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SHEFFER STROKE BG-ALGEBRAS

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Abstract. In this paper, Sheffer stroke BG-algebra is defined and its features are investigated. It is indicated that the axioms of a Sheffer stroke BG-algebra are independent. It is stated the connection between a Sheffer stroke BG-algebra and a BG-algebra by defining a unary operation on a Sheffer stroke BG-algebra. After describing a subalgebra and a normal subset of a Sheffer stroke BG-algebra, the relationship of these structures is shown.

Keywords:BG-algebras, Sheffer stroke, Sheffer stroke BG-algebras

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1. Introduction

Y. Imai and K. Iséki presented a novel algebraic structure named BCK algebra in 1966. Today, many authors study BCK algebras and this algebra is applied to many branches of mathematics, such as group theory, functional analysis, probability theory, topology, fuzzy set theory, ect. K. Iséki introduced the new idea which is called BCI algebra in 1980 [4]. BCK/ BCI algebra is a significant class of logical algebras and is researched by many researchers. Moreover, the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

Neggers and Kim developed a new notion called a B-algebras [8]. B-algebras is connected several classes of algebras of interest such as BCK/BCI-algebras. In addition, BG-algebras which is a generalization of B-algebras was presented by C. B. Kim and H. S. Kim [5].

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An algebraic structure BG-algebra was constructed on a non-empty set X with a binary operation $*$ and a constant 0 satisfying some axioms. In 2004, S. S. Ahn and H. D. Lee discussed BG-algebra and some of its properties such as fuzzy subalgebras of BG-algebras [1]. R. Muthuraj et al worked on anti Q-fuzzy BG-ideals in BG-algebra in 2010 [7] and D. K. Basnet investigated on fuzzy ideals of BG-algebras in 2011 [2].

The Sheffer stroke operation, which was first introduced by H. M. Sheffer [13], engages many scientists' attention, because any Boolean function or axiom can be expressed by means of this operation [6]. It reducts axiom systems of many algebraic structures. So, many researchers want to use this operation on their studies. For example, interval Sheffer stroke basic algebras [9], relation between Sheffer stroke operation and Hilbert algebras [10], filters of strong Sheffer stroke non-associative MV-algebras [11], (Fuzzy) filters of Sheffer stroke BL-algebras [12] and Sheffer operation in ortholattices [3] are given as some research on Sheffer stroke operation in recent years.

After giving basic definitions and notions about a Sheffer stroke and a BG-algebra, it is defined a Sheffer stroke BG-algebra. By presenting fundamental notions about this algebraic structure, it is proved that the axiom system of a Sheffer stroke BG-algebra is independent. Sheffer stroke B-algebra is defined and it is indicated that the axioms of a Sheffer stroke B-algebra are independent. The relationship between a Sheffer stroke BG-algebra and a Sheffer stroke B-algebra is indicated. It is shown that the connection between a Sheffer stroke BG-algebra and a BG-algebra and Cartesian product of two Sheffer stroke BG-algebras is a Sheffer stroke BG-algebra. A subalgebra and a normal subset of a Sheffer stroke BG-algebra is defined and the relationship between this structures is demonstrated. Finally, it is shown that the Sheffer stroke BG-algebra is a group-derived under one condition.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a BG-algebra.

Definition 2.1. [3] *Let $\mathcal{A} = \langle A, | \rangle$ be a groupoid. The operation $|$ is said to be Sheffer stroke if it satisfies the following conditions:*

$(S1)$ $a_1|a_2 = a_2|a_1$,

$(S2)$ $(a_1|a_1)|(a_1|a_2) = a_1$,

$(S3)$ $a_1|((a_2|a_3)|(a_2|a_3)) = ((a_1|a_2)|(a_1|a_2))|a_3$,

$(S4)$ $(a_1|((a_1|a_1)|(a_2|a_2)))|(a_1|((a_1|a_1)|(a_2|a_2))) = a_1$.

Definition 2.2. [5] *A BG-algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying the following axioms:*

$(BG.1)$ $a_1 * a_1 = 0,$

$(BG.2)$ $a_1 * 0 = a_1,$

$(BG.3)$ $(a_1 * a_2) * (0 * a_2) = a_1,$

for all $a_1, a_2 \in A$.

A BG-algebra is called bounded if it has the greatest element.

