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PLANE CURVES WITH SAME EQUI-AFFINE AND EUCLIDEAN INVARIANTS

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Abstract. In the present paper, we consider and solve the problem of finding parametric plane curves with the same equi-affine and Frenet curvatures. We then classify the parametric plane curves with prescribed equi-affine curvature by solving certain ordinary differential equations. Our classification generalizes the plane curves with constant equi-affine curvature. Several examples are also given by figures.

Keywords: Plane curve, Equiaffine transformation, Equiaffine curvature, Euclidean curvature

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1. Introduction

Affine Differential Geometry, since nineteenth century, has been investigated and developed by a larger group of geometers led by Pick, Tzitzeica, Berwald, Blaschke among others. See [13] for this process in detail. This branch of Geometry is based on the study of the invariant properties of affine $n-$space $\mathbb{R}^n$ under the (equi-)affine transformations.

The theory of curves in $\mathbb{R}^n$ has had a great interest from past to present [1, 2, 4, 5, 10, 11, 16, 18, 20, 21, 23, 24]. In this paper, we mainly consider the problem of finding parametric plane curves with the same equi-affine and Frenet curvatures. For example, a unit circle in

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the Euclidean setting has the same constant equi-affine and Frenet curvatures, i.e. 1. We find the motivation for this study from [3, equation 28], [17, Theorem 4, Theorem 5], [22, Remark 6]. Although, in these cited papers, all authors constructed certain relations between the equi-affine and Frenet curvatures for a 2d curve, as far as we know, to solve our main problem has been overlooked till now.

In the context of affine curves, another interest has been to find the parametric equations of the curves with prescribed affine curvatures, see [9, 8, 14, 25]. As a secondary purpose of this paper, we follow this mainstream and classify parametric plane curves with prescribed equi-affine curvatures by solving certain vector ordinary differential equations (ODEs).

The framework of this paper can be explained as follows in detail.

Let $\mathbb{R}^2$ be the affine plane equipped with a fixed area form $|\cdot\ \cdot|$ such that $|\mathbf{u}\ \mathbf{v}| = u_1v_2 - u_2v_1$, for some vectors $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$. The *equi-affine group* of $\mathbb{R}^2$ is generated by the action of the special linear group $SL(2, \mathbb{R})$ and the group of translations of $\mathbb{R}^2$. An *equi-affine transformation* of $\mathbb{R}^2$ is given in matrix form

$$\bar{\mathbf{x}} = A\mathbf{x} + \mathbf{b}, \tag{1.1}$$

where $\bar{\mathbf{x}}, \mathbf{x}, \mathbf{b} \in \mathbb{R}^{2\times 1}$ and $A \in SL(2, \mathbb{R})$. Point out that the area of a parallelogram is preserved by (1.1) and hence it is so-called *area-preserving affine transformation* [15].

(1.1) turns to a *Euclidean transformation* of $\mathbb{R}^2$ if $A \in SO(2)$ [6]. By an *equi-affine* (resp. a *Euclidean*) *invariant* we mean a property of $\mathbb{R}^2$ that remains unchanged under the equi-affine (resp. Euclidean) group.

Let $\mathbf{x} = \mathbf{x}(\sigma) = (x(\sigma), y(\sigma))$, $\sigma \in I \subset \mathbb{R}$, be a smooth parametric curve in $\mathbb{R}^2$. The equi-affine arc-length parameter σ and curvature κ_a of $\mathbf{x}$ are equi-affine invariants of $\mathbb{R}^2$ while the Euclidean arc-length parameter s and Frenet curvature κ_f of $\mathbf{x}$ are Euclidean invariants. The Fundamental Theorem of equi-affine (Euclidean) plane curves implies that a plane curve with constant equi-affine (Frenet) curvature is a quadratic curve (a straight line or a circle) [15, 19]. An equi-affine plane curve with constant equi-affine curvature is homogeneous, i.e. the orbit of a point under a 1-parameter group of the transformations given by (1.1). The converse is true as well [7].

We point out that a unit circle in the Euclidean setting has the same constant equi-affine and Frenet curvatures (i.e. $\kappa_a = \kappa_f = 1$) as well as the same arc-length parameters. Naturally the following question occurs: *is there any plane curve* $\mathbf{x}$ *with* $\kappa_a = \kappa_f$ *besides the unit circle in the Euclidean setting?* In the mean while, we state that the plane curve $\mathbf{x}$ is a unit circle in the Euclidean setting if and only if its equi-affine and Euclidean arc-length parameters are same (see Lemma 3.1.) We answer to this question (see Theorem 3.2) by assuming that $\mathbf{x}$ has different equi-affine and Euclidean arc-length parameters, because it turns to a unit circle in the Euclidean setting otherwise.

It is worth to specify that, in centro-affine context, Liu [14, Proposition 4.1] obtained a characterization for a plane curve in terms of its Frenet curvature that the centro-affine and Euclidean arc-length parameters are same.

