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SOME RESULTS IN THE THEORY OF QUASILINEAR SPACES

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Abstract. In this study, we present some new consequences and exercises of homogenized quasilinear spaces. We also research on the some characteristics of the homogenized quasilinear spaces. Then, we introduce the concept of equivalent norm on a quasilinear space. As in the linear functional analysis, we obtained some results with equivalent norms defined in normed quasilinear spaces.

Keywords: Quasilinear space, Normed quasilinear space, Inner product quasilinear space, Homogenized quasilinear space, Equivalent norms.

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1. Introduction

In the 1986, Aseev [1] presented the quasilinear spaces and normed quasilinear spaces which are generalization of linear spaces and normed linear spaces, respectively. The biggest difference between quasilinear space and linear space is that it has a partial order relation. He gave some properties and some results which are quasilinear provisions of some conclusions in classical linear functional analysis. Later, in [1], he presented the some new concepts in normed quasilinear spaces. Further, in ([7], [10], [11], [12], [2], [9], [8] etc.), they have proposed a series of new concepts and new results of quasilinear spaces. In [7], they introduced the concept of proper quasilinear space which is a new notion of quasilinear functional analysis.

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In the same study, they presented concept of dimension of a quasilinear space which are very meaningful to improvement of quasilinear algebra.

In the light of all these studies, in [6], we extended the notion of inner product spaces to the quasilinear conditions. After giving this new definition, we obtained some new concepts on inner product quasilinear spaces such as Hilbert quasilinear spaces and some orthogonality concepts. Further, in [6], we examined the sample of quasilinear spaces "$I\mathbb{R}^n$" interval space and we presented the quasilinear spaces Is, Ic_0, Il_∞ and Il_2. Also, we have studied to clarify geometric properties of inner product quasilinear spaces in [13]. Furthermore, we tried to enlarge the results in quasilinear functional analysis in [3], [4] and [5].

In this paper, we present some new conclusions of homogenized quasilinear space. Also, we obtain some results with considerable advantages about features of homogenized quasilinear spaces. Furthermore, we obtain some results with equivalent norms in a normed quasilinear space.

2. Preliminaries

In this section, we give some definitions and results on quasilinear spaces given by Aseev [1].

Definition 2.1. [1] *A quasilinear space over a field $\mathbb{R}$ is a set Q with a partial order relation "$\preceq$" with the operations of addition $Q \times Q \to Q$ and scalar multiplication $\mathbb{R} \times Q \to Q$ satisfying the following conditions:*

(Q1) $q \preceq q$,

(Q2) $q \preceq z$, *if* $q \preceq w$ *and* $w \preceq z$,

(Q3) $q = w$, *if* $q \preceq w$ *and* $w \preceq q$,

(Q4) $q + w = w + q$,

(Q5) $q + (w + z) = (q + w) + z$,

(Q6) there exists an element $\theta \in Q$ *such that* $q + \theta = q$,

(Q7) $\alpha \cdot (\beta \cdot q) = (\alpha \cdot \beta) \cdot q$,

(Q8) $\alpha \cdot (q + w) = \alpha \cdot q + \alpha \cdot w$,

(Q9) $1 \cdot q = q$,

(Q10) $0 \cdot q = \theta$,

(Q11) $(\alpha + \beta) \cdot q \preceq \alpha \cdot q + \beta \cdot q$,

(Q12) $q + z \preceq w + v$, *if* $q \preceq w$ *and* $z \preceq v$,

(Q13) $\alpha \cdot q \preceq \alpha \cdot w$, *if* $q \preceq w$

for every $q, w, z, v \in Q$ and every $\alpha, \beta \in \mathbb{R}$.

The considerable instance which is a quasilinear space is the set of all closed intervals of $\mathbb{R}$ with the relation " $\subseteq$ ", algebraic sum operation $M + N = \{m + n : m \in M,\ n \in N\}$ and the real-scalar multiplication $\lambda \cdot M = \{\lambda m : m \in M\}$. We indicate this set by $\Omega_C(\mathbb{R})$. Also, the set of all compact subsets of $\mathbb{R}$ is $\Omega(\mathbb{R})$.

Let Q be a quasilinear space and $W \subseteq Q$. Then W is called a *subspace of* Q, whenever W is a quasilinear space with the same partial order relation and the restriction of the operations on Q to W. An element $q \in Q$ is said to be symmetric if $-q = q$, where $-q = (-1) \cdot q$, and Q_d denotes the set of all symmetric elements of Q.

Theorem 2.1. *W is a subspace of a quasilinear space Q if and only if, for every, $q, w \in W$ and $\alpha, \beta \in \mathbb{R}$, $\alpha \cdot q + \beta \cdot w \in W$* [12].