Lemma 2.1. [5] *In a BG-algebra A, the following properties hold for all $a_1, a_2, a_3 \in A$:*

(i) $a_1 * a_2 = a_3 * a_2$ *implies* $a_1 = a_3$,

(ii) $0 * (0 * a_1) = a_1$,

(iii) *If* $a_1 * a_2 = 0$, *then* $a_1 = a_2$,

(iv) *If* $0 * a_1 = 0 * a_2$, *then* $a_1 = a_2$,

(v) $(a_1 * (0 * a_1)) * a_1 = a_1$.

Definition 2.3. [5] *A nonempty subset S of a BG-algebra A is called a BG-subalgebra if $a_1 * a_2 \in S$, for all $a_1, a_2 \in S$.*

Definition 2.4. [8] *Let A be a BG-algebra. A nonempty subset N of A is said to be normal if $(a_1 * x) * (a_2 * y) \in N$ for any $a_1 * a_2$, $x * y \in N$.*

Definition 2.5. [8] *A B-algebra is a non-empty set A with a constant 0 and a binary operation $*$ satisfying the following axioms:*

(i) $a_1 * a_1 = 0,$

(ii) $a_1 * 0 = a_1,$

(iii) $(a_1 * a_2) * a_3 = a_1 * (a_3 * (0 * a_2)),$

for all $a_1, a_2, a_3 \in A$.

3. Sheffer stroke BG-Algebras

In this part, we define a Sheffer Stroke BG-algebra and give some properties.

Definition 3.1. *A Sheffer stroke BG-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$ such that 0 is the constant in A and the following axioms are satisfied:*

$(sBG.1)$ $(a_1|(a_1|a_1))|(a_1|(a_1|a_1) = 0$,

$(sBG.2)$ $(0|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = a_1|a_1,$

for all $a_1, a_2 \in A$.

Let A be a Sheffer stroke BG-algebra, unless otherwise is indicated.

Lemma 3.1. *The axioms (sBG.1) and (sBG.2) are independent.*

Proof.

(1) Independence of (sBG.1):

We construct an example for this axiom which is false while (sBG.2) is true. Let $(\{0,1\}, |_1)$ be the groupoid defined as follows:

$\vert_1$	0	1
0	1	1
1	0	0

Then $|_1$ satisfies (sBG.2) but not (sBG.1) when $a_1 = 1$.

(2) Independence of (sBG.2):

Let $(\{0,1\}, |_2)$ be the groupoid defined as follows:

$\vert_2$	0	1
0	1	1
1	1	0

Then $|_2$ satisfies (sBG.1) but not (sBG.2) when $a_1 = 1$ and $a_2 = 1$.

Lemma 3.2. *Let A be a Sheffer stroke BG-algebra. Then the following features hold for all $a_1, a_2, a_3 \in A$:*

(1) $(0|0)|(a_1|a_1) = a_1$,

(2) $(a_1|(0|0))|(a_1|(0|0)) = a_1$,

(3) $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = (a_3|(a_2|a_2))|(a_3|(a_2|a_2))$ *implies* $a_1 = a_3$,

(4) $(0|(0|(a_1|a_1))) = a_1|a_1$,

(5) *If* $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0$ *then* $a_1 = a_2$,

(6) *If* $(0|(a_1|a_1)) = (0|(a_2|a_2))$ *then* $a_1 = a_2$,

(7) $(((a_1|(0|(a_1|a_1)))|(a_1|(0|(a_1|a_1))))|(a_1|a_1)) = a_1|a_1$,

(8) $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$.

Proof.

(1) By using (sBG.1), (S1) and (S2), we obtain

$$\begin{aligned}(0|0)|(a_1|a_1) &= (((a_1|(a_1|a_1))|(a_1|(a_1|a_1)))|((a_1|(a_1|a_1))|(a_1|(a_1|a_1))))|(a_1|a_1)\\ &= (a_1|(a_1|a_1))|(a_1|a_1)\\ &= (a_1|a_1)|(a_1|(a_1|a_1))\\ &= a_1.\end{aligned}$$

(2) By using (S1), (S2) and (1), we have

$$\begin{aligned}(a_1|(0|0))|(a_1|(0|0)) &= (((a_1|(a_1)|(a_1|a_1))|(0|0))|((a_1|(a_1)|(a_1|a_1))|(0|0)))\\ &= ((0|0)|((a_1|a_1)|(a_1|a_1)))|((0|0)|((a_1|a_1)|(a_1|a_1)))\\ &= (a_1|a_1)|(a_1|a_1)\\ &= a_1.\end{aligned}$$

(3) Let $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = (a_3|(a_2|a_2))|(a_3|(a_2|a_2))$. By using (sBG.1), we get

$$\begin{aligned}a_1|a_1 &= (0|(a_2|a_2))|((a_1|(a_2|a_2))|(a_1|a_2|a_2))\\ &= (0|(a_2|a_2))|((a_3|(a_2|a_2))|(a_3|a_2|a_2))\\ &= a_3|a_3.\end{aligned}$$

By using (S2), we have $a_1 = (a_1|a_1)|(a_1|a_1) = (a_3|a_3)|(a_3|a_3) = a_3$.

