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SOME RESULTS ON GEODESICS AND F-GEODESICS IN TANGENT BUNDLE WITH φ-SASAKIAN METRIC OVER PARA-KÄHLER-NORDEN MANIFOLD

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Abstract. In this paper, we investigate some geodesics and F-geodesics problems on tangent bundle and φ-unit tangent bundle $T_1^{\varphi}M$ equipped with the φ-Sasaki metric over para-Kähler-Norden manifold (M^{2m}, φ, g).

Keywords: Para-Kähler-Norden manifold, φ-Sasaki metric, geodesics, F-geodesics.

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1. Introduction

On the tangent bundle of a Riemannian manifold one can define natural Riemannian metrics. Their construction makes use of the Levi-Civita connection. Among them, the so called Sasaki metric [15] is of particular interest. That is why the geometry of tangent bundle equipped with this metric has been studied by many authors. The rigidity of Sasaki metric has incited some researchers to construct and study other metrics on tangent bundle. Among them, we mention [20, 22]. The geometry of tangent bundle remains a rich area of research in differential geometry to this day.

Geodesics on the tangent bundle has been studied by many authors. In particular the oblique geodesics, non-vertical geodesics and their projections onto the base manifold. Sasaki [16] and Sato [17] gave a complete description of the curves and vector fields along them which generated non-vertical geodesics on the tangent bundle and unit the tangent bundle

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respectively. They proved that the projected curves have constant geodesic curvatures (Frenet curvatures). Nagy [9] generalized these results to the case of locally symmetric base manifold. Yampolsky [18] also did the same studies on the tangent bundle and unit the tangent bundle with the Berger-type deformed Sasaki metric over Kählerian manifold, in the cases of locally symmetric base manifold and of the constant holomorphic curvature base manifold. Also, we refer to [2, 11, 13, 21].

The notion of F-planar curves generalizes the magnetic curves and implicitly the geodesics (see [5, 8]), but the notion of F-geodesic, which is slightly different from that of F-planar curve [1]. Recently, a number of articles on magnetic curves, F-planar curves and F-geodesics have been published in the mathematical literature (see [3, 4, 10]).

In previous works, [20, 22], we proposed the φ-Sasaki metric on the tangent bundle over para-Kähler-Norden manifold (M^{2m}, φ, g), where we studied the para-Kähler-Norden properties on the tangent bundle and the geometry of φ-Sasaki metric on tangent bundle respectively. In this paper, after the introduction and generalities, in Section 3, we study the geodesics on φ-unit tangent bundle with respect to the φ-Sasaki metric, where we establish necessary and sufficient conditions under which a curve be a geodesic with respect to this metric (Theorem 3.1, Corollary3.1 and Corollary3.2), then we discuss the Frenet curvatures of the projected of the non-vertical geodesic (Theorem 3.2, Theorem 3.3, Corollary3.3 and Theorem 3.4). In section 4, we investigate the F-geodesics and F-planar curves on tangent bundle with respect to the φ-Sasaki metric (Theorem 4.1, Theorem 4.2 and Theorem 4.3). In the last section, we study the F-geodesics and F-planar curves on the φ-unit tangent bundle with respect to the φ-Sasaki metric (Theorem 5.1, Theorem 5.2 and Theorem 5.4).

2. Generalities on the φ-Sasaki metric

Let TM be the tangent bundle over an m-dimensional Riemannian manifold (M^m, g) and the natural (bundle) projection $\pi : TM \to M$. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, \xi^i)_{i=\overline{1,m}}$ on TM. Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

The Levi Civita connection ∇ defines a direct sum decomposition

$$T_{(x,\xi)}TM = V_{(x,\xi)}TM \oplus H_{(x,\xi)}TM.$$

of the tangent bundle to TM at any $(x, \xi) \in TM$ into vertical subspace

$$V_{(x,\xi)}TM \quad = \quad Ker(d\pi_{(x,\xi)}) = \{a^i \frac{\partial}{\partial \xi^i}|_{(x,\xi)},\ a^i \in \mathbb{R}\},$$

and the horizontal subspace

$$H_{(x,\xi)}TM \quad = \quad \{a^i \frac{\partial}{\partial x^i}|_{(x,\xi)} - a^i \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}|_{(x,\xi)},\ a^i \in \mathbb{R}\}.$$

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Let $Z = Z^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of Z are defined by

$$\begin{aligned} {}^V Z &= Z^i \frac{\partial}{\partial \xi^i}, \\ {}^H Z &= Z^i (\frac{\partial}{\partial x^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}). \end{aligned}$$

We have ${}^H(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial x^i} - \xi^j \Gamma^k_{ij} \frac{\partial}{\partial \xi^k}$ and ${}^V(\frac{\partial}{\partial x^i}) = \frac{\partial}{\partial \xi^i}$, then $({}^H(\frac{\partial}{\partial x^i}), {}^V(\frac{\partial}{\partial x^i}))_{i=\overline{1,m}}$ is a local adapted frame on TTM.

An almost product structure φ on a manifold M is a $(1,1)$-tensor field on M such that $\varphi^2 = id_M$, $\varphi \neq \pm id_M$ (id_M is the identity tensor field of type $(1,1)$ on M). The pair (M, φ) is called an almost product manifold.

An almost para-complex manifold is an almost product manifold (M, φ), such that the two eigenbundles TM^+ and TM^- associated to the two eigenvalues $+1$ and -1 of φ, respectively, have the same rank. Note that the dimension of an almost para-complex manifold is necessarily even.

