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TRANSLATION-FACTORABLE SURFACES WITH VANISHING CURVATURES IN GALILEAN 3-SPACE

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Abstract. In this paper, we define two types of Translation-Factorable (TF-) surfaces in the Galilean 3-space. Then, we obtain the complete classification of these surfaces with vanishing Gaussian curvature and mean curvature and also, we give some explicit graphics of these surfaces.

Keywords: Flat surfaces, minimal surfaces, translation surface, factorable surfaces, Galilean space.

2010 Mathematics Subject Classification: 53A35, 53A40.

1. Introduction

The history of the view of what constitutes geometry has been changed radically on a number of occasions. For centuries, it was thought that the single aim of geometry is the through investigation of the properties of ordinary 3-dimensional Euclidean space. That view was broadened by Gauss in 1816, by Bolyai 1824 and by Lobachevski in 1826, independently. Furthermore, the explorations and views of Riemann [15] and Klein [11] being a synthesis of the geometric views of Cayley showed that there exist other (non-Euclidean) geometric systems. Until this time, many technical and popular resources have been written about the geometry of non-Euclidean space. Among these space, there are Minkowski space [13], Galilean and pseudo-Galilean space [1, 10, 14, 16] and so on.

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It should be noted that one of the pioneer books of the Galilean geometry is the Yaglom's book [19]. In that book was discussed on the physical basis of this geometry closely related with Galilean's principle of relativity, i.e., Newtonian mechanics. In the last decade, the Galilean and pseudo-Galilean space were used by several researchers as an ambient space for the well-known Euclidean concepts (see in [3, 4, 5, 6, 7, 9, 12, 17, 18]).

In this paper, we first introduce the notations that we are going to use and give a brief summary of basic definitions in theory of surfaces in Galilean 3-space. Then, we define two types of Translation-Factorable (TF-) surfaces in Galilean 3-space, by considering the definition of these surfaces given in [8] in Euclidean and Lorentzian 3-space. Also, we give the complete classification of such surfaces with vanishing Gaussian curvature and mean curvature and also some explicit graphics of them.

2. Preliminaries

First, we would like to give a brief summary of basic definitions, facts and equations in the theory of surfaces of Galilean 3-space (see for detail, [14, 16, 19]).

The Galilean 3−space $\mathbb{G}^3$ arises in a Cayley-Klein way by pointing out an absolute figure $\{\omega, f, J\}$ in the 3-dimensional real projective space $\mathbb{P}_3(\mathbb{R})$ where ω is the ideal (absolute) plane, f is the absolute line and J is the fixed elliptic involution of points of f. Then the homogeneous coordinates $(x_0 : x_1 : x_2 : x_3)$ are introduced such that ω is given by $x_0 = 0$, f is given by $x_0 = x_1 = 0$ and J, by $(0 : 0 : x_2 : x_3) \mapsto (0 : 0 : -x_3 : x_2)$.

In affine coordinates defined by $(x_0 : x_1 : x_2 : x_3) = (1 : x_1 : x_2 : x_3)$, the distance between two points $P_i = (x_i, y_i, z_i)$ with $i \in \{1, 2\}$ is defined by the formula

$$d_{P_1P_2} = \begin{cases} |x_2 - x_1| & \text{if} \quad x_1 \neq x_2, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2} & \text{if} \quad x_1 = x_2 \end{cases}$$

The group of motions of $\mathbb{G}^3$ is a six-parameter group. Regarding this group of motions, except the absolute plane, there exist two classes of planes in $\mathbb{G}^3$: Euclidean planes that contain f where the induced metric is Euclidean and isotropic planes that do not contain f whose induced metric is isotropic. Also, there are four types of lines in $\mathbb{G}^3$: isotropic lines that intersect f, non-isotropic lines that do not intersect f, non-isotropic lines in ω and the absolute line f, [6].

