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GENERALIZED TANAKA-WEBSTER CONNECTION ON β-KENMOTSU MANIFOLDS

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Abstract. This research paper aims to study the postulates of the generalized Tanaka-Webster connection (briefly, gTWc) on β-Kenmotsu manifolds. We find the curvature properties of a β-Kenmotsu manifold concerning gTWc, and studied the conditions for the ϕ-projectively flat, ϕ-conformally flat and ϕ-concircularly flat β-Kenmotsu manifolds along with the same connection. Also, we have discussed the ξ-flat properties on same curvatures for the β-Kenmotsu manifold admitting gTWc. At the end we provide an example to verify some of our results.

Keywords: β-Kenmotsu manifold, generalized Tanaka-Webster connection, curvature tensor, η-Einstein manifold.

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1. Introduction

The generalized Tanaka-Webster connection (gTWc) is a canonical affine connection defined on a non-degenerated pseudo-Hermitian CR-manifold. The gTWc was introduced by Tanno [23] as a generalization of the connections defined at the end of 1976 by Tanaka in [22] and Webster in [25]. These connections coincide with the Tanaka-Webster connection (TWc) if the associated CR-structure is integrable. Many geometers studied some characterizations of the gTWc on various manifolds. Recently, S.Y. Perktas et al. [18], Ghosh and De [5, 7],

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Gautam et al. [6], Ayar and Cavusoglu [2], and many others have studied the properties of this connection on distinct structures. Also, see [12, 16].

Kenmotsu [13], introduced a new class of almost contact Riemannian manifolds, known as the Kenmotsu manifold. As it is well known, odd-dimensional spheres permit Sasakian structures, but odd-dimensional hyperbolic spaces do not admit Sasakian structures but do have Kenmotsu structures. Kenmotsu manifolds are normal almost contact Riemannian manifolds. The basic fundamental properties of the local structure of such manifolds were investigated by many geometers. In general, the Kenmotsu manifolds are locally isometric to warped product spaces with one-dimensional bases. Oubina [17] introduced the notion of trans-Sasakian manifolds of type (α, β), which is the generalization of Kenmotsu manifolds and Sasakian manifolds, and are closely related to the locally conformal Kähler manifolds. A trans-Sasakian manifold of type $(0, 0)$, $(\alpha, 0)$ and $(0, \beta)$ are, respectively called, the cosympletic, α-Sasakian and β-Kenmotsu manifold, where α and β be some scalar functions. In particular, if $\alpha = 0$, $\beta = 1$; $\alpha = 0$, β is non-zero constant and $\alpha = 1$, $\beta = 0$ then a trans Sasakian manifold will be a Kenmotsu; homothetic Kenmotsu manifold and Sasakian manifold, respectively. β-Kenmotsu manifolds have been studied by several authors, like Bozdag et al. [3], Hui and Chakraborty [11], Kumar [15], Shaikh and Hui [19] and Mobin et al.[1]. We recommend the papers [8, 9, 10, 20, 21, 24] for more related stidies and references therein.

2. Preliminaries

In this section, we review basic definitions and results that are needed to state and prove our results.

A $(2n+1)$-dimensional smooth differentiable manifold $\mathcal{M}$ is said to be an almost contact metric structure (ϕ, ξ, η, g) if the following conditions are satisfying

$$\phi^2 X = -X + \eta(X)\xi, \ \ \eta(\xi) = 1, \ \ \phi\xi = 0, \ \ \eta \circ \phi = 0, \tag{2.1}$$

$$g(X, Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

$$g(X, \phi Y) = -g(\phi X, Y), \tag{2.3}$$

$$g(X, \xi) = \eta(X) \tag{2.4}$$

for all X, Y, Z on $\mathcal{M}$, where ϕ is a $(1, 1)$-tensor field, ξ is a vector field and η is a 1-form. An almost metric manifold $\mathcal{M}$ is said to be a β-Kenmotsu manifold if it satisfies

$$(\nabla_X \phi)Y = \beta[g(\phi X, Y)\xi - \eta(Y)\phi X], \tag{2.5}$$

$$(\nabla_X \eta)Y = \beta[g(X,Y) - \eta(X)\eta(Y)], \tag{2.6}$$

$$\nabla_X \xi = \beta[X - \eta(X)\xi], \tag{2.7}$$

where ∇ is a Levi-Civita connection.