An almost para-complex structure φ is integrable if the Nijenhuis tensor:

$$N_\varphi(X,Y) = [\varphi X, \varphi Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y] + [X,Y]$$

vanishes identically on M. On the other hand, in order that an almost para-complex structure be integrable, it is necessary and sufficient that we can introduce a torsion free linear connection ∇ such that $\nabla \varphi = 0$ [14].

An almost para-complex Norden manifold (M^{2m}, φ, g) is a $2m$-dimensional differentiable manifold M with an almost para-complex structure φ and a Riemannian metric g such that:

$$g(\varphi X, Y) = g(X, \varphi Y) \quad \Leftrightarrow \quad g(\varphi X, \varphi Y) = g(X,Y),$$

for any vector fields X and Y on M, in this case g is called a pure metric with respect to φ or para-Norden metric (B-metric)[14].

Also note that

$$G(X,Y) = g(\varphi X, Y), \tag{2.1}$$

is a bilinear, symmetric tensor field of type $(0,2)$ on (M,φ) and pure with respect to the paracomplex structure φ, which is called the twin (or dual) metric of g, and it plays a role similar to the Kähler form in Hermitian Geometry. Some properties of twin Norden metric are investigated in [6, 14].

A para-Kähler-Norden manifold is an almost para-complex Norden manifold (M^{2m},φ,g) such that φ is integrable i.e. $\nabla\varphi = 0$, where ∇ is the Levi-Civita connection of g [12, 14].

It is well known that if (M^{2m},φ,g) is a para-Kähler-Norden manifold, the Riemannian curvature tensor is pure [14].

Definition 2.1. [20] *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold. On the tangent bundle TM, we define a φ-Sasaki metric noted g^φ by*

$$\begin{array}{lrcl}
(1) & g^\varphi({}^HX,{}^HY)_{(x,\xi)} & = & g_x(X,Y),\\
(2) & g^\varphi({}^HX,{}^VY)_{(x,\xi)} & = & 0,\\
(3) & g^\varphi({}^VX,{}^VY)_{(x,\xi)} & = & g_x(X,\varphi Y) = G_x(X,Y),
\end{array}$$

for any vector fields X and Y on M and $(x,\xi)\in TM$, where G is the twin Norden metric of g defined by (2.1).

Theorem 2.1. [20] *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric g^φ. If ∇ (resp $\widetilde{\nabla}$) denote the Levi-Civita connection of (M^{2m},φ,g) (resp (TM,g^φ)), then we have*

$$\begin{array}{lrcl}
(1) & (\widetilde{\nabla}_{{}^HX}{}^HY)_{(x,\xi)} & = & {}^H(\nabla_XY)_{(x,\xi)} - \frac{1}{2}{}^V(R_x(X,Y)\xi),\\
(2) & (\widetilde{\nabla}_{{}^HX}{}^VY)_{(x,\xi)} & = & {}^V(\nabla_XY)_{(x,\xi)} + \frac{1}{2}{}^H(R_x(\varphi\xi,Y)X),\\
(3) & (\widetilde{\nabla}_{{}^VX}{}^HY)_{(x,\xi)} & = & \frac{1}{2}{}^H(R_x(\varphi\xi,X)Y),\\
(4) & (\widetilde{\nabla}_{{}^VX}{}^VY)_{(x,\xi)} & = & 0,
\end{array}$$

for all vector fields X and Y on M and $(x,\xi)\in TM$, where R denote the curvature tensor of (M^{2m},φ,g).

The φ-unit tangent sphere bundle over a para-Kähler-Norden manifold (M^{2m},φ,g), is the hypersurface

$$T_1^\varphi M = \big\{(x,\xi)\in TM,\, g(\xi,\varphi\xi) = 1\big\}.$$

The unit normal vector field to $T_1^{\varphi}M$ is given by

$$\mathcal{N} = {}^{V}\xi.$$

The tangential lift ${}^{T}X$ with respect to g^{φ} of a vector $X \in T_xM$ to $(x,\xi) \in T_1^{\varphi}M$ as the tangential projection of the vertical lift of X to (x,ξ) with respect to $\mathcal{N}$, that is

$${}^{T}X = {}^{V}X - g^{\varphi}_{(x,\xi)}({}^{V}X, \mathcal{N}_{(x,\xi)})\mathcal{N}_{(x,\xi)} = {}^{V}X - g_x(X,\varphi\xi){}^{V}\xi.$$

The tangent space $T_{(x,\xi)}T_1^{\varphi}M$ of $T_1^{\varphi}M$ at (x,ξ) is given by

$$T_{(x,\xi)}T_1^{\varphi}M = \{{}^{H}X + {}^{T}Y \,/\, X \in T_xM, Y \in \xi^{\perp} \subset T_xM\}.$$

where $\xi^{\perp} = \big\{Y \in T_xM,\ g(Y,\varphi\xi) = 0\big\}$, see [22].

Theorem 2.2. [22] *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. If $\widehat{\nabla}$ denote the Levi-Civita connection of φ-Sasaki metric on $T_1^{\varphi}M$, then we have the following formulas.*

$$\begin{aligned}
1.\ \widehat{\nabla}_{{}^{H}X}{}^{H}Y &= {}^{H}(\nabla_X Y) - \frac{1}{2}{}^{T}(R(X,Y)\xi),\\
2.\ \widehat{\nabla}_{{}^{H}X}{}^{T}Y &= {}^{T}(\nabla_X Y) + \frac{1}{2}{}^{H}(R(\varphi\xi,Y)X),\\
3.\ \widehat{\nabla}_{{}^{T}X}{}^{H}Y &= \frac{1}{2}{}^{H}(R(\varphi\xi,X)Y),\\
4.\ \widehat{\nabla}_{{}^{T}X}{}^{T}Y &= -g(Y,\varphi\xi){}^{T}X,
\end{aligned}$$

for all vector fields X and Y on M, where ∇ is the Levi-Civita connection and R is its curvature tensor of (M^{2m},φ,g).