(4) In (sBG.2), we substitute $[a_2 := a_1]$ and by using (sBG.1) and (S1), we obtain

$$\begin{aligned}a_1|a_1 &= (0|(a_1|a_1))|((a_1|(a_1|a_1))|(a_1|(a_1|a_1)))\\ &= (0|(a_1|a_1))|0\\ &= 0|(0|(a_1|a_1)).\end{aligned}$$

(5) By using (sBG.1) and (3), we get $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = 0 = (a_2|(a_2|a_2))$ $|(a_2|(a_2|a_2))$. Then $a_1 = a_2$.

(6) In (sBG.2) we substitute $[a_2 := a_1]$ and by using (sBG.1), we obtain

$$\begin{aligned} a_1|a_1 &= (0|(a_1|a_1))|(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) \\ &= (0|(a_1|a_1))|0 \\ &= (0|(a_2|a_2))|0 \\ &= (0|(a_2|a_2))|(a_2|(a_2|a_2))|(a_2|(a_2|a_2)) \\ &= a_2|a_2. \end{aligned}$$

Thus, $a_1 = (a_1|a_1)|(a_1|a_1) = (a_2|a_2)|(a_2|a_2) = a_2$ from (S2).

(7) In (sBG.2) we substitute $[a_2 := (0|(a_1|a_1))|(0|(a_1|a_1))]$ and by using (S1), (S2) and (4), we get

$$\begin{aligned} a_1|a_1 &= (((a_1|(0|(a_1|a_1)))|(a_1|(0|(a_1|a_1))))|(0|(0|(a_1|a_1))) \\ &= (((a_1|(0|(a_1|a_1)))|(a_1|(0|(a_1|a_1))))|(a_1|a_1). \end{aligned}$$

(8) Substituting $[a_2 := (a_1|a_1)]$ in (S2), we obtain

$$(a_1|a_1)|(a_1|(a_1|a_1)) = a_1.$$

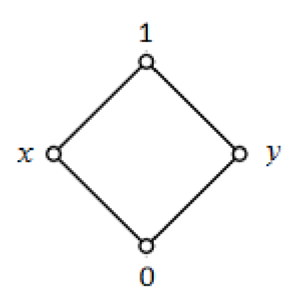
By using (S1), we get $(a_1|(a_1|a_1))|(a_1|a_1) = a_1$.

Definition 3.2. *A Sheffer stroke B-algebra is an algebra $(A, |, 0)$ of type $(2, 0)$, where A is a non-empty set, 0 is the constant in A and $|$ is Sheffer stroke on A, such that the following identities are satisfied for all $a_1, a_2, a_3 \in A$:*

$(sB.1)$ $(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0,$

$(sB.2)$ $((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = (a_1|(a_3|(0|(a_2|a_2))))$.

Example 3.1. *Consider $(A, |, 0)$ with the following Hasse diagram, where $A = \{0, x, y, 1\}$:*



The binary operation $|$ on A has Cayley table as follow:

$\mid$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

Then this structure is a Sheffer stroke B-algebra.

Lemma 3.3. *The axioms (sB.1) and (sB.2) are independent.*

Proof.

(1) Independence of (sB.1):

We construct an example for this axiom which is false while (sB.2) is true. Let $(\{0,1\}, |_3)$ be the groupoid defined as follows:

$\vert_3$	0	1
0	1	1
1	0	0

Then $|_3$ satisfies (sB.2) but not (sB.1) when $a_1 = 1$.

(2) Independence of (sB.2):

Let $(\{0,1\}, |_4)$ be the groupoid defined as follows:

$\vert_4$	0	1
0	1	1
1	1	0

Then $|_4$ satisfies (sB.1) but not (sB.2) when $a_1 = 1$, $a_2 = 1$ and $a_3 = 0$.