If $\beta = 1$, then $\mathcal{M}$ is called a Kenmotsu manifold, and if β is constant then $\mathcal{M}$ are named homothetic Kenmotsu manifolds and provide a large variety of Kenmotsu manifolds. In a β-Kenmotsu manifold $\mathcal{M}$, the following relations hold:

$$\mathcal{R}(X,Y)\xi = -\beta^2[\eta(Y)X - \eta(X)Y] + (X\beta)\{Y - \eta(Y)\xi\} - (Y\beta)\{X - \eta(X)\xi\},$$

$$\mathcal{R}(\xi,X)Y = \{\beta^2 + \xi\beta\}[\eta(Y)X - g(X,Y)\xi],$$

$$Ric(X,\xi) = -\{2n\beta^2 + \xi\beta\}\eta(X) - (2n-1)(X\beta),$$

$$Ric(\phi X,\phi Y) = Ric(X,Y) + \{2n\beta^2 + \xi\beta\}\eta(X)\eta(Y) + (2n-1)(X\beta)\eta(Y), \tag{2.8}$$

where $X(\beta) = g(X, D\beta)$, D is the gradient operator of g.
An $\mathcal{M}$ is said to be η-Einstein if its Ricci tensor $Ric(\neq 0)$ satisfies

$$Ric(X,Y) = ag(X,Y) + b\eta(X)\eta(Y),$$

for any vector fields X and Y on $\mathcal{M}$, where a and b are smooth functions on $\mathcal{M}$.

The gTWc $\hat{\nabla}$ for a contact metric manifold $\mathcal{M}$ is given by [23],

$$\hat{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi + \eta(X)\phi Y \tag{2.9}$$

for all X, Y on $\mathcal{M}$.

3. β-Kenmotsu Manifolds concerning $\hat{\nabla}$

In this section, we prove that the gTWc $\hat{\nabla}$ is a metric connection; and moreover, we obtain an expression of the torsion tensor $\hat{T}$ on the manifold.

Let $\mathcal{M}$ be a $(2n+1)$-dimensional β-Kenmotsu manifold. The gTWc $\hat{\nabla}$ on an $\mathcal{M}$ is given by

$$\hat{\nabla}_X Y = \nabla_X Y - \beta\eta(Y)X + \beta g(X,Y)\xi + \eta(X)\phi Y, \tag{3.10}$$

where (2.6),(2.7) and (2.9) being used.

Now putting $Y = \xi$ in (3.10) and using (2.1), (2.2) and (2.4), we get

$$\hat{\nabla}_X \xi = 0. \tag{3.11}$$

From (2.9) and (2.3), we find

$$(\hat{\nabla}_X \eta)Y = 0. \tag{3.12}$$

Also, from (2.9) and (2.5), we find

$$(\hat{\nabla}_X g)(Y, Z) = 0. \tag{3.13}$$

Thus, in the view of (3.11), (3.12) and (3.13), we can state the following:

Proposition 3.1. *In an $\mathcal{M}$, ξ and η are parallel with respect to $\hat{\nabla}$, which is a metric connection.*

Proposition 3.2. *In an $\mathcal{M}$, the integral curves of a vector field ξ are geodesic concerning the gTWc $\hat{\nabla}$.*

Now, since the connection $\hat{\nabla}$ is metric, so the torsion tensor $\hat{T}$ of $\hat{\nabla}$ is given by

$$\hat{T}(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X. \tag{3.14}$$

From (3.10) and (3.14), we get

$$\hat{T}(X, Y) = \beta\{\eta(X)Y - \eta(Y)X\} + \eta(X)\phi Y - \eta(Y)\phi X. \tag{3.15}$$

Since, we know

$$\begin{aligned} g(\hat{\nabla}_X Y, Z) &= g(\nabla_X Y, Z) + \frac{1}{2}[g(\hat{T}(X, Y), Z) - g(\hat{T}(X, Z), Y) \\ &\quad - g(\hat{T}(Y, Z), X)]. \end{aligned} \tag{3.16}$$

Using (3.15) in (3.16), we get (3.10). Hence, we can state:

Theorem 3.1. *The gTWc $\hat{\nabla}$ associated with the connection ∇ is a unique affine connection, which is metric and its torsion is of the form $\hat{T}(X, Y) = \beta\{\eta(X)Y - \eta(Y)X\} + \eta(X)\phi Y - \eta(Y)\phi X$.*

4. Curvature properties of β-Kenmotsu manifolds concerning $\hat{\nabla}$

In the currect section, we establish the relationships between R and $\hat{\mathcal{R}}$; Ric and $\hat{Ric}$; and s and $\hat{s}$ with respect to ∇ and $\hat{\nabla}$.

