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A NOTE ON POINTWISE SEMI-SLANT CONFORMAL SUBMERSIONS

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Abstract. As a generalization of pointwise slant submersions, we investigate pointwise semi-slant conformal submersions from almost Hermitian manifolds onto Riemannian manifolds in the present work. With the investigation of the distributions' leaves geometry, we explore integrability conditions for distributions. In this study, we additionally explore the notion of pluriharminicty.

Keywords: Almost Hermitian manifolds, Riemannian submersion, Pointwise semi-slant conformal submersions, Conformal submersions.

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1. Introduction

The theory of submersions and immersions had originally been developed and proposed by B. O'Neill [27] and A. Gray [14]. They studied the geometrical properties of Riemannian manifolds and discovered certain Riemannian equations for them. When discussing the characteristics between differentiable structures in differential geometry, submersions theory becomes an intriguing subject. Mathematics and physics identically study Riemannian submersions because of their many applications, most prominent among them being Yang-Mills and Kaluza-Klein theories.(see [9], [42], [25], [21]). In 1976, B. Watson [41] glanced into Riemannian submersions from almost Hermitian manifolds onto Riemannian manifolds. Afterwards, B. Sahin [34] investigated the geometry of Riemannian submersions and geometric properties. He defined anti-invariant Riemannian submersions onto Riemannian manifolds

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by using an almost Hermitian manifold. He establishes that their vertical distribution is anti-invariant under the almost complex structure of the total manifold. Numerous writers examined and developed this work by examining anti-invariant submersions [4], [34], semi-invariant submersions [35], slant submersions [12], [36], and semi-slant submersions [18], [28], among other topics. Tastan, Sahin, and Yanan [40] defined and examined hemi-slant submersions from almost Hermitian manifolds as a generalization case of semi-invariant and semi-slant submersions.

In this contribution, T. W. Lee and B. Sahin [24] extended their concept of slant submersion a step further by expanding it to include pointwise slant submersions from almost Hermitian manifolds onto Riemannian manifolds. In doing so, they discovered a technique for illustrating examples for this kind of submersions. Additionally, they established characterizations for pointwise slant submersions. B. Fuglede [15] and T. Ishihara [22] introduced the concept of conformal submersion as a generalisation of Riemannian submersions and talked about some of their geometric characteristics. It is clear that conformal submersion with dilation $\lambda = 1$ is a Riemannian submersion. Gudmundsson and Wood [17] investigated conformal holomorphic submersion as a generalisation of holomorphic submersion. The necessary and sufficient conditions for harmonic morphisms of conformal holomorphic submersions have been established. Later on, conformal anti-invariant submersions , [37], [31], conformal semi-invariant submersions [5], conformal slant submersions [3], and conformal semi-slant submersions [2] have been studied and defined by Akyol and Sahin. Conformal hemi-slant submersions [38], [39], conformal bi-slant submersions [6], and quasi bi-slant conformal submersions [7] have all been studied geometrically recently, and several decomposition theorems have been covered. They also extended the notion of pluriharmonicity to almost contact metric manifolds, from almost Hermitian manifolds.

In this paper, we investigate pointwise semi-slant conformal submersions from Almost Hermtian manifold onto a Riemannian manifold. The structure of the paper is as follows. Section 2 introduces almost contact manifolds, precisely Kaehler manifold with the properties required for this study. In the third section of our paper, we define pointwise semi-slant conformal submersion and report a few intriguing findings. The prerequisites for distribution integrability and the totally geodesicness of its leaves were covered in detail in Section 4. Lastly, the notion of J-pluriharmonicity is discussed at the end of the study.

Note: In this paper, we will use abbreviation as follows:

Pointwise semi-slant conformal submersion- $\mathcal{PWSSCS}$

Almost Hermitian manifold- AHM

Kaehler manifold- KM

Riemannian manifold- RM

Horizontal conformal submersion -HCS

2. Preliminaries

We shall provide a few fundamental ideas and consequences that are highly productive for our paper.

Definition 2.1. [8] *Let* Π *be a Riemannian submersions between two Riemannian manifolds. Then* Π *is called a horizontally conformal submersion (HCS), if there is a positive function* λ *such that*

$$g_1(\hat{U}_1, \hat{V}_1) = \frac{1}{\lambda^2} g_2(\Pi_* \hat{U}_1, \Pi_* \hat{V}_1) \tag{2.1}$$

for any $\hat{U}_1, \hat{V}_1 \in \Gamma(\ker \Pi_*)^\perp$. *If the dilation function* $\lambda = 1$ *then, HCS become RS.*

Let $\Pi : (\Theta_1, g_1, J) \to (\Theta_2, g_2)$ be a Riemannian submersion. A vector field $\hat{X}$ on Θ_1 is called a basic vector field if $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$ and Π-related with a vector field $\hat{X}$ on Θ_2 i.e $\Pi_*(\hat{X}(q)) = \hat{X}\Pi(q)$ for $q \in \Theta_1$.

The formulae given by B . O'Neill of two $(1, 2)$ tensor fields $\mathscr{T}$ and $\mathfrak{A}$ are

$$\mathfrak{A}_{E_1} F_1 = \mathfrak{H} \nabla_{\mathfrak{H} E_1} \mathscr{V} F_1 + \mathscr{V} \nabla_{\mathfrak{H} E_1} \mathfrak{H} F_1, \tag{2.2}$$

$$\mathscr{T}_{E_1} F_1 = \mathfrak{H} \nabla_{\mathscr{V} E_1} \mathscr{V} F_1 + \mathscr{V} \nabla_{\mathscr{V} E_1} \mathfrak{H} F_1, \tag{2.3}$$

for any $E_1, F_1 \in \Gamma(T\Theta_1)$ and ∇ is Levi-Civita connection of g_1. From equations (2.2) and (2.3), we can deduce

$$\nabla_{\hat{U}_1} \hat{V}_1 = \mathscr{T}_{\hat{U}_1} \hat{V}_1 + \mathscr{V} \nabla_{\hat{U}_1} \hat{V}_1 \tag{2.4}$$

$$\nabla_{\hat{U}_1} \hat{X}_1 = \mathscr{T}_{\hat{U}_1} \hat{X}_1 + \mathfrak{H} \nabla_{\hat{U}_1} \hat{X}_1 \tag{2.5}$$

$$\nabla_{\hat{X}_1} \hat{U}_1 = \mathfrak{A}_{\hat{X}_1} \hat{U}_1 + \mathscr{V}_1 \nabla_{\hat{X}_1} \hat{U}_1 \tag{2.6}$$

$$\nabla_{\hat{X}_1} \hat{Y}_1 = \mathfrak{H} \nabla_{\hat{X}_1} \hat{Y}_1 + \mathfrak{A}_{\hat{X}_1} \hat{Y}_1 \tag{2.7}$$

for any vector fields $\hat{U}_1, \hat{V}_1 \in \Gamma(\ker \Pi_*)$ and $\hat{X}_1, \hat{Y}_1 \in \Gamma(\ker \Pi_*)^\perp$ [13].

It is obvious that $\mathscr{T}$ and $\mathfrak{A}$ are skew-symmetric, that is

$$g(\mathfrak{A}_{\hat{X}} E_1, F_1) = -g(E_1, \mathfrak{A}_{\hat{X}} F_1), \; g(\mathscr{T}_{\hat{V}} E_1, F_1) = -g(E_1, \mathscr{T}_{\hat{V}} F_1), \tag{2.8}$$

for any vector fields $E_1, F_1 \in \Gamma(T_p\Theta_1)$. Since $\mathscr{T}_{\hat{V}}$ is skew-symmetric, we say that Π has totally geodesic fibres if and only if $\mathscr{T} = 0$. For the special case when Π is HCS, we have

Proposition 2.1. *Let $\Pi : (\Theta_1, g_M) \to (\Theta_2, g_2)$ be a HCS with dilation λ and $\hat{X}, Y$ be the horizontal vectors, then*

$$A_{\hat{X}}\hat{Y} = \frac{1}{2}\{\mathscr{V}[\hat{X}, \hat{Y}] - \lambda^2 g(\hat{X}, \hat{Y}) grad_{\mathscr{V}}(\frac{1}{\lambda^2})\} \tag{2.9}$$

measures the obstruction integrability of the horizontal distribution

The second fundamental form of smooth map Π is provided by the formula

$$(\nabla\Pi_*)(\hat{U}_1, \hat{V}_1) = \nabla^{\Pi}_{\hat{U}_1}\Pi_*\hat{V}_1 - \Pi_*\nabla_{\hat{U}_1}\hat{V}_1, \tag{2.10}$$

if $(\nabla\Pi_*)(\hat{U}_1, \hat{V}_1) = 0$ for all $\hat{U}_1, \hat{V}_1 \in \Gamma(T_p\Theta_1)$, then Π is said to be a totally geodesic map where ∇ and ∇^{Π_*} are Levi-Civita and pullback connections.

Lemma 2.1. *Let $\Pi : \Theta_1 \to \Theta_2$ be a HCS. Then, we have*

(i) $(\nabla\Pi_*)(\hat{X}_1, \hat{Y}_1) = \hat{X}_1(ln\lambda)\Pi_*(\hat{Y}_1) + \hat{Y}_1(ln\lambda)\Pi_*(\hat{X}_1) - g_1(\hat{X}_1, \hat{Y}_1)\Pi_*(grad\ ln\lambda,$

(ii) $(\nabla\Pi_*)(\hat{U}_1, \hat{V}_1) = -\Pi_*(\mathscr{T}_{\hat{U}_1}\hat{V}_1)$

(iii) $(\nabla\Pi_*)(\hat{X}_1, \hat{U}_1) = -\Pi_*(\nabla_{\hat{X}_1}\hat{U}_1) = -\Pi_*(\mathfrak{A}_{\hat{X}_1}\hat{U}_1)$

for any horizontal vector fields $\hat{X}_1, \hat{Y}_1$ and vertical vector fields $\hat{U}_1, \hat{V}_1$ [8].

