International Journal of Maps in Mathematics
Volume 7, Issue 2, 2024, Pages:236-257
E-ISSN: 2636-7467
www.journalmim.com

STUDY OF SOME CURVES ALONG CONFORMAL SUBMERSION

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Abstract. In this article, we study bi-f-harmonic curves, hyperelastic curves, helices and circles along conformal Riemannian submersion. We investigate the behavior of an arbitrary horizontal curve on the total manifold under the conformal submersion. Moreover, we show that a totally geodesic Riemannian submersion takes a horizontal bi-f-harmonic curve, helix and circle to a bi-f-harmonic curve, helix and circle on target manifold, respectively. In addition, we also find the conditions for which Riemannian submersion takes a horizontal bi-f-harmonic curve, helix and circle to a bi-f-harmonic curve, helix and circle on target manifold, respectively.

Keywords: Bi-f-harmonic curve, bi-harmonic curve, helix, circle, hyperelastic curves, elastic curve, totally geodesic conformal submersion and totally umbilical conformal submersion.
2010 Mathematics Subject Classification: 53B20, 53C42, 53C43, 58E20.

1. Introduction

In 1964, J. Eells and J. H. Sampson [6], introduced the concept of bi-harmonic maps by generalizing the harmonic maps. Harmonic maps have important applications in various areas of mathematics and physics with nonlinear partial differential equations. A harmonic map $\alpha : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ between the Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ is a

Received:2024.02.28 *Revised:2024.05.13* *Accepted:2024.05.28*

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critical point of the energy functional,

$$E(\alpha) = \frac{1}{2}\int_{\Gamma_N} |d\alpha|^2 v_{g_N},$$

where Γ_N is some compact domain of N and $\tau(\alpha) = Trace_{g_N}\nabla d\alpha$ is the tension field of α. The harmonic map equation is an Euler-Lagrange equation of the functional $\tau(\varphi) \equiv Trace_{g_N}\nabla d\varphi = 0$, where $\tau(\varphi) = Trace_{g_N}\nabla d\varphi$ is the tension field of φ [6]. The bi-harmonic map α between the Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ is a critical point of the bi-energy functional, $E_2(\alpha) = \frac{1}{2}\int_{\Gamma_N} |\tau(\alpha)|^2 v_{g_N}$, where Γ_N is a compact domain of N. The bi-harmonic map equation is an Euler-Lagrange equation of the functional,

$$\tau_2(\alpha) \equiv Trace_{g_N}(\nabla^\alpha\nabla^\alpha - \nabla^\alpha_{\nabla^N})\tau(\alpha) - Trace_{g_N}R^{\bar{N}}(d\alpha, \tau(\alpha))d\alpha = 0,$$

where $R^{\bar{N}} = [\nabla^{\bar{N}}_X, \nabla^{\bar{N}}_Y]Z - \nabla^{\bar{N}}_{[X,Y]}Z$, is a Riemann curvature tensor of $(\bar{N}, g_{\bar{N}})$ [16]. In 1991 [5], the author introduced the bi-harmonic submanifolds of Euclidean space and stated a conjecture " any bi-harmonic submanifold of Euclidean space is harmonic, thus minimal". If the definition of bi-harmonic maps for Riemannian immersion in Euclidean space is used, then the Chen's definition of a bi-harmonic submanifold coincides with the definition given by the bi-energy functional.

Bi-f-harmonic maps are the generalization of harmonic maps and f-harmonic maps. There are two methods to formalize the link between bi-harmonic maps and f-harmonic maps. In the first method of formalization, the authors extended the bi-energy functional in [32, 39] to the bi-f-energy functional and got bi-f-harmonic maps. Further, for the second formalization, the f-energy functional is extended to the f-bi-energy functional. In [22], the author introduced the f-bi-harmonic maps by generalizing the bi-harmonic maps. The bi-f-harmonic equation for curves in Euclidean space, hyperbolic space, sphere and hypersurfaces of manifolds were studied in [30].

In [34], authors studied the charcterization of submanifold by taking the hyperelastic curves along an immersion. The following properties of Riemannian submersions were studied in [10, 25, 19]. In 1974, the authors proved that if a circle is mapped by immersion from a submanifold to the ambient manifold, then the submanifold is said to be totally umbilical with a parallel mean curvature vector field [26].

In the sixties O' Neill and Gray introduced the concept of Riemannian submersions between Riemannian manifolds [11, 25]. A differential map G between two Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ is known as a submersion if the rank of G_* is equal to the dimension of the targeted manifold. Also, if the submersion is isometry between (N, g_N) and $(\bar{N}, g_{\bar{N}})$, then G is called a Riemannian submersion. Conformal submersion and the fundamental equations of conformal submersion were studied in [28, 12]. In [37, 38], authors study the totally umbilical, geodesic and minimal fibers by using conformal submersions. Horizontally conformal submersion is a generalization of the Riemannian submersion [9, 14]. Horizontally conformal map is useful for the characterization of harmonic morphisms [4] and has many applications in medical imaging (brain imaging) and computer graphics.

Hyperelastic curves in a Riemannian manifold are solutions to a constrained variable problem and are characterized by Euler-Lagrange equations. A parametrized curve by its arclength is said to be a hyperelastic curve if it is a critical point of the following curvature energy action defined on a suitable space of curves in a Riemannian manifold

$$\mathcal{F}_{\gamma}^{r} = \int (\kappa^{r} + \mu) ds, \tag{1.1}$$

where κ denotes the curvature of γ [3, 36, 31]. If $\mu = 0$, then these curves are called free hyperelastic curves. In 2021, B. Sahin, G. O. Tukel and T. Turhan, studied the effect of hyperelastic curves on the geometry of isometric immersions in [33]. The functional $\mathcal{F}_{\lambda}^{r}$ is the classical Euler-Bernoulli's bending (or elastic) energy functional for $r = 2$. Immersed curves which are critical for the bending energy functional satisfying some boundary conditions are said to be elastic curves (or elastica) [20]. The existence, classification or stability problems of elastic curves or their generalizations in Riemannian manifolds attracted the attention of many researchers. There are the following examples in the literature worked by D. Singer et al. [15, 21, 20, 35]. In 1984, J. Langer and D. Singer proved that there exist closed elastic curves of a fixed length in a compact Riemannian manifold [20].

