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THE APPROXIMATION OF BIVARIATE GENERALIZED BERNSTEIN-DURRMEYER TYPE GBS OPERATORS

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Abstract. In the present paper, we introduce the generalized Bernstein-Durrmeyer type operators and obtain some approximation properties of these operators studied in the space of continuous functions of two variables on a compact set. The rate of convergence of these operators are given by using the modulus of continuity. The order of approximation using Lipschitz function and Peetre's K- functional are given. Further, we introduce Bernstein-Durrmeyer type GBS (Generalized Boolean Sum) operator by means of Bögel continuous functions which is more extensive than the space of continuous functions. We obtain the degree of approximation for these operators by using the mixed modulus of smoothness and mixed K-functional. Finally, we show comparisons by some illustrative graphics in Maple for the convergence of the operators to some functions.

Keywords: Bernstein-Durrmeyer operators, Modulus of continuity, Peetre's K- functional, GBS operators, B-continuous function, B-differentiable function, Mixed modulus of smoothness, Mixed K-functional.

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1. Introduction

Let $f(x)$ be a function defined on the closed interval $[0,1]$ the expression

$$B_n f(x) = B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{1.1}$$

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is called Bernstein polynomial of order n of the function $f(x)$. The polynomials $B_nf(x)$ were introduced by S. Bernstein (see [5]) to give an especially simple proof of Weierstrass approximation theorem. The generalizations of Bernstein polynomials (1.1) were investigated in [15]- [12]. In 1988, [15] the function of two real variables function f be given over the unit square

$$s : [0,1] \times [0,1]$$

then the bivariate Bernstein polynomial of degree (n, m), corresponding to the function f, is defined by means of the formula

$$B_{n,m}(x) = B_{n,m}(f; x, y) = \sum_{k=0}^{n}\sum_{j=0}^{m} f\left(\frac{k}{n}, \frac{j}{m}\right)\binom{n}{k}\binom{m}{j} x^k(1-x)^{n-k}y^j(1-y)^{m-j}. \quad (1.2)$$

There are many investigations devoted to the problem of approximating continuous functions by classical Bernstein polynomials, as well as by two-dimensional Bernstein polynomials and their generalizations.

In 1967, Durrmeyer [11] introduced the following positive linear operators of the classical Bernstein operators, which modify with each function f integrable on the interval $[0, 1]$ the polynomial

$$M_n(f(x)) = (n+1)\sum_{k=0}^{n} p_{n,k}(x)\int_0^1 p_{n,k}(t)f(t)dt,$$

which $p_{n,k}(x) = \left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$. D. C. Morales and V. Gupta [9] studied two families of Bernstein-Durrmeyer type operators. The Baskakov Durrmeyer operators were introduced in 1985 and many properties of such operators were studied comprehensively. Gupta [13] presented the approximation properties of these operators. In 2007 [1] local approximation properties of a variant of the Bernstein-Durrmeyer operators were given.

In this paper, firstly we introduce bivariate generalized Bernstein-Durrmeyer operators. We investigate the properties of approximation of generalized Bernstein-Durrmeyer polynomials and the order of approximation using Lipschitz function and Peetre's K- functional. Then, we define the Generalized Boolean Sum (GBS) operators of generalized Bernstein-Durrmeyer type and study the degree of approximation in terms of the mixed modulus of smoothness.

2. Construction of the Bivariate Generalized Bernstein-Durrmeyer Type Operators

Let $\mathbb{D}=[-1,1]\times[-1,1]$, $(x,y)\in\mathbb{D}$, $n,m\in\mathbb{N}$ and f defined on the interval $C(\mathbb{D})$. We define the linear positive operators $D_{n,m}(f;x,y)$ in the following way:

$$D_{n,m}(f;x,y)=\frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y)\int_{-1}^{1}\int_{-1}^{1}\phi_{n,m}^{k,j}(t,u)f(t,u)dtdu \tag{2.3}$$

where

$$\phi_{n,m}^{k,j}(x,y)=\varphi_n^k(x)\varphi_m^j(y)$$

and

$$\varphi_n^k(x)=\frac{1}{2^n}\binom{n}{k}(1+x)^k(1-x)^{n-k}.$$

Lemma 2.1. *For $\forall(x,y)\in\mathbb{D}$ and $\forall n,m\in\mathbb{N}$, Bernstein-Durrmeyer operators (2.3) satisfy the following equalities:*

$$D_{n,m}(1;x,y)=1 \tag{2.4}$$

$$D_{n,m}(t;x,y)=x-\frac{2x}{n+2}$$

$$D_{n,m}(u;x,y)=y-\frac{2y}{m+2}$$

$$\begin{aligned}D_{n,m}(t^2+u^2;x,y)=&x^2-\frac{(6n+6)x^2-4nx}{(n+2)(n+3)}+\frac{2-2n}{(n+2)(n+3)}\\&+y^2-\frac{(6m+6)y^2-4my}{(m+2)(m+3)}+\frac{2-2m}{(m+2)(m+3)}\end{aligned} \tag{2.5}$$

$$\begin{aligned}D_{n,m}(t^3+u^3;x,y)=&x^3-\frac{12n^2+24n+24}{(n+2)(n+3)(n+4)}x^3+\frac{6n^2+6n}{(n+2)(n+3)(n+4)}x\\&+\frac{12n+48}{(n+2)(n+3)(n+4)}+y^3-\frac{12m^2+24m+24}{(m+2)(m+3)(m+4)}y^3\\&+\frac{6m^2+6m}{(m+2)(m+3)(m+4)}y+\frac{12m+48}{(m+2)(m+3)(m+4)}\end{aligned}$$

$$\begin{aligned}D_{n,m}(t^4+u^4;x,y)=&x^4-\frac{20n^3+60n^2+160n+120}{(n+2)(n+3)(n+4)(n+5)}x^4+\frac{12n^3-16n^2+4n}{(n+2)(n+3)(n+4)(n+5)}x^2\\&+\frac{-4n^3-16n^2+32n}{(n+2)(n+3)(n+4)(n+5)}x+y^4-\frac{20m^3+60m^2+160m+120}{(m+2)(m+3)(m+4)(m+5)}y^4\\&+\frac{12m^3-16m^2+4m}{(m+2)(m+3)(m+4)(m+5)}y^2+\frac{-4m^3-16m^2+32m}{(m+2)(m+3)(m+4)(m+5)}y.\end{aligned}$$

From Lemma 2.1, we obtained the following lemma.