Furthermore, by solving a vector ODE of Euler-Cauchy type [12, p. 69] we obtain the parametric plane curves with $\kappa_a(\sigma) = a(b\sigma + c)^{-2}$, for some constants a, b, c with $b^2+c^2 \neq 0$. When a or b is equal to zero, these reduce to the plane curves with constant equi-affine curvature and thus our case is more general.

2. Preliminaries

Theorem 2.1. *We provide basic differential geometric objects of plane curves from* [6, 7, 15, 19].

2.1. **Equi-affine plane curves.** Let $\mathbf{x} = \mathbf{x}(t) = (x(t), y(t))$, $t \in I \subset \mathbb{R}$, be a *non-degenerate* smooth parametric curve in $\mathbb{R}^2$, namely $|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}| \neq 0$ for any t, where $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt}$ and $\ddot{\mathbf{x}} = \frac{d^2\mathbf{x}}{dt^2}$. This yields that nowhere $\mathbf{x}$ has inflection points. *Equi-affine arc-length function* σ is defined by

$$\sigma(t) = \int_{t_0}^{t} \sqrt[3]{|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}|}\,dt. \tag{2.1}$$

Denote $\mathbf{x}' = \frac{d\mathbf{x}}{d\sigma}$ and $\mathbf{x}'' = \frac{d^2\mathbf{x}}{d\sigma^2}$. It follows

$$\left|\mathbf{x}'\ \mathbf{x}''\right| = 1 \text{ for all } \sigma, \tag{2.2}$$

in which the parameter σ is said to be *equi-affine arc-length.* Taking derivative of (2.2) with respect to σ yields $|\mathbf{x}'\ \mathbf{x}'''| = 0$, which means that $\mathbf{x}'$ and $\mathbf{x}'''$ are linearly dependent. Then there exist a function κ_a of σ such that $\mathbf{x}''' = -\kappa_a \mathbf{x}'$. Therefore the following occurs

$$\kappa_a(\sigma) = \left|\mathbf{x}''\ \mathbf{x}'''\right|, \tag{2.3}$$

called *equi-affine curvature* of $\mathbf{x}$. Because κ_a is given by determinant, it is invariant of equi-affine transformations of $\mathbb{R}^2$. It is also clear that the following vector ODE holds

$$\mathbf{x}''' + \kappa_a \mathbf{x}' = 0. \tag{2.4}$$

The Fundamental Theorem of equi-affine plane curves states that for a given smooth function $\kappa_a(\sigma)$, $\sigma \in I$, there exist a unique equi-affine plane curve $\mathbf{x}$ admitting σ as equi-affine arc-length and κ_a as equi-affine curvature up to an equi-affine transformation of $\mathbb{R}^2$. In this regard, if $\kappa_a(\sigma)$ is a constant function then the solutions of (2.4) yield that, up to suitable equi-affine transformations, $\mathbf{x}$ is either a parabola ($\kappa_a = 0$) or an ellipse ($\kappa_a > 0$) or a hyperbola ($\kappa_a < 0$) given in explicit forms $y = \frac{1}{2}x^2$ and $\kappa_a x^2 + \kappa_a^2 y^2 = 1$. Point out that $\mathbf{x}$ turns to a unit circle in the Euclidean setting when $\kappa_a = 1$ identically.

2.2. **Euclidean plane curves.** Let $\mathbf{x} = \mathbf{x}(t) = (x(t), y(t))$, $t \in I \subset \mathbb{R}$, be a *regular* smooth parametric curve in the Euclidean plane $\mathbb{E}^2$, namely $\|\dot{\mathbf{x}}\| \neq 0$ for any t, where $\|\cdot\|$ stands for the Euclidean norm. *Euclidean arc-length function* s of $\mathbf{x}$ is given by

$$s(t) = \int_{t_0}^{t} \|\dot{\mathbf{x}}\| \, dt,$$

in which $\frac{ds}{dt}$ is strictly positive and the inverse of s exists. Therefore $\mathbf{x}\left(s^{-1}(t)\right)$ is so-called *unit-speed* curve, i.e. $\left\| \frac{d\mathbf{x}\left(s^{-1}(t)\right)}{dt} \right\| = 1$, and the parameter s is said to be *Euclidean arc-length*. If $\mathbf{x} = \mathbf{x}(s)$ is a unit-speed curve then its *Frenet curvature* is given by $\kappa_f(s) = \left\| \frac{d^2\mathbf{x}}{ds^2} \right\|$. In this sense, there is a smooth function θ of s, called *turning angle* of $\mathbf{x}$, such that

$$\frac{d\mathbf{x}}{ds} = (\cos\theta(s), \sin\theta(s)). \tag{2.5}$$

Here we easily get $\kappa_f(s) = \left|\frac{d\theta}{ds}\right|$. Note that $\frac{d\theta}{ds}$ is also called the *signed Frenet curvature* of $\mathbf{x}$.