A smooth curve parametrized by its arc-length on a Riemannian manifold N is said to be circle if it satisfies $\nabla^2_{\dot{\beta}}\dot{\beta} = -\kappa^2\dot{\beta}$, where κ is a non-negative constant curvature of β and $\nabla_{\dot{\beta}}$ is the covariant differentiation along β with respect to the Riemannian connection ∇ on N. In [26], Nomizu-Yano proved that β is a circle iff the following is satisfies

$$\nabla^2_{\dot{\beta}}\dot{\beta} + g(\nabla_{\dot{\beta}}\dot{\beta}, \nabla_{\dot{\beta}}\dot{\beta})\dot{\beta} = 0,$$

where g is the Riemannian metric on N and $\nabla^2_{\dot{\beta}}\dot{\beta} = \nabla_{\dot{\beta}}\nabla_{\dot{\beta}}\dot{\beta}$. Many authors studied circles on Riemannian manifolds and they showed that it is possible to obtain certain properties of a submanifolds by observing the extrinsic structure of circles on this submanifold, [34, 2, 7, 13, 17, 23, 24, 27, 29]. In 1963, S. Kobayashi and K. Nomizu showed that an ordinary helix

$c = c(s)$ satisfies the following equation, $\nabla^3_{\dot{\beta}}\dot{\beta} + K^2\nabla_{\dot{\beta}}\dot{\beta} = 0$, where $K^2 = \kappa^2 + \tau^2$ is a positive constant. Conversely, if a curve $c = c(s)$ satisfies the above condition, then it is an ordinary helix or a geodesic, [18].

The structure of the article is as follows: In Section 2, we recall some basic concepts about conformal Riemannian submersion, totally geodesic fibers and the second fundamental form of Riemannian submersion. In Section 3, some conditions are derived for the case where the curve either in the base manifold or in the target manifold is a bi-f-harmonic curve. In section 3, we show that a totally geodesic conformal submersion between two Riemannian manifolds takes a bi-harmonic curve to a bi-harmonic curve. In section 4, we prove that the conformal submersion takes a curve to a helix iff the curve is of constant curvature. In the same section, we also find the conditions for a curve to become a circle in a targeted manifold by conformal submersion. In the final section, we study the hyperelastic curves along the conformal submersions.

2. Preliminaries

Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a differentiable map between the Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ of dimensions n_1 and n_2, respectively such that $n_1 > n_2$. Then G is said to be a Riemannian submersion if rank of G is maximal and differential G_* preserves the lengths of horizontal vectors. A Riemannian submersion $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ is said to be a conformal submersion if the restriction of G_* to the horizontal distribution of G is a conformal map, i.e. there exist a smooth function $\lambda : N \to R^+$ such that

$$g_{\bar{N}}(G_*(X), G_*(Y)) = \lambda^2(p) g_N(X, Y),$$

for all $X, Y \in \Gamma(kerG_*)^\perp$ and $p \in N$.

A curve $\beta : I \to N$ on (N, g) is said to be a bi-f-harmonic curve if and only if β satisfies the condition [30],

$$(ff''' + f'f'')\dot{\beta} + (3ff'' + 2f'^2)\nabla_{\dot{\beta}}\dot{\beta} + 4ff'\nabla^2_{\dot{\beta}}\dot{\beta} + f^2\nabla^3_{\dot{\beta}}\dot{\beta}$$
$$+ f^2 R(\nabla_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta} = 0, \tag{2.2}$$

where $f : I \to (0, \infty)$ is a smooth function, ∇ is a Levi-Civita connection and R is a Riemannian curvature tensor on N. Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a Riemannian submersion between (N, g) and $(\bar{N}, \bar{g})$. Then β is said to be a horizontal curve if $\dot{\beta}(t) \in (kerG_*)^\perp$; $\forall\, t \in I$. If $\nabla^{\bar{N}}$ is the Levi-Civita connection on $(\bar{N}, \bar{g})$, then the second fundamental form of

G is given by

$$(\nabla G_*)(X,Y) = \overset{\bar{N}}{\nabla}{}^G_X G_*(Y) - G_*(\nabla^N_X Y), \quad \forall\, X,Y \in \Gamma(TN), \tag{2.3}$$

where $\overset{\bar{N}}{\nabla}{}^G$ is the pullback connection of $\nabla^{\bar{N}}$. Now, if $X,Y \in \Gamma((kerG_*)^{\perp})$, then the second fundamental form of Riemannian submersion is

$$(\nabla G_*)(X,Y) = 0. \tag{2.4}$$

Also, if $X,Y \in \Gamma((kerG_*)^{\perp})$ and $V \in \Gamma((rangeG_*)^{\perp})$, then

$$\nabla^{\bar{N}}_{G_*(X)} V = -S_V G_*(X) + \nabla^{G\perp}_X V, \tag{2.5}$$

where $S_V G_*(X)$ is the tangential component of $\nabla^{\bar{N}}_{G_*(X)} V$. Since (∇G_*) is symmetric and S_V is a symmetric linear transformation of $rangeG_*$, therefore

$$g_{\bar{N}}(S_V G_*(X), G_*(Y)) = g_{\bar{N}}(V, (\nabla G_*)(X,Y)). \tag{2.6}$$

From equations (2.3) and (2.4), we get

$$\begin{aligned} R^{\bar{N}}(G_*(X), G_*(Y))G_*(Z) &= -S_{(\nabla G_*)(Y,Z)}G_*(X) + S_{(\nabla G_*)(X,Z)}G_*(Y) \\ &+G_*(R^N(X,Y)Z) + (\tilde{\nabla}_X(\nabla G_*))(Y,Z) - (\tilde{\nabla}_Y(\nabla G_*))(X,Z), \end{aligned} \tag{2.7}$$

where $\tilde{\nabla}$ is the covariant derivative of the second fundamental form. The O' Neill tensors [34] A and T are given by

$$A_P P' = h\nabla_{hP} vP' + v\nabla_{hP} hP', \tag{2.8}$$

$$T_P P' = h\nabla_{vP} vP' + v\nabla_{vP} hP', \tag{2.9}$$

for all $P, P' \in \Gamma(TN)$, where ∇ is the Levi-civita connection on N. For $P \in \Gamma(TN)$, T is vertical such that $T_P = T_{vP}$ and A is horizontal such that $A_P = A_{hP}$. Also, if $U, W \in \Gamma(kerG_*)$, then we have $T_U W = T_W U$.