Let (Θ, g) be an AHM. This means that Θ admits a tensor field J of type $(1, 1)$ on Θ such that

$$J^2 = -I,\ g(J\hat{X}, J\hat{Y}) = g(\hat{X}, \hat{Y}) \text{ for all } \hat{X}, \hat{Y} \in \Gamma(T\Theta). \tag{2.11}$$

An AHM Θ is called KM if

$$(\nabla_{\hat{X}}J)\hat{Y} = 0,\ \text{ for all } \hat{X}, \hat{Y} \in \Gamma(T\Theta) \tag{2.12}$$

where ∇ is the Levi-Civita connection on Θ. The covariant derivative of J is defined by

$$(\nabla_{\hat{X}}J)\hat{Y} = \nabla_{\hat{X}}J\hat{Y} - J\nabla_{\hat{X}}\hat{Y} \tag{2.13}$$

for all vector fields $\hat{X}, \hat{Y}$ in Θ.

Here, we recall the definitions which will be helpful for our text.

Definition 2.2. *Let Π be a Riemannian submersion from AHM $(\bar{\Theta_1}, g_1, J)$ onto RM $(\bar{\Theta_2}, g_2)$. If for any non-zero vector $\hat{X} \in \Gamma(\ker\Pi_*)$, the angle $\theta(\hat{X})$ between $J\hat{X}$ and the space $\ker\Pi_*$ is constant,i.e., it is independent of the choice of point $p \in \bar{\Theta_1}$, and choice of tangent vector*

$\hat{X}$ in $\ker \Pi_*$, then we said Π is slant submersion. In this case, the angle θ is called the slant angle of submersion.

Now, we recall the definition of pointwise slant submersion defined by T.W. Lee and B. Sahin [24]

Definition 2.3. *Let Π be a Riemannian submersion from AHM $(\bar{\Theta_1}, g_1, J)$ onto RM $(\bar{\Theta_2}, g_2)$. If at each given point $q \in \bar{\Theta_2}$, the wirtinger angle $\theta(\hat{X})$ between $J\hat{X}$ and the space $\ker \Pi_*$ is independent of choice of the non-zero vector $\hat{X} \in \Gamma(\ker \Pi_*)$, then we say that Π is a pointwise slant submersion. In this case, the angle θ can be regarded as a function on $\bar{\Theta_1}$, which is called slant function of the pointwise slant submersion.*

3. Pointwise semi-slant conformal submersions ($\mathcal{PWSSCS}$)

In this section, we will review the definition that will aid us in discussing and investigating the concept of pointwise semi-slant conformal submersions $\mathcal{PWSSCS}$ from almost Hermitian manifolds.

Definition 3.1. *Let $\Pi : (\bar{\Theta_1}, g_1, J) \to (\bar{\Theta_2}, g_2)$ be a HCS where $(\bar{\Theta_1}, g_1, J)$ is a AHM and $(\bar{\Theta_2}, g_2)$ is a RM. A HCS Π is called a $\mathcal{PWSSCS}$ if there exists a distribution $\mathfrak{D}$ such that $\ker \Pi_* = \mathfrak{D} \oplus \mathfrak{D}^\theta$, $J(\mathfrak{D}) = \mathfrak{D}$ and for any given point $q \in \bar{\Theta_1}$ and $\hat{X} \in (\mathfrak{D}^\theta)_q$, the angle $\theta = \theta(\hat{X})$ between $J\hat{X}$ and space $(\mathfrak{D}^\theta)_q$ is independent of choice of non-zero vector $\hat{X} \in (\mathfrak{D}^\theta)_q$, where $\mathfrak{D}^\theta$ is the orthogonal complement of $\mathfrak{D}$ in $\ker \Pi_*$. In this case, the angle θ can be regarded as a slant function and called pointwise semi-slant function of submersion.*

If we suppose m_1 and m_2 are the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, then we have the following:

(i) If $m_1 = 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then Π is a pointwise slant submersion.

(i) If $m_1 \neq 0$ and $m_2 = 0$, then Π is a invariant submersion

(ii) If $m_1 \neq 0$, $m_2 \neq 0$ and $0 < \theta < \frac{\pi}{2}$, then Π is a pointwise semi-slant submersion.

We are providing the example of $\mathcal{PWSSCS}$ for support of our study.

Let Π be a $\mathcal{PWSSCS}$ from an AHM $(\bar{\Theta_1}, g_1, J)$ onto a RM $(\bar{\Theta_2}, g_2)$. Then, for any $\hat{W} \in (\ker \Pi_*)$, we have

$$\hat{W} = \mathbb{P}\hat{W} + \mathbb{Q}\hat{W} \tag{3.14}$$

where $\mathbb{P}$ and $\mathbb{Q}$ are the projections morphism onto $\mathfrak{D}$ and $\mathfrak{D}^\theta$. Now, for any $\hat{W} \in (\ker \Pi_*)$, we have

$$J\hat{W} = \psi\hat{W} + \zeta\hat{W} \tag{3.15}$$

where $\psi\hat{W} \in \Gamma(\ker \Pi_*)$ and $\zeta\hat{W} \in \Gamma(\ker \Pi_*)^\perp$. From equations (3.14) and (3.15), we have

$$\begin{aligned} J\hat{U} =& J(\mathbb{P}\hat{W}) + J(\mathbb{Q}\hat{W}) \\ =& \psi(\mathbb{P}\hat{W}) + \zeta(\mathbb{P}\hat{W}) + \psi(\mathbb{Q}\hat{W}) + \zeta(\mathbb{Q}\hat{W}). \end{aligned}$$

Since $J\mathfrak{D} = \mathfrak{D}$ and $\zeta(\mathbb{P}\hat{W}) = 0$, we have

$$J\hat{U} = \psi(\mathbb{P}\hat{W}) + \psi(\mathbb{Q}\hat{W}) + \zeta(\mathbb{Q}\hat{W}).$$

Now, we have the following decomposition

$$(\ker \Pi_*)^\perp = \zeta\mathfrak{D}^\theta \oplus \mu, \tag{3.16}$$

where μ is the orthogonal complement to $\zeta\mathfrak{D}^\theta$ in $(\ker \Pi_*)^\perp$ such that μ is invariant with respect to J. Now, for any $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$, we have

$$J\hat{X} = \mathfrak{B}\hat{X} + \mathfrak{C}\hat{X} \tag{3.17}$$

where $\mathfrak{B}\hat{X} \in \Gamma(\ker \Pi_*)$ and $\mathfrak{C}\hat{X} \in \Gamma(\ker \Pi_*)^\perp$.

Lemma 3.1. *Let $(\bar{\Theta_1}, g_1, J)$ be an KM and $(\bar{\Theta_2}, g_2)$ be a RM. If $\Pi : \bar{\Theta_1} \to \bar{\Theta_2}$ is a $\mathcal{PWSSCS}$, then we have*

$$-\hat{W} = \psi^2\hat{W} + \mathbb{P}\zeta\hat{W},\ \zeta\psi\hat{W} + \mathfrak{C}\zeta\hat{W} = 0,\ -\hat{Y} = \zeta\mathfrak{B}\hat{Y} + \mathfrak{C}^2\hat{Y},\ \psi\mathfrak{B}\hat{Y} + \mathfrak{B}\mathfrak{C}\hat{Y},$$

for any vector field $\hat{W} \in \Gamma(\ker \Pi_*)$ and $\hat{Y} \in \Gamma(\ker \Pi_*)^\perp$.

Proof. On considering the equations (2.11), (3.15) and (3.17), the proof of Lemma exists. □

Since $\Pi : \bar{\Theta_1} \to \bar{\Theta_2}$ is a $\mathcal{PWSSCS}$, let us present some helpful investigations that will be applied in this paper.

Lemma 3.2. *Let Π be a $\mathcal{PWSSCS}$ from an AHM $(\bar{\Theta_1}, g_1, J)$ onto a RM $(\bar{\Theta_2}, g_2)$, then we have*

$$\psi^2\hat{W} = (-\cos^2\theta)\hat{W}, \tag{3.18}$$

for any vector fields $\hat{W} \in \Gamma(\mathfrak{D}^\theta)$.

Lemma 3.3. *Let Π be a $\mathcal{PWSSCS}$ from an AHM $(\bar{\Theta_1}, g_1, J)$ onto a RM $(\bar{\Theta_2}, g_2)$, then we have*

(i) $g_1(\psi\hat{Z}, \psi\hat{W}) = \cos^2\theta\, g_1(\hat{Z}, \hat{W})$,

(ii) $g_1(\zeta\hat{Z}, \zeta\hat{W}) = \text{operatornamesin}^2\theta\, g_1(\hat{Z}, \hat{W})$,

for any vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The proof of the preceding Lemmas is identical to the proof of Theorem (2.2) of [11]. As a result, we omit the proofs. □

Let us suppose that $(\bar{\Theta}_2, g_2)$ be a RM and (Θ_1, g_1, J) be an AHM. We now analyse how the Hermitian structure on Θ_1 influences the tensor fields $\mathscr{T}$ and $\mathfrak{A}$ of $\mathcal{PWSSCS}$ $\Pi : (\Theta_1, g_1, J) \to (\Theta_2, g_2)$.

