International Journal of Maps in Mathematics

Volume 7, Issue 2, 2024, Pages:224-235

E-ISSN: 2636-7467

www.journalmim.com

ON THE OPERATOR EQUATION $ABA = ACA$ AND ITS GENERALIZATION ON NON-ARCHIMEDEAN BANACH SPACES

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Abstract. Let X and Y be non-Archimedean Banach spaces over a non-Archimedean valued field $\mathbb{K}$. In this paper, we study some properties of $A \in \mathcal{L}(X,Y)$ and $B, C \in \mathcal{L}(Y,X)$ such that $ABA = ACA$ and many basic operator properties in common of $AC - I_Y$ and $BA - I_X$ are given. In particular, $N(I_Y - AC)$ is a complemented subspace of Y if and only if $N(I_X - BD)$ is a complemented subspace of X. Moreover, the approach is generalized for considering relationships between the properties of $I_Y - AC$ and $I_X - BD$. Finally, several illustrative examples are provided.

Keywords: Non-Archimedean Banach spaces, operator equation, operators theory.

2010 Mathematics Subject Classification: Primary 47A05, 47A10, Secondary 47S10.

1. Introduction

In classical operators theory, the operator equations $ABA = A^2$ and $BAB = B^2$ were studied by sereval researchers, for more details see [10, 11, 12, 14]. Recently, Barnes established some consequences on common operator properties of continuous linear operators AB and BA on complex Banach spaces, for more details, we refer to [1]. Moreover, Corach et al. [2] established some common properties of $AC - I_Y$ and $BA - I_X$ when $ABA = ACA$. In [17], Zeng and Zhong proceeded to examine common properties of AC and BA from the attitude of classical spectral theory considering $A \in \mathcal{L}(X,Y), B, C \in \mathcal{L}(Y,X)$ such that $ABA = ACA$, where X and Y were assumed to be Banach spaces over $\mathbb{C}$. In particular, they gave an affirmative answer to one question raised by the authors [2], by showing that $AC - I_Y$ is of

Received:2023.11.21 *Revised:2024.04.06* *Accepted:2024.04.25*

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closed range if and only if $BA - I_X$ is of closed range. Yan and Fang [16] examined the joint properties of BD and AC in terms of regularity when $A, D \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $ACD = DBD$ and $DBA = ACA$. In non-Archimedean operators theory, Ettayb [4] studied specific properties of operator equations $ABA = A^2$ and $BAB = B^2$ on a non-Archimedean Banach space X and many basic operator properties in common of $I_X - AB$ and $I_X - BA$ were described. In particular, if X, Y are non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$, $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, then $N(I_Y - AB)$ is a complemented subspace of Y if and only if $N(I_X - BA)$ is a complemented subspace of X. Recently, Ettayb [6] examined the common properties of AC and BD whenever $A, D \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ such that $ACD = DBD$ and $DBA = ACA$ where X and Y were supposed to be non-Archimedean Banach spaces over a non-Archimedean valued field $\mathbb{K}$.

Non-Archimedean spectral theory played a crucial role in non-Archimedean functional analysis which has had numerous applications in non-Archimedean applied mathematics and physics, including non-Archimedean differential, pseudo-differential equations and quantum physics. For additional details see [7, 8, 15]. This work is motivated by many studies on non-Archimedean operators theory and spectral theory of continuous linear operators, e.g. [4, 6, 5, 7, 8, 15]. Throughout this paper, $\mathbb{K}$ is a complete non-Archimedean valued field with a non-trivial valuation $|\cdot|$, X and Y are non-Archimedean Banach spaces over $\mathbb{K}$ and $\mathcal{L}(X, Y)$ will denote the collection of all continuous linear operators from X into Y. For $X = Y$, we put $\mathcal{L}(X, X) = \mathcal{L}(X)$. $(\mathbb{Q}_p, |.|_p)$ is the field of p-adic numbers. For more details, we refer to [3, 9, 13]. Let $A \in \mathcal{L}(X)$, $R(A)$, $N(A)$, A^*, $\sigma(A)$, $\sigma_p(A)$ and $\rho(A)$ denote the range, the kernel, the adjoint, the spectrum, the point spectrum and the resolvent set of A respectively.