From equations (2.8) and (2.9), we get

$$\nabla_V W = T_U V + v\nabla_V W, \tag{2.10}$$

$$\nabla_X V = A_X V + v\nabla_X V, \tag{2.11}$$

$$\nabla_Y Z = A_Y Z + H\nabla_Y Z, \tag{2.12}$$

for all $V, W \in \Gamma(kerG_*)$ and $Y, Z \in \Gamma(kerG_*)^{\perp}$. The covariant derivative of ∇G_* and S are

$$(\tilde{\nabla}_X(\nabla G_*))(Y,Z) = \nabla_X^{G^{\perp}}(\nabla G_*)(Y,Z) - (\nabla G_*)(\nabla_X^N Y, Z) - (\nabla G_*)(Y, \nabla_X^N Z), \quad (2.13)$$

and

$$(\tilde{\nabla}_X S)_V G_*(Y) = G_*(\nabla_X^N {}^*G_*(S_V G_*(Y))) - S_{\nabla_X^{G^{\perp}} V} G_*(Y) - S_V Q \overset{\bar{N}}{\nabla}{}_X^G G_*(Y), \quad (2.14)$$

respectively. Here Q is a projection morphism on $rangeG_*$ and ${}^*G_*$ is an adjoint map of G_*. From equations (2.13) and (2.14), we obtain

$$g_{\bar{N}}((\tilde{\nabla}_X(\nabla G_*))(Y,Z), V) = g_{\bar{N}}((\tilde{\nabla}_X S)_V G_*(Y), G_*(Z)). \quad (2.15)$$

Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$. Then G is called a conformal submersion with totally geodesic fibers if and only if T vanishes identically.

3. Characterization of bi-f-harmonic curves

Let $\beta : I \to N$ be a curve in an n_1-dimensional Riemannian manifold N with an orthonormal frame $\{W_0, W_1, ...W_{n_1-1}\}$ in ΓTN, where $W_0 = T$, $W_1 = N$ and $W_2 = U$ are the unit tangent vector, the unit normal vector and the unit binormal vector of α, respectively. Then the Frenet equations are given by

$$\nabla_T W_j = -\kappa_j W_{j-1} + \kappa_{j+1} W_{j+1}, \qquad 0 \le j \le m-1, \quad (3.16)$$

where $\kappa_0 = \kappa_{n_1} = 0$, $\kappa_1 = \kappa = ||\nabla_T T||$ is curvature and $\tau = \kappa_2 = -\langle \nabla_T W_1, W_2 \rangle$ is torsion of β on N, respectively. Next, we introduce the concept horizontal bi-f-harmonic curve.

Definition 3.1. *Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a conformal submersion between the Riemannian manifolds (N, g) and $(\bar{N}, \bar{g})$. Then a horizontal curve on (N, g) with (2.2) is said to be a horizontal bi-f-harmonic curve on (N, g).*

Lemma 3.1. *Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a conformal submersion between (N, g) and $(\bar{N}, \bar{g})$. Now, if $\bar{\beta} = G \circ \beta$ is a curve on $(\bar{N}, \bar{g})$ and β is a horizontal curve on (N, g), then*

$$(i) \quad \bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta}), \quad (3.17)$$

$$(ii) \quad \bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta})\dot{\beta}) - 2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}} \dot{\beta})$$
$$+ (\nabla G_*)(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, A_{\dot{\beta}} \dot{\beta}), \quad (3.18)$$

where $\overset{\wedge}{\nabla}$ and $\bar{\nabla}$ are the Levi-Civita connections of N and $\bar{N}$, respectively.

Proof. Let β be a horizontal curve with curvature κ on Riemannian manifold (N, g) and $\bar{\beta} = G \circ \beta$ is a curve with curvature $\bar{\kappa}$ on $(\bar{N}, \bar{g})$. Then a vector field $G_*(\dot{\beta})$ along $\bar{\beta}$ is defined by

$$G_*(\dot{\beta}) = G_{*\beta}\dot{\beta},$$

where $\dot{\beta}(s) = \dot{\beta}$ is a vector field along $\beta(s) = \beta$.

(i) From equations (2.3), (2.4) and (2.5), we have

$$\bar{\nabla}^2_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}). \tag{3.19}$$

Taking the covariant derivative of (3.22) and using (2.3), (2.4) and (2.5), we get the required condition.

(ii) From equations (2.3), (2.4) and (2.5), we get the required equation. □

Definition 3.2. *A Riemannian submersion $G : (N, g) \to (\bar{N}, \bar{g})$ between Riemannian manifolds (N, g) and $(\bar{N}, \bar{g})$ is said to be totally geodesic conformal submersion if second fundamental form of G is identically zero. i.e.*

$$(\nabla G_*)(X, Y) = 0, \ \forall \ X, Y \in \Gamma(TN). \tag{3.20}$$

Lemma 3.2. *Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a totally geodesic conformal submersion between Riemannian manifolds (N, g) and $(\bar{N}, \bar{g})$. If β is a horizontal curve with curvature κ on (N, g) and $\bar{\beta} = G \circ \beta$ is a bi-f-harmonic curve on $(\bar{N}, \bar{g})$, then the curvature of $\bar{\beta}$ is given by*

$$\kappa = \frac{1}{f^{\frac{4}{3}}}\left(\frac{2}{3}\int f^{\frac{2}{3}}(ff''' + f'f'')ds + C\right)^{\frac{1}{2}}, \tag{3.21}$$

where C is some constant.

