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PROJECTIVE AND PROJECTIVE-PERMUTATION INVARIANTS FOR POINT SHAPE RECOGNITION

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ABSTRACT. In the present paper, the complete system of projective invariants of a point shape and the complete system of invariants under simultaneous projective and permutation transformations of a point shape are obtained.

Keywords: projective invariant, Cross ratio, Projective-permutation invariant.

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1. Introduction

Projective invariants and projective-permutation invariants have an important role in computer vision for recognition of shapes (see books [6, 13, 14, 17, 18] and papers [2, 4, 5, 7, 8, 10, 11, 16]). The projectively invariant descriptors of objects in the object recognition problems can be computed from relations between points, lines and conics that are coplanar on object surfaces in 3D. (see [4]). By [1, Corollary 6.1.4] and [6, Lemma 5.8.2], the cross-ratio is a complete system of projective invariants of a regular point shape of size 4. The volume cross ratios of points and theirs invariants in the projective space are introduced in [23, Section 27].

An extension of the cross-ratio (an harmonic ratio) to n-space is given in the paper [2]. In the paper Burns, Weiss and Riseman [3], it is proved that there is no a non-trivial function that is view-invariant for all possible (non-degenerate) $3D$ point sets of size n for any n. The

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non-existence of such a general-case view invariant is shown for the true perspective, weak perspective and orthographic models. Moreover, complete classifications of joint invariants of points for the groups in the Euclidean,affine and projective spaces are given in [15].

Let n, m be natural numbers such that $n < m$. In the present paper, we give a definition of a regular nD point set of size m and obtain a complete system of projective invariants for the system of all regular nD point sets of size m. We investigate fundamental relations between elements of the complete system of projective invariants. Similar results have obtained for the complete system of invariants under simultaneous projective and permutation transformations (p^2-invariants, for short) of a $2D$ and $3D$ point set of size m. The problem on complete systems of p^2-invariants of a nD point set of size m in computer vision is considered in papers ([5, 7, 12, 21, 22]). This problem investigated also in projective geometry, algebraic geometry (theory of hyperelliptic curves) and the invariant theory of binary forms (see [25]).

Our paper is organized as follows. In section 2, we give the definition of a regular nD point of size m and obtain the complete system of projectively invariants for the system of all regular nD points of size m (Theorem 1). We describe the system of fundamental relations between elements of the complete system of projectively invariants (Theorem 2). We prove that the complete system is a minimal complete system of projectively invariants. In section 3, we obtain the complete system of p^2-invariants for the system of all regular nD points of size m (Theorem 3).

2. Projectively invariants of a point shape and their complete and the minimal complete systems

Let $\mathbb{R}$ be the field of real numbers, n and m are natural numbers, $n \geq 2, m > n+1$. The general linear group $GL(n, \mathbb{R})$ is the set $n \times n$ invertible matrices with elements in $\mathbb{R}$. The special linear group $SL(n, \mathbb{R})$ is the set $n \times n$ matrices with determinant 1. $\mathbb{R}_*$ be a group with respect to the multiplication in $\mathbb{R}$. Let $(\mathbb{R}_*)^m$ be the m time direct product of the group $\mathbb{R}_*$. We denote the direct product of groups $(\mathbb{R}_*)^m$ and $GL(n, \mathbb{R})$ by $P(m, n)$. Let $(\mathbb{R}^n)^m$ be the m time direct sum of the n-dimensional real linear space $\mathbb{R}^n$. We define an action Ψ of the group $P(m, n)$ on the space $(\mathbb{R}^n)^m$ by the following: for $q = ((r_1, r_2, \ldots, r_m), g) \in P(m, n)$, $r_i \in \mathbb{R}, g \in GL(n, \mathbb{R})$, and $X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$, we put

$$\Psi(q, X) = ((r_1, r_2, \ldots, r_m), g), X) = (r_1 g x_1, r_2 g x_2, \ldots, r_m g x_m).$$

The following definitions 1-5 and proposition 2 are known in the literature. (See some papers ([9, 19], [20, p.11]). Proposition 1 is given in [18].