Lemma 3.4. *Let $\Pi : \bar{\Theta}_1 \to \bar{\Theta}_2$ be $\mathcal{PWSSCS}$ with semi-slant function θ where, $(\bar{\Theta}_1, g_1, J)$ KM and $(\bar{\Theta}_2, g_2)$ be a RM, then we have*

$$\mathfrak{A}_{\hat{X}}\mathfrak{C}\hat{Y} + \mathscr{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y} = \mathfrak{B}\mathfrak{H}\nabla_{\hat{X}}\hat{Y} + \psi\mathfrak{A}_{\hat{X}}\hat{Y} \tag{3.19}$$

$$\mathfrak{H}\nabla_{\hat{X}}\mathfrak{C}\hat{Y} + \mathfrak{A}_{\hat{X}}\mathfrak{B}\hat{Y} = \mathfrak{C}\mathfrak{H}\nabla_{\hat{X}}\hat{Y} + \zeta\mathfrak{A}_{\hat{X}}\hat{Y} \tag{3.20}$$

$$\mathscr{V}\nabla_{\hat{X}}\psi\hat{V} + \mathfrak{A}_{\hat{X}}\zeta\hat{V} = \mathfrak{B}\mathfrak{A}_{\hat{X}}\hat{V} + \psi\mathscr{V}\nabla_{\hat{X}}\hat{V} \tag{3.21}$$

$$\mathfrak{A}_{\hat{X}}\psi\hat{V} + \mathfrak{H}\nabla_{\hat{X}}\zeta\hat{V} = \mathfrak{C}\mathfrak{A}_{\hat{X}}\hat{V} + \zeta\mathscr{V}\nabla_{\hat{X}}\hat{V} \tag{3.22}$$

$$\mathscr{V}\nabla_{\hat{V}}\mathfrak{B}\hat{X} + \mathscr{T}_{\hat{V}}\mathfrak{C}\hat{X} = \psi\mathscr{T}_{\hat{V}}\hat{X} + \mathfrak{B}\mathfrak{H}\nabla_{\hat{V}}\hat{X} \tag{3.23}$$

$$\mathscr{T}_{\hat{V}}\mathfrak{B}\hat{X} + \mathfrak{H}\nabla_{\hat{V}}\mathfrak{C}\hat{X} = \zeta\mathscr{T}_{\hat{V}}\hat{X} + \mathfrak{C}\mathfrak{H}\nabla_{\hat{V}}\hat{X} \tag{3.24}$$

$$\mathscr{V}\nabla_{\hat{U}}\psi\hat{V} + \mathscr{T}_{\hat{U}}\zeta\hat{V} - \psi\mathscr{V}\nabla_{\hat{U}}\hat{V} = \mathfrak{B}\mathscr{T}_{\hat{U}}\hat{V} \tag{3.25}$$

$$\mathscr{T}_{\hat{U}}\psi\hat{V} + \mathfrak{H}\nabla_{\hat{U}}\zeta\hat{V} = \mathfrak{C}\mathscr{T}_{\hat{U}}\hat{V} + \zeta\mathscr{V}\nabla_{\hat{U}}\hat{V}, \tag{3.26}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(\ker \Pi_*)$ and $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^{\perp}$.

Proof. By using (2.12), (2.13) and (2.7) (3.17), we get first two relations (3.19) and (3.20). Similarly, by considering equations (2.12), (2.13) (2.7), (2.4)-(2.7) and (3.15) (3.17), the desired results holds good. □

We will now go through some key conclusions that can be utilised to examine the geometry of $\mathcal{PWSSCS}$ $\Pi : \Theta_1 \to \Theta_2$. From the direct calculations, we can conclude the following:

$$(\nabla_{\hat{U}}\psi)\hat{V} = \mathscr{V}\nabla_{\hat{U}}\psi\hat{V} - \psi\mathscr{V}\nabla_{\hat{U}}\hat{V} \tag{3.27}$$

$$(\nabla_{\hat{U}}\zeta)\hat{V} = \mathfrak{H}\nabla_{\hat{U}}\zeta\hat{V} - \zeta\mathscr{V}\nabla_{\hat{U}}\hat{V} \tag{3.28}$$

$$(\nabla_{\hat{X}}\mathfrak{B})\hat{Y} = \mathscr{V}\nabla_{\hat{X}}\mathfrak{B}\hat{Y} - \mathfrak{B}\mathfrak{H}\nabla_{\hat{X}}\hat{Y} \tag{3.29}$$

$$(\nabla_{\hat{X}}\mathfrak{C})\hat{Y} = \mathfrak{H}\nabla_{\hat{X}}\mathfrak{C}\hat{Y} - \mathfrak{H}\nabla_{\hat{X}}\hat{Y}, \tag{3.30}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(\ker \Pi_*)$ and $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^{\perp}$.

Lemma 3.5. *Let (Θ_1, g_1, J) be a KM and (Θ_2, g_2) be a RM. If $\Pi : \Theta_1 \to \Theta_2$ is a $\mathcal{PWSSCS}$ with semi-slant function θ, then we have*

$$(\nabla_{\hat{U}}\psi)\hat{V} = \mathfrak{B}\mathscr{T}_{\hat{U}}\hat{V} - \mathscr{T}_{\hat{U}}\zeta\hat{V}$$

$$(\nabla_{\hat{U}}\zeta)\hat{V} = \mathfrak{C}\mathscr{T}_{\hat{U}}\hat{V} - \mathscr{T}_{\hat{U}}\psi\hat{V}$$

$$(\nabla_{\hat{X}}\mathfrak{B})\hat{Y} = \psi\mathfrak{A}_{\hat{X}}\hat{Y} - \mathfrak{A}_{\hat{X}}\mathfrak{C}\hat{Y}$$

$$(\nabla_{\hat{X}}\mathfrak{C})\hat{Y} = \zeta\mathfrak{A}_{\hat{X}}\hat{Y} - \mathfrak{A}_{\hat{X}}\mathfrak{B}\hat{Y},$$

for all vector fields $\hat{U}, \hat{V} \in \Gamma(\ker \Pi_*)$ and $\hat{X}, \hat{Y} \in \Gamma(\ker \Pi_*)^{\perp}$.

Proof. By using equations (2.13), (2.4)- (2.7) and equations (3.27)-(3.30), we can obtain the results. □

The tensor fields ψ and ζ, if they are parallel with regard to the Levi- Civita connection ∇ of Θ_1, then we obtain

$$\mathfrak{B}\mathscr{T}_{\hat{U}}\hat{V} = \mathscr{T}_{\hat{U}}\zeta\hat{V},\ \mathfrak{C}\mathscr{T}_{\hat{U}}\hat{V} = \mathscr{T}_{\hat{U}}\psi\hat{V}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(T\Theta_1)$.

4. CONDITIONS FOR INTEGRABILITY AND TOTALLY GEODESICNESS

In this section, we discuss the geometry of $\mathcal{PWSSCS}$ $\Pi : (\Theta_1, g_1, J) \to (\Theta_2, g_2)$ from KM onto RM in terms of integrability of invariant and slant distribution. Apart from this, we also examine the necessary and sufficient conditions for the leaves of distribution to be define totally geodesic foliation on Θ_1. We start the condition for integrability for invariant distribution as follows :

Theorem 4.1. *Let $\Pi : \bar{\Theta_1} \to \bar{\Theta_2}$ be $\mathcal{PWSSCS}$ with semi-slant function θ where, $(\bar{\Theta_1}, g_1, J)$ is a KM and $(\bar{\Theta_2}, g_2)$ be a RM. Then the invariant distribution $\mathfrak{D}$ is integrable if and only if*

$$\mathscr{V}\nabla_{\hat{U}}\psi\hat{Z} + \mathscr{T}_{\hat{U}}\zeta\hat{Z} \in \Gamma(\mathfrak{D}^{\theta}) \text{ and } \mathscr{V}\nabla_{\hat{V}}\psi\hat{Z} + \mathscr{T}_{\hat{V}}\zeta\hat{Z} \in \Gamma(\mathfrak{D}^{\theta}), \tag{4.31}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{Z} \in \Gamma(\mathfrak{D}^{\theta})$.

Proof. For all vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{Z} \in \Gamma(\mathfrak{D}^{\theta})$ and by using equations (2.11), (2.12) and (2.13), we have

$$\begin{aligned} g_1([\hat{U}, \hat{V}], \hat{Z}) &= g_1(\nabla_{\hat{U}}J\hat{V}, J\hat{Z}) - g_1(\nabla_{\hat{V}}J\hat{U}, J\hat{Z}) \\ &= -g_1(\nabla_{\hat{U}}J\hat{Z}, J\hat{V}) + g_1(\nabla_{\hat{V}}J\hat{Z}, J\hat{U}). \end{aligned}$$

Taking account the fact from equations (2.4) and (2.5) in both part of the above equation in right hand side, takes the form

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\nabla_{\hat{U}}\psi\hat{Z},\psi\hat{V}) - g_1(\nabla_{\hat{U}}\zeta\hat{Z},\psi\hat{V}).$$

By using equation (3.15) in above relation, we have

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{Z}) = -g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{Z},\psi\hat{V}) - g_1(\mathscr{T}_{\hat{U}}\zeta\hat{Z},\psi\hat{V}).$$

In above equation, change the role of $\hat{U}$ and $\hat{V}$, we may yield

$$g_1([\hat{U},\hat{V}],\hat{Z}) = -g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{Z} + \mathscr{T}_{\hat{U}}\zeta\hat{Z},\psi\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{V}}\psi\hat{Z} + \mathscr{T}_{\hat{V}}\zeta\hat{Z},\psi\hat{U}).$$

□

Theorem 4.2. *Let* Π *be* $\mathcal{PWSSCS}$ *with semi-slant function* θ *from KM* (Θ_1, g_1, J) *onto a RM* (Θ_2, g_2)*. Then* $\mathfrak{D}^\theta$ *is integrable if and only if*

$$\psi(\mathscr{T}_{\hat{Z}}\zeta\hat{W} - \mathscr{T}_{\hat{W}}\zeta\hat{Z}) = (\mathscr{T}_{\hat{W}}\zeta\psi\hat{Z} - \mathscr{T}_{\hat{Z}}\zeta\psi\hat{W}), \tag{4.32}$$

for any vector fields $\hat{Z},\hat{W} \in \Gamma(\mathfrak{D}^\theta)$ *and* $\hat{U} \in \Gamma(\mathfrak{D})$*.*

