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RULED SURFACES WITH $T_1N_1B_1$-SMARANDACHE BASE CURVE OBTAINED FROM THE SUCCESSOR FRAME

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Abstract. In this study, ruled surfaces formed by the movement of the Frenet vectors of the successor curve along the Smarandache curve obtained from the tangent and principal normal vectors of the successor curve of a curve are definened. Then, the Gaussian and mean curvatures of each ruled surface were calculated. It has been shown that the ruled surface formed by the tangent vector of the successor curve moving along the Smarandache curve is a developable ruled surface. In addition, it was found that the surface formed by the principal normal vector of the succesor curve along the Smarandache curve is a minimal developable ruled surface if the principal curve is planar. Conditions are given for other surfaces to be developable or minimal surfaces.

Keywords: Smarandache ruled surfaces, Sucessor curve, mean curvature, Gaussian curvature.

2010 Mathematics Subject Classification: 53A04, 53A05.

1. Introduction

The image of a function with two real variables in three-dimensional space is a surface. Surfaces are used in many fields, such as architecture and engineering [26]. In 1795, Monge defined the ruled surface as the surface formed by the movement of the line along the curve. Any ruled surface is formed as a result of the continuous movement of a line along any curve. These curves are called the base curve and the director curve, respectively. The curvature

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of surfaces was defined by Gauss in the 19th century, and therefore it was named Gaussian curvature [19]. Gaussian curvatures are related to the dimensions of the surface [27]. Since the average curvature of the surface is a ratio, it is independent of the size of the surface. Thus far, many studies [1, 3, 6, 7, 8, 9, 10, 12, 13] on the Gaussian curvatures of surfaces have been conducted.

There are many special curves in differential geometry. One of them is the successor curve. This curve is defined as, there is a new curve, such that the tangent of one curve the principal normal of the other curve, by Menninger [14] in 2014. Later, Masal [11] investigated the relationships between the position vectors of this curve and defined Successor planes. Thus far, many studies have been conducted on this concept [5, 30]. Another special curve is the Smarandache curve defined in Minkowski space [2, 21, 28, 29].

In recent years, many studies have been carried out on ruled surfaces whose base curve is Smarandache curve. Some of these studies can be accessed from [4, 16, 17, 18, 22, 23, 24, 25].

In this paper, we present some special ruled surfaces with $T_1N_1B_1$-Smarandache curves obtained from their successor frames. We then investigate the properties of these ruled surfaces by means of Gaussian and mean curvatures. We obtain the conditions that which of these surfaces developable and which of these minimal. At the end, we visualise the main idea by providing four examples.

2. Preliminaries

This section provides some basic notions needed to be the following sections. Throughout this paper, let $\alpha = \alpha(s)$ and $\beta = \beta(s)$ be two differentiable unit speed curve in E^3 and their Frenet aparatus be $\{T, N, B, \kappa, \tau\}$ and $\{T_1, N_1, B_1, \kappa_1, \tau_1\}$, respectively. Then,

$$\begin{aligned} T &= \alpha', \;\; N = \frac{\alpha''}{\|\alpha''\|}, \;\; B = T \wedge N, \;\; \kappa = \|\alpha''\|, \;\; \tau = \langle N', B \rangle, \\ T' &= \kappa N, \;\; N' = -\kappa T + \tau B, \;\; B' = -\tau N. \end{aligned}$$

The surface formed by a line moving depending on the parameter of a curve is called a ruled surface, and its parametric expression is $X(s, \nu) = \alpha(s) + \nu r(s)$. Here, ν is a constant. Besides, α and r are referred to as the base curve and the director curve of X, respectively. The normal vector field N_X, the Gaussian curvature K_X, and the mean curvature H_X of $X(s, \nu)$ are as follows:

$$N_X = \frac{X_s \wedge X_\nu}{\|X_s \wedge X_\nu\|}, \tag{2.1}$$

$$K_X = \frac{eg - f^2}{EG - F^2}, \quad H_X = \frac{Eg - 2fF + eG}{2(EG - F^2)}, \tag{2.2}$$

Here,

$$E = \langle X_s, X_s \rangle, \;\; F = \langle X_s, X_\nu \rangle, \;\; G = \langle X_\nu, X_\nu \rangle, \tag{2.3}$$

$$e = \langle X_{ss}, N_X \rangle, \;\; f = \langle X_{s\nu}, N_X \rangle, \;\; g = \langle X_{\nu\nu}, N_X \rangle. \tag{2.4}$$

Definition 2.1. [11, 14] *If the unit tangent vector of α is the principal normal vector of β, then β is called Successor curve of α.*

Theorem 2.1. [11, 14] *Let β be the Successor curve of α. Frenet apparatus of β curve is as follows:*

$$T_1 = -\cos\theta N + \sin\theta B, \quad N_1 = T, \quad B_1 = \sin\theta N + \cos\theta B, \quad \kappa_1 = \kappa\cos\theta, \; \tau_1 = \kappa\sin\theta.$$

where, θ is the angle between binormal vectors B and B_1 and $\theta(s) = \theta_0 + \int \tau(s)ds$.

Definition 2.2. [29] *A regular curve in Minkowski space, whose position vector is obtained by Frenet frame vectors on another regular curve, is called a Smarandache Curve.*

Let β be the Successor curve of α. It can be observed that the unit curve γ, inspired in [11], produces Smarandache curves, for all $s \in I \subseteq \mathbb{R}$, such that

$$\gamma(s) = \frac{aT + bN + cB}{\sqrt{a^2 + b^2 + c^2}}, \;\; a, b, c \in \mathbb{R}.$$

Here, if a, b, and c are nonzero the Smarandache curves produced by $\gamma(s)$ are denoted by $\{T\,N\,B\}$-Smarandache Curves. This paper consider $\{T\,N\,B\}$-Smarandache Curves.

3. Ruled Surfaces with $T_1N_1B_1$-Smarandache Base Curve Obtained From the Successor Frame

In this section, firstly we define some special ruled surfaces with $T_1N_1B_1$-Smarandache base curve obtained from the successor frame. Then we examine the properties of these ruled surfaces by means of Gaussian and mean curvatures. And we give the conditions of being developable or minimal surface.

Definition 3.1. *Let the successor curve of the α curve be β. The ruled surface formed by targent vector T_1 the vector along the $T_1N_1B_1$ Smarandache curve obtained from the T_1 targent vector, N_1 principal normal vector and B_1 binormal vector of the β curve as follows:*

$$\begin{aligned} \Phi(s,v) &= \tfrac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vT_1 \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B) + v(-\cos\theta N + \sin\theta B). \end{aligned} \tag{3.5}$$

Theorem 3.1. *Let the successor curve of the curve α be β. The Gaussian and mean curvature of the $\Phi(s,v)$ ruled surface are as follows:*

$$K_\Phi = \frac{-3\cos^2\theta\sin^2\theta\big(((1+v\sqrt{3})\cos\theta-\sin\theta)^2+\sin^2\theta\big)}{\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$H_\Phi = \frac{\sqrt{3}\kappa\sin\theta\Big(2\cos^2\theta-\big(((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2+1)\Big)-\sqrt{3}\tau(1+v\sqrt{3})}{2\kappa\big(\sin^2\theta+((1+v\sqrt{3})^2\cos\theta-\sin\theta)^2\big)^{\frac{3}{2}}}.$$

Proof. Partial derivatives of equation (3.5) are,

$$\Phi_s = \frac{\kappa}{\sqrt{3}}\Big(((1+v\sqrt{3})\cos\theta-\sin\theta\big)T+N\Big),\quad \Phi_v=-\cos\theta N+\sin\theta B,\quad \Phi_{sv}=\kappa\cos\theta T,$$

$$\Phi_{ss} = \frac{\begin{array}{l}\Big(\kappa'\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)-\kappa^2-\kappa\tau\big((1+v\sqrt{3})\sin\theta+\cos\theta\big)\Big)T\\ +\Big(\kappa'+\kappa^2\big(\sin\theta-(1-v\sqrt{3})\cos\theta\big)\Big)N+\kappa\tau B\end{array}}{\sqrt{3}},\quad \Phi_{vv}=0.$$

Thus, from equation (2.1) the normal of the surface N_Φ is given as

$$N_\Phi = \frac{\sin\theta T-\sin\theta\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)N-\cos\theta\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)B}{\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\Phi=\frac{\kappa^2}{3}\big(((1+v\sqrt{3})\cos\theta-\sin\theta)^2+1\big),\quad F_\Phi=-\frac{\kappa\cos\theta}{\sqrt{3}},\quad G_\Phi=1,$$

$$e_\Phi=-\frac{\kappa^2\sin\theta\big(\sin\theta-(1+v\sqrt{3})\cos\theta+1\big)+\kappa\tau(1+v\sqrt{3})}{\sqrt{3}\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_\Phi=\frac{\kappa\cos\theta\sin\theta}{\Big(\sin^2\theta+\big((1+v\sqrt{3})\cos\theta-\sin\theta\big)^2\Big)^{\frac{1}{2}}},\quad g_\Phi=0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

Corollary 3.1. *Let the successor curve of the α curve be β. If α curve is planar and $\theta=\pi+k\pi$ $(k\in\mathbb{N})$, the ruled surface $\Phi(s,v)$ is the minimal developable surface.*

Definition 3.2. *Let the successor curve of the α curve be β. The ruled surface formed by principal normal normal vector N_1 along the $T_1N_1B_1$ Smarandache curve obtained from the T_1 targent vector, N_1 principal normal vector and B_1 binormal vector of the β curve as* *follows:*

$$\begin{aligned} Q(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1)+vN_1 \\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)+vT. \end{aligned} \tag{3.6}$$

Theorem 3.2. *Let the successor curve of the α curve be β. The Gaussian and mean curvature of the $Q(s,v)$ ruled surface are as follows:*

$$K_Q=0,\quad H_Q=\frac{\sqrt{3}\tau}{2\kappa(1+v\sqrt{3})}.$$

Proof. Partial derivatives of equation (3.6) are,

$$Q_s=\frac{\kappa}{\sqrt{3}}\big((\cos\theta-\sin\theta)T+(1+v\sqrt{3})N\big),\qquad Q_v=T,\qquad Q_{sv}=\kappa N\qquad Q_{vv}=0,$$

$$Q_{ss}=\frac{\big(\kappa'(\cos\theta-\sin\theta)-\kappa\tau(\cos\theta+\sin\theta)-\kappa^2(1+v\sqrt{3})\big)T+\big(\kappa^2(\cos\theta-\sin\theta)+\kappa'(1+v\sqrt{3})\big)N+\kappa\tau(1+v\sqrt{3})B}{\sqrt{3}}.$$

Thus, from equation (2.1) the normal of the surface N_Q is given as $N_Q=-B$. Moreover, in equaitons (2.3) and (2.4) the coefficients of fundamental forms are

$$\begin{aligned} &E_Q=\frac{\kappa^2}{3}\big((\cos\theta-\sin\theta)^2+(1+v\sqrt{3})^2\big),\quad F_Q=\frac{\kappa}{\sqrt{3}}(\cos\theta-\sin\theta),\quad G_Q=1,\\ &e_Q=-\frac{\kappa\tau}{\sqrt{3}}(1+v\sqrt{3}),\quad f_Q=g_Q=0. \end{aligned}$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

Corollary 3.2. *Let the successor curve of the α curve be β. If α curve is planar, the ruled surface $Q(s,v)$ is the minimal developable surface.*

Definition 3.3. *Let the successor curve of the α curve be β. The ruled surface formed by binormal vector B_1 the vector along the $T_1N_1B_1$ Smarandache curve obtained from the T_1 targent vector, N_1 principal normal vector and B_1 binormal vector of the β curve as follows:*

$$\begin{aligned} M(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1)+vB_1 \\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)+v(\sin\theta N+\cos\theta B). \end{aligned} \tag{3.7}$$

Theorem 3.3. *Let the successor curve of the α curve be β. The Gaussian and mean curvature of the $M(s,v)$ ruled surface are as follows:*

$$\begin{aligned} K_M &= \frac{-3\sin^2\theta\cos^2\theta}{\Big(\cos^2\theta+\big(\cos\theta-(1+v\sqrt{3})\sin\theta\big)^2\Big)^2},\\ H_M &= \frac{-\sqrt{3}\kappa\cos\theta\big(2\sin^2+(\cos\theta-(1+v\sqrt{3})\sin\theta)^2+1\big)-\sqrt{3}\tau(1+v\sqrt{3})}{2\kappa\Big(\cos^2\theta+\big(\cos\theta-(1+v\sqrt{3})\sin\theta\big)\Big)^{\frac{3}{2}}}. \end{aligned}$$