The Riemannian curvature tensor with respect to $\hat{\nabla}$ on $\mathcal{M}$ is given by

$$\hat{\mathcal{R}}(X, Y)Z = \hat{\nabla}_X \hat{\nabla}_Y Z - \hat{\nabla}_Y \hat{\nabla}_X Z - \hat{\nabla}_{[X,Y]} Z. \tag{4.17}$$

By using (3.10), (4.17) takes the form

$$\begin{aligned}\hat{\mathcal{R}}(X,Y)Z &= \mathcal{R}(X,Y)Z + X(\beta)\{g(Y,Z)\xi - \eta(Z)Y\} \\ &\quad -Y(\beta)\{g(X,Z)\xi - \eta(Z)X\} + \beta^2\{g(Y,Z)X - g(X,Z)Y\},\end{aligned} \tag{4.18}$$

where $\mathcal{R}(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$.

The inner product of (4.18) with W yields

$$\begin{aligned}\hat{\mathcal{R}}(X,Y,Z,W) &= \mathcal{R}(X,Y,Z,W) + X(\beta)\{g(Y,Z)\eta(W) - \eta(Z)g(Y,W)\} \\ &\quad -Y(\beta)\{g(X,Z)\eta(W) - \eta(Z)g(X,W)\} \\ &\quad +\beta^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\},\end{aligned} \tag{4.19}$$

where $\hat{\mathcal{R}}(X,Y,Z,W) = g(\hat{\mathcal{R}}(X,Y)Z,W)$.

Let $\{e_i,\xi\}_{i=1}^{2n+1}$ be the set of orthonormal basis of tangent space at each point of the manifold, then contracting (4.19) over X and W, we get

$$\hat{Ric}(Y,Z) = Ric(Y,Z) + 2n\beta^2 g(Y,Z). \tag{4.20}$$

From (4.20) it follows that

$$\hat{\mathcal{Q}}Z = \mathcal{Q}Z + 2n\beta^2 Z, \tag{4.21}$$

where $\hat{Ric}(Y,Z) = g(\hat{\mathcal{Q}}Y,Z)$.

Also, the scalar curvature $\hat{s}$ is given by,

$$\hat{s} = s + 2n(2n+1)\beta^2. \tag{4.22}$$

Hence, we can state:

Lemma 4.1. *In an $\mathcal{M}$ admitting $\hat{\nabla}$ and β=constant, we have*

- *The curvature tensor $\hat{\mathcal{R}}$ is given by* (4.18),
- *The Ricci tensor $\hat{Ric}$ is given by* (4.20) *and it is symmetric,*
- *The Ricci operator $\hat{\mathcal{Q}}$ is given by* (4.21),
- *The scalar curvature $\hat{s}$ is given by* (4.22).

Lemma 4.2. *In an $\mathcal{M}$ admitting $\hat{\nabla}$, we have*

- $\hat{\mathcal{R}}(X,Y)\xi = 0$,
- $\hat{\mathcal{R}}(X,Y)Z + \hat{\mathcal{R}}(Y,X)Z = 0$,
- $\hat{\mathcal{R}}(X,Y)Z + \hat{\mathcal{R}}(Y,Z)X + \hat{\mathcal{R}}(Z,X)Y = 0$,

- *$\hat{Ric}(Y,\xi) = 0$ if β is constant. Otherwise, $\hat{Ric}(Y,\xi) = -(\xi\beta)\eta(Y) - (2n-1)(X\beta)$, for all $X, Y, Z \in \chi(\mathcal{M})$.*

5. Projective curvature tensor in β-Kenmotsu manifolds concerning $\hat{\nabla}$

Let $\mathcal{M}$ be a $(2n+1)$-dimensional Riemannain manifold. If there exists a one to one correspondence between each coordinate neighbourhood of $\mathcal{M}$ and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then $\mathcal{M}$ is said to be locally projectively flat. For $n \geq 1$, $\mathcal{M}$ is locally projectively flat if and only if the projective curvature tensor vanishes. The projective curvature tensor P_1 with respect to the Levi-Civita connection ∇ is defined by [28]

$$P_1(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{1}{2n}\{Ric(Y,Z)X - Ric(X,Z)Y\}, \tag{5.23}$$

for all X, Y on $\mathcal{M}$, where $\mathcal{R}$ are Ric are the Riemannian curvature tensor and the Ricci tensor, respectively.