Definition 2.1. *Let $\Gamma, \Omega \in (\mathbb{R}^n)^m$. If there exists $q \in P(m,n)$ such that $\Omega = \Psi(q, \Gamma)$, then the elements Γ and Ω are called $P(m,n)$-equivalent, a relationship which is written symbolically in this paper as $\Gamma \overset{P(m,n)}{\sim} \Omega$.*

Definition 2.2. *A real rational function $f(x_1, x_2, \dots, x_k)$ of elements $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is called projectively invariant if*

$$f(\Psi(q, X)) = f(X).$$

for all $q \in P(m,n)$.

Definition 2.3. *A set $M \subseteq (R^n)^m$ is $P(m,n)$-invariant if $\Psi(q, X) \in M$ for all $X \in M$ and for all $q \in P(m,n)$.*

Definition 2.4. *Let M be a $P(m,n)$-invariant subset of $(\mathbb{R}^n)^m$. Let $f_i : M \to R$ for $i = 1, 2, \dots, k$ be the projectively invariant rational functions.*

A system $\{f_1, f_2, \dots, f_k\}$ of is called a complete system of $P(m,n)$-invariants on the set M if $f_i(\Gamma) = f_i(\Omega)$ for all $i \in \{1, 2, \dots, k\}$ and for $\Gamma, \Omega \in M$ imply $\Gamma \overset{P(m,n)}{\sim} \Omega$.

Proposition 2.1. *Let M be a $P(m,n)$-invariant set of $(\mathbb{R}^n)^m$. Then every projectively invariant rational function on M if a function of the system $\{f_1, f_2, \dots, f_k\}$.*

Definition 2.5. *A complete system of projectively invariant rational functions*

$W = \{f_1, f_2, \dots, f_k\}$ is called a minimal complete system of projectively invariant rational functions if $W \setminus \{f_i\}$ is not complete for any $i \in \{1, 2, \dots, k\}$.

Proposition 2.2. *$W = \{f_1, f_2, \dots, f_k\}$ is a minimal complete system iff f_i is not function of the subsystem $\{f_1, f_2, \dots, f_{i-1}, f_{i+1}, \dots, f_m\}$ for all $i = 1, 2, \dots, k$.*

Let $[x_1 x_2 \cdots x_n]$ be the determinant of vectors $x_1, x_2, \cdots, x_n \in \mathbb{R}^n$. Assume that $x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2} \in \mathbb{R}^n$ vectors such that $[x_1 x_2 \cdots x_{n-1} x_k] \neq 0$ for all $i = n, n+1, n+2$ and $[x_2 x_3 \cdots x_n x_j] \neq 0$ for $j = n+1, n+2$. Consider the following cross-invariant of vectors $x_1, x_2, \cdots, x_n, x_{n+1}, x_{n+2} \in \mathbb{R}^n$:

$$\frac{[x_1 x_2 \cdots x_{n-1} x_{n+1}][x_2 x_3 \cdots x_n x_{n+2}]}{[x_1 x_2 \cdots x_{n-1} x_{n+2}][x_2 x_3 \cdots x_n x_{n+1}]}.$$

We denote it by $\langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \rangle$. It is known that it is projectively invariant.

Definition 2.6. *$X = (x_1, x_2, \ldots, x_m) \in (\mathbb{R}^n)^m$ is called regular if $[x_{p_1} x_{p_2} \cdots x_{p_{n-1}} x_{p_n}] \neq 0$ for all natural numbers $p_1, p_2, \ldots, p_n$ such that $1 \leq p_1 < p_2 < \cdots < p_n \leq m$.*

If $X = (x_1, x_2, \ldots, x_m)$ is regular, from $X \overset{P(m,n)}{\sim} Y$, we have $Y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^n)^m$. Hence $Y = (y_1, y_2, \ldots, y_m)$ is also regular. Hence the set of all regular elements is a $P(m,n)$-invariant subset of $(\mathbb{R}^n)^m$.