Lemma 2.2. *If the operator $D_{n,m}$ is defined by (2.3), then for $\forall(x,y)\in\mathbb{D}$ and $n,m\in\mathbb{N}$*

$$D_{n,m}((t-x)^2;x,y)=\frac{(-2n+6)x^2+4nx+2-2n}{(n+2)(n+3)} \tag{2.6}$$

$$D_{n,m}((u-y)^2;x,y)=\frac{(-2m+6)y^2+4my+2-2m}{(m+2)(m+3)} \tag{2.7}$$

$$\begin{aligned}D_{n,m}((t-x)^4;x,y)=&\frac{72n^3+852n^2+1916n+1680}{(n+2)(n+3)(n+4)(n+5)}x^4+\frac{24n}{(n+2)(n+3)}x^3\\&+\frac{-24n^3-272n^2-830n+840}{(n+2)(n+3)(n+4)(n+5)}x^2+\frac{-4n^3-64n^2-464n-960}{(n+2)(n+3)(n+4)(n+5)}x\end{aligned}$$

$$\begin{aligned}D_{n,m}((u-y)^4;x,y)=&\frac{72m^3+852m^2+1916m+1680}{(m+2)(m+3)(m+4)(m+5)}y^4+\frac{24m}{(m+2)(m+3)}y^3\\&+\frac{-24m^3-272m^2-830m+840}{(m+2)(m+3)(m+4)(m+5)}y^2+\frac{-4m^3-64m^2-464m-960}{(m+2)(m+3)(m+4)(m+5)}y.\end{aligned}$$

Let $C(\mathbb{D})$ is a continuous functions space on the $\mathbb{D}=[-1,1]\times[-1,1]$. $C(\mathbb{D})$ is a linear normed space with the norm

$$\|f\|_{C(\mathbb{D})}=\max_{x\in[-1,1]\times[-1,1]}|f(x,y)|.$$

If $f_{n,m}$ is a sequence on the space $C(\mathbb{D})$, for $f\in C(\mathbb{D})$

$$\lim_{n,m\to\infty}\|f_{n,m}-f\|=0,$$

then it is called uniformly convergence to the function f.

Lemma 2.3. *Let $n\in\mathbb{N}$, for every fixed $x_0\in[-1,1]$, there exists a positive constant $M_1(x_0)$ such that $D_{n,n}\left((t-x_0)^4;x_0,y\right)\le M_1(x_0)n^{-1}$.*

Theorem 2.1. *If $T_{n,m}$ is a sequence of linear positive operators satisfying the conditions*

$$\begin{aligned}\lim_{n,m\to\infty}\|T_{n,m}(1;x,y)-1\|_{C(\mathbb{X})}&=0,\\ \lim_{n,m\to\infty}\|T_{n,m}((t-x);x,y)-x\|_{C(\mathbb{X})}&=0,\\ \lim_{n,m\to\infty}\|T_{n,m}((u-y);x,y)-y\|_{C(\mathbb{X})}&=0,\\ \lim_{n,m\to\infty}\left\|T_{n,m}(t^2+u^2;x,y)-(x^2+y^2)\right\|_{C(\mathbb{X})}&=0,\end{aligned}$$

then for any function $f\in C(\mathbb{X})$, which is bounded in $\mathbb{R}^2$ and $\mathbb{X}$ is a compact set,

$$\lim_{n,m\to\infty}\|T_{n,m}(f;x,y)-f(x,y)\|_{C(\mathbb{X})}=0.$$

In the following theorem we show that the linear positive operator $D_{n,m}$ convergences to f uniformly with the help of Theorem 2.1 given by Volkov [18].

Theorem 2.2. *Let $f \in C(\mathbb{D})$, the operators $D_{n,m}$ defined by (2.3) converge uniformly to f on $\mathbb{D} \subset \mathbb{R}^2$ as $n, m \to \infty$.*

Proof. From (2.4)-(2.5), we obtain

$$\begin{aligned} \lim_{n,m\to\infty} \|D_{n,m}(1;x,y) - 1\|_{C(\mathbb{D})} &=0, \\ \lim_{n,m\to\infty} \|D_{n,m}((t-x);x,y) - x\|_{C(\mathbb{D})} &=0, \\ \lim_{n,m\to\infty} \|D_{n,m}((u-y);x,y) - y\|_{C(\mathbb{D})} &=0, \\ \lim_{n,m\to\infty} \left\|D_{n,m}(t^2+u^2;x,y) - (x^2+y^2)\right\|_{C(\mathbb{D})} &=0. \end{aligned}$$

The proof is obvious from Volkov's Theorem.

2.1. **Degree of Approximation by $D_{n,m}$.**

Definition 2.1. *Let $f \in C(\mathbb{D})$ be a continuous function and δ a positive number. For $x, y \in \mathbb{D}$, the full continuity modulus of the function $f(x,y)$ is*

$$\omega(f;\delta) = \max_{\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}\leq\delta} |f(x_1,y_1) - f(x_2,y_2)|$$

and its partial continuity moduli with respect to x and y are defined by

$$\begin{aligned} \omega^{(1)}(f;\delta) &= \max_{-1\leq y\leq 1} \max_{|x_1-x_2|\leq\delta} |f(x_1,y) - f(x_2,y)| \\ \omega^{(2)}(f;\delta) &= \max_{-1\leq x\leq 1} \max_{|y_1-y_2|\leq\delta} |f(x,y_1) - f(x,y_2)|. \end{aligned}$$

It is also known that $\lim_{\delta\to 0}\omega(f;\delta) = 0$ and $\omega(f;\lambda\delta) \leq (\lambda+1)\omega(f;\delta)$ for any $\lambda \geq 0$. The same properties are satisfied by partial continuity moduli.

Theorem 2.3. *Let $f \in C(\mathbb{D})$, the following inequalities hold:*

$$\|D_{n,m}(f;x,y) - f\|_{C(\mathbb{D})} \leq 3\left(\omega^{(1)}\left(f;\frac{1}{\sqrt{n}}\right) + \omega^{(2)}\left(f;\frac{1}{\sqrt{n}}\right)\right) \tag{2.8}$$

$$\|D_{n,m}(f;x,y) - f\|_{C(\mathbb{D})} \leq 3\omega\left(f;\sqrt{\frac{1}{n}+\frac{1}{m}}\right). \tag{2.9}$$

Proof. From (2.3)-(2.4) and using the properties of the modulus of continuity we obtain

$$\begin{aligned}|D_{n,m}(f;x,y)-f(x,y)| \leq& |D_{n,m}(f(t,u)-f(t,y);x,y)|+|D_{n,m}(f(t,y)-f(x,y);x,y)|\\ \leq& D_{n,m}\left(|f(t,u)-f(t,y)|\right)+D_{n,m}\left(|f(t,y)-f(x,y)|\right)\\ \leq& \omega^{(1)}\left(f;\delta_n\right)\left\{1+\frac{1}{\delta_n}\frac{n+1}{2}\sum_{k=0}^{n}\varphi_n^k(x)\int_{-1}^{1}|t-x|\,\varphi_n^k(t)dt\right\}\\ &+\omega^{(2)}\left(f;\delta_m\right)\left\{1+\frac{1}{\delta_m}\frac{m+1}{2}\sum_{j=0}^{m}\varphi_m^j(y)\int_{-1}^{1}|u-y|\,\varphi_m^j(u)du\right\}\end{aligned}$$

where δ_n, δ_m are the sequences which tend to zero as $n,m\to\infty$. Applying the Cauchy-Schwartz inequality we obtain

$$\begin{aligned}&|D_{n,m}(f;x,y)-f(x,y)|\\ &\leq \omega^{(1)}\left(f;\delta_n\right)\left\{1+\frac{1}{\delta_n}\frac{n+1}{2}\sum_{k=0}^{n}\varphi_n^k(x)\left(\int_{-1}^{1}(t-x)^2\varphi_n^k(t)dt\right)^{1/2}\left(\int_{-1}^{1}\varphi_n^k(t)dt\right)^{1/2}\right\}\\ &+\omega^{(2)}\left(f;\delta_m\right)\left\{1+\frac{1}{\delta_m}\frac{m+1}{2}\sum_{j=0}^{m}\varphi_m^j(y)\left(\int_{-1}^{1}(u-y)^2\varphi_m^j(u)du\right)^{1/2}\left(\int_{-1}^{1}\varphi_m^j(u)du\right)^{1/2}\right\}.\end{aligned}$$