Proof. Let $G : (N, g) \to (\bar{N}, \bar{g})$ be a conformal submersion between Riemannian manifolds (N, g) and $(\bar{N}, \bar{g})$. Then for any horizontal curve β on (N, g) and bi-f-harmonic curve $\bar{\beta} = G \circ \beta$ on $(\bar{N}, \bar{g})$, we have

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0.\end{aligned} \tag{3.22}$$

From Lemma 3.1 and equation (3.22), we have

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) \\ + f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0.\end{aligned} \tag{3.23}$$

Now using second part of Lemma (3.1) in equation (3.23), we get

$$(ff''' + f'f'')G_*(\dot\beta) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta) + 4ff'G_*(\overset{\wedge}{\nabla}^{2}_{\dot\beta}\dot\beta)$$
$$+f^2G_*(\overset{\wedge}{\nabla}^{3}_{\dot\beta}\dot\beta) + f^2G_*(R(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)\dot\beta) - 2f^2(\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta)$$
$$+f^2(\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) + f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta) = 0. \tag{3.24}$$

Taking inner-product of equation (3.24) with $G_*(\dot\beta)$ both sides, we obtain

$$\lambda^2 f^2 g_N(\overset{\wedge}{\nabla}^{3}_{\dot\beta}\dot\beta, \dot\beta) + \lambda^2 f^2 g_N(R(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)\dot\beta, \dot\beta) - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$\lambda^2(ff''' + f'f'') + \lambda^2(3ff'' + 2f'^2)g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) + \lambda^2 4ff' g_N(\overset{\wedge}{\nabla}^{2}_{\dot\beta}\dot\beta, \dot\beta)$$
$$+f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.25}$$

Substituting the values of $g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) = -g_N(A_{\dot\beta}\dot\beta, \dot\beta)$, $g_N(\overset{\wedge}{\nabla}^{2}_{\dot\beta}\dot\beta, \dot\beta) = -\kappa^2 - g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta)$ $- g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)$ and $g_N(\overset{\wedge}{\nabla}^{3}_{\dot\beta}\dot\beta, \dot\beta) = -3\kappa\kappa' - g_N(v\nabla_{\dot\beta}v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) - g_N(v\nabla_{\dot\beta}A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) -$ $g_N(A_{\dot\beta}\overset{\wedge}{\nabla}^{2}_{\dot\beta}\dot\beta, \dot\beta)$ in equation (3.25), we obtain

$$\lambda^2(ff''' + f'f'') - \lambda^2(3ff'' + 2f'^2)g_N(A_{\dot\beta}\dot\beta, \dot\beta) - \lambda^2 4ff' g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta)$$
$$-\lambda^2 4ff'\kappa^2 - \lambda^2 4ff' g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) - \lambda^2 3\kappa\kappa' f^2 - \lambda^2 f^2 g_N(v\nabla_{\dot\beta}v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta)$$
$$-\lambda^2 f^2 g_N(v\nabla_{\dot\beta}A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) - f^2 g_N(A_{\dot\beta}\overset{\wedge}{\nabla}^{2}_{\dot\beta}\dot\beta, \dot\beta) - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$+f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.26}$$

Using the orthogonal condition in equation (3.26), we have

$$(ff''' + f'f'') - 4ff'\kappa^2 - 3\kappa\kappa' f^2 - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$+f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.27}$$

Since G is totally geodesic, then equation (3.27) reduces to (by using mapple),

$$\kappa = \frac{1}{f^{\frac{4}{3}}}(\frac{2}{3}\int f^{\frac{2}{3}}(ff''' + f'f'')ds + C)^{\frac{1}{2}}. \tag{3.28}$$

□

Theorem 3.1. *Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a totally geodesic conformal submersion between Riemannian manifolds (N, g_N) and $(\bar N, g_{\bar N})$. Then G maps horizontal bi-f-harmonic curve on (N, g_N) to bi-f-harmonic curve on $(\bar N, g_{\bar N})$.*

Proof. Substituting the values of $\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$, $\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta})$ and $\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta})$ in equation (3.22), we get

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = (ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) \\ &+4ff'G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}) + f^2G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + f^2\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}). \qquad (3.29)\end{aligned}$$

Then using second part of Lemma 3.1 in the equation (3.29), we obtain

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = (ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) \\ &+4ff'G_*(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}) + f^2G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + f^2G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &-2f^2(\nabla G_*)(\dot{\beta}, H\nabla_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) + f^2(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \qquad (3.30)\end{aligned}$$

Now from equations (2.4) and (3.30), we get

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*\{(ff''' + f'f'')(\dot{\beta}) + (3ff'' + 2f'^2)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) \\ &+4ff'(\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}) + f^2(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + f^2(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta})\} - 2f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &+f^2(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \qquad (3.31)\end{aligned}$$

Using the fact that β is a horizontal bi-f-harmonic curve on (N, g_N), equation (3.31) reduces to

$$\begin{aligned}&(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) + \\ &f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(0) - 3f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ &+f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \qquad (3.32)\end{aligned}$$

Since G is a totally geodesic conformal submersion, therefore

$$\begin{aligned}(ff''' + f'f'')G_*(\dot{\beta}) &+ (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) \\ &+f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \qquad (3.33)\end{aligned}$$

□

Theorem 3.2. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$. Now, if β is a bi-f-harmonic curve on (N, g_N) and $\bar{\beta} = G \circ \beta$ is a bi-f-harmonic curve on $(\bar{N}, g_{\bar{N}})$, then either*

$$ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' < 0 \text{ or } g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, H'), G_*(\dot{\beta})) \geq 0. \tag{3.34}$$

Proof. Let $\bar{\beta}$ be a bi-f-harmonic curve on $(\bar{N}, g_{\bar{N}})$, then

$$(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) + 4ff'\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta})$$
$$+f^2\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + f^2\bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{3.35}$$

Substituting the values of $\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$, $\bar{\nabla}^2_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})$ and $\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ in equation (3.35), we get

$$(ff''' + f'f'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})$$
$$+f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + f^2\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{3.36}$$

Then substituting $\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta})$ $+ (\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta})$ in equation (3.36), we have

$$(ff''' + ff'')G_*(\dot{\beta}) + (3ff'' + 2f'^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 4ff'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + f^2G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$$
$$+f^2G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2f^2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) + f^2(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})$$
$$+f^2(\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}) = 0. \tag{3.37}$$