Proof. Partial derivatives of equation (3.7) are,

$$M_s = \frac{\kappa}{\sqrt{3}}\big((\cos\theta - (1+v\sqrt{3})\sin\theta)T + N\big), \quad M_v = \sin\theta N + \cos\theta B,$$

$$M_{sv} = -\kappa\sin\theta T, \quad M_{vv} = 0,$$

$$M_{ss} = \frac{\begin{array}{c}\big(\kappa'(\cos\theta - (1+v\sqrt{3})\sin\theta) - \kappa\tau(\sin\theta + (1-v\sqrt{3})\cos\theta) - \kappa^2\big)T \\ +\big(\kappa' + \kappa^2(\cos\theta - (1+v\sqrt{3})\sin\theta)\big)N + \kappa\tau B\end{array}}{\sqrt{3}}.$$

Thus, from equation (2.1) the normal of the surface N_M is given as

$$N_M = \frac{\cos\theta T - \cos\theta\big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)N + \sin\theta\big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)B}{\Big(\cos^2\theta + \big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_M = \frac{\kappa^2}{3}\big((\cos\theta - (1+v\sqrt{3})\sin\theta)^2 + 1\big), \quad F_M = \frac{\kappa\sin\theta}{\sqrt{3}}, \quad G_M = 1,$$

$$e_M = \frac{-\kappa\tau(1+v\sqrt{3}) - \kappa^2\cos\theta\big((\cos\theta - (1+v\sqrt{3})\sin\theta)^2 + 1\big)}{\sqrt{3}\Big(\cos^2\theta + \big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_M = \frac{-\kappa\sin\theta\cos\theta}{\Big(\cos^2\theta + \big(\cos\theta - (1+v\sqrt{3})\sin\theta\big)^2\Big)^{\frac{1}{2}}}, \quad g_M = 0.$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

Corollary 3.3. *If the $\theta = \pi + k\pi \ (k \in \mathbb{N})$, the ruled surface $M(s,v)$ is a developable surface.*

Definition 3.4. *Let the successor curve of the α curve be β. The ruled surface formed by T_1N_1 the vector along the $T_1N_1B_1$ Smarandache curve obtained from the T_1 targent vector, N_1 principal normal vector and B_1 binormal vector of the β curve as follows:*

$$\begin{aligned}\mu(s,v) &= \tfrac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \tfrac{v}{\sqrt{2}}(T_1 + N_1) \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B) + \tfrac{v}{\sqrt{2}}(T - \cos\theta N + \sin\theta B).\end{aligned} \tag{3.8}$$

Theorem 3.4. *Let the successor curve of the α curve be β. The Gaussian and mean curvature of the $\mu(s,v)$ ruled surface are as follows:*

$$K_\mu = \frac{-6\sin^4\theta}{\begin{array}{c}\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big(-\sqrt{2}\sin^2\theta + (\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta\big)^2\big(\sqrt{2}\cos\theta\sin\theta - (\sqrt{2}+v\sqrt{3})(\cos^2\theta+1)\big)^2\Big). \\ \Big(((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta)^2 + (\sqrt{2}+v\sqrt{3})^2 - \sin^2\theta\Big)\end{array}},$$

$$H_\mu = \frac{2\sqrt{6}\kappa\sin^3\theta - \sqrt{6}\kappa\sin\theta\Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2\Big) - 2\sqrt{6}\tau(\sqrt{2}+v\sqrt{3})^2}{\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\cos^2\theta\big)^2\Big)^{\frac{1}{2}} . \; 2\kappa\Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2 - \sin^2\theta\Big)}.$$

Proof. Partial derivatives of equation (3.8) are,

$$\mu_s = \frac{\kappa}{\sqrt{6}}\big(((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta)T + (\sqrt{2}+v\sqrt{3})N\big), \quad \mu_{sv} = \frac{\kappa}{\sqrt{2}}(\cos\theta T + N),$$

$$\mu_v = \frac{1}{\sqrt{2}}(T - \cos\theta N + \sin\theta B), \quad \mu_{vv} = 0,$$

$$\mu_{ss} = \frac{\Big((\sqrt{2}+v\sqrt{3})\big(\kappa'\cos\theta - \kappa\tau\sin\theta - \kappa^2\big) - \sqrt{2}\big(\kappa'\sin\theta + \kappa\tau\cos\theta\big)\Big)T + \Big(\kappa^2\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big) + (\sqrt{2}+v\sqrt{3})\kappa'\Big)N + (\sqrt{2}+v\sqrt{3})\kappa\tau B}{\sqrt{6}}.$$

Thus, from equation (2.1) the normal of the surface N_μ is given as

$$N_\mu = \frac{(\sqrt{2}+v\sqrt{3})\sin\theta T - \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)N - \big((\sqrt{2}+v\sqrt{3})(\cos^2\theta+1) - \sqrt{2}\cos\theta\sin\theta\big)B}{\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})(\cos^2\theta+1) - \sqrt{2}\cos\theta\sin\theta\big)^2\Big)^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\mu = \frac{\kappa^2}{6}\big(((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta)^2 + (\sqrt{2}+v\sqrt{3})^2\big), \quad F_\mu = -\frac{\kappa\sin\theta}{\sqrt{6}}, \quad G_\mu = 1, \quad g_\mu = 0,$$

$$e_\mu = -\frac{2\kappa\tau(\sqrt{2}+v\sqrt{3})^2 + \kappa^2\sin\theta\Big(\big((\sqrt{2}+v\sqrt{3})\cos\theta - \sqrt{2}\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2\Big)}{\sqrt{6}\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\cos^2\theta\big)^2\Big)^{\frac{1}{2}}},$$

$$f_\mu = \frac{\kappa\sin^2\theta}{\Big((\sqrt{2}+v\sqrt{3})^2\sin^2\theta + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\sin^2\theta\big)^2 + \big((\sqrt{2}+v\sqrt{3})\cos\theta\sin\theta - \sqrt{2}\cos^2\theta\big)^2\Big)^{\frac{1}{2}}}.$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

Corollary 3.4. *If the $\theta = k\pi \ (k \in \mathbb{N})$, the ruled surface $\mu(s,v)$ is a developable surface.*

Definition 3.5. *Let the successor curve of the α curve be β. The ruled surface formed by T_1B_1 the vector along the $T_1N_1B_1$ Smarandache curve obtained from the T_1 targent vector, N_1 principal normal vector and B_1 binormal vector of the β curve as follows:*

$$\begin{aligned} \psi(s,v) &= \tfrac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \tfrac{v}{\sqrt{2}}(T_1 + B_1) \\ &= \tfrac{1}{\sqrt{3}}(T + (\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B) + \tfrac{v}{\sqrt{2}}\big((\sin\theta - \cos\theta)N + (\sin\theta + \cos\theta)B\big). \end{aligned} \tag{3.9}$$

Theorem 3.5. *Let the successor curve of the α curve be β. The Gaussian and mean curvature of the $\psi(s,v)$ ruled surface are as follows:*

$$K_\psi = \frac{-6(\cos^2\theta - \sin^2\theta)^2}{\Big(2(\sin\theta+\cos\theta)^2 + (\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big). (1-\sin 2\theta)\big((\sqrt{2}+v\sqrt{3})^2-1)+2\big)},$$

$$H_\psi = \frac{3\kappa(\sin\theta+\cos\theta)\big((\sqrt{2}+v\sqrt{3})^2(\sin 2\theta-1)+2(\sin\theta-\cos\theta)^2-2\big) - 3\tau(\sqrt{2}+v\sqrt{3})}{\sqrt{6}\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big).\Big)^{\frac{1}{2}} (1-\sin 2\theta)\big((\sqrt{2}+v\sqrt{3})^2-1)+2\big)}.$$

Proof. Partial derivatives of equation (3.8) are,

$$\psi_s = \frac{\kappa}{\sqrt{6}}\big((\sqrt{2}+v\sqrt{3})(\cos\theta-\sin\theta)T+\sqrt{2}N\big), \quad \psi_{sv} = \frac{\kappa}{\sqrt{2}}(\cos\theta-\sin\theta)T,$$

$$\psi_v = \frac{1}{\sqrt{2}}(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B, \quad \psi_{vv}=0,$$

$$\psi_{ss} = \frac{\Big(\big((\sqrt{2}+v\sqrt{3})(\kappa'(\cos\theta-\sin\theta)-\kappa\tau(\sin\theta+\cos\theta)\big)-\sqrt{2}\kappa^2\Big)T + \Big(\sqrt{2}\kappa'+\kappa^2(\sqrt{2}+v\sqrt{3})(\cos\theta-\sin\theta)\Big)N+\sqrt{2}\kappa\tau B}{\sqrt{6}}.$$

Thus, from equation (2.1) the normal of the surface N_ψ is given as

$$N_\psi = \frac{\sqrt{2}(\sin\theta+\cos\theta)T+(\sqrt{2}+v\sqrt{3})(\sin^2\theta-\cos^2\theta)N+(\sqrt{2}+v\sqrt{3})(\sin 2\theta-1)B}{\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big)^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\psi = \frac{\kappa^2}{6}\big((\sqrt{2}+v\sqrt{3})^2(1-\sin 2\theta)+2\big), \quad F_\psi = \frac{\kappa}{\sqrt{6}}(\sin\theta-\cos\theta), \quad G_\psi = 1,$$

$$e_\psi = -\frac{\kappa\tau(4-2v\sqrt{6})+\kappa^2(\sin\theta+\cos\theta)\big(2+(\sqrt{2}+v\sqrt{3})^2(1-\sin 2\theta)\big)}{\sqrt{6}\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big)^{\frac{1}{2}}},$$

$$f_\psi = -\frac{-\kappa(\cos^2\theta-\sin^2\theta)}{\Big(2(\sin\theta+\cos\theta)^2+(\sqrt{2}+v\sqrt{3})^2\big((\sin^2\theta-\cos^2\theta)^2+(\sin 2\theta-1)^2\big)\Big)^{\frac{1}{2}}}, \quad g_\psi = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

Corollary 3.5. *If the $\theta = \frac{\pi}{4}+\frac{k\pi}{2}$ $(k\in\mathbb{Z})$, the ruled surface $\psi(s,v)$ is a developable surface.*

Definition 3.6. *Let the successor curve of the α curve be β. The ruled surface formed by N_1B_1 the vector along the $T_1N_1B_1$ Smarandache curve obtained from the T_1 targent vector, N_1 principal normal vector and B_1 binormal vector of the β curve as follows:*

$$\begin{aligned}\eta(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1)+\tfrac{v}{\sqrt{2}}(N_1+B_1)\\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)+\tfrac{v}{\sqrt{2}}(T+\sin\theta N+\cos\theta B).\end{aligned}\tag{3.10}$$

Theorem 3.6. *Let the successor curve of the α curve be β. The Gaussian and mean curvature of the $\eta(s,v)$ ruled surface are as follows:*

$$K_\eta=\frac{-6\cos^4\theta}{\begin{pmatrix}((\sqrt{2}+v\sqrt{3})\cos\theta)^2-\big(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2\\+\big(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}\cdot\Big(\big(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\big)^2+(\sqrt{2}+v\sqrt{3})^2-\cos^2\theta\Big)},$$

$$H_\eta=-\frac{6\tau(\sqrt{2}+v\sqrt{3})^2+3\kappa\cos\theta\Big((\sqrt{2}+v\sqrt{3})^2+\big(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\big)^2-2\cos^2\theta\Big)}{\sqrt{6}\kappa\begin{pmatrix}((\sqrt{2}+v\sqrt{3})\cos\theta)^2-\big(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2\\+\big(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}\cdot\Big(\big(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\big)^2+(\sqrt{2}+v\sqrt{3})^2-\cos^2\theta\Big)}.$$

Proof. Partial derivatives of equation (3.10) are,

$$\eta_s=\frac{\kappa}{\sqrt{6}}\Big(\big(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\big)T+(\sqrt{2}+v\sqrt{3})N\Big),\quad \eta_{sv}=-\frac{\kappa}{\sqrt{2}}(\sin\theta T-N),$$
$$\eta_v=\frac{1}{\sqrt{2}}(T+\sin\theta N+\cos\theta B),\quad \eta_{vv}=0,$$

$$\eta_{ss}=\frac{\begin{aligned}&\Big(\kappa'\big(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\big)-\kappa\tau\big((\sqrt{2}+v\sqrt{3})\cos\theta+\sqrt{2}\sin\theta\big)-\kappa^2(\sqrt{2}+v\sqrt{3})\Big)T\\&+\Big(\kappa'(\sqrt{2}+v\sqrt{3})+\kappa^2\big(\sqrt{2}\cos\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\big)\Big)N+\kappa\tau(\sqrt{2}+v\sqrt{3})B\end{aligned}}{\sqrt{6}}.$$