Proof. By using equation (2.11), (2.12) and (2.13), we may yield

$$g_1([\hat{Z},\hat{W}],\hat{U}) = g_1(\nabla_{\hat{Z}}J\hat{W},J\hat{U}) - g_1(\nabla_{\hat{W}}J\hat{Z},J\hat{U}),$$

for every vector fields $\hat{Z},\hat{W} \in \Gamma(\mathfrak{D}^\theta)$ and $\hat{U} \in \Gamma(\mathfrak{D})$. By using equation (3.15), we can write

$$g_1([\hat{Z},\hat{W}],\hat{U}) = -g_1(\nabla_{\hat{Z}}\psi\hat{W},J\hat{U}) - g_1(\nabla_{\hat{W}}\psi\hat{Z},J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\hat{W},J\hat{U}) - g_1(\nabla_{\hat{W}}\zeta\hat{Z},J\hat{U}).$$

Now, considering the equation (2.11) and equation (2.5) in third and fourth terms, above equation takes the form

$$g_1([\hat{Z},\hat{W}],\hat{U}) = g_1(\nabla_{\hat{Z}}J\psi\hat{W},\hat{U}) + g_1(\nabla_{\hat{W}}J\psi\hat{Z},\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\hat{W},J\hat{U}) - g_1(\mathscr{T}_{\hat{W}}\zeta\hat{Z},J\hat{U}). \tag{4.33}$$

Taking account the fact from (3.15) in first term, we get $g_1(\nabla_{\hat{Z}}J\psi\hat{W},\hat{U}) = -g_1(\nabla_{\hat{Z}}\psi^2\hat{W},\hat{U}) - g_1(\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{U})$. By using Lemma 3.2, $g_1(\nabla_{\hat{Z}}J\psi\hat{W},\hat{U}) = \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{W},\hat{U}) - g_1(\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{U})$. The same calculation in second term, we get $-g_1(\nabla_{\hat{W}}J\psi\hat{Z},\hat{U}) = -\cos^2\theta g_1(\nabla_{\hat{W}}\hat{Z},\hat{U}) + g_1(\nabla_{\hat{W}}\zeta\psi\hat{Z},\hat{U})$. On combining these calculations, finally equation (4.33) takes the form

$$g_1([\hat{Z},\hat{W}],\hat{U}) = \cos^2\theta g_1([\hat{Z},\hat{W}],\hat{U}) - g_1(\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{U}) + g_1(\nabla_{\hat{W}}\zeta\psi\hat{Z},\hat{U})$$
$$+ g_1(\mathscr{T}_{\hat{Z}}\zeta\hat{W},J\hat{U}) - g_1(\mathscr{T}_{\hat{W}}\zeta\hat{Z},J\hat{U}).$$

Finally, by using equation (2.5), we can write

$$\sin^2\theta g_1([\hat{Z},\hat{W}],\hat{U}) = g_1(\mathscr{T}_{\hat{W}}\zeta\psi\hat{Z},\hat{U}) - g_1(\mathscr{T}_{\hat{Z}}\zeta\psi\hat{W},\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\hat{W} - \mathscr{T}_{\hat{W}}\zeta\hat{Z}, J\hat{U}).$$

From which, we can conclude the result. □

Since $\Pi : (\Theta_1, g_1, J) \rightarrow (\Theta_2, g_2)$ be a $\mathcal{PWSSCS}$ which ensure the availability of slant distributions. After discussing the integrability conditions of distributions, we are going to examine the necessary and sufficient condition for which the leaves of distributions defined totally geodesic foliation on Θ_1.

Theorem 4.3. *Let Π be $\mathcal{PWSSCS}$ with semi-slant function θ from KM $(\bar{\Theta_1}, g_1, J)$ onto a RM $(\bar{\Theta_2}, g_2)$. Then $\mathfrak{D}$ is defines totally geodesic foliation on $\bar{\Theta_1}$ if and only if*

$$\mathscr{T}_{\hat{U}}\zeta\psi\hat{Z} = -\psi(\mathscr{T}_{\hat{U}}\zeta\hat{Z}) \text{ and } g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{V}, \mathfrak{B}\hat{X}) + g_1(\mathscr{T}_{\hat{U}}\psi\hat{V}, \mathfrak{C}\hat{X}) = 0, \tag{4.34}$$

for any vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D}), \hat{Z} \in \Gamma(\mathfrak{D}^\theta)$ and $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$.

Proof. By considering $g_1(\nabla_{\hat{U}}\hat{V}, \hat{Z})$, for any vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{Z} \in \Gamma(\mathfrak{D}^\theta)$. Since, $\hat{V}$ and $\hat{Z}$ are orthogonal to each other, this can be write as $g_1(\nabla_{\hat{U}}\hat{V}, \hat{Z}) = -g_1(\nabla_{\hat{U}}\hat{Z}, \hat{V})$. Operating almost complex structure J on both side and using equations (2.11), (2.12), (2.13) and (3.15), we have

$$g_1(\nabla_{\hat{U}}\hat{V}, \hat{Z}) = -g_1(\nabla_{\hat{U}}\psi\hat{Z}, JV) - g_1(\nabla_{\hat{U}}\zeta\hat{Z}, JV).$$

Further, in the light of equations (3.15) and (2.5), we get

$$g_1(\nabla_{\hat{U}}\hat{V}, \hat{Z}) = -g_1(\nabla_{\hat{U}}\psi^2\hat{Z}, \hat{V}) + g_1(\nabla_{\hat{U}}\zeta\psi\hat{Z}, \hat{V}) - g_1(\mathscr{T}_{\hat{U}}\zeta\hat{Z}, JV).$$

Since, Π is a $\mathcal{PWSSCS}$ with semi-slant function θ, then by using Lemma 3.2 in first term of above equation, finally this will takes the form

$$\sin^2\theta g_1(\nabla_{\hat{U}}\hat{V}, \hat{Z}) = g_1(\nabla_{\hat{U}}\zeta\psi\hat{Z}, \hat{V}) - g_1(\mathscr{T}_{\hat{U}}\zeta\hat{Z}, JV).$$

From this we can get the first part of theorem. For next one, we consider $g_1(\nabla_{\hat{U}}\hat{V}, \hat{X})$ for any vector fields $\hat{U}, \hat{V} \in \Gamma(\mathfrak{D})$ and $\hat{X} \in \Gamma(\ker \Pi_*)^\perp$. By using equation (2.11), (2.12), (2.13) and (3.17), (3.15), this term will takes the form as $g_1(\nabla_{\hat{U}}\hat{V}, \hat{X}) = g_1(\nabla_{\hat{U}}\psi\hat{V}, \mathfrak{B}\hat{X} + \mathfrak{C}\hat{X})$. Finally, considering equation (2.4), we can write

$$g_1(\nabla_{\hat{U}}\hat{V}, \hat{X}) = g_1(\mathscr{V}\nabla_{\hat{U}}\psi\hat{V}, \mathfrak{B}\hat{X}) + g_1(\mathscr{T}_{\hat{U}}\psi\hat{V}, \mathfrak{C}\hat{X}).$$

From which the second part of theorem holds good. □

Since, Π is $\mathcal{PWSSCS}$ with semi-slant function θ from (Θ_1, g_1, J) onto (Θ_2, g_2). The slant distribution is mutually othogonal to invariant distribution. After discussion geometry of leaves of invariant distribution, it is quite interesting to study the leaves of slant distribution geometrical point of view in following manner.

Theorem 4.4. *Let $\Pi : \Theta_1 \to \Theta_2$ be $\mathcal{PWSSCS}$ with semi-slant function θ where, (Θ_1, g_1, J) a KM and (Θ_2, g_2) a RM. Then $\mathfrak{D}^\theta$ is defines totally geodesic foliation on Θ_1 if and only if*

$$\psi(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W} + \mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W} + \mathscr{T}_{\hat{Z}}\zeta\hat{W}\mathbb{Q}) \in \Gamma(\mathfrak{D}^\theta) \tag{4.35}$$

and

$$\begin{aligned}
&g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z}, \psi\hat{W}) - g_1(\mathscr{T}_{\hat{Z}}\psi\mathbb{P}\hat{Z}, \zeta\hat{W}) + g_1(\nabla_{\hat{X}}\mathbb{P}\psi\mathbb{Q}\hat{Z}, \hat{W})\\
&= g_1(\mathscr{T}_{\hat{X}}\zeta\mathbb{Q}\hat{Z}, \psi\hat{W}) - \sin^2\theta g_1([\hat{Z}, \hat{X}], \hat{W}) - 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z}, \hat{W})\\
&\ + g_1(\hat{X}, grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z}, \zeta\hat{W}) + g_1(\zeta\mathbb{Q}\hat{Z}, grad\ln\lambda)g_1(\hat{X}, \zeta\hat{W})\\
&\ - g_1(\zeta\hat{W}, grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z}, \zeta\hat{W}),
\end{aligned} \tag{4.36}$$

for any vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta), \hat{U} \in \Gamma(\mathfrak{D})$ and $\hat{X} \in \Gamma(\ker\Pi_*)^\perp$.

