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ON e^*-TOPOLOGICAL RINGS

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Abstract. The main purpose of this manuscript is to introduce the concept of e^*-topological ring. This class appears as a generalised version of the class of β-topological rings. In addition, we have discussed the relation between the concept of e^*-topological ring and some other types of topological rings existing in the literature. Also, some fundamental results about e^*-topological rings are revealed. Moreover, we give some counterexamples regarding our results.

Keywords: topological ring, e^*-open, e^*-topological ring, e^*-continuous

2010 Mathematics Subject Classification: 54H13

1. Introduction

In topology, it is sometimes necessary to use algebra to find solutions to some problems, such as determining whether two topological spaces are homeomorphic. For instance, if the fundamental groups of two topological spaces are not isomorphic, then the topological spaces can not be homeomorphic. Thanks to fundamental groups of topological spaces, we can decide that two topological spaces are not homeomorphic but not all. This situation leads to the definition of different concepts in the related field. One of these concepts is the concept of topological ring. To better understand topological rings, the concept of topological groups should be well known. A topological group is a group $\mathbb{X}$ that is also a topological space such that the addition and the inversion are continuous as functions $\psi : \mathbb{X} \to \mathbb{X}, x \mapsto -x$ and

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$\varphi : \mathbb{X} \times \mathbb{X} \to \mathbb{X}, (x, y) \mapsto x + y$, where $\mathbb{X} \times \mathbb{X}$ carries the product topology. The concept of topological ring was first introduced in [4, 5] by Kaplansky. A topological ring is a ring $\mathbb{X}$ that is also a topological space such that both the addition and the multiplication are continuous as functions $\varphi : \mathbb{X} \times \mathbb{X} \to \mathbb{X}$, where $\mathbb{X} \times \mathbb{X}$ carries the product topology. That means $\mathbb{X}$ is an additive topological group and a multiplicative topological semigroup.

The types of open sets in the literature such as α-open [9], semi-open [8], pre-open [10], β-open [1], etc. allow a generalization of the notion of topological ring. Studying the features of these generalised versions and investigating their relations with topological rings are just some of the different advances in the literature. Some of the recent advancements in this direction are β-topological rings [2], irresolute topological rings [12] and α-irresolute topological rings [11].

In 2021, Billawaria et al. studied β-topological ring which is a more general notion than the notion of topological ring [2]. They have revealed some fundamental properties of β-topological rings. Also, the authors gave some other useful results on β-topological rings.

In this paper, we introduce the notion of e^*-topological ring by utilizing e^*-open sets defined by Ekici in [3]. Also, we obtain some of its fundamental properties. Moreover, we compare between this notion and some notions existing in the literature. In addition, we give some counterexamples regarding our results obtained in the scope of this study. Furthermore, we provide an example of e^*-topological ring which is not a β-topological ring.

2. Preliminaries

Throughout this paper, $(\mathbb{X}, \mu)$ and $(\mathbb{Y}, \rho)$ (or briefly $\mathbb{X}$ and $\mathbb{Y}$) always mean topological spaces. For a subset E of a topological space $\mathbb{X}$, the interior of E and the closure of E are denoted by $int(E)$ and $cl(E)$, respectively. The family of all open (resp. closed) sets of $\mathbb{X}$ will be denoted by $O(\mathbb{X})$ (resp. $C(\mathbb{X})$). In addition, the family of all open sets of $\mathbb{X}$ containing a point a of $\mathbb{X}$ is denoted by $O(\mathbb{X}, a)$. Recall that a subset E of a space $\mathbb{X}$ is called regular open [13] (resp. regular closed [13]) if $E = int(cl(E))$ (resp. $E = cl(int(E))$). The family of all regular open subsets of $\mathbb{X}$ is denoted by $RO(\mathbb{X})$. The family of all regular open sets of $\mathbb{X}$ containing a point a of $\mathbb{X}$ is denoted by $RO(\mathbb{X}, a)$.

The union of all regular open sets of $\mathbb{X}$ contained in E is called the δ-interior [14] of E and is denoted by $\delta\text{-}int(E)$. A subset E of a space $\mathbb{X}$ is said to be δ-open [14] if $A = \delta\text{-}int(A)$. Also, a subset E of a space $\mathbb{X}$ is said to be δ-closed if its complement is δ-open. The intersection of all regular closed sets of $\mathbb{X}$ containing E is called the δ-closure [14] of E and is denoted by $\delta\text{-}cl(E)$.

A subset E of a space $\mathbb{X}$ is called e^*-open if $E \subseteq cl(int(\delta\text{-}cl(E)))$. The complement of an e^*-open set is called e^*-closed. The intersection of all e^*-closed sets of $\mathbb{X}$ containing E is called the e^*-closure of E and is denoted by $e^*\text{-}cl(E)$. Dually, the union of all e^*-open sets of $\mathbb{X}$ contained in E is called the e^*-interior of E and is denoted by $e^*\text{-}int(E)$. The family of all e^*-open subsets (resp. e^*-closed) $\mathbb{X}$ denoted by $e^*O(\mathbb{X})$ (resp. $e^*C(\mathbb{X})$). The family of all e^*-open (resp. e^*-closed) sets of $\mathbb{X}$ containing a point a of $\mathbb{X}$ denoted by $e^*O(\mathbb{X}, a)$ (resp. $e^*C(\mathbb{X}, a)$).

Definition 2.1. [4] *Let $(\mathbb{X}, +, \cdot)$ be a ring and μ be a topology on $\mathbb{X}$. The quadruple $(\mathbb{X}, +, \cdot, \mu)$ is called a topological ring if the following three conditions hold:*

i) For every $a, b \in \mathbb{X}$ and every open set $M \in O(\mathbb{X}, a + b)$, there exist $K \in O(\mathbb{X}, a)$ and $L \in O(\mathbb{X}, b)$ such that $K + L \subseteq M$,

ii) For every $a \in \mathbb{X}$ and every $L \in O(\mathbb{X}, -a)$, there exists $K \in O(\mathbb{X}, a)$ such that $-K \subseteq L$,

iii) For every $a, b \in \mathbb{X}$ and every $M \in O(\mathbb{X}, ab)$, there exist $K \in O(\mathbb{X}, a)$ and $L \in O(\mathbb{X}, b)$ such that $KL \subseteq M$.