The goal of this work is to develop the theory of some operator equations of continuous linear operators in non-Archimedean Banach spaces.

2. Preliminaries

We continue with the following preliminaries.

Definition 2.1. [3] *A field $\mathbb{K}$ is said to be non-Archimedean if it is endowed with an absolute value $|\cdot| : \mathbb{K} \to \mathbb{R}^+$ such that:*

(i) $|\alpha| = 0$ *if, and only if,* $\alpha = 0$;

(ii) *For all* $\alpha, \mu \in \mathbb{K}$, $|\alpha\mu| = |\alpha||\mu|$;

(iii) *For each* $\alpha, \mu \in \mathbb{K}$, $|\alpha + \mu| \leq \max\{|\alpha|, |\mu|\}$.

Definition 2.2. [9] *$\mathbb{K}$ is said to be spherically complete if each sequence of balls $(B_n)_{n\geq 1}$ of $\mathbb{K}$ such that $B_{n+1} \subset B_n$ for every $n \geq 1$, we have $\bigcap_{n\geq 1} B_n \neq \emptyset$.*

Definition 2.3. [3] *Let X be a vector space over $\mathbb{K}$. A function $\|\cdot\| : X \to \mathbb{R}_+$ is called a non-Archimedean norm if:*

(i) *For each $u \in X$, $\|u\| = 0$ if and only if $u = 0$;*
(ii) *For all $u \in X$ and $\lambda \in \mathbb{K}$, $\|\lambda u\| = |\lambda|\|u\|$;*
(iii) *For each $u, y \in X$, $\|u + y\| \leq \max(\|u\|, \|y\|)$.*

Definition 2.4. [3] *A non-Archimedean Banach space is a complete non-Archimedean normed space.*

Example 2.1. [3] *The space $c_0(\mathbb{K})$ is the space of all sequences $(x_i)_{i\in\mathbb{N}}$ in $\mathbb{K}$ such that $\lim_{i\to\infty} x_i = 0$. Hence $(c_0(\mathbb{K}), \|\cdot\|)$ is a non-Archimedean Banach space where for any $(x_i)_{i\in\mathbb{N}} \in c_0(\mathbb{K})$, $\| (x_i)_{i\in\mathbb{N}} \| = \sup_{i\in\mathbb{N}} |x_i|$.*

Definition 2.5. [13] *Let $P \in \mathcal{L}(X)$. P is said to be a projection if $P^2 = P$.*

Remark 2.1. [13] *If P is a projection on X, then $R(P)$ is the kernel of $I - P$ so that $R(P)$ is a closed linear subspace of X.*

The following lemma holds.

Lemma 2.1. [13] *Let $P \in \mathcal{L}(X)$ be a projection. Hence, we have the following:*

(i) *$I_X - P$ is a projection;*
(ii) *If $P \neq 0$, then $\|P\| \geq 1$;*
(iii) *If $P \neq 0$ and $P \neq I_X$, then $\|P\| = \|I_X - P\|$;*
(iv) *If Q is projection such that $PQ = QP$, then PQ is projection.*

We have the following definition.

Definition 2.6. [9] *A subspace D of X is said to be complemented if there is a continuous projection $P \in \mathcal{L}(X)$ such that $R(P) = D$. In such case, $D = R(P)$ and $D_1 = N(P)$ are closed subspaces and $X = D \oplus D_1$.*

Remark 2.2. [9] *If D, D_1 are closed subspaces of X such that $X = D \oplus D_1$, then D is complemented in X and D_1 is a complement of D.*

For more details, see [9, 13].

Proposition 2.1. [6] *Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, we have:*

(i) $N(B) \cap N(I_Y - AB) = \{0\}$;

(ii) $B(N(I_Y - AB)) = N(I_X - BA)$.

The following theorem is valid.

Theorem 2.1. [6] *Let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$. Hence $R(I_Y - AB)$ is closed if and only if $R(I_X - BA)$ is closed.*

Theorem 2.2. [4] *Let X and Y be non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$ and let $A \in \mathcal{L}(X, Y)$ and $B \in \mathcal{L}(Y, X)$, hence $N(I_Y - AB)$ is a complemented subspace of Y if and only if $N(I_X - BA)$ is a complemented subspace of X.*

3. Main Results

We have the following results.

