International Journal of Maps in Mathematics

Volume 7, Issue 2, 2024, Pages:324-334

E-ISSN: 2636-7467

www.journalmim.com

AFFINE TRANSLATION SURFACES WITH CONSTANT MEAN CURVATURE IN MINKOWSKI 3-SPACE

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Abstract. In this paper, we obtain classification results for spacelike affine translation surfaces with constant mean curvature in three dimensional Minkowski space $\mathbb{E}^3_1$.

Keywords: Minkowski 3-space, affine translation surface, mean curvature, spacelike surface.

2010 Mathematics Subject Classification: 53A05; 53A10; 53C42.

1. Introduction

In 3-dimensional spaces, a regular surface parameterized as $\Psi(u,v) = (u, v, z(u,v))$ is called a translation surface if usually $z(u,v)$ is of the form

$$z(u,v) = f(u) + g(v),$$

where f and g are differentiable functions of u and v, respectively. Scherk [10] discovered the first non-trivial minimal translation surface in Euclidean 3-space $\mathbb{E}^3$, famously known as the Scherk surface, and is given by

$$z(u,v) = \frac{1}{c}\log\left|\frac{\cos(cu)}{\cos(cv)}\right|,$$

where $c(\neq 0)$ is a constant. Planes and Scherk surfaces are the only minimal translation surfaces in $\mathbb{E}^3$. More than a century later, Liu proved that the circular cylinder is the only

Received:2024.04.15 *Revised:2024.06.29* *Accepted:2024.07.10*

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translation surface with non-zero constant mean curvature [5]. The study of constant mean curvature translation surfaces has gathered significant attention. For some of the studies, we refer the reader to see [3, 4, 5, 8, 9, 12]. A natural extension of the translation surface appears in the form of an affine translation surface, which is a surface parameterized by $\Psi(u,v) = (u, v, z(u,v))$, where now

$$z(u,v) = f(u) + g(au + v),$$

and $a(\neq 0)$ is a constant. Liu and Yu proved that the non-trivial minimal affine translation surface in $\mathbb{E}^3$ is given by

$$z(u,v) = \frac{1}{b}\log\left|\frac{\cos\left(b\sqrt{1+a^2}u\right)}{\cos\left(b(v+au)\right)}\right|,$$

where $b(\neq 0)$ is a constant. This surface is known as the affine Scherk surface [7]. For other related works on affine translation surfaces, we refer the reader to see [1, 2, 6, 11].

In connection to the non-zero constant mean curvature of affine translation surfaces, Liu and Jung [6] obtained the classification results in $\mathbb{E}^3$. Now, a Minkowski space is one of the most trivial indefinite space forms, and it marks its great significance as the trivial solution to the vacuum Einstein Field Equations without a cosmological constant. Inspired by the previous developments in the theory of constant mean curvature surfaces, we seek to classify spacelike affine translation surfaces with constant mean curvature in Minkowski 3-space $\mathbb{E}^3_1$.

Consider $\Psi(u,v)$ to be a regular spacelike surface in Minkowski 3-space $\mathbb{E}^3_1$. The coefficients of the 1^{st} fundamental form E, F, G of $\Psi(u,v)$ are given by

$$E = \langle \Psi_u, \Psi_u \rangle, F = \langle \Psi_u, \Psi_v \rangle, G = \langle \Psi_v, \Psi_v \rangle,$$

and the coefficients of the 2^{nd} fundamental form L, M, N of $\Psi(u,v)$ are given by

$$L = \langle \Psi_{uu}, \hat{n} \rangle, M = \langle \Psi_{uv}, \hat{n} \rangle, N = \langle \Psi_{vv}, \hat{n} \rangle,$$

where $\hat{n}$ is the unit normal vector and $\langle *, * \rangle = du^2 + dv^2 - dz^2$ is the Minkowski metric. The mean curvature H of the surface $\Psi(u,v)$ is given by

$$H(u,v) = \frac{EN - 2FM + GL}{2\big(EG - F^2\big)}. \tag{1.1}$$

For a spacelike surface $\Psi(u,v)$ in $\mathbb{E}^3_1$, we have $EG - F^2 > 0$, and for a timelike surfaces $EG - F^2 < 0$. In regards to the regular surface $\Psi(u,v)$ embedded in $\mathbb{E}^3_1$, following two types of affine translation surfaces exist:

