International Journal of Maps in Mathematics

Volume 7, Issue 2, 2024, Pages:335-355

E-ISSN: 2636-7467

www.journalmim.com

ALGORITHMIC APPROACH TO BITONIC ALGEBRAS AND THEIR GRAPHS

SULE AYAR OZBAL, REFET POLAT *, SAADET ESKIIZMIRLILER, AND MEHMET KURT

Abstract. Under the aim of this paper, we establish the terms of graphs related with bitonic-algebras, which is a bitonic-graph where the vertices are the elements of bitonic algebra and where the edges are the companian of two vertices, that is two elements from bitonic algebra. We designate the upper sets of elements in a bitonic algebra and studied properties of these sets. We state algorithms to check whether the given set is a bitonic algebra or a commutative bitonic algebra or not. Additionally, we mention the codes of these algorithms. Moreover, we associate the algorithms of graphs of a bitonic algebra and state properties of these graphs obtained.

Keywords: Bitonic algebras, upper sets, graphs, graphs of algebras

2010 Mathematics Subject Classification: 03G25, 05C25, 13P25, 33F99, 68W30

1. Introduction

In recent mathematical articles and studies, it has been an important matter that the artificial intelligence is to make a computer simulate a human being in dealing with certainty and uncertainty in information. At this stage, logic plays an important role to act the foundation of this mission. Classical logic is a base for information processing dealing with certain information whereas nonclassical logic including many-valued logic and fuzzy logic

Received:2024.03.29 *Revised:2024.07.01* *Accepted:2024.07.24*

* Corresponding author

Sule Ayar Ozbal ⋄ sule.ayar@yasar.edu.tr ⋄ https://orcid.org/0000-0001-5933-5858

Refet Polat ⋄ refet.polat@yasar.edu.tr ⋄ https://orcid.org/0000-0001-9761-8787

Saadet Eskiizmirliler ⋄ saadet.eskiizmirliler@yasar.edu.tr ⋄ https://orcid.org/0000-0001-5736-1278

Mehmet Kurt ⋄ kurt.mehmet@gmail.com ⋄ https://orcid.org/0000-0002-7308-5661.

use classical logic to handle information with various facets of uncertainty, such as fuzziness and randomness. For this reason, for computer science to deal with fuzzy information and uncertain information, non classical logic has been used as a useful and a formal instrument.

As required for this reason, BCK-algebras are considered as a generalization of the notion of algebra of sets with the set subtraction as the only fundamental nonnullary operation and on the other hand the notion of implication algebra. They are defined by Imai and Iseki in [19]. Later, Komori [24] introduced the new class of algebras called BCC-algebras as he gave the name of this work to prove the class of all BCK-algebras does not form a variety. The notion of dual BCC-algebras is a generalization of many different algebras such as DBCK-algebras ([9, 23, 40]), Hilbert-algebras([13, 16, 17, 27]), Heyting-algebras (or Brouwerian lattices)([11, 8]), implication algebras ([1]) and lattice implication algebras ([37, 38])that ensure the property: (P) $x \leq y$ implies $z * x \leq z * y$ and $y * z \leq x * z$. Lastly, Yon and Özbal in [39] defined the bitonic algebras and with the help of derivations they studied properties of this algebra. Additionally, Özbal studied on filters of bitonic algebras to investigate the relations between filters and upper sets in [5].

With the help of graphs to deeply study algebraic systems, graphs have been considered a significant method and topic in many mathematical papers and studies in recent times. For instance, in 1998 Beck [7] studied rings and algebras in this manner. In this study, by presenting the zero-divisor graph of a commutative ring a correlation between graph theory and commutative ring theory is constructed. Then, many mathematicians extend this graphs in classical structures more definitely, in commutative ring [3], commutative semirings [35] and semigroups [25], near-rings [36], Cayley Vague Graphs [28]. These studies consider graphs in classical and non-classical algebras. The total graph of a commutative ring is studied in [6] and investigated the total graph of a commutative semiring with non-zero identity. Also, the annihilator graph of a commutative ring is considered in [2] and the area of zero-divisor graphs of commutative rings is focused in [12]. Brešar et al.[10] defined the cover incomparability graphs of posets and the directed graphs of lattices is examined in [32]. Nowadays, many mathematicians have focused on graph of logical algebras because of the reason that these algebras are related to information systems and many other different branches of computer sciences. For example, Jun and Lee [20] studied zero-divisor graph in BCK/BCI-algebras whereas Hu and Li [18] obtained some properties on graphs of BCH-algebras. Additionally, Gürsoy et al. introduced an alternative construction of graphs on MV-algebras in [15] and they obtained a suitable representation of MV-algebras by using graphs. Similarly, Kırcalı

Gürsoy focused on the notion of graphs on Wajsberg algebras and stated that commutative W-graphs are also symmetric graphs in [22]. And, many other graph operations are studied in [21], such as coloring of a commutative ring in [4], and these will be applied on graphs of bitonic algebras in the future work.