Proof. Let us consider for any vector fields $\hat{Z}, \hat{W} \in \Gamma(\mathfrak{D}^\theta)$ and $\hat{U} \in \Gamma(\mathfrak{D})$. In light of equations (2.11), (2.12) and (2.13) after operating almost complex structure J on both side, we have

$$g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) = g_1(\nabla_{\hat{Z}}J\hat{W}, J\hat{U}).$$

By using decomposition (3.14), $g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) = g_1(\nabla_{\hat{Z}}J(\mathbb{P}\hat{W} + \mathbb{Q}\hat{W}), J\hat{U})$. Taking account the fact from equation (3.15), we have

$$\begin{aligned}
g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) &= g_1(\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W}, J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{P}\hat{W}, J\hat{U})\\
&\quad + g_1(\nabla_{\hat{Z}}\psi\mathbb{Q}\hat{W}, J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W}, J\hat{U}).
\end{aligned}$$

Considering the equations (2.4) and (2.5) and since $\mathfrak{D}$ is invariant under almost structure J, i.e., $J\mathfrak{D} = \mathfrak{D}$, we may yields

$$\begin{aligned}
g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) =& g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W}, J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W}, J\hat{U})\\
&- g_1(\nabla_{\hat{Z}}\psi^2\mathbb{Q}\hat{W}, \hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W}, J\hat{U}).
\end{aligned} \tag{4.37}$$

By using Lemma 3.2 in third term of above equation, which can be write as $-g_1(\nabla_{\hat{Z}}\psi^2\mathbb{Q}\hat{W}, \hat{U}) = g_1(\nabla_{\hat{Z}}(\cos^2\theta)\mathbb{Q}\hat{W}, \hat{U})$. Then the equation (4.37), will takes the form as

$$\begin{aligned}
g_1(\nabla_{\hat{Z}}\hat{W}, \hat{U}) =& g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W}, J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W}, J\hat{U})\\
&+ g_1(\nabla_{\hat{Z}}(\cos^2\theta)\mathbb{Q}\hat{W}, \hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W}, J\hat{U}).
\end{aligned}$$

Since, Π is a $\mathcal{PWSSCS}$ with semi-slant function θ, then we can write the above equation as:

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{U}) =& g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W},J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W},J\hat{U}) \\ &+ g_1(\nabla_{\hat{Z}}(\cos^2\theta)\mathbb{Q}\hat{W},\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W},J\hat{U}). \end{aligned}$$

With simple steps of calculations, finally we get

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{U}) =& g_1(\mathscr{V}\nabla_{\hat{Z}}\psi\mathbb{P}\hat{W},J\hat{U}) + g_1(\mathscr{T}_{\hat{Z}}\zeta\mathbb{P}\hat{W},J\hat{U}) + g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{W},J\hat{U}) \\ &+ 2\sin\theta\cos\theta\hat{Z}(\theta)g_1(\mathbb{Q}\hat{W},\hat{U}) + \cos^2\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{W},\hat{U}). \end{aligned}$$

From which the first part of theorem holds good. For the other part of theorem, let us suppose for any vector fields $\hat{Z},\hat{W}\in\Gamma(\mathfrak{D}^\theta)$ and $\hat{X}\in\Gamma(\ker\Pi_*)^\perp$. We start with considering the term $g_1(\nabla_{\hat{Z}}\hat{W},\hat{X})$, by using basic calcaltions, this term can be write as $g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\nabla_{\hat{X}}\hat{Z},\hat{W})$. By using equation (2.11), (2.12) and (2.13), this term takes the form as

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\nabla_{\hat{X}}J\hat{Z},J\hat{W}).$$

In the light of equations (2.4) and since $\mathfrak{D}$ is invariant under J, we get

$$g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) = -g([\hat{Z},\hat{X}],\hat{W}) - g_1(\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},J\hat{W}) - g_1(\nabla_{\hat{X}}\psi\mathbb{Q}\hat{Z},J\hat{W}) - g_1(\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},J\hat{W}).$$

By using equations (2.11), (2.4) and (2.5), we have

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) =& - g([\hat{Z},\hat{X}],\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\psi^2Q\hat{Z},\hat{W}) \\ &+ g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}). \end{aligned} \tag{4.38}$$

Since, Π is a $\mathcal{PWSSCS}$ with semi-slant function θ, then with simple steps of calculations, the fourth term of above equation take place as

$$\begin{aligned} g_1(\nabla_{\hat{X}}\psi^2Q\hat{Z},\hat{W}) =& - g_1(\nabla_{\hat{X}}(-\cos^2\theta)\mathbb{Q}\hat{Z},\hat{W}) \\ &= 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{Z},\hat{W}). \end{aligned}$$

By using the above equation in (4.38), we may write as

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) =& - g([\hat{Z},\hat{X}],\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) \\ &+ g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) \\ &+ 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{Z},\hat{W}). \end{aligned} \tag{4.39}$$

Now, the first and last term can be write as:

$$-g([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{Z},\hat{W}) = \sin^2\theta g_1([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{X},\hat{W}). \tag{4.40}$$

Since, Π is a $\mathcal{P}\hat{\mathcal{W}}\mathcal{SSCS}$, then by using equation (4.40) in (4.39), we can write

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) =& 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) + \sin^2\theta g_1([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{X},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}). \end{aligned} \tag{4.41}$$

Now, using the horizontal conformality of Π from Lemma 2.1 and equations (2.1), (2.10) in the last term of above equation, can be written as

$$\begin{aligned} -g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) =& \frac{1}{\lambda^2}g_1(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})) - \frac{1}{\lambda^2}g_1((\nabla\Pi_*)(\hat{X},\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})) \\ & + \frac{1}{\lambda^2}g_1(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})) - g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) \\ & - g_1(\zeta\mathbb{Q}\hat{Z},grad\ln\lambda)g_1(\hat{X},\zeta\hat{W}) + g_1(\zeta\hat{W},grad\ln\lambda)g_1(\hat{X},\zeta\mathbb{Q}\hat{Z}). \end{aligned}$$

Now, by using the above relation, equation (4.41) finally turns into

$$\begin{aligned} g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}) =& 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{Z},\hat{W}) + \sin^2\theta g_1([\hat{Z},\hat{X}],\hat{W}) - \cos^2\theta g_1(\nabla_{\hat{Z}}\hat{X},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{Z},\zeta\hat{W}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{Z},\psi\hat{W}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{Z},\hat{W}) \\ & - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{Z},\psi\hat{W}) - g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{Z},\zeta\hat{W}) \\ & - g_1(\zeta\mathbb{Q}\hat{Z},grad\ln\lambda)g_1(\hat{X},\zeta\hat{W}) + g_1(\zeta\hat{W},grad\ln\lambda)g_1(\hat{X},\zeta\mathbb{Q}\hat{Z}) \\ & + \frac{1}{\lambda^2}g_1(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{Z}),\Pi_*(\zeta\hat{W})). \end{aligned}$$

Hence, this proves the theorem completely. □

The study of geometry of leaves of horizontal and vertical distributions of $\mathcal{PWSSCS}$ is very important. We start our discussion with necessary and sufficient conditions for vertical distribution $\ker\Pi_*$ is totally geodesic.

Theorem 4.5. *Let us suppose that Π be a $\mathcal{PWSSCS}$ with semi-slant function θ from KM (Θ_1, g_1, J) onto a RM (Θ_2, g_2). Then $\ker\Pi_*$ is defines totally geodesic foliation if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) + g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) \\ & = \cos^2\theta\, g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{U},\hat{V}) - 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{U},\hat{V}) + g_1([\hat{U},\hat{X}],\hat{V}) \\ & + g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) + g_1(\zeta\mathbb{Q}\hat{U},grad\ln\lambda)g_1(\hat{X},\zeta\hat{V}) \\ & - g_1(\zeta\hat{V},grad\ln\lambda)g_1(\hat{X},\zeta\mathbb{Q}\hat{U}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}), \end{aligned} \tag{4.42}$$

for any vector fields $\hat{U},\hat{V} \in \Gamma(\ker\Pi_*)$ and $\hat{X} \in \Gamma(\ker\Pi_*)^{\perp}$.

Proof. We start the proof of theorem with considering the term $g_1(\nabla_{\hat{U}}\hat{V},\hat{X})$. From simple steps of calculations with basic definition, this turns into $g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = -g_1([\hat{U},\hat{X}],\hat{V}) - g_1(\nabla_{\hat{X}}\hat{U},\hat{V})$. Operating J, which is a almost complex structure with using equation (2.11), (2.12) and (2.13) on second term, this will take place

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = -g_1([\hat{U},\hat{X}],\hat{V}) - g_1(\nabla_{\hat{X}}J\hat{U},JV),$$

for any vertical vector fields $\hat{U},\hat{V}$ and horizontal vector field $\hat{X}$. In the light of decomposition (3.14) and (3.15) the second term of above equation, we can write

$$g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = -g_1([\hat{U},\hat{X}],\hat{V}) - g_1(\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},JV) - g_1(\nabla_{\hat{X}}\psi\mathbb{Q}\hat{U},JV) - g_1(\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},JV). \tag{4.43}$$

In the light of equation (3.15) and (2.6), second term of above equation become $-g_1(\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},JV) = g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V})$. Similarly, from equation (2.11), (2.12) and (2.6), third term turns as $-g_1(\nabla_{\hat{X}}\psi\mathbb{Q}\hat{U},JV) = g_1(\nabla_{\hat{X}}\psi^2\mathbb{Q}\hat{U},\hat{V}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V})$. In last term, taking account the fact from decomposition (3.15) and equation (2.7), this will take place as $-g_1(\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},JV) = -g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\hat{V})$. Put the values of all these terms in equation (4.43), we get

$$\begin{aligned}g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) =& -g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) + g_1(\nabla_{\hat{X}}\psi^2\mathbb{Q}\hat{U},\hat{V})\\ &+ g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}).\end{aligned}$$