(i) Affine translation surface of type 1:

$$\Psi(u,v) = (u, v, z(u,v)) \tag{1.2}$$

such that

$$z(u,v) = f(u) + g(au+v). \tag{1.3}$$

(ii) Affine translation surface of type 2:

$$\Psi(u,v) = (u(v,z), v, z) \tag{1.4}$$

such that

$$u(v,z) = h(v) + t(bv+z), \tag{1.5}$$

where $a(\neq 0), b(\neq 0)$ are constants and f, g, h, t are smooth functions. We note that whenever $a = 0$ or $b = 0$, affine translation surfaces reduce simply to translation surfaces.

2. Affine translation surfaces with non-zero constant mean curvature

Theorem 2.1. *Let $\Psi(u,v) = (u, v, z(u,v))$ be a spacelike affine translation surface of type 1 in $\mathbb{E}^3_1$. If $\Psi(u,v)$ has a non-zero constant mean curvature, then $z(u,v)$ is given by*

$$z(u,v) = \pm\frac{\sqrt{1+a^2-c^2}}{2H\left(1+a^2\right)}\sqrt{1+a^2+4H^2(b-au-v)^2} + \left(\frac{u-av}{1+a^2}\right)c + p$$

such that $c^2 < 1+a^2$; or

$$z(u,v) = cv \pm \frac{\sqrt{1-c^2}}{2H}\sqrt{1+4H^2(b-u)^2} + q$$

such that $c^2 < 1$; where a, b, c, p, q are all constants.

Proof. We know that the mean curvature of a spacelike surface $\Psi(u,v) = (u, v, z(u,v))$ in $\mathbb{E}^3_1$ is given by

$$H(u,v) = \frac{\left(z_u^2-1\right)z_{vv} - 2z_u z_v z_{uv} + \left(z_v^2-1\right)z_{uu}}{2\left(1-{z_u}^2-{z_v}^2\right)^{\frac{3}{2}}}, \tag{2.6}$$

where z_u, z_v denotes partial differentiation of z w.r.t. u and v, respectively. We obtain the following partial derivatives of $z(u, v)$ from (1.3)

$$\begin{cases} z_u = f' + ag', \\ z_v = g', \\ z_{uu} = f'' + a^2 g'', \\ z_{vv} = g'', \\ z_{uv} = ag'', \end{cases} \tag{2.7}$$

where $f' = \dfrac{df}{du}$ and $g' = \dfrac{dg}{dx}$ for $x = au + v$. Using (2.7) in (2.6), gives us

$$-f'' - \left(1 + a^2\right)g'' + \left(g'^2 f'' + f'^2 g''\right) = 2HT^3, \tag{2.8}$$

where $T^2 = 1 - \left(f' + ag'\right)^2 - g'^2$ and $H(\neq 0)$ is a constant. Eqn (2.8) writes as

$$-\left(1 - g'^2\right)f'' - \left(1 + a^2 - f'^2\right)g'' = 2HT^3. \tag{2.9}$$

Now, we have the following two cases:

Case I. When $f'' = 0$, we have $f' = c$, where c is a constant. Substituting $f' = c$ in (2.9) gives us

$$-\left(1 + a^2 - c^2\right)g'' = 2H\left[1 - \left(c + ag'\right)^2 - g'^2\right]^{\frac{3}{2}}. \tag{2.10}$$

Thus, we have

$$g'' = -\frac{2H\left(1 + a^2\right)^{\frac{3}{2}}}{1 + a^2 - c^2}\left[\frac{1 + a^2 - c^2}{\left(1 + a^2\right)^2} - \left(g' + \frac{ac}{1 + a^2}\right)^2\right]^{\frac{3}{2}}. \tag{2.11}$$