Thus, from equation (2.1) the normal of the surface N_η is given as

$$N_\eta=\frac{\begin{aligned}&((\sqrt{2}+v\sqrt{3})\cos\theta)T-\big(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)N\\&+\big(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)B\end{aligned}}{\begin{pmatrix}((\sqrt{2}+v\sqrt{3})\cos\theta)^2+\big(\sqrt{2}\cos^2\theta-(\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2\\+\big(\sqrt{2}\cos\theta\sin\theta-(\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\eta = \frac{\kappa^2}{6}\Big(\big(\sqrt{2}\cos\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\big)^2 + (\sqrt{2}+v\sqrt{3})^2\Big), \quad F_\eta = \frac{\kappa\cos\theta}{\sqrt{6}}, \quad G_\eta = 1,$$

$$e_\eta = \frac{-2\kappa\tau(\sqrt{2}+v\sqrt{3})^2 - \kappa^2\cos\theta\Big((\sqrt{2}+v\sqrt{3})^2 + \big(\sqrt{2}\cos\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\big)^2\Big)}{\sqrt{6}\begin{pmatrix}\big((\sqrt{2}+v\sqrt{3})\cos\theta\big)^2 + \big(\sqrt{2}\cos^2\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2\\ +\big(\sqrt{2}\cos\theta\sin\theta - (\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}},$$

$$f_\eta = \frac{-\kappa\cos^2\theta}{\begin{pmatrix}\big((\sqrt{2}+v\sqrt{3})\cos\theta\big)^2 + \big(\sqrt{2}\cos^2\theta - (\sqrt{2}+v\sqrt{3})\sin\theta\cos\theta\big)^2\\ +\big(\sqrt{2}\cos\theta\sin\theta - (\sqrt{2}+v\sqrt{3})(\sin^2\theta+1)\big)^2\end{pmatrix}^{\frac{1}{2}}}, \quad g_\eta = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. ☐

Corollary 3.6. *If the* $\theta = \frac{\pi}{2} + k\pi$ $(k \in \mathbb{N})$, *the ruled surface* $\eta(s,v)$ *is a minimal developable surface.*

Definition 3.7. *Let* β *be the Successor curve of* α. *The ruled surface formed by* $T_1N_1B_1$ *the vector along the* $T_1N_1B_1$ *Smarandache curve obtained from the* T_1 *targent vector,* N_1 *principal normal vector and* B_1 *binormal vector of the* β *curve as follows:*

$$\begin{aligned}\Gamma(s,v) &= \tfrac{1}{\sqrt{3}}(T_1+N_1+B_1) + \tfrac{v}{\sqrt{3}}(T_1+N_1+B_1)\\ &= \tfrac{1}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B) + \tfrac{v}{\sqrt{3}}(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B)\end{aligned} \tag{3.11}$$

Theorem 3.7. *Let* β *be the Successor curve of* α. *The Gaussian and mean curvature of the* $\Gamma(s,v)$ *ruled surface are as follows:*

$$K_\Gamma = 0, \quad H_\Gamma = -\frac{\sqrt{3}\tau + \sqrt{3}\kappa\big((\sin\theta+\cos\theta)(\sin\theta-\cos\theta)\cos 2\theta}{2\kappa(1+v)(2-\sin 2\theta)\sqrt{2+\sin 2\theta}}.$$

Proof. Partial derivatives of equation (3.11) are,

$$\Gamma_s = \frac{\kappa(1+v)}{\sqrt{3}}\big((\cos\theta-\sin\theta)T+N\big), \quad \Gamma_{sv} = \frac{\kappa}{\sqrt{3}}\big((\cos\theta-\sin\theta)T+N\big),$$

$$\Gamma_v = \frac{1}{\sqrt{3}}\big(T+(\sin\theta-\cos\theta)N+(\sin\theta+\cos\theta)B\big), \quad \Gamma_{vv} = 0,$$

$$\Gamma_{ss} = \frac{(1+v)\big(\kappa'(\cos\theta-\sin\theta)-\kappa\tau(\sin\theta+\cos\theta)-\kappa^2\big)T + \big(\kappa'+\kappa^2(\sin\theta-\cos\theta)\big)N + \kappa\tau B}{\sqrt{3}}.$$

Thus, from equation (2.1) the normal of the surface N_Γ is given as

$$N_\Gamma = \frac{(\sin\theta+\cos\theta)T - \cos 2\theta N + \sin 2\theta B}{\sqrt{2+\sin 2\theta}}.$$

Moreover, in equations (2.3) and (2.4) the coefficients of fundamental forms are

$$E_\Gamma = \frac{\kappa^2(1+v)^2(2-\sin 2\theta)}{3}, \quad F_\Gamma = 0, \quad G_\Gamma = 1,$$
$$e_\Gamma = -\frac{\kappa\tau(1+v)+\kappa^2(1+v)\big((\sin\theta+\cos\theta)(\cos\theta-\sin\theta)\cos 2\theta\big)}{\sqrt{6+\sqrt{3}\sin 2\theta}}, \quad f_\Gamma = 0, \quad g_\Gamma = 0$$

respectively. Thus, by using equation (2.2) the Gaussian and mean curvatures are found. □

Example 3.1. *Let β Salkowski curve* [15] *be the Successor curve of α. The equation of this curve for $m=\frac{1}{3}$ is as follows:*

$$\beta(s) = \frac{3}{\sqrt{10}}\left(\begin{array}{c} -\frac{\sqrt{10}-1}{4\sqrt{10}+8}\big(\sin(\frac{\sqrt{10}+2}{\sqrt{10}})s\big) - \frac{\sqrt{10}-1}{4\sqrt{10}-8}\big(\sin(\frac{\sqrt{10}-2}{\sqrt{10}})s\big) - \frac{1}{2}\sin s, \\ -\frac{\sqrt{10}-1}{4\sqrt{10}+8}\big(\cos(\frac{\sqrt{10}+2}{\sqrt{10}})s\big) + \frac{\sqrt{10}-1}{4\sqrt{10}-8}\big(\cos(\frac{\sqrt{10}-2}{\sqrt{10}})s\big) + \frac{1}{2}\cos s, \frac{3}{4}\cos(\frac{2s}{\sqrt{10}}) \end{array}\right)$$

The Successor frames of β curve $\{T_1, N_1, B_1\}$ are as follows:

$$T_1(s) = \left(\ -\cos s\cos\tfrac{s}{\sqrt{10}} - \tfrac{1}{\sqrt{10}}\sin s\sin\tfrac{s}{\sqrt{10}},\ \ -\sin s\cos\tfrac{s}{\sqrt{10}} + \tfrac{1}{\sqrt{10}}\cos s\sin\tfrac{s}{\sqrt{10}},\ \ \tfrac{3}{\sqrt{10}}\sin\tfrac{s}{\sqrt{10}}\ \right),$$

$$N_1(s) = \left(\ \tfrac{3}{\sqrt{10}}\sin s,\ \ -\tfrac{3}{\sqrt{10}}\cos s,\ \ -\tfrac{1}{\sqrt{10}}\ \right),$$

$$B_1(s) = \left(\ -\cos s\sin\tfrac{s}{\sqrt{10}} - \tfrac{1}{\sqrt{10}}\sin s\cos\tfrac{s}{\sqrt{10}},\ \ -\sin s\cos\tfrac{s}{\sqrt{10}} + \tfrac{1}{\sqrt{10}}\cos s\cos\tfrac{s}{\sqrt{10}},\ \ \tfrac{3}{\sqrt{10}}\cos\tfrac{s}{\sqrt{10}}\ \right)$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figures 1- 7;

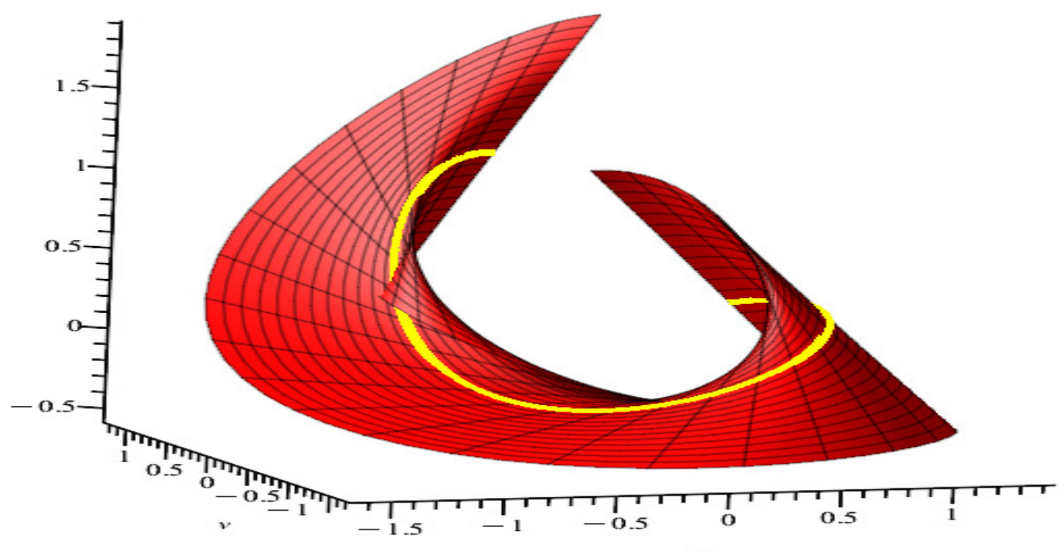

FIGURE 1. The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{3}}(T_1+N_1+B_1)+vT_1$