The Fundamental Theorem of Euclidean plane curves states that for given smooth function $\kappa_f(s)$, $s \in I$, there exist a unique Euclidean plane curve $\mathbf{x}$ admitting s as Euclidean arc-length and κ_f as signed Frenet curvature up to a Euclidean transformation of $\mathbb{E}^2$. In this regard, if $\kappa_f(s)$ is a constant function then the solutions of (2.5) yield that, up to suitable Euclidean transformations, $\mathbf{x}$ is either a straight line ($\kappa_f = 0$) or a circle ($\kappa_f \neq 0$) with radius $\frac{1}{\kappa_f}$.

3. Plane curves with $\kappa_a = \kappa_f$

Throughout the section, for a plane curve, the equi-affine arc-length parameter is denoted by σ, the Euclidean arc-length parameter by s, the equi-affine curvature by κ_a and the Frenet curvature by κ_f.

Lemma 3.1. *Let $\mathbf{x}$ be a non-degenerate smooth parameterized curve in $\mathbb{R}^2$ by the same equi-affine and Euclidean arc-length parameters. Then, up to a Euclidean transformation, it is the quadratic curve with $\kappa_a = 1$ parameterized by $\mathbf{x}(\sigma) = (\cos\sigma, \sin\sigma)$.*

Remark 3.1. *In Euclidean setting it is the unit circle and its Frenet curve is also* $\kappa_f = 1$.

Proof. Let σ denote both the equi-affine and Euclidean arc-length parameters of $\mathbf{x}$. It then follows

$$\left\|\mathbf{x}'\right\| = \left|\mathbf{x}' \ \mathbf{x}''\right| = 1, \tag{3.1}$$

where $\mathbf{x}' = \frac{d\mathbf{x}}{d\sigma}$ and $\mathbf{x}'' = \frac{d^2\mathbf{x}}{d\sigma^2}$. Denoting the curve $\mathbf{x}$ as $\mathbf{x}(\sigma) = (x(\sigma), y(\sigma))$ and using (3.1) we get

$$\left(x'\right)^2 + \left(y'\right)^2 = 1 \tag{3.2}$$

and

$$x'y'' - x''y' = 1. \tag{3.3}$$

Differentiating (3.2) with respect to σ we have

$$x'x'' + y'y'' = 0 \text{ or } x'' = \frac{-y'y''}{x'}, \tag{3.4}$$

where $x' \neq 0$ due to (3.3). Substituting (3.4) into (3.3) gives

$$y'' = x'. \tag{3.5}$$

By differentiating (3.5) with respect to σ we find

$$y' = x + c, \tag{3.6}$$

for a constant of integration c. Using (3.5) and (3.6) into (3.4) implies that

$$x'' + x = -c. \tag{3.7}$$

By solving (3.7), we derive

$$x(\sigma) = \lambda_1 \cos\sigma + \lambda_2 \sin\sigma - c. \tag{3.8}$$

It follows from (3.6) and (3.8) that

$$y(\sigma) = \lambda_1 \sin\sigma - \lambda_2 \cos\sigma + d,$$

for a constant of integration d. (3.2) immediately implies

$$\lambda_1^2 + \lambda_2^2 = 1$$

and therefore the curve $\mathbf{x}$ can be parametrically written as

$$\mathbf{x}(\sigma) = \left(\lambda_1 \cos\sigma + \lambda_2 \sin\sigma - c,\ \lambda_1 \sin\sigma - \lambda_2 \cos\sigma + d\right).$$

Up to a Euclidean transformation we complete the proof.

Next we observe the non-degenerate plane curves whose the equi-affine and Frenet curvatures are same. Obviously, this curvature cannot be zero in our case. Remark also that the equi-affine arc-length of such a curve is related by its Frenet curvature as follows

$$\sigma(s) = \int_{s_0}^{s} \sqrt[3]{\kappa_f(t)}dt.$$

Therefore we have the following result.

Theorem 3.1. *Let* $\mathbf{x}$ *be a non-degenerate smooth parametric curve in* $\mathbb{R}^2$ *with the same equi-affine and signed Frenet curvatures* $\kappa = \kappa(\sigma)$. *Then, up to suitable equi-affine transformations, it is either a quadratic curve with* $\kappa = 1$ *(namely a unit circle in Euclidean setting) or parameterized by*

$$\mathbf{x}(\sigma) = \left(\int \cos\left(\int \kappa^{\frac{2}{3}} d\sigma\right) \kappa^{\frac{-1}{3}} d\sigma, \int \sin\left(\int \kappa^{\frac{2}{3}} d\sigma\right) \kappa^{\frac{-1}{3}} d\sigma\right), \tag{3.9}$$

where σ *is the equi-affine arc-length parameter of* $\mathbf{x}$ *given by one of the following*

$$\sigma = \frac{1}{3}\left(1+\kappa^{\frac{-1}{3}}\right)\sqrt{-1+2\kappa^{\frac{-1}{3}}}, \tag{3.10}$$

and

$$\sigma = c^{-1}\sqrt{-1+2\kappa^{\frac{-1}{3}}+c\kappa^{\frac{-2}{3}}} - c^{\frac{-3}{2}}\ln\left|1+c\kappa^{\frac{-1}{3}}+\sqrt{c}\sqrt{-1+2\kappa^{\frac{-1}{3}}+c\kappa^{\frac{-2}{3}}}\right|, \tag{3.11}$$

for some constant $c \neq 0$.