Lemma 3.1. *Let $A \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ with $ABA = ACA$. Hence $R(AC - I_Y)$ is closed in Y if and only if $R(BA - I_X)$ is closed in X.*

Proof. If $R(AC - I_Y)$ is closed in Y, hence let $(x_n)_{n\in\mathbb{N}} \subset R(BA - I_X)$ with $x_n \to x$ for some $x \in X$ as $n \to \infty$. Then there is a sequence $(z_n)_{n\in\mathbb{N}} \subset X$ with $x_n = (BA - I_X)z_n$ for any $n \in \mathbb{N}$. Thus

$$\begin{aligned}
Ax &= \lim_{n\to\infty} Ax_n \\
&= \lim_{n\to\infty} A((BA - I_X)z_n) \\
&= \lim_{n\to\infty} (ABA - A)z_n \\
&= \lim_{n\to\infty} (ACA - A)z_n \\
&= \lim_{n\to\infty} (AC - I_Y)Az_n.
\end{aligned}$$

From $R(AC-I_Y)$ is closed in Y, there is $y \in X$ with $(AC-I_Y)y = Ax$. Thus $y = ACy - Ax$.

$$\begin{aligned} x &= BAx - (BA - I_X)x \\ &= B(AC - I_Y)y - (BA - I_X)x \\ &= (BAC - B)(ACy - Ax) - (BA - I_X)x \\ &= BACACy - BACAx - BACy + BAx - (BA - I_X)x \\ &= BABACy - BABAx - BACy + BAx - (BA - I_X)x \\ &= (BA - I_X)(BACy - BAx - x). \end{aligned}$$

Consequently, $x \in R(BA - I_X)$. Then $R(BA - I_X)$ is closed in X. Conversely, assume that $R(BA - I_X)$ is closed in X. Then, from Theorem 2.1, $R(BA - I_X)$ is closed in X if and only if $R(AB - I_Y)$ is closed in Y. Thus $R(CA - I_X)$ is closed in X. By Theorem 2.1, $R(AC - I_Y)$ is closed in Y. Consequently, $R(AC - I_Y)$ is closed in Y if and only if $R(BA - I_X)$ is closed in X. □

Lemma 3.2. *If $A \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ such that $ABA = ACA$. Hence for each $n \in \mathbb{N}$, $R((AC - I_Y)^n)$ is closed in Y if and only if $R((BA - I_X)^n)$ is closed in X.*

Proof. Set

$$(\forall n \in \mathbb{N})\ B_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} B(AB)^{k-1}$$

and

$$(\forall n \in \mathbb{N})\ C_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} C(AC)^{k-1}.$$

Since $ABA = ACA$, for each $n \in \mathbb{N}$, $AB_nA = AC_nA$. Moreover, for each $n \in \mathbb{N}$,

$$\begin{aligned} I - AC_n &= I - \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} AC(AC)^{k-1} \\ &= I + \sum_{k=1}^{n+1} (-1)^{k} \binom{n+1}{k} (AC)(AC)^{k-1} \\ &= I + \sum_{k=1}^{n+1} \binom{n+1}{k} (-AC)^{k} \\ &= \sum_{k=0}^{n+1} \binom{n+1}{k} (-AC)^{k} \\ &= (I - AC)^{n+1}. \end{aligned}$$

Similarly, we have for all $n \in \mathbb{N}$, $(I - BA)^{n+1} = I - B_n A$. By Lemma 3.1, for each $n \in \mathbb{N}$, $R((AC - I_Y)^n)$ is closed in Y if and only if $R((BA - I_X)^n)$ is closed in X. □

Lemma 3.3. *If $A \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ with $ABA = ACA$. Hence for each $n \in \mathbb{N}$,*

(i) $AR((BA - I_X)^n) \subset R((AC - I_Y)^n)$;

(ii) $AN((BA - I_X)^n) \subset N((AC - I_Y)^n)$;

(iii) $BACN((AC - I_Y)^n) \subset N((BA - I_X)^n)$;

(iv) $BACR((AC - I_Y)^n) \subset R((BA - I_X)^n)$.