Taking the inner-product of equation (3.37) with $G_*(\dot{\beta})$ both sides, we obtain

$$(ff''' + f'f'')g_{\bar{N}}(G_*(\dot{\beta}), G_*(\dot{\beta})) + (3ff'' + 2f'^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$$
$$+4ff'g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + f^2g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + f^2g_{\bar{N}}(G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}), G_*(\dot{\beta}))$$
$$-2f^2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + f^2g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta}))$$
$$+f^2g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{3.38}$$

Using the definition of conformal submersion in equation (3.38), we obtain

$$(ff''' + f'f'')\lambda^2 + (3ff'' + 2f'^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) + 4ff'\lambda^2 g_N(\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta, \dot\beta)$$
$$+f^2\lambda^2 g_N(\overset{\wedge}{\nabla}{}^3_{\dot\beta}\dot\beta, \dot\beta) + f^2\lambda^2 g_N(R(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)\dot\beta, \dot\beta) - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$+f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.39}$$

Substituting $g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta$, $g_N(\overset{\wedge}{\nabla}{}^2_{\dot\beta}\dot\beta, \dot\beta)$ and $g_N(\overset{\wedge}{\nabla}{}^3_{\dot\beta}\dot\beta, \dot\beta)$ in equation (3.39), we obtain

$$(ff''' + f'f'')\lambda^2 - 4ff'\kappa^2\lambda^2 - 3\kappa\kappa' f^2\lambda^2 - 2f^2 g_{\bar N}((\nabla G_*)(\dot\beta, A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta), G_*(\dot\beta))$$
$$+f^2 g_{\bar N}((\nabla G_*)(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta), G_*(\dot\beta)) + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, A_{\dot\beta}\dot\beta), G_*(\dot\beta)) = 0. \tag{3.40}$$

Then using the definition of totally umbilical i.e. $A_{\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta}\dot\beta = g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)H'$, $A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta = g_N(\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta)H'$ and $A_{\dot\beta}\dot\beta = g_N(\dot\beta, \dot\beta)H'$ in equation (3.40), we have

$$(ff''' + f'f'')\lambda^2 - 4ff'\kappa^2\lambda^2 - 3\kappa\kappa'\lambda^2 + f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) = 0. \tag{3.41}$$

Since equation (3.41) is a quadratic equation in λ, therefore

$$\lambda = \frac{0 \pm \sqrt{-4(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2)f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta))}}{2(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2)}. \tag{3.42}$$

Since λ is a positive real valued function, therefore

$$4(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2)f^2 g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) \leq 0. \tag{3.43}$$

Thus from equations (3.42) and (3.43), we can conclude that either $(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2) < 0$ and $g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) \geq 0$ or $(ff''' + f'f'' - 4ff'\kappa^2 - 3\kappa\kappa' f^2) > 0$ and $g_{\bar N}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, H'), G_*(\dot\beta)) \leq 0$, to make λ always positive. ☐

3.1. **Characterization of bi-harmonic curves.** A bi-harmonic curve (bi-1-harmonic curve) is a special case of bi-f-harmonic curve for $f = 1$. Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar N, g_{\bar N})$ such that $\bar\beta$ is the bi-harmonic curve on $(\bar N, g_{\bar N})$, then

$$\bar\nabla^3_{G_*(\dot\beta)} G_*(\dot\beta) + \bar R(G_*(\nabla_{\dot\beta}\dot\beta), G_*(\dot\beta))G_*(\dot\beta) = 0.$$

Theorem 3.3. *Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar N, g_{\bar N})$. If β is a horizontal curve with curvature κ on (N, g) and $\bar\beta = G \circ \beta$ is a bi-harmonic curve on $(\bar N, \bar g)$, then κ is constant.*

Proof. Let $\bar{\beta}$ is a bi-harmonic curve on $(\bar{N}, g_{\bar{N}})$, then taking $f = 1$ in equation (3.23), we have

$$G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + \bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{3.44}$$

Using second part of Lemma 3.1 in equation (3.44), we get

$$\begin{aligned} G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 2(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) \\ +(\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}) = 0. \end{aligned} \tag{3.45}$$

Taking inner-product of equation (3.45) with $G_*(\dot{\beta})$, we obtain

$$\begin{aligned} g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}(G_*(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}), G_*(\dot{\beta})) - 2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ +g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \end{aligned} \tag{3.46}$$

Using the definition of conformal submersion and $g_N(R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}, \dot{\beta}) = 0$ in equation (3.46), we get

$$\begin{aligned} \lambda^2 g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - 2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ +g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \end{aligned} \tag{3.47}$$

Substituting $g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -3\kappa\kappa' - g_N(v\nabla^3_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and $g_N(v\nabla^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0$ in equation (3.47), we obtain

$$\begin{aligned} -\lambda^2 3\kappa\kappa' - 2g_{\bar{N}}((\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) \\ +g_{\bar{N}}((\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \end{aligned} \tag{3.48}$$

Since G be a totally geodesic conformal submersion i.e. second fundamental form is identically zero, therefore equation (3.48) reduces to

$$-\lambda^2 3\kappa\kappa' = 0, \implies \kappa = constant. \tag{3.49}$$

□

Theorem 3.4. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a totally geodesic conformal submersion between Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$. Then G maps horizontal bi-harmonic curve on (N, g_N) to bi-harmonic curve on $(\bar{N}, g_{\bar{N}})$.*

Proof. Taking $f = 1$ and substituting $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta})$ in equation (3.33), we get

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = \overset{\wedge}{\nabla}{}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}). \tag{3.50}$$

Using the second part of Lemma 3.1 in equation (3.50), we get

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta} + R(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})\dot{\beta}) - 3(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}) + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \tag{3.51}$$

Using the fact that β is a horizontal bi-harmonic curve on (N, g_N), equation (3.51) reduces to

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = -3(\nabla G_*)(\dot{\beta}, A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta} + (\nabla G_*)(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, A_{\dot{\beta}}\dot{\beta}). \tag{3.52}$$

Since G is a totally geodesic conformal submersion map, therefore

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + \bar{R}(G_*(\nabla_{\dot{\beta}}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0 \tag{3.53}$$

Hence $\bar{\beta}$ is a bi-harmonic curve on $(\bar{N}, g_{\bar{N}})$.