Definition 2.2. [1] *Let Q be a quasilinear space. A function $\|.\|_Q : Q \longrightarrow \mathbb{R}$ is named a norm if the following circumstances hold:*

$(NQ1)$ $\|q\|_Q > 0$ *if* $q \neq 0$,

$(NQ2)$ $\|q + w\|_Q \leq \|q\|_Q + \|w\|_Q$,

$(NQ3)$ $\|\alpha \cdot q\|_Q = |\alpha| \cdot \|q\|_Q$,

$(NQ4)$ *if* $q \preceq w$, *then* $\|q\|_Q \leq \|w\|_Q$,

$(NQ5)$ *if for any $\varepsilon > 0$ there exists an element $q_\varepsilon \in Q$ such that, $q \preceq w + q_\varepsilon$ and $\|q_\varepsilon\|_Q \leq \varepsilon$ then $q \preceq w$ for any elements $q, w \in Q$ and any real number $\alpha \in \mathbb{R}$.*

Let Q be a normed quasilinear space. Hausdorff metric on Q is defined by the equality

$$h_Q(q, w) = \inf\left\{r \geq 0 : q \preceq w + z_1^r,\ w \preceq q + z_2^r, \|z_i^r\| \leq r\right\}.$$

Since $q \preceq w + (q - w)$ and $w \preceq q + (w - q)$, the quantity $h_Q(q, w)$ is well-defined for any elements $q, w \in Q$, and

$$h_Q(q, w) \leq \|q - w\|_Q.$$

Example 2.1. *Let X be a Banach space. A norm on $\Omega(X)$ is defined by*

$$\|A\|_{\Omega(X)} = \sup_{a \in A} \|a\|_X.$$

Then $\Omega(X)$ and $\Omega_C(X)$ are normed quasilinear spaces. The Hausdorff metric is described as ordinary:

$$h_{\Omega_C(X)}(A, B) = \inf\{r \geq 0 : A \subset B + S_r(\theta),\ B \subset A + S_r(\theta)\},$$

where $S_r(\theta)$ *demonstrates a closed ball of radius* r *about* $\theta \in X$ [1].

Definition 2.3. *Let* Q *be a quasilinear space,* $M \subseteq Q$ *and* $m \in M$*. The set*

$$F_m^M = \{z \in M_r : z \preceq m\}$$

is called floor in M of m*. If* $M = Q$*, then it is called* floor of m *and written* F_m *in place of* F_m^M [7].

Definition 2.4. *Let* Q *be a quasilinear space and* $M \subseteq Q$*. Then the set*

$$\mathcal{F}_M^Q = \bigcup_{m \in M} F_m^Q$$

is called floor in Q of M *and is indicated by* $\mathcal{F}_M^Q$ [7].

Definition 2.5. *Let* Q *be a quasilinear space.* Q *is called solid-floored quasilinear space whenever*

$$y = \sup\{m \in Q_r : m \preceq y\}$$

for all $y \in Q$*. Other than,* Q *is called non solid-floored quasilinear space* [7].

Example 2.2. $\Omega(\mathbb{R})$ *and* $\Omega_C(\mathbb{R})$ *are solid-floored quasilinear space. However, singular subspace of* $\Omega_C(\mathbb{R})$ *is non-solid floored quasilinear space. For example,*

$$\sup\left\{m : m \in \left(\left(\Omega_C(\mathbb{R})\right)_s \cup \{0\}\right)_r,\ m \subseteq y\right\} = \{0\} \neq y$$

for element $y = [-2, 2] \in (\Omega_C(\mathbb{R}))_s \cup \{0\}$*. Also, we can not find any element* $m \in \left((\Omega_C(\mathbb{R}))_s \cup \{0\}\right)_r$ *such that* $m \subseteq z$ *for* $z = [1, 3] \in (\Omega_C(\mathbb{R}))_s \cup \{0\}$.

Definition 2.6. *Let* Q *be a quasilinear space.* Consolidation of floor of Q *is the smallest solid-floored quasilinear space* $\widehat{Q}$ *containing* Q*, that is, if there exists another solid-floored quasilinear space* W *containing* Q *then* $\widehat{Q} \subseteq W$ [13].

$\widehat{Q} = Q$ for some solid-floored quasilinear space Q. Besides, $\widehat{\Omega_C(\mathbb{R}^n)_s} = \Omega_C(\mathbb{R}^n)$. For a quasilinear space Q, the set

$$F_y^{\widehat{Q}} = \left\{z \in \left(\widehat{Q}\right)_r : z \preceq y\right\}.$$

is the *floor of* Q *in* $\widehat{Q}$.

