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RICCI-YAMABE SOLITONS ON THE LIE GROUP $H_2 \times \mathbb{R}$

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Abstract. In this paper, we study Ricci-Yamabe and gradient Yamabe solitons on the Lie group $H_2 \times \mathbb{R}$ with a left-invariant metric. We prove that the kind of the Ricci-Yamabe soliton is only related with one variable existing in the definition.

Keywords: $H_2 \times \mathbb{R}$ Lie group, Ricci-Yamabe soliton, gradient Ricci-Yamabe soliton.

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1. Introduction

Homogenous geometries play crucial role in the theory of manifolds. Their importance come from the well-known Thurston conjecture. This conjecture says that every compact orientable 3-dimensional manifold has a canonical decomposition into parts, each of which involves a canonical geometric structure from among the eight maximal simple connected homogenous Riemannian three-dimensional geometries [8]. These model spaces are E^3, H^3, S^3, $S^2 \times \mathbb{R}$, Nil, $\widetilde{SL_2\mathbb{R}}$, Sol and $H_2 \times \mathbb{R}$. In this paper, we deal with the latter one.

The geometry of different types of solitons on manifolds has been the focus of attention of many mathematicians during the last years (see for examples [1],[2],[3]). Yamabe flow was introduced by Hamilton and Yamabe solitons are special solutions of the Yamabe flow, [5]. Given an $n(n \geq 2)$, dimensional Riemannian manifold (M, g) such that $\{g(t)\}$ is the 1-parameter family of metrics and $r(t)$ is its scalar curvature. In this case, the equation of

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Yamabe flow is defined by

$$\frac{\partial g(t)}{\partial t} = -r(t)g(t).$$

Similarly, the equation of Ricci flow is given by

$$\frac{\partial g(t)}{\partial t} = -2Ric(t)g(t),$$

where $Ric(t)$ is the Ricci tensor [6]. In 2019, scalar combination of the Yamabe and Ricci flow was introduced by Güler and Crasmareanu as follows:

$$\frac{\partial g(t)}{\partial t} = \beta_2 r(t)g(t) - 2\beta_1 Ric(t)g(t).$$

This equation is known as Ricci-Yamabe flow and special solutions of the Ricci-Yamabe flow are famous as Ricci-Yamabe solitons [7]. Due to the sign of the scalars β_1 and β_2 the Ricci-Yamabe flow becomes a Riemannian, a semi-Riemannian or a singular Riemannian flow.

In this paper, we show that the Lie group $H_2 \times \mathbb{R}$ involves a vector field satisfying a Ricci-Yamabe soliton. We also prove that there does not exist a gradient Ricci-Yamabe soliton. Throughout the paper, all geometric objects (curves, manifolds, vector fields, functions etc.) are assumed to be smooth.

2. Preliminaries

2.1. **Ricci-Yamabe and Gradient Ricci-Yamabe Solitons.** A connected Riemannian manifold $(M, g, \beta_1, \beta_2, \beta_3)$ of dimension n $(n \geq 2)$ is said to be a Ricci-Yamabe soliton if it satisfies

$$L_X g + 2\beta_1 Ric = -(2\beta_3 - \beta_2 r)g, \tag{2.1}$$

where $L_X g$ is the Lie derivative of the metric g in the direction of the vector field X, $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$, Ric and r denote the Ricci tensor and the scalar curvature of M, respectively. A Ricci-Yamabe soliton $(M, g, \beta_1, \beta_2, \beta_3)$ is said to be steady, expanding or shrinking and Ricci-Yamabe soliton if $\beta_3 = 0$, $\beta_3 > 0$ and $\beta_3 < 0$ respectively. If a function $f : M \to \mathbb{R}$ satisfies $X = gradf$, then we say that the Ricci-Yamabe soliton is a gradient Ricci-Yamabe soliton.