Let $\vec{X} = (x_1, x_2, x_3)$ be a vector in $\mathbb{G}^3$. If $x_1 = 0$, then $\vec{X}$ is called as isotropic; otherwise, it is said to be non-isotropic. Note that, the x_1-axis is non-isotropic while the x_2-axis and the x_3-axis are isotropic, for standart coordinates (x_1, x_2, x_3). Moreover, a plane of the form

$x_1 = const.$ is called an Euclidean plane, otherwise isotropic. For two vectors $\vec{X} = (x_1, x_2, x_3)$ and $\vec{Y} = (y_1, y_2, y_3)$, the Galilean scalar product is given by

$$\langle X, Y \rangle = \begin{cases} x_1 y_1 & \text{if} \quad x_1 \neq 0 \quad \text{or} \quad y_1 \neq 0, \\ x_2 y_2 + x_3 y_3 & \text{if} \quad x_1 = y_1 = 0. \end{cases}$$

The norm of vector $\vec{X}$ in $\mathbb{G}^3$ is defined by $\left\|\vec{X}\right\| := \sqrt{\left\langle \vec{X}, \vec{X} \right\rangle}$. If $\left\|\vec{X}\right\| = 1$, then $\vec{X}$ is called as unit vector. Also, the Galilean cross product of the vectors $\vec{X}$ and $\vec{Y}$ of which at least one is non-isotropic is defined by

$$\vec{X} \times \vec{Y} = \Big(0, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1\Big). \tag{2.1}$$

Assume that U is an open set of $\mathbb{R}^2$ and S is a C^r-surface such that $r \geq 2$, immersed in $\mathbb{G}^3$ parametrized by

$$\varphi : U \to \mathbb{R}^2, \quad \varphi(u_1, u_2) = \left(\varphi_1(u_1, u_2), \varphi_2(u_1, u_2), \varphi_3(u_1, u_2)\right). \tag{2.2}$$

Let us denote $\dfrac{\partial \varphi}{\partial u_i} = \varphi_{,i}$, $\dfrac{\partial \varphi_k}{\partial u_i} = (\varphi_k)_{,i}$ and $\dfrac{\partial^2 \varphi_k}{\partial u_i \partial u_j} = (\varphi_k)_{,ij}$ where $1 \leq k \leq 3$ and $1 \leq i, j \leq 2$. Then a surface is admissible (i.e., without Euclidean tangent planes) if and only if $(\varphi_1)_{,i} \neq 0$ for some $i = 1, 2$. Let $S \subset \mathbb{G}^3$ be a regular admissible surface. We define the side tangential vector field by

$$\sigma = \frac{(\varphi_1)_{,1}\varphi_{,2} - (\varphi_1)_{,2}\varphi_{,1}}{W} \tag{2.3}$$

and a unit normal vector N as

$$N = \frac{\varphi_{,1} \times \varphi_{,2}}{W} \tag{2.4}$$

where the function $W = \|\varphi_{,1} \times \varphi_{,2}\|$, [17].

Now, we introduce the coefficients of the second fundamental form

$$L_{ij} = \left\langle \frac{\varphi_{,ij}(\varphi_1)_{,1} - (\varphi_1)_{,ij}\varphi_{,1}}{(\varphi_1)_{,1}}, N \right\rangle = \left\langle \frac{\varphi_{,ij}(\varphi_1)_{,2} - (\varphi_1)_{,ij}\varphi_{,2}}{(\varphi_1)_{,2}}, N \right\rangle. \tag{2.5}$$

Consequently, the Gaussian curvature K and the mean curvature H of M are defined by

$$K = \frac{L_{11}L_{22} - L_{12}^2}{W^2}, \tag{2.6}$$

$$H = \frac{1}{2}\sum_{i,j=1}^{2} g^{ij} L_{ij}, \tag{2.7}$$

where

$$g^1 = \frac{(\varphi_1)_{,2}}{W}, \quad g^2 = \frac{(\varphi_1)_{,1}}{W} \quad \textit{and} \quad g^{ij} = g^i g^j \quad \textit{for} \quad i, j = 1, 2. \tag{2.8}$$

Note that, if M has zero curvatures, i.e., $K = 0$ or $H = 0$, then it is called as flat or minimal, respectively.

2.1. **Translation-Factorable Surfaces in Galilean 3-space.** In this section, we first would like to state the following definitions given in [3, 4, 17] :

Definition 2.1. *Let M^2 be an admissible surface in Galilean space. Then M is called a factorable surface if it can be locally written as one of the following:*

$$x(s,t) = (s, t, f(s)g(t)), \tag{2.9}$$

or

$$x(s,t) = (f(s)g(t), s, t), \tag{2.10}$$

which are called as first and second kind, respectively. Here f, g are smooth functions of one variable.