3. Geodesics on φ-unit tangent bundle with the φ-Sasaki metric

Let $\Gamma = (\gamma(t),\xi(t))$ be a naturally parameterized curve on the tangent bundle TM (i.e. t is an arc length parameter on Γ), where γ is a curve on M and ξ is a vector field along this curve. Denote $\gamma'_t = \frac{d\gamma}{dt}$, $\gamma''_t = \nabla_{\gamma'_t}\gamma'_t$, $\xi'_t = \nabla_{\gamma'_t}\xi$, $\xi''_t = \nabla_{\gamma'_t}\xi'_t$ and $\Gamma'_t = \frac{d\Gamma}{dt}$. Then

$$\Gamma'_t = {}^{H}\gamma'_t + {}^{V}\xi'_t. \tag{3.2}$$

Lemma 3.1. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and $\Gamma = (\gamma(t),\xi(t))$ be a curve on $T_1^{\varphi}M$. Then we have*

$$\Gamma'_t = {}^{H}\gamma'_t + {}^{T}\xi'_t, \tag{3.3}$$

Proof. Using (3.2), we have

$$\Gamma'_t = {}^H\gamma'_t + {}^V\xi'_t = {}^H\gamma'_t + {}^T\xi'_t + g(\xi'_t, \varphi\xi)^V\xi.$$

Since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$ then $g(\xi, \varphi\xi) = 1$, on the other hand

$$0 = \gamma'_t g(\xi, \varphi\xi) = 2g(\xi'_t, \varphi\xi),$$

i.e.

$$g(\xi'_t, \varphi\xi) = 0. \tag{3.4}$$

Hence, the proof of the lemma is completed. □

From (3.3), we have

$$1 = |\gamma'_t|^2 + g(\xi'_t, \varphi\xi'_t), \tag{3.5}$$

where $|\,.\,|$ mean the norm of vectors with respect to the (M^{2m}, φ, g).

Theorem 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a curve on $T_1^{\varphi}M$. Then Γ is a geodesic on $T_1^{\varphi}M$ if and only if*

$$\begin{cases} \gamma''_t = R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = 0 \end{cases} \tag{3.6}$$

Moreover,

$$\begin{cases} |\gamma'_t| = 1 \\ |\xi'_t| = \kappa = const \end{cases} \tag{3.7}$$

i.e. t is an arc length parameter on γ.

Proof. Using formula (3.3)and Theorem 2.2, we compute the derivative $\widehat{\nabla}_{\Gamma'_t}\Gamma'_t$.

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \widehat{\nabla}_{({}^H\gamma'_t + {}^T\xi'_t)}({}^H\gamma'_t + {}^T\xi'_t) \\ &= \widehat{\nabla}_{{}^H\gamma'_t}{}^H\gamma'_t + \widehat{\nabla}_{{}^H\gamma'_t}{}^T\xi'_t + \widehat{\nabla}_{{}^T\xi'_t}{}^H\gamma'_t + \widehat{\nabla}_{{}^T\xi'_t}{}^T\xi'_t \\ &= {}^H\gamma''_t + {}^H(R(\varphi\xi, \xi'_t)\gamma'_t) + {}^T\xi''_t. \end{aligned}$$

If we put $\widehat{\nabla}_{\Gamma'_t}\Gamma'_t$ equal to zero, we find (3.6).
From (3.4) we have, $0 = \gamma'_t g(\xi'_t, \varphi\xi) = g(\xi''_t, \varphi\xi) + g(\xi'_t, \varphi\xi'_t)$ then,

$$g(\xi'_t, \varphi\xi'_t) = -g(\xi''_t, \varphi\xi),$$

using the second equation of the formula (3.6), we get $g(\xi'_t, \varphi\xi'_t) = 0$, then, by (3.5), we find,

$$|\gamma'_t| = 1.$$

On the other hand, as well

$$\gamma'_t|\xi'_t|^2 = \gamma'_t g(\xi'_t, \xi'_t) = 2g(\xi''_t, \xi'_t) = 0,$$

then $|\xi'_t| = \kappa = const$, □

A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is said to be a horizontal lift of the curve γ on M if and only if $\xi'_t = 0$ [19]. In general, the horizontal lift $\Gamma = (\gamma(t), \xi(t))$ of the curve γ on M does not belong to $T_1^{\varphi}M$, we have $\xi'_t = 0$, then $0 = 2g(\xi'_t, \varphi\xi) = \gamma'_t g(\xi, \varphi\xi)$, hence $g(\xi, \varphi\xi) = const \neq 1$ (in general). Then, we have the following corollary.

Corollary 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. If $\Gamma = (\gamma(t), \xi(t))$ is a horizontal lift of γ and $\Gamma \in T_1^{\varphi}M$, then Γ is a geodesic on $T_1^{\varphi}M$ if and only if γ is a geodesic on M.*

The curve $\Gamma = (\gamma(t), \gamma'_t(t))$ is called a natural lift of the curve γ on TM [19]. Likewise as for the natural lift, in the general case it does not belong to $T_1^{\varphi}M$. If γ is a geodesic on M. we have $\gamma''_t = 0$, then $0 = 2g(\gamma''_t, \varphi\gamma'_t) = \gamma'_t g(\gamma'_t, \varphi\gamma'_t)$, hence $g(\gamma'_t, \varphi\gamma'_t) = const \neq 1$ (in general). Then, we get the following corollary.