Hence we get

$$\begin{aligned}&|D_{n,m}(f;x,y)-f(x,y)|\leq\omega^{(1)}\left(f;\delta_n\right)\left\{1+\frac{1}{\delta_n}\frac{n+1}{2}\left(\sum_{k=0}^{n}\varphi_n^k(x)\right)^{1/2}\left(\int_{-1}^{1}(t-x)^2\varphi_n^k(t)dt\right)^{1/2}\right\}\\ &+\omega^{(2)}\left(f;\delta_m\right)\left\{1+\frac{1}{\delta_m}\frac{m+1}{2}\left(\sum_{j=0}^{m}\varphi_m^j(y)\right)^{1/2}\left(\int_{-1}^{1}(u-y)^2\varphi_m^j(u)du\right)^{1/2}\right\}\\ &=\omega^{(1)}\left(f;\delta_n\right)\left\{1+\frac{1}{\delta_n}\left(D_{n,m}((t-x)^2;x,y)\right)^{1/2}\right\}+\omega^{(2)}\left(f;\delta_m\right)\left\{1+\frac{1}{\delta_m}\left(D_{n,m}((u-y)^2;x,y)\right)^{1/2}\right\}.\end{aligned}$$

From (2.6) and (2.7), we obtain (2.8). Using (2.3), (2.4) and letting

$$\delta=\sqrt{(t-x)^2+(u-y)^2}$$

we have

$$|f(t,u)-f(x,y)|\leq\omega\left(f;\delta_{nm}\right)\left(\frac{\sqrt{(t-x)^2+(u-y)^2}}{\delta_{nm}}+1\right).$$

Hence, we obtain

$$
\begin{aligned}
|D_{n,m}(f;x,y)-f(x,y)| \leq & D_{n,m}(|f(t,u)-f(x,y)|\,;x,y) \\
\leq & \omega\left(f;\delta_{nm}\right)\left\{1+\frac{1}{\delta_{nm}}D_{n,m}\left(\sqrt{(t-x)^2+(u-y)^2};x,y\right)\right\} \\
\leq & \omega\left(f;\delta_{nm}\right)\Bigg\{1+\frac{1}{\delta_{nm}}\frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y) \\
& \int_{-1}^{1}\int_{-1}^{1}\left(\sqrt{(t-x)^2+(u-y)^2}\right)\phi_{n,m}^{k,j}(t,u)dtdu\Bigg\}
\end{aligned}
$$

applying the Cauchy-Schwartz inequality, we obtain

$$
\begin{aligned}
|D_{n,m}(f;x,y)-f(x,y)| \leq & \omega\left(f;\delta_{nm}\right)\Bigg\{1+\frac{1}{\delta_{nm}}\Bigg(\frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y) \\
& \int_{-1}^{1}\int_{-1}^{1}\left((t-x)^2+(u-y)^2\right)^2\phi_{n,m}^{k,j}(t,u)dtdu\Bigg)^{1/2}\Bigg\} \\
\leq & \omega\left(f;\delta_{nm}\right)\left\{1+\frac{1}{\delta_{nm}}\left(D_{n,m}\left((t-x)^2+(u-y)^2;x,y\right)\right)^{1/2}\right\}.
\end{aligned}
$$

With (2.6) and (2.7) we get desired result (2.9).

Now, we give the order of approximation using Lipschitz function and Peetre's K- functional.

Corollary 2.1. *If f additionally satisfies a Lipschitz condition*

$$|f(x_1,y_1)-f(x_2,y_2)| \leq K\left((x_1-x_2)^2+(y_1-y_2)^2\right)^{\alpha/2}, 0 \prec \alpha \leq 1$$

then the inequality

$$|D_{n,n}(f;x,y)-f(x,y)| \leq K'\left(\frac{1}{n}+\frac{1}{m}\right)^{\alpha/2},$$

where $K'=3K$.

Corollary 2.2. *If f additionally satisfies a Lipschitz condition*

$$|f(x_1,y)-f(x_2,y)| \leq K_1\,|x_1-x_2|^{\alpha/2}$$

and

$$|f(x,y_1)-f(x,y_2)| \leq K_2\,|y_1-y_2|^{\gamma/2}$$

then the inequality

$$|D_{n,n}(f;x,y)-f(x,y)| \leq K_1'\left(\frac{1}{n}\right)^{\alpha/2}+K_2'\left(\frac{1}{m}\right)^{\alpha/2},$$

where $K_1'=3K_1$, $K_2'=3K_2$ holds.

Let $C^2(\mathbb{D})$ be the space of all functions $f \in C(\mathbb{D})$ such that $\frac{\partial^i f}{\partial x^i}, \frac{\partial^i f}{\partial y^i} \in C(\mathbb{D})$ for $i = 1, 2$. The norm on the space $C^2(\mathbb{D})$ is defined as

$$\|f\|_{C^2(\mathbb{D})} = \|f\|_{C(\mathbb{D})} + \sum_{i=1}^{2} \left(\left\| \frac{\partial^i f}{\partial x^i} \right\|_{C(\mathbb{D})} + \left\| \frac{\partial^i f}{\partial y^i} \right\|_{C(\mathbb{D})} \right).$$

Definition 2.2. *Let $f \in C(\mathbb{D})$. The Peetre's K-functional is defined by*

$$\mathcal{K}(f;\delta) = \inf_{g \in C^2(\mathbb{D})} \left\{ \|f-g\|_{C(\mathbb{D})} + \delta \|g\|_{C^2(\mathbb{D})},\ \delta > 0 \right\}. \tag{2.10}$$

Theorem 2.4. *For the function $f \in C(\mathbb{D})$, we get*

$$|D_{n,m}(f;x,y) - f(x,y)| \le 2\mathcal{K}(f;\delta_{n,m}(x,y)),$$

where $\delta_{n,m}(x,y) = \max\left(\frac{2}{n+2}, \frac{2}{m+2}\right)$.