Definition 2.7. *Let Q be a quasilinear space. A mapping $\langle \ , \ \rangle : Q \times Q \to \Omega(\mathbb{R})$ is called an inner product on Q if for any $q, w, z \in Q$ and $\alpha \in \mathbb{R}$ the following conditions hold:*

$(IPQ1)$ *If $q, w \in Q_r$ then $\langle q, w \rangle \in \Omega_C(\mathbb{R})_r \equiv \mathbb{R}$,*

$(IPQ2)$ $\langle q + w, z \rangle \subseteq \langle q, z \rangle + \langle w, z \rangle$,

$(IPQ3)$ $\langle \alpha \cdot q, w \rangle = \alpha \cdot \langle q, w \rangle$,

$(IPQ4)$ $\langle q, w \rangle = \langle w, q \rangle$,

$(IPQ5)$ $\langle q, q \rangle \geq 0$ *for* $q \in X_r$ *and* $\langle q, q \rangle = 0 \Leftrightarrow q = 0$,

$(IPQ6)$ $\|\langle q, w \rangle\|_{\Omega(\mathbb{R})} = \sup \left\{ \|\langle a, b \rangle\|_{\Omega(\mathbb{R})} : a \in F_q^{\widehat{Q}}, b \in F_w^{\widehat{Q}} \right\}$,

$(IPQ7)$ *if* $q \preceq w$ *and* $u \preceq v$ *then* $\langle q, u \rangle \subseteq \langle w, v \rangle$,

$(IPQ8)$ *if for any $\varepsilon > 0$ there exists an element $q_\varepsilon \in Q$ such that $q \preceq w + q_\varepsilon$ and $\langle q_\varepsilon, q_\varepsilon \rangle \subseteq S_\varepsilon(\theta)$ then $q \preceq w$.*

A quasilinear space with an inner product is called an inner product quasilinear space [6].

Example 2.3. $\Omega_C(\mathbb{R})$, *is an example of inner product quasilinear space with*

$$\langle A, B \rangle = \{ab : a \in A, b \in B\}.$$

For any two elements q, w of an inner product quasilinear space Q, we have

$$\|\langle q, w \rangle\|_{\Omega(\mathbb{R})} \leq \|q\|_Q \|w\|_Q .$$

Every inner product quasilinear space Q is a normed quasilinear space with the norm described by

$$\|q\| = \sqrt{\|\langle q, q \rangle\|_{\Omega(\mathbb{R})}}$$

for every $q \in Q$.

Definition 2.8. *An element q of the inner product quasilinear space Q is said to be orthogonal to an element $w \in Q$ if*

$$\|\langle q, w \rangle\|_{\Omega(\mathbb{R})} = 0.$$

From here, we can call that q and w are orthogonal and we show $q \perp w$ [6].

An orthonormal set $M \subset Q$ is an orthogonal set in Q whose elements have norm 1, that is, for every q, $w \in M$

$$\|< q, w >\|_{\Omega(\mathbb{R})} = \begin{cases} 0, & q \neq w \\ 1, & q = w \end{cases} .$$

Definition 2.9. $A^{\perp}$, *is called the orthogonal complement of A and is showed by*

$$A^{\perp}=\{q\in Q:\|\langle q,w\rangle\|_{\Omega(\mathbb{R})}=0,\ w\in A\}.$$

For any subset A of an inner product quasilinear space Q, $A^{\perp}$ is a closed subspace of Q [6].

Example 2.4. *Let $X=(X_1,X_2,\ldots,X_n)\in I\mathbb{R}^n$ and $Y=(Y_1,Y_2,\ldots,Y_n)\in I\mathbb{R}^n$. The algebraic sum operation on $I\mathbb{R}^n$ is defined by*

$$X+Y=(X_1+Y_1,X_2+Y_2,\ldots,X_n+Y_n)$$

and multiplication by a real number $\alpha\in\mathbb{R}$ is defined by

$$\alpha\cdot X=(\alpha\cdot X_1,\alpha\cdot X_2,\ldots,\alpha\cdot X_n).$$

If we will be assumed that the partial order on $I\mathbb{R}^n$ is given by

$$X\leq Y\Leftrightarrow X_i\preceq Y_i \qquad 1\leq i\leq n$$

then $I\mathbb{R}^n$ is quasilinear space according to the above processes. Furthermore, different two norm on $I\mathbb{R}^n$ are defined by

$$\|X\|=\|(X_1,X_2,\ldots,X_n)\|=\sup_{1\leq i\leq n}\|X_i\|_{I\mathbb{R}}$$

and

$$\|X\|_2=\left(\sum_{i=1}^{n}\|X_i\|_{I\mathbb{R}}^2\right)^{\frac{1}{2}}.$$

The quasilinear space $I\mathbb{R}^n$, with the inner product

$$\langle X,Y\rangle=\sum_{i=1}^{n}\langle X_i,Y_i\rangle_{I\mathbb{R}}$$

is an inner product quasilinear space.