The cross-ratio obtained from $\langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \rangle$ by transposition of elements x_1 and x_j, where $1 \leq j \leq n-1$, will be denoted by $T_j \langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_{n+2} \rangle$. Thus

$$T_j \langle x_1 x_2 x_3 \cdots x_{j-1} x_j x_{j+1} \ldots x_n x_{n+1} x_{n+2} \rangle = \langle x_j x_2 x_3 \cdots x_{j-1} x_1 x_{j+1} \ldots x_n x_{n+1} x_{n+2} \rangle$$

for all $j = 1, 2, \ldots, n-1$.
If $X = (x_1, x_2, \ldots, x_m)$ is regular, then $T_j \langle x_1 x_2 x_3 \ldots x_n x_{n+1} x_{n+2} \rangle \neq 0$ and $T_j \langle x_1 x_2 x_3 \ldots x_n x_{n+1} x_{n+2} \rangle \neq \infty$ for all $j = 1, 2, \ldots, n-1$.

Theorem 2.1. *Regular elements $X = (x_1, x_2, \ldots, x_m), Y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^2)^m$ are $P(m,n)$-equivalent if and only if*

$$T_j \langle x_1 x_2 x_3 \ldots x_{n-1} x_n x_{n+1} x_k \rangle = T_j \langle y_1 y_2 x_3 \ldots y_{n-1} y_n y_{n+1} y_k \rangle \tag{2.1}$$

for all $j = 1, 2, \ldots n-1$ and for all $k = n+2, \ldots, m$.

Proof. Since the function $\langle x_1 x_2 x_3 \cdots x_n x_{n+1} x_k \rangle$ is projectively invariant, $P(m,n)$-equivalence of $X = (x_1, x_2, \ldots, x_m)$ and $Y = (y_1, y_2, \ldots, y_m)$ implies (2.1). Prove the converse assertion.

Assume that (2.1) holds. We consider vectors $e_1, e_2, \ldots, e_n, e_{n+1} \in \mathbb{R}^n$, where

$$e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots e_n = (0, 0, \ldots, 1), e_{n+1} = (1, 1, \ldots, 1).$$

By the fundamental theorem of projective geometry ([1, p.97]), elements $g \in GL(n, \mathbb{R})$ and $r_1, r_2, \ldots, r_n, r_{n+1} \in \mathbb{R}_*$ exist such that $r_i g x_i = e_i, i = 1, 2, \ldots n, n+1$. Similarly, elements $h \in GL(n, \mathbb{R})$ and $q_1, q_2, \ldots, q_n, q_{n+1} \in \mathbb{R}_*$ exist such that $q_i h y_i = e_i, i = 1, 2, \ldots, n, n+1$. Since the function $\langle x_1 x_2 x_3 \ldots x_{n-1} x_n x_{n+1} x_k \rangle$ is projectively invariant, we have

$$\langle (r_1 g x_1)(r_2 g x_2) \cdots (r_n g x_n)(r_{n+1} g x_{n+1})(g x_k) \rangle = \langle e_1 e_2 \cdots e_n e_{n+1} (g x_k) \rangle = \langle x_1 x_2 \cdots x_n x_{n+1} x_k \rangle =$$
$$\langle y_1 y_2 \cdots y_n y_{n+1} y_k \rangle = \langle (q_1 h y_1)(q_2 h y_2) \cdots (q_n h y_n)(q_{n+1} h y_{n+1})(h y_k) \rangle = \langle e_1 e_2 \cdots e_n e_{n+1} (h y_k) \rangle$$

for all $k = n+2, \ldots m$. Hence $\langle e_1 e_2 \cdots e_n e_{n+1} (g x_k) \rangle = \langle e_1 e_2 \cdots e_n e_{n+1} (h y_k) \rangle$. Similarly, we obtain

$$T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (g x_k) \rangle = T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (h y_k) \rangle \tag{2.2}$$

for all $j = 1, 2, \dots n-1$ and for all $k = n+2, \dots, m$.