Corollary 3.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. If $\Gamma = (\gamma(t), \gamma'_t(t))$ is a natural lift of γ and $\Gamma \in T_1^{\varphi}M$, then Γ is a geodesic on $T_1^{\varphi}M$ if and only if γ is a geodesic on M.*

Remark 3.1. *As a reminder, note that locally we have:*

$$\gamma''_t = \sum_{l=1}^{2m}\Big(\frac{d^2\gamma^l}{dt^2} + \sum_{i,j=1}^{2m}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^l\Big)\frac{\partial}{\partial x^l}, \tag{3.8}$$

and

$$\xi'_t = \sum_{l=1}^{2m}\Big(\frac{d\xi^l}{dt} + \sum_{i,j=1}^{2m}\frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^l\Big)\frac{\partial}{\partial x^l}. \tag{3.9}$$

Example 3.1. *Let $(\mathbb{R}^2, \varphi, g)$ be a para-Kähler-Norden manifold such that*

$$g = e^{2x}dx^2 + e^{2y}dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1.$$

1) *Let γ be a curve on $\mathbb{R}^2$, such that $\gamma(t) = (x(t), y(t))$, from* (3.8)*, γ is a geodesic if and only if $\gamma_t'' = 0$ or equivalently γ satisfies the system of differential equations,*

$$\frac{d^2\gamma^l}{dt^2} + \sum_{i,j=1}^{2} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma_{ij}^l = 0 \quad \Leftrightarrow \quad \begin{cases} \dfrac{d^2x}{dt^2} + (\dfrac{dx}{dt})^2 = 0 \\ \\ \dfrac{d^2y}{dt^2} + (\dfrac{dy}{dt})^2 = 0 \end{cases}$$

$$\Leftrightarrow \quad \begin{cases} x(t) = \ln(c_1 t + c_2) \\ \\ y(t) = \ln(c_3 t + c_4) \end{cases}$$

where c_1, c_2, c_3 and c_4 are real constants, hence

$$\gamma(t) = (\ln(c_1 t + c_2), \ln(c_3 t + c_4)),\ \gamma_t'(t) = \frac{c_1}{c_1 t + c_2}\frac{\partial}{\partial x} + \frac{c_3}{c_3 t + c_4}\frac{\partial}{\partial y}.$$

On the other hand we have

$$g(\gamma_t', \varphi\gamma_t') = 1 \Leftrightarrow c_1 = \pm\sqrt{1 + c_3^2},$$

become $\Gamma_1 = (\gamma(t), \gamma_t'(t)) \in T_1^{\varphi}\mathbb{R}^2$. Hence from Corollary 3.2*, the curve Γ_1 is a geodesic on $T_1^{\varphi}\mathbb{R}^2$.*

2) *If $\Gamma_2 = (\gamma(t), \xi(t))$ is horizontal lift of γ, such that $\xi(t) = (u(t), v(t))$, from* (3.9)*, we have,*

$$\frac{d\xi^l}{dt} + \sum_{i,j=1}^{2} \frac{dx^j}{dt}\xi^i\Gamma_{ij}^l = 0 \Leftrightarrow \begin{cases} \dfrac{du}{dt} + \dfrac{dx}{dt}u = 0 \\ \dfrac{dv}{dt} + \dfrac{dy}{dt}v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = \dfrac{c_5}{c_1 t + c_2} \\ v(t) = \dfrac{c_6}{c_3 t + c_4} \end{cases}$$

where c_5 and c_6 are real constants, hence

$$\xi(t) = \frac{c_5}{c_1 t + c_2}\frac{\partial}{\partial x} + \frac{c_6}{c_3 t + c_4}\frac{\partial}{\partial y},$$

but when

$$g(\xi, \varphi\xi) = 1 \Leftrightarrow c_5 = \pm\sqrt{1 + c_6^2},$$

become $\Gamma_2 = (\gamma(t), \xi(t)) \in T_1^{\varphi}\mathbb{R}^2$. Hence from Corollary 3.1*, the curve Γ_2 is a geodesic on $T_1^{\varphi}\mathbb{R}^2$.*

Let Γ be a curve on $T_1^{\varphi}M$, the cure $\pi \circ \Gamma$ is called the projection (projected curve) of the curve Γ on M, where $\pi : T_1^{\varphi}M \to M$ is a bundle projection.

Theorem 3.2. *Let (M^{2m}, φ, g) be a locally symmetric para-Kähler-Norden manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a geodesic on $T_1^{\varphi}M$, then all Frenet curvatures of the projected curve $\pi \circ \Gamma$ are constants.*

Proof. Using the first equation of (3.7), we have $|\gamma_t'| = 1 = const$.
On the other hand we have

$$\begin{aligned}\gamma_t''' &= (\nabla_{\gamma_t'} R)(\xi_t', \varphi\xi)\gamma_t' + R(\xi_t'', \varphi\xi)\gamma_t' + R(\xi_t', \varphi\xi_t')\gamma_t' + R(\xi_t', \varphi\xi)\gamma_t'' \\ &= R(\xi_t', \varphi\xi)\gamma_t''.\end{aligned}$$

Since

$$\gamma_t' g(\gamma_t'', \gamma_t'') = 2g(\gamma_t''', \gamma_t'') = 2g(R(\xi_t', \varphi\xi)\gamma_t'', \gamma_t'') = 0,$$

hence, $|\gamma_t''| = const$.
Continuing the process, by recurrence we obtain the following

$$\gamma_t^{(p+1)} = R(\xi_t', \varphi\xi)\gamma_t^{(p)}, \quad p \geq 1 \tag{3.10}$$

and

$$\gamma_t' g(\gamma_t^{(p)}, \gamma_t^{(p)}) = 2g(\gamma_t^{(p+1)}, \gamma_t^{(p)}) = 2g(R(\xi_t', \varphi\xi)\gamma_t^{(p)}, \gamma_t^{(p)}) = 0.$$