Making the following substitutions in (2.11)

$$\alpha = \frac{2H\left(1 + a^2\right)^{\frac{3}{2}}}{1 + a^2 - c^2}, \quad \beta = \frac{ac}{1 + a^2} \quad \text{and} \quad \gamma^2 = \frac{1 + a^2 - c^2}{\left(1 + a^2\right)^2},$$

results in

$$\frac{g''}{\left[\gamma^2 - \left(g' + \beta\right)^2\right]^{\frac{3}{2}}} = -\alpha. \tag{2.12}$$

Integrating (2.12) and isolating the expression for g' gives us

$$g' = \pm\frac{\gamma^3(c_1 - \alpha x)}{\sqrt{1 + \gamma^4(c_1 - \alpha x)^2}} - \beta, \tag{2.13}$$

where c_1 is a constant. By integrating (2.13), we obtain

$$g(x) = \pm\frac{1}{\alpha\gamma}\sqrt{1+\gamma^4(c_1-\alpha x)^2} - \beta x + c_2, \tag{2.14}$$

where c_2 is a constant. Substituting the values of α, β and γ in (2.14) gives us

$$g(x) = \pm\frac{\sqrt{1+a^2-c^2}}{2H\big(1+a^2\big)}\sqrt{1+a^2+4H^2(c_3-x)^2} - \left(\frac{ac}{1+a^2}\right)x + c_2, \tag{2.15}$$

where c_3 is a constant. Also, $f' = c$ gives us

$$f(u) = cu + c_4, \tag{2.16}$$

where c_4 is a constant. Thus from (1.3), (2.15) and (2.16), we have

$$\begin{aligned} z(u,v) = &\pm\frac{\sqrt{1+a^2-c^2}}{2H\big(1+a^2\big)}\sqrt{1+a^2+4H^2(c_3-au-v)^2} \\ &+\left(\frac{u-av}{1+a^2}\right)c + p, \end{aligned} \tag{2.17}$$

where p is a constant and $c^2 < 1+a^2$.

Case II. When $f'' \neq 0$. Differentiating (2.9) w.r.t. u gives us

$$\begin{aligned} &\big(1-g'^2\big)f''' + \big(1+a^2-f'^2\big)ag''' - 2\big(f'+ag'\big)g''f'' \\ &= -6HT\big[\big(f'+ag'\big)\big(f''+a^2g''\big) + ag'g''\big]. \end{aligned} \tag{2.18}$$

Now, differentiating (2.9) w.r.t. v gives us

$$\big(1+a^2-f'^2\big)g''' - 2g'g''f'' = -6HT\big[\big(f'+ag'\big)ag'' + g'g''\big]. \tag{2.19}$$

Eqn's (2.18) and (2.19) yield

$$\big(1-g'^2\big)f''' - 2f'f''g'' = -6HT\big(f'+ag'\big)f''. \tag{2.20}$$

Substituting the value of g'' from (2.9) in (2.20), we have

$$\big(1-g'^2\big)f''' - 2f'f''\left[\frac{2HT^3-\big(1-g'^2\big)}{\big(1+a^2-f'^2\big)}\right] = -6HT\big(f'+ag'\big)f''. \tag{2.21}$$

Thus, we obtain

$$\begin{aligned} &\big(1-g'^2\big)\left[\big(1+a^2-f'^2\big)f''' + 2f'f''^2\right] \\ &= -2HT\left[3\big(f'+ag'\big)\big(1+a^2-f'^2\big) - 2T^2f'\right]f''. \end{aligned} \tag{2.22}$$

Squaring both sides of (2.22) and substituting the value of T^2 gives us

$$\begin{aligned}&\left(1-g'^2\right)^2\left[\left(1+a^2-f'^2\right)f''' + 2f'f''^2\right]^2\\&=4H^2\left[1-\left(f'+ag'\right)^2-g'^2\right]\\&\quad\times\left[3\left(f'+ag'\right)\left(1+a^2-f'^2\right)-2\left\{1-\left(f'+ag'\right)^2-g'^2\right\}f'\right]^2 f''^2.\end{aligned} \tag{2.23}$$