2.2. **The Lie group $H_2 \times \mathbb{R}$.** We recall fundamental information about the Lie group $H_2 \times \mathbb{R}$ from [4]. Let $H_2 = \{(u, v) \in \mathbb{R}^2 \mid v > 0\}$ denotes the upper half model of the hyperbolic plane equipped with the metric $g_{H_2} = \frac{1}{v^2}(du^2 + dv^2)$. The hyperbolic space H_2 with the group structure occured by the composition of proper affine maps is a Lie group with the left invariant metric

$$g = \frac{1}{v^2}(du^2 + dv^2) + dw^2. \tag{2.2}$$

A left-invariant orthonormal frame field (e_1, e_2, e_3) is given by

$$e_1 = v\frac{\partial}{\partial u},\ e_2 = v\frac{\partial}{\partial v},\ e_3 = \frac{\partial}{\partial w}.$$

The Levi-Civita connection ∇ with respect to the this orthonormal frame is given by

$$\begin{aligned} \nabla_{e_1} e_1 = e_2,\ \nabla_{e_1} e_2 = -e_1,\ \nabla_{e_1} e_3 = 0, \\ \nabla_{e_2} e_1 = 0,\ \nabla_{e_2} e_2 = 0,\ \nabla_{e_2} e_3 = 0, \\ \nabla_{e_3} e_1 = 0, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = 0. \end{aligned} \tag{2.3}$$

The Lie brackets are obtained as

$$[e_2, e_3] = 0,\ [e_1, e_3] = 0,\ [e_1, e_2] = -e_1.$$

The non-zero components of the curvature tensor field R and the Ricci tensor Ric are

$$\begin{aligned} R(e_1, e_2)e_1 &= e_2,\ R(e_1, e_2)e_2 = -e_1, \\ Ric(e_1, e_1) &= Ric(e_2, e_2) = -1. \end{aligned}$$

The scalar curvature r of $H_2 \times \mathbb{R}$ is

$$r = -2.$$

3. Main results

We start this section by considering

$$X = t_1 e_1 + t_2 e_2 + t_3 e_3 \tag{3.4}$$

is a potential vector field on M, where $t_1,\ t_2$ and t_3 are differentiable functions of u, v and w. We label the coordinate basis by $\{\partial_u, \partial_v, \partial_w\}$.

Theorem 3.1. *Let us consider the Lie group $H_2 \times \mathbb{R}$ with the metric (2.2). Then the space $H_2 \times \mathbb{R}$ admits an expanding, steady or shrinking Ricci Yamabe soliton if and only if $\beta_1 > 0,\ \beta_1 < 0$ or $\beta_1 = 0$.*

Proof. From (2.2), (2.3) and (3.4) the Lie derivative of the metric tensor g is computed as

$$\begin{aligned} L_X g(e_1, e_1) &= -2(t_2 - \partial_u t_1), \\ L_X g(e_1, e_2) &= t_1 + \partial_v t_1 + \partial_u t_2, \\ L_X g(e_1, e_3) &= \partial_w t_1 + \partial_u t_3, \\ L_X g(e_2, e_2) &= 2\partial_v t_2, \\ L_X g(e_2, e_3) &= \partial_w t_2 + \partial_v t_3, \\ L_X g(e_3, e_3) &= 2\partial_w t_3. \end{aligned}$$

Putting (2.2), (2.3) and (3.4) in (2.1), we have

$$-(t_2 - \partial_u t_1) = \beta_1 - (\beta_3 + \beta_2), \tag{3.5}$$

$$t_1 + \partial_v t_1 + \partial_u t_2 = 2\beta_2, \tag{3.6}$$

$$\partial_w t_1 + \partial_u t_3 = 2\beta_2, \tag{3.7}$$

$$\partial_v t_2 = \beta_1 - (\beta_3 + \beta_2), \tag{3.8}$$

$$\partial_w t_2 + \partial_v t_3 = 2\beta_2, \tag{3.9}$$

$$\partial_w t_3 = -(\beta_3 + \beta_2). \tag{3.10}$$

Equation (3.10) gives $\partial_u \partial_w t_3 = 0$. Using this equation and deriving (3.7) with respect to w, we get

$$\partial_w^2 t_1 = 0. \tag{3.11}$$

From (3.11), we occur

$$t_1 = \varphi(u, v)w + \psi(u, v),$$

where φ and ψ are functions. Equation (3.5) gives $-\partial_v t_2 + \partial_u \partial_v t_1 = 0$. Having in mind this relation and taking derivative in (3.5) with respect to v and using (3.8), we obtain