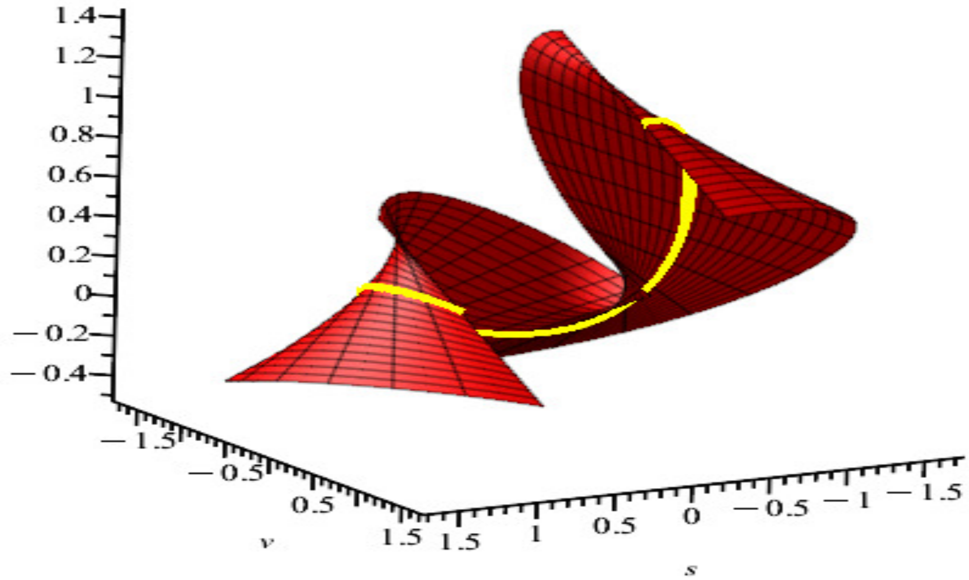

FIGURE 2. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

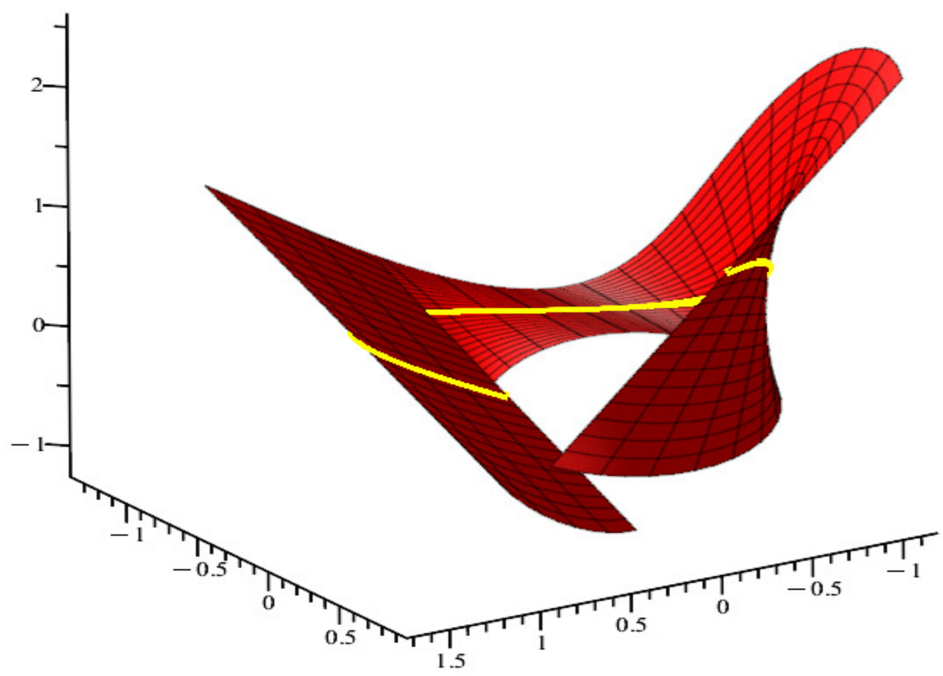

FIGURE 3. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

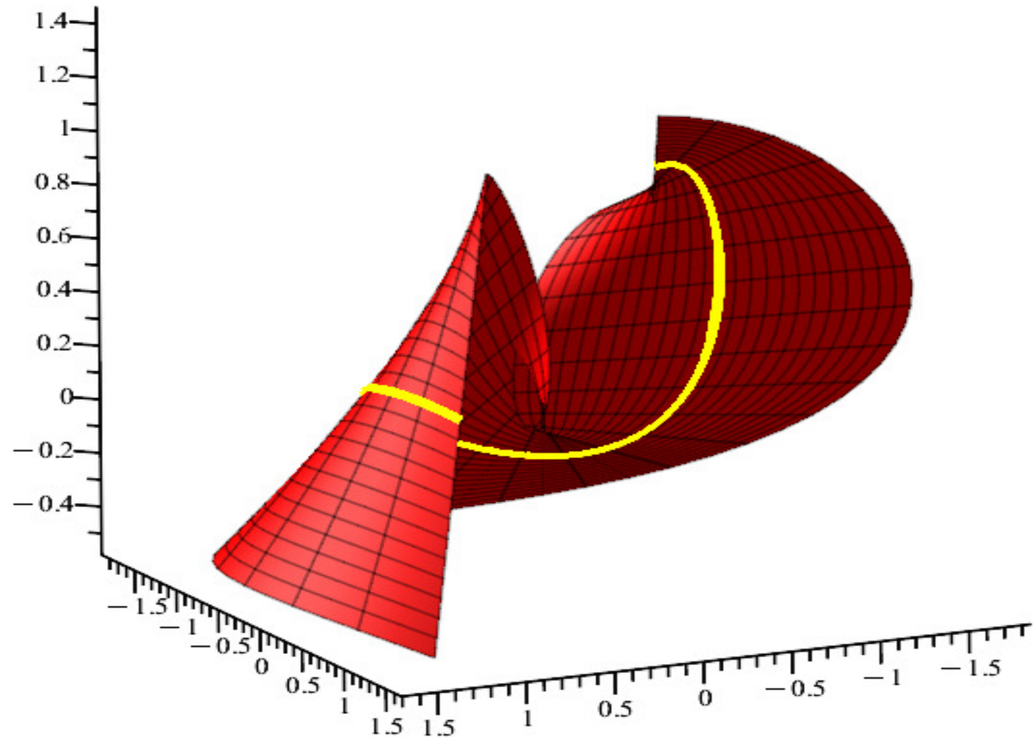

FIGURE 4. The ruled surface $\mu(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+N_1)$

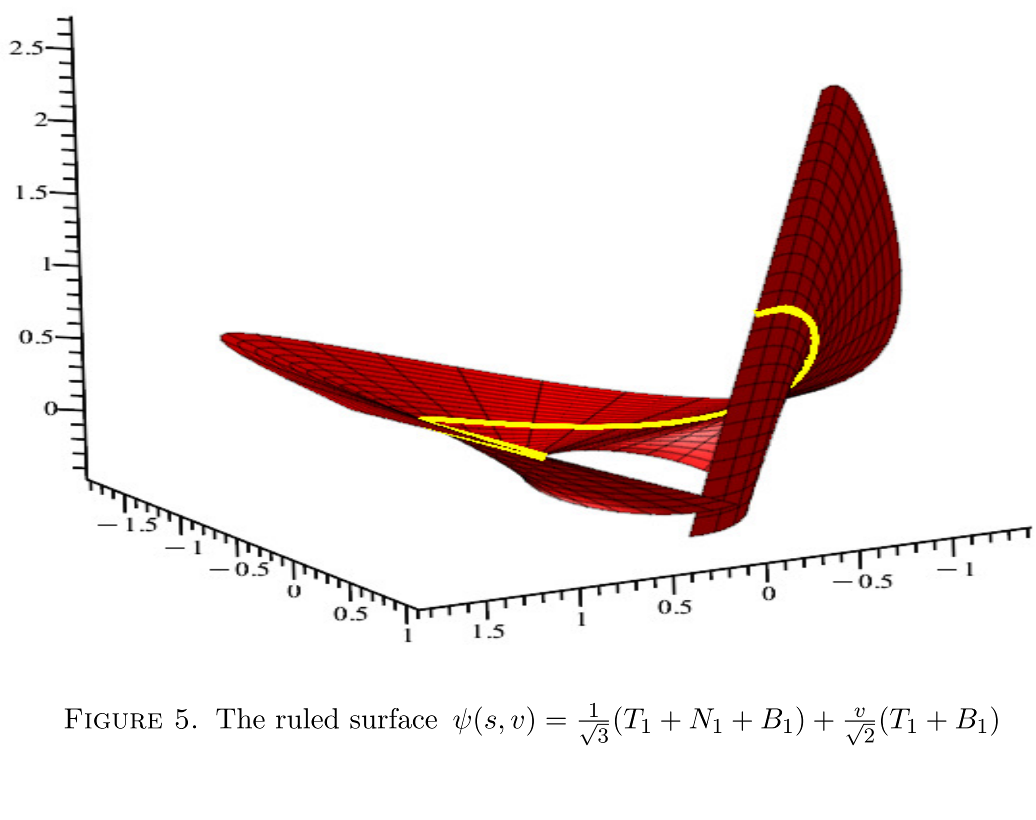

FIGURE 5. The ruled surface $\psi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

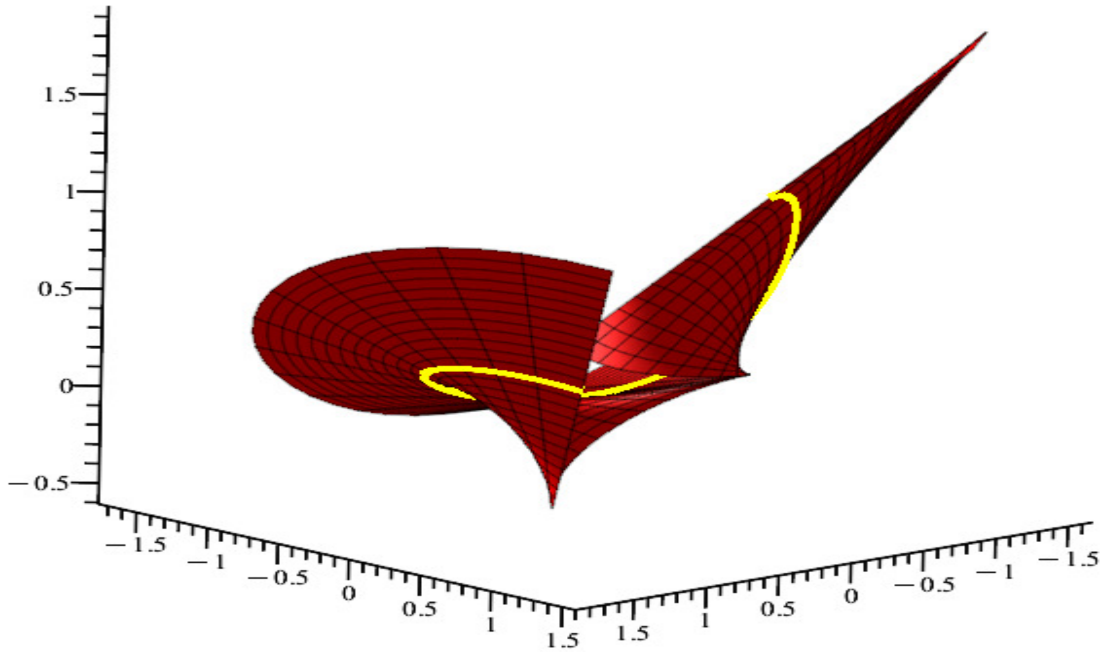

FIGURE 6. The ruled surface $\eta(s,v)\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

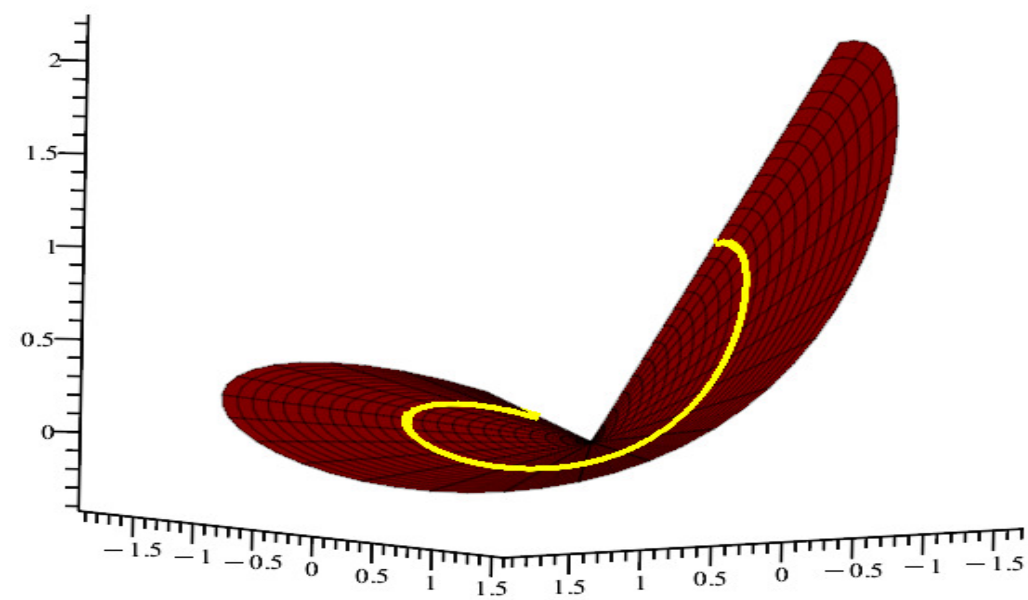

FIGURE 7. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 3.2. *Let the Salkowski curve in Example 3.1 be the main curve. From* [15] *and Theorem 2.1 the Successor frames are as follows:*

$$T_1(s) = \begin{pmatrix} -\cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right) + \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\cos s\sin\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right), \\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right) - \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(-\sin s\sin\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right), \\ \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}} + \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right) \end{pmatrix},$$

$$N_1(s) = \left(-\cos s\cos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}, -\sin s\cos\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}, \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}} \right),$$

$$B_1(s) = \begin{pmatrix} \sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\sin s\right) - \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\cos s\sin\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}\right), \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos s\right) - \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\sin s\sin\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}}\right), \\ -\sin\left(\int tan\frac{s}{\sqrt{10}}ds\right)\frac{1}{\sqrt{10}} + \cos\left(\int tan\frac{s}{\sqrt{10}}ds\right)\left(\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\right) \end{pmatrix}.$$

The graphs of the ruled surfaces obtained from these frames for $s \in [-\pi, \pi]$ *and* $v \in [-1, 1]$ *are shown figures 8-14;*

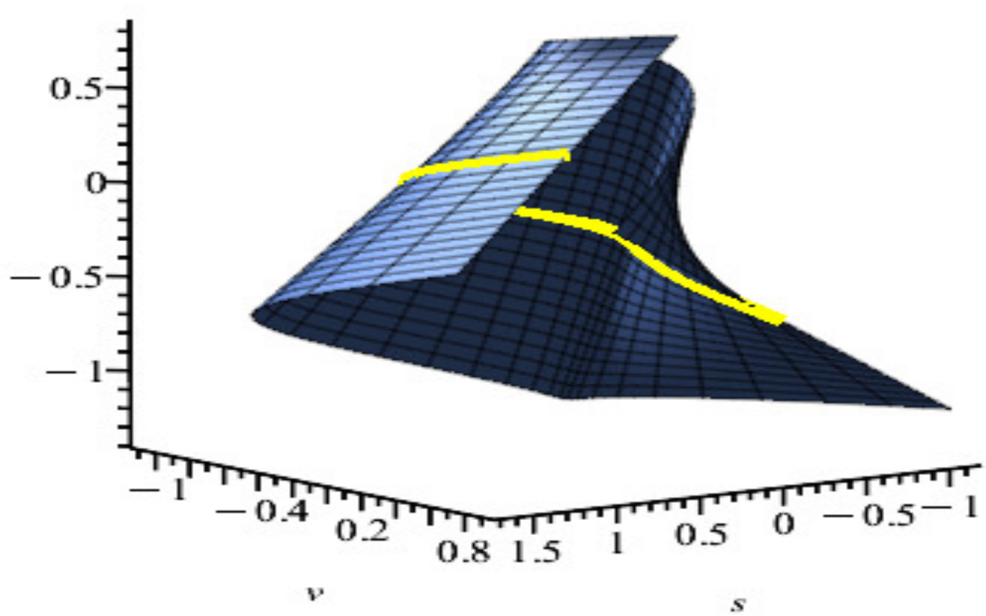

FIGURE 8. The ruled surface $\Phi(s, v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vT_1$