Since, Π is a $\mathcal{PWSSCS}$ with semi-slant function θ, using Lemma 3.2, above equation turns into

$$\begin{aligned}g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) =& -g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) - g_1(\nabla_{\hat{X}}(\cos^2\theta)\mathbb{Q}\hat{U},\hat{V})\\ &+ g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) - g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V})\\ =& -g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V})\\ &+ 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{U},\hat{V}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{U},\hat{V})\\ &- g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}).\end{aligned} \tag{4.44}$$

Considering equations (2.1) and (2.10), second last term of the above equation will be

$$-g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})) - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})).$$

By using the definition of horizontal conformality of Π from Lemma 3.2, we can write

$$-g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*(\zeta\mathbb{Q}\hat{U}),\Pi_*(\zeta\hat{V})) - \frac{1}{\lambda^2}g_2((\hat{X}(\ln\lambda)\Pi_*(\zeta\mathbb{Q}\hat{U}) + \zeta\mathbb{Q}\hat{U}(\ln\lambda))\Pi_*(\hat{X}) - g_1(\hat{X},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*(\zeta\hat{V})).$$

Now, by using above two equations in (4.44), finally we have

$$\begin{aligned} g_1(\nabla_{\hat{U}}\hat{V},\hat{X}) = &- g_1([\hat{U},\hat{X}],\hat{V}) + g_1(\mathfrak{A}_{\hat{X}}\psi\mathbb{P}\hat{U},\zeta\hat{V}) - g_1(\mathscr{V}\nabla_{\hat{X}}\psi\mathbb{P}\hat{U},\psi\hat{V}) + g_1(\nabla_{\hat{X}}\zeta\psi\mathbb{Q}\hat{U},\hat{V}) \\ &+ 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\mathbb{Q}\hat{U},\hat{V}) - \cos^2\theta g_1(\nabla_{\hat{X}}\mathbb{Q}\hat{U},\hat{V}) - g_1(\mathfrak{A}_{\hat{X}}\zeta\mathbb{Q}\hat{U},\psi\hat{V}) \\ &- g_1(\hat{X},grad\ln\lambda)g_1(\zeta\mathbb{Q}\hat{U},\zeta\hat{V}) - g_1(\zeta\mathbb{Q}\hat{U},grad\ln\lambda)g_1(\hat{X},\zeta\hat{V}) \\ &+ g_1(\hat{X},\zeta\mathbb{Q}\hat{U})g_1(\zeta\hat{V},grad\ln\lambda). \end{aligned}$$

□

This completes the proof.

Theorem 4.6. *Let Π be $\mathcal{PWSSCS}$ from a KM (Θ_1, g_1, J) onto a RM (Θ_2, g_2). Then the map Π is totally geodesic map if and only if*

(i) $\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\hat{W} + \zeta\psi\hat{Z}(\ln\lambda)\Pi_*\hat{Z} - g_1(\hat{Z},\zeta\psi\hat{W})\Pi_*(grad\ln\lambda),\Pi_*(\hat{X}))$
$= g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X}) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X}))$

(ii) $\cos^2\theta g_1(\mathscr{T}_{\hat{X}}\hat{Y},\hat{Z}) + \frac{1}{\lambda^2}\{g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\psi\hat{Y},\Pi_*(\hat{Z}) - g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\hat{Y},\Pi_*(\mathfrak{C}\hat{Z}))\} = 0$

(iii) $\operatorname{cosec}^2\theta g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W}) + \cot^2\theta\cos^2\theta g_1(\mathfrak{H}\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})$
$= -\frac{1}{\lambda^2}\{g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\mathbb{Q}\hat{U},\Pi_*(\hat{W})) + g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\mathbb{Q}\hat{U},\Pi_*(\mathfrak{C}\hat{W}))\}$,

for any $\hat{U},\hat{V}\in\Gamma(\mathfrak{D}), \hat{X},\hat{Y}\in\Gamma(\mathfrak{D}^{\theta})$ and $\hat{Z}\in\Gamma(\ker\Pi_*)^{\perp}, \hat{U}_1\in\Gamma(\ker\Pi_*)$.

Proof. Let us consider $g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X}))$, for any $\hat{Z},\hat{W}\in\Gamma(\mathfrak{D})$ and $\hat{X}\in\Gamma(\ker\Pi_*)^{\perp}$. On using equations (2.10) with definition 2.1, we may obtain $g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = -\lambda^2 g_1(\nabla_{\hat{Z}}\hat{W},\hat{X})$. This relation can be turn into

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = g_1(\nabla_{\hat{Z}}\hat{W},\hat{X}).$$

Taking account the fact that $J\hat{W} = \psi\mathfrak{D}$ if $\hat{W}\in\Gamma(\mathfrak{D})$ and from equations (2.11), (3.15) in the right hand side of above equation, we get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = -g_1(\nabla_{\hat{Z}}\psi\hat{W},J\hat{X}).$$

By using equations (2.4), (2.5) with (3.15), we can get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) = g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X}) - g_1(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{X}). \tag{4.45}$$

Since Π is $\mathcal{PWSSCS}$, by using definition 2.1, the second term in the right hand side of above equation can be turn into $-g_1(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\hat{W},\hat{X}) = \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\zeta\psi\hat{W}),\Pi_*(\hat{X})) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X}))$. By using this in (4.45), we may have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) =& g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X}) + \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\zeta\psi\hat{W}),\Pi_*(\hat{X}))\\ &- \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X})).\end{aligned}$$

Finally with using Lemma 3.3, we get

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{W}),\Pi_*(\hat{X})) =& \frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\hat{W} + \zeta\psi\hat{W}(\ln\lambda)\Pi_*\hat{Z}\\ &- g_1(\hat{Z},\zeta\psi\hat{W})\Pi_*(grad\ln\lambda)) + g_1(\mathscr{T}_{\hat{Z}}\psi^2\hat{W},\hat{X})\\ &- \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\hat{W},\Pi_*(\hat{X})),\end{aligned}$$

which is part (i). For part (ii), take into consideration $g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z}))$, for any $\hat{X},\hat{Y}\in\Gamma(\mathfrak{D}^{\theta})$ and $\hat{Z}\in\Gamma(\ker\Pi_*)^{\perp}$. From equations (2.10) with definition 2.1, we can write $g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) = -\lambda^2 g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})$. In the light of relation (2.11), (2.12) and (3.15), we get

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) = -g_1(\nabla_{\hat{X}}\psi\hat{Y},J\hat{Z}) - g_1(\nabla_{\hat{X}}\zeta\hat{Y},J\hat{Z}).$$

By using equations (2.11), (2.12), (3.17), above equations turn into

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) = g_1(\nabla_{\hat{X}}J\psi\hat{Y},\hat{Z}) - g_1(\nabla_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}+\mathfrak{C}\hat{Z}).$$

By using equation (2.5), we can write

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) =& g_1(\nabla_{\hat{X}}\psi^2\hat{Y},\hat{Z}) + g_1(\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z})\\ &- g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\hat{Y},\mathfrak{C}\hat{Z}) - g_1(\mathscr{T}_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}).\end{aligned}$$

Taking account the fact from equation (2.5) with Lemma 3.3, we may have

$$\begin{aligned}\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z})) =& g_1(\nabla_{\hat{X}}(\cos^2\theta)\hat{Y},\hat{Z}) + g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z})\\ &- g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\hat{Y},\mathfrak{C}\hat{Z}) - g_1(\mathscr{T}_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}).\end{aligned}\tag{4.46}$$

Since Π is a $\mathcal{PWSSCS}$ from a KM Θ_1, the the first term of equation (4.46) turn into as $g_1(\nabla_{\hat{X}}(\cos^2\theta)\hat{Y},\hat{Z}) = 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\hat{Y},\hat{Z}) + \cos^2\theta g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z})$, where the second term as $g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z}) = \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\psi\hat{Y},\Pi_(*\hat{Z})) - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\psi\hat{Y}),\Pi_*(\hat{Z}))$ and third term as $g_1(\mathfrak{H}\nabla_{\hat{X}}\zeta\psi\hat{Y},\hat{Z}) = -\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\hat{Y},\Pi_*(\mathfrak{C}\hat{Z})) + \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\hat{Y}),\Pi_*(\mathfrak{C}\hat{Z}))$ by using equation (2.10) and definition 2.1. □

With all these facts using in equation (4.46), we can write

$$
\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z}))\\
&= 2\sin\theta\cos\theta\hat{X}(\theta)g_1(\hat{Y},\hat{Z}) + \cos^2\theta g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z}) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\psi\hat{Y},(\Pi_*\hat{Z}))\\
&\quad - \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\psi\hat{Y}),\Pi_*(\hat{Z})) - \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\hat{Y},\Pi_(*\mathfrak{C}\hat{Z}))\\
&\quad + \frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\zeta\hat{Y}),\Pi_*(\mathfrak{C}\hat{Z})) - g_1(\mathscr{T}_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z}).
\end{aligned}
$$

Finally, by using the Lemma 3.3 in fourth and fifth terms, the above equations takes the form

$$
\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{X},\hat{Y}),\Pi_*(\hat{Z}))\\
&= \frac{1}{\lambda^2}g_2(\hat{X}(\ln\lambda)\Pi_*\zeta\hat{Y} + \zeta\hat{Y}(\ln\lambda)\Pi_*\hat{X} - g_1(\hat{X},\zeta\hat{Y})\Pi_*(grad\ln\lambda),\Pi_*(\mathfrak{C}\hat{Z}))\\
&- \frac{1}{\lambda^2}g_2(\hat{X}(\ln\lambda)\Pi_*\zeta\psi\hat{Y} + \zeta\psi\hat{Y}(\ln\lambda)\Pi_*\hat{X} - g_1(\hat{X},\zeta\psi\hat{Y})\Pi_*(grad\ln\lambda),\Pi_*(\hat{Z}))\\
&- \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\hat{Y},\Pi_(*\mathfrak{C}\hat{Z})) - g_1(\mathscr{T}_{\hat{X}}\zeta\hat{Y},\mathfrak{B}\hat{Z})\\
&+ \cos^2\theta g_1(\nabla_{\hat{X}}\hat{Y},\hat{Z}) + \frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{X}}\Pi_*\zeta\psi\hat{Y},\Pi_*(\hat{Z})).
\end{aligned}
$$