Definition 2.2. [2] *Let $(\mathbb{X}, +, \cdot)$ be a ring and μ be a topology on $\mathbb{X}$. The quadruple $(\mathbb{X}, +, \cdot, \mu)$ is called an β-topological ring if the following three conditions hold:*

i) For every $a, b \in \mathbb{X}$ and every $M \in O(\mathbb{X}, a+b)$, there exist $K \in \beta O(\mathbb{X}, a)$ and $L \in \beta O(\mathbb{X}, b)$ such that $K + L \subseteq M$,

ii) For every $a \in \mathbb{X}$ and every $L \in O(\mathbb{X}, -a)$, there exists $K \in \beta O(\mathbb{X}, a)$ such that $-K \subseteq L$,

iii) For every $a, b \in \mathbb{X}$ and every $M \in O(\mathbb{X}, ab)$, there exist $K \in \beta O(\mathbb{X}, a)$ and $L \in \beta O(\mathbb{X}, b)$ such that $KL \subseteq M$.

Definition 2.3. [3] *A function $f : (\mathbb{X}, \mu) \to (\mathbb{Y}, \rho)$ is said to be e^*-continuous if $f^{-1}[G] \in e^*O(\mathbb{X})$ for every $G \in O(\mathbb{Y})$.*

Lemma 2.1. [3] *A function $f : (\mathbb{X}, \mu) \to (\mathbb{Y}, \rho)$ is e^*-continuous if and only if for every $a \in \mathbb{X}$ and for every $H \in O(\mathbb{Y}, f(a))$, there exists $G \in e^*O(\mathbb{X}, a)$ such that $f[G] \subseteq H$.*

Definition 2.4. [3] *Let $(\mathbb{X}, \mu)$ be a topological space and $E \subseteq \mathbb{X}$. Then, the following statements hold:*

a) E is e^*-open if and only if $E = e^*\text{-}int(E)$,

b) E is e^*-closed if and only if $E = e^*\text{-}cl(E)$.

Lemma 2.2. *Let $(\mathbb{X},\mu)$ and $(\mathbb{Y},\rho)$ be two topological spaces. If $E \in e^*O(\mathbb{X})$ and $F \in e^*O(\mathbb{Y})$, then $E \times F \in e^*O(\mathbb{X}\times\mathbb{Y},\mu\star\rho)$.*

Proof. Let $E \in e^*O(\mathbb{X})$ and $F \in e^*O(\mathbb{Y})$.

$$\left.\begin{array}{l} E \in e^*O(\mathbb{X}) \Rightarrow E \subseteq cl(int(\delta\text{-}cl(E))) \\ F \in e^*O(\mathbb{Y}) \Rightarrow F \subseteq cl(int(\delta\text{-}cl(F))) \end{array}\right\} \Rightarrow E\times F \subseteq cl(int(\delta\text{-}cl(E))) \times cl(int(\delta\text{-}cl(F)))$$

$$\begin{array}{rcl} \Rightarrow E\times F & \subseteq & cl(int(\delta\text{-}cl(E))) \times cl(int(\delta\text{-}cl(F))) \\ & = & cl[int(\delta\text{-}cl(E)) \times int(\delta\text{-}cl(F))] \\ & = & cl(int[\delta\text{-}cl(E) \times \delta\text{-}cl(F)]) \\ & = & cl(int(\delta\text{-}cl(E\times F))) \end{array}$$

This means $E \times F \in e^*O(\mathbb{X}\times\mathbb{Y})$. □

3. e^*-Topological Rings

Now, we introduce and study the concept of e^*-topological ring by utilizing e^*-open sets.

Definition 3.1. *Let $(\mathbb{X},+,\cdot)$ be a ring and μ be a topology on $\mathbb{X}$. The quadruple $(\mathbb{X},+,\cdot,\mu)$ is called an e^*-topological ring if the following three conditions hold:*

i) For every $a,b\in\mathbb{X}$ and every open set $M \in O(\mathbb{X},a+b)$, there exist $K \in e^*O(\mathbb{X},a)$ and $L \in e^*O(\mathbb{X},b)$ such that $K+L\subseteq M$,

ii) For every $a\in\mathbb{X}$ and every open set $L \in O(\mathbb{X},-a)$, there exists $K\in e^*O(\mathbb{X},a)$ such that $-K\subseteq L$,

iii) For every $a,b\in\mathbb{X}$ and every open set $M\in O(\mathbb{X},ab)$, there exist $K\in e^*O(\mathbb{X},a)$ and $L\in e^*O(\mathbb{X},b)$ such that $KL\subseteq M$.

Remark 3.1. *It is clear that every β-topological ring is an e^*-topological ring since every β-open set is an e^*-open set. Nevertheless, the converse need not always to be true as shown in the following example.*

Example 3.1. *Let $\mathbb{X}=\{k,l,m,n\}$ and $\mu=\{\emptyset,\mathbb{X},\{k\},\{k,l\}\}$. Let the addition and the multiplication operations on $\mathbb{X}$ be as given in the following tables:*

$+$	k	l	m	n
k	k	l	m	n
l	l	m	n	k
m	m	n	k	l
n	l	k	l	m

$\cdot$	k	l	m	n
k	k	k	k	k
l	k	m	k	m
m	k	k	k	k
n	k	m	k	m

In this topological space, simple calculations show that $e^*O(\mathbb{X}) = 2^{\mathbb{X}}$ and $\beta O(\mathbb{X}) = \{\emptyset, \mathbb{X}, \{k\}, \{k,n\}, \{k,m\}, \{k,l\}, \{k,l,n\}, \{k,l,m\}, \{k,m,n\}\}$. Then, it is clear that $(\mathbb{X}, +, \cdot, \mu)$ is an e^*-topological ring but it is not a β-topological ring.

Example 3.2. *Let $(\mathbb{R}, +, \cdot)$ be the ring of real numbers and let $\mathcal{U}$ the usual topology on $\mathbb{R}$. Then, $(\mathbb{R}, +, \cdot, \mathcal{U})$ is an e^*-topological ring.*

Example 3.3. *Let $(\mathbb{X}, +, \cdot)$ be any ring and let μ the discrete topology on $\mathbb{X}$. Then, $(\mathbb{X}, +, \cdot, \mu)$ is an e^*-topological ring.*

Theorem 3.1. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring. Then, the following functions are e^*-continuous.*

a) $+ : \mathbb{X}^2 \to \mathbb{X}$ defined by $+(x,y) = x + y$ for all $(x,y) \in \mathbb{X}^2$,

b) $\cdot : \mathbb{X}^2 \to \mathbb{X}$ defined by $\cdot(x,y) = xy$ for all $(x,y) \in \mathbb{X}^2$,

c) $- : \mathbb{X} \to \mathbb{X}$ defined by $-(x) = -x$ for all $x \in \mathbb{X}$.