Definition 5.1. *A β-Kenmotsu manifold $\mathcal{M}$ is said to be ξ-projectively flat with respect to $\hat{\nabla}$ if*

$$\hat{P}_1(X,Y)\xi = 0,$$

where $\hat{P}_1(X,Y)Z$ is the projective curvature tensor of dimension $(2n+1)$ concerning $\hat{\nabla}$ and is given by

$$\hat{P}_1(X,Y)Z = \hat{\mathcal{R}}(X,Y)Z - \frac{1}{2n}\{\hat{Ric}(Y,Z)X - \hat{Ric}(X,Z)Y\}, \tag{5.24}$$

for all $X, Y, Z \in \chi(\mathcal{M})$.

Theorem 5.1. *An $\mathcal{M}$ of dimension $(2n+1)$ is ξ-projectively flat with respect to $\hat{\nabla}$ if and only if it is ξ-projectively flat with respect to ∇, provided β is constant.*

Proof. From (4.18), (4.20) and (5.24), we have

$$\begin{aligned} \hat{P}_1(X,Y)Z &= P_1(X,Y)Z + X(\beta)\{g(Y,Z)\xi - \eta(Z)Y\} \\ &\quad -Y(\beta)\{g(X,Z)\xi - \eta(Z)X\}, \end{aligned} \tag{5.25}$$

where $P_1(X,Y)Z$ is defined in (5.23). Now, putting $Z = \xi$ in (5.25), and considering β as a constant, we get

$$\hat{P}_1(X,Y)\xi = P_1(X,Y)\xi.$$

□

Definition 5.2. *A β-Kenmotsu manifold $\mathcal{M}$ satisfying the condition*

$$\phi^2(\hat{P}_1(\phi X, \phi Y)\phi Z) = 0$$

is called ϕ-projectively flat with respect to $\hat{\nabla}$. As we know that

$$\phi^2(\hat{P}_1(\phi X, \phi Y)\phi Z) = 0 \Longleftrightarrow g(\hat{P}_1(\phi X, \phi Y)\phi Z, \phi W) = 0 \tag{5.26}$$

for all $X, Y, Z, W \in \chi(\mathcal{M})$.

Theorem 5.2. *Let $\mathcal{M}$ be a $(2n+1)$-dimensional ϕ-projectively flat β-Kenmotsu manifold with respect to $\hat{\nabla}$ and β is constant. Then $\mathcal{M}$ is an η-Einstein manifold.*

Proof. Let $\mathcal{M}$ be a ϕ-projectively flat β-Kenmotsu manifold with respect to $\hat{\nabla}$, then (5.26) holds. Thus, from (5.24) and (5.26), we have

$$g(\hat{\mathcal{R}}(\phi X, \phi Y)\phi Z, \phi W) = \frac{1}{2n}\{\hat{Ric}(\phi Y, \phi Z)g(\phi X, \phi W) - \hat{Ric}(\phi X, \phi Z)g(\phi Y, \phi W)\},$$

which by using (4.18) and (4.20) turns to

$$\begin{aligned} g(\mathcal{R}(\phi X, \phi Y)\phi Z, \phi W) &= -\beta^2\{g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\} \\ &+\frac{1}{2n}\{Ric(\phi Y, \phi Z)g(\phi X, \phi W) + 2n\beta^2 g(\phi Y, \phi Z)g(\phi X, \phi W) \\ &-Ric(\phi X, \phi Z)g(\phi Y, \phi W) - 2n\beta^2 g(\phi X, \phi Z)g(\phi Y, \phi W)\}. \end{aligned} \tag{5.27}$$

Now choosing a set $\{e_i, \phi e_i, \xi\}(1 \leq i \leq 2n)$ as an orthogonal basis of $\mathcal{M}$, by contracting (5.27) over X and W, we obtain

$$\begin{aligned} Ric(\phi Y, \phi Z) &= -(2n\beta^2 + \xi\beta)g(\phi Y, \phi Z) \\ &+\frac{1}{2n}\{(2n-1)Ric(\phi Y, \phi Z) + 2n(2n-1)\beta^2 g(\phi Y, \phi Z)\}. \end{aligned}$$

This implies

$$Ric(\phi Y, \phi Z) = -(\beta^2 + \xi\beta)g(\phi Y, \phi Z). \tag{5.28}$$

By using (2.2) and (2.8) in (5.28), we have

$$Ric(Y, Z) = -(\beta^2 + \xi\beta)g(Y, Z) - (2n-1)\beta^2\eta(Y)\eta(Z) - (2n-1)Y(\beta)\eta(Z). \tag{5.29}$$