Proof. Let $g \in C^2(\mathbb{D})$ and $t, s \in [-1,1]$. If we use Taylor's theorem at point (x,y) for the function $g(t,s)$, we get

$$\begin{aligned} g(t,s) - g(x,y) =& \frac{\partial g(x,y)}{\partial x}(t-x) + \int_x^t (t-x)\frac{\partial^2 g(u,y)}{\partial u^2}du + \frac{\partial g(x,y)}{\partial y}(s-y) \\ &+ \int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv. \end{aligned}$$

From Lemma 2.1, we have $D_{n,m}(t-x;x,y) = -\frac{2x}{n+2}$ ve $D_{n,m}(u-y;x,y) = -\frac{2y}{m+2}$. Applying the operator $D_{n,m}$ on the above equation, we obtain

$$\begin{aligned} D_{n,m}(g;x,y) - g(x,y) =& -\frac{2x}{n+2}g_x + D_{n,m}\left(\int_x^t (t-u)\frac{\partial^2 g(u,y)}{\partial u^2}du; x, y \right) \\ & -\frac{2y}{m+2}g_y + D_{n,m}\left(\int_y^s (s-v)\frac{\partial^2 g(x,v)}{\partial v^2}dv; x, y \right). \end{aligned}$$

Hence,

$$\begin{aligned} &|D_{n,m}\ (g;x,y) - g(x,y)| \\ &\quad \le \left| \frac{2x}{n+2}g_x + \frac{2y}{m+2}g_y \right| + D_{n,m}\left(\left| \int_x^t |t-u| \left| \frac{\partial^2 g(u,y)}{\partial u^2} \right| du \right|; x, y \right) \\ &\quad + D_{n,m}\left(\left| \int_y^s |s-v| \left| \frac{\partial^2 g(x,v)}{\partial v^2} \right| dv \right|; x, y \right) \\ &\quad \le \left| \frac{2x}{n+2}g_x + \frac{2y}{m+2}g_y \right| + \frac{1}{2}\left| \frac{\partial^2 g}{\partial x^2} \right| \left| D_{n,m}\left((t-x)^2; x, y\right) \right| \\ &\quad + \frac{1}{2}\left| \frac{\partial^2 g(x,v)}{\partial v^2} \right| \left| D_{n,m}\left((u-y)^2; x, y\right) \right|. \end{aligned}$$

Using norm for $\forall x, y \in (D)$, we get

$$\begin{aligned}
\left\| D_{n,m}\left(g;x,y\right)-g(x,y)\right\|_{C(\mathbb{D})} \leq & \frac{2}{n+2}\left\| g_x\right\|_{C(\mathbb{D})}+\frac{2}{m+2}\left\| g_y\right\|_{C(\mathbb{D})} \\
& +\frac{1}{n+2}\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(\mathbb{D})}+\frac{1}{m+2}\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(\mathbb{D})} \\
\leq & \max\left(\frac{1}{n+2},\frac{1}{m+2}\right)\left(\left\| g_x\right\|_{C(\mathbb{D})}+\left\| g_y\right\|_{C(\mathbb{D})}\right. \\
& \left.+\left\|\frac{\partial^2 g}{\partial x^2}\right\|_{C(\mathbb{D})}+\left\|\frac{\partial^2 g}{\partial y^2}\right\|_{C(\mathbb{D})}\right) \\
\leq & \delta_{n,m}\left\| g\right\|_{C^2(\mathbb{D})},
\end{aligned}$$

where $\delta_{n,m}=\max\left(\frac{2}{n+2},\frac{2}{m+2}\right)$. Since $D_{n,m}$ is a linear operator and for $\forall\ f\in C(\mathbb{D})$, $g\in C^2(\mathbb{D})$, we have

$$\begin{aligned}
\left\| D_{n,m}\left(f;x,y\right)-f(x,y)\right\|_{C(\mathbb{D})} \leq & \left\| D_{n,m}\left(f-g;x,y\right)\right\|_{C(\mathbb{D})} \\
& +\left\| D_{n,m}\left(g;x,y\right)-g(x,y)\right\|_{C(\mathbb{D})}+\left\| f-g\right\|_{C(\mathbb{D})} \\
\leq & \left\| f-g\right\|_{C(\mathbb{D})}\left| D_{n,m}\left(1;x,y\right)\right| \\
& +\left\| D_{n,m}\left(g;x,y\right)-g(x,y)\right\|_{C(\mathbb{D})}+\left\| f-g\right\|_{C(\mathbb{D})}.
\end{aligned}$$

Hence

$$\left\| D_{n,m}\left(f;x,y\right)-f(x,y)\right\|_{C(\mathbb{D})}\leq 2\left(\left\| f-g\right\|_{C(\mathbb{D})}+\delta_{n,m}\left\| g\right\|_{C^2(\mathbb{D})}\right)$$

Taking the infimum on the right hand side, we get

$$\left| D_{n,m}\left(f;x,y\right)-f(x,y)\right|\leq 2\mathcal{K}\left(f;\delta_{n,m}(x,y)\right).$$

3. Construction of GBS Operator of Generalized Bernstein-Durrmeyer Type

In 1934, Bögel introduced the term B-continuous and B-differentiable function and established important result for these functions [6]-[7]. In 1966, Dobrescu and Matei [10] gave some approximation properties for bivariate Bernstein polynomials using a generalized boolean sum. The Test function theorem is given by Badea et al. [4] for Bögel continuous functions. Sidharth et al. introduced GBS operators of Bernstein–Schurer–Kantorovich type and studied the degree of approximation by means of the mixed modulus of smoothness and the mixed Peetre's K -functional in [17].

In this section, we introduce Bernstein-Durrmeyer type GBS (Generalized Boolean Sum) operator by means of Bögel continuous functions which is more extensive than the space of continuous functions. The degree of approximation for Bernstein-Durrmeyer type GBS operators are obtained by using the mixed modulus of smoothness and mixed K-functional.

Let X and Y be a compact real intervals and let $\Delta_{(x,y)} f\left[x_0, y_0; x, y\right]$ be mixed difference of f defined by

$$\Delta_{(x,y)} f\left[x_0, y_0; x, y\right] = f(x,y) - f(x,y_0) - f(x_0,y) + f(x_0,y_0)$$

for $(x,y),(x_0,y_0) \in X \times Y$. A function $f : X \times Y \to \mathbb{R}$ is called B-continuous (Bögel continuous) at $(x_0,y_0) \in X \times Y$, if

$$\lim_{(x,y)\to(x_0,y_0)} \Delta_{(x,y)} f\left[x_0, y_0; x, y\right] = 0$$

for $(x,y) \in X \times Y$. Let the function $f : X \times Y \to \mathbb{R}$ if there exist $M > 0$ such that

$$\left|\Delta_{(x,y)} f\left[x_0, y_0; x, y\right]\right| \leq M$$

for every $(x,y),(x_0,y_0) \in X \times Y$, then the function f is defined by B-bounded (Bögel bounded) on $X \times Y$.

Throughout this paper $B_b(X \times Y)$ denotes all B-bounded functions on $X \times Y$ and $C_b(X \times Y)$ denotes B-continuous functions on $X \times Y$. As usual $B(X \times Y)$ and $C(X \times Y)$ predicate the space of all bounded functions and the space of all continuous functions on $X \times Y$.

The mixed modulus of smoothness of $f \in C_b(X \times Y)$ is defined by

$$\omega_{mixed}\left(f;\delta_1,\delta_2\right) := \sup\left\{\left|\Delta_{(x,y)} f\left[x_0, y_0; x, y\right]\right| : |x - x_0| < \delta_1, |y - y_0| < \delta_2\right\} \tag{3.11}$$

for $(x,y),(x_0,y_0) \in X \times Y$ and for any $(\delta_1,\delta_2) \in (0,\infty)\times(0,\infty)$ with $\omega_{mixed} : [0,\infty)\times[0,\infty) \to \mathbb{R}$.