Proof. a) Let $(x,y) \in \mathbb{X}^2$ and $W \in O(\mathbb{X}, x+y)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, x+y) \Rightarrow (\exists U \in e^*O(\mathbb{X}, x))(\exists V \in e^*O(\mathbb{X}, y))(U + V \subseteq W) \\ O := U \times V \end{array}\right\} \overset{\text{Lemma 2.2}}{\Rightarrow}$$

$$\Rightarrow (O \in e^*O(\mathbb{X}^2, (x,y)))(+[O] = +[U \times V] = U + V \subseteq W).$$

b) Let $(x,y) \in \mathbb{X}^2$ and $W \in O(\mathbb{X}, xy)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, xy) \Rightarrow (\exists U \in e^*O(\mathbb{X}, x))(\exists V \in e^*O(\mathbb{X}, y))(UV \subseteq W) \\ O := U \times V \end{array}\right\} \overset{\text{Lemma 2.2}}{\Rightarrow}$$

$$\Rightarrow (O \in e^*O(\mathbb{X}^2, (x,y)))(\cdot[O] = \cdot[U \times V] = UV \subseteq W).$$

c) Let $V \in O(\mathbb{X})$. Our aim is to show that $-^{-1}[V] \in e^*O(\mathbb{X})$.

$$\left.\begin{array}{r} -^{-1}[V] = \{x \in \mathbb{X} : -(x) \in V\} = \{x \in \mathbb{X} : -x \in V\} = -V \\ V \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Teorem 3.2}}{\Rightarrow} -^{-1}[V] \in e^*O(\mathbb{X}).$$

□

Theorem 3.2. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring. Then, the following properties hold.*

a) If $G \in O(\mathbb{X})$, then $-G \in e^*O(\mathbb{X})$,

b) If $G \in O(\mathbb{X})$ and $a \in \mathbb{X}$, then $a + G \in e^*O(\mathbb{X})$,

c) If $G \in O(\mathbb{X})$ and $a \in \mathbb{X}$, then $G + a \in e^*O(\mathbb{X})$.

Proof. a) Let $G \in O(\mathbb{X})$.

$G \in O(\mathbb{X}) \Rightarrow -G \subseteq \mathbb{X} \Rightarrow e^*\text{-}int(-G) \subseteq -G \ldots (1)$

Now, let $b \in -G$. Our purpose is to show that $b \in e^*\text{-}int(-G)$.

$$\left.\begin{array}{r} b \in -G \Rightarrow -b \in G \\ G \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Definition 3.1}}{\Rightarrow} (\exists U \in e^*O(\mathbb{X}, b))(-U \subseteq G)$$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(U \subseteq -G)$

$\Rightarrow b \in e^*\text{-}int(-G)$

Then, we have $-G \subseteq e^*\text{-}int(-G) \ldots (2)$

$(1), (2) \Rightarrow e^*\text{-}int(-G) = -G \Rightarrow -G \in e^*O(\mathbb{X})$.

$b)$ Let $G \in O(\mathbb{X})$ and $a \in \mathbb{X}$. Our purpose is to show that $a + G \in e^*O(\mathbb{X})$. For this, we will show that $a + G = e^*\text{-}int(a + G)$. Now, let $b \in a + G$. If we prove $b \in e^*\text{-}int(a+G)$, then the proof complete.

$$\left.\begin{array}{r} b \in a + G \Rightarrow (\exists c \in G)(b = a + c) \\ G \in O(\mathbb{X}) \end{array}\right\} \Rightarrow -a + b \in G \in O(\mathbb{X})$$

$\overset{\text{Definition 3.1}}{\Rightarrow} (\exists U \in e^*O(\mathbb{X}, -a))(\exists V \in e^*O(\mathbb{X}, b))(-a + V \subseteq U + V \subseteq G)$

$\Rightarrow (\exists V \in e^*O(\mathbb{X}, b))(-a + V \subseteq G)$

$\Rightarrow (\exists V \in e^*O(\mathbb{X}, b))(V \subseteq a + G)$

$\Rightarrow b \in e^*\text{-}int(a + G)$.

$c)$ This follows (b) since the addition is commutative.

□

Corollary 3.1. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G \subseteq \mathbb{X}$. Then, the following statements hold.*

a) If $G \in O(\mathbb{X})$, then $-G \subseteq cl(int(\delta\text{-}cl(-G)))$,

b) If $G \in O(\mathbb{X})$, then $a + G \subseteq cl(int(\delta\text{-}cl(a + G)))$ for every $a \in \mathbb{X}$,

c) If $G \in O(\mathbb{X})$, then $G + a \subseteq cl(int(\delta\text{-}cl(G + a)))$ for every $a \in \mathbb{X}$.

Theorem 3.3. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G \subseteq \mathbb{X}$. Then, the following properties hold.*

a) If $G \in C(\mathbb{X})$, then $-G \in e^*C(\mathbb{X})$,

b) If $a \in \mathbb{X}$ and $G \in C(\mathbb{X})$, then $a + G \in e^*C(\mathbb{X})$,

c) If $a \in \mathbb{X}$ and $G \in C(\mathbb{X})$, then $G + a \in e^*C(\mathbb{X})$.

Proof. $a)$ Let $G \in C(\mathbb{X})$. Our purpose is to show that $-G \in e^*C(\mathbb{X})$. Now, let $b \in e^*\text{-}cl(-G)$. We will show that $b \in -G$, i.e. $-b \in G$. Let $W \in O(\mathbb{X}, -b)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, -b) \Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(-U \subseteq W) \\ b \in e^*\text{-}cl(-G) \end{array}\right\} \Rightarrow (U \subseteq -W)(U \cap (-G) \neq \emptyset)$$

$\Rightarrow \emptyset \neq U \cap (-G) \subseteq (-W) \cap (-G)$

$\Rightarrow W \cap G \neq \emptyset$

Then, we get $-b \in cl(G)$. Since $G \in C(\mathbb{X})$, we have $-b \in G$, i.e. $b \in -G$. Thus, we have $-G \subseteq e^*\text{-}cl(-G) \subseteq -G$, i.e. $-G = e^*\text{-}cl(-G)$. This means $-G \in e^*C(\mathbb{X})$.

$b)$ Let $b \in \mathbb{X}$ and $G \in C(\mathbb{X})$. Our purpose is to show that $a + G \in e^*C(\mathbb{X})$. Now, let $b \in e^*\text{-}cl(a + G)$. We will prove that $b \in a + G$, i.e. $-a + b \in G$. Let $W \in O(\mathbb{X}, -b + a)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, -a+b) \Rightarrow (\exists U \in e^*O(\mathbb{X}, -a))(\exists V \in e^*O(\mathbb{X}, y))(U + V \subseteq W) \\ b \in e^*\text{-}cl(a+G) \end{array}\right\} \Rightarrow$$

$\Rightarrow (U + V \subseteq W)(V \cap (a + G) \neq \emptyset)$

$\Rightarrow \emptyset \neq (-a + V) \cap G \subseteq (U + V) \cap G \subseteq W \cap G$

$\Rightarrow W \cap G \neq \emptyset$

Then, we have $-a + b \in cl(G)$. Since $G \in C(\mathbb{X})$, we get $-a + b \in G$. Hence, $b \in a + G$.