□

4. Helices and circles along the conformal submersion

Let $\beta : I \to N$ be a curve, then β is said to be a general helix if it satisfies the condition

$$\nabla^3_{\dot{\beta}}\dot{\beta} + K^2 \nabla_{\dot{\beta}}\dot{\beta} = 0,$$

where $K^2 = \kappa^2 + \tau^2$ is a positive constant. Conversely, if the curve $\beta = \beta(s)$ satisfies the above condition, then it is an ordinary helix or a geodesic [18].

Theorem 4.1. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$. Then, $\bar{\beta} = G \circ \beta$ is a helix on $(\bar{N}, g_{\bar{N}})$ iff β is a horizontal curve of constant curvature on (N, g_N).*

Proof. Let $\bar{\beta}$ be a helix on $(\bar{N}, g_{\bar{N}})$, then

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0. \tag{4.54}$$

Using $\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta})$ in equation (4.54), we get

$$G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2+\tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) = 0. \tag{4.55}$$

Taking inner-product of equation (4.55) with $G_*(\dot{\beta})$, we obtain

$$g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + (\kappa^2+\tau^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{4.56}$$

Using the definition of conformal submersion in equation (4.56), we have

$$\lambda^2 g_N(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\kappa^2+\tau^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{4.57}$$

Substituting $g_N(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -3\kappa\kappa' - g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and $g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ in equation (4.57), we get

$$\begin{aligned} &-\lambda^2 3\kappa\kappa' - \lambda^2 g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - \lambda^2 g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) \\ &-\lambda^2 g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - (\kappa^2+\tau^2)\lambda^2 g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \end{aligned} \tag{4.58}$$

Using the condition of orthogonality in equation (4.58), we have

$$\lambda^2 3\kappa\kappa' = 0 \implies \kappa = C(constant).$$

Conversely, assume that β be a curve of constant curvature on (N, g_N) and $\bar{\beta} = G\circ\beta$ is a curve on $(\bar{N}, g_{\bar{N}})$, where $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion and using equation (3.17). Then, we have

$$\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + (\kappa^2+\tau^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2+\tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}). \tag{4.59}$$

Taking inner-product of equation (4.59) with $G_*(\dot{\beta})$ both sides, we have

$$\begin{aligned} g_{\bar{N}}(\bar{\nabla}^3_{G_*(\dot{\beta})}G_*(\dot{\beta}) + (\kappa^2+\tau^2)\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}), G_*(\dot{\beta})) &= g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}) + (\kappa^2+\tau^2)G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ &= g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + (\kappa^2+\tau^2)g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) \\ &= \lambda^2 g_N(\overset{\wedge}{\nabla}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (\kappa^2+\tau^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) \\ &= -\lambda^2 3\kappa\kappa' - \lambda^2 g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - \lambda^2 g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) \\ &= -\lambda^2 g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - (\kappa^2+\tau^2)\lambda^2 g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \end{aligned}$$

Therefore

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0.$$

Hence $\bar{\beta}$ is a helix. □

Theorem 4.2. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a totally geodesic conformal submersion between Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$. Then G maps horizontal helix on (N, g_N) to a helix on $(\bar{N}, g_{\bar{N}})$.*

Proof. From equation (4.54) and using relation $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta})$, we get

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta} + (\kappa^2 + \tau^2)\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}). \tag{4.60}$$

Since β is a horizontal helix on (N, g_N), therefore equation (4.60) reduces to

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0.$$

Hence, $\bar{\beta}$ is a helix on $(\bar{N}, g_{\bar{N}})$. □

Corollary 4.1. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion map between two Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ such that β is a helix on (N, g_N). If $\bar{\beta} = G \circ \beta$ is a helix on $(\bar{N}, g_{\bar{N}})$, then β is a helix of constant curvature on (N, g_N).*

Proof. Since $\bar{\beta}$ is a helix on $(\bar{N}, g_{\bar{N}})$, so

$$\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) + (\kappa^2 + \tau^2)\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = 0. \tag{4.61}$$

Substituting the values of $\bar{\nabla}^3_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta})$ and $\overset{\wedge}{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta})$ in equation (4.61), we have

$$G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta}) + (\kappa^2 + \tau^2) G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}) = 0. \tag{4.62}$$

Taking the inner-product of equation (4.62) with $G_*(\dot{\beta})$ both sides, we get

$$g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta}), G_*(\dot{\beta})) + (\kappa^2 + \tau^2) g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}), G_*(\dot{\beta})) = 0. \tag{4.63}$$

Using the definition of conformal submersion in equation (4.63), we obtain

$$\lambda^2 + g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + (\kappa^2 + \tau^2)\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = 0. \tag{4.64}$$

Substituting the values of $g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}} \dot{\beta}, \dot{\beta})$ and $g_N(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta})$ in equation (4.64), we get the required result. □

Theorem 4.3. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion map between two Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ such that β is a circle on (N, g_N). If $\bar{\beta} = G \circ \beta$ is a circle on $(\bar{N}, g_{\bar{N}})$, then curvature $\kappa = \pm 1$, where κ is curvature of β.*

Proof. Let $\bar{\beta}$ is a circle on $(\bar{N}, g_{\bar{N}})$, then

$$\bar{\nabla}^2_{G_*(\dot{\beta})} G_*(\dot{\beta}) + g_{\bar{N}}(\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}), \bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta})) G_*(\dot{\beta}) = 0. \tag{4.65}$$

Substituting the values of $\bar{\nabla}^2_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta})$ and $\bar{\nabla}_{G_*(\dot{\beta})} G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta})$ in equation (4.65), we get

$$G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}) + g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}), G_*(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta})) G_*(\dot{\beta}) = 0. \tag{4.66}$$

Using the definition of conformal submersion in equation (4.66), we get

$$G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}) + \lambda^2(p) g_N(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}) G_*(\dot{\beta}) = 0. \tag{4.67}$$

Substituting $g_N(\overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}) = 1$ in equation (4.67), we obtain

$$G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}) + \lambda^2 G_*(\dot{\beta}) = 0. \tag{4.68}$$