Motivated by these works, in this paper we study the associated graphs of bitonic algebras that are the generalizations of dual BCC-algebras in a different manner than those mentioned above.

2. Preliminaries

During this section, firstly, we give some fundamental definitions, lemmas, theorems about bitonic algebras that will be used as a tool. Secondly, we remind some graph theory concepts used in this study.

2.1. **Bitonic Algebras.** A *dual BCC-algebra* is an algebraic system $(X, *, 1)$ satisfying the following axioms for all $x, y, z \in X$:

(D1) $(x * y) * ((y * z) * (x * z)) = 1$,

(D2) $1 * x = x$,

(D3) $x * 1 = 1$,

(D4) $x * x = 1$,

(D5) $x * y = 1$ and $y * x = 1$ imply $x = y$.

Definition 2.1. [39] *A bitonic algebra is an algebraic system $(A, *, 1)$ satisfying the following axioms for every $a, b, c \in A$:*

(B1) $a * 1 = 1$,

(B2) $1 * a = a$,

(B3) $a * b = 1$ *and* $b * a = 1$ *implies* $a = b$,

(B4) $a * b = 1$ *implies* $(c * a) * (c * b) = 1$ *and* $(b * c) * (a * c) = 1$.

where A is a set, 1 an element in A and $*$ a binary operation on A.

Lemma 2.1. [39] *In a bitonic algebra $(A, *, 1)$ for all $a, b, c \in A$ the followings hold:*

(1) $a * a = 1$,

(2) $a * b = b * c = 1$ *implies* $a * c = 1$,

(3) $a * (b * a) = 1$.

Corollary 2.1. [39] *If a binary relation "$\leq$" on A where $(A, *, 1)$ be a bitonic algebra is defined by*

$$a \leq b \iff a * b = 1$$

for any $a, b \in A$, then it is clear that by (B3) and Lemma 2.1 $\leq$ is a partial order on A.

Lemma 2.2. [39] *Let $(A, *, 1)$ be a bitonic algebra. Then for all $a, b, c \in A$:*

(1) *$a \leq b$ refers $c * a \leq c * b$ and $b * c \leq a * c$,*

(2) *$a \leq b * a$.*

Example 2.1. [39] *Let $*$ be a binary operation on $N = \{1, x, y, z, w\}$ with the table given below:*

Table 1. Cayley Table for N

$*$	1	x	y	z	w
1	1	x	y	z	w
x	1	1	y	z	w
y	1	x	1	z	w
z	1	1	1	1	x
w	1	1	1	z	1

$(N, *, 1)$ is a bitonic algebra and the Hasse diagram can be given as

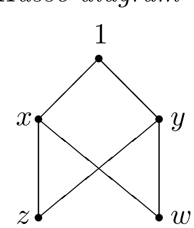

Figure 1. Hasse diagram of the bitonic algebra N in Example 2.1

Definition 2.2. *Let $(A, *, 1)$ be a bitonic algebra. The binary operation "$\Diamond$" on A is defined by $a \Diamond b = (a * b) * b$ for every $a, b \in A$.*

Lemma 2.3. [39] *For the binary operation $\Diamond$ on $(A, *, 1)$ as a bitonic algebra*

(1) *$b \leq a \Diamond b$,*

(2) $a \leq b$ *implies* $a \Diamond b = b$,

(3) $1 \Diamond a = 1$ *and* $a \Diamond 1 = 1$

hold for every $a, b \in A$.

Definition 2.3. [39] $S \neq \emptyset$ *as a subset of a bitonic algebra* A *is called a bitonic subalgebra of* A *if "*$x * y \in S$*" for all* $x, y \in S$*, and* $F \neq \emptyset$ *as a subset of* A *is called a filter of* A *if*

(F1) $1 \in F$,

(F2) $x \in F$ *with* $x * y \in F$ *refers* $y \in F$ *for any* $x, y \in F$.

Definition 2.4. [39] *A bitonic algebra* $(A, *, 1)$ *is said to be commutative if*

$$(a * b) * b = (b * a) * a \text{ for every } a, b \in A.$$

Example 2.2. *Let* $*$ *be a binary operation on* $C = \{1, a, b\}$ *with the table given below then*

Table 2. Cayley Table for C

$*$	1	a	b
1	1	a	b
a	1	1	b
b	1	a	1

$(C, *, 1)$ *is a commutative bitonic algebra.*

2.2. **Some Basic Concepts on Graph Theory.** Here, we consider some basic definitions and concepts in graph theory.