$c)$ This follows (b) since the addition is commutative. $\square$

Corollary 3.2. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G \subseteq \mathbb{X}$. Then, the following statements hold.*

$a)$ If $G \in C(\mathbb{X})$, then $int(cl(int(-G))) \subseteq -G$,

$b)$ If $G \in C(\mathbb{X})$, then $int(cl(int(a + G))) \subseteq a + G$ for all $a \in \mathbb{X}$,

$c)$ If $G \in C(\mathbb{X})$, then $int(cl(int(G + a))) \subseteq G + a$ for all $a \in \mathbb{X}$.

4. MAIN RESULTS

In this section, we obtain some basic properties of e^*-topological ring. In addition, this section contains the definition of e^*-topological rings with unit and many fundamental results on this new notion.

Theorem 4.1. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring. Then, the following functions are e^*-continuous:*

$a)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a + b$ for all $b \in \mathbb{X}$,

$b)$ $f : \mathbb{X} \to \mathbb{X}$ defined by $f(a) = -a$ for all $a \in \mathbb{X}$,

$c)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = b + a$ for all $b \in \mathbb{X}$,

$d)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a + b + a$ for all $b \in \mathbb{X}$.

Proof. $a)$ Let $H \in O(\mathbb{X})$. Our aim is to show that $f_a^{-1}[H] \in e^*O(\mathbb{X})$.

$$\left.\begin{array}{r} f_a^{-1}[H] = \{b \in \mathbb{X} | f_a(b) \in H\} = \{b \in \mathbb{X} | b \in -a + H\} = -a + H \\ H \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow}$$

$\Rightarrow f_a^{-1}[H] \in e^*O(\mathbb{X})$.

$b)$ Let $H \in O(\mathbb{X})$. Our purpose is to show that $f^{-1}[H] \in e^*O(\mathbb{X})$.

$$\left.\begin{array}{r} f^{-1}[H] = \{a \in \mathbb{X} | f(a) \in H\} = \{a \in \mathbb{X} | -a \in H\} = -H \\ H \in O(\mathbb{X}) \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} f^{-1}[H] \in e^*O(\mathbb{X}).$$

$c)$ This follows (b) since the addition is commutative.

$d)$ This follows (b) and (c) since the addition is commutative.

□

Definition 4.1. *A bijective function $f : (\mathbb{X}, \mu) \to (\mathbb{Y}, \rho)$ which is e^*-continuous and whose inverse is e^*-continuous is called an e^*-homeomorphism.*

Corollary 4.1. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring. Then, the following functions are e^*-homeomorphism.*

$a)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a + b$ for all $b \in \mathbb{X}$,

$b)$ $f : \mathbb{X} \to \mathbb{X}$ defined by $f(a) = -a$ for all $a \in \mathbb{X}$,

$c)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = b + a$ for all $b \in \mathbb{X}$,

$d)$ For a fixed $a \in \mathbb{X}$, $f_a : \mathbb{X} \to \mathbb{X}$ defined by $f_a(b) = a + b + a$ for all $b \in \mathbb{X}$.

Definition 4.2. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring. If $(\mathbb{X}, +, \cdot)$ is a ring with unit, then $(\mathbb{X}, +, \cdot, \mu)$ is said to be an e^*-topological ring with unit. The notation $\mathbb{X}^*$ will be used to denote the set of all invertible elements in $(\mathbb{X}, +, \cdot)$.*

Theorem 4.2. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring with unit and $G \subseteq \mathbb{X}$. Then, the following properties hold.*

$a)$ If $G \in O(\mathbb{X})$, then Gs is e^*-open in $\mathbb{X}$ for each $s \in \mathbb{X}^*$,

$b)$ If $G \in O(\mathbb{X})$, then sG is e^*-open in $\mathbb{X}$ for each $s \in \mathbb{X}^*$.

Proof. $a)$ Let $G \in O(\mathbb{X})$ and $s \in \mathbb{X}^*$. We will prove $Gs \in e^*O(\mathbb{X})$. If we prove $Gs \subseteq e^*\text{-}int(Gs)$, then the proof complete. Let $b \in Gs$.

$$\left.\begin{array}{r} \left.\begin{array}{r} b \in Gs \Rightarrow (\exists k \in G)(b = ks) \\ s \in T^* \end{array}\right\} \Rightarrow (\exists k \in G)(bs^{-1} = k) \\ G \in O(\mathbb{X}) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(\exists V \in e^*O(\mathbb{X}, s^{-1}))(Us^{-1} \subseteq UV \subseteq G)$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, b))(U \subseteq Gs)$

$\Rightarrow y \in e^*\text{-}int(Gs)$

Then, we have $Gs \subseteq e^*\text{-}int(Gs) \subseteq Gs$ which means $Gs \in e^*O(\mathbb{X})$.

b) It is proved similarly to (a). □

Theorem 4.3. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring with unit and $G \subseteq \mathbb{X}$. Then, the following properties hold.*

a) If $G \in C(\mathbb{X})$, then $Gs \in e^*C(\mathbb{X})$ for each $s \in \mathbb{X}^*$,

b) If $G \in C(\mathbb{X})$, then $sG \in e^*C(\mathbb{X})$ for each $s \in \mathbb{X}^*$.

Proof. Let $G \in C(\mathbb{X})$ and $s \in \mathbb{X}^*$.

$$\left.\begin{array}{r} b \notin sG \Rightarrow (\forall k \in G)(b \neq sk) \\ \text{Hypothesis} \end{array}\right\} \Rightarrow (\forall k \in cl(G))(b \neq sG)$$

$\Rightarrow (\forall k \in cl(G))(s^{-1}b \neq k)$

$\Rightarrow s^{-1}y \notin cl(G)$

$\Rightarrow (\exists U \in O(\mathbb{X}, s^{-1}b))(U \cap G = \emptyset)$

$\Rightarrow (\exists K \in e^*O(\mathbb{X}, s^{-1}))(\exists M \in e^*O(\mathbb{X}, s^{-1}))(s^{-1}M \cap G \subseteq KM \cap G \subseteq U \cap G = \emptyset)$

