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ON SHEFFER STROKE BH-ALGEBRAS

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Abstract. In this study, a Sheffer stroke BH-algebra is introduced and its features are examined. After showing that the axioms of a Sheffer stroke BH-algebra are independent, the connection between a Sheffer stroke BH-algebra and a BH-algebra is stated. After describing a subalgebra and a normal subset of a Sheffer stroke BH-algebra, the relationship between these structures is shown. A filter of a Sheffer stroke BH-algebra is defined and the quotient of a Sheffer stroke BH-algebra is constructed. Then a homomorphism between Sheffer stroke BH-algebras is introduced and its properties are studied.

Keywords: (Sheffer stroke) BH-algebra, Sheffer stroke, homomorphism

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1. Introduction

The notion of BCK-algebras was first formulated in 1966 [9] by Y. Imai and K. Iséki as a generalization of the notation of set-theoretic difference and propositional calculus, where this notion was originated from two different ways: one of the motivation was set theory, another motivation was classical and theory, the other from classical and non-classical propositional calculus. In the same year, K. Iséki introduced the notion of a BCI–algebra [10]. It is known that the BCI-algebra is a generalization of a BCK-algebra. Q. P. Hu and X. Li introduced a

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large class of abstract algebras: BCH-algebras [7, 8]. The class of BCI-algebras is a proper subclass of the class of BCH-algebras. In 1998, Y. B. Jun, E. H. Roh and H.S. Kim introduced a new concept, called a BH-algebra, which is generalization of BCH/BCI/BCK-algebras [11]. They argued further properties of BH-algebras. In 2011, H. H. Abbass and H. M. Saeed introduced the notions of (a closed ideal and closed BCH-algebra) with respect to an element of a BCH-algebra[3]. In 2012, H. H. Abbass and H. A. Dahham introduced the notion of completely closed ideal of a BH-algebra [2]. In 2014, H. H. Abbass and S. A. Neamah introduced the notion of a fuzzy implicative ideal with respect to an element of a BH-algebra [4].

The Sheffer stroke operation was originally introduced by H. M. Sheffer [20]. Since any Boolean function or operation can be stated by only this operation [12], it attracts the attention of many researchers. It also leads to reduction of axiom systems of many structures. Also, some applications of this operation have appeared in algebraic structures such as Sheffer stroke BG-algebras [13], Sheffer stroke BCK-algebras [14], the Sheffer stroke operation reducts of basic algebra [15], a construction of very true operator on Sheffer stroke MLT-algebras [17], congruences of Sheffer Stroke Basic Algebras [16], a view on state operators in Sheffer stroke basic algebras [18] and Bosbach state operators on Sheffer stroke MTL-algebras [19].

After giving main definitions and concepts of a Sheffer stroke and a BH-algebra, a Sheffer stroke BH-algebra is defined. It is proved that the axiom system of a Sheffer stroke BH-algebra is independent. By presenting fundamental notions about this algebraic structure, the connection between a Sheffer stroke BH-algebras is a BH-algebra is given. It is shown that Cartesian product of two Sheffer stroke BH-algebras is a Sheffer stroke BH-algebra. After defining a subalgebra and a normal subset, the relationship between a subalgebra and a normal subset on a Sheffer stroke BH-algebra is shown. A filter in a Sheffer stroke BH-algebra is defined. It is proved that the family of all filters of a Sheffer stroke BH-algebra forms a complete lattice. Then a homomorphism on a Sheffer stroke BH-algebra is defined and it is shown that the notion of a filter on a Sheffer stroke BH-algebra is preserved under the homomorphism. It is presented that a quotient of a Sheffer stroke BH-algebra is a Sheffer stroke BH-algebra. furthermore, a kernel of a homomorphism is constructed and proved that the kernel is a filter under a condition.

2. Preliminaries

In this part, we give the basic definitions and notions about a Sheffer stroke and a BH-algebra.

Definition 2.1. [5] *Let $\mathcal{A} = \langle A, |\rangle$ be a groupoid. The operation $|$ is said to be a Sheffer stroke if it satisfies the following conditions:*
(S1) $\breve{a}_1|\breve{a}_2 = \breve{a}_2|\breve{a}_1$,
(S2) $(\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_2) = \breve{a}_1$,
(S3) $\breve{a}_1|((\breve{a}_2|\breve{a}_3)|(\breve{a}_2|\breve{a}_3)) = ((\breve{a}_1|\breve{a}_2)|(\breve{a}_1|\breve{a}_2))|\breve{a}_3$,
(S4) $(\breve{a}_1|((\breve{a}_1|\breve{a}_1)|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|((\breve{a}_1|\breve{a}_1)|(\breve{a}_2|\breve{a}_2))) = \breve{a}_1$.

Definition 2.2. [11] *A BH-algebra is an algebra $(A, *, 0)$ of type (2, 0) satisfying the following conditions:*
$(BH.1)$ $\breve{a}_1 * \breve{a}_1 = 0$,
$(BH.2)$ $\breve{a}_1 * \breve{a}_2 = 0$ *and* $\breve{a}_2 * \breve{a}_1 = 0$ *imply* $\breve{a}_1 = \breve{a}_2$,
$(BH.3)$ $\breve{a}_1 * 0 = \breve{a}_1$
for all $\breve{a}_1, \breve{a}_2 \in A$.