Proof. (i) If $x \in R((BA - I_X)^n)$, hence there is $u \in X$ with $x = (BA - I_X)^n u$. Then

$$Ax = A(BA - I_X)^n u = (AC - I_Y)^n Au \in R((AC - I_Y)^n).$$

Thus $AR((BA - I_X)^n) \subset R((AC - I_Y)^n)$.

(ii) Let $x \in N((BA - I_X)^n)$, then $(BA - I_X)^n x = 0$. Hence

$$(AC - I_Y)^n Ax = A(BA - I_X)^n x = 0.$$

Consequently, $Ax \in N((AC - I_Y)^n)$. Thus $AN((BA - I_X)^n) \subset N((AC - I_Y)^n)$.

(iii) Let $y \in N((AC - I_Y)^n)$, then $(AC - I_Y)^n y = 0$. Hence

$$(BA - I_X)^n BACy = BAC(AC - I_Y)^n y = 0.$$

Consequently, $BACy \in N((BA - I_X)^n)$. Then

$$BACN((AC - I_Y)^n) \subset N((BA - I_X)^n).$$

(iv) If $z \in R((AC - I_Y)^n)$. Hence there is $y \in Y$ with $z = (AC - I_Y)^n y$. Then

$$BACz = BAC(AC - I_Y)^n y = (BA - I_X)^n BACy \in R((BA - I_X)^n).$$

Consequently, $BACR((AC - I_Y)^n) \subset R((BA - I_X)^n)$.

□

Lemma 3.4. *If $A \in \mathcal{L}(X, Y)$ and $B, C \in \mathcal{L}(Y, X)$ with $ABA = ACA$. Hence for each $n \in \mathbb{N}$, $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in Y if and only if $R(BA - I_X) + N((BA - I_X)^n)$ is closed in X.*

Proof. Assume that $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in Y. Let $(x_n)_{n \in \mathbb{N}} \subset R(BA - I_X) + N((BA - I_X)^n)$ with $x_n \to x$ for some $x \in X$ as $n \to \infty$. Hence there are sequences

$(z_n)_{n\in\mathbb{N}} \subset R(BA - I_X)$ and $(w_n)_{n\in\mathbb{N}} \subset N((BA - I_X)^n)$ with $x_n = z_n + w_n$ for any $n \in \mathbb{N}$. Thus

$$\begin{aligned} Ax &= \lim_{n\to\infty} Ax_n \\ &= \lim_{n\to\infty} A(z_n + w_n). \end{aligned}$$

From (i) and (ii) of Lemma 3.3, we get $Az_n \in R(AC - I_Y)$ and $Aw_n \in N((AC - I_Y)^n)$. From $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in Y, there are $z \in Y$ and $w \in N((AC - I_Y)^n)$ with $(AC - I_Y)z + w = Ax$. Thus $z = ACz - Ax + w$. Consequently,

$$\begin{aligned} x &= BAx - (BA - I_X)x \\ &= B((AC - I_Y)z + w) - (BA - I_X)x \\ &= (BAC - B)z + Bw - (BA - I_X)x \\ &= (BAC - B)(ACz - Ax + w) + Bw - (BA - I_X)x \\ &= BACACz - BACAx + BACw - BACz + BAx - Bw + Bw - (BA - I_X)x \\ &= BACACz - BACAx + BACw - BACz + BAx - (BA - I_X)x \\ &= BABACz - BABAx + BACw - BACz + BAx - (BA - I_X)x \\ &= (BA - I_X)(BACz - BAx - x) + BACw. \end{aligned}$$

Since $w \in N((AC - I_Y)^n)$ and from (iii) of Lemma 3.3, we have $x \in R(BA - I_X) + N((BA - I_X)^n)$. Then $R(BA - I_X) + N((BA - I_X)^n)$ is closed in X. Conversely, assume that $R(BA - I_X) + N((BA - I_X)^n)$ is closed in X. By Theorem 2.1, $R(AB - I_Y) + N((AB - I_Y)^n)$ is closed in Y. Then $R(CA - I_X) + N((CA - I_X)^n)$ is closed in X. From Theorem 2.1, $R(AC - I_Y) + N((AC - I_Y)^n)$ is closed in Y. □

The following theorem holds.