Let $gx_k = (a_{1k}, a_{2k}, \dots a_{nk}), hy_k = (b_{1k}, b_{2k}, \dots b_{nk}), k = n+2 \dots, m$. Using regularity of $(r_1 gx_1, \dots r_{n+1} gx_{n+1}, gx_{n+2}, \dots, gx_m)$ and $(q_1 hy_1, \dots q_{n+1} hy_{n+1}, hy_{n+2}, \dots, hy_m)$, we obtain $[e_1 e_2 \dots e_{n-1} gx_k] = a_{nk} \neq 0$ and $[e_1 e_2 \dots e_{n-1} gy_k] = b_{nk} \neq 0$ for all $k = n+2, \dots m$. It is easy to see that

$$T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (gx_k) \rangle = \frac{a_{jk}}{a_{nk}} \tag{2.3}$$

and

$$T_j \langle e_1 e_2 e_3 \cdots e_{n-1} e_n e_{n+1} (gy_k) \rangle = \frac{b_{jk}}{b_{nk}}, \tag{2.4}$$

for all $j = 1, 2, \dots n-1$ and for all $k = n+2, \dots, m$.

Equations (2.2), (2.3) and (2.4) imply $\frac{a_{jk}}{a_{nk}} = \frac{b_{jk}}{b_{nk}}$ for all $j = 1, 2, \dots n-1; k = n+2, \dots m$. Put $d_{jk} = \frac{a_{jk}}{a_{nk}} = \frac{b_{jk}}{b_{nk}}$ for all $j = 1, 2, \dots n-1; k = n+2, \dots m$. Then $gx_k = a_{nk}(d_{1k}, d_{2k}, \dots, d_{n-1k}, 1)$ and $hy_k = b_{nk}(d_{1k}, d_{2k}, \dots, d_{n-1k}, 1)$ for all $k = n+2, \dots m$.

This means that
$(e_1, e_2, \dots e_n, e_{n+1}, gx_{n+2}, \dots, gx_m) \overset{P(m,n)}{\sim} (e_1, e_2, \dots e_n, e_{n+1}, hy_{n+2}, \dots, hy_m)$.

Using $(x_1, x_2, \dots x_n, x_{n+1}, x_{n+2}, \dots, x_m) \overset{P(m,n)}{\sim} (e_1, e_2, \dots e_n, e_{n+1}, gx_{n+2}, \dots, gx_m)$, $(e_1, e_2, \dots e_n, e_{n+1}, gx_{n+2}, \dots, gx_m) \overset{P(m,n)}{\sim} (e_1, e_2, \dots e_n, e_{n+1}, hy_{n+2}, \dots, hy_m)$ and $(e_1, e_2, \dots e_n, e_{n+1}, hy_{n+2}, \dots, hy_m) \overset{P(m,n)}{\sim} (y_1, y_2, \dots y_n, y_{n+1}, y_{n+2}, \dots, y_m)$, we obtain $X \overset{P(m,n)}{\sim} Y$. □

Remark 2.1. *Theorem* 2.1 *means that the system of projectively invariants*

$$T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle, \tag{2.5}$$

for all $j = 1, 2, \dots n-1$ *and for all* $k = n+2, \dots m$ *is a complete system of projectively invariants on the set of all regular elements of* $(\mathbb{R}^n)^m$.

Corollary 2.1. *Every projectively invariant function* $f(x_1, x_2, \dots, x_m)$ *on the set of all regular elements* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ *is a function of elements of the system* (2.5)

Proof. It follows from ([20, Theorem 1 and 1.1]). □

Now we find all fundamental relations between elements of the complete system (2.5) If $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is regular, then

$$T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle \neq 0 \tag{2.6}$$

for all $j = 1, 2, \dots n-1$ and for all $k = n+2, \dots m$.