Definition 2.5. [14] *A graph is a pair* $G = (V, E)$ *of sets satisfying* $E \subseteq V \times V$*; thus, the elements of* E *are ordered pairs of* V *. To avoid notational ambiguities, we shall always assume tacitly that* $V \cap E = \emptyset$*. The elements of* V *are the vertices (or nodes, or points) of the graph* G*, the elements of* E *are its edge edges (or lines). The usual way to picture a graph is by drawing a dot for each vertex and joining two of these dots by a line if the corresponding two vertices form an edge.*

Example 2.3. *Let* $V = \{1, 2, 3, 4, 5\}$ *be a vertex set and* $E = \{(1, 2), (2, 3), (3, 5), (3, 4), (4, 5)\}$ *be a edge set of* $G = (V, E)$. *We can illustrate this graph as Figure 2.*

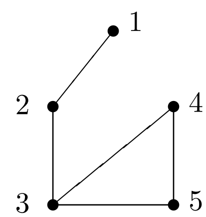

FIGURE 2. The graph of $G = (V, E)$ for Example 2.3

Definition 2.6. [14] *A path is a non-empty graph* $P = (V, E)$ *of the form*

$$V = \{x_0, x_1, \ldots, x_k\},\ E = \{(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_k)\}$$

where the x_i *are all distinct. The vertices* x_0 *and* x_k *are linked by* P *and are called its ends; the vertices* $x_1, x_2, \ldots, x_{k-1}$ *are inner vertices of* P. *The number of edges of a path is its length, and the path of length* k *ise denoted by* P^k.

Definition 2.7. [14] *The distance* $d_G(x, y)$ *in* G *of two vertices* x, y *is the length of a shortest* $x - y$ *path in* G*; if no such path exist, we set* $d(x, y) = \infty$. *The greatest distance between any two vertices in* G *is the diameter of* G*, diameter denoted by diam(G). The diameter of* G *is said to be zero if there is only one vertex in* G.

Definition 2.8. [26, 34] *A connected graph with more than one vertex has a diameter of one if and only if each pair of distinct vertices forms an edge and it is called a complete graph.*

3. Some New Algorithms on Bitonic Algebra

In this section, we will introduce the algorithms that test whether the definitions given in the previous section are ensured by any sets. These algorithms are coded in VB-Script Language in Ms EXCEL Program to make our studies on these algebras easier than usual. After explaining these mentioned algorithms we will share the codes belongs to this language.

Algorithm 1: Determining Bitonic Algebra

Data: Set A, * operation, operator table

Result: (A, *, 1) is a bitonic algebra or not

```
initialization;
if A ≠ ∅ OR 1 ∉ A then
    Return false
foreach x in A do
    if x * 1 ≠ 1 then
        Return false
    if 1 * x ≠ x then
        Return false
foreach x, y in A do
    if x * y = 1 AND y * x = 1 then
        if x ≠ y then
            Return false
foreach x, y in A do
    if x * y = 1 then
        foreach z in A do
            if (z * x) * (z * y) ≠ 1 OR (y * z) * (x * z) ≠ 1 then
                Return false
```

In Algorithm 1, it is checked whether the given structure is a bitonic algebra or not by using Definition 2.1 of bitonic algebras. The inputs of this algorithm are the set A and the Cayley Table of the operator $*$. Firstly, the algorithm examines whether the given set A is an empty set, and 1 belongs to A. If the given set A is empty or does not contain 1, then the algorithm returns $FALSE$. Then, it examines for every x in A, whether the conditions $B1$ and $B2$ of Definition 2.1 are satisfied. If any of these conditions are not satisfied then the algorithm returns $FALSE$.

The following step of the algorithm is to check the condition $B3$ of Definition 2.1. According to this step, whenever $x * y = 1$ or $y * x = 1$ is satisfied for any x, y in A, it is checked whether these x and y are different elements. If these elements are different than each other then the algorithm returns $FALSE$. The last step of the algorithm checks whether the last part $B4$ of Definition 2.1 is satisfied or not. According to this step, it is examined whether $x*y = 1$ is satisfied or not for any elements x, y in A. If this is satisfied for any x, y in A, then for any element in A (say z in A) the algorithm works out for the solution of $(z * x) * (z * y)$ and $(y * z) * (x * z)$. If one these results is different than 1, then the algorithm $FALSE$.

```
'The Cayley Table is being imported into a two-dimensional array
For i = 2 To m + 1
    For j = 2 To m + 1
        m1(i - 1, j - 1) = Cells(i, j)
    Next j
Next i

'The conditions B1 and B2 are being checked
control = 0
For i = 1 To m
    If m1(1, i) <> Cells(1, i + 1) Then control = 1
    If m1(i, 1) <> 1 Then control = 1
Next i

If control <> 0 Then
    MsgBox "B1 OR B2 are not obtained"
    GoTo bit
End If

'The condition B3 is being checked
For i = 1 To m
    For j = 1 To m
        If m1(i, j) = 1 Then
            If m1(j, i) = 1 Then
                If i <> j Then
                    MsgBox "B3 is not obtained"
```

```
                    control = 1
                    GoTo bit
                End If
            End If
        End If
    Next j
Next i

'The condition B4 is being checked
For i = 1 To m
    For j = 1 To m
        If m1(i, j) = 1 Then
            For k = 1 To m
                If (m1(m1(k, i), m1(k, j)) = 1 And m1(m1(j, k), m1(i, k)) = 1) <> True Then
                    MsgBox "B4 is not appropriate (" & k & "*" & i & ")*(" & k & "*" & j & ") or one of these  (" & j & "*" & k & ")*(" & i & "*" & k & ") is not 1"
                    control = 1
                    GoTo bit
                End If
            Next k
        End If
    Next j
Next i

If control = 0 Then
    Cells(12, 1) = "Bitonic Algebra"
    Else
    Cells(12, 1) = "Not a Bitonic Algebra"
    GoTo bit
End If
bit:
```

Listing 1. Codes for Algorithm 1

We consider a code for this table that will be easily runned in EXCEL to list its entries.