Thus, we get

$$|\gamma_t^{(p)}| = const, \quad p \geq 1. \tag{3.11}$$

Let $\nu_1 = \gamma_t'$ be the first vector in the Frenet frame $\nu_1, \ldots, \nu_{2m-1}$ along γ and let $k_1, \ldots, k_{2m-1}$ the Frenet curvatures of γ. Then from the Frenet formulas

$$\begin{cases} (\nu_1)_t' &= k_1\nu_2 \\ (\nu_i)_t' &= -k_{i-1}\nu_{i-1} + k_i\nu_{i+1}, \quad 2 \leq i \leq 2m-2 \\ (\nu_{2m-1})_t' &= -k_{2m-2}\nu_{2m-2} \end{cases}$$

we obtain

$$\gamma_t'' = (\nu_1)_t' = k_1\nu_2. \tag{3.12}$$

Now (3.11) implies $k_1 =$ const. Next, in a similar way, we have

$$\gamma_t''' = k_1(\nu_2)_t' = -k_1^2\nu_1 + k_1k_2\nu_3. \tag{3.13}$$

and again (3.11) implies $k_2 =$ const.
By continuing the process, we finish the proof. □

Proposition 3.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. If $\Gamma = (\gamma(t), \xi(t))$ is a curve on $T_1^{\varphi}M$, then we have*

(1) *$\Phi = (\gamma(t), \varphi\xi(t))$ is a curve on $T_1^{\varphi}M$.*

(2) *Φ is a geodesic on $T_1^{\varphi}M$ if and only if Γ is a geodesic on $T_1^{\varphi}M$.*

Proof. (1) We put $\mu(t) = \varphi\xi(t)$, since $\Gamma = (\gamma(t), \xi(t)) \in T_1^{\varphi}M$, then $g(\xi, \varphi\xi) = 1$.
On the other hand, $g(\mu, \varphi\mu) = g(\varphi\xi, \varphi(\varphi\xi)) = g(\varphi\xi, \xi) = 1$ i.e.

$$\Phi(t) = (\gamma(t), \mu(t)) \in T_1^{\varphi}M.$$

(2) In a similar way proof of (3.6), and using $\mu_t' = \varphi\xi_t'$ and $\mu_t'' = \varphi\xi_t''$, we have

$$\begin{aligned} \widehat{\nabla}_{\Phi_t'}\Phi_t' &= {}^H(\gamma_t'' + R(\varphi\mu, \mu_t')\gamma_t') + {}^T\mu_t'' \\ &= {}^H\big(\gamma_t'' + R(\xi, \varphi\xi_t')\gamma_t'\big) + {}^T(\varphi\xi_t''). \end{aligned}$$

Since the Riemannian curvature tensor is pure, we get

$$\widehat{\nabla}_{\Phi_t'}\Phi_t' = {}^H\big(\gamma_t'' + R(\varphi\xi, \xi_t')\gamma_t'\big) + {}^T(\varphi\xi_t''),$$

hence,

$$\begin{aligned} \widehat{\nabla}_{\Phi_t'}\Phi_t' = 0 \quad &\Leftrightarrow \quad \begin{cases} \gamma_t'' &= -R(\varphi\xi, \xi_t')\gamma_t' \\ \varphi\xi_t'' &= 0 \end{cases} \\ &\Leftrightarrow \quad \begin{cases} \gamma_t'' &= R(\xi_t', \varphi\xi)\gamma_t' \\ \xi_t'' &= 0 \end{cases} \\ &\Leftrightarrow \quad \widehat{\nabla}_{\Gamma_t'}\Gamma_t' = 0. \end{aligned}$$

□

From Theorem 3.2 and Proposition 3.1, we have the following theorem

Theorem 3.3. *Let (M^{2m}, φ, g) be a locally symmetric para-Kähler-Norden manifold, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and $\Gamma = (\gamma(t), \xi(t))$ be a geodesic on $T_1^{\varphi}M$ then, all Frenet curvatures of the projected curve $\pi \circ \Phi$ are constants, where $\Phi = (\gamma(t), \varphi\xi(t))$.*

Now we study the geodesics on φ-unit tangent bundle with the φ-Sasaki metric over para-Kähler-Norden manifold of constant sectional curvature. From Theorem 3.1, we have the following

Corollary 3.3. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold of constant curvature $c\neq 0$, $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and $\Gamma=(\gamma(t),\xi(t))$ be a curve on $T_1^{\varphi}M$. Then Γ is a geodesic on $T_1^{\varphi}M$ if and only if*

$$\begin{cases} \gamma''_t &= cg(\varphi\xi,\gamma'_t)\xi'_t - cg(\xi'_t,\gamma'_t)\varphi\xi \\ \xi''_t &= 0 \end{cases} \tag{3.14}$$

Theorem 3.4. *Let $(\mathbb{R}^{2m},\varphi,<,>)$ be a para-Kähler-Norden real euclidean space , $T_1^{\varphi}\mathbb{R}^{2m}$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. Any geodesics $\Gamma=(\gamma(t),\xi(t))$ on $T_1^{\varphi}\mathbb{R}^{2m}$ is the following form*

$$\begin{cases} \gamma^i(t) &= a^i t + b^i \\ \xi^i(t) &= c^i t + d^i \end{cases} \tag{3.15}$$

where $\gamma(t)=(\gamma^1(t),\ldots,\gamma^{2m}(t)),\xi(t)=(\xi^1(t),\ldots,\xi^{2m-1}(t))$ and a^i,b^i,c^i,d^i are real constants.