Theorem 2.2. *Let* $\{c_{jk}, j = 1, 2, \dots n-1; k = n+2, \dots, m\}$ *be a system of real numbers such that* $c_{jk} \neq 0$ *for all* $j = 1, 2, \dots n-1; k = n+2, \dots, m$. *Then a regular element* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ *exists such that*

$$T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle = c_{jk} \tag{2.7}$$

for all $j = 1, 2, \dots n-1$ *and for all* $k = n+2, \dots m$

Proof. Let $e_1, e_2, \dots, e_n, e_{n+1} \in \mathbb{R}^n$ be vectors in Theorem (2.1). Consider the element $X = (x_1, x_2, \dots, x_n, x_{n+1}, \dots x_m) \in (\mathbb{R}^n)^m$, where

$x_1 = e_1 = (1, 0, 0, \dots, 0), x_2 = e_2 = (0, 1, 0, \dots 0), \dots, x_n = e_n = (0, 0, 0, \dots 1), x_{n+1} = e_{n+1} = (1, 1, 1, \dots, 1)$, $x_k = (c_{1k}, c_{2k}, \dots, c_{n-1k}, 1)$ for all $k = n+2, \dots m$. It is easy to see that (2.7) hold for X. Since $c_{jk} \neq 0$ for all $j = 1, 2, \dots n-1; k = n+2, \dots, m$, (2.7) implies that X is regular. □

Corollary 2.2. *The system* (2.5) *is a system of functionally independent projectively invariants on the set of of all regular elements* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$.

Proof. It follows from Theorem 2.2 □

Corollary 2.3. *The system* (2.5) *is a minimal complete system of projectively invariants on the set of of all regular elements* $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$.

Proof. It follows from Proposition 2.1 and Corollary 2.2. □

3. Projective-Permutation invariants of a point shape

Let $S(n, m)$ be the group of all permutations of the numbers $n+2, n+3, \dots, m$ and $P(m, n) \times S(n, m)$ is a direct product of groups $P(m, n)$ and $S(n, m)$. We define an action β of the group $P(m, n) \times S(n, m)$ on the space $(\mathbb{R}^n)^m$ as follows: for $q = ((r_1, r_2, \dots, r_m), g, h) \in P(m, n) \times S(n, m)$, $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$, $h \in S(n, m)$,

$$h = \begin{pmatrix} 1 & 2 & \dots & n+1 & n+2 \dots & m \\ 1 & 2 & \dots & n+1 & h(n+2) \dots & h(m) \end{pmatrix}, \tag{3.8}$$

we put $\beta(q, X) = ((r_1, \dots, r_m), g, h), X) = (r_1 g x_{h(1)}, r_2 g x_{h(2)}, \dots, r_m g x_{h(m)})$, where $h(j) = j$ for $j = 1, 2, \dots, n+1$.

Definition 3.1. *Elements* $A, B \in (\mathbb{R}^n)^m$ *is called* $P(m, n) \times S(n, m)$*-equivalent if there exists* $q \in P(m, n) \times S(n, m)$ *such that* $B = \beta(q, A)$. *In this case, we write* $A \overset{P(m,n)\times S(n,m)}{\sim} B$.

Definition 3.2. *A rational function $f(x_1,\dots,x_m)$ of $X=(x_1,\dots,x_m)\in(\mathbb{R}^n)^m$ is called $P(m,n)\times S(n,m)$-invariant if $f(\alpha(q,X))=f(X)$ for all $q\in P(m,n)\times S(n,m)$.*