Now, if β is constant, then (5.29) reduces to

$$Ric(Y, Z) = -\beta^2 g(Y, Z) - (2n-1)\beta^2\eta(Y)\eta(Z).$$

Thus $\mathcal{M}$ is an η-Einstein manifold. □

6. Concircular curvature tensor in β-Kenmotsu manifolds concerning $\hat{\nabla}$

A transformation of a $(2n+1)$-dimensional Riemannian manifold $\mathcal{M}$, which transforms every geodesic circle of $\mathcal{M}$ into a geodesic circle, is called a concircular transformation [14, 27]. A concircular transformation is always a conformal transformation [14]. Here geodesic circle means a curve in $\mathcal{M}$ whose first curvature is constant and second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is generalization of inversive geometry in the sense that the change of metric is more general than induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor with respect to the Levi-Civita connection and is defined by

$$P_2(X,Y)Z = \mathcal{R}(X,Y)Z - \frac{s}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{6.30}$$

for all X, Y and Z on $\mathcal{M}$, where s is the the scalar curvature with respect to the Levi-Civita connection.

Definition 6.1. *A β-Kenmotsu manifold $\mathcal{M}$ satisfying the condition*

$$\phi^2(\hat{P}_2(\phi X,\phi Y)\phi Z) = 0$$

is called ϕ-concircularly flat with respect to $\hat{\nabla}$, where $\hat{P}_2(X,Y)Z$ is the concircular curvature tensor of dimension $(2n+1)$ with respect to $\hat{\nabla}$ and is given by

$$\hat{P}_2(X,Y)Z = \hat{\mathcal{R}}(X,Y)Z - \frac{\hat{s}}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\}. \tag{6.31}$$

As we know that

$$\phi^2(\hat{P}_2(\phi X,\phi Y)\phi Z) = 0 \Longleftrightarrow g(\hat{P}_2(\phi X,\phi Y)\phi Z,\phi W) = 0, \tag{6.32}$$

for all X,Y,Z,W on $\mathcal{M}$.

Theorem 6.1. *Let $\mathcal{M}$ be a $(2n+1)$-dimensional ϕ-concircularly flat β-Kenmotsu manifold with respect to $\hat{\nabla}$ and β is constant. Then $\mathcal{M}$ is an η-Einstein manifold.*

Proof. If $\mathcal{M}$ is a ϕ-concircularly flat with respect to $\hat{\nabla}$, then (6.32) holds. Thus, from (6.31) and (6.32), we have

$$\begin{aligned} g(\hat{\mathcal{R}}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{\hat{s}}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &\quad - g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{6.33}$$

By using (4.18) and (2.2) in (6.33), we have

$$\begin{aligned} g(\mathcal{R}(\phi X,\phi Y)\phi Z,\phi W) &= -\beta^2\{g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &= \frac{s+2n(2n+1)\beta^2}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &\quad -g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \end{aligned} \tag{6.34}$$

Now choosing $\{e_i,\phi e_i,\xi\}(1\leq i\leq 2n)$ as a set of orthogonal basis of $\mathcal{M}$ and contracting (6.34) over X and W, we obtain

$$Ric(\phi Y,\phi Z)=(\frac{s}{2n}+(\beta^2-\xi\beta))g(\phi Y,\phi Z). \tag{6.35}$$

By using (2.2) and (2.8) in (6.35), we have

$$\begin{aligned} Ric(Y,Z) &= (\frac{s}{2n}+(\beta^2-\xi\beta))g(Y,Z)-(\frac{s}{2n}+(2n+1)\beta^2)\eta(Y)\eta(Z) \\ &\quad -(2n-1)Y(\beta)\eta(Z). \end{aligned} \tag{6.36}$$

Now, if β is constant, then (6.36) reduces to

$$Ric(Y,Z)=(\frac{s}{2n}+\beta^2)g(Y,Z)-(\frac{s}{2n}+(2n+1)\beta^2)\beta^2\eta(Y)\eta(Z).$$