4. F-geodesics on tangent bundle with the φ-Sasaki metric

Let (M^m,g) be an Riemannian manifold and F be a $(1,1)$-tensor field on (M^m,g). A curve γ on M is called F-planar if its speed remains, under parallel translation along the curve γ, in the distribution generated by the vector γ'_t and $F\gamma'_t$ along γ. This is equivalent to the fact that the tangent vector γ'_t satisfies

$$\gamma''_t = \varrho_1(t)\gamma'_t + \varrho_2 F\gamma'_t, \tag{4.16}$$

where ϱ_1 and ϱ_2 are some functions of the parameter t, see [5, 7, 8].
We say that a curve γ on M is an F-geodesic if γ satisfies:

$$\gamma''_t = F\gamma'_t, \tag{4.17}$$

One can see that an F-geodesic is an F-planar curve, but in general an F-planar curve is not always an F-geodesic, the F-geodesic generalize the geodesics, see [1, 3].

Let $\widetilde{\nabla}$ be the Levi-Civita connection of φ-Sasaki metric on tangent bundle TM, given in the Theorem 2.1.

Theorem 4.1. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma=(\gamma(t),\xi(t))$ on TM is an ${}^HF$-planar with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2 F\gamma'_t + R(\xi'_t,\varphi\xi)\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2 F\xi'_t \end{cases}$$

Proof. Using Theorem 2.1 and (3.2), we find

$$\begin{aligned}
\widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= \widetilde{\nabla}_{({}^H\gamma'_t + {}^V\xi'_t)}({}^H\gamma'_t + {}^V\xi'_t) \\
&= \widetilde{\nabla}_{{}^H\gamma'_t}{}^H\gamma'_t + \widetilde{\nabla}_{{}^H\gamma'_t}{}^V\xi'_t + \widetilde{\nabla}_{{}^V\xi'_t}{}^H\gamma'_t + \widetilde{\nabla}_{{}^V\xi'_t}{}^V\xi'_t \\
&= {}^H(\gamma''_t + R(\varphi\xi, \xi'_t)\gamma'_t) + {}^V\xi''_t \qquad (4.18)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1\Gamma'_t + \varrho_2{}^HF\Gamma'_t \\
&= \varrho_1({}^H\gamma'_t + {}^V\xi'_t) + \varrho_2{}^HF({}^H\gamma'_t + {}^V\xi'_t) \\
&= \varrho_1{}^H\gamma'_t + \varrho_2{}^HF{}^H\gamma'_t + \varrho_1{}^V\xi'_t + \varrho_2{}^HF{}^V\xi'_t \\
&= {}^H(\varrho_1\gamma'_t + \varrho_2F\gamma'_t) + {}^V(\varrho_1\xi'_t + \varrho_2F\xi'_t). \qquad (4.19)
\end{aligned}$$

From (4.18) and (4.19), the result immediately follows. □

Corollary 4.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is an ${}^H\varphi$-planar with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t \end{cases}$$

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 4.1, we obtain the following result.

Theorem 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is an ${}^HF$-geodesic with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = F\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = F\xi'_t \end{cases}$$

Corollary 4.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and TM its tangent bundle equipped with the φ-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is an ${}^H\varphi$-geodesic with respect to $\widetilde{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varphi\xi'_t \end{cases}$$

Theorem 4.3. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric and F be a $(1,1)$-tensor field on M. If $\Gamma=(\gamma(t),\xi(t))$ is a horizontal lift of a curve γ, then Γ is an ${}^HF$-planar curve (resp. ${}^HF$-geodesic) if and only if γ is an F-planar curve (resp. F-geodesic).*

Proof. Let γ be an F-planar with respect to ∇ on M, i.e. γ satisfies

$$\gamma''_t=\varrho_1\gamma'_t+\varrho_2F\gamma'_t,$$

where ϱ_1 and ϱ_2 are some functions of the parameter t. Since $\Gamma=(\gamma(t),\xi(t))$ is a horizontal lift of a curve γ then $\xi'_t=0$ and from (3.2), we have $\Gamma'_t={}^H\gamma'_t$. Using (4.18), we get,

$$\begin{aligned}\widetilde{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^H\gamma''_t\\ &= {}^H(\varrho_1\gamma'_t+\varrho_2F\gamma'_t)\\ &= \varrho_1{}^H\gamma'_t+\varrho_2{}^HF{}^H\gamma'_t\\ &= \varrho_1\Gamma'_t+\varrho_2{}^HF\Gamma'_t.\end{aligned}$$

i.e. Γ be an ${}^HF$-planar with respect to $\widetilde{\nabla}$. In the case of $\varrho_1=0$ and $\varrho_2=1$, we get that Γ is an ${}^HF$-geodesic if and only γ is an F-geodesic. □

Corollary 4.3. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold, TM its tangent bundle equipped with the φ-Sasaki metric. If $\Gamma=(\gamma(t),\xi(t))$ is a horizontal lift of a curve γ, then Γ is an ${}^H\varphi$-planar curve (resp., ${}^H\varphi$-geodesic) if and only if γ is an φ-planar curve (resp., φ-geodesic).*

Example 4.1. *Let $(\mathbb{R}^2,\varphi,g)$ be a para-Kähler-Norden manifold such that*

$$g=dx^2+dy^2,\quad \varphi=\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Let $\Gamma=(\gamma(t),\xi(t))$ such that $\gamma(t)=(x(t),y(t))$ and $\xi(t)=(u(t),v(t))$

1) *From the Corollary 4.2, Γ is an ${}^H\varphi$-geodesic if and only if the*

$$\left\{\begin{array}{l}\gamma''_t=\varphi\gamma'_t\\ \xi''_t=\varphi\xi'_t\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}x''=x'\\ y''=-y'\\ u''=u'\\ v''=-v'\end{array}\right.\Leftrightarrow\left\{\begin{array}{l}x(t)=a_1e^t+a_2\\ y(t)=a_3e^{-t}+a_4\\ u(t)=b_1e^t+b_2\\ v(t)=b_3e^{-t}+b_4\end{array}\right.$$

where a_i and b_i, $i = \overline{1,4}$ are real constants.