In 1988-90's, Badea obtained the basic properties of the mixed modulus of smoothness ω_{mixed} and these properties are similar to usual modulus of continuity. Also; the mixed modulus of smoothness provide the next inequality for $\delta_1, \delta_2 > 0$

$$\omega_{mixed}\left(f;\lambda_1\delta_1,\lambda_2\delta_2\right) \leq (1+\lambda_1)(1+\lambda_2)\omega_{mixed}\left(f;\delta_1,\delta_2\right). \tag{3.12}$$

Let give the concept of Bögel differentiable function. A function $f : X \times Y \subset \mathbb{R}^2 \to \mathbb{R}$ is called B-differentiable function at the point $(x_0,y_0) \in X \times Y$ if the limit

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\Delta_{(x,y)} f\left[x_0, y_0; x, y\right]}{(x-x_0)(y-y_0)}$$

exists and is finite. The limit is call to be the B-differential of f at the point (x_0,y_0) and is denoted by $T_{xy}f(x_0,y_0) := T_B(f;x_0,y_0)$. The space of all B-differentiable functions is denoted by $T_B(X \times Y)$.

Let $f \in C_b(\mathbb{D})$, the mixed K-functional definition is given by

$$\mathcal{K}_{mixed}(f;t_1,t_2) = \inf_{g_1,g_2,h}\Big\{\|f-g_1-g_2-h\|_\infty + t_1\left\|T_B^{2,0}g_1\right\|_\infty + t_2\left\|T_B^{0,2}g_2\right\|_\infty + t_1t_2\left\|T_B^{2,2}h\right\|_\infty\Big\},$$

where $g_1 \in C_B^{2,0}$, $g_2 \in C_B^{0,2}$, $h \in C_B^{2,2}$ and for $0 \le p,q \le 2$ $C_B^{p,q}$ denotes the space of the functions $f \in C_b(\mathbb{D})$ with continuous mixed partial derivates $T_B^{a,b}f$, $0 \le a \le p$, $0 \le b \le q$. The partial derivates are

$$T_x f(x_0,y_0) := T_B^{1,0}(f;x_0,y_0) = \lim_{x\to x_0}\frac{\Delta_x f([x_0,x];y_0)}{(x-x_0)}$$

and

$$T_y f(x_0,y_0) := T_B^{0,1}(f;x_0,y_0) = \lim_{y\to y_0}\frac{\Delta_y f(x_0;[y_0,y])}{(y-y_0)}$$

where

$$\Delta_x f([x_0,x];y_0) = f(x,y_0) - f(x_0,y_0)$$

and

$$\Delta_y f(x_0;[y_0,y]) = f(x_0,y) - f(x_0,y_0).$$

Definition 3.1. *For $f \in C(\mathbb{D})$ and $m,n \in \mathbb{N}$, we define the Generalized Boolean Sum (GBS) operator of generalized Bernstein-Durrmeyer type operator $D_{n,m}$ as follows:*

$$S_{n,m}(f(t,s);x,y) = \frac{n+1}{2}\frac{m+1}{2}\sum_{k=0}^{n}\sum_{j=0}^{m}\phi_{n,m}^{k,j}(x,y)\int_{-1}^{1}\int_{-1}^{1}\phi_{n,m}^{k,j}(t,u) \times (f(t,y)+f(x,s)-f(t,s))\,dtdu, \tag{3.13}$$

for $(x,y) \in \mathbb{D}$ where the operator $S_{n,m}$ is well defined on the space $C_b(\mathbb{D})$ and $f \in C_b(\mathbb{D})$.

3.1. Degree of Approximation by $S_{n,m}$.

Theorem 3.1. *For every $f \in C_b(\mathbb{D})$, the operator (3.13) satisfy the following inequality at each point $(x,y) \in \mathbb{D}$*

$$|S_{n,m}(f;x,y) - f(x,y)| \le 9\omega_{mixed}\left(f;n^{-1/2},m^{-1/2}\right).$$

Proof. Using the definition of $\omega_{mixed}(f;\delta_1,\delta_2)$ and for $\delta_1,\delta_2 > 0$ taking the inequality

$$\omega_{mixed}(f;\lambda_1\delta_1,\lambda_2\delta_2) \le (1+\lambda_1)(1+\lambda_2)\omega_{mixed}(f;\delta_1,\delta_2)$$

we can write

$$\begin{aligned}\left|\Delta_{(x,y)}f\left[t,s;x,y\right]\right| \leq& \omega_{mixed}\left(f;|t-x|,|s-y|\right)\\ \leq& \left(1+\frac{|t-x|}{\delta_1}\right)\left(1+\frac{|s-y|}{\delta_2}\right)\omega_{mixed}\left(f;\delta_1,\delta_2\right)\end{aligned} \tag{3.14}$$

for every $(x,y),(t,s)\in\mathbb{D}$ and for any $(\delta_1,\delta_2)>0$. From the definition of $\Delta_{(x,y)}f\left[t,s;x,y\right]$, we have

$$f(x,s)+f(t,y)-f(t,s)=f(x,y)-\Delta_{(x,y)}f\left[t,s;x,y\right]. \tag{3.15}$$

If we apply this equality the operator $D_{n,m}$ and take the definition $S_{n,m}$, we can write

$$S_{n,m}\left(f;x,y\right)=f(x,y)D_{n,m}\left(1;x,y\right)-D_{n,m}\left(\Delta_{(x,y)}f\left[t,s;x,y\right];x,y\right).$$

From (2.4), we have $D_{n,m}\left(1;x,y\right)=1$. Taking (3.14) into account and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|S_{n,m}\left(f;x,y\right)-f(x,y)\right| \leq& D_{n,m}\left(\Delta_{(x,y)}f\left[t,s;x,y\right];x,y\right)\\ \leq& \Bigg(D_{n,m}\left(1;x,y\right)+\delta_1^{-1}\sqrt{D_{n,m}\left((t-x)^2;x,y\right)}\\ &+\delta_2^{-1}\sqrt{D_{n,m}\left((s-y)^2;x,y\right)}\\ &+\delta_1^{-1}\delta_2^{-1}\sqrt{D_{n,m}\left((t-x)^2;x,y\right)D_{n,m}\left((s-y)^2;x,y\right)}\Bigg)\\ &\times\omega_{mixed}\left(f;\delta_1,\delta_2\right).\end{aligned}$$

From Lemma 2.2 and for every $(x,y)\in\mathbb{D}$, we have

$$D_{n,m}((t-x)^2;x,y)\leq\frac{4}{n}$$

and

$$D_{n,m}((u-y)^2;x,y)\leq\frac{4}{m}.$$

Therefore, choosing $\delta_1=n^{-1/2}$ ve $\delta_2=m^{-1/2}$ we get

$$\left|S_{n,m}\left(f;x,y\right)-f(x,y)\right|\leq 9\omega_{mixed}\left(f;n^{-1/2},m^{-1/2}\right).$$

Theorem 3.2. *Let take $T_Bf\in B(\mathbb{D})$ with the function $f\in T_b(\mathbb{D})$. Then, for every $(x,y)\in\mathbb{D}$, we get*

$$|S_{n,m}(f;x,y)-f(x,y)|\leq M.\left[\|T_Bf\|_\infty+\omega_{mixed}\left(T_Bf;n^{-1/2},m^{-1/2}\right)\right](nm)^{-1/2} \tag{3.16}$$

where M is any positive constant.