This is the proof of part (ii). For (iii) part, we consider

$$\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W}) = -g_1(\Pi_*\nabla_{\hat{Z}}\hat{U}_1,\Pi_*\hat{W}),$$

for any $\hat{U}_1 \in \Gamma(\ker\Pi_*)$ and $\hat{Z},\hat{W} \in \Gamma(\ker\Pi_*)^{\perp}$. By using equations (2.11), (2.12), (3.14) and (3.15), we can write

$$
\begin{aligned}
\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W}) = &- g_1(\nabla_{\hat{Z}}\mathbb{P}\hat{U},\hat{W}) + g_1(\nabla_{\hat{Z}}\psi^2Q\hat{U},\hat{W}) - g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W})\\
&- g_1(\mathfrak{H}\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{C}\hat{W}) - g_1(\mathfrak{A}_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}).
\end{aligned}
$$

Since $\mathcal{PWSSCS}$, then by using Lemma 3.3 and definition of horizontal conformality 2.1, the above equation turn into

$$
\begin{aligned}
\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W}) = &- g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W}) - \sin 2\theta\hat{Z}(\theta)g_1(\mathbb{Q}\hat{U},\hat{W}) + \cos^2\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})\\
&- \frac{1}{\lambda^2}g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U}),\Pi_*(\hat{W})) - \frac{1}{\lambda^2}g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U}),\Pi_*(\mathfrak{C}\hat{W}))\\
&- g_1(\mathfrak{A}_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}).
\end{aligned}
\tag{4.47}
$$

The second term on right hand side of above equations become 0 since $\mathbb{Q}\hat{U}$ and $\hat{W}$ both are orthogonal, whereas the third term reduces with equations (2.11), (2.12) and Lemma 3.3 as,

$-\cos^2\theta(g_1(\nabla_{\hat{Z}}\psi^2 Q\hat{U},\hat{W})+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W}))+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}+\mathfrak{C}\hat{W})$. With this value equation (4.47) reduces to

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W})\\
&=-g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W})-\cos^2\theta g_1(\nabla_{\hat{Z}}\psi^2 Q\hat{U},\hat{W})+\cos^4\theta g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W})\\
&\quad+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}+\mathfrak{C}\hat{W})+\cos^2\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})\\
&\quad-\frac{1}{\lambda^2}g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U}),\Pi_*(\hat{W}))-\frac{1}{\lambda^2}g_2(\Pi_*(\mathfrak{H}\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U}),\Pi_*(\mathfrak{C}\hat{W}))\\
&\quad-g_1(\mathfrak{A}_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}).
\end{aligned}\tag{4.48}$$

Since Π is a $\mathcal{PWSSCS}$ from KM onto RM, by using the formula of second fundamental form of Π and Lemma 3.3, sixth term in the right hand side of above equations reduces to $\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}\hat{U}+\zeta\psi\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\psi\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\hat{W})$ and the seventh term as $\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\mathbb{Q}\hat{U}+\zeta\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\mathfrak{C}\hat{W})$. Putting these values in equation (4.48), we have

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W})\\
=&\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}\hat{U}+\zeta\psi\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\psi\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\hat{W})\\
&+\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\mathbb{Q}\hat{U}+\zeta\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\mathfrak{C}\hat{W})\\
&-g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W})-\cos^2\theta g_1(\nabla_{\hat{Z}}\psi^2 Q\hat{U},\hat{W})+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\psi\mathbb{Q}\hat{U},\hat{W})-g_1(\mathfrak{A}_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W})\\
&+\cos^2\theta g_1(\nabla_{\hat{Z}}\zeta\mathbb{Q}\hat{U},\mathfrak{B}\hat{W}+\mathfrak{C}\hat{W})+\cos^4\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W})\\
&-\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\mathbb{Q}\hat{U},\Pi_*\hat{W})-\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\mathbb{Q}\hat{U},\Pi_*\mathfrak{C}\hat{W}).
\end{aligned}$$

Finally, by using definition of horizontal conformality with Lemma 3.3 and equation (2.7), we can write

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla\Pi_*)(\hat{Z},\hat{U}_1),\Pi_*\hat{W})\\
&=\sin^2\theta\{\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\psi\mathbb{Q}\hat{U}+\zeta\psi\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\psi\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\hat{W})\\
&+\frac{1}{\lambda^2}g_2(\hat{Z}(\ln\lambda)\Pi_*\zeta\mathbb{Q}\hat{U}+\zeta\mathbb{Q}\hat{U}(\ln\lambda)\Pi_*\hat{Z}-g_1(\hat{Z},\zeta\mathbb{Q}\hat{U})\Pi_*(grad\ln\lambda),\Pi_*\mathfrak{C}\hat{W})\}\\
&-\sin^2\theta\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\psi\mathbb{Q}\hat{U},\Pi_*\hat{W})-\sin^2\theta\frac{1}{\lambda^2}g_2(\nabla^{\Pi}_{\hat{Z}}\Pi_*\zeta\mathbb{Q}\hat{U},\Pi_*\mathfrak{C}\hat{W})\\
&-g_1(\mathfrak{A}_{\hat{Z}}\mathbb{P}\hat{U},\hat{W})+\cos^4\theta g_1(\nabla_{\hat{Z}}\mathbb{Q}\hat{U},\hat{W}),
\end{aligned}$$

from which we can get part (iii) of Theorem.

5. Pluriharmonicity

In this section, we discussed the concept of J-pluriharmonicity on AHMs which was once studied and defined by Y. Ohnita [26]. Let Π be a $\mathcal{PWSSCS}$ from KM (Θ_1, g_1, J) onto a RM (Θ_2, g_2). Then $\mathcal{PWSSCS}$ is J-pluriharmonic, $\mathfrak{D}$-J-pluriharmonic, $\mathfrak{D}^\theta$-J-pluriharmonic, $(\mathfrak{D}-\mathfrak{D}^\theta)$-$\phi$ pluriharmonic, $\ker\Pi_*$-J-pluriharmonic, $(\ker\Pi_*)^\perp$-J-pluriharmonic and $((\ker\Pi_*)^\perp - \ker\Pi_*)$-$\phi$-pluriharmonic if

$$(\nabla\Pi_*)(\hat{X},\hat{Y}) + (\nabla\Pi_*)(J\hat{X},J\hat{Y}) = 0, \tag{5.49}$$

for any $\hat{X},\hat{Y} \in \Gamma(\mathfrak{D})$, for any $\hat{X},\hat{Y} \in \Gamma(\mathfrak{D}^\theta)$, for any $\hat{X} \in \Gamma(\mathfrak{D}), \hat{Y} \in \Gamma(\mathfrak{D}^\theta)$, for any $\hat{X},\hat{Y} \in \Gamma(\ker\Pi_*)$, for any $\hat{X},\hat{Y} \in \Gamma(\ker\Pi_*)^\perp$ and for any $\hat{X} \in \Gamma(\ker\Pi_*)^\perp, \hat{Y} \in \Gamma(\ker\Pi_*)$.

Theorem 5.1. *Let Π be a $\mathcal{PWSSCS}$ from KM (Θ_1, g_1, J) onto a RM (Θ_2, g_2). Suppose that Π is $\mathfrak{D}^\theta$-J-pluriharmonic. Then $\mathfrak{D}^\theta$ defines totally geodesic foliation on Θ_1 if and only if*

$$\begin{aligned}\nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1 + \nabla^{\Pi}_{\zeta\hat{X}_1}\Pi_*\zeta\hat{Y}_1 =&\Pi_*(\mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \mathfrak{A}_{\zeta\hat{X}_1}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{P}\psi\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{P}\psi\hat{Y}_1)\\ &+ \Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{Q}\psi\hat{Y}_1 + \mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{Q}\psi\hat{Y}_1)\\ &- \cos^2\theta\,\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi\zeta\mathbb{Q}\psi\hat{Y}_1 + \mathscr{T}_{\psi\hat{X}_1}\mathbb{Q}\psi\hat{Y}_1),\end{aligned}$$

for any $\hat{\hat{X}}_1, \hat{Y}_1 \in \Gamma(\mathfrak{D}^\theta)$.