For $j=1,2,\dots n-1; k=n+2,\dots m$, we put $T_{jk}(X)=T_j\left\langle x_1x_2x_3\dots x_{n-1}x_nx_{n+1}x_k\right\rangle$. We denote the algebra of polynomials of all $T_{jk}(X)$ by $A(n,m)$. Let $\{z_j : j=1,2,\dots n-1\}$ be independent variables. We consider the following function

$$\prod_{k=1}^{m-n-1}\left(1+\sum_{i=1}^{n-1}T_k(X)z_i\right)$$

and define the polynomials $U_{r_1r_2\dots r_{n-1}}(X)\in A(n,m)$ by the following equality

$$\prod_{k=1}^{m-n-1}\left(1+\sum_{i=1}^{n-1}T_k(X)z_i\right)=1+\sum_{1\le\sum_{i=1}^{n-1}r_i\le m-n-1}U_{r_1r_2\dots r_{n-1}}(X)z_1^{r_1}z_2^{r_2}\cdots z_{n-1}^{r_{n-1}}. \tag{3.9}$$

It is obvious that every function $U_{r_1r_2\dots r_{n-1}}(X)$ is $P(m,n)\times S(n,m)$-invariant.

Theorem 3.1. *Regular elements $X=(x_1,x_2,\dots,x_m), Y=(y_1,y_2,\dots,y_m)\in(\mathbb{R}^n)^m$ are $P(m,n)\times S(n,m)$-equivalent if and only if*

$$U_{r_1r_2\dots r_{n-1}}(X)=U_{r_1r_2\dots r_{n-1}}(Y) \tag{3.10}$$

for $1\le\sum_{i=1}^{n-1}r_i\le m-n-1$.

Proof. Since $U_{r_1r_2\dots r_{n-1}}(X)$ is $P(m,n)\times S(n,m)$-invariant, $P(m,n)\times S(n,m)$-equivalence of $X=(x_1,x_2,\dots,x_m)$ and $Y=(y_1,y_2,\dots,y_m)$ implies (3.10). Prove the converse assertion.

Assume that (3.10) holds. Then (3.9) and (3.10) imply the equation

$$\prod_{k=1}^{m-n-1}(1+\sum_{i=1}^{n-1}T_k(X)z_i)=\prod_{k=1}^{m-n-1}(1+\sum_{i=1}^{n-1}T_k(Y)z_i). \tag{3.11}$$

By the theorem on the unique factorization in the algebra $A(n,m)$ (see [24, p.91-94]), a permutation (3.8) exists such that

$$\sum_{i=1}^{n-1}T_k(X)z_i=\sum_{i=1}^{n-1}T_k(Y)z_i$$

for all $k=n+2,\dots,m$.

This equality implies

$$\begin{cases} T_j\left\langle x_1x_2x_3\dots x_{n-1}x_nx_{n+1}x_k\right\rangle=T_j\left\langle y_1y_2y_3\dots y_{n-1}y_ny_{n+1}y_{h(k)}\right\rangle, \\ j=1,2,\dots n-1; k=n+2,\dots,m. \end{cases}$$

By Theorem 1, these equalities imply $P(m,n)$-equivalence of elements $X=(x_1,x_2,\dots,x_m)$and $hY=(y_{h(1)},y_{h(2)},\dots,y_{h(m)})$, where $h(j)=j$ for $j=1,2,\dots\dots,n+1$. This means $P(m,n)\times S(n,m)$-equivalence of elements $X=(x_1,x_2,\dots,x_m)$ and $Y=(y_1,y_2,\dots,y_m)$. □

Let $S(m)$ be the group of all permutations of the numbers $1, 2, \dots, m$ and $P(m,n)\times S(m)$ is a direct product of groups $P(m,n)$ and $S(m)$. We define an action β of the group $P(m,n)\times S(m)$ on the space $(R^n)^m$ as follows: for $q = ((r_1, r_2, \dots, r_m), g, h) \in P(m,n)\times S(m)$, $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$, where

$$h = \begin{pmatrix} 1 & 2 & \dots & m \\ h(1) & h(2) & \dots & h(m) \end{pmatrix}, \tag{3.12}$$

we put

$$\beta(q, X) = ((r_1, r_2, \dots, r_m), g, h), X) = (r_1 g x_{h(1)}, r_2 g x_{h(2)}, \dots, r_m g x_{h(m)}).$$