Theorem 3.1. *Every Sheffer stroke B-algebra is a Sheffer stroke BG-algebra.*

Proof. Since the axioms (sB.1) and (sBG.1) are the same, we show only (sBG.2). By using (S1), (S2), (sB.2) and Lemma 3.2 (2), we have

$$\begin{aligned}
(0|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) &= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|((0|(a_2|a_2))|(0|(a_2|a_2))| \\
&\quad (0|(a_2|a_2))|(0|(a_2|a_2))) \\
&= (a_1|((((0|(a_2|a_2))|(0|(a_2|a_2))))|(0|(a_2|a_2)))) \\
&= (a_1|((0|(a_2|a_2))|((0|(a_2|a_2))|(0|(a_2|a_2))))) \\
&= (a_1|(0|0)) \\
&= a_1|a_1.
\end{aligned}$$

Theorem 3.2. *Let* $(A, |, 0)$ *be a Sheffer stroke BG-algebra. If we define*

$$a_1 * a_2 := (a_1|(a_2|a_2))|(a_1|a_2|a_2),$$

then $(A, *, 0)$ *is a BG-algebra.*

Proof. By using (S1), (S2), (sBG.1), (sBG.2), Lemma 3.2 (2), we have:

$(BG.1)$: $a_1 * a_1 = (a_1|(a_1|a_1))|(a_1|(a_1|a_1)) = 0.$

$(BG.2)$: $a_1 * 0 = (a_1|(0|0))|(a_1|(0|0)) = a_1.$

$(BG.3)$:

$$\begin{aligned}
(a_1 * a_2) * (0 * a_2) &= ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|((0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))| \\
&\quad (0|(a_2|a_2)))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|((0|(a_2|a_2))|(0|(a_2|a_2))| \\
&\quad (0|(a_2|a_2))|(0|(a_2|a_2))) \\
&= (((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_2|a_2)))|(((a_1|(a_2|a_2))|(a_1|(a_2|a_2))) \\
&\quad |(0|(a_2|a_2))) \\
&= (0|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1|(a_2|a_2))|(0|(a_2|a_2))|(a_1|(a_2|a_2))|(a_1 \\
&\quad |(a_2|a_2)) \\
&= (a_1|a_1)|(a_1|a_1) \\
&= a_1.
\end{aligned}$$

Theorem 3.3. *Let* $(A, *, 0, 1)$ *be a bounded BG-algebra. If we define*

$$a_1|a_2 := (a_1 * a_2^0)^0,$$

where $a_1^0 = a_1|a_1$*, then* $(A, |, 0)$ *is a Sheffer stroke BG-algebra.*

Proof. (i) By using (BG.1), we have

$$\begin{aligned}
(a_1|(a_1|a_1))|(a_1|(a_1|a_1)) &= (a_1|a_1^0)|(a_1|a_1^0) \\
&= (a_1 * a_1)^0|(a_1 * a_1)^0 \\
&= ((a_1 * a_1)^0)^0 \\
&= a_1 * a_1 \\
&= 0.
\end{aligned}$$

(ii) By using (BG.2), we obtain

$$
\begin{aligned}
((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(a_2|a_2)) &= ((a_1 * a_2)^0|(a_1 * a_2)^0)|(0 * a_2)^0 \\
&= ((a_1 * a_2)^0)^0|(0 * a_2)^0 \\
&= (a_1 * a_2)|(0 * a_2)^0 \\
&= ((a_1 * a_2) * (0 * a_2))^0 \\
&= (a_1)^0 \\
&= a_1|a_1.
\end{aligned}
$$

Theorem 3.4. *Let $(A, |_A, 0_A)$ and $(B, |_B, 0_B)$ be Sheffer stroke BG-algebras. Then, $(A \times B, |_{A\times B}, 0_{A\times B})$ is a Sheffer stroke BG-algebra, where the operation $|_{A\times B}$ is defined by*

$$(a_1, b_1)|_{A\times B}(a_2, b_2) = (a_1|_A a_2, b_1|_B b_2)$$

and $0_{A\times B} = (0_A, 0_B)$.

Definition 3.3. *A non-empty subset S of a Sheffer stroke BG-algebra A is called a Sheffer stroke BG-subalgebra of A if $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in S$ for all $a_1, a_2 \in S$.*

Theorem 3.5. *Let $(A, |, 0)$ be a Sheffer stroke BG-algebra and $\emptyset \neq S \subseteq A$. Then the following are equivalent:*

(a) S is a subalgebra of A,

(b) $(a_1|(0|(a_2|a_2)))|(a_1|(0|(a_2|a_2))) \in S$, $(0|(a_2|a_2))|(0|(a_2|a_2)) \in S$ for any $a_1, a_2 \in S$.