We notice that the above expression can be expanded as a polynomial in the powers of g'. The coefficients of g' in the above expression are functions of u, and the expression itself is identically zero, so each term must be zero. But, the coefficient of g' with the highest degree, i.e., 6 in (2.23), is $-16H^2(1+a^2)^3 f'^2 f''^2$, which is non-zero. Thus, it follows that g' is a constant (Liu and Jung have used the same argument in [6]). Substituting $g' = c$ in (2.9) yields

$$z(u,v) = cv \pm \frac{\sqrt{1-c^2}}{2H}\sqrt{1+4H^2(b-u)^2}+q, \tag{2.24}$$

where b, c, q are constants and $c^2 < 1$. Thus, the proof of the theorem is complete. □

Theorem 2.2. *Let $\Psi(v,z) = (u(v,z), v, z)$ be a spacelike affine translation surface of type 2 in $\mathbb{E}^3_1$. If $\Psi(v,z)$ has a non-zero constant mean curvature, then $u(v,z)$ is given by*

$$u(v,z) = \pm\frac{\sqrt{1-b^2+c^2}}{2H\left(1-b^2\right)}\sqrt{4H^2\left(a-bv-z\right)^2-\left(1-b^2\right)}+\left(\frac{v+bz}{1-b^2}\right)c+p,$$

such that $1-b^2>0$; or

$$u(v,z) = \pm\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2(a-v)^2-1}+cz+q,$$

where a, b, c, p, q are all constants.

Proof. The mean curvature $H(v,z)$ of a spacelike surface $r(v,z) = (u(v,z), v, z)$ in $\mathbb{E}^3_1$ is given by

$$H(v,z) = \frac{\left(u_v^2+1\right)u_{zz}-2u_vu_zu_{vz}+\left(u_z^2-1\right)u_{vv}}{2\left(u_z^2-u_v^2-1\right)^{\frac{3}{2}}}, \tag{2.25}$$

where u_v, u_z denotes partial differentiation of u w.r.t. v and z, respectively. We obtain the following partial derivatives of $u(v,z)$ from (1.5)

$$\begin{cases} u_v = h' + bt', \\ u_z = t', \\ u_{vv} = h'' + b^2 t'', \\ u_{zz} = t'', \\ u_{vz} = bt'', \end{cases} \tag{2.26}$$

where $h' = \dfrac{dh}{dv}$ and $t' = \dfrac{dt}{dy}$ for $y = bv + z$. Using (2.26) in (2.25), gives us

$$-h'' + \big(1 - b^2\big)t'' + \big(t'^2 h'' + h'^2 t''\big) = 2HT^3, \tag{2.27}$$

where $T^2 = t'^2 - \big(h' + bt'\big)^2 - 1$ and $H(\neq 0)$ is a constant. Eqn (2.27) writes as

$$\big(-1 + t'^2\big)h'' + \big(1 - b^2 + h'^2\big)t'' = 2HT^3, \tag{2.28}$$

Now, we have the following two cases:

Case I. When $h' = c$ is a constant. It follows from (2.28)

$$(1 - b^2 + c^2)t'' = 2H\left[t'^2 - \big(c + bt'\big)^2 - 1\right]^{\frac{3}{2}}. \tag{2.29}$$

Thus, we have

$$t'' = \frac{2H\big(1 - b^2\big)^{\frac{3}{2}}}{1 - b^2 + c^2}\left[\left(t' - \frac{bc}{1 - b^2}\right)^2 - \frac{1 - b^2 + c^2}{\big(1 - b^2\big)^2}\right]^{\frac{3}{2}}. \tag{2.30}$$

Making the following substitutions in (2.30)

$$\alpha = \frac{2H\big(1 - b^2\big)^{\frac{3}{2}}}{1 - b^2 + c^2}, \quad \beta = -\frac{bc}{1 - b^2} \quad \text{and} \quad \gamma^2 = \frac{1 - b^2 + c^2}{\big(1 - b^2\big)^2},$$

results in

$$\frac{t''}{\left[\big(t' + \beta\big)^2 - \gamma^2\right]^{\frac{3}{2}}} = \alpha. \tag{2.31}$$