Definition 2.2. *Let M^2 be an admissible surface in Galilean space. Then M is called a translation surface if it can be locally written as one of the following:*

$$x(s,t) = (s, t, f(s) + g(t)), \tag{2.11}$$

or

$$x(s,t) = (f(s) + g(t), s, t), \tag{2.12}$$

which are called as first and second kind, respectively. Here f, g are smooth functions of one variable.

Note that as can be seen from the definitions of translation surfaces or factorable surfaces given above, there exist some distinction into two types of them coming from the fact that the x-direction and another direction in the yz-plane play distinct roles due to the degeneracy of the metric. Now by considering these definitions, we would like to give the definition of translation-factorable (TF-) surface in Galilean 3-space, firstly defined in [8] in Euclidean and Lorentzian 3-space, as follows:

Definition 2.3. *Let M^2 be an admissible surface in Galilean 3-space. Then M is called a translation-factorable (TF-) surface if it can be locally written as one of the following:*

$$\varphi(s,t) = (s, t, Bf(s)g(t) + A(f(s) + g(t))), \tag{2.13}$$

or

$$\varphi(s,t) = (Bf(s)g(t) + A(f(s) + g(t)), s, t)\,, \tag{2.14}$$

which are called as first and second type, respectively. Here f and g are real functions and A, B are non-zero constants.

Remark 2.1. *From Definition 2.3, one can observe that the surface M given by* (2.13) *and* (2.14) *becomes a factorable surface when $A = 0$, $B \neq 0$. Similarly, if one takes $B = 0$ and $A \neq 0$, then surface is a translation surface.*

Hence, we are going to consider the case $AB \neq 0$.

3. Classification of Translation-Factorable surfaces with vanishing curvature in $\mathbb{G}^3$

In this section, we obtain the Gaussian and the mean curvature of TF-surfaces in $\mathbb{G}^3$. Then, we obtain the complete classification of flat and minimal TF-surfaces.

3.1. **Type I TF-surfaces with zero curvature.** Let M^2 be a type I TF-surface in $\mathbb{G}^3$ given by (2.13). Then, we have

$$\varphi_s = (1, 0, (Bg(t) + A)f'(s)), \tag{3.15}$$

$$\varphi_t = (0, 1, g'(t)(Bf(s) + A)). \tag{3.16}$$

In addition by using (2.4), we obtain

$$N = \frac{1}{\sqrt{1 + g'(t)^2(Bf(s) + A)^2}}(0, -g'(t)(Bf(s) + A), 1). \tag{3.17}$$

Here by $'$, we have denoted derivatives with respect to corresponding parameters. For readability, here and in the rest of the paper, we will drop the explicit dependence of the functions on the variables and simply write $f = f(s)$ and $g = g(t)$. Now, by combining (3.15)-(3.17) with (2.5) and (2.8), respectively, we get

$$L_{11} = \frac{f''(Bg + A)}{W}, \quad L_{12} = \frac{Bf'g'}{W}, \quad L_{22} = \frac{g''(Bf + A)}{W}, \tag{3.18}$$

and

$$g^{11} = 0, \quad g^{12} = 0, \quad g^{22} = \frac{1}{W^2}, \tag{3.19}$$

where $W^2 = 1 + g'(t)^2(Bf(s) + A)^2$. Consequently, (2.6) and (2.7) give

$$K = \frac{f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2}{(1 + g'^2(Bf + A)^2)}, \tag{3.20}$$

$$H = \frac{g''(Bf + A)}{2(1 + g'^2(Bf + A)^2)^{3/2}}, \tag{3.21}$$

respectively.

Now, we would like to investigate the vanishing curvature problem for TF-surfaces. First, we examine a type I TF-surface in Galilean 3-space, whose Gaussian curvature is identically zero.