Taking inner-product of equation (4.68) with $G_*(\dot{\beta})$, gives us

$$g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}(G_*(\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{4.69}$$

Again using the definition of conformal submersion in equation (4.69), we have

$$\lambda^2 g_N(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + \lambda^2 g_N(\dot{\beta}, \dot{\beta}) = 0. \tag{4.70}$$

Substituting the values of $g_N(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = -\kappa^2 - g_N(v\nabla_{\dot{\beta}} A_{\dot{\beta}} \dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta})$ and $g_N(\dot{\beta}, \dot{\beta}) = 1$ in equation (4.70), we get

$$-\lambda^2 \kappa^2 - \lambda^2 g_N(v\nabla_{\dot{\beta}} A_{\dot{\beta}} \dot{\beta}, \dot{\beta}) - \lambda^2 g_N(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) + \lambda^2 = 0. \tag{4.71}$$

Since $g_N(v\nabla_{\dot{\beta}} A_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = 0$ and $g_N(A_{\dot{\beta}} \overset{\wedge}{\nabla}_{\dot{\beta}} \dot{\beta}, \dot{\beta}) = 0$. Thus from equation (4.71), we get the required result.

□

Theorem 4.4. *Let $G : (N, g) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion map between two Riemannian manifolds (N, g) and $(\bar{N}, g_{\bar{N}})$. If β is a circle on (N, g) and $\bar{\beta} = G \circ \beta$ is a circle on $(\bar{N}, g_{\bar{N}})$, then either $\lambda = \pm\kappa$ or $\lambda = 0$, where κ is curvature of β on N.*

Proof. Considering the definition of conformal submersion in equation (4.66), we get

$$G_*(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta) + \lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta)G_*(\dot\beta) = 0. \tag{4.72}$$

Taking inner-product of equation (4.72) with $G_*(\dot\beta)$, gives us

$$g_{\bar N}(G_*(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta), G_*(\dot\beta)) + \lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta)g_{\bar N}(G_*(\dot\beta), G_*(\dot\beta)) = 0. \tag{4.73}$$

Since, $g_N(\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \overset{\wedge}{\nabla}_{\dot\beta}\dot\beta) = 1$ and $g_N(\dot\beta, \dot\beta) = 1$, therefore equation (4.73) reduces to

$$\lambda^2 g_N(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta, \dot\beta) + \lambda^2\lambda^2 = 0. \tag{4.74}$$

Taking $g_N(\overset{\wedge}{\nabla}^2_{\dot\beta}\dot\beta, \dot\beta) = -\kappa^2 - g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) - g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)$ in equation (4.74), then we have

$$-\lambda^2(\kappa^2 + g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) + g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta)) + \lambda^2\lambda^2 = 0. \tag{4.75}$$

Substituting $g_N(v\nabla_{\dot\beta}A_{\dot\beta}\dot\beta, \dot\beta) = 0$ and $g_N(A_{\dot\beta}\overset{\wedge}{\nabla}_{\dot\beta}\dot\beta, \dot\beta) = 0$ in equation (4.75), we get

$$-\kappa^2\lambda^2 + \lambda^2\lambda^2 = 0. \tag{4.76}$$

As equation (4.76) is quadratic in λ^2, therefore

$$\lambda^2 = \frac{\kappa^2 \pm \sqrt{\kappa^4}}{2}. \tag{4.77}$$

Thus, from equation (4.77), we can say that either $\lambda = \pm\kappa$ or $\lambda = 0$.

□

5. Hyperelastic curve along the conformal submersion

In this section, we study the hyperelastic curve along the conformal submersion.

Theorem 5.1. *Let $G : (N, g_N) \to (\bar N, g_{\bar N})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar N, g_{\bar N})$. If β is a hyperelastic curve on (N, g_N) and $\bar\beta = G \circ \beta$ is a hyperelastic curve on $(\bar N, g_{\bar N})$, then*

$$\begin{aligned}(-2(r-2)\kappa^{r-1}\kappa' - 3\kappa^{r-1}\kappa')\lambda^2 + \lambda^2(\dot\beta(\tfrac{2r-1}{r}\kappa^r + b))\\ +\kappa^{r-2}g_{\bar N}((\nabla G_*)(-\tfrac{\lambda^2}{2}(\dot\beta\tfrac{1}{\lambda^2}), \dot\beta), G_*(\dot\beta)) = 0,\end{aligned} \tag{5.78}$$

where $r \geq 2$ is a natural number.

Proof. Let $\bar{\beta}$ is a hyperelastic curve on $(\bar{N}, g_{\bar{N}})$, then from [33], we have

$$\bar{\nabla}^2_{G_*(\dot{\beta})}(\kappa^{r-2}\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta})) + \bar{\nabla}_{G_*(\dot{\beta})}(\mu G_*(\dot{\beta})) + \kappa^{r-2}\bar{R}(\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) = 0. \tag{5.79}$$

Substituting $\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta}) = G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})$, $\bar{\nabla}^2_{G_*(\dot{\beta})}(\kappa^{r-2}\bar{\nabla}_{G_*(\dot{\beta})}G_*(\dot{\beta})) = \kappa^{r-2}G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + (r-2)(r-3)\kappa^{r-4}(\kappa')^2G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa''G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta})$ and $\bar{\nabla}_{G_*(\dot{\beta})}(\mu G_*(\dot{\beta})) = G_*(\nabla_{\dot{\beta}}\mu\dot{\beta})$ in equation (5.79), we have

$$(r-2)(r-3)\kappa^{r-4}\kappa'^2G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + (r-2)\kappa^{r-3}\kappa''G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}) + 2(r-2)\kappa^{r-3}\kappa'G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}) + \kappa^{r-2}G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}) + \kappa^{r-2}\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}) + G_*(\nabla_{\dot{\beta}}\mu\dot{\beta}) = 0. \tag{5.80}$$

Taking inner-product of equation (5.80) with $G_*(\dot{\beta})$ both sides, we get

$$((r-2)(r-3)\kappa^{r-4}\kappa'^2 + (r-2)\kappa^{r-3}\kappa'')g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + 2(r-2)\kappa^{r-3}\kappa'g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}(G_*(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}(\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}(G_*(\nabla_{\dot{\beta}}\mu\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{5.81}$$