The quasilinear spaces $I\mathbb{R}^n$ and $\Omega_C(\mathbb{R}^n)$ are different from each other. For instance; while the set $A=\left\{(q,w):q^2+w^2\leq 1\right\}$ is element of $\Omega_C(\mathbb{R}^2)$, it is not element of $I\mathbb{R}^2$. Further, $B=([1,3],\{4\})\in I\mathbb{R}^2$ but $B\notin\Omega_C(\mathbb{R}^2)$. Thus, $I\mathbb{R}^n$ and $\Omega_C(\mathbb{R}^n)$ are two distinct instances of quasilinear spaces.

3. Main Results

In this section, we give the concept of homogenized quasilinear space by [5]. Then, we give new findings about this concept.

Definition 3.1. *Let Q be a quasilinear space. Q is called* ***homogenized quasilinear space*** *if for all $q \in Q$ and $\alpha\beta \geq 0$ the following circumstance is satisfied:*

$$(\alpha+\beta)\cdot q = \alpha \cdot q + \beta \cdot q.$$

Obviously, every vector space is a homogenized quasilinear space. However, the inverse is false.

Theorem 3.1. $\Omega_C(Q)$ *is a homogenized quasilinear space for every normed quasilinear space Q. However, $\Omega(Q)$ is not homogenized quasilinear space.*

Proof. Since $\Omega_C(Q)$ is a quasilinear space, we have $(\alpha+\beta)\cdot A \subseteq \alpha \cdot A + \beta \cdot A$ from (Q11) for every $A \in \Omega_C(Q)$. Now, we only prove the converse. Let $a \in \alpha \cdot A + \beta \cdot A$ for every $A \in \Omega_C(Q)$. Then, we obtain

$$a = \alpha \cdot q + \beta \cdot w$$

for a $q, w \in A$. From here, we can write

$$a = (\alpha+\beta)\left[\frac{\alpha}{\alpha+\beta}\cdot q + \frac{\beta}{\alpha+\beta}\cdot w\right].$$

If $t = \frac{\alpha}{\alpha+\beta}$ and $k = \frac{\beta}{\alpha+\beta}$, there is two different cases since $\alpha\beta \geq 0$:

i) If $\alpha \leq \alpha+\beta$ for $\alpha, \beta \in \mathbb{R}^+$, then we get $\frac{\alpha}{\alpha+\beta} \leq 1$ and $0 \leq \frac{\alpha}{\alpha+\beta}$.

ii) If $\alpha+\beta \leq \alpha$ for $\alpha, \beta \in \mathbb{R}^-$, then we get $1 \geq \frac{\alpha}{\alpha+\beta}$ and $0 \leq \frac{\alpha}{\alpha+\beta}$.

From i) and ii), we obtain $0 \leq t \leq 1$. Further, clearly $t+k=1$. According to the definition of convexity on quasilinear spaces, we get $\frac{\alpha}{\alpha+\beta}\cdot q + \frac{\beta}{\alpha+\beta}\cdot w \in A$. So, we show that

$$a = (\alpha+\beta)\cdot z \in A$$

for a $z \in A$.

Example 3.1. $\Omega(\mathbb{R})$ *is a non-homogenized quasilinear space. Consider the element $A = \{1,2,3\} \in \Omega(\mathbb{R})$. Clearly, $2\cdot A = \{2,4,6\}$. But $A+A = \{2,3,4,5,6\}$. Therefore $2\cdot A \neq A+A$ for $\alpha = 1$ and $\beta = 1$. This shows us that $\Omega(\mathbb{R})$ is not a homogenized quasilinear space.*

Theorem 3.2. *Let Q be a homogenized inner product quasilinear space and $q \in Q_d$. Then there exists at least one $w \in X$ such that $q = w - w$.*

Proof. We know that $(\alpha+\beta)\cdot w = \alpha\cdot w+\beta\cdot w$ for every $w \in Q$ and $\alpha, \beta \in \mathbb{R}^+$. Further $q = -q$ and $q = q$ since q is a symmetric element of Q. Same time, we get $q + q = q - q$. From here, we obtain $q = \frac{q}{2} - \frac{q}{2}$ since $2 \cdot q = q - q$. This complete the proof.

Proposition 3.1. *Let Q be a homogenized quasilinear space and $q \in Q$. Then F_q is convex subset of Q.*

Proof. Let Q be a homogenized quasilinear space. From Definition 2.3, we have

$$F_q = \{a \in Q_r : a \preceq q\}$$

for a $q \in Q$. Thus, we obtain

$$a \preceq q \text{ and } b \preceq q$$

for every $a, b \in F_q$. From (Q13), we have

$$\gamma \cdot a \preceq \gamma \cdot q \text{ and } (1-\gamma) \cdot b \preceq (1-\gamma) \cdot q$$

for every $0 \preceq \gamma \preceq 1$. Hence,

$$\gamma \cdot a + (1-\gamma) \cdot b \preceq \gamma \cdot q + (1-\gamma) \cdot q.$$

Since, Q is a homogenized quasilinear space,

$$\gamma \cdot q + (1-\gamma) \cdot q = (\gamma + 1 - \gamma) \cdot q = q$$

for every $0 \preceq \gamma \preceq 1$. Therefore, we get

$$\gamma \cdot a + (1-\gamma) \cdot b \preceq q.$$

Thus, $\gamma \cdot a + (1-\gamma) \cdot b \in F_q$.