Theorem 3.1. *Let M^2 be a type I TF-surface defined by* (2.13) *in the Galilean 3-space. Then, M^2 is a flat surface if and only if it belongs to one of the following families:*

(1) *M^2 is a part of an isotropic plane,*

(2) *M^2 is an admissible cylindrical surface in $\mathbb{G}^3$ parametrized by*

$$\varphi(s,t) = (s, t, C_1 g(t) + C_2), \tag{3.22}$$

where C_1, C_2 are non-zero constant and g is arbitrary function or

$$\varphi(s,t) = (s, t, C_1 f(s) + C_2) \tag{3.23}$$

where C_1, C_2 are non-zero constant and f is arbitrary function.

(3) *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}. \tag{3.24}$$

(4) *f and g are given by*

$$f(s) = -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((1 - C)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \quad g(t) = -\frac{A}{B} + B^{\frac{1}{C-1}}\Big((1 - \frac{1}{C})(c_1 t + c_2)\Big)^{\frac{C}{C-1}}, \tag{3.25}$$

where $C \neq 1$ is non-zero constant.

Proof. Let M^2 be a type I TF-flat surface. Thus, from (3.20), we have

$$f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2 = 0. \tag{3.26}$$

Let us consider on the following possibilities:

Case (1): $f' = 0$ and $g' = 0$. Then, the equation (3.26) is trivially satisfied. By considering these assumptions in (2.13), respectively, we obtain M^2 is an open part of plane. Thus, we have Case (1) of Theorem 3.1.

Case (2): Either $f' = 0$ or $g' = 0$. First, assume that $f' = 0$, i.e., f be constant. In case, the equation (3.26) is trivially satisfied. But, in case g is a arbitrary smooth function. Thus, we get (3.22). Similarly, by considering the assumption of g as $g' = 0$, we can get (3.23) in Theorem 3.1.

Case (3): Let $f'' = 0$ or $g'' = 0$, but not both. First, assume that $f'' = 0$, i.e., f be a linear function. In this case, one get $g' = 0$ to provide the equation (3.26). Second, let $g'' = 0$. Then by the similar way, $f' = 0$ must be. Note that one can easily see that these cases are covered by Case (2).

Case (4): Let f', g', f'' and g'' be non-zero. Then, the equation (3.26) can be rewritten as

$$\frac{f''(A+Bf)}{B(f')^2} = \frac{B(g')^2}{g''(A+Bg)} = C, \tag{3.27}$$

for non-zero constant C. We are going to consider the following cases seperately:

Case (4a): $C = 1$. In this case (3.27) implies that

$$f''(A+Bf) = B(f')^2 \quad \text{and} \quad B(g')^2 = g''(A+Bg), \tag{3.28}$$

from which, we get (3.24) in Case (3) in Theorem 3.1.

Case (4b): $C \neq 1$. In this case we solve (3.27) to obtain (3.25).

Conversely, a direct computation yields that the Gaussian curvature of each of surfaces given in Theorem 3.1 vanishes identically.

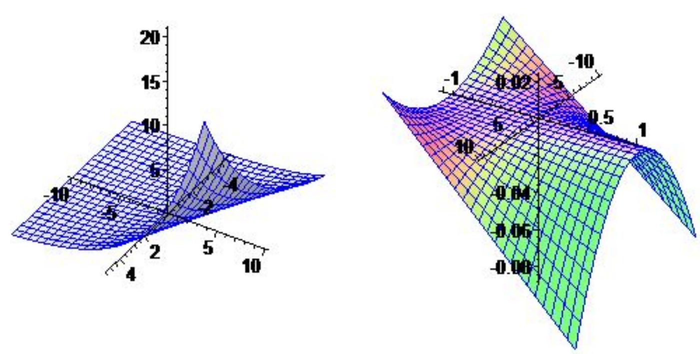

FIGURE 1. A type I TF-flat surfaces parametrized by (3.24) and (3.25), respectively.

Now, we examine a type I TF-surface in Galilean 3-space, whose mean curvature is identically zero.

Theorem 3.2. *Let M^2 be a type I TF-surface defined by* (2.13) *in the Galilean 3-space. Then, M^2 is a minimal surface if and only if it is either*

(1) *an open part of the plane* $z = -\frac{A^2}{B}$ *or*

(2) *a ruled surface of type* C *in* $\mathbb{G}^3$ *parametrized by*

$$\varphi(s,t) = (s, 0, H(s))) + t\,(0, 1, F(s)) \tag{3.29}$$

where $F(s) = a(Bf + A)$ *and* $H(s) = Af + b(A + B)$.