Remark 3.2. *In (3.11) it is not easy to express the curvature function* κ *in terms of* σ, *however (3.10) can be simplified as follows: put* $y = 1+\kappa^{\frac{-1}{3}}$ *and then (3.10) turns to the following algebraic equation of degree 3*

$$2y^3 - 3y^2 - 9\sigma^2 = 0,$$

in which the real root is

$$y = \frac{1}{2}\left[-1+\left(-1+18\sigma^2+6\sqrt{-\sigma^2+9\sigma^4}\right)^{\frac{-1}{3}} + \left(-1+18\sigma^2+6\sqrt{-\sigma^2+9\sigma^4}\right)^{\frac{1}{3}}\right].$$

Therefore we deduce

$$\kappa(\sigma) = \left[\frac{-3}{2}+\frac{1}{2}\left(-1+18\sigma^2+6\sqrt{-\sigma^2+9\sigma^4}\right)^{\frac{-1}{3}} + \frac{1}{2}\left(-1+18\sigma^2+6\sqrt{-\sigma^2+9\sigma^4}\right)^{\frac{1}{3}}\right]^{-3}.$$

Proof. A plane curve is completely determined by its signed Frenet curvature $\kappa = \kappa(s)$, namely

$$\mathbf{x}(s) = \left(\int \cos\left(\int \kappa ds\right) ds, \int \sin\left(\int \kappa ds\right) ds\right). \tag{3.12}$$

Let a derivative with respect to σ be denoted by a dash $'$. Differentiating (3.12) three times with respect to σ gives the following equations

$$\mathbf{x}' = \left(\cos\left(\int \kappa ds\right), \sin\left(\int \kappa ds\right)\right) s' \tag{3.13}$$

$$\begin{aligned}\mathbf{x}'' = \Big(&-\kappa\left(s'\right)^2 \sin\left(\textstyle\int \kappa ds\right) + s'' \cos\left(\textstyle\int \kappa ds\right), \\ &\kappa\left(s'\right)^2 \cos\left(\textstyle\int \kappa ds\right) + s'' \sin\left(\textstyle\int \kappa ds\right)\Big)\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\mathbf{x}''' = \Big(&-\left[\kappa'\left(s'\right)^2 + 3\kappa s' s''\right] \sin\left(\textstyle\int \kappa ds\right) + \left[-\kappa^2\left(s'\right)^3 + s'''\right] \cos\left(\textstyle\int \kappa ds\right), \\ &\left[-\kappa^2\left(s'\right)^3 + s'''\right] \sin\left(\textstyle\int \kappa ds\right) + \left[\kappa'\left(s'\right)^2 + 3\kappa s' s''\right] \cos\left(\textstyle\int \kappa ds\right)\Big).\end{aligned} \tag{3.15}$$

Substituting (3.13) and (3.15) into (2.4) gives

$$\begin{aligned}&\mathbf{x}''' + \kappa\mathbf{x}' = \\ &= \Big(-\left[\kappa'\left(s'\right)^2 + 3\kappa s' s''\right] \sin\left(\textstyle\int \kappa(s)\, ds\right) + \left[-\kappa^2\left(s'\right)^3 + s''' + \kappa s'\right] \cos\left(\textstyle\int \kappa(s)\, ds\right), \\ &\left[-\kappa^2\left(s'\right)^3 + s''' + \kappa s'\right] \sin\left(\textstyle\int \kappa(s)\, ds\right) - \left[\kappa'\left(s'\right)^2 + 3\kappa s' s''\right] \cos\left(\textstyle\int \kappa(s)\, ds\right)\Big) = 0.\end{aligned} \tag{3.16}$$

By using the linearly independence of Sine and Cosine in (3.16) we find

$$\kappa'\left(s'\right)^2 + 3\kappa s' s'' = 0$$

and

$$-\kappa^2\left(s'\right)^3 + \kappa s' + s''' = 0. \tag{3.17}$$

On the other hand from (2.3), (3.12) and (3.13) we conclude

$$\kappa\left(s'\right)^3 = 1, \tag{3.18}$$

which leads to

$$ds = \kappa^{\frac{-1}{3}} d\sigma. \tag{3.19}$$

By (3.12) and (3.19) we have (3.9). Assume now in (3.18) that both κ and s' are constants. Put $\kappa = \kappa_0$. It follows from (2.5) that $\mathbf{x}$ turns to a circle in the Euclidean setting with radius $\frac{1}{|\kappa_0|}$. In order for such a curve to have constant equi-affine curvature, $\mathbf{x}$ must be a quadratic curve with $\kappa_a = 1$. Otherwise, namely neither κ nor s' is constant, it then follows from (3.17) and (3.18) that

$$\kappa\left[\kappa^{\frac{-1}{3}} - 1\right] + \left(\kappa^{\frac{-1}{3}}\right)'' = 0. \tag{3.20}$$