The above equation shows that $\mathcal{M}$ is an η-Einstein manifold. □

7. Conformal curvature tensor in β-Kenmotsu manifolds concerning $\hat{\nabla}$

If the Riemannian metric g on a manifold $\mathcal{M}$ is conformally related with a flat Euclidean metric, then g is called conformally flat. A Riemannian manifold equipped with a conformally flat Riemannian metric is named a conformally flat manifold. By using conformal transformation, Weyl [26] introduced a generalized curvature tensor which vanishes whenever the metric is conformally flat. Due to this reason it is called confomal curvature tensor. It is well-known that a Riemannian manifold $\mathcal{M}$ of dimension $(2n+1)$ is conformally flat if and only if the Weyl conformal curvature tensor field P_3 vanishes for the dimension >3. The conformal curvature tensor P_3 in a $(2n+1)$-dimensional Riemannian manifold is defined by

$$\begin{aligned} P_3(X,Y)Z &= \mathcal{R}(X,Y)Z-\frac{1}{2n-1}\{Ric(Y,Z)X-Ric(X,Z)Y+g(Y,Z)\mathcal{Q}X \\ &- g(X,Z)\mathcal{Q}Y\}+\frac{s}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}, \end{aligned} \tag{7.37}$$

for all vector fields X,Y,Z on $\mathcal{M}$, where $\mathcal{R}$, Ric, $\mathcal{Q}$, and s be the Riemannian curvature tensor, the Ricci tensor, the Ricci operator, and the scalar curvature, respectively.

Definition 7.1. *A β-Kenmotsu manifold $\mathcal{M}$ is ξ-conformally flat with respect to $\hat{\nabla}$ if*

$$\hat{P}_3(X,Y)\xi = 0,$$

where $\hat{P}_3(X,Y)Z$ is the conformal curvature tensor of dimension $(2n+1)$ with respect to $\hat{\nabla}$ and is given by

$$\begin{aligned}\hat{P}_3(X,Y)Z &= \hat{\mathcal{R}}(X,Y)Z - \frac{1}{(2n-1)}\{\hat{Ric}(X,Z)X - \hat{Ric}(X,Z)Y + g(Y,Z)\hat{\mathcal{Q}}X \\ &\quad -g(X,Z)\hat{\mathcal{Q}}Y\} + \frac{\hat{s}}{2n(2n-1)}\{g(Y,Z)X - g(X,Z)Y\} \qquad (7.38)\end{aligned}$$

for all X,Y,Z on $\mathcal{M}$.

Theorem 7.1. *A $(2n+1)$-dimensional β-Kenmotsu manifold with respect to $\hat{\nabla}$ is ξ-conformally flat iff it is ξ-conformally flat with respect to ∇, provided β is constant.*

Proof. From (4.18), (4.20) and (7.38), we have

$$\begin{aligned}\hat{P}_3(X,Y)Z &= P_3(X,Y)Z + X(\beta)\{g(Y,Z)\xi - \eta(Z)Y\} \qquad (7.39)\\ &\quad -Y(\beta)\{g(X,Z)\xi - \eta(Z)X\},\end{aligned}$$

where $P_3(X,Y)Z$ is defined by (7.37). By putting $Z=\xi$ in (7.39), and considering β as a constant, we get

$$\hat{P}_3(X,Y)\xi = P_3(X,Y)\xi.$$

This completes the proof. □

Definition 7.2. *A β-Kenmotsu manifold $\mathcal{M}$ is called ϕ-conformally flat with respect to $\hat{\nabla}$ if*

$$\phi^2(\hat{P}_3(\phi X,\phi Y)\phi Z) = 0 \Longleftrightarrow g(\hat{P}_3(\phi X,\phi Y)\phi Z,\phi W) = 0, \qquad (7.40)$$

for all $X,Y,Z,W \in \chi(\mathcal{M})$.

Theorem 7.2. *Let $\mathcal{M}$ be a $(2n+1)$-dimensional ϕ-conformally flat β-Kenmotsu manifold with respect to $\hat{\nabla}$ and β is constant. Then $\mathcal{M}$ is an η-Einstein manifold.*

Proof. If $\mathcal{M}$ is a ϕ-conformaly flat, then in the view of equation (7.38) and (7.40), we have

$$\begin{aligned}g(\hat{\mathcal{R}}(\phi X,\phi Y)\phi Z,\phi W) &= \frac{1}{2n}\{\hat{Ric}(\phi Y,\phi Z)g(\phi X,\phi W) - \hat{Ric}(\phi X,\phi Z)g(\phi Y,\phi W) \\ &\quad +g(\phi Y,\phi Z)\hat{Ric}(\phi X,\phi W) - g(\phi X,\phi Z)\hat{Ric}(\phi Y,\phi W)\} \qquad (7.41)\\ &\quad -\frac{\hat{s}}{2n(2n-1)}\{g(\phi Y,\phi Z)g(\phi X,\phi W) - g(\phi X,\phi Z)g(\phi Y,\phi W)\}.\end{aligned}$$