Algorithm 2: Generating $\Diamond$-Operation Table on a Bitonic Algebra

Data: (A, 1, *) Bitonic Algebra, * Operation Table

Result: $\Diamond$-Table[,]

$\Diamond$-Table[,]=*null*

foreach x, y **in** A **do**

  $\Diamond$-Table[x,y]=(x*y)*y

The Cayley table of the $\Diamond$ operator is obtained with the help of the Algorithm 2. This time, the input of this algorithm is a bitonic algebra. In the beginning, an empty Cayley table of the $\Diamond$ operator is formed. Then, the solution of $(x * y) * y$ is examined for every element x, y in A.

One of the main purpose of this algorithm is that this algorithm is coded in VBScript language and these codes are given in Listing 2.

```
For i = 1 To m
    For j = 1 To m
        m1v(i, j) = m1(m1(i, j), j)
    Next j
Next i
```

Listing 2. Codes for Algorithm 2

Algorithm 3: Determining Commutativity of a Bitonic Algebra

Data: $\Diamond$-Table[,] of (A, 1, *) Bitonic Algebra

Result: (A, *, 1) is a Commutative Bitonic Algebra or not

initialization;

foreach x, y **in** A **do**

  if $x \Diamond y \neq y \Diamond x$ **then**

    Return $false$

In this Algorithm 3, it is checked whether the given bitonic algebra is commutative or not. The inputs of this algorithm are a bitonic algebra and the cayley table of $\Diamond$ operation defined in this bitonic algebra. If this table is symmetric then the given bitonic algebra is commutative. For this reason, it is examined whether $x\Diamond y$ and $y\Diamond x$ are the same or not for any x, y in the given bitonic algebra. If a non-identical condition is detected, then the algorithm outputs $FALSE$.

This algorithm is coded in VBScript language and these codes are given in Listing 3

```
controlv = 0
For i = 1 To m - 1
    For j = i + 1 To m
    If (m1v(i, j) <> m1v(j, i)) Then controlv = 1
    Next j
Next i

If controlv = 0 Then
    Cells(12, 13) = "Commutative Bitonic Algebra"
    Else
    Cells(12, 13) = "Not a Commutative Bitonic Algebra"
    GoTo bit
End If
bit:
```

Listing 3. Codes for Algorithm 3

4. Graphs on Bitonic Algebras

To consider graphs of bitonic algebras, firstly we focus on upper sets of these algebras. In this section, initially we give the definition of graph of bitonic algebras and introduce the algorithm lists the entries of adjacency matrix on a bitonic algebra. Moreover, we consider the codes of this algorithm to list the entries of the adjaceny matrix on a bitonic algebra.

Definition 4.1. $(A, *, 1)$ *as a commutative bitonic algebra corresponds to an undirected graph* $G(A)$*, where* $V(G(A))$ *consists of the elements of* A *and two distinct elements* $a, b \in A$ *are called adjacent if and only if* $a \Diamond b = 1$*.* G *is said to be a* $A-$*graph under these conditions.*

Algorithm 4: Generating Adjacency Matrix on a Bitonic Algebra

Data: $\Diamond$-Table[,] of (A, 1, *) Bitonic Algebra

Result: $Adj(G(A))[\ ,\ ]$ Adjacency Matrix

initialization;

foreach x, y **in** A **do**

  if $x \Diamond y = 1$ **then**

    $Adj(G(A))[x, y] = 1$

  else

    $Adj(G(A))[x, y] = 0$

In this Algorithm 4, the adjacency matrix for any bitonic algebra given according to the Definition 4.1 is created. The inputs of this algorithm are a bitonic algebra and the Cayley table of $\Diamond$ operation defined in this algebra. In the beginning, the solution of $x \Diamond y$ for any elements x, y in the given bitonic algebra is studied. If this solution is 1, then the corresponding element in a two-dimensional array in the adjacency matrix is assigned as 1, otherwise it is assigned as 0.