Letting $\kappa = (y+1)^{-3}$ into (3.20) yields

$$y'' + \frac{y}{(y+1)^3} = 0. \tag{3.21}$$

After putting $p = y'$ and $\frac{dp}{dy} = \frac{y''}{y'}$ into (3.21) we deduce

$$p\frac{dp}{dy} + \frac{y}{(y+1)^3} = 0. \tag{3.22}$$

Solving (3.22) gives

$$p(y) = \sqrt{c + \frac{1+2y}{(1+y)^2}}. \tag{3.23}$$

Because $y = \kappa^{\frac{-1}{3}} - 1$, (3.23) follows

$$d\sigma = \frac{d\left(\kappa^{\frac{-1}{3}}\right)}{\sqrt{c + 2\kappa^{\frac{1}{3}} - \kappa^{\frac{2}{3}}}}. \tag{3.24}$$

By solving (3.24) we obtain (3.10) and (3.11) according to $c = 0$ or $c \neq 0$.

4. Plane curves with prescribed equi-affine curvature

As we can see from (2.4), classifying parametric plane curves with prescribed equi-affine curvature directly reduces to solve vector ODE with variable coefficient. In general, solving such equations is not easy and one of the well-known ODEs with variable coefficient is of Euler-Cauchy type. If we put the equi-affine curvature as

$$\kappa_a(\sigma) = a(b\sigma + c)^{-2},\ b^2 + c^2 \neq 0, \tag{4.1}$$

for some constants a, b, c, then (2.4) leads to a vector ODE of Euler-Cauchy type. Moreover, our choice (4.1) generalizes plane curves with constant equi-affine curvature as a secondary purpose of this paper because $\mathbf{x}$ turns to a parabola if $a = 0$, an ellipse $b = 0$ and $ac^{-2} > 0$, and a hyperbola $b = 0$ and $ac^{-2} < 0$.

In this section, we try to classify parametric plane curves whose the equi-affine curvature is given by (2.4). Since we want to generalize plane curves with constant equi-affine curvature we may assume that $ab \neq 0$. Putting $p = ab^{-2}$, (4.1) turns to $\kappa_a(\sigma) = p\sigma^{-2}$ up to a suitable translation of σ. Therefore we have the following result

Theorem 4.1. *Let the interval I do not contain zero and a plane curve $\mathbf{x} : I \to \mathbb{R}^2$ have the equi-affine curvature $\kappa_a(\sigma) = p\sigma^{-2}$, $p \neq 0$. Then, up to suitable equi-affine transformations, it has one of the following parametric expressions*

(1) *if* $p = -2$,

$$\mathbf{x}(\sigma) = \frac{1}{3}\left(\sigma^3, -\ln\sigma\right);$$

(2) *if* $p < \frac{1}{4}$ *and* $p \neq -2$, $p \neq 0$,

$$\mathbf{x}(\sigma) = \left(\frac{2}{3+\sqrt{1-4p}} \sigma^{\frac{3+\sqrt{1-4p}}{2}}, \frac{2}{1-4p-3\sqrt{1-4p}} \sigma^{\frac{3-\sqrt{1-4p}}{2}} \right);$$

(3) *if* $p = \frac{1}{4}$,

$$\mathbf{x}(\sigma) = \left(\frac{2}{3}\sigma^{\frac{3}{2}}, \frac{2}{9}\sigma^{\frac{3}{2}} \left(-2 + 3\ln\sigma\right) \right);$$

(4) *if* $p > \frac{1}{4}$,

$$\begin{aligned}\mathbf{x}(\sigma) = \tfrac{2\sigma^{\frac{3}{2}}}{5+16p} &\left(3\cos\left(\sqrt{4p-1}\ln\sigma\right) + 2\sqrt{4p-1}\sin\left(\sqrt{4p-1}\ln\sigma\right)\right) \\ &-2\cos\left(\sqrt{4p-1}\ln\sigma\right) + \tfrac{3}{\sqrt{4p-1}}\sin\left(\sqrt{4p-1}\ln\sigma\right)\Big).\end{aligned}$$

Proof. By (2.4) we write the following vector ODE

$$\mathbf{x}''' + \frac{p}{\sigma^2}\mathbf{x}' = 0, \ p \neq 0, \tag{4.2}$$

where $\mathbf{x}' = \frac{d\mathbf{x}}{d\sigma}$, etc. Let $\mathbf{x}' = \mathbf{y}$ and $\mathbf{x}''' = \mathbf{y}''$, then (4.2) implies to the ODE of Euler-Cauchy type

$$\sigma^2\mathbf{y}'' + p\mathbf{y} = 0, \tag{4.3}$$

which can be reduced to the vector linear ODE with constant coefficient

$$\ddot{\mathbf{y}} - \dot{\mathbf{y}} + p\mathbf{y} = 0, \tag{4.4}$$

where $\dot{\mathbf{y}} = \frac{d\mathbf{y}}{du}$, $\ddot{\mathbf{y}} = \frac{d^2\mathbf{y}}{du^2}$ and $\sigma = e^u$. The characteristic equation of (4.4) follows

$$\lambda^2 - \lambda + p = 0,$$

in which the roots are $\lambda_{1,2} = \frac{1\pm\sqrt{1-4p}}{2}$. According to the sign of the discriminant $1-4p$, we have to distinguish three cases:

(1) $p < \frac{1}{4}$. We write the solution of (4.4) as

$$\mathbf{y}(\sigma) = c_1\sigma^{\frac{1+\sqrt{1-4p}}{2}} + c_2\sigma^{\frac{1-\sqrt{1-4p}}{2}} \tag{4.5}$$

for some constant vectors $c_1, c_2 \in \mathbb{R}^2$. We have two cases:

(a) $p = -2$. Integrating (4.5) gives

$$\mathbf{x}(\sigma) = \frac{1}{3}c_1\sigma^3 + c_2\ln\sigma + c_0,$$

for a constant vector $c_0 \in \mathbb{R}^2$. The fact that $|\mathbf{x}' \ \mathbf{x}''| = 1$ for each $\sigma \in I$ implies $|c_1 \ c_2| = \frac{-1}{3}$ and hence we may set $c_0 = (0,0)$, $c_1 = (1,0)$ and $c_2 = \left(0, \frac{-1}{3}\right)$. This proves the first statement of the theorem.

(b) $p \neq -2$. Then integrating (4.5) leads to

$$\mathbf{x}(\sigma) = \frac{2}{3+\sqrt{1-4p}} c_1 \sigma^{\frac{3+\sqrt{1-4p}}{2}} + \frac{2}{3-\sqrt{1-4p}} c_2 \sigma^{\frac{3-\sqrt{1-4p}}{2}} + c_0,$$

for a constant vector $c_0 \in \mathbb{R}^2$. The condition that $|\mathbf{x}' \ \mathbf{x}''| = 1$ for each $\sigma \in I$ gives $|c_1 \ c_2| = \frac{-1}{\sqrt{1-4p}}$ and hence we may set $c_0 = (0,0)$, $c_1 = (1,0)$ and $c_2 = \left(0, \frac{-1}{\sqrt{1-4p}}\right)$, which gives the proof of the second statement of the theorem.

(2) $p = \frac{1}{4}$. Then the solution of (4.4) follows

$$\mathbf{y}(\sigma) = \sigma^{\frac{1}{2}} \left[c_1 + c_2 \ln \sigma\right], \tag{4.6}$$

for some constant vectors $c_1, c_2 \in \mathbb{R}^2$. Integrating (4.6) yields

$$\mathbf{x}(\sigma) = \frac{2}{3}\sigma^{\frac{3}{2}} c_1 + \frac{2}{9}\sigma^{\frac{3}{2}} \left(-2 + 3\ln\sigma\right) c_2 + c_0,$$

for a constant vector $c_0 \in \mathbb{R}^2$. Because $|\mathbf{x}' \ \mathbf{x}''| = 1$ for each $\sigma \in I$ we get $|c_1 \ c_2| = 1$ and may set $c_0 = (0,0)$, $c_1 = (1,0)$ and $c_2 = (0,1)$. Therefore we derive the proof of the third statement of the theorem.

(3) $p > \frac{1}{4}$. (4.4) leads to

$$\mathbf{y}(\sigma) = \sigma^{\frac{1}{2}} \left[\cos\left(\sqrt{4p-1}\ln\sigma\right) c_1 + \sin\left(\sqrt{4p-1}\ln\sigma\right) c_2\right], \tag{4.7}$$

for some constant vectors $c_1, c_2 \in \mathbb{R}^2$. By integrating (4.7) we conclude

$$\mathbf{x}(\sigma) = \tfrac{2\sigma^{\frac{3}{2}}}{5+16p} \left\{\left[3\cos\left(\sqrt{4p-1}\ln\sigma\right) + 2\sqrt{4p-1}\sin\left(\sqrt{4p-1}\ln\sigma\right)\right] c_1 - \right.$$
$$\left. \left[-2\sqrt{4p-1}\cos\left(\sqrt{4p-1}\ln\sigma\right) + 3\sin\left(\sqrt{4p-1}\ln\sigma\right)\right] c_2\right\} + c_0,$$

for a constant vector $c_0 \in \mathbb{R}^2$. Because $|\mathbf{x}' \ \mathbf{x}''| = 1$ for each $\sigma \in I$ we have $|c_1 \ c_2| = \frac{1}{\sqrt{4p-1}}$ and may set $c_0 = (0,0)$, $c_1 = (1,0)$ and $c_2 = \left(0, \frac{1}{\sqrt{4p-1}}\right)$. This completes the proof.