Remark 3.1. *Floor of an element of an inner product quasilinear space Q is convex if and only if this inner product quasilinear space Q is homogenized. If Q is not homogenized in the Proposition 3.1, then F_q is not convex since $(\alpha+\beta) \cdot q \neq \alpha \cdot q + \beta \cdot q$.*

Example 3.2. *$I\mathbb{R}^n$ is a homogenized inner product quasilinear space. In* [6]*, we showed that $I\mathbb{R}^n$ is an inner product quasilinear space with*

$$\langle X, Y \rangle = \sum_{i=1}^{n} \langle X_i, Y_i \rangle_{I\mathbb{R}}.$$

For every $X \in I\mathbb{R}^n$ and $\alpha\beta \geq 0$, we can write

$$\begin{aligned}(\alpha+\beta)\cdot X &= (\alpha+\beta)\cdot(X_1, X_2, \ldots, X_n) \\ &= ((\alpha+\beta)\cdot X_1, (\alpha+\beta)\cdot X_2, \ldots, (\alpha+\beta)\cdot X_n).\end{aligned}$$

Then, we obtain

$$\begin{aligned}(\alpha+\beta)\cdot X &= (\alpha\cdot X_1+\beta\cdot X_1, \alpha\cdot X_2+\beta\cdot X_2, \ldots, \alpha\cdot X_n+\beta\cdot X_n) \\ &= (\alpha\cdot X_1, \alpha\cdot X_2, \ldots, \alpha\cdot X_n)+(\beta\cdot X_1, \beta\cdot X_2, \ldots, \beta\cdot X_n) \\ &= \alpha\cdot X+\beta\cdot X\end{aligned}$$

since $I\mathbb{R}$ is a homogenized quasilinear space.

Example 3.3. *All interval sequence spaces Is, all bounded interval sequence spaces $Il_\infty = \{X = (X_n) \in I\mathbb{R}^\infty : |(X_n)| \leq \infty\}$ and all convergent interval sequence spaces*

$$Ic_0 = \{X = (X_n) \in I\mathbb{R}^\infty : (X_n) \to 0\}$$

are further example of homogenized quasilinear spaces.

Before giving the equivalent norms on the qasilinear spaces, we will give an example to cartesian product of quasilinear spaces.

Example 3.4. *Let Q be the Cartesian product of quasilinear spaces $Q_1, Q_2, ..., Q_n$, that is, $Q = Q_1 \times Q_2 \times ... \times Q_n$. The space Q is a quasilinear space with the algebraic sum operation*

$$(q_1, q_2, ..., q_n)+(w_1, w_2, ..., w_n) = (q_1+w_1, q_2+w_2+...+q_n+w_n),$$

real scalar multiplication

$$\alpha\cdot(q_1, q_2, ..., q_n) = (\alpha\cdot q_1, \alpha\cdot q_2, ..., \alpha\cdot q_n)$$

and order relation

$$(q_1, q_2, ..., q_n) \preceq (w_1, w_2, ..., w_n) \Leftrightarrow q_1 \preceq w_1, q_2 \preceq w_2, ..., q_n \preceq w_n$$

for every $(q_1, q_2, ...q_n), (w_1, w_2, ...w_n) \in Q_1 \times Q_2 \times ... \times Q_n = Q$.

Example 3.5. *Let Q and W be the normed quasilinear spaces with $\|\cdot\|_1$ and $\|\cdot\|_2$, respectively. Define $Q \times W = \{z = (q, w) : q \in Q$ and $w \in W\}$. The functions*

$$\|z\| = \max(\|q\|_1, \|w\|_2) \tag{3.1}$$

$$\|z\|_0 = \|q\|_1 + \|w\|_2 \tag{3.2}$$

defines norms on $Q \times W$. Then $Q \times W$ is normed quasilinear space.

Proposition 3.2. *Let $\|\cdot\|_1$ be a norm on quasilinear space Q and $\|\cdot\|_2$ be a norm on quasilinear space W. From Example 3.5, we have $Z = Q \times W$ is normed quasilinear space with norms (3.1) and (3.2). Let $\{(q_n, w_n)\}$ be sequence in $Q \times W$. The following conditions are satisfied:*

i) The sequence $\{(q_n, w_n)\}$ is convergent to $\{(q, w)\}$ in Z if and only if $\{q_n\}$ is convergent to q in Q and $\{w_n\}$ is convergent to w in W.

ii) The sequence $\{(q_n, w_n)\}$ is Cauchy sequence in Z if and only if $\{q_n\}$ is Cauchy sequence in Q and $\{w_n\}$ is Cauchy sequence in W.