2) *From the Corollary 4.1, Γ is an ${}^H\varphi$-planar if and only if the*

$$\begin{cases} \gamma_t'' = \varrho_1\gamma_t' + \varrho_2\varphi\gamma_t' \\ \xi_t'' = \varrho_1\xi_t' + \varrho_2\varphi\xi_t' \end{cases} \Leftrightarrow \begin{cases} x'' = (\varrho_1+\varrho_2)x' \\ y'' = (\varrho_1-\varrho_2)y' \\ u'' = (\varrho_1+\varrho_2)u' \\ v'' = (\varrho_1-\varrho_2)v' \end{cases}$$

$$\Leftrightarrow \begin{cases} x(t) = \pm\int(e^{\int(\varrho_1+\varrho_2)dt})dt \\ y(t) = \pm\int(e^{\int(\varrho_1-\varrho_2)dt})dt \\ u(t) = \pm\int(e^{\int(\varrho_1+\varrho_2)dt})dt \\ v(t) = \pm\int(e^{\int(\varrho_1-\varrho_2)dt})dt \end{cases}$$

For example: $\varrho_1(t) = \dfrac{1}{t+1}$ and $\varrho_2(t) = \dfrac{1}{t-1}$, we find

$$\begin{cases} x(t) = c_1t^3 - 3c_1t + c_2 \\ y(t) = c_3\ln(t-1)^2 + c_3t + c_4 \\ u(t) = d_1t^3 - 3d_1t + d_2 \\ v(t) = d_3\ln(t-1)^2 + d_3t + d_4 \end{cases}$$

where c_i and d_i, $i = \overline{1,4}$ are real constants.

Example 4.2. *Let $(\mathbb{R}^2, \varphi, g)$ be a para-Kähler-Norden manifold such that*

$$g = e^{2x}dx^2 + e^{2y}dy^2, \quad \varphi = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \ a, b \in \mathbb{R}^*.$$

The non-null Christoffel symbols of the Riemannian connection are:

$$\Gamma_{11}^1 = \Gamma_{22}^2 = 1.$$

Let $\Gamma = (\gamma(t), \xi(t))$ be a horizontal lift of a curve γ, such that $\gamma(t) = (x(t), y(t))$ and $\xi(t) = (u(t), v(t))$ then $\xi_t' = 0$, from (3.9) we have,

$$\frac{d\xi^h}{dt} + \sum_{i,j=1}^{2}\frac{d\gamma^j}{dt}\xi^i\Gamma_{ij}^h = 0 \Leftrightarrow \begin{cases} u' + x'u = 0 \\ v' + y'v = 0 \end{cases} \Leftrightarrow \begin{cases} u(t) = \dfrac{\lambda_1}{e^{x(t)}} \\ v(t) = \dfrac{\lambda_2}{e^{y(t)}} \end{cases}$$

where λ_1, λ_2 are real constants.

γ is an F-geodesic if and only if $\gamma_t'' = F\gamma_t'$, from (3.8) we have

$$\begin{cases} x'' + (x')^2 = ax' \\ y'' + (y')^2 = by' \end{cases} \Leftrightarrow \begin{cases} x(t) = \ln(\frac{\mu_1}{a}e^{at} + \mu_2) \\ y(t) = \ln(\frac{\mu_3}{b}e^{bt} + \mu_4) \end{cases}$$

where $\mu_1, \mu_2, \mu_3, \mu_4$ *are real constants.*

The horizontal lift $\Gamma = (\ln(\frac{\mu_1}{a}e^{at} + \mu_2), \ln(\frac{\mu_3}{b}e^{bt} + \mu_4), \dfrac{\lambda_1}{\frac{\mu_1}{a}e^{at} + \mu_2}, \dfrac{\lambda_2}{\frac{\mu_3}{b}e^{bt} + \mu_4})$ *is an* ${}^{H}F$*-geodesic on* $T\mathbb{R}^2$.

γ *is an* F*-planar if and only if* $\gamma''_t = \varrho_1\gamma'_t + \varrho_2 F\gamma'_t$*, where* ϱ_1 *and* ϱ_2 *are some functions of the parameter* t*, from* (3.8) *we have*

$$\begin{cases} x'' + (x')^2 = (\varrho_1 + a\varrho_2)x' \\ y'' + (y')^2 = (\varrho_2 + b\varrho_2)y' \end{cases} \Leftrightarrow \begin{cases} x(t) = \ln(\pm\int(e^{\int(\varrho_1 + a\varrho_2)dt})dt) \\ y(t) = \ln(\pm\int(e^{\int(\varrho_1 + b\varrho_2)dt})dt) \end{cases}$$

For example: If $\varrho_1(t) = \dfrac{-1}{t}$ *and* $\varrho_2(t) = \dfrac{1}{t}$*, we find*

$$\begin{cases} x(t) = \ln(\frac{\alpha_1}{a}t^a + \alpha_2) \\ y(t) = \ln(\frac{\alpha_3}{b}t^b + \alpha_4) \\ u(t) = \dfrac{\lambda_1}{\frac{\alpha_1}{a}t^a + \alpha_2} \\ v(t) = \dfrac{\lambda_2}{\frac{\alpha_3}{b}t^b + \alpha_4} \end{cases}$$

where α_i*,* $i = \overline{1,4}$ *are real constants, then* $\Gamma = (x(t), y(t), u(t), v(t))$ *is an* ${}^{H}\varphi$*-planar on* $T\mathbb{R}^2$.