Proof. Let M^2 be a type I TF-minimal surface. Thus, from (3.21), it is clear that is sufficient that

$$g''(Bf + A) = 0. \tag{3.30}$$

Let us consider on the following possibilities:

Case (1): $f = -\frac{A}{B}$. Then the surface given in (2.13) can be reparametrized as $\varphi(s,t) = \left(s, t, -\frac{A^2}{B}\right)$ which is an open part of the plane $z = -\frac{A^2}{B}$. Thus, we have Case (1) of Theorem 3.2.

Case (2): $g'' = 0$. Then, the function $g(t)$ is a linear function, i.e., $g(t) = at + b, a, b \in \mathbb{R}$. Then the surface given in (2.13) can be parametrized as in (3.29).

The converse follows from a direct computation.

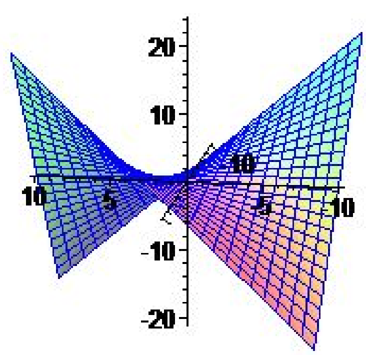

FIGURE 2. A type I TF-minimal surfaces parametrized by (3.29).

Now, we would like to do the calculations for the second type TF-surfaces.

3.2. **Type II TF-surfaces with zero curvature.** Let M^2 be an admissible type II TF-surface in $\mathbb{G}^3$ given by (2.14). Then, we have

$$\varphi_s = ((Bg + A)f', 1, 0), \tag{3.31}$$

$$\varphi_t = (g'(Bf + A), 0, 1). \tag{3.32}$$

Moreover, by substituting these into (2.4) we obtain

$$N = \frac{1}{\sqrt{f'^2(Bg + A)^2 + g'^2(Bf + A)^2}}(0, -f'(Bg + A), -g'(Bf + A)). \tag{3.33}$$

Here by $'$, we have denoted derivatives with respect to corresponding parameters. Now, by combining the above with (2.5) and (2.8), respectively, we get

$$L_{11} = \frac{f''(Bg + A)}{W}, \quad L_{12} = \frac{Bf'g'}{W}, \quad L_{22} = \frac{g''(Bf + A)}{W}, \tag{3.34}$$

and

$$g^{11} = \frac{(g')^2(Bf + A)^2}{W^2}, \quad g^{12} = \frac{f'g'(Bf + A)(Bg + A)}{W^2}, \quad g^{22} = \frac{(f')^2(Bg + A)^2}{W^2}, \tag{3.35}$$

where $W = \sqrt{f'^2(Bg + A)^2 + g'^2(Bf + A)^2}$. Consequently, (2.6) and (2.7) give

$$K = \frac{f''g''(Bf + A)(Bg + A) - B^2(f')^2(g')^2}{(f'^2(Bg + A)^2 + g'^2(Bf + A)^2)} \tag{3.36}$$

and

$$H = \frac{f''g'^2(Bf + A)^2(Bg + A) + g''f'^2(Bg + A)^2(Bf + A) - 2Bf'^2g'^2(Bf + A)(Bg + A)}{2(f'^2(Bg + A)^2 + g'^2(Bf + A)^2)^{3/2}}, \tag{3.37}$$

respectively.

Remark 3.1. *By comparing Eq. (3.20) and Eq. (3.36) implies that the Gaussian curvatures of type I and type II of TF-surfaces in Galilean 3-space seem to be really very similar. Thus the following classification of type II TF- flat surfaces can be proved as in Theorem 3.1.*

Theorem 3.3. *Let M^2 be a type II TF-surface defined by* (2.14) *in the Galilean 3-space. Then, M^2 is a flat surface if and only if it belongs to one of the following families:*

(1) *M^2 is a part of an isotropic plane,*
(2) *M^2 is an admissible surface in $\mathbb{G}^3$ parametrized by*

$$\varphi(s, t) = (C_1 g(t) + C_2, s, t), \tag{3.38}$$

where C_1, C_2 are non-zero constant or

$$\varphi(s, t) = (C_1 f(s) + C_2, s, t) \tag{3.39}$$

where C_1, C_2 are non-zero constant.