Definition 3.3. *Elements $A, B \in (\mathbb{R}^n)^m$ is called $P(m,n)\times S(m)$-equivalent if there exists $q \in P(m,n)\times S(m)$ such that $B = \beta(q, A)$. In this case, we write $A \overset{P(m,n)\times S(m)}{\sim} B$.*

Definition 3.4. *A rational function $f(x_1, x_2, \dots, x_m)$ of elements $X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is called $P(m,n)\times S(m)$-invariant if $f(\alpha(q,X)) = f(X)$ for all $q \in P(m,n)\times S(m)$.*

Definition 3.5. *$X = (x_1, x_2, \dots, x_m) \in (\mathbb{R}^n)^m$ is called strongly regular if $[x_{p_1} x_{p_2} \cdots x_{p_n}] \neq 0$ for all natural numbers $p_1, p_2, \dots, p_n$ such that $1 \le p_1 < p_2 < \cdots < p_n \le m$.*

If $X = (x_1, x_2, \dots, x_m)$ is strongly regular, $Y = (y_1, y_2, \dots, y_m) \in (\mathbb{R}^n)^m$ and $X \overset{P(m,n)}{\sim} Y$ then $Y = (y_1, y_2, \dots, y_m)$ is also strongly regular. Hence the set of all strongly regular elements is a $P(m,n)$-invariant subset in $(\mathbb{R}^n)^m$.

For $j = 1, 2, \dots n-1; k = n+2, \dots m$, we put $T_{jk}(X) = T_j \langle x_1 x_2 x_3 \dots x_{n-1} x_n x_{n+1} x_k \rangle$, where $X = (x_1, x_2, \dots, x_m)$. We denote by A_{nm} the algebra of real polynomials of $T_{jk}(X); j = 1, 2, \dots n-1; k = n+2, \dots m$. Let t and $\{z_{jk} | j = 1, 2, \dots n-1; k = n+2, \dots m\}$ be independent variables. We consider the following function

$$\prod_{h\in S(m)} \left[t - \sum_{k=n+2}^{m} (\sum_{i=1}^{n-1} T_{ik}(h(X)) z_{ik}) \right].$$

We define the polynomials $U_{r_1 r_2 \dots r_{m-n-1}}(X) \in A_{nm}$ by the following equality

$$\prod_{h\in S(m)} \left[t - \sum_{k=n+2}^{m} (\sum_{i=1}^{n-1} T_{ik}(h(X)) z_{ik}) \right] = \tag{3.13}$$

$$t^{m!} + \sum_{l=1}^{m!-1} (-1)^l t^{m!-l} \sum_{\sum_{j=1}^{m-n-1} (\sum_{i=1}^{n-1} r_{ij} = l)} U_{r_{ij}}(X) \cdot \prod_{k=n+2}^{m} (z_{1k}^{r_{1k}} z_{2k}^{r_{2k}} \cdots z_{n-1k}^{r_{n-1k}}),$$

where $m! = 1 \cdot 2 \cdot \dots \cdot m$ and $i = 1, \dots, n-1; j = 1, \dots, m-n-1$. It is obvious that every function $U_{r_{ij}}(X)$ is $P(m,n)\times S(m)$-invariant.

Theorem 3.2. *Strongly regular elements* $X = (x_1, x_2, \ldots, x_m), Y = (y_1, y_2, \ldots, y_m) \in (\mathbb{R}^n)^m$ *are* $P(m,n) \times S(m)$*-equivalent if and only if*

$$\begin{cases} U_{r_{ij}}(X) = U_{r_{ij}}(Y), \\ 1 \leq r_1 + r_2 \cdots + r_{n-1} \leq m - n - 1; i = 1, \ldots, n-1; j = 1, \ldots, m-n-1. \end{cases} \tag{3.14}$$

Proof. A proof is similar to the proof of Theorem 3.1. □

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