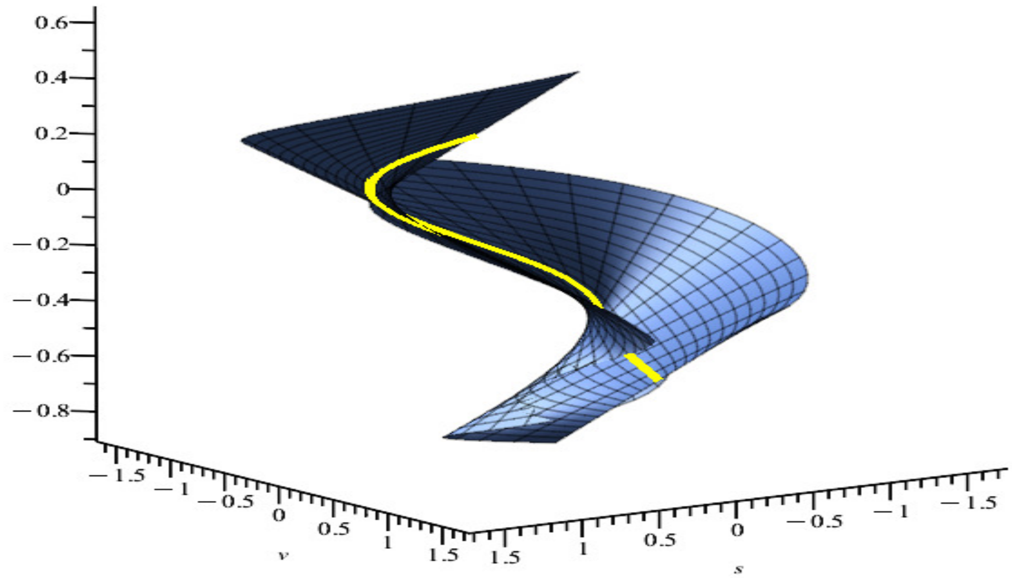

FIGURE 9. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

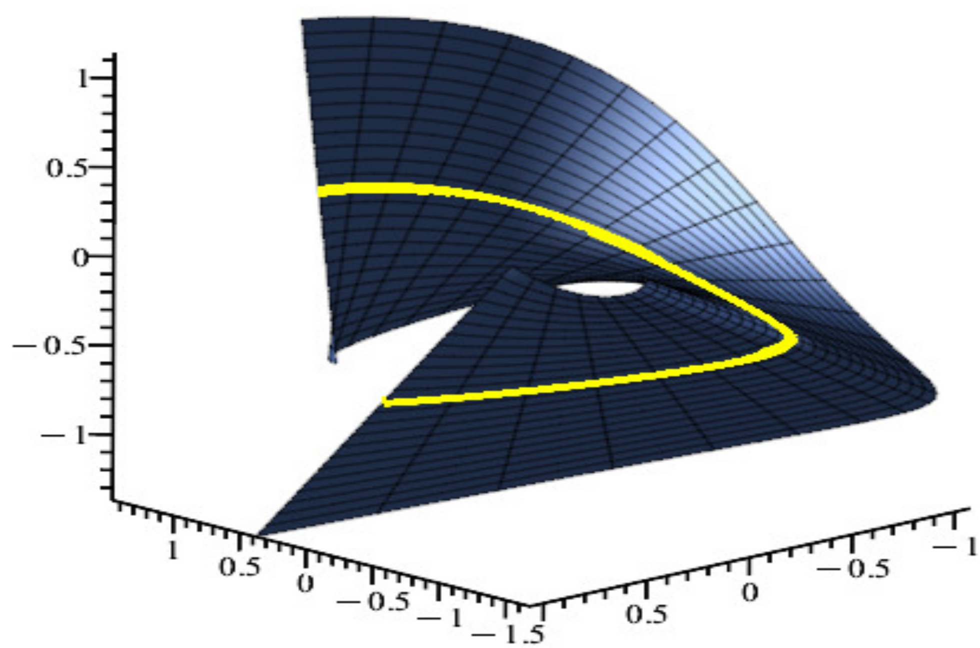

FIGURE 10. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

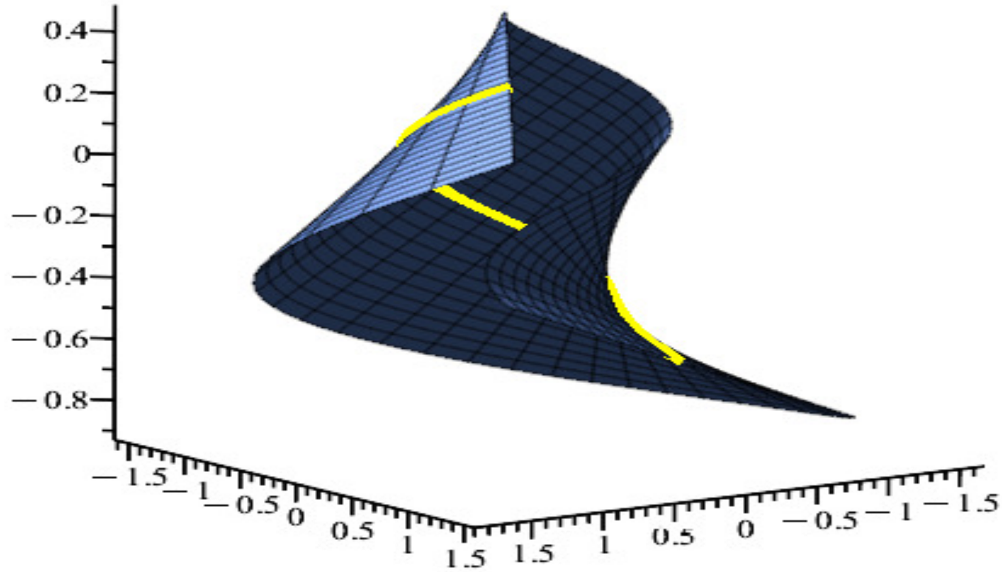

FIGURE 11. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

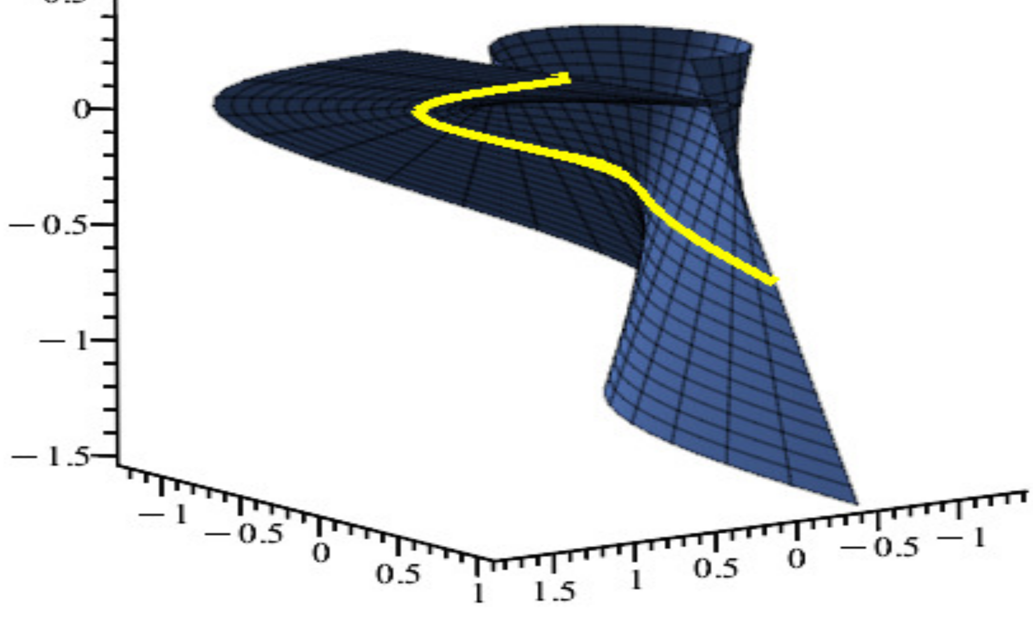

FIGURE 12. The ruled surface $\psi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

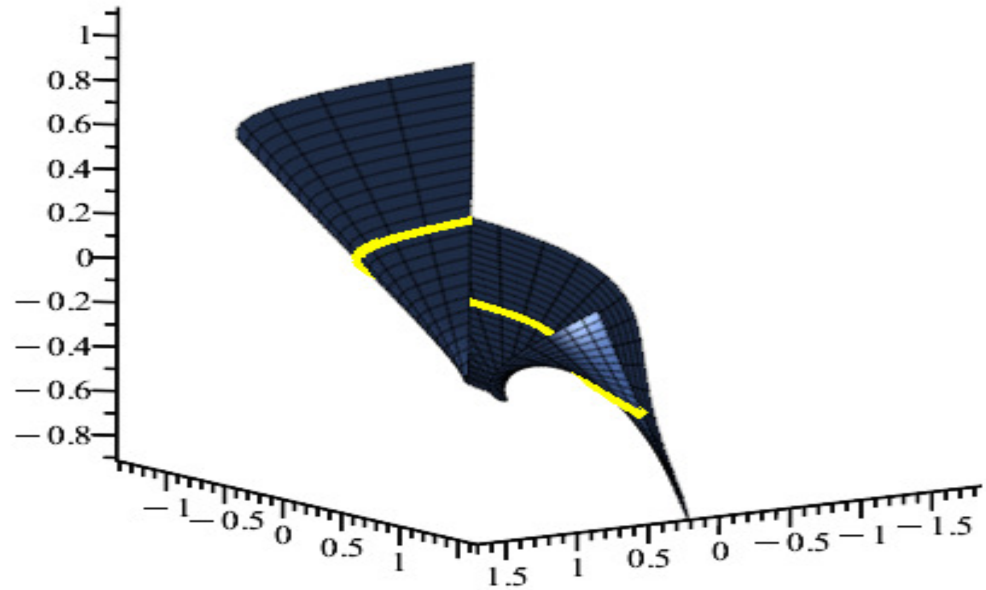

FIGURE 13. The ruled surface $\eta(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(N_1 + B_1)$

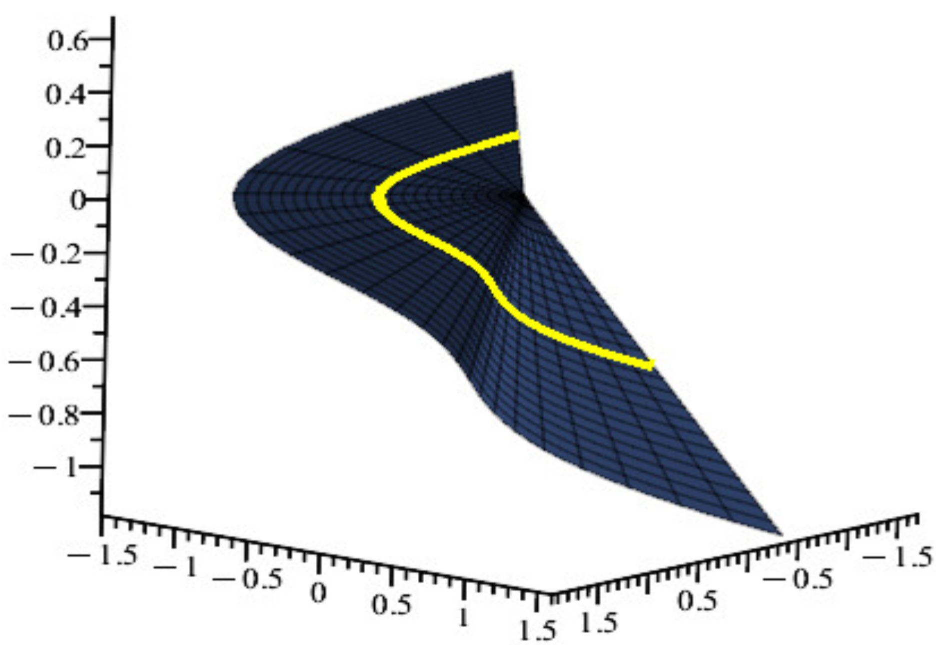

FIGURE 14. The ruled surface $\Gamma(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{3}}(T_1 + N_1 + B_1)$