Proof. Suppose that $(q_n, w_n) \to (q, w) \in Z$. Then corresponding to each $\epsilon > 0$, $\exists$ $n_0 \in \mathbb{N}$ such that the following inequalities hold for $n > n_0$:

$$(q_n, w_n) \preceq (q, w) + a^{\epsilon}_{1,n}, \ \ (q, w) \preceq (q_n, w_n) + a^{\epsilon}_{2,n}, \ \ \left\|a^{\epsilon}_{i,n}\right\| \le \epsilon.$$

Here, $a^{\epsilon}_{1,n} = \left(b^{\epsilon}_{1,n}, c^{\epsilon}_{1,n}\right)$ and $a^{\epsilon}_{2,n} = \left(b^{\epsilon}_{2,n}, c^{\epsilon}_{2,n}\right)$. Since Z is quasilinear space, we get

$$q_n \preceq q + b^{\epsilon}_{1,n}, \ \ q \preceq q_n + b^{\epsilon}_{2,n}$$

and

$$w_n \preceq w + c^{\epsilon}_{1,n}, \ \ w \preceq w_n + c^{\epsilon}_{2,n}.$$

Also, since $\left\|a^{\epsilon}_{i,n}\right\| = \max\left(\left\|b^{\epsilon}_{i,n}\right\|_1, \left\|c^{\epsilon}_{i,n}\right\|_2\right) \le \epsilon$ or $\left\|a^{\epsilon}_{i,n}\right\|_0 = \left\|b^{\epsilon}_{i,n}\right\|_1 + \left\|c^{\epsilon}_{i,n}\right\|_2 \le \epsilon$, we obtain $\left\|b^{\epsilon}_{i,n}\right\|_1 \le \epsilon$ and $\left\|c^{\epsilon}_{i,n}\right\|_2 \le \epsilon$ according to (3.1) and (3.2). This proves that the sequence $\{q_n\}$ is convergent to q in Q and the sequence $\{w_n\}$ is convergent to w in W. The opposite can be shown in a similar way.

Let $\{(q_n, w_n)\}$ be a Cauchy sequence in Z. For an arbitrary $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$(q_n, w_n) \preceq (q_m, w_m) + a^{\epsilon}_{1,n}, \ \ (q_m, w_m) \preceq (q_n, w_n) + a^{\epsilon}_{2,n}, \quad \left\|a^{\epsilon}_{i,n}\right\| \le \epsilon$$

for all $m, n > n_0$, and thus also

$$q_n \preceq q_m + b^{\epsilon}_{1,n}, \ \ q_m \preceq q_n + b^{\epsilon}_{2,n}$$

and

$$w_n \preceq w_m + c^{\epsilon}_{1,n}, \ \ w_m \preceq w_n + c^{\epsilon}_{2,n}.$$

Further, we obtain $\left\|b_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$ and $\left\|c_{i,n}^{\epsilon}\right\|_2 \leq \epsilon$ for two norms defined in (3.1) and (3.2) since $\left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$. Now, let $\{q_n\}$ is Cauchy sequence in Q and $\{w_n\}$ is Cauchy sequence in W. Then for any $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$q_n \preceq q_m + b_{1,n}^{\epsilon},\ q_m \preceq q_n + b_{2,n}^{\epsilon}, \quad \left\|b_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$$

and

$$q_n \preceq q_m + c_{1,n}^{\epsilon},\ q_m \preceq q_n + c_{2,n}^{\epsilon}, \quad \left\|c_{i,n}^{\epsilon}\right\|_2 \leq \epsilon$$

for all $n, m > n_0$. Since Q and W are quasilinear space, we get

$$\begin{aligned} (q_n, w_n) &\preceq \left(q_m + b_{1,n}^{\epsilon}, w_m + c_{1,n}^{\epsilon}\right) = (q_m, w_m) + \left(b_{1,n}^{\epsilon}, c_{1,n}^{\epsilon}\right), \\ (q_m, w_m) &\preceq \left(q_n + b_{2,n}^{\epsilon}, w_n + c_{2,n}^{\epsilon}\right) = (q_n, w_n) + \left(b_{2,n}^{\epsilon}, c_{2,n}^{\epsilon}\right). \end{aligned}$$

Consequently, we obtain $\left\|\left(b_{i,n}^{\epsilon}, c_{i,n}^{\epsilon}\right)\right\| \leq \epsilon$ because $\left\|b_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$ and $\left\|c_{i,n}^{\epsilon}\right\|_2 \leq \epsilon$. This completes the proof.