Example 4.1. *Let the following plane curves with prescribed equi-affine curvature be parameterized by*

(1) $\mathbf{x}(\sigma) = \frac{1}{3}\left(\sigma^3, -\ln\sigma\right)$, $\kappa_a(\sigma) = -2\sigma^{-2}$ *for* $\sigma \in \left[\frac{1}{2}, \pi\right]$,
(2) $\mathbf{x}(\sigma) = \left(\frac{2}{7}\sigma^{\frac{7}{2}}, \frac{1}{2}\sigma^{\frac{-1}{2}}\right)$, $\kappa_a(\sigma) = \frac{-15}{4}\sigma^{-2}$ *for* $\sigma \in \left[\frac{1}{2}, 1\right]$,
(3) $\mathbf{x}(\sigma) = \left(\frac{2}{3}\sigma^{\frac{3}{2}}, \frac{2}{9}\sigma^{\frac{3}{2}}\left(-2 + 3\ln\sigma\right)\right)$, $\kappa_a(\sigma) = \frac{1}{4}\sigma^{-2}$ *for* $\sigma \in \left[\frac{1}{2}, \pi\right]$,
(4) $\mathbf{x}(\sigma) = \frac{2\sigma^{\frac{3}{2}}}{13}\left(3\cos(\ln\sigma) + 2\sin(\ln\sigma), -2\cos(\ln\sigma) + 3\sin(\ln\sigma)\right)$, $\kappa_a(\sigma) = \frac{1}{2}\sigma^{-2}$ *for* $\sigma \in \left[\frac{1}{2}, \pi\right]$.

These curves can be plotted as below:

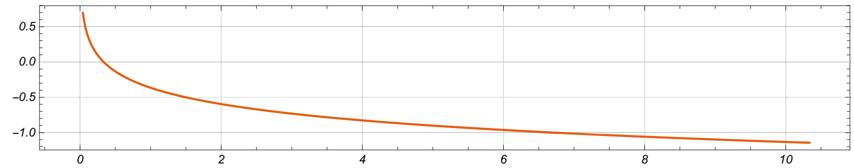

FIGURE 1. Plane curve with $\kappa_a(\sigma) = -2\sigma^{-2}$ for $\sigma \in \left[\frac{1}{2}, \pi\right]$.

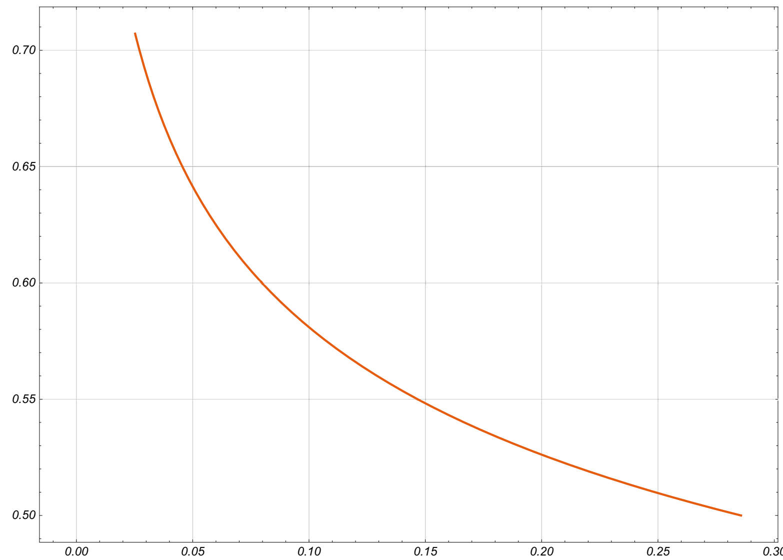

FIGURE 2. Plane curve with $\kappa_a(\sigma) = \frac{-15}{4}\sigma^{-2}$ for $\sigma \in \left[\frac{1}{2}, 1\right]$.

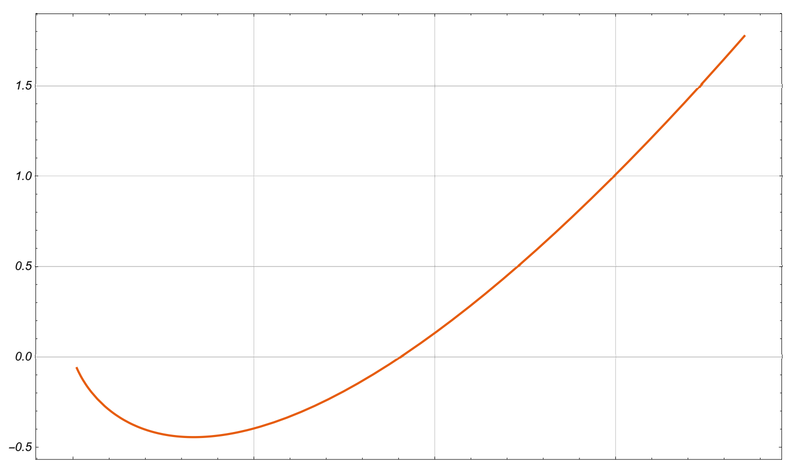

FIGURE 3. Plane curve with $\kappa_a(\sigma) = \frac{1}{4}\sigma^{-2}$ for $\sigma \in \left[\frac{1}{2}, \pi\right]$.