Coding in this algorithm VBScript language in MS EXCEL programe is one the most important part of this paper and these codes are given is Listing 4.

```
For i = 1 To m
    For j = 1 To m
        If m1v(i, j) = 1 Then
            m1vadj(i, j) = 1
            Else
            m1vadj(i, j) = 0
        End If
    Next j
Next i
```

LISTING 4. Codes for Algorithm 4

Example 4.1. *Let* $(A, *, 1)$ *be the bitonic algebra mentione in Example 2.2. Using the Definition 4.1, the adjacency matrix of the graph of A is*

$$Adj(G(A))= \begin{array}{c|ccc} & 1 & a & b \\ \hline 1 & 1 & 1 & 1 \\ a & 1 & 0 & 1 \\ b & 1 & 1 & 0 \end{array}$$

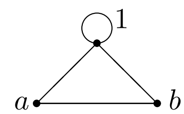

FIGURE 3. The graph of the bitonic algebra A given in Example 2.2 with the Hasse Diagram

Example 4.2. *Let* $(A, *, 1)$ *be the bitonic algebra given in Example 2.1. Using the Definition 4.1, the adjacency matrix of the graph of A is:*

$$Adj(G(A))= \begin{array}{c|ccccc} & 1 & x & y & z & w \\ \hline 1 & 1 & 1 & 1 & 1 & 1 \\ x & 1 & 0 & 1 & 1 & 1 \\ y & 1 & 1 & 0 & 1 & 1 \\ z & 1 & 0 & 0 & 0 & 0 \\ w & 1 & 0 & 0 & 1 & 0 \end{array}$$

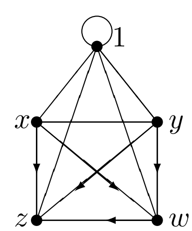

FIGURE 4. The graph of the bitonic algebra A given in Example 2.1

Lemma 4.1. *Vertex* 1 *in a* A*-graph is adjacent to all vertices in* $G(A)$.

Proof. By definition of a bitonic algebra $a\Diamond 1 = (a * 1) * 1 = 1 * 1 = 1$ for ever $a \in A$. So, vertex 1 and vertex a are connected with an edge in every A-graph. $\square$

Lemma 4.2. *A-graph $G(A)$ is connected by having $diam(G(A)) \leq 2$.*

Proof. Let $a, b \in A$ be any two distinct vertices on $G(A)$. Assume that $a \Diamond b = 1$. Therefore, we get $d(a, b) = 1$, so have $diam(G(A)) \leq 2$. Now, assume that $a \Diamond b \neq 1$. For a bitonic algebra we have $a \Diamond 1 = (a * 1) * 1 = 1 * 1 = 1$ and $b \Diamond 1 = (b * 1) * 1$. That is to say that a is adjacent to 1 and b is adjacent to 1. Hence, we get $d(a, b) \leq 2$ meaning that $diam(G(A)) \leq 2$. □

Definition 4.2. *$(A, *, 1)$ as a bitonic algebra is said to be $self-distributive$ if $a * (b * c) = (a * b) * (a * c)$ for all $a, b, c \in A$.*

Example 4.3. *The bitonic algebra $(A, *, 1)$ given in Example 1 is not a self-distributive algebra. But, if $*$ on $A = \{1, x\}$ is defined as a binary relation whose table is*

$*$	1	x
1	1	x
x	1	1

It is easy to see that $(A, *, 1)$ a self-distributive bitonic algebra.

We will consider the upper set of a as an element in a bitonic algebra A by

$$U(1, a) = \{x \in A | 1 * (a * x) = 1\}$$

for each $a \in A$.

For the rest of the paper $(A, *, 1) = A$ is given as a self-distributive bitonic algebra.

Proposition 4.1. *For any $a \in A$, the upper set $U(1, a)$ is a filter of A.*

Proof. Let $a \in A$. Then by $(B1)$, $(B2)$ we have for all $a \in A$ $a * 1 = 1$ and $1 * (a * 1) = 1$. Therefore, $1 \in U(1, a)$. Now, let $y \in U(1, a)$ and $y * x \in U(1, a)$ for any $x, y \in A$. Then we have $a * y = 1$ and $a * (y * x) = 1$. Since A is a self-distributive bitonic algebra we have

$$a * (y * x) = (a * y) * (a * x) = 1 * (a * x) = (a * x) = 1.$$

Therefore, $x \in U(1, a)$. Hence we get $U(1, a)$ is a filter of A. □

Proposition 4.2. *Let $B, C \subseteq A$, then*

(1) If $B \subseteq C$ then $U(1, C) \subseteq U(1, B)$,

(2) $U(1, B \cup C) = U(1, B) \cap U(1, C)$,

(3) $U(1,B) \cup U(1,C) \subseteq U(1,B \cap C)$.

Proof. Let $B, C \subseteq A$.

(1) Let $B \subseteq C$ and suppose that $x \in U(1,C)$. So we have $1 * (c * x) = 1$ and $c * x = 1$ for all $c \in C$. But we know that $B \subseteq C$ so every element of B is in C therefore, $b * x = 1 * (b * x) = 1$ for all $b \in B \subseteq C$. Therefore, $x \in U(1,B)$ that is to say that $U(1,C) \subseteq U(1,B)$.

(2) We know that $B \subseteq B \cup C$ and $C \subseteq B \cup C$, and by part 1 we have $U(1,B \cup C) \subseteq U(1,B)$ and $U(1,C)$. Then $U(1,B \cup C) \subseteq U(1,B) \cap U(1,C)$.