Theorem 3.3. *Let $Q_1, Q_2, ..., Q_n$ be Banach quasilinear spaces over the same scalar field $\mathbb{R}$ with norm $\|\cdot\|_i$ $(1 \leq i \leq n)$, respectively. Then the product space $Q = Q_1 \times Q_2 \times ... \times Q_n$ is Banach quasilinear space with norm*

$$\|q\| = \max_{1 \preceq k \preceq n} \left(\|q_k\|_k\right).$$

Proof. Let $q^k = \left(\left(q_1^1, q_2^1, ..., q_n^1\right), \left(q_1^2, q_2^2, ..., q_n^2\right), ..., \left(q_1^k, q_2^k, ..., q_n^k\right), ...\right)$ be a Cauchy sequence in Q. For $\epsilon > 0$, there exists a number n_0 such that for $k, m > n_0$ there are elements $a_{1,n}^{\epsilon}, b_{2,n}^{\epsilon} \in Q$ for which

$$\begin{aligned} \left(q_1^k, q_2^k, ..., q_n^k\right) &\preceq \left(q_1^m, q_2^m, ..., q_n^m\right) + (a_i)_{1,k,m}^{\epsilon}, \\ \left(q_1^m, q_2^m, ..., q_n^m\right) &\preceq \left(q_1^k, q_2^k, ..., q_n^k\right) + (a_i)_{2,k,m}^{\epsilon}, \\ \left\|(a_i)_{j,k,m}^{\epsilon}\right\| &\leq \epsilon. \end{aligned}$$

From here, we get

$$\left\|\left(q_1^k, q_2^k, ..., q_n^k\right) - \left(q_1^m, q_2^m, ..., q_n^m\right)\right\| = \max_{1 \preceq i \preceq n} \left\|q_i^k - q_i^m\right\|_i \to 0$$

$(k, m \to \infty)$. Hence, $\left\|q_i^k - q_i^m\right\|_i \to 0$ for every $1 \leq i \leq n$ when $k, m \to \infty$. This proves that the $\left(q_i^k\right)$ is a Cauchy sequence in Q_i for every $1 \leq i \leq n$. Since Q_i is Banach, $\left(q_i^k\right)$ converges to a q_i in Q_i, $(k \to \infty)$. Note that this implies that for $\epsilon > 0$ there exists a n_0 such that for $k > n_0$:

$$q_i^k \preceq q_i + (a_i)_{1,k}^{\epsilon}, \quad q_i \preceq q_i^k + (a_i)_{2,k}^{\epsilon}, \qquad \left\|(a_i)_{j,k}^{\epsilon}\right\|_i \leq \epsilon$$

for every $1 \preceq i \preceq n$. Since

$$\begin{aligned} \left\| q^k - q \right\| &= \left\| \left(q_1^k, q_2^k, ..., q_n^k \right) - (q_1, q_2, ..., q_n) \right\| \\ &= \max_{1 \preceq i \preceq n} \left(\left\| q_i^k - q_i \right\|_i \right) \\ &\leq \epsilon, \end{aligned}$$

we have $q^k \to q \in Q, (k \to \infty)$. Consequently, Q is Banach quasilinear space.

Proposition 3.3. *If $Q_1, Q_2, ..., Q_n$ are solid-floored quasilinear space then $Q = Q_1 \times Q_2 \times ... \times Q_n$ is solid-floored quasilinear space.*

Proof. Let Q_i is solid-floored quasilinear space for every $1 \leq i \leq n$. From the Definition 2.5, we have

$$q_i = \sup \{ w_i \in (Q_i)_r : w_i \preceq q_i \}$$

for every $q_i \in Q_i$. Since Q is a quasilinear space, we obtain

$$(w_1, w_2, ..., w_n) \preceq (q_1, q_2, ..., q_n)$$

such that $(w_1, w_2, ..., w_n) = w \in Q_r$ and $(q_1, q_2, ..., q_n) = q \in Q$. From here, we have

$$q = \sup \{ (w_1, w_2, ..., w_n) \in Q_r : (w_1, w_2, ..., w_n) \preceq (q_1, q_2, ..., q_n) \} .$$

Now, we introduce the concept of equivalent norms on the same quasilinear space. Also, we concentrate on the Hausdorff metric properties for two equivalent norms that are defined on a quasilinear space.