Substituting $g_{\bar{N}}(\bar{R}(G_*(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))G_*(\dot{\beta}), G_*(\dot{\beta})) = -2g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$ and using the definition of conformal submersion in equation (5.81), we get

$$\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})((r-2)(r-3)\kappa^{r-4}\kappa'^2 + \kappa^{r-2}\lambda^2 g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) + (r-2)\kappa^{r-3}\kappa'') + 2(r-2)\kappa^{r-3}\kappa'\lambda^2 g_N(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - 2\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + \lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\mu\dot{\beta}, \dot{\beta}) + \kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) = 0. \tag{5.82}$$

Substituting the values of
$g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta})$, $g_N(\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -\kappa^2 - g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ and
$g_N(\overset{\wedge}{\nabla}{}^3_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = -3\kappa\kappa' - g_N(v\nabla_{\dot{\beta}}v\nabla_{\dot{\beta}}A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(v\nabla_{\dot{\beta}}A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) - g_N(A_{\dot{\beta}}\overset{\wedge}{\nabla}{}^2_{\dot{\beta}}\dot{\beta}, \dot{\beta})$ in equation

(5.82), we obtain

$$-2(r-2)\kappa^{r-1}\kappa'\lambda^2 - 3\kappa^{r-1}\kappa'\lambda^2 - 2\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$$

$$+\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta})) + \kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}|, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta}))$$

$$+\lambda^2 g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\mu\dot{\beta}, \dot{\beta}) = 0. \tag{5.83}$$

Since $\overset{\wedge}{\nabla}_{\dot{\beta}}\mu\dot{\beta} = (\dot{\beta}(\frac{2r-1}{r}\kappa^r + b))\dot{\beta} + (\frac{2r-1}{r}\kappa^r + b)\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}$, where $\mu = \frac{2r-1}{r}\kappa^r + b$, therefore from equation (5.83),

$$-2(r-2)\kappa^{r-1}\kappa'\lambda^2 - 3\kappa^{r-1}\kappa'\lambda^2 - 2\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta}\dot{\beta}, \dot{\beta}), G_*(\dot{\beta}))$$

$$+\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(A_{\dot{\beta}}\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \lambda^2 g_N((\dot{\beta}(\tfrac{2r-1}{r}\kappa^r + b))\dot{\beta}$$

$$+(\tfrac{2r-1}{r}\kappa^r + b)\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{5.84}$$

Using the totally umbilical conditions $A_{\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}}\dot{\beta} = g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})H'$, $\forall\ \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta},\ \dot{\beta} \in \Gamma(kerG_*)^{\perp}$, where $H' = -\frac{\lambda^2}{2}(\nabla_{\dot{\beta}}\frac{1}{\lambda^2})$ and $A_{\dot{\beta}}\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta} = g_N(\dot{\beta}, \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta})H'$ in equation (5.84), we get

$$(-2(r-2)\kappa^{r-1}\kappa' - 3\kappa^{r-1}\kappa')\lambda^2 - \kappa^{r-2}g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta})g_{\bar{N}}((\nabla G_*)(H', \dot{\beta}), G_*(\dot{\beta}))$$

$$+\kappa^{r-2}g_N(\dot{\beta}, \dot{\beta})g_{\bar{N}}((\nabla G_*)(H', \overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}), G_*(\dot{\beta})) + \lambda^2(\dot{\beta}(\tfrac{2r-1}{r}\kappa^r + b))g_N(\dot{\beta}, \dot{\beta})$$

$$+\lambda^2(\tfrac{2r-1}{r}\kappa^r + b)g_N(\overset{\wedge}{\nabla}_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0. \tag{5.85}$$

Substituting $H' = -\frac{\lambda^2}{2}(\dot{\beta}\frac{1}{\lambda^2})$ and $g_N(A_{\dot{\beta}}\dot{\beta}, \dot{\beta}) = 0$ in equation (5.85), we obtain

$$-\lambda^2(2(r-2)\kappa^{r-1}\kappa' + 3\kappa^{r-1}\kappa') + \lambda^2(\dot{\beta}(\tfrac{2r-1}{r}\kappa^r + b))$$

$$+\kappa^{r-2}g_{\bar{N}}((\nabla G_*)(-\tfrac{\lambda^2}{2}(\dot{\beta}\tfrac{1}{\lambda^2}), \dot{\beta}), G_*(\dot{\beta})) = 0. \tag{5.86}$$

Hence the proof. □

Corollary 5.1. *Let $G : (N, g_N) \to (\bar{N}, g_{\bar{N}})$ be a conformal submersion between Riemannian manifolds (N, g_N) and $(\bar{N}, g_{\bar{N}})$ such that β is a elastic curve on (N, g_N). If $\bar{\beta} = G \circ \beta$ is a elastic curve on $(\bar{N}, g_{\bar{N}})$, then*

$$g_{\bar{N}}((\nabla G_*)(-\frac{\lambda^2}{2}(\dot{\beta}\frac{1}{\lambda^2}), \dot{\beta})G_*(\dot{\beta})) = 0. \tag{5.87}$$

Proof. Substituting $r = 2$ in equation (5.78), we have

$$-3\kappa\kappa'\lambda^2 + g_{\bar{N}}((\nabla G_*)(-\frac{\lambda^2}{2}(\dot{\beta}\frac{1}{\lambda^2}), \dot{\beta}), G_*(\dot{\beta})) + \lambda^2(\dot{\beta}(\frac{3}{2}\kappa^2 + b)) = 0. \tag{5.88}$$

Substituting the value of $\dot{\beta}(\frac{3}{2}\kappa^2 + b) = 3\kappa\kappa'$ in equation (5.88), we get the required result. □

Acknowledgments. The authors wish to express their sincere thanks and gratitute to the referees for their valuable suggestions toward the improvement of the paper. The first author would like to thanks 'Incentive Grant' under IoE Scheme of BHU and the second would like to thanks university grant commission of India for their financial support Ref. No. 1179/ (CSIR-UGC NET JUNE 2019).

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