Theorem 3.1. *Let X and Y be non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$. If $A, D \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ with $ACD = DBD$ and $DBA = ACA$. Hence $N(I_Y - AC)$ is complemented in Y if and only if $N(I_X - BD)$ is complemented in X.*

Proof. Assume that $N(I_Y - AC)$ is a complemented subspace of Y, hence there is a bounded projection $Q \in \mathcal{L}(Y)$ with $R(Q) = N(I_Y - AC)$, thus $(I_Y - AC)Q = 0$, hence $Q = ACQ$. Set $P = BQACD \in \mathcal{L}(X)$. By $DBQ = DBACQ = ACACQ = ACQ = Q$, hence

$$P^2 = (BQACD)(BQACD) = BQACQACD = BQACD = P.$$

Note that

$$(I_X - BD)P = (I_X - BD)(BQACD) = BQACD - BDBQACD = 0.$$

Then $R(P) \subseteq N(I_X - BD)$. If $x \in N(I_X - BD)$. Thus $Dx = DBDx = ACDx$, hence $Dx \in N(I_Y - AC) = R(Q)$. Thus $QDx = Dx$, then

$$Px = BQACDx = BQACQDx = BQDx = BDx = x,$$

hence $N(I_X - BD) \subseteq R(P)$. Then P is a projection with $R(P) = N(I_X - BD)$. Conversely, suppose that W is a projection with $R(W) = N(I_X - BD)$. Put $Z = ACDWBACAC$. By $BDW = W$, it follows that

$$\begin{aligned}
Z^2 &= (ACDWBACAC)(ACDWBACAC) \\
&= ACDWB(ACA)CACDWBACAC \\
&= ACDWB(DBA)C(ACD)WBACAC \text{ from } ACA = DBA \text{ and } ACD = DBD \\
&= ACDWBDB(ACD)BDWBACAC \\
&= ACDWBDB(DBD)BDWBACAC \\
&= ACDWBDBDBDBDWBACAC \\
&= ACDWBACAC \\
&= Z.
\end{aligned}$$

From

$$\begin{aligned}
(I_Y - AC)Z &= (I_Y - AC)(ACDWBACAC) \\
&= ACDWBACAC - ACACDWBACAC \\
&= ACDWBACAC - ACDBDWBACAC \\
&= ACDWBACAC - ACDWBACAC \\
&= 0,
\end{aligned}$$

$R(Z) \subseteq N(I_Y - AC)$. Let $x \in N(I_Y - AC)$. Hence $x = ACx$. By $BACx = BACACx = BDBACx$ and $BACx \in N(I_X - BD) = R(W)$, we get $WBACx = BACx$. Thus

$$\begin{aligned} Zx &= ACDWBACACx \\ &= ACDWBACx \\ &= ACDBACx \\ &= ACACACx \\ &= x, \end{aligned}$$

hence $N(I_Y - AC) \subseteq R(Z)$. Then Z is the projection onto $N(I_Y - AC)$. □

Theorem 3.2. *Let X and Y be non-Archimedean Banach spaces over a spherically complete field $\mathbb{K}$. Let $A, D \in \mathcal{L}(X, Y)$, $B, C \in \mathcal{L}(Y, X)$ such that $ACD = DBD$ and $DBA = ACA$. Hence $R(I_Y - AC)$ is complemented in Y if and only if $R(I_X - BD)$ is complemented in X.*

Proof. Suppose that Q is the projection with $R(Q) = R(I_Y - AC)$. Put $P = I_X - BAC(I_Y - Q)D$. From $(I_Y - Q)(I_Y - AC) = 0$ and $(I_Y - Q)AC = I_Y - Q$, we get

$$\begin{aligned} P^2 &= [I_X - BAC(I_Y - Q)D][I_X - BAC(I_Y - Q)D] \\ &= I_X - BAC(I_Y - Q)D - BAC(I_Y - Q)D + BAC(I_Y - Q)DBAC(I_Y - Q)D \\ &= I_X - BAC(I_Y - Q)D - BAC(I_Y - Q)D + BAC(I_Y - Q)ACAC(I_Y - Q)D \\ &= I_X - BAC(I_Y - Q)D - BAC(I_Y - Q)D + BAC(I_Y - Q)D \\ &= I_X - BAC(I_Y - Q)D \\ &= P. \end{aligned}$$