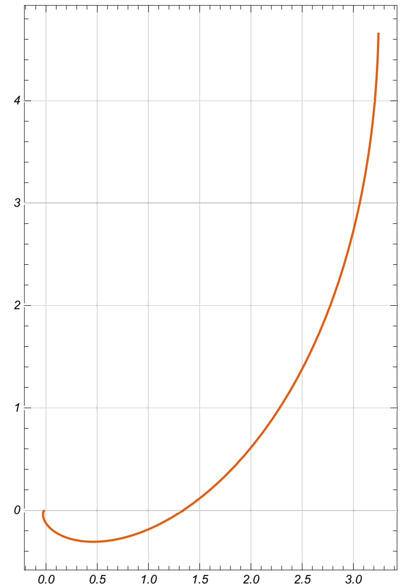

FIGURE 4. Plane curve with $\kappa_a(\sigma) = \frac{1}{2}\sigma^{-2}$ for $\sigma \in \left[\frac{1}{2}, \pi\right]$.

REFERENCES

[1] Calabi, E., Olver, P.J. and Tannenbau.(1996). A.Affine geometry, curve flows, and invariant numerical approximations, Adv. Math.124,154–196.

[2] Davis, D.(2006).Generic affine differential geometry of curves in R^n, Proc. Royal Soc. Edinburgh A 136, 1195–1205.

[3] Flash, T. and Handzel, A.(2007). Affine differential geometry analysis of human arm movements, Biol. Cybern. 96, 577-601.

[4] Gardner, R.B. and Wilkens, G.R.(1997). The fundamental theorems of curves and hypersurfaces in centroaffine geometry, Bull. Belg. Math. Soc. Simon Stevin 4, 379–401.

[5] Giblin, P.J. and Sano,T.(2012). Generic equi-centro-affine differential geometry of plane curves, Top. Appl., 159 , 476–483.

[6] Gray, A.(1998). Modern Differential Geometry of Curves and Surfaces with Mathematica, Second Ed. CRC Press.

[7] Guggenheimer, H.W.(1963). Differential Geometry, McGraw-Hill, New York.

[8] Hu, N.(2011). Centroaffine space curves with constant curvatures and homogeneous surfaces, J. Geom. 102 , 103–114.

[9] Hu, N.(2012). Affine Geometry of Space Curves and Homogeneous Surfaces, PH.D. thesis, Dept. Math., G.S. Sci., Hokkaido University, Japan.

[10] Izumiya, S. and Sano, T.(1998). Generic affine differential geometry of plane curves, Proc. Royal Soc. Edinburgh A 41 , 315–324.

[11] Kreyszig, E. and Pendl, A.(1975). Spherical curves and their analogues in affine differential geometry, Proc. Am. Math. Soc. 48 1975, 423–428.

[12] Kreyszig, E.(2006). Advanced Engineering Mathematics, John Wiley and Sons, Inc, Daners.

[13] Li, A.-M., Simon, U., Zhao, G. and Hu,Z.(2015). Global Affine Differential Geometry of Hypersurfaces, 2nd edn. W. de Gruyter, Berlin .

[14] Liu, H.(2014). Curves in affine and semi-Euclidean spaces, Results. Math. 65 , 235–249.

[15] Nomizu, K. and Sasaki, T.(1994). Affine Differential Geometry. Geometry of Affine Immersions. Cambridge Tracts in Mathematics, vol. 111. Cambridge University Press, Cambridge.

[16] Olver, P.J.(2010). Moving frames and differential invariants in centro-affine geometry, Lobachevskii J. Math. 31 , 77-89.

[17] Olszak, K. and Olszak, Z.(2017). The equi-affine and Frenet curvatures of curves in pseudo-Riemannian 2-manifolds, Colloq. Math. 150 , 103-112.

[18] Peksen, O. and Khadjev, D.(2004). On invariants of curves in centro-affine geometry, J. Math. Kyoto Univ. 44, 603–613.

[19] Pressley, A.(2010). Elemantary Differential Geometry, Berlin, Germany: Springer Verlag.

[20] Sagiroglu, Y. and Peksen, O.(2010). The equivalence of centro-equiaffine curves, Turk. J. Math. 34(1) , 95–104

[21] Sagiroglu, Y. and Peksen, O.(2010). The equivalence of equi-affine curves, Turk. J. Math. 34 , 95–104.

[22] Sapiro, G. and Tannenbaum, A.(1994). On affine plane curve evolution, J. Funct. Anal. 119 , 79-120.

[23] Tuncer, Y., Kocayigit, H. and Karacan, M.K.(2019). Naturel mates of equiaffine space curves in affine 3-space, Thermal Science . https://doi.org/10.2298/TSCI190901404T.

[24] Verpoort, S.(2011). Curvature functionals for curves in the equi-affine plane, Czechoslovak Math. J. 61(2) , 419–435.

[25] Yang,Y. and Yu, Y.(2020). Moving frames and differential invariants on fully affine planar curves, Bull. Malays. Math. Sci. Soc. 43 , 3229-3258.

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