$\Rightarrow (\exists M \in e^*O(\mathbb{X}, b))(s^{-1}M \cap G = \emptyset)$

$\Rightarrow (\exists M \in e^*O(\mathbb{X}, b))(M \cap sG = \emptyset)$

$\Rightarrow b \notin e^*\text{-}cl(sG)$

Then, we have $sG \subseteq e^*\text{-}cl(sG) \subseteq sG$ which means $sG \in e^*C(\mathbb{X})$.

b) It is proved similarly to (a). □

Theorem 4.4. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring with unit and $G \subseteq \mathbb{X}$. Then, the following properties hold:*

a) $s \cdot e^*\text{-}cl(G) \subseteq cl(sG)$ for each $s \in \mathbb{X}$,

b) $int(sG) \subseteq s \cdot e^*\text{-}int(G)$ for each $s \in \mathbb{X}$.

c) $s \cdot int(G) \subseteq e^*\text{-}int(sG)$ for each $s \in \mathbb{X}^*$,

d) $e^*\text{-}cl(sG) \subseteq s \cdot cl(G)$ for each $s \in \mathbb{X}^*$,

e) $e^*\text{-}cl(G) \cdot s \subseteq cl(Gs)$ for each $s \in \mathbb{X}$,

f) $int(G) \cdot s \subseteq e^*\text{-}int(Gs)$ for each $s \in \mathbb{X}^*$.

Proof. a) Let $a \in s \cdot e^*\text{-}cl(G)$. Our purpose is to show that $a \in cl(sG)$. Now, let $U \in O(\mathbb{X}, a)$.

$$\left.\begin{array}{r} a \in s \cdot e^*\text{-}cl(G) \Rightarrow (\exists b \in e^*\text{-}cl(G))(a = sb) \\ U \in O(\mathbb{X}, a) \end{array}\right\} \Rightarrow$$

$\Rightarrow (b \in e^*\text{-}cl(G))(\exists K \in e^*O(\mathbb{X}, s))(\exists L \in e^*O(\mathbb{X}, b))(KL \subseteq U)$

$\Rightarrow (\exists K \in e^*O(\mathbb{X}, s))(\exists L \in e^*O(\mathbb{X}, b))(KL \subseteq U)(L \cap G \neq \emptyset)$

$\Rightarrow \emptyset \neq KL \cap sG \subseteq U \cap sG$

$\Rightarrow \emptyset \neq U \cap sG$

Then, we have $a \in cl(sG)$.

$b)$ Let $a \in int(sG)$. Our purpose is to show that $a \in s \cdot e^*\text{-}int(G)$.

$$\begin{aligned} a \in int(sG) &\Rightarrow (a \in sG)(int(sG) \in O(\mathbb{X}, a)) \\ &\Rightarrow (\exists b \in G)(a = sb)(int(sG) \in O(\mathbb{X}, a)) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, s))(\exists V \in e^*O(\mathbb{X}, b))(sV \subseteq UV \subseteq int(sG) \subseteq sG) \\ &\Rightarrow (\exists V \in e^*O(\mathbb{X}, b))(V \subseteq G) \\ &\Rightarrow b \in e^*\text{-}int(G) \\ &\Rightarrow a = sb \in s \cdot e^*\text{-}int(G). \end{aligned}$$

$c)$ Let $a \in s \cdot int(G)$. Our purpose is to show that $a \in e^*\text{-}int(sG)$.

$$\begin{aligned} a \in s \cdot int(G) &\Rightarrow s^{-1}a \in int(G) \\ &\Rightarrow int(G) \in O(\mathbb{X}, s^{-1}a) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, s^{-1}))(\exists V \in e^*O(\mathbb{X}, a))(s^{-1}V \subseteq UV \subseteq int(G) \subseteq G) \\ &\Rightarrow (\exists V \in e^*O(\mathbb{X}, a))(V \subseteq sG) \\ &\Rightarrow a \in e^*\text{-}int(sG). \end{aligned}$$

$d)$ Let $a \in e^*\text{-}cl(sG)$ and $W \in O(\mathbb{X}, s^{-1}a)$.

$$\left.\begin{array}{r} W \in O(\mathbb{X}, s^{-1}a) \Rightarrow (\exists U \in e^*O(\mathbb{X}, s^{-1}))(\exists V \in e^*O(\mathbb{X}, a))(s^{-1}V \subseteq UV \subseteq W) \\ a \in e^*\text{-}cl(sG) \end{array}\right\} \Rightarrow$$

$\Rightarrow (\exists U \in e^*O(\mathbb{X}, s^{-1}))(\exists V \in e^*O(\mathbb{X}, a))(s^{-1}V \subseteq UV \subseteq W)(V \cap sG \neq \emptyset)$

$\Rightarrow \emptyset \neq UV \cap A \subseteq W \cap G$

$\Rightarrow W \cap G \neq \emptyset$

Then, we have $s^{-1}a \in cl(G)$ which means $a \in s \cdot cl(G)$.

$e)$ Let $a \in e^*\text{-}cl(G) \cdot s$. Our purpose is to show that $a \in cl(Gs)$. Now, let $U \in O(\mathbb{X}, a)$.

$$\left.\begin{array}{r} a \in e^*\text{-}cl(G) \cdot s \Rightarrow (\exists b \in e^*\text{-}cl(G))(a = bs) \\ U \in O(\mathbb{X}, a) \end{array}\right\} \Rightarrow$$

$\Rightarrow (b \in e^*\text{-}cl(G))(\exists K \in e^*O(\mathbb{X}, b))(\exists L \in e^*O(\mathbb{X}, s))(KL \subseteq U)$

$\Rightarrow (\exists K \in e^*O(\mathbb{X}, b))(\exists L \in e^*O(\mathbb{X}, s))(KL \subseteq U)(K \cap G \neq \emptyset)$

$\Rightarrow \emptyset \neq KL \cap Gs \subseteq U \cap Gs$

$\Rightarrow \emptyset \neq U \cap Gs$

Then, we have $a \in cl(Gs)$.

$f)$ Let $a \in int(G) \cdot s$. Our purpose is to show that $a \in e^*\text{-}int(Gs)$.

$$\begin{aligned} a \in int(G) \cdot s &\Rightarrow as^{-1} \in int(G) \\ &\Rightarrow int(G) \in O(\mathbb{X}, as^{-1}) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, a))(\exists V \in e^*O(\mathbb{X}, s^{-1}))(Us^{-1} \subseteq UV \subseteq int(G) \subseteq G) \\ &\Rightarrow (\exists U \in e^*O(\mathbb{X}, a))(U \subseteq Gs) \\ &\Rightarrow a \in e^*\text{-}int(Gs). \end{aligned}$$ □

Theorem 4.5. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring with unit and $s \in \mathbb{X}^*$. Then, the following functions are e^*-continuous.*

a) $f_s : \mathbb{X} \to \mathbb{X}$ *defined by* $f_s(a) = sa$ *for all* $a \in \mathbb{X}$,

b) $f_s : \mathbb{X} \to \mathbb{X}$ *defined by* $f_s(a) = as$ *for all* $a \in \mathbb{X}$,

c) $f_s : \mathbb{X} \to \mathbb{X}$ *defined by* $f_s(a) = sas$ *for all* $a \in \mathbb{X}$.