$$\partial_u \partial_v t_1 = \beta_1 - (\beta_3 + \beta_2). \tag{3.12}$$

Taking derivative in (3.6) with respect to v, we find $\partial_v t_1 + \partial_v^2 t_1 + \partial_v \partial_u t_2 = 0$. Using this relation in (3.12), we find

$$\partial_v t_1 + \partial_v^2 t_1 = 0. \tag{3.13}$$

Substituting t_1 in equations (3.12) and (3.13), we get

$$\begin{cases} \partial_u \partial_v \varphi w + \partial_u \partial_v \psi = \beta_1 - (\beta_3 + \beta_2), \\ (\partial_v \varphi + \partial_v^2 \varphi) w + \partial_v \psi + \partial_v^2 \psi = 0. \end{cases} \tag{3.14}$$

Taking derivative in (3.14) with respect to w, we get

$$\begin{cases} \partial_v \varphi + \partial_v^2 \varphi = 0, \\ \partial_v \psi + \partial_v^2 \psi = 0, \\ \partial_u \partial_v \varphi = 0, \\ \partial_u \partial_v \psi = \beta_1 - (\beta_3 + \beta_2). \end{cases} \tag{3.15}$$

Now, if we take derivative in $(3.15)_2$ with respect to u and use in $(3.15)_4$, we find

$$\beta_1 = \beta_3 + \beta_2. \tag{3.16}$$

Integrating $(3.15)_1$ and $(3.15)_2$ give us

$$\begin{cases} \varphi(u, v) = \gamma_1 e^{-v} + \varphi_1(u), \\ \psi(u, v) = \gamma_2 e^{-v} + \psi_1(u), \end{cases} \tag{3.17}$$

where $\gamma_i \in \mathbb{R}$ and φ_1, ψ_1 are functions. Therefore,

$$t_1 = (\gamma_1 e^{-v} + \varphi_1(u)) w + \gamma_2 e^{-v} + \psi_1(u). \tag{3.18}$$

Substituting t_1 in (3.6), we deduce

$$(\varphi_1(u) + \varphi_1''(u)) w + \psi_1(u) + \psi_1''(u) = 2\beta_2. \tag{3.19}$$

Deriving equation (3.19) with respect to w, we find

$$\begin{cases} \varphi_1(u) + \varphi_1''(u) = 0, \\ \psi_1(u) + \psi_1''(u) = 0. \end{cases} \tag{3.20}$$

Integrating (3.20) with respect to u, we find

$$\begin{cases} \varphi_1(u) = \gamma_3 \cos u + \gamma_4 \sin u, \\ \psi_1(u) = \gamma_5 \cos u + \gamma_6 \sin u, \end{cases} \tag{3.21}$$

where $\gamma_i \in \mathbb{R}$. Hence,

$$t_1 = (\gamma_1 e^{-v} + \gamma_3 \cos u + \gamma_4 \sin u) w + \gamma_2 e^{-v} + \gamma_5 \cos u + \gamma_6 \sin u. \tag{3.22}$$

From (3.5), we have

$$t_2 = (-\gamma_3 \sin u + \gamma_4 \cos u) w - \gamma_5 \sin u + \gamma_6 \cos u. \tag{3.23}$$

Equation (3.10) yields

$$t_3 = -(\beta_3 + \beta_2)w + \xi(u, v),$$

where ξ is a function. Substituting $t_1,\ t_2,\ t_3$ in (3.7) and (3.9) lead to

$$\begin{cases} \partial_u \xi = 2\beta_2 - (\gamma_1 e^{-v} + \gamma_3 \cos u + \gamma_4 \sin u) \\ \partial_v \xi = 2\beta_2 - (\gamma_3 \sin u + \gamma_4 \cos u). \end{cases} \tag{3.24}$$