(3) *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}. \tag{3.40}$$

(4) *f and g are given by*

$$f(s) = -\frac{A}{B} + B^{\frac{C}{C-1}}\Big((1-C)(c_1 s + c_2)\Big)^{\frac{1}{1-C}}, \quad g(t) = -\frac{A}{B} + B^{\frac{1}{C-1}}\Big((1-\frac{1}{C})(c_1 t + c_2)\Big)^{\frac{C}{C-1}}, \tag{3.41}$$

where $C \neq 1$ and c_1, c_2 are non-zero constant.

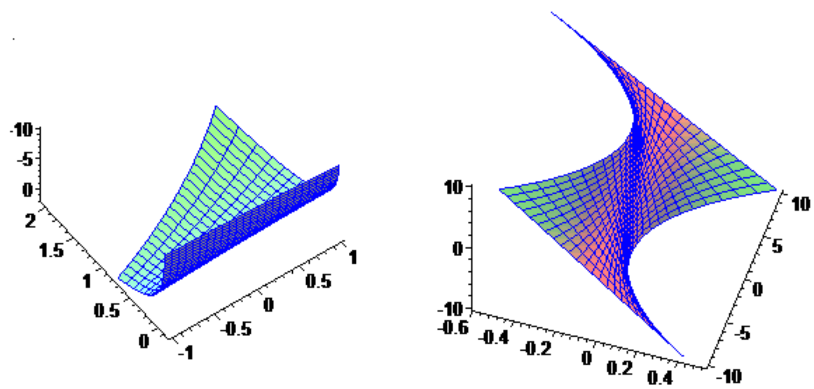

FIGURE 3. A type II TF-flat surfaces parametrized by (3.40) and (3.41), respectively.

Finally, we would like to give the following classification theorem for a type II TF- minimal surface:

Theorem 3.4. *Let M^2 be a type II TF-surface defined by (2.14) in the Galilean 3-space. Then, M^2 is a minimal surface if and only if it belongs to one of the following families:*

(1) *M^2 is an open part of plane,*

(2) *M^2 is an admissible surface in $\mathbb{G}^3$ parametrized by*

$$\varphi(s,t) = (s, t, C_1 g(t) + C_2), \tag{3.42}$$

where C_1, C_2 are non-zero constant and g is arbitrary function or

$$\varphi(s,t) = (s, t, C_1 f(s) + C_2) \tag{3.43}$$

where C_1, C_2 are non-zero constant and f is arbitrary function.

(3) *f and g are given by*

$$f(s) = -\frac{1}{B}e^{B(c_1 s + c_2)} + \frac{A}{B}, \quad g(t) = -\frac{1}{B}e^{B(c_1 t + c_2)} + \frac{A}{B}, \tag{3.44}$$

(4) *f and g are given by either*

(a)

$$f(s)=-\frac{A}{B}+B^{\frac{C}{1-C}}\Big((1-C)(c_1s+c_2)\Big)^{\frac{1}{1-C}},\quad g(t)=-\frac{A}{B}+B^{\frac{2-C}{C-1}}\Big((C-1)(c_1t+c_2)\Big)^{\frac{1}{C-1}},\tag{3.45}$$

or

(b)

$$f(s)=-\frac{A}{B}+B^{\frac{2-C}{C-1}}\Big((C-1)(c_1s+c_2)\Big)^{\frac{1}{C-1}},\quad g(t)=-\frac{A}{B}+B^{\frac{C}{1-C}}\Big((1-C)(c_1t+c_2)\Big)^{\frac{1}{1-C}},\tag{3.46}$$

where c_1, c_2 are non-zero constant and $C\neq 1$.

