}(\tilde{L}_X g)(U, V) = (\tilde{\tau} - \delta)g(U, V), \tag{5.49}$$

that is,

$$\frac{1}{2}\{g(\tilde{\nabla}_U X, V) + g(U, \tilde{\nabla}_V X)\} = (\tilde{\tau} - \delta)g(U, V). \tag{5.50}$$

Putting $X = \xi$ in (5.50), we obtain $\tilde{\tau} = \delta$, which implies that the following:

Theorem 5.3. *The scalar curvature $\tilde{\tau}$ of a trans-Sasakian 3-manifold bearing a Yamabe soliton (M, ξ, δ, g) with respect to the Schouten-van Kampen connection is equal to δ.*

So we give the followings:

Corollary 5.4. *A trans-Sasakian 3-manifold bearing a Yamabe soliton (M, ξ, δ, g) with respect to the Schouten-van Kampen connection is of constant scalar curvature with respect to the Schouten-van Kampen connection.*

Corollary 5.5. *If a trans-Sasakian 3-manifold bearing a Yamabe soliton (M, ξ, δ, g) with respect to the Schouten-van Kampen connection, then the Riemannian metric g is a Yamabe metric.*

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