Integrating (2.31) and isolating the expression for t' gives us

$$t' = \pm\frac{\gamma^3(c_1 - \alpha y)}{\sqrt{\gamma^4(c_1 - \alpha y)^2 - 1}} - \beta, \tag{2.32}$$

where c_1 is a constant. Thereby integrating (2.32), we obtain

$$t(y) = \pm\frac{1}{\alpha\gamma}\sqrt{\gamma^4(c_1 - \alpha y)^2 - 1} - \beta y + c_2, \tag{2.33}$$

where c_2 is a constant. Substituting the values of α, β and γ in (2.33) gives us

$$t(y) = \pm\frac{\sqrt{1-b^2+c^2}}{2H\big(1-b^2\big)}\sqrt{4H^2(c_3-y)^2-\big(1-b^2\big)}+\frac{bc}{1-b^2}y+c_2. \tag{2.34}$$

Also, $h'=c$ gives us

$$h(v) = cv + c_4, \tag{2.35}$$

where c_4 is a constant. Thus, from (1.5), (2.34) and (2.35), we have

$$\begin{aligned} u(v,z) = \pm & \frac{\sqrt{1-b^2+c^2}}{2H\big(1-b^2\big)}\sqrt{4H^2\big(c_3-bv-z\big)^2-\big(1-b^2\big)} \\ & + \left(\frac{v+bz}{1-b^2}\right)c+p, \end{aligned} \tag{2.36}$$

where p is a constant.

Case II. When $h'' \neq 0$. Differentiating (2.28) w.r.t. v gives us

$$\begin{aligned} & \big(-1+t'^2\big)h''' + \big(1-b^2+h'^2\big)bt''' + 2\big(h'+bt'\big)h''t'' \\ & = 6HT\big[bt't'' - \big(h'+bt'\big)\big(h''+b^2t''\big)\big]. \end{aligned} \tag{2.37}$$

Now, differentiating (2.28) w.r.t. z gives us

$$\big(1-b^2+h'^2\big)t''' + 2h''t't'' = 6HT\big[t't'' - \big(h'+bt'\big)bt''\big]. \tag{2.38}$$

Eqn's (2.37) and (2.38) yield

$$\big(-1+t'^2\big)h''' + 2h'h''t'' = -6HT\big(h'+bt'\big)h''. \tag{2.39}$$

Substituting the value of t'' from (2.28) in (2.39), we have

$$\big(-1+t'^2\big)h''' + 2h'h''\left[\frac{2HT^3-\big(-1+t'^2\big)}{\big(1-b^2+h'^2\big)}\right] = -6HT\big(h'+bt'\big)h''. \tag{2.40}$$

Thus, we obtain

$$\begin{aligned} & \big(-1+t'^2\big)\left[\big(1-b^2+h'^2\big)h''' - 2h'h''^2\right] \\ & = 2HT\left[-3\big(h'+bt'\big)\big(1-b^2+h'^2\big) - 2T^2h'\right]h'' \end{aligned} \tag{2.41}$$

Squaring both sides of (2.41) and substituting the value of T^2 gives us

$$\begin{aligned} & \big(-1+t'^2\big)^2\left[\big(1-b^2+h'^2\big)h''' - 2h'h''^2\right]^2 \\ & = 4H^2\left[t'^2-\big(h'+bt'\big)^2-1\right] \\ & \times\left[-3\big(h'+bt'\big)\big(1-b^2+h'^2\big) - 2\big\{t'^2-\big(h'+bt'\big)^2-1\big\}h'\right]^2 h''^2. \end{aligned} \tag{2.42}$$

The coefficient of t'^6 in (2.42) is $16H^2(1-b^2)^3h'^2h''^2$, which is non-zero and the concluding argument in Theorem 2.1 results in t' being constant. Substituting $t' = c$ in (2.28) yields

$$u(v,z) = \pm\frac{\sqrt{1+c^2}}{2H}\sqrt{4H^2(a-v)^2 - 1} + cz + q, \tag{2.43}$$

where a, c, q is a constant. Thus, the proof of the theorem is complete. □

3. Maximal affine translation surfaces

Theorem 3.1. *Let $\Psi(u,v) = (u, v, z(u,v))$ be a maximal affine translation surface of type 1 in $\mathbb{E}^3_1$. Then, $\Psi(u,v)$ is either a planar surface or $z(u,v)$ is given by*

$$z(u,v) = \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{1+a^2}u + c_1\right]}{\cosh\left[c(au+v)+c_2\right]}\right| + c_3,$$

where a, c, c_1, c_2, c_3 are constants and $c \neq 0$.