Now, conversely, let $x \in U(1,B) \cap U(1,C)$. Then we have $1 * (b * x) = 1$ that is $b * x = 1$ for all $b \in B$ and similarly $1 * (c * x) = 1$ that is $(c * x) = 1$ for all $c \in C$. Therefore, for any $a \in B \cup C$ we have $a \in B$ or $a \in C$, and hence $1 * (a * x) = 1$ that is $(a * x) = 1$ for all $a \in B \cup C$. And so we get $x \in U(1, B \cup C)$ gives us $U(1,B) \cap U(1,C) \subseteq U(1,B \cup C)$. Therefore, $U(1,B \cup C) = U(1,B) \cap U(1,C)$.

(3) We have $B \cap C \subseteq B$ and $B \cap C \subseteq C$ and also by part 1 of this proposition we have $U(1,B) \subseteq U(1,B \cap C)$ and $U(1,C) \subseteq U(1,B \cap C)$. Therefore, we get $U(1,B) \cup U(1,C) \subseteq U(1,B \cap C)$.

□

Proposition 4.3. *If $B \neq \emptyset$ is a subset of A then,* $U(1,B) = \bigcap_{b \in B} U(1,b)$.

Proof. Let B be a non-empty subset of a self-distributive bitonic algebra A. We have $B = \bigcup_{b \in B}\{b\}$, and by Proposition 4.2 (2) we have

$$U(1,B) = U(1, \bigcup_{b \in B}\{b\}) = \bigcap_{b \in B} U(1,b).$$

□

Proposition 4.4. *If $a \leq b$ for any $a, b \in A$ then $U(1,a) \subseteq U(1,b)$.*

Proof. Let $x \in U(1,a)$. Then we get $1 * (a * x) = 1$ that is $a * x = 1$. And, by our assumption we have $a \leq b$ and by Lemma 2.2 (1) $(b * x) \leq (a * x)$. Hence we get $(b * x) = 1$ since $(a * x) = 1$. Therefore, $x \in U(1,b)$. □

Definition 4.3. *Let A be bitonic algebra $(A, *, 1)$ that is $self-distributive$. Define a relation $\sim$ on A as $a \sim b \Leftrightarrow U(1,a) = U(1,b)$.*

Lemma 4.3. *The relation forms an equivalence relation on a bitonic algebra* $(A, *, 1)$ *that is* $self - distributive$.

Lemma 4.4. *For every* $a, b, c \in (A, *, 1)$

$$a * (b * c) = b * (a * c).$$

Proof. Let $a, b, c \in A$. Then

$$a * (b * c) = (a * b) * (a * c),$$

and since $b \leq a * b$, $(a * b) * (a * c) \leq b * (a * c)$. This implies $a * (b * c) \leq b * (a * c)$.

Interchanging the role of a and b, we can show $b * (a * c) \leq a * (b * c)$. Hence $a * (b * c) = b * (a * c)$. □

On the definition of *upper set*, since $1 * a = a$ for every $a \in A$, we should define the upper set by

$$U(a) = \{x \in A \mid a * x = 1\}.$$

Definition 4.4. *Let* $U(x)$ *and* $U(y)$ *be the upper sets of the bitonic algebra* $(A, *, 1)$ *for all* $x, y \in A$. *Then we can define* "$\diamondsuit$" *among the upper sets* $U(x)$ *and* $U(y)$ *as follow*

$$U(x) \diamondsuit U(y) = \{z \in A | a \diamondsuit b = z \ \ for \ \ all \ \ a \in U(x) \ \ and \ \ for \ \ all \ \ b \in U(y)\}.$$

Then we can define the graph $G_U(A)$, *where* $V(G_U(A))$ *consists of the elements of* $U(A)$, *and two distinct elements* $U(a), U(b) \in U(A)$ *are called adjacent if and only if* $U(a) \diamondsuit U(b) = \{1\}$.

Example 4.4. *Let* A *be the bitonic algebra given in Example 2.1. Then we can consider the upper sets for every elements of* A *as given below.*

$U(1) = \{1\}, U(x) = \{1, x\}, U(y) = \{1, y\}, U(z) = \{1, x, y, z\}, U(w) = \{1, x, y, w\}$.

Then, we can give the table of $\diamondsuit$ *among these upper sets as follow.*

$\diamondsuit$	$U(1)$	$U(x)$	$U(y)$	$U(z)$	$U(w)$
$U(1)$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
$U(x)$	$\{1\}$	$\{1,x\}$	$\{1\}$	$\{1,x\}$	$\{1,x\}$
$U(y)$	$\{1\}$	$\{1\}$	$\{1,y\}$	$\{1,y\}$	$\{1,y\}$
$U(z)$	$\{1\}$	$\{1,x\}$	$\{1,y\}$	$\{1,x,y,z\}$	$\{1,x,y,w\}$
$U(w)$	$\{1\}$	$\{1,x\}$	$\{1,y\}$	$\{1,x,y\}$	$\{1,x,y,w\}$