Proof. $a)$ Let $U \in O(\mathbb{X})$. Our purpose is to show that $f_s^{-1}[U] \in e^*O(\mathbb{X})$. For this, we will prove $f_s^{-1}[U] = e^*\text{-}int(f_s^{-1}[U])$. We have always $e^*\text{-}int(f_s^{-1}[U]) \subseteq f_s^{-1}[U] \ldots (1)$

Now, let $b \in f_s^{-1}[U]$.

$$\left.\begin{array}{r} b \in f_s^{-1}[U] = s^{-1}U \\ U \in O(\mathbb{X}) \Rightarrow U = int(U) \end{array}\right\} \Rightarrow b \in s^{-1} \cdot int(U) \overset{\text{Theorem 4.4}}{\Rightarrow} \left.\begin{array}{r} b \in e^*\text{-}int(s^{-1}U) \\ s^{-1}U = f_s^{-1}[U] \end{array}\right\} \Rightarrow$$

$\Rightarrow b \in f_s^{-1}[U]$

Then, we have $f_s^{-1}[U] \subseteq e^*\text{-}int(f_s^{-1}[U]) \ldots (2)$

$(1), (2) \Rightarrow f_s^{-1}[U] = e^*\text{-}int(f_s^{-1}[U]) \Rightarrow f_s^{-1}[U] \in e^*O(\mathbb{X})$.

$b)$ This follows Theorem 4.4.

$c)$ This follows (a) and (b). □

Corollary 4.2. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring with unit and $s \in \mathbb{X}^*$. Then, the following functions are e^*-homeomorphism.*

a) $f_s : \mathbb{X} \to \mathbb{X}$ *defined by* $f_s(a) = sa$ *for all* $a \in \mathbb{X}$,

b) $f_s : \mathbb{X} \to \mathbb{X}$ *defined by* $f_s(a) = as$ *for all* $a \in \mathbb{X}$.

c) $f_s : \mathbb{X} \to \mathbb{X}$ *defined by* $f_s(a) = sas$ *for all* $a \in \mathbb{X}$.

Theorem 4.6. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G \subseteq \mathbb{X}$. Then, the following properties hold for each $a \in \mathbb{X}$.*

a) $a + e^*\text{-}cl(G) \subseteq cl(a + G)$,

b) $e^*\text{-}cl(a + G) \subseteq a + cl(G)$,

c) $a + int(G) \subseteq e^*\text{-}int(a + G)$, *d)* $int(a+G) \subseteq a + e^*\text{-}int(G)$.

Proof. *a)* Let $b \in a + e^*\text{-}cl(G)$. Our purpose is to prove that $b \in cl(a+G)$. Now, let $U \in O(\mathbb{X}, b)$. If we prove $U \cap (a+G) \neq \emptyset$, then the proof complete.

$$\left.\begin{array}{r} b \in a + e^*\text{-}cl(G) \Rightarrow (\exists c \in e^*\text{-}cl(G))(b = a + c) \\ U \in O(\mathbb{X}, b) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists K \in e^*O(\mathbb{X}, a))(\exists L \in e^*O(\mathbb{X}, c))(\emptyset \neq (K+L) \cap (a+G) \subseteq U \cap (a+G))$$

$$\Rightarrow U \cap (a+G) \neq \emptyset.$$

b) Let $b \in e^*\text{-}cl(a+G)$. Our purpose is to show that $b \in a + cl(G)$. Now, let $U \in O(\mathbb{X}, -a+b)$.

$$U \in O(\mathbb{X}, -a+b) \Rightarrow (\exists K \in e^*O(\mathbb{X}, -a))(\exists L \in e^*O(\mathbb{X}, b))(-a + L \subseteq K + L \subseteq U)$$

$$\Rightarrow \emptyset \neq (-a+L) \cap G \subseteq U \cap G$$

Therefore, $-a + b \in cl(G)$ which means $b \in a + cl(G)$.

c) Let $b \in a + int(G)$. Our purpose is to show that $b \in e^*\text{-}int(a+G)$.

$$\begin{array}{rcl} b \in a + int(G) & \Rightarrow & -a + b \in int(G) \in O(\mathbb{X}) \\ & \Rightarrow & (\exists U \in e^*O(\mathbb{X}, -a))(\exists V \in e^*O(\mathbb{X}, b))(-a + V \subseteq U + V \subseteq int(G) \subseteq G) \\ & \Rightarrow & (\exists V \in e^*O(\mathbb{X}, b))(V \subseteq a + G) \\ & \Rightarrow & b \in e^*\text{-}int(a+G). \end{array}$$

d) Let $b \in int(a+G)$. Our purpose is to show that $b \in a + e^*\text{-}int(G)$.

$$\begin{array}{rcl} b \in int(a+G) & \Rightarrow & (\exists U \in O(\mathbb{X}, b))(U \subseteq a + G) \\ & \Rightarrow & (\exists U \in O(\mathbb{X}, b))(-a + U \subseteq G) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & (-a + U \in e^*O(\mathbb{X}, -a+b))(-a + U \subseteq G) \\ & \Rightarrow & -a + b \in e^*\text{-}int(G) \\ & \Rightarrow & b \in a + e^*\text{-}int(G). \end{array}$$

□

Theorem 4.7. *Let* $(\mathbb{X}, +, \cdot, \mu)$ *be an* e^**-topological ring and* $G \subseteq \mathbb{X}$. *Then, we have the following properties.*

a) $-e^*\text{-}cl(G) \subseteq cl(-G)$,

b) $e^*\text{-}cl(-G) \subseteq -cl(G)$,

c) $-int(G) \subseteq e^*\text{-}int(-G)$,

d) $int(-G) \subseteq -e^*\text{-}int(G)$.