Proof. Let the function $f\in T_b(\mathbb{D})$. From [8], we have the identity

$$\Delta_{(x,y)}f\left[t,s;x,y\right]=(t-x)(s-y)T_Bf\left(\varsigma,\rho\right),\ x<\varsigma<t,y<\rho<s. \tag{3.17}$$

From the definition $\Delta_{(x,y)}f\left[t,s;x,y\right]$ and appliying T_Bf to each side of the equality (3.15), we get

$$T_Bf\left(\varsigma,\rho\right)=\Delta_{(x,y)}T_Bf\left(\varsigma,\rho\right)+T_Bf\left(\varsigma,y\right)+T_Bf\left(x,\rho\right)-T_Bf\left(x,y\right).$$

Taking $T_Bf\in B(\mathbb{D})$ and above equation into account, we can write

$$\begin{aligned}
&\left|D_{n,m}\ \left(\Delta_{(x,y)}f\left[t,s;x,y\right];x,y\right)\right| \\
&\quad=\left|D_{n,m}\left((t-x)(s-y)T_Bf\left(\varsigma,\rho\right);x,y\right)\right| \\
&\quad\leq D_{n,m}\left(|t-x||s-y|\left|\Delta_{(x,y)}T_Bf\left(\varsigma,\rho\right)\right|;x,y\right) \\
&\qquad+D_{n,m}\left(|t-x||s-y|\left(|T_Bf\left(\varsigma,y\right)|+|T_Bf\left(x,\rho\right)|+|T_Bf\left(x,y\right)|\right);x,y\right) \\
&\quad\leq D_{n,m}\left(|t-x||s-y|\omega_{mixed}\left(T_Bf;|\varsigma-x|,|\rho-y|\right);x,y\right) \\
&\qquad+3\left\|T_Bf\right\|_\infty D_{n,m}\left(|t-x||s-y|;x,y\right).
\end{aligned}$$

Also, since the mixed modulus of smoothness ω_{mixed} is nondecreasing, we have

$$\begin{aligned}
\omega_{mixed}\left(T_Bf;|\varsigma-x|,|\rho-y|\right)&\leq\omega_{mixed}\left(T_Bf;|t-x|,|s-y|\right) \\
&\leq\left(1+\delta_1^{-1}|t-x|\right)\left(1+\delta_2^{-1}|s-y|\right)\omega_{mixed}\left(f;\delta_1,\delta_2\right).
\end{aligned}$$

Substituting in the above equality and applying the linearity of the operator $D_{n,m}$ and using the inequality of Cauchy-Schwarz, we get

$$\begin{aligned}
|S_{n,m}(f;x,y)-f(x,y)| =& \left|D_{n,m}\left(\Delta_{(x,y)}f\left[t,s;x,y\right];x,y\right)\right| \\
\leq& 3\left\|T_B f\right\|_\infty \sqrt{D_{n,m}\left((t-x)^2(s-y)^2;x,y\right)} \\
&+ \left[D_{n,m}\left(|t-x||s-y|;x,y\right)\right. \\
&+ \delta_1^{-1}D_{n,m}\left((t-x)^2|s-y|;x,y\right) \\
&+ \delta_2^{-1}D_{n,m}\left(|t-x|(s-y)^2;x,y\right) \\
&\left.+\delta_1^{-1}\delta_2^{-1}D_{n,m}\left((t-x)^2(s-y)^2;x,y\right)\right]\omega_{mixed}\left(f;\delta_1,\delta_2\right) \\
\leq& 3\left\|T_B f\right\|_\infty \sqrt{D_{n,m}\left((t-x)^2(s-y)^2;x,y\right)} \\
&+ \left[\sqrt{D_{n,m}\left((t-x)^2(s-y)^2;x,y\right)}\right. \\
&+ \delta_1^{-1}\sqrt{D_{n,m}\left((t-x)^4(s-y)^2;x,y\right)} \\
&+ \delta_2^{-1}\sqrt{D_{n,m}\left((t-x)^2(s-y)^4;x,y\right)} \\
&\left.+\delta_1^{-1}\delta_2^{-1}D_{n,m}\left((t-x)^2(s-y)^2;x,y\right)\right]\omega_{mixed}\left(f;\delta_1,\delta_2\right).
\end{aligned}$$

From Lemma 2.2, we have

$$D_{n,m}((t-x)^2;x,y) \leq \frac{4}{n}$$

and

$$D_{n,m}((u-y)^2;x,y) \leq \frac{4}{m}.$$

For $(x,y),(t,s) \in \mathbb{D}$, $p,q \in 1,2$ and taking

$$D_{n,m}\left((t-x)^{2p}(s-y)^{2q};x,y\right) = D_{n,m}\left((t-x)^{2p};x,y\right)D_{n,m}\left((s-y)^{2q};x,y\right)$$

into account, choosing $\delta_1 = n^{-1/2}$ ve $\delta_2 = m^{-1/2}$, we get the desired result (3.16).

In the following theorem, we evaluate the order of approximation of the sequence $\{S_{n,m}(f)\}$ to the function $f \in C_b(\mathbb{D})$ in terms of mixed K-functional.

Theorem 3.3. *Let the operator $S_{n,m}$ given in (3.13). Then, for every $f \in C_b(\mathbb{D})$ we get*

$$|S_{n,m}(f;x,y)-f(x,y)| \leq 2\mathcal{K}_{mixed}\left(f;\frac{2}{n},\frac{2}{m}\right). \tag{3.18}$$

Proof. For the function $g_1 \in C_B^{2,0}(\mathbb{D})$ using Taylor formula, we get

$$g_1(t,s) = g_1(x,y) + (t-x)T_B^{1,0}g_1(x,y) + \int_x^t (t-u)T_B^{2,0}g_1(u,y)du$$

([6]). Since the operator $S_{n,m}$ reproduces linear functions

$$S_{n,m}\left(g_1;x,y\right)=g_1(x,y)+S_{n,m}\left(\int_x^t (t-u)T_B^{2,0}g_1(u,y)du;x,y\right)$$

and the definition of $S_{n,m}$ operator for $g_1\in C_B^{2,0}(\mathbb{D})$, we get

$$\begin{aligned}\left|S_{n,m}\left(g_1;x,y\right)-g_1(x,y)\right|=&\left|D_{n,m}\left(\int_x^t (t-u)\left[T_B^{2,0}g_1(u,y)-T_B^{2,0}g_1(u,s)\right]du;x,y\right)\right|\\ \leq& D_{n,m}\left(\left|\int_x^t |t-u|\left|T_B^{2,0}g_1(u,y)-T_B^{2,0}g_1(u,s)\right|du;x,y\right|\right)\\ \leq&\left\|T_B^{2,0}g_1\right\|_\infty D_{n,m}\left((t-x)^2;x,y\right)\\ <&\left\|T_B^{2,0}g_1\right\|_\infty\cdot\frac{4}{n}.\end{aligned}$$