Definition 3.2. *A norm $\|\cdot\|$ on a normed quasilinear space Q is said to be equivalent to a norm $\|\cdot\|_0$ on Q if there are positive real numbers a and b such that for all $q \in Q$ we have*

$$a \|q\|_0 \leq \|q\| \leq b \|q\|_0 .$$

Example 3.6. *The following norms on $I\mathbb{R}^2 = \{(X_1, X_2) : X_1, X_2 \in \Omega_C(\mathbb{R})\}$ are equivalent:*

$$\begin{aligned} \|(x, y)\| &= \|x\| + \|y\| \\ \|(x, y)\|_1 &= \max \{ \|x\|, \|y\| \} . \end{aligned}$$

Theorem 3.4. *Let Q be a quasilinear space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. The sequence $\{q_n\}$ is convergent to q in normed quasilinear space $(Q, \|\cdot\|)$ if and only if $\{q_n\}$ is convergent to q in $(Q, \|\cdot\|_1)$.*

Proof. Suppose that $\{q_n\} \to q$ in normed quasilinear space $(Q, \|\cdot\|)$. Then for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that:

$$q_n \preceq q + q_{1,n}^{\epsilon},\ q \preceq q_n + q_{2,n}^{\epsilon}, \quad \left\|q_{i,n}^{\epsilon}\right\| \leq \frac{\epsilon}{M}$$

$\forall n \geq N$ and $M \in \mathbb{N}^{+}$. Since the norms $\|\cdot\|$ and $\|\cdot\|_1$ are equivalent, we have

$$\left\|q_{i,n}^{\epsilon}\right\|_1 \leq M \left\|q_{i,n}^{\epsilon}\right\| \leq \epsilon.$$

Hence $\{q_n\} \to q$ in $(Q, \|\cdot\|_1)$.

Conversely, let $\{q_n\} \to q$ in $(Q, \|\cdot\|_1)$. Then for every $\epsilon > 0$ there exists an index N such that

$$q_n \preceq q + q_{1,n}^{\epsilon},\ q \preceq q_n + q_{2,n}^{\epsilon}, \quad \left\|q_{i,n}^{\epsilon}\right\|_1 \leq \epsilon$$

$\forall n \geq N$. Since the norms are equivalent, we get

$$m\,\|q\| \leq \|q\|_1 \leq \epsilon.$$

Hence, $\{q_n\}$ is convergent to q in $(Q, \|\cdot\|)$.

Theorem 3.5. *Let Q be a quasilinear space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. The sequence $\{q_n\}$ is Cauchy sequence in normed quasilinear space $(Q, \|\cdot\|)$ if and only if $\{q_n\}$ is Cauchy sequence in $(Q, \|\cdot\|_1)$.*

Proof. Let $\{q_n\}$ be a Cauchy sequence in $(Q, \|\cdot\|)$. For an arbitrary $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$q_n \preceq q_m + a_{1,n}^{\epsilon},\ q_m \preceq q_n + a_{2,n}^{\epsilon}, \quad \left\|a_{i,n}^{\epsilon}\right\| \leq \frac{\epsilon}{M}$$

for all $n, m > n_0$. Similar way to the above theorem, we obtain $\left\|a_{i,n}^{\epsilon}\right\|_1 \leq M \left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$. This proves that the sequence $\{q_n\}$ is Cauchy sequence in $(Q, \|\cdot\|_1)$. The proof of opposite can be proved by similar way.

Theorem 3.6. *Let Q be a quasilinear space and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. $(Q, \|\cdot\|)$ is complete if and only if $(Q, \|\cdot\|_1)$ is complete.*

Proof. Let $(Q, \|\cdot\|)$ be a complete and $\|\cdot\|$ and $\|\cdot\|_1$ be equivalent norms on Q. If $\{q_n\}$ is a Cauchy sequence in $(Q, \|\cdot\|_1)$, then for an arbitrary $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}$ such that

$$q_n \preceq q_m + a_{1,n}^{\epsilon},\ q_m \preceq q_n + a_{2,n}^{\epsilon}, \quad \left\|a_{i,n}^{\epsilon}\right\| \leq \epsilon$$

for all $n, m > n_0$. From Theorem 3.5, we have $\{q_n\}$ is a Cauchy sequence in $(Q, \|\cdot\|_1)$. We obtain $q_n \to q \in Q$ from the completeness of $(Q, \|\cdot\|)$. From Theorem 3.4, we get $\{q_n,\ n \in \mathbb{N}\}$ is convergent to q in $(Q, \|\cdot\|_1)$ which proves completeness of $(Q, \|\cdot\|_1)$. The converse can be proved similarly.

Corollary 3.1. *If two norms $\|\cdot\|$ and $\|\cdot\|_0$ on a quasilinear space Q are equivalent, then $\|q_n - q\| \to 0$ if and only if $\|q_n - q\|_0 \to 0$ for any sequence (q_n) in Q and any $q \in Q$.*

If Q is finite dimensional normed quasilinear space, then any two norms on Q_r are equivalent since Q_r is a normed linear subspace of Q.

4. Conclusion

In this paper, we define the notion of homogenized quasilinear space as a new concept in quasilinear spaces. We also research on the some properties of the homogenized quasilinear spaces. Then, we introduce the concept of equivalent norm on a quasilinear space. As in the linear functional analysis, we obtained some results related to equivalent norms defined in normed quasilinear spaces.

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