Proof. For any $\hat{\hat{X}}_1, \hat{Y}_1 \in \Gamma(\mathfrak{D}^\theta)$ and using the pluriharmonicity of J with equation (2.10), we get

$$\begin{aligned}0 &= (\nabla\Pi_*)(\hat{X}_1,\hat{Y}_1) + (\nabla\Pi_*)(J\hat{X}_1,J\hat{Y}_1)\\ &= -\Pi_*\nabla_{\hat{\hat{X}}_1}\hat{Y}_1 + \nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1 - \Pi_*\nabla_{J\hat{X}_1}J\hat{Y}_1.\end{aligned}$$

Now, from the above equation, we can write

$$\Pi_*\nabla_{\hat{\hat{X}}_1}\hat{Y}_1 = \nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1 - \Pi_*\nabla_{J\hat{X}_1}J\hat{Y}_1.$$

The second term in the right hand side of above equation with using equation (3.15), takes the form as $\Pi_*\nabla_{\psi\hat{X}_1}\psi\hat{Y}_1 + \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1 + \Pi_*\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1 + \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1$. Now, equation (5) can be write as

$$\begin{aligned}\Pi_*\nabla_{\hat{\hat{X}}_1}\hat{Y}_1 =&\nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1 - \Pi_*\nabla_{\psi\hat{X}_1}\psi\hat{Y}_1 - \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1\\ &- \Pi_*\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1 - \Pi_*\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1.\end{aligned}$$

Taking account the fact that Π is $\mathcal{PWSSCS}$ with using equations (2.5), (2.6), (2.10) and (3.14), we have

$$\begin{aligned}\Pi_*\nabla_{\hat{X}_1}\hat{Y}_1 =& -\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\zeta\hat{Y}_1+\mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1+\mathfrak{A}_{\zeta\hat{X}_1}\psi\hat{Y}_1+\mathscr{V}\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1)\\ &+\{\zeta\hat{X}_1(\ln\lambda)\Pi_*\zeta\hat{Y}_1+\zeta\hat{Y}_1(\ln\lambda)\Pi_*\zeta\hat{X}_1-g_1(\zeta\hat{X}_1,\zeta\hat{Y}_1)\Pi_*(grad\ln\lambda)\}\\ &-\nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1-\nabla^{\Pi}_{\zeta\hat{X}_1}\Pi_*\zeta\hat{Y}_1+\Pi_*(J\nabla_{\psi\hat{X}_1}J(\mathbb{P}\psi\hat{Y}_1+\mathbb{Q}\psi\hat{Y}_1)).\end{aligned}$$

Operating J in the last term in the right hand side of above equation with Lemma 3.2 and equations (2.5) and (2.6), we may have

$$\begin{aligned}\Pi_*\nabla_{\hat{X}_1}\hat{Y}_1 =&\{\zeta\hat{X}_1(\ln\lambda)\Pi_*\zeta\hat{Y}_1+\zeta\hat{Y}_1(\ln\lambda)\Pi_*\zeta\hat{X}_1-g_1(\zeta\hat{X}_1,\zeta\hat{Y}_1)\Pi_*(grad\ln\lambda)\}\\ &+\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{P}\psi\hat{Y}_1+\mathscr{V}\nabla_{\psi\hat{X}_1}\psi^2\mathbb{P}\psi\hat{Y}_1+\mathscr{T}_{\psi\hat{X}_1}\zeta\psi\mathbb{P}\psi\hat{Y}_1+\mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{P}\psi\hat{Y}_1)\\ &+\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi^2\mathbb{Q}\psi\hat{Y}_1+\mathscr{V}\nabla_{\psi\hat{X}_1}\psi^2\mathbb{Q}\psi\hat{Y}_1+\mathscr{T}_{\psi\hat{X}_1}\zeta\psi\mathbb{Q}\psi\hat{Y}_1+\mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\psi\mathbb{Q}\psi\hat{Y}_1)\\ &-\cos^2\theta\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\psi\zeta\mathbb{Q}\psi\hat{Y}_1+\mathscr{V}\nabla_{\psi\hat{X}_1}\psi\zeta\mathbb{Q}\psi\hat{Y}_1+\mathscr{T}_{\psi\hat{X}_1}\mathbb{Q}\psi\hat{Y}_1+\mathscr{V}\nabla_{\psi\hat{X}_1}\mathbb{Q}\psi\hat{Y}_1)\\ &+\Pi_*(\mathscr{T}_{\psi\hat{X}_1}\zeta\hat{Y}_1+\mathfrak{H}\nabla_{\psi\hat{X}_1}\zeta\hat{Y}_1+\mathfrak{A}_{\zeta\hat{X}_1}\psi\hat{Y}_1+\mathscr{V}\nabla_{\zeta\hat{X}_1}\psi\hat{Y}_1)\\ &-\nabla^{\Pi}_{J\hat{X}_1}\Pi_*J\hat{Y}_1-\nabla^{\Pi}_{\zeta\hat{X}_1}\Pi_*\zeta\hat{Y}_1.\end{aligned}$$

□

Theorem 5.2. *Let Π be a $\mathcal{PWSSCS}$ from KM (Θ_1, g_1, J) onto a RM (Θ_2, g_2). Suppose that Π is $((\ker\Pi_*)^\perp-\ker\Pi_*)$-$J$-pluriharmonic. Then the horizontal distribution $(\ker\Pi_*)^\perp$ defines totally geodesic foliation on Θ_1 if and only if*

$$\begin{aligned}&\Pi_*\nabla_{\hat{Y}_1}\hat{W}-\nabla^{\Pi}_{J\hat{X}_1}\Pi_*(J\hat{W})+\cos^4\theta\Pi_*\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\mathbb{Q}\hat{W}-\cos^2\theta\Pi_*\zeta\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\zeta\mathbb{Q}\hat{W}\\ &=\sin^2\theta\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\psi\mathbb{Q}\hat{W})+\zeta\psi\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1)-g_1(\zeta\psi\mathbb{Q}\hat{W},\mathfrak{C}\hat{Y}_1)\Pi_*grad\ln\lambda\}\\ &+\cos^2\theta J\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\mathbb{Q}\hat{W})+\zeta\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1)-g_1(\zeta\mathbb{Q}\hat{W},\mathfrak{C}\hat{Y}_1)\Pi_*grad\ln\lambda\}\\ &-\Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W}+\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W}+\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}+\mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W})\\ &-\sin^2\theta\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\psi\mathbb{Q}\hat{W})+\cos^2\theta J\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\mathbb{Q}\hat{W}),\end{aligned}$$

for any $\hat{Y}_1\in\Gamma(\ker\Pi_*)^\perp$ and $\hat{W}\in\Gamma(\ker\Pi_*)$.

Proof. For any $\hat{Y}_1\in\Gamma(\ker\Pi_*)^\perp, \hat{W}\in\Gamma(\ker\Pi_*)$ and using equations (2.10), (3.14), (3.15) with considering the fact that the pluriharminicity of J, we can write

$$\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W}=-\Pi_*(\nabla_{\mathfrak{B}\hat{Y}_1}\psi\hat{W}+\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}+\nabla_{\mathfrak{C}\hat{Y}_1}\psi\hat{W})-\Pi_*\nabla_{\hat{Y}_1}\hat{W}+\nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W}.$$

Now, by using equations (2.4), (2.6), (3.14), (3.15) and from the Lemma 3.2, above equation can takes the form as

$$\begin{aligned}\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W} =& -\Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W} - \Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\psi\mathbb{Q}\hat{W}\\ &+ \Pi_*(J\nabla_{\mathfrak{B}\hat{Y}_1}J\psi\hat{W} + J\nabla_{\mathfrak{C}\hat{Y}_1}J\psi\hat{W}) - \cos^2\theta\Pi_*(J\nabla_{\mathfrak{C}\hat{Y}_1}JQ\hat{W})\\ &+ \Pi_*(-\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} - \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} - \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W})\\ &+ \Pi_*(-\mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} - \mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} - \mathscr{V}\nabla_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}).\end{aligned}$$

By using the horizontal conformality of Π, Lemma 3.2, equations (3.15) and (2.10) with some simple steps of calculations, we may have

$$\begin{aligned}\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W} =& -\cos^4\theta\Pi_*(\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\mathbb{Q}\hat{W}) + \cos^2\theta\Pi_*(\zeta\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\zeta\mathbb{Q}\hat{W}) + \sin^2\theta(\nabla\Pi_*)(\mathfrak{C}\hat{Y}_1, \zeta\psi\mathbb{Q}\hat{W})\\ &- \sin^2\theta\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \cos^2\theta J(\nabla\Pi_*)(\mathfrak{C}\hat{Y}_1, \zeta\mathbb{Q}\hat{W}) + \cos^2\theta J\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\mathbb{Q}\hat{W})\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W}\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W})\\ &- \Pi * (\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} - \mathscr{V}\nabla_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}).\end{aligned}$$

Finally, by using the Lemma 2.1, the above equation takes the form

$$\begin{aligned}&\Pi_*\nabla_{\mathfrak{C}\hat{Y}_1}\zeta\hat{W}\\ =&\sin^2\theta\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \zeta\psi\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1) - g_1(\mathfrak{C}\hat{Y}_1, \zeta\psi\mathbb{Q}\hat{W})\Pi_*(grad\ln\lambda)\}\\ &\cos^2\theta J\{\mathfrak{C}\hat{Y}_1(\ln\lambda)\Pi_*(\zeta\mathbb{Q}\hat{W}) + \zeta\mathbb{Q}\hat{W}(\ln\lambda)\Pi_*(\mathfrak{C}\hat{Y}_1) - g_1(\mathfrak{C}\hat{Y}_1, \zeta\mathbb{Q}\hat{W})\Pi_*(grad\ln\lambda)\}\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W} + \mathfrak{H}\nabla_{\mathfrak{B}\hat{Y}_1}\zeta\hat{W}) - \Pi_*\nabla_{\hat{Y}_1}\hat{W} + \nabla^{\Pi}_{J\hat{Y}_1}\Pi_*J\hat{W} - \cos^4\theta\Pi_*(\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\mathbb{Q}\hat{W})\\ &- \Pi_*(\mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{P}\psi\hat{W} + \mathscr{T}_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W} + \mathscr{V}\nabla_{\mathfrak{B}\hat{Y}_1}\mathbb{Q}\psi\hat{W}) + \cos^2\theta\Pi_*(\zeta\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\zeta\mathbb{Q}\hat{W})\\ &- \Pi * (\mathfrak{A}_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W} - \mathscr{V}\nabla_{\mathfrak{C}\hat{Y}_1}\psi\mathbb{P}\hat{W}) - \sin^2\theta\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\psi\mathbb{Q}\hat{W}) + \cos^2\theta J\nabla^{\Pi}_{\mathfrak{C}\hat{Y}_1}\Pi_*(\zeta\mathbb{Q}\hat{W}).\end{aligned}$$

From the above equation, we can get the proof of Theorem 5.2. □

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