Example 3.3. *Let β^* anti Salkowski curve [15] be the Successor curve of α. The equation of this curve for $m=\frac{1}{3}$ is as follows:*

$$\beta^*(s)=\frac{\sqrt{10}}{40}\begin{pmatrix} -\frac{5}{2\sqrt{10}}\left(\frac{3}{\sqrt{10}}\cos(\frac{1}{5}+\cos(\frac{2}{\sqrt{10}})s)\right)+\frac{6}{5}\sin s\sin\frac{2}{\sqrt{10}}s, \\ -\frac{5}{2\sqrt{10}}\left(\frac{3}{\sqrt{10}}\sin(\frac{1}{5}+\cos(\frac{2}{\sqrt{10}})s)\right)+\frac{6}{5}\cos s\sin\frac{2}{\sqrt{10}}s,\ \ -\frac{9\sqrt{10}}{40}\left(\frac{2}{\sqrt{10}}s+\sin(\frac{2}{\sqrt{10}})s\right) \end{pmatrix}.$$

The Successor frames of β^* curve $\{T_1^*, N_1^*, B_1^*\}$ are as follows:

$$T_1^*(s)=\left(\ -\cos s\sin\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}},\ \ -\sin s\sin\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\cos s\cos\frac{s}{\sqrt{10}},\ \ -\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}}\ \right),$$

$$N_1^*(s)=\left(\ \frac{3}{\sqrt{10}}\sin s,\quad -\frac{3}{\sqrt{10}}\cos s,\ \ \frac{1}{\sqrt{10}}\ \right),$$

$$B_1^*(s)=\left(\ -\cos s\cos\frac{s}{\sqrt{10}}-\frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}},\ \ -\sin s\cos\frac{s}{\sqrt{10}}+\frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}},\ \ \frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\ \right).$$

The graphs of the ruled surfaces obtained from these frames for $s\in[-\pi,\pi]$ and $v\in[-1,1]$ are shown figure {15-21};

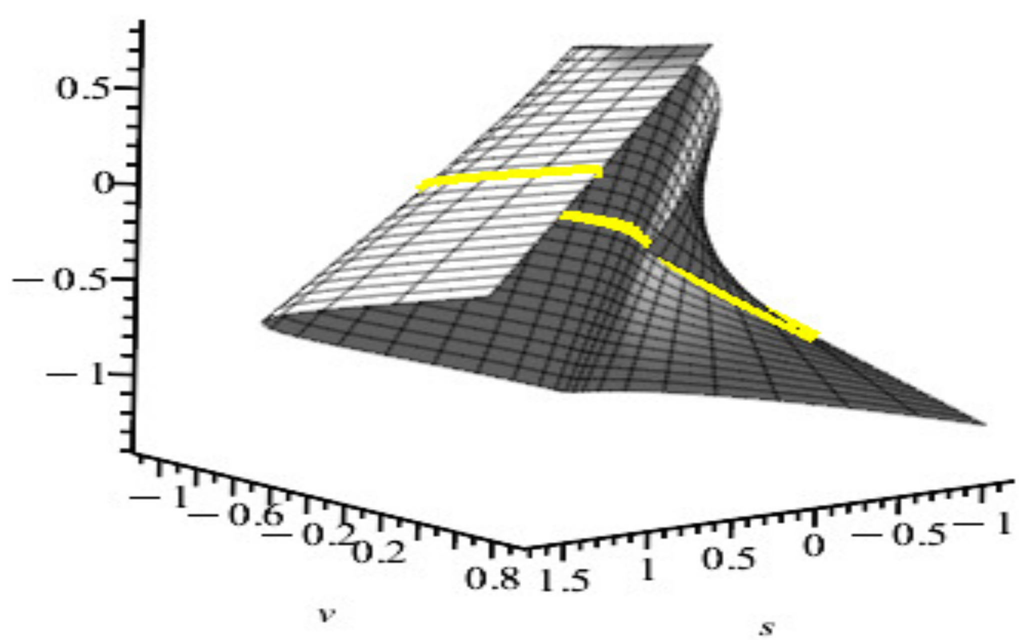

FIGURE 15. The ruled surface $\Phi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+vT_1$

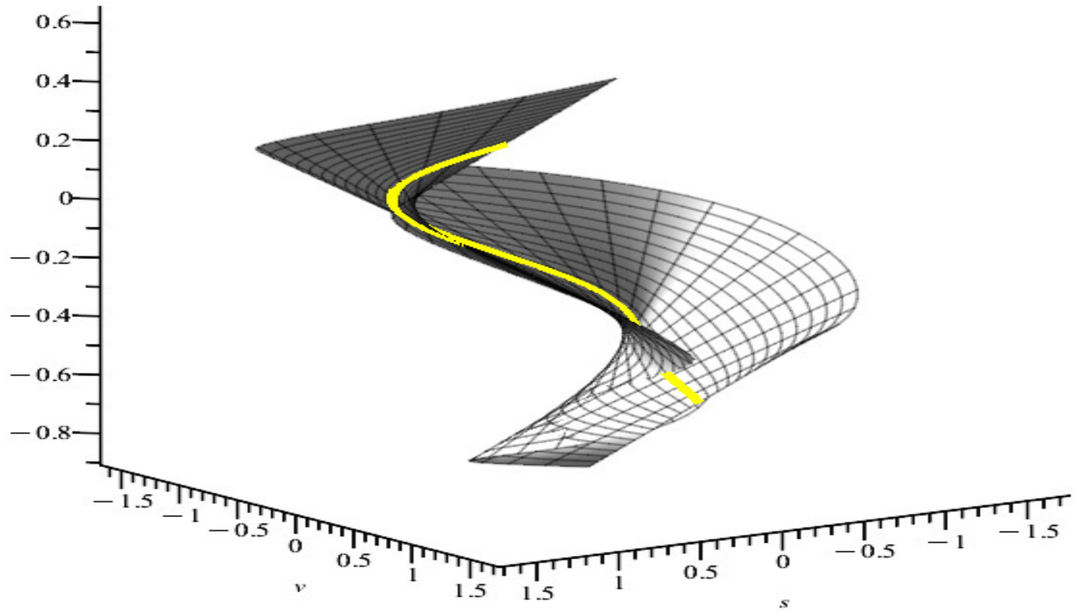

FIGURE 16. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

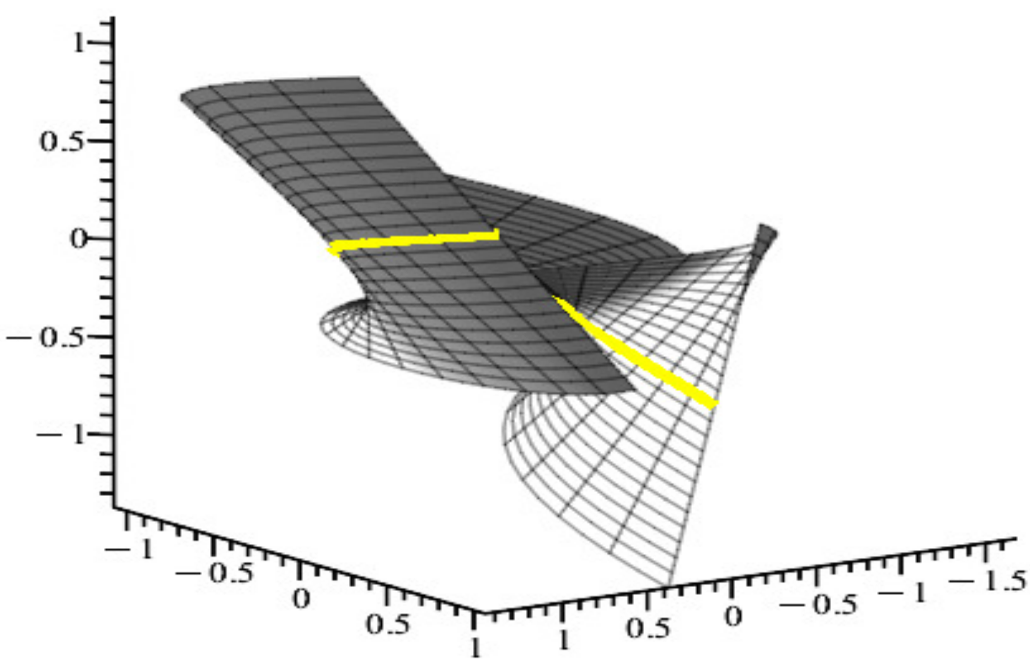

FIGURE 17. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

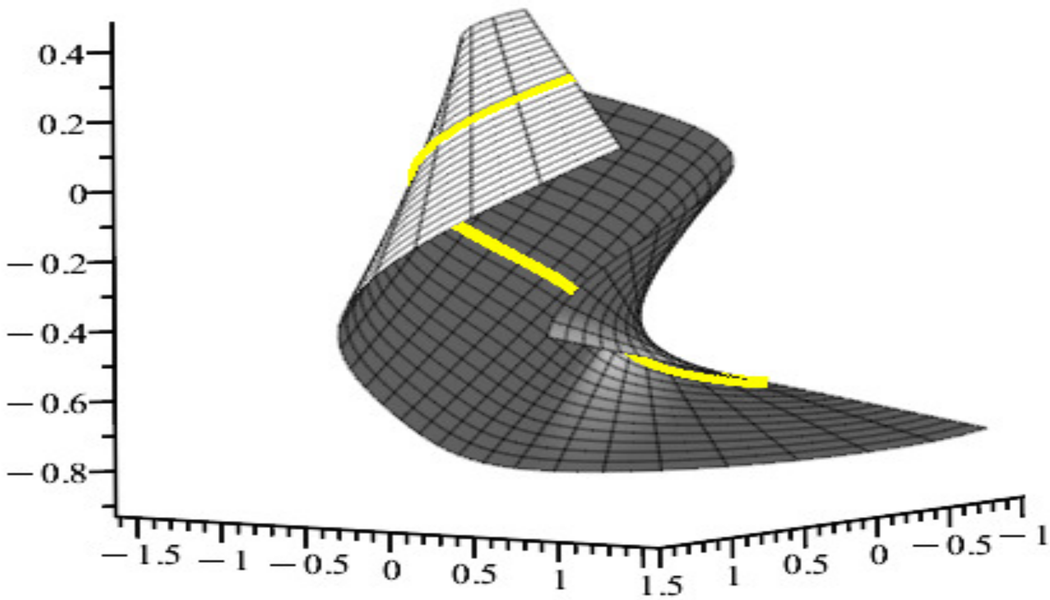

FIGURE 18. The ruled surface $\mu(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+N_1)$

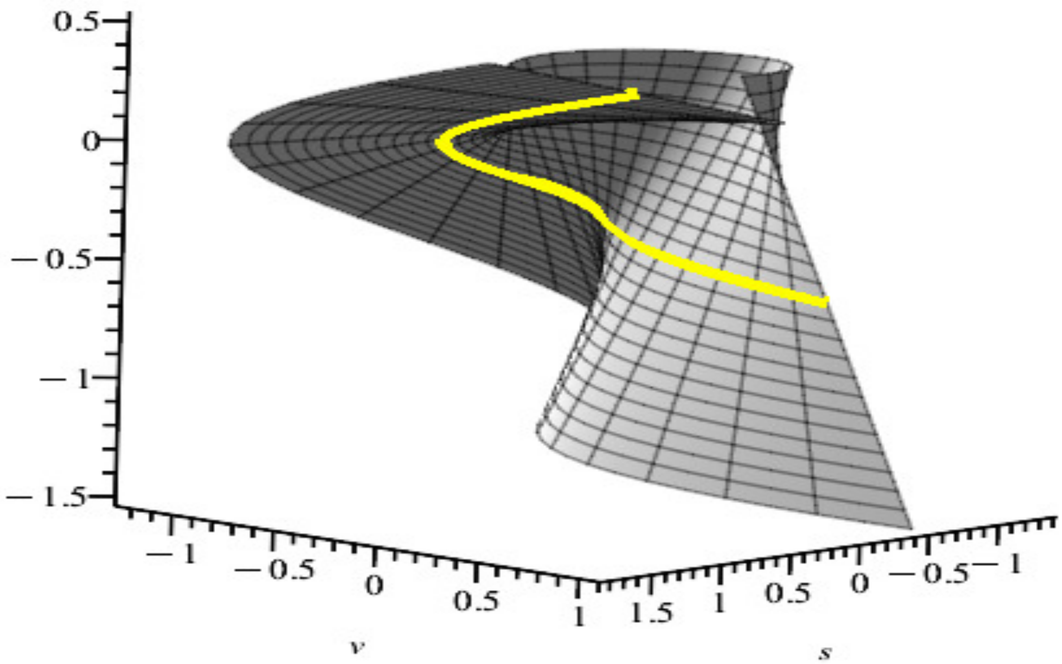

FIGURE 19. The ruled surface $\psi(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(T_1+B_1)$