Then the graph $G_U(A)$, where $V(G_U(A))$ consists of the elements of $U(A)$ and two distinct elements $U(x), U(y) \in U(A)$ are called adjacent if and only if $U(x)\diamondsuit U(y) = \{1\}$ that we can read from this table. Therefore, the adjacency matrix of the graph of $G_U(A)$ can be given as follows:

Adj($G_U(A)$)=

	$U(1)$	$U(x)$	$U(y)$	$U(z)$	$U(w)$
$U(1)$	1	1	1	1	1
$U(x)$	1	0	1	0	0
$U(y)$	1	1	0	0	0
$U(z)$	1	0	0	0	0
$U(w)$	1	0	0	0	0

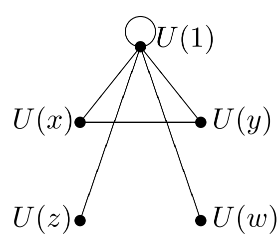

FIGURE 5. The graph of the bitonic algebra A given in Example 2.1 with the Hasse Diagram

Algorithm 5: Generating U(1, x) Upper Sets on a Bitonic Algebra

Data: (A, 1, *) Bitonic Algebra, * Operation Table

Result: (U(1, x) Upper Sets for all $x \in A$

initialization;

foreach x **in** A **do**

  U(1, x)=*null*

  foreach y **in** A **do**

    if *1 * (x * y) = 1* **then**

      add element y to U(1, x)

In Algorithm 5, the upper sets are examined according to the Definition 4.2 given for a bitonic algebra. The input of this algorithm is a bitonic algebra. Here, according to the given definition, the upper sets of every elements (except 1) are examined separately. Let x be the element whose upper set is examined. In the algorithm, firstly an empty $U(1, x)$ element is created. Then, for every element y in the bitonic algebra given, the operation $1 * (x * y)$ is calculated. If this is equivalent to 1, then y is assigned to the upper set $U(1, x)$.

This algorithm is also coded in MS EXCEL programe in VBScript language and these codes are given in Listing 5.

```

For a = 2 To m
    'the sets are written in a cell in EXCEL"
    Cells(a, 37) = "U(1," & a & ") = "
    counter = 0
    For x = 1 To m
  If m1(1, m1(a, x)) = 1 Then
    If sayac <> 0 Then
        Cells(a, 38) = Cells(a, 38) & "," & x
    End If
    If the counter = 0 Then
        Cells(a, 38) = Cells(a, 38) & x
        counter = counter + 1
    End If
```

```
  End If
    Next x
Next a

```

Listing 5. Codes for Algorithm 5

5. Conclusion and Future Works

In this work, firstly we evoke basic knowledge about bitonic algebras and graphs. Then, algorithms are enhanced to check whether any given set is a bitonic algebra or not. These algorithms check the properties or definitions given in preliminaries for a bitonic algebra and additionally these algorithms are coded in VB-Script Language. In the third section, with the help of the operators defined for bitonic algebras, the graphs based on bitonic algebras are defined and some examples are stated. In recent years, the studies consider the relations between algebraic structures and graphs gain very importance. A new point of view is gained on daily life problems by graph modeling of theoretical findings in algebraic structures. Additionally, the Sheffer Stroke Operation that reducts axiom systems of many algebraic structures [30],[31], [29] and fuzzy concepts are used to study different notions of algebraical systems [33]. Because of this reason, our work to search for the relationship between bitonic algebra and different graph types continues based on this study. We will consider fuzzy graphs of bitonic algebras and also extend our work to the graphs of Sheffer stroke bitonic algebras.

References

[1] Abbott, J. C. (1966). Algebras of implication and semi-lattices. Séminaire Dubreil. Algèbre et théorie des nombres, 20(2), 1-8.

[2] Afkhami, M., Khashyarmanesh, K., and Rajabi, Z. (2017). Some results on the annihilator graph of a commutative ring. Czechoslovak Mathematical Journal, 67, 151-169.

[3] Anderson, D. F., Badawi, A. (2008). The total graph of a commutative ring. Journal of algebra, 320(7), 2706-2719.

[4] Anderson, D. D., Naseer, M. (1993). Beck′s coloring of a commutative ring. Journal of algebra, 159(2), 500-514.

[5] Ayar Ozbal, Ş. (2022). On Filters of Bitonic Algebras. Symmetry, 14(8), 1509.

[6] Atani, S. E., Saraei, F. E. K. (2013). The total graph of a commutative semiring. Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică, 21(2), 21-33.

[7] Beck, I. (1988). Coloring of commutative rings. Journal of algebra, 116(1), 208-226.

[8] Birkhoff, G. (1940). Lattice theory (Vol. 25). American Mathematical Soc..

[9] Borzooei, R. A., and Khosravi Shoar, S. (2006). Implication algebras are equivalent to the dual implicative BCK-algebras. Scientiae Mathematicae Japonicae, 63(3), 429-432.

[10] Brešar, B., Changat, M., Klavžar, S., Kovše, M., Mathews, J., and Mathews, A. (2008). Cover-incomparability graphs of posets. Order, 25, 335-347.