By using (4.18) and (4.20), (7.41) takes the form

$$\begin{aligned} g(\mathcal{R}(\phi X,\phi Y)\phi Z,\phi W) &= -\beta^2\{g(\phi Y,\phi Z)g(\phi X,\phi W)-g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &+\frac{1}{2n}\{Ric(\phi Y,\phi Z)g(\phi X,\phi W)+2n\beta^2 g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-Ric(\phi X,\phi Z)g(\phi Y,\phi W)-2n\beta^2 g(\phi X,\phi Z)g(\phi Y,\phi W) \quad (7.42) \\ &+g(\phi Y,\phi Z)Ric(\phi X,\phi W)+2n\beta^2 g(\phi Y,\phi Z)g(\phi X,\phi W) \\ &-g(\phi X,\phi Z)Ric(\phi Y,\phi W)-2n\beta^2 g(\phi X,\phi Z)g(\phi Y,\phi W)\} \\ &-\frac{s+2n(2n+1)\beta^2}{2n}g(\phi Y,\phi Z). \end{aligned}$$

Now choosing $\{e_i,\phi e_i,\xi\}(1\leq i\leq 2n)$ as a set of orthogonal basis of $\mathcal{M}$ and contracting (7.42) over X and W, we obtain

$$Ric(\phi Y,\phi Z)=(\frac{s}{2n}-(2n-1)(\beta^2+\xi\beta))g(\phi Y,\phi Z). \quad (7.43)$$

Now using (2.2) and (2.8) in (7.43), we have

$$\begin{aligned} Ric(Y,Z) &= (\frac{s}{2n}-(2n-1)(\beta^2+\xi\beta))g(Y,Z) \\ &-(\frac{s}{2n}+\beta^2-2(n-1)(\xi\beta))\eta(Y)\eta(Z)-(2n-1)Y(\beta)\eta(Z). \quad (7.44) \end{aligned}$$

Now, if β is constant, then (7.44) reduces to

$$Ric(Y,Z)=(\frac{s}{2n}-(2n-1)\beta^2)g(Y,Z)-(\frac{s}{2n}+\beta^2)\eta(Y)\eta(Z).$$

The above equation shows that $\mathcal{M}$ is an η-Einstein manifold. □

8. Example

In this section, an example has been stated to verify some results of the paper.

We assume a 3-dimensional manifold $\mathcal{M}=\{(u,v,w)\in\mathbb{R}^3\}$, where (u,v,w) are the usual coordinates in $\mathbb{R}^3$. We choose the linearly independent vector fields at each point of $\mathcal{M}$ as [20]

$$\epsilon_1=w^2\frac{\partial}{\partial u},\ \epsilon_2=w^2\frac{\partial}{\partial v},\ \epsilon_3=\frac{\partial}{\partial w}.$$

Let the Riemannian metric g is defined by

$$g(\epsilon_i,\epsilon_j)=\begin{cases} 1 & \text{if} \quad i=j \\ 0 & \text{if} \quad i\neq j \end{cases} \ ; i,j=1,2,3.$$

Let the 1-form η is defined by

$$\eta(X)=g(X,\epsilon_3),$$

for any X on $\mathcal{M}$. Let the $(1,1)$-tensor field ϕ is defined by

$$\phi(\epsilon_1) = -\epsilon_2, \quad \phi(\epsilon_2) = \epsilon_1, \quad \phi(\epsilon_3) = 0.$$

Using the linearity of ϕ and g, we have

$$\phi^2 X = -X + \eta(X)\epsilon_3, \quad \eta(\epsilon_3) = 1, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$$

for any X, Y on $\mathcal{M}$. Thus for $\epsilon_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost contact metric structure on $\mathcal{M}$. For the connection ∇, we have

$$[\epsilon_1, \epsilon_2] = 0, \quad [\epsilon_1, \epsilon_3] = -\frac{2}{w}\epsilon_1, \quad [\epsilon_2, \epsilon_3] = -\frac{2}{w}\epsilon_2.$$