Proof. a) Let $b \notin cl(-G)$.

$$\begin{aligned} b \notin cl(-G) &\Rightarrow (\exists U \in O(\mathbb{X}, b))(U \cap (-G) = \emptyset) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, -b))((-U) \cap G = \emptyset) \\ &\Rightarrow -b \notin e^*\text{-}cl(G) \\ &\Rightarrow b \notin -e^*\text{-}cl(G). \end{aligned}$$

b) Let $b \notin -cl(G)$.

$$\begin{aligned} b \notin -cl(G) &\Rightarrow -b \notin cl(G) \\ &\Rightarrow (\exists U \in O(\mathbb{X}, -b))(U \cap G = \emptyset) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, b))((-U) \cap (-G) = \emptyset) \\ &\Rightarrow b \notin e^*\text{-}cl(-G). \end{aligned}$$

c) Let $b \in -int(G)$.

$$\begin{aligned} b \in -int(G) &\Rightarrow -b \in int(G) \\ &\Rightarrow (\exists U \in O(\mathbb{X}, -b))(U \subseteq G) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, b))(-U \subseteq -G) \\ &\Rightarrow b \in e^*\text{-}int(-G). \end{aligned}$$

d) Let $b \in int(-G)$.

$$\begin{aligned} b \in int(-G) &\Rightarrow (\exists U \in O(\mathbb{X}, b))(U \subseteq -G) \\ &\Rightarrow (-U \in e^*O(\mathbb{X}, -b))(-U \subseteq G) \\ &\Rightarrow -b \in e^*\text{-}int(G) \\ &\Rightarrow b \in -e^*\text{-}int(G). \quad \square \end{aligned}$$

Theorem 4.8. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G \subseteq \mathbb{X}$. Then, we have the following properties for all $a \in \mathbb{X}$.*

a) $a + int(cl(\delta\text{-}int(G))) \subseteq cl(a + G)$,

b) $int(cl(\delta\text{-}int(a + G))) \subseteq a + cl(G)$,

c) $a + int(G) \subseteq cl(int(\delta\text{-}cl(a + G)))$,

d) $int(a + G) \subseteq a + cl(\delta\text{-}cl(G))$.

Proof. a) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$(G \subseteq X)(a \in \mathbb{X}) \Rightarrow cl(a + G) \in C(\mathbb{X}) \overset{\text{Theorem 3.2}}{\Rightarrow} -a + cl(a + G) \in e^*C(\mathbb{X})$$

$$\Rightarrow int(cl(\delta\text{-}int(e^*\text{-}cl(G)))) \subseteq int(cl(\delta\text{-}int(-a + cl(a + G)))) \subseteq -a + cl(a + G)$$

$$\Rightarrow int(cl(\delta\text{-}int(G))) \subseteq int(cl(\delta\text{-}int(e^*\text{-}cl(G)))) \subseteq -a + cl(a + G)$$

$$\Rightarrow a + int(cl(\delta\text{-}int(G))) \subseteq cl(a + G).$$

b) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$\left.\begin{array}{r} G \subseteq \mathbb{X} \Rightarrow cl(G) \in C(\mathbb{X}) \\ a \in \mathbb{X} \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} a + cl(G) \in e^*C(\mathbb{X})$$

$$\Rightarrow int(cl(\delta\text{-}int(a+G))) \subseteq int(cl(\delta\text{-}int(a+cl(G)))) \subseteq a + cl(a+G).$$

c) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$\left.\begin{array}{r} G \subseteq \mathbb{X} \Rightarrow int(G) \in O(\mathbb{X}) \\ a \in \mathbb{X} \end{array}\right\} \overset{\text{Theorem 3.2}}{\Rightarrow} a + int(G) \in e^*O(\mathbb{X})$$

$$\Rightarrow a + int(G) \subseteq cl(int(\delta\text{-}cl(a+int(G)))) \subseteq cl(int(\delta\text{-}cl(a+G))).$$

d) Let $G \subseteq \mathbb{X}$ and $a \in \mathbb{X}$.

$$(G \subseteq \mathbb{X})(a \in \mathbb{X}) \Rightarrow int(a+G) \in O(\mathbb{X}) \overset{\text{Theorem 3.2}}{\Rightarrow} -a + int(a+G) \in e^*O(\mathbb{X})$$

$$\Rightarrow -a + int(a+G) \subseteq cl(int(\delta\text{-}cl(-a+int(a+G)))) \subseteq cl(int(\delta\text{-}cl(G))).$$ □

Theorem 4.9. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G \subseteq \mathbb{X}$. Then, we have the following properties.*

a) $-int(cl(\delta\text{-}int(G))) \subseteq cl(-G)$,

b) $int(cl(\delta\text{-}int(-G))) \subseteq -cl(G)$,

c) $-int(G) \subseteq cl(int(\delta\text{-}cl(-G)))$,

d) $int(-G) \subseteq -cl(int(\delta\text{-}cl(G)))$.

Proof. *a*) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{lcl} G \subseteq \mathbb{X} & \Rightarrow & cl(-G) \in C(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -cl(-G) \in e^*C(\mathbb{X}) \\ & \Rightarrow & int(cl(\delta\text{-}int(G))) \subseteq int(cl(\delta\text{-}int(-cl(-G)))) \subseteq -cl(-G) \\ & \Rightarrow & -int(cl(\delta\text{-}int(G))) \subseteq cl(-G). \end{array}$$

b) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{lcl} G \subseteq \mathbb{X} & \Rightarrow & cl(G) \in C(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -cl(G) \in e^*C(\mathbb{X}) \\ & \Rightarrow & int(cl(\delta\text{-}int(-G))) \subseteq int(cl(\delta\text{-}int(-cl(G)))) \subseteq -cl(G). \end{array}$$

c) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{lcl} G \subseteq \mathbb{X} & \Rightarrow & int(G) \in O(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -int(G) \in e^*O(\mathbb{X}) \\ & \Rightarrow & -int(G) \subseteq cl(int(\delta\text{-}cl(-int(G)))) \subseteq cl(int(\delta\text{-}cl(-G))). \end{array}$$

d) Let $G \subseteq \mathbb{X}$.

$$\begin{array}{lcl} G \subseteq \mathbb{X} & \Rightarrow & int(-G) \in O(\mathbb{X}) \\ & \overset{\text{Theorem 3.2}}{\Rightarrow} & -int(-G) \in e^*O(\mathbb{X}) \\ & \Rightarrow & -int(-G) \subseteq cl(int(\delta\text{-}cl(-int(-G)))) \subseteq cl(int(\delta\text{-}cl(G))) \\ & \Rightarrow & int(-G) \subseteq -cl(int(\delta\text{-}cl(G))). \quad \square \end{array}$$

Theorem 4.10. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and $G, H \subseteq \mathbb{X}$. Then, $e^*\text{-}cl(G) + e^*\text{-}cl(H) \subseteq cl(G + H)$.*

Proof. Let $c \in e^*\text{-}cl(G) + e^*\text{-}cl(H)$. Our purpose is to show that $c \in cl(G + H)$. Now, let $W \in O(\mathbb{X}, c)$.