Integrating $(3.24)_1$with respect to u, we obtain

$$\xi(u, v) = 2\beta_2 u - \gamma_1 u e^{-v} - \gamma_3 \sin u + \gamma_4 \cos u + \gamma_7,$$

and putting ξ in $(3.24)_2$, we see that

$$\gamma_1 u e^{-v} = 2\beta_2 - \gamma_3 \sin u - \gamma_4 \cos u.$$

This gives us

$$\gamma_1 = \gamma_3 = \gamma_4 = \beta_2 = 0.$$

Finally, we find

$$\begin{cases} t_1 = \gamma_2 e^{-v} + \gamma_5 \cos u + \gamma_6 \sin u, \\ t_2 = -\gamma_5 \sin u + \gamma_6 \cos u, \\ t_3 = -\beta_3 w + \gamma_7, \end{cases} \tag{3.25}$$

where $\gamma_i \in \mathbb{R}$.

This shows that $X = t_1 e_1 + t_2 e_2 + t_3 e_3$ given by (3.25) satisfies (2.1). We also found that $\beta_1 = \beta_3 + \beta_2$ and $\beta_2 = 0$. Therefore we proved the theorem. □

Theorem 3.2. *Let us consider the Lie group $H_2 \times \mathbb{R}$ with the metric (2.2). Then the space $H_2 \times \mathbb{R}$ does not admit a gradient Ricci-Yamabe soliton.*

Proof. Suppose that $X = grad y$ is a gradient vector field on M with potential function y. Then X is given by

$$grad y = v^2 \partial_u y \partial_u + v^2 \partial_v y \partial_v + \partial_w y \partial_w.$$

From (3.25), we see that the Lie group $H_2 \times \mathbb{R}$ is a gradient Yamabe soliton if and only if the function y fulfills the following equations:

$$\begin{cases} \partial_u y = \frac{\gamma_2}{v} e^{-v} + \frac{\gamma_5}{v} \cos u + \frac{\gamma_6}{v} \sin u, \\ \partial_v y = \frac{-\gamma_5}{v} \sin u + \frac{\gamma_6}{v} \cos u, \\ \partial_w y = -\beta_3 w + \gamma_7. \end{cases} \tag{3.26}$$

Integrating above system we find

$$y(u,v,w) = \ln v[\gamma_5 \sin u - \gamma_6 \cos u] - \frac{\beta_3^2}{2}\gamma_7 w^2 + \gamma_8, \ \gamma_i \in \mathbb{R}.$$

But the function y does not fulfill $(3.26)_1$. This ends the proof. □

4. Conclusion

Ricci and Yamabe solitons have many applications for several sciences such as differential geometry and theoretical physics. Similar to these solitons, their scalar combinations, which are called Ricci-Yamabe solitons, have been studied increasingly since the work of Güler and Crasmareanu [7]. In this paper, we investigate Ricci-Yamabe solitons in the Lie group $H_2 \times \mathbb{R}$. Our calculations in this paper may provide an insight for further studies about Ricci-Yamabe solitons on other Thurston geometries.

References

[1] Altunbaş, M. (2023). Ricci solitons on tangent bundles with the complete lift of a projective semi-symmetric connection. Gulf Journal of Mathematics, 14(2), 8-15.

[2] Altunbaş, M. (2023). Conformal Yamabe Solitons on Tangent Bundles with Complete Lifts of Some Special Connections. Proceedings of the Bulgarian Academy of Sciences, 76(8), 1176-1186.

[3] Altunbaş, M. (2023). Ricci solitons of three-dimensional Lorentzian Bianchi-Cartan-Vranceanu spaces. Turkish Journal of Mathematics and Computer Science, 15(2), 270-276.

[4] Belarbi, L. (2020). Ricci solitons of the $\mathrm{H}_2 \times R$ group. Elec. Res. Archive, 28(1), 157-163.

[5] Hamilton, R.S. (1988). The Ricci flow on surfaces, Mathematics and general relativity. Contemporary Mathematics 71, 237-261.

[6] Hamilton, R.S. (1982). Three-manifolds with positive Ricci curvature. Journal of Differential Geometry, 17(2), 255-306.

[7] Güler, S., Crasmareanu, M. (2019). Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy. Turkish Journal of Mathematics, 43(5), 2631-2641.

[8] Thurston, W. (1997). Three dimensional geometry and topology, vol 35 of Princeton Math. Series. Princeton University Press, Princeton, USA.

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