5. F-geodesics on φ-unit tangent bundle with the φ-Sasaki metric

Let $\widehat{\nabla}$ be the Levi-Civita connection of φ-Sasaki metric on φ-unit tangent bundle $T_1^{\varphi}M$, given in the Theorem 2.2.

Theorem 5.1. *Let* (M^{2m}, φ, g) *be a para-Kähler-Norden manifold and* $T_1^{\varphi}M$ *its* φ*-unit tangent bundle equipped with the* φ*-Sasaki metric and* F *be a* $(1,1)$*-tensor field on* M*. A curve* $\Gamma = (\gamma(t), \xi(t))$ *on* $T_1^{\varphi}M$ *is an* ${}^{H}F$*-planar with respect to* $\widehat{\nabla}$ *if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2 F\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2 F\xi'_t \end{cases}$$

Proof. From the proof of Theorem 3.1, we find

$$\widehat{\nabla}_{\Gamma'_t}\Gamma'_t = {}^{H}(\gamma''_t + R(\varphi\xi, \xi'_t)\gamma'_t) + {}^{T}\xi''_t. \tag{5.20}$$

On the other hand,

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1\Gamma'_t + \varrho_2{}^{H}F\Gamma'_t \\ &= \varrho_1({}^{H}\gamma'_t + {}^{T}\xi'_t) + \varrho_2{}^{H}F({}^{H}\gamma'_t + {}^{T}\xi'_t). \end{aligned}$$

From (3.4), we have ${}^{T}\xi'_t = {}^{V}\xi'_t$, then

$$\begin{aligned}\widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= \varrho_1{}^{H}\gamma'_t + \varrho_2{}^{H}F{}^{H}\gamma'_t + \varrho_1{}^{V}\xi'_t + \varrho_2{}^{H}F{}^{V}\xi'_t \\ &= {}^{H}(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^{V}(\varrho_1\xi'_t + \varrho_2 F\xi'_t) \\ &= {}^{H}(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) + {}^{T}(\varrho_1\xi'_t + \varrho_2 F\xi'_t) \qquad (5.21)\end{aligned}$$

From (5.20) and (5.21), the result immediately follows. □

Corollary 5.1. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-planar with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varrho_1\gamma'_t + \varrho_2\varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varrho_1\xi'_t + \varrho_2\varphi\xi'_t \end{cases}$$

In the particular case when $\varrho_1 = 0$ and $\varrho_2 = 1$ in the Theorem 5.1, we obtain the following result.

Theorem 5.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and F be a $(1,1)$-tensor field on M. A curve $\Gamma = (\gamma(t), \xi(t))$ on TM is an ${}^{H}F$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = F\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = F\xi'_t \end{cases}$$

Corollary 5.2. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}\varphi$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = \varphi\gamma'_t + R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = \varphi\xi'_t \end{cases}$$

Theorem 5.3. *Let (M^{2m}, φ, g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. A curve $\Gamma = (\gamma(t), \xi(t))$ on $T_1^{\varphi}M$ is an ${}^{H}(R(\xi'_t, \varphi\xi))$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = 2R(\xi'_t, \varphi\xi)\gamma'_t \\ \xi''_t = R(\xi'_t, \varphi\xi)\xi'_t \end{cases}$$

Corollary 5.3. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold of constant sectional curvature $c\neq 0$ and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric. A curve $\Gamma=(\gamma(t),\xi(t))$ on $T_1^{\varphi}M$ is an ${}^H(R(\xi'_t,\varphi\xi))$-geodesic with respect to $\widehat{\nabla}$ if and only if the*

$$\begin{cases} \gamma''_t = 2c(g(\varphi\xi,\gamma'_t)\xi'_t - g(\xi'_t,\gamma'_t)\varphi\xi) \\ \xi''_t = -cg(\xi'_t,\xi'_t)\varphi\xi \end{cases}$$

Theorem 5.4. *Let (M^{2m},φ,g) be a para-Kähler-Norden manifold and $T_1^{\varphi}M$ its φ-unit tangent bundle equipped with the φ-Sasaki metric and F be a $(1,1)$-tensor field on M. If $\Gamma=(\gamma(t),\xi(t))$ is a horizontal lift of γ and $\Gamma\in T_1^{\varphi}M$, then Γ is an ${}^HF$-planar curve (resp., ${}^HF$-geodesic) if and only if γ is an F-planar curve (resp., F-geodesic).*

Proof. Let γ be an F-planar with respect to ∇ on M, i.e. γ satisfies

$$\gamma''_t = \varrho_1\gamma'_t + \varrho_2 F\gamma'_t,$$

where ϱ_1 and ϱ_2 are some functions of the parameter t. Since $\Gamma=(\gamma(t),\xi(t))$ is a horizontal lift of a curve γ then $\xi'_t=0$ and from (5.20), we have,

$$\begin{aligned} \widehat{\nabla}_{\Gamma'_t}\Gamma'_t &= {}^H\gamma''_t \\ &= {}^H(\varrho_1\gamma'_t + \varrho_2 F\gamma'_t) \\ &= \varrho_1{}^H\gamma'_t + \varrho_2{}^HF{}^H\gamma'_t \\ &= \varrho_1\Gamma'_t + \varrho_2{}^HF\Gamma'_t. \end{aligned}$$

i.e. Γ be an ${}^HF$-planar with respect to $\widehat{\nabla}$. In the case of $\varrho_1=0$ and $\varrho_2=1$, we get that Γ is an ${}^HF$-geodesic if and only γ is an F-geodesic. □

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