$$\left.\begin{array}{r} c \in e^*\text{-}cl(G) + e^*\text{-}cl(H) \Rightarrow (\exists a \in e^*\text{-}cl(G))(\exists b \in e^*\text{-}cl(H))(c = a + b) \\ W \in O(\mathbb{X}, c) \end{array}\right\} \Rightarrow$$

$$\Rightarrow (\exists U \in O(\mathbb{X}, a))(\exists V \in O(\mathbb{X}, b))(U + V \subseteq W)(U \cap G \neq \emptyset)(V \cap H \neq \emptyset)$$

$$\Rightarrow (W \in O(\mathbb{X}, c))((U \cap G) + (V \cap H) \neq \emptyset)(U + V \subseteq W)$$

$$\Rightarrow (W \in O(\mathbb{X}, c))(\exists t \in \mathbb{X})(t \in (U \cap G) + (V \cap H))(U + V \subseteq W)$$

$$\Rightarrow (W \in O(\mathbb{X}, c))(\exists u \in U \cap G)(\exists v \in V \cap H)(t = u + v)(U + V \subseteq W)$$

$$\Rightarrow (W \in O(\mathbb{X}, c))(u \in U)(u \in G)(v \in V)(v \in H)(U + V \subseteq W)$$

$$\Rightarrow (W \in O(\mathbb{X}, c))(u + v \in U + V)(u + v \in G + H)(U + V \subseteq W)$$

$$\Rightarrow (W \in O(\mathbb{X}, c))(u + v \in (U + V) \cap (G + H) \subseteq W \cap (G + H))$$

$$\Rightarrow (W \in O(\mathbb{X}, c))(W \cap (G + H) \neq \emptyset). \quad \square$$

Remark 4.1. *The following example shows that the converse of inclusion given in Theorem 4.10 need not to be true in general.*

Example 4.1. *Let $\mathbb{X} = \{k, l, m, n\}$ and $\mu = \{\emptyset, \mathbb{X}, \{k\}, \{k, l\}\}$. Let the addition and multiplication operations on $\mathbb{X}$ be as given in Example 3.1. For the subsets $G = \{k\}$ and $H = \{m\}$, we have $cl(G + H) = cl(\{m\}) = \{m, n\}$ and $e^*\text{-}cl(G) + e^*\text{-}cl(H) = e^*\text{-}cl(\{k\}) + e^*\text{-}cl(\{m\}) = \{k\} + \{m\} = \{m\}$. It is obvious that $cl(\{m\}) = \{m, n\} \nsubseteq \{m\} = e^*\text{-}cl(G) + e^*\text{-}cl(H)$.*

Theorem 4.11. *Let $(\mathbb{X}, +, \cdot, \mu)$ be an e^*-topological ring and let $(\mathbb{Y}, +, \cdot, \rho)$ be a topological ring. If $f : \mathbb{X} \to \mathbb{Y}$ is a ring homomorphism and continuous at $0_{\mathbb{X}}$, then f is e^*-continuous.*

Proof. Let f be a homomorphism and continuous at $0_{\mathbb{X}}$. Our purpose is to show that f is e^*-continuous. Now, let $V \in O(\mathbb{X}, f(a))$.

$$\left.\begin{array}{r} V \in O(\mathbb{X}, f(a)) \Rightarrow f(a) = f(a + 0_{\mathbb{X}}) \in V \in O(T) \\ f \text{ is homomorphism} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow f(a) + f(0_{\mathbb{X}}) \in V \in O(\mathbb{X}) \Rightarrow -f(a) + V \in O(\mathbb{X}, f(0_{\mathbb{X}})) \\ f \text{ is continuous in } 0_{\mathbb{X}} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists W \in O(\mathbb{X}, 0_{\mathbb{X}}))(f[W] \subseteq -f(a) + V) \Rightarrow (\exists W \in O(\mathbb{X}, 0_{\mathbb{X}}))(f(a) + f[W] \subseteq V) \\ f \text{ is homomorphism} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{r} \Rightarrow (\exists W \in O(\mathbb{X}, 0_{\mathbb{X}}))(f[a + W] \subseteq V) \\ U := a + W \end{array}\right\} \Rightarrow (U \in e^*O(\mathbb{X}, a))(f[U] \subseteq V).$$

□

5. Conclusion

The idea of obtaining more general results than those existing in the literature has led mathematicians to introduce new concepts such as topological groups, topological rings, topological fields, and topological vector spaces. In this article, we have introduced a new concept, called e^*-topological ring, by utilizing e^*-open sets. This new concept comes across as a more general concept than the concept of β-topological rings. On the other hand, the results given in this study coincide with the results given [2] in regular topological spaces, since the collection of all β-open sets is equal to the collection of all e^*-open sets in regular spaces. We obtained not only many results related to this new notion but also gave some counterexamples. We believe that the results obtained in this study will find an important place in future research on topological rings.

References

[1] Abd El-Monsef, M.E. El-Deeb, S.N. & Mahmoud, R.A. (1983). β-open sets and β-continuous mappings, Bull. Fac. Sci. Assiut Univ., 12, 77-90.

[2] Billawria, S. & Sharma, Sh. (2021). On β-topological rings, J. Algebra Relat. Top., 9(1), 51-60.

[3] Ekici, E. (2009). On e^*-open sets and (D, S)-sets, Math. Morav., 13, 29-36.

[4] Kaplansky, I. (1947). Topological rings, Amer. J. Math., 69, 153-182.

[5] Kaplansky, I. (1948). Topological rings, Bull. Amer. Math. Soc., 54, 809-826.

[6] Stone, M.H. (1937). Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc., 41, 375-381.

[7] Velicko, N.V. (1968). H-closed topological spaces, Amer. Math. Soc. Transl., 78, 103-118.

[8] Levine, N. (1963). Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70, 36-41.

[9] Njastad, O. (1965). On some classes of nearly open sets, Pacific J. Math., 15, 961-970.

[10] Mashhour, A.S. Abd El-Monsef, M.E. & El-Deeb, S.N. (1982). On precontinuous and weak precontinuous mappings, Proc. Math Phys. Soc. Egypt, 53, 47-53.

[11] Ram, M. Sharma, S. Billawria, S. & Landol, T. (2019). On α-irresolute topological rings, Int. J. of Math. Trends and Technology, (2) 65, 1-5.

[12] Salih, H.M.M. (2018). On irresolute topological rings, J. Adv. Stud. Topol., 9(2), 130-134.

[13] Stone, M.H. (1937). Applications of the theory of Boolean rings to general topology, Trans. Amer. Math. Soc. 41, 375-381.

[14] Veličko, N.V. (1968). *H*-closed topological spaces, Amer. Math. Soc. Transl. 78(2), 103-118.

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