Proof. $(a) \Rightarrow (b)$: Since $S \neq \emptyset$, there exists an element $a_1 \in S$ and $0 = (a_1|(a_1|a_1))|(a_1|(a_1|a_1)) \in S$. Since S is closed under $|$, $(0|(a_2|a_2))|(0|(a_2|a_2)) \in S$ and thus $(a_1|((0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2)))|(a_1|((0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))) \in S$. From (S2), we get $(a_1|(0|(a_2|a_2))|(a_1|(0|(a_2|a_2)) \in S$.

$(b) \Rightarrow (a)$: By using Lemma 3.2 (4), $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = (a_1|((0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))))|(a_1|((0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))))$, $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in S$ for any $a_1, a_2 \in S$.

Definition 3.4. *Let A be a Sheffer stroke BG-algebra. A non-empty subset N of A is said to be normal subset of A if*

$$(((a_1|(x|x))|(a_1|(x|x)))|(a_2|(y|y)))|(((a_1|(x|x))|(a_1|(x|x)))|(a_2|(y|y))) \in N,$$

for any $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)), (x|(y|y))|(x|(y|y)) \in N$.

Theorem 3.6. *Every normal subset N of a Sheffer stroke BG-algebra A is a Sheffer stroke subalgebra of A.*

Proof. If $a_1, a_2 \in N$ then $(a_1|(0|0))|(a_1|(0|0)), (a_2|(0|0))|(a_2|(0|0)) \in N$. Since N is normal , then $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) = ((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(0|0))|((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(0|(0|0)) \in N$. Therefore, N is a Sheffer stroke subalgebra.

Lemma 3.4. *Let N be a Sheffer stroke normal subalgebra of a Sheffer stroke BG-algebra A and let $a_1, a_2 \in N$. If $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N$ then $(a_2|(a_1|a_1))|(a_2|(a_1|a_1)) \in N$.*

Proof. Let $(a_1|(a_2|a_2))|(a_1|(a_2|a_2)) \in N$. Since $(a_2|(a_2|a_2))|(a_2|(a_2|a_2)) = 0 \in N$ and N is normal, $(a_2|(a_1|a_1))|(a_2|(a_1|a_1)) = (a_2|(a_1|a_1))|(a_2|(a_1|a_1))|((a_2|(a_2|a_2))|(a_2|(a_2|a_2))|(a_2|(a_2|a_2))|(a_2|(a_2|a_2)))|(a_2|(a_1|a_1))|(a_2|(a_1|a_1))|((a_2|(a_2|a_2))|(a_2|(a_2|a_2))|(a_2|(a_2|a_2))|(a_2|(a_2|a_2))) \in N$.

Theorem 3.7. *Let $(A, |, 0)$ be a Sheffer stroke BG-algebra with the identity*

$$((a_1|(a_2|a_2))|(a_1|(a_2|a_2)))|(a_3|a_3) = a_1|(0|(((0|(a_2|a_2))|(0|(a_2|a_2)))|(a_3|a_3))),$$

for all $a_1, a_2, a_3 \in A$. Then $(A, |, 0)$ is group-derived.

Proof. Define a binary operation "o" on A by

$$a_1 \ o \ a_2 = (a_1|(0|(a_2|a_2)))|(a_1|(0|(a_2|a_2))).$$

Then by using (S2), (sBG.1), Lemma 3.2 (2) and (4), we get

$$\begin{aligned} a_1 \ o \ 0 &= (a_1|(0|(0|0)))|(a_1|(0|(0|0))) \\ &= (a_1|((0|(0|0))|(0|(0|0))|(0|(0|0))|(0|(0|0))))|(a_1|((0|(0|0))|(0|(0|0))|(0|(0|0)) \\ &\quad |(0|(0|0)))) \\ &= (a_1|(0|0))|(a_1|(0|0)) \\ &= a_1 \end{aligned}$$

and

$$\begin{aligned} 0 \ o \ a_1 &= (0|(0|(a_1|a_1)))|(0|(0|(a_1|a_1))) \\ &= (a_1|a_1)|(a_1|a_1) \\ &= a_1. \end{aligned}$$