[11] Curry HB. (1963).Foundations of Mathematics Logics. McGraw-Hill.

[12] DF, A., PS, L. (1999). The zero-divisor graph of a commutative ring. Journal of Algebra, 217, 434-447.

[13] Diego A. (1966). Sur les algébres de Hilbert. Collection de Logique Mathématique, Sér , A21.

[14] Diestel R. (2000). Graph Theory. Springer-Verlag .

[15] Gursoy, A., Kircali Gursoy, N., Oner, T., and Şenturk, I. (2021). An alternative construction of graphs by associating with algorithmic approach on MV-algebras. Soft Computing, 25(21), 13201-13212.

[16] Halaš, R. (2002). Remarks on commutative Hilbert algebras. Mathematica Bohemica, 127(4), 525-529.

[17] Henkin, L. (1951). An algebraic characterization of quantifiers. Journal of Symbolic Logic, 16(4).

[18] Hu, QINGPIN. (1983). On BCH-algebras. In Math. Seminar Notes (Vol. 11, pp. 313-320).

[19] Imai, Y., Iseki, K. (1966). On axiom systems of propositional calculi. XIV. Proceedings of the Japan Academy, 42(1), 19-22.

[20] Jun, Y. B., Lee, K. J. (2011). Graphs based on BCK/BCI-algebras. International journal of mathematics and mathematical sciences, 2011.

[21] Kandilci, S., Bacak-Turan, G., and Polat, R. (2013). Graph operations and neighbor rupture degree. Journal of Applied Mathematics, 2013.

[22] Kircali Gursoy, N. (2023). Graphs of Wajsberg Algebras via Complement Annihilating. Symmetry, 15(1), 121.

[23] Kim, K. H., Yon, Y. H. (2007). Dual BCK-algebra and MV-algebra. Scientiae Mathematicae Japonicae, 66(2), 247-254.

[24] Komori, Y. (1984). The class of BBC-algebras is not a variety. Math. Japon., 29, 391-394.

[25] Maimani, H. R., Yassemi, S. (2011). On the zero-divisor graphs of commutative semigroups. Houston J. Math, 37, 733-740.

[26] Marcus DA. (2020). Graph Theory, American Mathematical Society. 53.

[27] Marsden, E. L. (1972). Compatible elements in implicative models. Journal of Philosophical Logic, 156-161.

[28] Muhammad A, Sovan S, Madhumangal P. (2017). Cayley Vague Graphs, The Journal of Fuzzy Mathematics 25(2).

[29] Oner, T., Şenturk, I. (2017). The Sheffer stroke operation reducts of basic algebras. Open Mathematics, 15(1), 926-935.

[30] Oner, T., Kalkan, T., and Saeid, A. B. (2022). Class of Sheffer stroke BCK-algebras. Analele ştiinţifice ale Universităţii" Ovidius" Constanţa. Seria Matematică, 30(1), 247-269.

[31] Oner, T., Kalkan, T., and Cakar, M. (2021). Sheffer stroke BCH-algebras. Journal of International Mathematical Virtual Institute, 11(1), 119-135.

[32] Şenturk, I., Gursoy, N. K. (2022). An algorithmic observation of directed graphs on lattices. Journal of International Mathematical Virtual Institute, 12(1).

[33] Oner, T., Katican, T., and Saeid, A. B. (2022). Study strong Sheffer stroke non-associative MV-algebras by fuzzy filters. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics, 71(1), 165-187.

[34] Trudeau, R. J. (2013). Introduction to graph theory. Courier Corporation.

[35] Talebi, Y., Darzi, A. (2017). On graph associated to co-ideals of commutative semirings. Commentationes Mathematicae Universitatis Carolinae, 58(3), 293-305.

[36] Tamizh Chelvam, T., Nithya, S. (2013). Zero-divisor graph of an ideal of a near-ring. Discrete Mathematics, Algorithms and Applications, 5(01), 1350007.

[37] Xu, Y. (1993). Lattice implication algebras. J. of Southwest Jiaotong Univ., 28(1), 20-27.

[38] Yang, X. U., Ke-yun, QIN. (1992). Lattice H implication algebras and lattice implication algebra classes. J Hebei Mining and Civil Engineering Institute, 3, 139-143.

[39] Yon, Y. H., Ozbal, Ş. A. (2018). On derivations and generalized derivations of bitonic algebras. Applicable analysis and discrete mathematics, 12(1), 110-125.

[40] Yon, Y., Kim, K. H. (2012). On Heyting algebras and dual BCK-algebras. Bulletin of the Iranian Mathematical Society, 38(1), 159-168.

Faculty of Science and Letters, Department of Mathematics, Yasar University, Izmir, Türkiye

Faculty of Science and Letters, Department of Mathematics, Yasar University, Izmir, Türkiye

Central Academic Planning Directorate, Yasar University, Izmir, Türkiye

Department of Mathematics, Ege University, Izmir, Türkiye