Proof. Taking $H = 0$ in (2.9) gives us

$$-\left(1-g'^2\right)f'' - \left(1+a^2-f'^2\right)g'' = 0, \tag{3.44}$$

which writes as

$$\frac{f''}{1+a^2-f'^2} = \frac{-g''}{1-g'^2} = \lambda, \tag{3.45}$$

where λ is a constant and $\left(1+a^2-f'^2\right)\left(1-g'^2\right) \neq 0$.

Depending on λ, we have the following 2 cases:

Case I. $\lambda = 0$, gives us

$$f'' = 0, \quad g'' = 0, \tag{3.46}$$

which leads to a planar surface in $\mathbb{E}^3_1$.

Case II. $\lambda \neq 0$, gives us

$$f(u) = \frac{1}{c}\log\left|2\cosh\left[c\sqrt{1+a^2}u + c_1\right]\right| + c_3, \tag{3.47}$$

$$g(au+v) = \frac{-1}{c}\log\left|2\cosh\left[c(au+v)+c_2\right]\right| + c_4, \tag{3.48}$$

where a, c, c_1, c_2, c_3, c_4 are constants. Thus, we have

$$z(u,v) = \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{1+a^2}u + c_1\right]}{\cosh\left[c(au+v)+c_2\right]}\right| + p, \tag{3.49}$$

where $c \neq 0$ and p is a constant. Thus, the proof of the theorem is complete. □

Theorem 3.2. *Let $\Psi(v,z) = (u(v,z), v, z)$ be a maximal generalized affine translation surface of type 2 in $\mathbb{E}_1^3$. Then, $\Psi(v,z)$ is either a planar surface or $u(v,z)$ is given by*

$$u(v,z) = \frac{1}{c}\log\left|\frac{\cos\left[c\sqrt{1-b^2}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \quad b^2 < 1$$

or

$$u(v,z) = \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{b^2-1}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \quad b^2 > 1;$$

where b, c, c_1, c_2, c_3 are constants and $c \neq 0$.

Proof. For $H = 0$, it follows from (2.28)

$$\big(-1+t'^2\big)h'' + \big(1-b^2+h'^2\big)t'' = 0. \tag{3.50}$$

Thus, we have

$$\frac{-h''}{1-b^2+h'^2} = \frac{t''}{-1+t'^2} = \lambda, \tag{3.51}$$

where λ is a constant and $\left(1-b^2+h'^2\right)\left(-1+t'^2\right) \neq 0$. $\lambda = 0$ leads to a planar surface in $\mathbb{E}_1^3$ and when $\lambda \neq 0$, we have the following cases:

Case I. If $b^2 < 1$; (3.51) yields

$$\begin{aligned} u(v,z) &= h(v) + t(bv+z), \\ &= \frac{1}{c}\log\left|\frac{\cos\left[c\sqrt{1-b^2}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \end{aligned} \tag{3.52}$$

where b, c, c_1, c_2, c_3 are constants and $c \neq 0$.

Case II. If $b^2 > 1$; (3.51) yields

$$\begin{aligned} u(v,z) &= h(v) + t(bv+z), \\ &= \frac{1}{c}\log\left|\frac{\cosh\left[c\sqrt{b^2-1}v + c_1\right]}{\cosh\left[c(bv+z)+c_2\right]}\right| + c_3, \end{aligned} \tag{3.53}$$

where b, c, c_1, c_2, c_3 are constants and $c \neq 0$. Thus, the proof of the theorem is complete. □

Acknowledgements. The third author is supported by the NBHM-DAE, India, Project number: (02011/44/2023/RDII/14757).

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