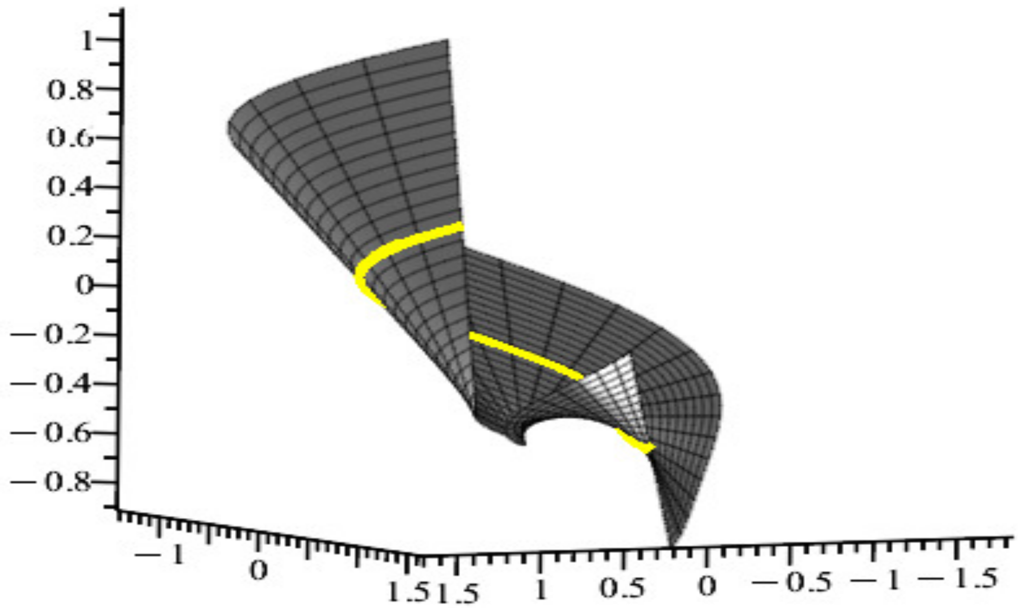

FIGURE 20. The ruled surface $\eta(s,v)\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

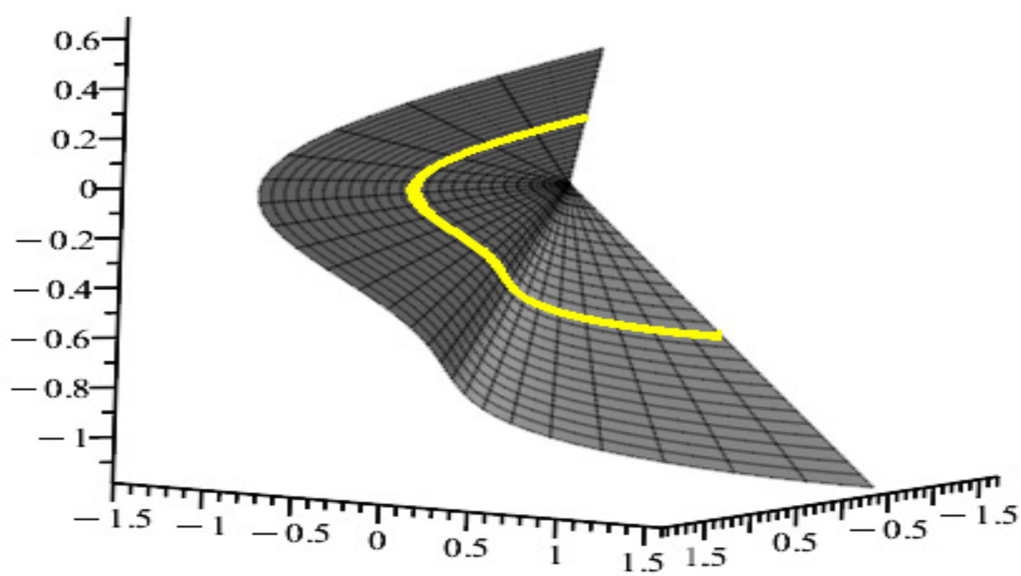

FIGURE 21. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

Example 3.4. *Let the Salkowski curve in Example 3.3 be the main curve. From* [15] *and Theorem 2.1 the Successor frames are as follows:*

$$T_1^*(s) = \begin{pmatrix} -\cos(s+c)\big(\frac{3}{\sqrt{10}}\sin s\big) + \sin(s+c)\big(-\cos s\cos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\big), \\ \cos(s+c)\big(\frac{3}{\sqrt{10}}\cos s\big) + \sin(s+c)\big(-\sin s\cos\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cos s\sin\frac{s}{\sqrt{10}}\big), \\ -\cos(s+c)\frac{1}{\sqrt{10}} + \sin(s+c)\big(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\big) \end{pmatrix},$$

$$N_1^*(s) = \begin{pmatrix} -\cos s\sin\frac{s}{\sqrt{10}} + \frac{1}{\sqrt{10}}\sin s\cos\frac{s}{\sqrt{10}}, -\sin s\sin\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\cos s\cos\frac{3}{\sqrt{10}}, \\ -\frac{3}{\sqrt{10}}\cos\frac{s}{\sqrt{10}} \end{pmatrix},$$

$$B_1^*(s) = \begin{pmatrix} \sin(s+c)\big(\frac{3}{\sqrt{10}}\sin s\big) + \cos(s+c)\big(-\cos s\cos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\big), \\ \sin(s+c)\big(\frac{3}{\sqrt{10}}\cos s\big) + \cos(s+c)\big(-\cos s\cos\frac{s}{\sqrt{10}} - \frac{1}{\sqrt{10}}\sin s\sin\frac{s}{\sqrt{10}}\big), \\ \sin(s+c)\frac{1}{\sqrt{10}} + \cos(s+c)\big(\frac{3}{\sqrt{10}}\sin\frac{s}{\sqrt{10}}\big) \end{pmatrix}.$$

The graphs of the ruled surfaces obtained from these frames for $s \in [-\pi, \pi]$ *and* $v \in [-1, 1]$ *are shown figures {22-28};*

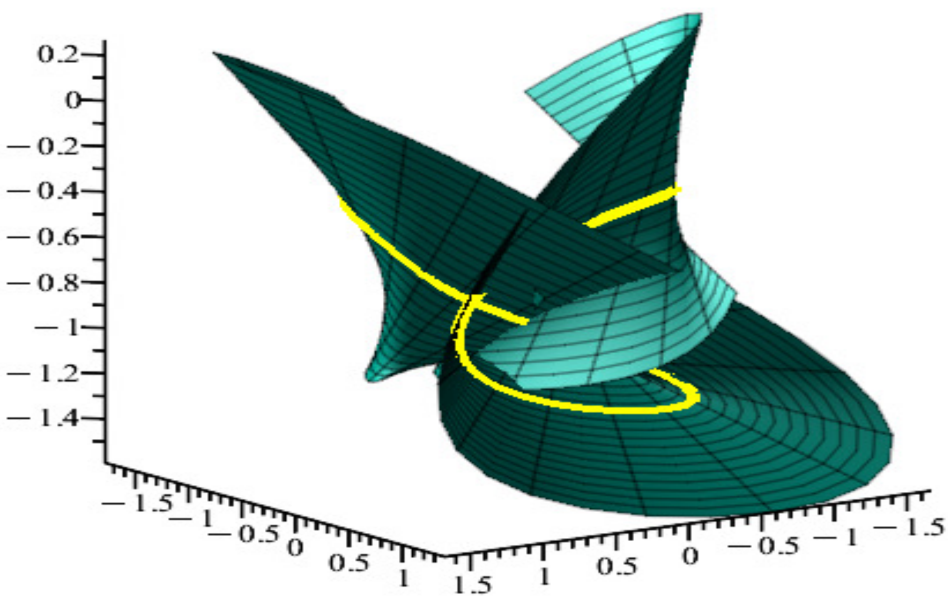

FIGURE 22. The ruled surface $\Phi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vT_1$

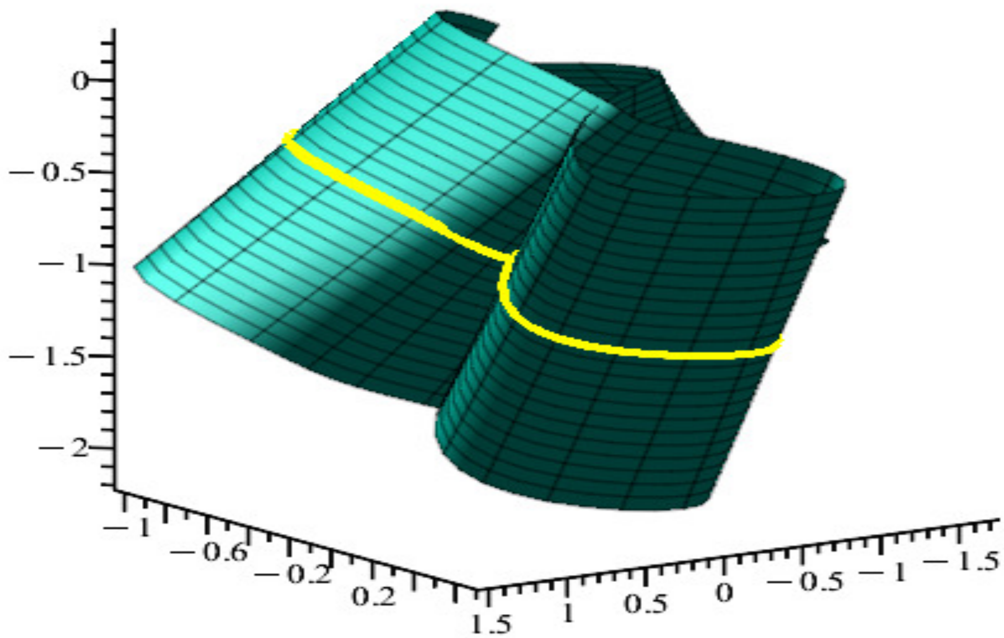

FIGURE 23. The ruled surface $Q(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vN_1$

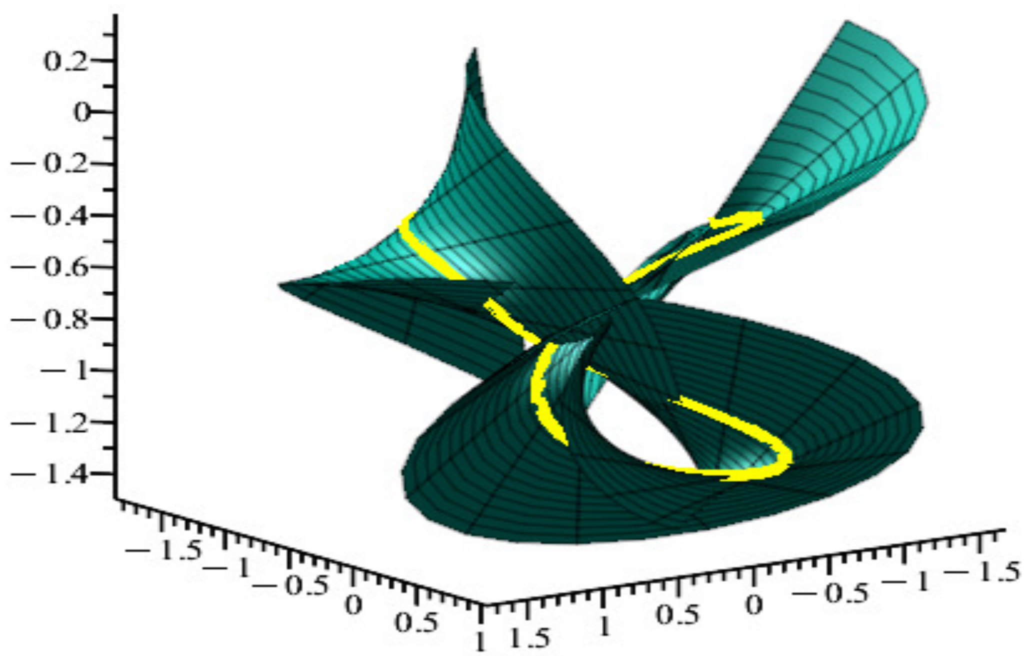

FIGURE 24. The ruled surface $M(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + vB_1$