For $g_2\in C_B^{0,2}(\mathbb{D})$,

$$\begin{aligned}\left|S_{n,m}\left(g_2;x,y\right)-g_2(x,y)\right|=&\left|D_{n,m}\left(\int_y^s (s-v)\left[T_B^{0,2}g_2(v,y)-T_B^{0,2}g_2(v,s)\right]dv;x,y\right)\right|\\ \leq& D_{n,m}\left(\left|\int_y^s |s-v|\left|T_B^{0,2}g_2(v,y)-T_B^{0,2}g_2(v,s)\right|dv;x,y\right|\right)\\ \leq&\left\|T_B^{0,2}g_2\right\|_\infty D_{n,m}\left((s-y)^2;x,y\right)\\ <&\left\|T_B^{0,2}g_2\right\|_\infty\cdot\frac{4}{m}.\end{aligned}$$

For $h\in C_B^{2,2}(\mathbb{D})$, we get

$$\begin{aligned}h(t,s)=&h(x,y)+(t-x)T_B^{1,0}h(x,y)+(s-y)T_B^{0,1}h(x,y)+(t-x)(s-y)T_B^{1,1}h(x,y)\\ &+\int_x^t (t-u)T_B^{2,0}h(u,y)du+\int_y^s (s-v)T_B^{0,2}h(x,v)dv\\ &+\int_x^t (s-y)(t-u)T_B^{2,1}h(u,y)du+\int_y^s (t-x)(s-v)T_B^{1,2}h(x,v)dv\\ &+\int_x^t\int_y^s (t-u)(s-v)T_B^{2,2}h(u,v)dvdu.\end{aligned}$$

Since $S_{n,m}\left((t-x);x,y\right)=0$, $S_{n,m}\left((s-y);x,y\right)=0$ and the definition of the operator $S_{n,m}$

$$\begin{aligned}\left|S_{n,m}\left(h;x,y\right)-h(x,y)\right|\leq&\left|D_{n,m}\left(\int_x^t\int_y^s (t-u)(s-v)T_B^{2,2}h(u,v)dvdu;x,y\right)\right|\\ \leq& D_{n,m}\left(\left|\int_x^t\int_y^s (t-u)(s-v)T_B^{2,2}h(u,v)dvdu\right|;x,y\right)\\ \leq& D_{n,m}\left(\int_x^t\int_y^s |t-u||s-v|\left|T_B^{2,2}h(u,v)\right|dvdu;x,y\right)\\ \leq&\frac{1}{4}\left\|T_B^{2,2}h\right\|_\infty D_{n,m}\left((t-x)^2(s-y)^2;x,y\right)\\ \leq& 4\left\|T_B^{2,2}h\right\|_\infty\frac{1}{n}\frac{1}{m}.\end{aligned}$$

Therefore, we get

$$
\begin{aligned}
|S_{n,m}(f;x,y)-f(x,y)| \leq & |(f-g_1-g_2-h)(x,y)|+|(g_1-S_{n,m}g_1)(x,y)| \\
& +|(g_2-S_{n,m}g_2)(x,y)|+|(h-S_{n,m}h)(x,y)| \\
& +|S_{n,m}((f-g_1-g_2-h);x,y)| \\
\leq & 2\|f-g_1-g_2-h\|_\infty+4\left\|T_B^{2,0}g_1\right\|_\infty \frac{1}{n} \\
& +4\left\|T_B^{0,2}g_2\right\|_\infty \frac{1}{m}+4\left\|T_B^{2,2}h\right\|_\infty \frac{1}{n}\frac{1}{m}
\end{aligned}
$$

for $f \in C_b(\mathbb{D})$. Since the definition of the mixed K-functional and taking the infimum over all $g_1 \in C_B^{2,0}(\mathbb{D})$, $g_2 \in C_B^{0,2}(\mathbb{D})$, $h \in C_B^{2,2}(\mathbb{D})$ we get the desired result (3.18).

3.2. **Numerical Examples.** The convergence of the operators by illustrative graphics in Maple to certain functions for two dimensional cases are given and some numerical values are calculated as follows. For $n,m=1,2,5,10$ and the function $f(x,y)=x^2y+y^2$, the convergence of the operators $D_{n,m}$ is shown in Fig 1. For $n,m=1,2,5,10$ and the function $f(x,y)=1-x^3+y^3$, the convergence of the operators $D_{n,m}$ is shown in Fig 2. It is seen that if the values of n,m increase, the convergence of $D_{n,m}$ to the function f becomes better. Finally, one can see that the convergence of the GBS operator $S_{n,m}$ has better approach than the operator $D_{n,m}$ for the function $f(x,y)=(1+x+y)sin(x+y)$ in Fig 3.

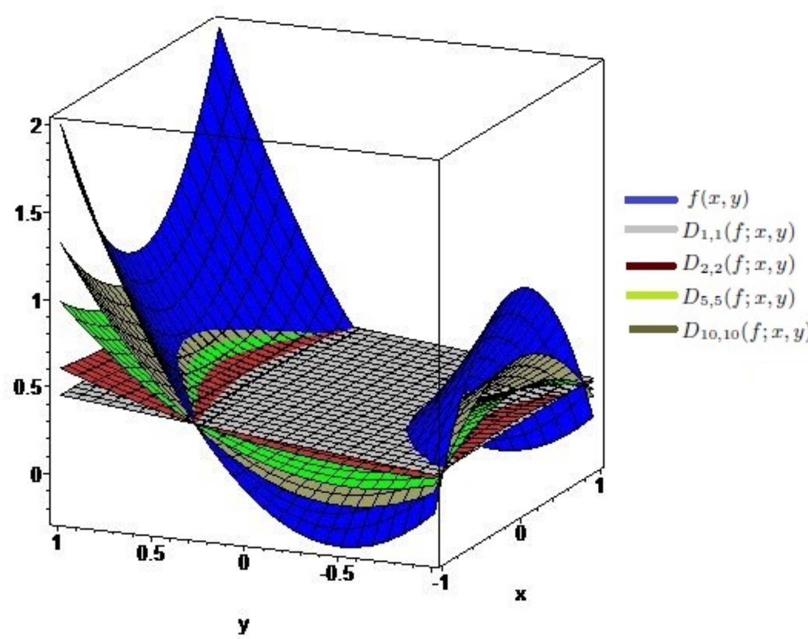

FIGURE 1. The convergence of the $D_{n,m}$ operators for $f(x,y)=x^2y+y^2$ and $n,m=1,2,5,10$.