Then 0 acts like an identity element on A. Since

$$\begin{aligned}
a_1\ o\ (0|(a_1|a_1))|(0|(a_1|a_1)) &= (a_1|(0|((0|(a_1|a_1))|(0|(a_1|a_1))|(0|(a_1|a_1))|(0|(a_1|a_1)))))\\
&\quad |(a_1|(0|((0|(a_1|a_1))|(0|(a_1|a_1))|(0|(a_1|a_1))|(0|(a_1|a_1)))))\\
&= (a_1|(0|(0|a_1|a_1)))|(a_1|(0|(0|a_1|a_1)))\\
&= (a_1|((0|(0|a_1|a_1))|(0|(0|a_1|a_1))|(0|(0|a_1|a_1))|(0|(0|a_1|a_1))))\\
&\quad |(a_1|((0|(0|a_1|a_1))|(0|(0|a_1|a_1))|(0|(0|a_1|a_1))|(0|(0|a_1|a_1))))\\
&= (a_1|(a_1|a_1))|(a_1|(a_1|a_1))\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
((0|(a_1|a_1))|(0|(a_1|a_1))\ o\ a_1 &= ((0|(a_1|a_1))|(0|(a_1|a_1)))|(0|(a_1|a_1))|((0|(a_1|a_1))|\\
&\quad (0|(a_1|a_1)))|(0|(a_1|a_1))\\
&= 0,
\end{aligned}$$

we obtain that $(0|(a_1|a_1))|(0|(a_1|a_1))$ behaves like a multiplicative inverse for a_1 with respect to the operation "o". We claim that $(A;o)$ is a semi-group. Indeed,

$$\begin{aligned}
a_1\ o\ (a_2\ o\ a_3) &= (a_1|(0|(a_2|(0|(a_3|a_3)))))|(a_1|(0|(a_2|(0|(a_3|a_3)))))\\
&= (a_1|(0|((0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2))))|(0|(a_3|a_3))))|(a_1|(0|((0|(0|(a_2|a_2)))\\
&\quad |(0|(0|(a_2|a_2))))|(0|(a_3|a_3))))\\
&= (a_1|(0|((0|(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2)))|(0|(0|(a_2|a_2))|\\
&\quad (0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))))|(0|(a_3|a_3))))|(a_1|(0|((0|(0|(a_2|a_2))\\
&\quad |(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2)))|(0|(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))\\
&\quad |(0|(a_2|a_2))))|(0|(a_3|a_3))))\\
&= ((a_1|(0|(a_2|a_2)))|(a_1|(0|(a_2|a_2))))|(0|(a_3|a_3))|((a_1|(0|(a_2|a_2)))\\
&\quad |(a_1|(0|(a_2|a_2))))|(0|(a_3|a_3))\\
&= (a_1\ o\ a_2)\ o\ a_3.
\end{aligned}$$

Note that

$$\begin{aligned} a_1 \ o \ (a_2)^{-1} &= (a_1|(0|((0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2)))))|(a_1|(0|((0|(a_2|a_2)) \\ &\quad |(0|(a_2|a_2))|(0|(a_2|a_2))|(0|(a_2|a_2))))) \\ &= (a_1|(0|(0|(0|(a_2|a_2)))))|(a_1|(0|(0|(0|(a_2|a_2))))) \\ &= (a_1|(((0|(0|(0|(a_2|a_2))))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2))))| \\ &\quad (a_1|(((0|(0|(0|(a_2|a_2))))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2)))|(0|(0|(a_2|a_2))))) \\ &= (a_1|(a_2|a_2))|(a_1|(a_2|a_2)). \end{aligned}$$

Hence $(A; |, 0)$ is also a group-derived Sheffer stroke BG-algebra. This completes the proof.

4. Conclusion

In this study, we introduce a Sheffer stroke BG-algebra, Cartesian product, a subalgebra, a normal subset and their some properties. After giving basic definitions and notions about Sheffer stroke and a BG-algebra, we describe a Sheffer stroke BG-algebra and a Sheffer stroke B-algebra and present basic notions about this algebraic structures. We indicate that the axiom systems of a Sheffer stroke BG-algebra and a Sheffer stroke B-algebra are independent. We show that a Sheffer stroke BG-algebra is a BG-algebra and that Cartesian product of two Sheffer stroke BG-algebras is a Sheffer stroke BG-algebra. After defining a subalgebra and a normal subset, we present the relationship between a subalgebra and a normal subset on Sheffer stroke BG-algebra. Finally, we show that the Sheffer stroke BG-algebra is a group-derived under one condition.

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