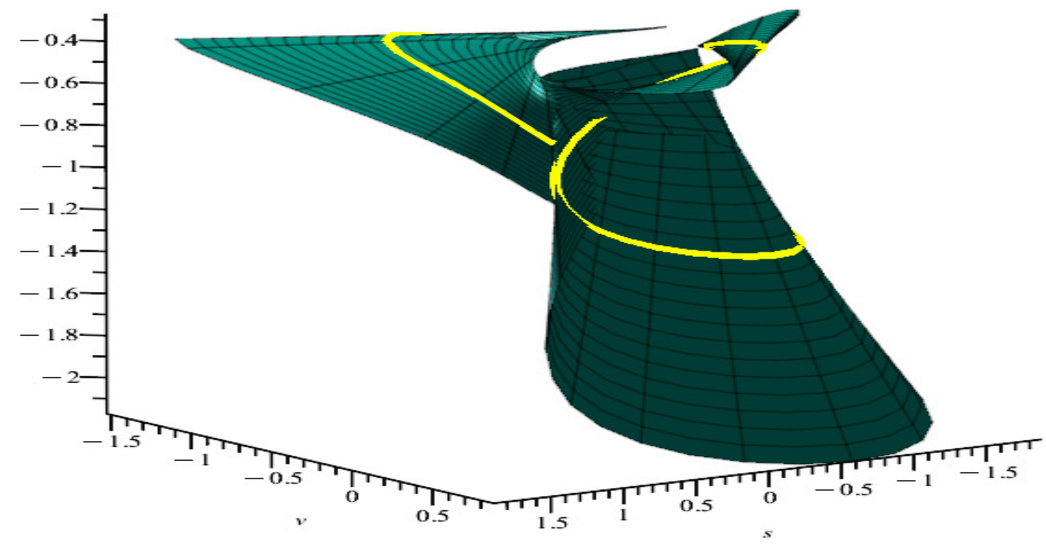

FIGURE 25. The ruled surface $\mu(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + N_1)$

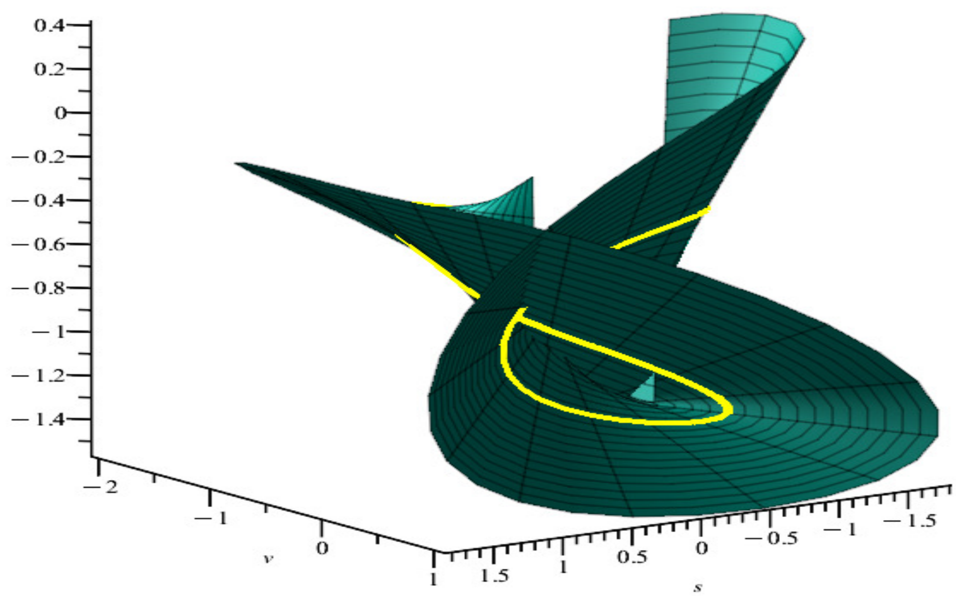

FIGURE 26. The ruled surface $\psi(s,v) = \frac{1}{\sqrt{3}}(T_1 + N_1 + B_1) + \frac{v}{\sqrt{2}}(T_1 + B_1)$

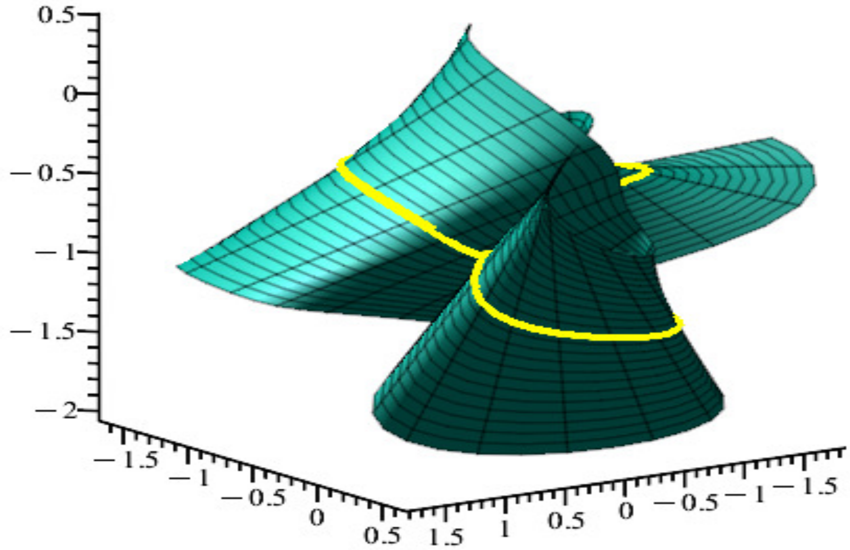

FIGURE 27. The ruled surface $\eta(s,v)\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{2}}(N_1+B_1)$

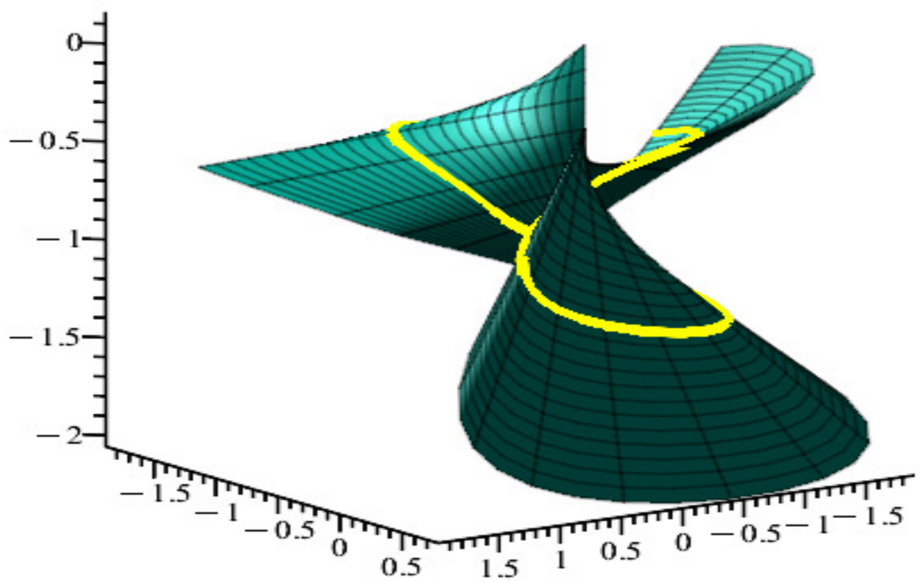

FIGURE 28. The ruled surface $\Gamma(s,v)=\frac{1}{\sqrt{3}}(T_1+N_1+B_1)+\frac{v}{\sqrt{3}}(T_1+N_1+B_1)$

4. Conclusion

This study defined ruled surfaces which one their base curve are $T_1N_1B_1$-Smarandache curve. There base curves targent vector, normal vector and binormal vector is successor curves Frenet aparatus. The Gaussian and mean curvatures of the surfaces were obatined using the coefficients of the first and the second fundamental forms. The conditions for the surfaces to be developable and minimal were given. These surfaces were drawn. This paper can be studied in Euclidean, Lorentz and dual space. New ruled surfaces can be defined and similar work can be done, by changing the base curve. Also, the singularity of surfaces can be examined.

References

[1] Akutagawa, K., & Nishikawa, S. (1990). The gauss map and space-like surfaces with prescribed mean curvature in minkowski 3-space. Tohoku Mathematical Journal Second Series, 42, 67-82.

[2] Ali, A. T. (2010). Special smarandache curves in the euclidean space. International Journal of Mathematical Combinatorics, 2, 30-36.

[3] Bilici, M., & Köseoğlu, G. (2023). Tubular Involutive Surfaces with Frenet Frame in Euclidean 3-space. Maejo International Journal of Science and Technology, 17(02), 96-106.

[4] Eren, K., & Şenyurt, S. (2022). On ruled surface with sannia frame in euclidean 3- space. Kyunpook Math. J., 62, 509-531.

[5] Erişir, T., & Öztaş, H. K. (2022). Spinor equations of successor curve. Universal of Mathematics and Applications, 5(1), 32-41.

[6] Grill, L., & Şenyurt, S., & Mazlum, S. G. (2020). Gaussian curvatures of parallel ruled surfaces. Applied Mathematical Sciences, 14, 171-183.

[7] Li, Y., & Çalışkan, A. (2023). Quaternionic Shape Operator and Rotation Matrix on Ruled Surfaces. Axioms, 2(5), 486.

[8] Li, Y., & Alkhaldi, A. H.,& Ali, A., & Abdel-Baky, R. A., & KhalifaSaad, M. (2023). Investigation of Ruled Rurfaces and Their Singularities According to Blaschke Frame in Euclidean 3-space. AIMS Mathematics, 8(6), 13875-13888.

[9] Li, Y., & Eren, K.,& Ali, A., & Ersoy, S. (2023). On Simultaneous Characterizations of Partner-Ruled Surfaces in Minkowski 3-space. AIMS Mathematics, 8(9), 22256–22273.

[10] Li, Y., & Şenyurt, S.,& Özduran, A., & Canlı, R. A., & D. (2022). The Characterizations of Parallel q-Equidistant Ruled Surfaces. Symmetry, 14(9), 1879.

[11] Masal, M. (2018). Curves according to the cuccessor frame in euclidean 3-space. Sakarya University Journal of Science, 6, 1868-1873.

[12] Massey, W. S. (1962). Surfaces of gaussian curvature zero in euclidean 3-space. Tohoku Mathematical Journal Second Series, 14, 73-79.

[13] Mazlum, S. G. & Grill, L. (2023). The invariants of dual parallel equidistant ruled surfaces. Symmetry, 15, 206.

[14] Menninger, A. (2014). Characterization of the slant helix as successor curves of the general helix. International Electronic Journal of Geometry, 2 , 84-91.

[15] Monterde, J. (2009). Salkowski curves revisited: a family of curves with constant curvature and non-constant torsion. Computer Aided Geometric Design, 93, 271-278.

[16] Ouarab, S. (2021). Smarandache ruled surfaces according to frenet-serret frame of a regular curve in E^3. Hindawi Abstract and Applied Analysis, 1-8.

[17] Ouarab, S. (2021). Smarandache ruled surfaces according to darboux Frame in E^3. Hindawi Journal of Mathematics, 10 pages.

[18] Ouarab, S. (2021). NC-Smarandache Ruled Surface and NW-Smarandache Ruled Surface According to Alternative Moving Frame in E^3. Hindawi Journal of Mathematics, 6 pages.

[19] Pottmann, H.,& Eigensatz, M., & Vaxman, A., & Wallner, J. (2015). Architectural geometry. Computers and Graphics, 47, 145-164.

[20] Pressley, A. (2010). Elementary Differential Geometry. Springer Science and Business Media.

[21] Şenyurt, S., & Sivas, S. (2013). An application of smarandache curve. Ordu Univ. J. Sci. Tech., 3, 46-60.

[22] Şenyurt, S., & Ayvacı, K. H.,& Canlı, D. (2021). Some characterizations of spherical indicatrix curves generated by sannia frame. Konuralp Journal of Mathematics, 9(2), 222-232.

[23] Şenyurt, S., & Canlı, D., & Can, E. (2022). Some special smarandache ruled surfaces by frenet frame in $E^3 - I$. Turkısh Journal of Science, 7(1), 31-42.

[24] Şenyurt, S., & Canlı, D., & Can, E., Mazlum, S. G. (2022). Some special smarandache ruled surfaces by frenet frame in $E^3 - II$. Honam Mathematical Journal, 44, 594-617.

[25] Şenyurt, S., &Uzun, G., & Akdağ, K. (2023). Ruled Surfaces Created by Successor Frames on the $\{T_1N_1\}$-Smarandache Curve Obtained from the Tangent and Normal Vectors of the Successor Curve. S. Küçükkolbaşı (Ed.), 2nd International E-Conference on Mathematical and Statistical Science: A Selcuk Meeting, Konya, pp.264.

[26] Stillwell, J. (2010). Mathematics and Its History. Undergraduate Texts in Mathematics, Third Edition Springer.

[27] Struik, D. J. (1961). Lectures on Classical Differential Geometry. Addison-Wesley Publishing Company.

[28] Taşköprü, K., & Tosun, M. (2014). Smarandache curves on S^2. Bol. Soc. Paran. Mat., 32(1) , 51–59.

[29] Turgut, N. & Yılmaz, S. (2008). Smarandache curves in minkowski space-time. International Journal of Mathematical Combinatorics, 3, 51-55.

[30] Uzun, G. (2023). Successor curves and equidistant ruled surfaces on the dual. Ph. D. Thesis, Ordu University, Ordu.

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