TABLE 1. Mean errors of figure 1

(n,m)	$maximize\,|D_{n,m}(x,y)-f(x,y)|$
n,m=5	1,0204
n,m=15	0,5113
n,m=25	0,3390
n,m=50	0,1836
n,m=100	0,0957
n,m=150	0,0647

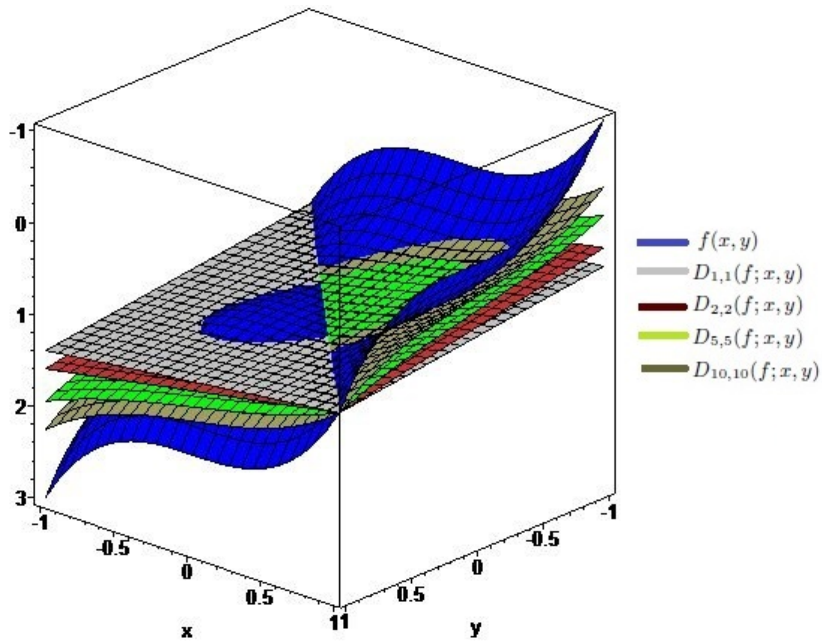

FIGURE 2. The convergence of the $D_{n,m}$ operators for $f(x,y)=1-x^3+y^3$ and $n,m=1,2,5,10$.

TABLE 2. Mean errors of figure 2

(n,m)	$maximize \, |D_{n,m}(x,y) - f(x,y)|$
n,m=5	1,0476
n,m=10	0,7362
n,m=50	0,2139
n,m=100	0,1131
n,m=500	0,0066

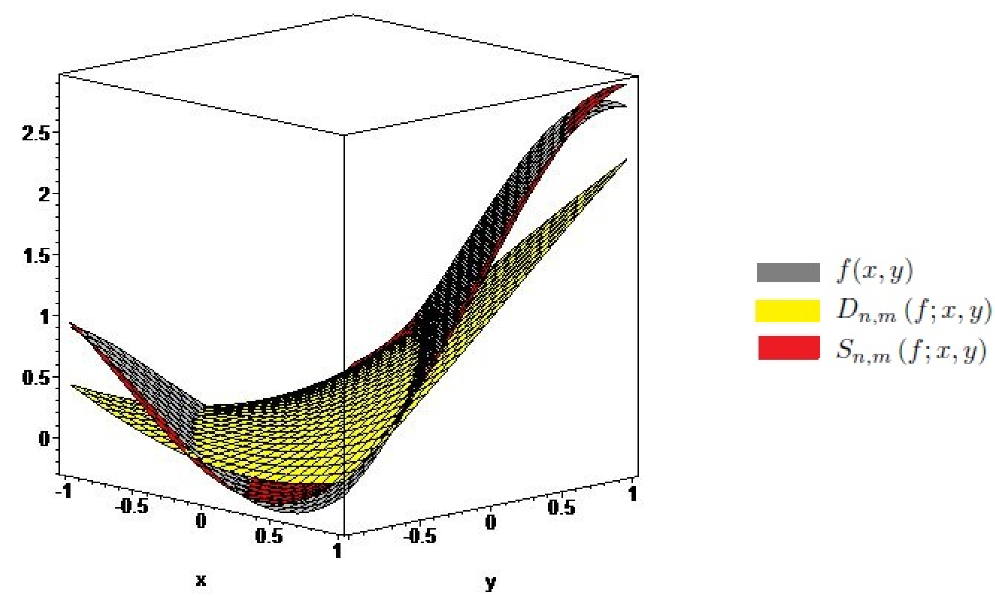

FIGURE 3. The convergence of the $D_{n,m}$ operators and the $S_{n,m}$ operators for $f(x,y) = (1+x+y)sin(x+y)$ and $n,m=5$.

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REFERENCES

[1] Abel, U., Gupta, V. & Mohapatra, R. N. (2008). Local approximation by a variant of Bernstein-Durrmeyer operators. Nonlinear Analysis, **68**, 3372-3381.

[2] Acar, T. & Kajla, A. (2018). Degree of approximation for bivariate generalized Bernstein type operators. Results Math., doi:10.1007/s00025-018-0838-1.

[3] Aydın Arı, D., & Kırcı Serenbay, S. (2019). Approximation by a generalization of the Jakimovski -Leviatan operators. Filomat, 33(8).

[4] Badea, C., Badea, I., & Gonska, H.H. (1986). A test function theorem and apporoximation by pseudopolynomials. Bull. Aust. Math. Soc. 34, 53–64.

[5] Bernstein, S. (1912). Démonstration du théoréme de Weierstrass, fondeé sur le calcul des probabilités. Commun. Soc. Math. Kharkov, 13(2), 1-2.

[6] Bögel, K. (1934). Mehrdimensionale differentiation von funktionen mehrerer veranderlicher. J. Reine Angew. Math., 170, 197-217.

[7] Bögel, K. (1935). Über die mehrdimensionale differentiation integration und beschrankte Variation. J. Reine Angew. Math., 173, 5-29.

[8] Bögel, K. (1962–1963). Uber die mehrdimensionale differentiation. Jber. Deutsch. Math-Verein. 65, 45–71.

[9] Cárdenas-Morales, D., & Gupta, V. (2014). Two families of Bernstein-Durrmeyer type operators. Applied Mathematics and Computation, 248, 342-353.

[10] Dobrescu, E. & Matei, I. (1966). The approximation by Berntein type polynomials of bidimensionally continuous functions. An. Univ. Timisoara Ser. Sti. Mat.-Fiz., 4, 85–90 (Romanian).

[11] Durrmeyer, J. L. (1967). Une formule d'inversion de la transformeé de Laplace: applications á la théorie des moments, Thése de 3e cycle. Faculté des Sciences de l'Université de Paris.

[12] Gurdek, M., Rempulska L., & Skorupa, M. (1999). The Baskakov operators for functions of two variables. Coolect. Math. 50(3), 289-302.

[13] Gupta, V. (2014). Different Durrmeyer Variants of Baskakov Operators. Topics in Mathematical Analysis and Applications, 94, 419-446.

[14] İzgi, A. (2012). Approximation by a class of new type Bernstein polynomials of one two variables. Global Journal of Pure and Applied Mathematics, 8(5), 55-71.

[15] Martinez, F. L. (1989). Some properties of two-dimensional Bernstein polynomials. Journal of approximation theory, 59(3), 300-306.

[16] Pop, O.T. & Barbosu, D. (2008). GBS operators of Durrmeyer–Stancu type. Miskolc Math. Notes, 9, 53–60.

[17] Sidharth, M., Ispir, N. & Agrawal, P.N. (2016). Approximation of B-continuous and B-differentiable functions by GBS operators of q-Bernstein–Schurer–Stancu type. Turk. J. Math., 40(6), 1298–1315.

[18] Volkov, V. I. (1957). On the convergence of sequences of linear positive operators in the space of continuous functions of two variables. Dokl. Akad. Nauk SSSR, 115, 17-19.

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