By using the Koszul's formula, we find

$$\begin{aligned} \nabla_{\epsilon_1}\epsilon_1 &= \tfrac{2}{w}\epsilon_3, & \nabla_{\epsilon_1}\epsilon_2 &= 0, & \nabla_{\epsilon_1}\epsilon_3 &= -\tfrac{2}{w}\epsilon_1, \\ \nabla_{\epsilon_2}\epsilon_1 &= 0, & \nabla_{\epsilon_2}\epsilon_2 &= \tfrac{2}{w}\epsilon_3, & \nabla_{\epsilon_2}\epsilon_3 &= -\tfrac{2}{w}\epsilon_2, \\ \nabla_{\epsilon_3}\epsilon_1 &= 0, & \nabla_{\epsilon_3}\epsilon_2 &= 0, & \nabla_{\epsilon_3}\epsilon_3 &= 0. \end{aligned} \tag{8.45}$$

From the above values, it is clear that (ϕ, ξ, η, g) is a β-Kenmotsu structure on $\mathcal{M}$, hence $\mathcal{M}(\phi, \xi, \eta, g)$ is a 3-dimensional β-Kenmotsu manifold satisfying the conditions (2.5)-(2.7), where $\beta = -\frac{2}{w}$. Using the results from equation (8.45), we can obtain the non-vanishing components of the Riemannian curvature tensor with respect to ∇ as follows:

$$\begin{aligned} \mathcal{R}(\epsilon_1, \epsilon_2)\epsilon_1 &= \tfrac{4}{w^2}\epsilon_2, & \mathcal{R}(\epsilon_1, \epsilon_2)\epsilon_2 &= -\tfrac{4}{w^2}\epsilon_1, & \mathcal{R}(\epsilon_1, \epsilon_3)\epsilon_1 &= \tfrac{4}{w^2}\epsilon_3, \\ \mathcal{R}(\epsilon_1, \epsilon_3)\epsilon_3 &= -\tfrac{4}{w^2}\epsilon_1, & \mathcal{R}(\epsilon_2, \epsilon_3)\epsilon_3 &= -\tfrac{4}{w^2}\epsilon_2, & \mathcal{R}(\epsilon_2, \epsilon_3)\epsilon_2 &= \tfrac{4}{w^2}\epsilon_3. \end{aligned} \tag{8.46}$$

The Ricci tensor concerning to ∇ are

$$Ric(\epsilon_i, \epsilon_i) = \begin{cases} -\frac{8}{w^2}, i = 1, 2, 3, \\ 0, \ otherwise. \end{cases} \tag{8.47}$$

Thus, the scalar curvature s with respect to the ∇ given by

$$s = -\frac{24}{w^2}. \tag{8.48}$$

By using the values of (8.45) in (3.10), we obtain

$$\hat{\nabla}_{\epsilon_i}\epsilon_j = \begin{cases} -\epsilon_2, \ i = 3, \ j = 1, \\ \epsilon_1, \ i = 3, \ j = 2, \\ 0, \ otherwise. \end{cases} \tag{8.49}$$

From the above results given in (8.49), we can easily calculate

$$\hat{\mathcal{R}}(\epsilon_i,\epsilon_j)\epsilon_k=0,\quad \hat{Ric}(\epsilon_i,\epsilon_j)=0,\quad \hat{\mathcal{Q}}=0,\quad \hat{s}=0,\qquad for\quad 1\leq i,j,k\leq 3. \tag{8.50}$$

In view of (8.50), it can be easily seen from (5.24) and (7.38) that

$$\begin{aligned}\hat{P}_1(\epsilon_1,\epsilon_2)\epsilon_3=\hat{P}_1(\epsilon_1,\epsilon_3)\epsilon_3=\hat{P}_1(\epsilon_2,\epsilon_3)\epsilon_3=0,\\ \hat{P}_3(\epsilon_1,\epsilon_2)\epsilon_3=\hat{P}_3(\epsilon_1,\epsilon_3)\epsilon_3=\hat{P}_3(\epsilon_2,\epsilon_3)\epsilon_3=0,\end{aligned} \tag{8.51}$$

respectively.

Also by using (8.46), (8.47) and (8.48) from (5.23) and (7.37), we find

$$\begin{aligned}P_1(\epsilon_1,\epsilon_2)\epsilon_3=P_1(\epsilon_1,\epsilon_3)\epsilon_3=P_1(\epsilon_2,\epsilon_3)\epsilon_3=0,\\ P_3(\epsilon_1,\epsilon_2)\epsilon_3=P_3(\epsilon_1,\epsilon_3)\epsilon_3=P_3(\epsilon_2,\epsilon_3)\epsilon_3=0,\end{aligned} \tag{8.52}$$

respectively.

Thus, the first relations of the equations (8.51) and (8.52) and the second relations of the equations (8.51) and (8.52) verifies Theorem 5.1 and Theorem 7.1, respectively.

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