Proof. Let M^2 be a type II TF-minimal surface. Thus, from (3.37), we have

$$f''{g'}^2(Bf+A)^2(Bg+A)+g''{f'}^2(Bg+A)^2(Bf+A)-2B{f'}^2{g'}^2(Bf+A)(Bg+A)=0.\tag{3.47}$$

Let us consider on the following possibilities:

Case (1): $f'=0$ and $g'=0$. Then, the equation (3.26) is trivially satisfied. By considering these assumptions in (2.14), respectively, we obtain M^2 is an open part of plane. Thus, we have Case (1) of Theorem 3.4.

Case (2): Either $f'=0$ or $g'=0$. First, assume that $f'=0$, i.e., f be constant. In case, the equation (3.30) is trivially satisfied. But, in case g is a arbitrary smooth function. Thus, we have (3.42) in Case (2) of Theorem 3.4. Similarly, by considering the assumption of g as $g'=0$, we can get (3.43) in Case (2) of Theorem 3.4.

Case (3): Let $f''=0$ or $g''=0$, but not both. First, assume that $f''=0$ and $g''\neq 0$. Hence, by considering this assumption in (3.47) yields

$$g''(Bg+A)^2-2B{g'}^2(Bg+A)=0,\tag{3.48}$$

from which we have two possibilities; $g=-\frac{A}{B}$ or

$$g''(Bg+A)-2B{g'}^2=0$$

is valid. But the first statement contradicts with the hypothesis. Hence, we will only deal with the second statement, whose solution is $g(t)=-\dfrac{1}{B^2(c_1t+c_2)}-\dfrac{A}{B}$ where $c_1\neq 0$. Thus, the surface is covered by in Case (4a) in Theorem 3.4 taking $C=0$.

Second, let $g''=0$. Thus, the surface is covered in exactly the same way as in the previous case, as in Case (4b) in Theorem 3.4.

Case (3): f', g' and both f'' and g'' be non-zero. Then, the equation (3.47) can be rewritten as

$$\frac{f''(A+Bf)}{B(f')^2}+\frac{g''(A+Bg)}{B(g')^2}=2. \tag{3.49}$$

Now, we are going to consider the following cases seperately:

Case (3a): $\dfrac{f''(A+Bf)}{B(f')^2}=1$ and $\dfrac{g''(A+Bg)}{B(g')^2}=1$. From there, we solve these equations to find (3.44) in Case (3) Theorem 3.4.

Case (3b): Let $\dfrac{f''(A+Bf)}{B(f')^2}=C\neq 1$. From (3.49), one gets $\dfrac{g''(A+Bg)}{B(g')^2}=2-C$. By solving these ODEs, we obtain the functions f,g given in (3.45).

Case (3c): Let $\dfrac{g''(A+Bg)}{B(g')^2}=C\neq 1$. Similarly, one gets $\dfrac{f''(A+Bf)}{B(f')^2}=2-C$. Thus, we obtain the functions f,g given in (3.46).

Conversely, a direct computation yields that the Gaussian curvature of each of surfaces given in Theorem 3.4 vanishes identically.

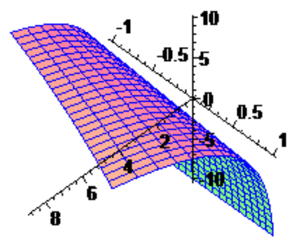

FIGURE 4. A type II TF-minimal surfaces parametrized by (3.44).

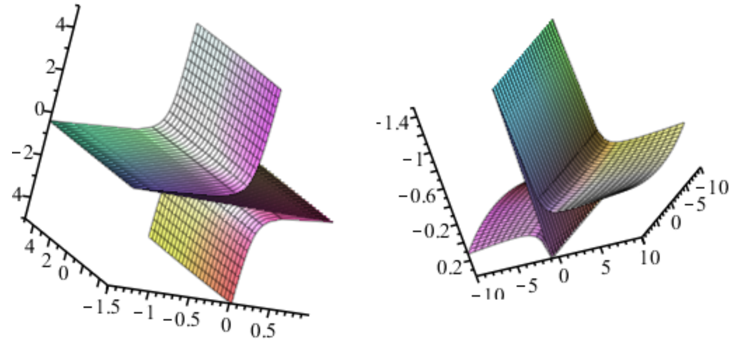

FIGURE 5. A type II TF-minimal surfaces parametrized by (3.45) and (3.46).

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