Thus $P^2 = P$. From $R(Q) = R(I_Y - AC)$, we get

$$R(BACQD) \subseteq R(BAC(I_Y - AC)) = R((I_X - BD)BAC) \subseteq R(I_X - BD).$$

Moreover,

$$\begin{aligned} P &= I_X - BAC(I_Y - Q)D \\ &= I_X - BACD + BACQD \\ &= I_X - BDBD + BACQD \\ &= (I_X - BD)(I_X + BD) + BACQD, \end{aligned}$$

and thus $R(P) \subseteq R(I_X - BD)$. If $x \in R(I_X - BD)$, hence there is $u \in X$ with $x = (I_X - BD)u$. From $Dx = D(I_X - BD)u = (I_Y - AC)Du \in R(Q)$, we get

$$Px = [I_X - BAC(I_Y - Q)D]x = x,$$

hence $R(I_X - BD) \subseteq R(P)$. Then $R(I_X - BD)$ is complemented in X. Conversely, suppose that Q is a projection with $R(Q) = R(I_X - BD)$. Set $W = I_Y - ACD(I_X - Q)BAC$. We demonstrate that W is a projection such that $R(W) = R(I_Y - AC)$. From $(I_X - Q)(I_X - BD) = 0$ and $(I_X - Q)BD = I_X - Q$, hence

$$\begin{aligned} W^2 &= [I_Y - ACD(I_X - Q)BAC]^2 \\ &= I_Y - 2ACD(I_X - Q)BAC + ACD(I_X - Q)BACACD(I_X - Q)BAC \\ &= I_Y - 2ACD(I_X - Q)BAC + ACD(I_X - Q)BDBACD(I_X - Q)BAC \\ &= I_Y - 2ACD(I_X - Q)BAC + ACD(I_X - Q)BDBDBD(I_X - Q)BAC \\ &= I_Y - ACD(I_X - Q)BAC \\ &= W. \end{aligned}$$

Thus $W^2 = W$. One can see that

$$\begin{aligned} W &= I_Y - ACD(I_X - Q)BAC \\ &= I_Y - ACDBAC + ACDQBAC \\ &= I_Y - ACACAC + ACDQBAC, \end{aligned}$$

and

$$\begin{aligned} R(W) &\subseteq R(I_Y - ACACAC + ACDQBAC) \\ &\subseteq R[(I_Y - AC)(I_Y + AC + ACAC)] + R[ACD(I_X - BD)] \\ &\subseteq R(I_Y - AC) + R[(I_Y - AC)ACD] \\ &\subseteq R(I_Y - AC). \end{aligned}$$

For each $y \in R(I_Y - AC)$, there is $w \in Y$ with $y = (I_Y - AC)w$. Hence $BACy = BAC(I_Y - AC)w = (I_X - BD)BACw \in R(Q)$, then

$$Wy = [I_Y - ACD(I_X - Q)BAC]y = y.$$

Thus $R(I_Y - AC) \subseteq R(W)$. Hence $R(I_Y - AC)$ is complemented in Y. □

We finish with the following examples.

Example 3.1. *Let* $A, B, C \in \mathcal{L}(c_0(\mathbb{K}))$ *be given respectively by*

$$A(x_1, x_2, x_3, x_4, \cdots) = (0, x_2, 0, x_4, \cdots),$$

$$B(x_1, x_2, x_3, x_4, \cdots) = (0, x_1, x_2, x_4, \cdots)$$

and

$$C(x_1, x_2, x_3, x_4, \cdots) = (0, 0, x_1, x_4, \cdots).$$

It is easy to see that $ABA = ACA$.

Example 3.2.

(i) *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

It is easy to see that $ABA = ACA = 0_{\mathcal{M}_2(\mathbb{Q}_p)}$.

(ii) *Let*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} \text{ and } C = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \in \mathcal{M}_2(\mathbb{Q}_p).$$

Then $ABA = ACA$.

(iii) *Let* $a, b, c \in \mathbb{Q}_p$. *Let*

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} a & b & c \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{Q}_p).$$

One can see that if $a = c = 0$, $b = 1$, *then* $B = C$ *and* $ABA = ACA$.

Moreover in the case $B \neq C$, *we have* $ABA = ACA$.

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