A BH-algebra is called bounded if it has the greatest element.

Definition 2.3. [11] *A nonempty subset S of a BH-algebra A is called a BH-subalgebra if $\breve{a}_1 * \breve{a}_2 \in S$, for all $\breve{a}_1, \breve{a}_2 \in S$.*

Definition 2.4. [1] *Let A be a BH-algebra. A nonempty subset N of A is said to be normal if $(\breve{a}_1 * x) * (\breve{a}_2 * y) \in N$, for any $\breve{a}_1 * \breve{a}_2$, $x * y \in N$.*

Definition 2.5. [1] *A filter of a BH-algebra A is a non-empty subset F of A satisfying the following conditions:*
(F_1) *If* $\breve{a}_1 \in F$ *and* $\breve{a}_2 \in F$, *then* $\breve{a}_2 * (\breve{a}_2 * \breve{a}_1) \in F$ *and* $\breve{a}_1 * (\breve{a}_1 * \breve{a}_2) \in F$,
(F_2) *If* $\breve{a}_1 \in F$ *and* $\breve{a}_1 * \breve{a}_2 = 0$ *then* $\breve{a}_2 \in F$.

Definition 2.6. [6] *A BH-algebra A is called an associative BH-algebra if $(\breve{a}_1 * \breve{a}_2) * \breve{a}_3 = \breve{a}_1 * (\breve{a}_2 * \breve{a}_3)$, for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$.*

Definition 2.7. [3] *Let A be a BH-algebra. Then the set $A_+ = \{\breve{a}_1 \in A | 0 * \breve{a}_1 = 0\}$ is called the BCA-part of A.*

3. Sheffer stroke BH-algebras

In this section, we provide fundamental definitions and concepts regarding a Sheffer stroke and a BH-algebra.

Definition 3.1. *A Sheffer stroke BH-algebra is a structure* $(A, |, 0)$ *of type* $(2, 0)$, *where* 0 *is the constant on* A *and the following axioms hold for all* $\breve{a}_1, \breve{a}_2 \in A$:
$(sBH.1)$ $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = 0$,
$(sBH.2)$ $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ *and* $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = 0$ *imply* $\breve{a}_1 = \breve{a}_2$.

Let A be a Sheffer stroke BH-algebra unless otherwise stated.

Lemma 3.1. *The axioms (sBH.1) and (sBH.2) are independent:*

Proof. Consider the groupoid $(\{0, 1\}, |_p)$.
(1) Independence of (sBH.1):

Table 1. Operation table for independence of (sBH.1)

$\vert_p$	0	1
0	1	1
1	0	0

Then $|_p$ satisfies (sBH.2) but not (sBH.1) since $(1|_p(1|_p1))|_p(1|_p(1|_p1)) = (0|_p0) = 1 \neq 0$.

(2) Independence of (sBH.2):

Table 2. Operation table for independence of (sBH.2)

$\vert_q$	0	1
0	0	1
1	1	0

Then $|_q$ satisfies (sBH.1) but not (sBH.2) since $(0|_q(1|_q1))|_q(0|_q(1|_q1)) = 0|_q0 = 0$ and $(1|_q(0|_q0))|_q(1|_q(0|_q0)) = 1|_q1 = 0$ but $1 \neq 0$.

□

Example 3.1. *Consider a set $A = \{0, x, y, 1\}$, and define a Sheffer stroke $|$ by Table 3 and its Hasse diagram is given in Figure 1.*

TABLE 3

$\|$	0	x	y	1
0	1	1	1	1
x	1	y	1	y
y	1	1	x	x
1	1	y	x	0

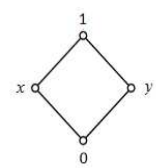

FIGURE 1. Hasse diagram

Then $(A, |)$ is a Sheffer stroke BH-algebra.

Lemma 3.2. *Let A be a Sheffer stroke BH-algebra. Then the following features hold for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$:*

(1) $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|\breve{a}_1) = \breve{a}_1$,

(2) $(0|0)|(\breve{a}_1|\breve{a}_1) = \breve{a}_1$,

(3) $(\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) = \breve{a}_1$,

(4) $\breve{a}_1|0 = 0|0$,

(5) $\breve{a}_1|((\breve{a}_2|(\breve{a}_3|\breve{a}_3))|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))) = \breve{a}_2|((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))$,

(6) $(\breve{a}_1|((\breve{a}_2|(\breve{a}_3|\breve{a}_3))|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))))|((\breve{a}_2|(\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))) = 0|0$,

(7) $\breve{a}_1|(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2))) = 0|0$,

(8) $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|\breve{a}_1) = 0|0$,

(9) $((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) = 0|0$.

Proof. (1) Substituting $[\breve{a}_2 := (\breve{a}_1|\breve{a}_1)]$ in (S2), we obtain

$$(\breve{a}_1|\breve{a}_1)|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = \breve{a}_1.$$

Then we have $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|\breve{a}_1) = \breve{a}_1$ from (S1).

(2) By (sBH.1), (S2) and (1), we obtain

$$\begin{aligned}(0|0)|(\breve{a}_1|\breve{a}_1) &= (((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1))))|(\breve{a}_1|\breve{a}_1)\\ &= (\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|\breve{a}_1)\\ &= \breve{a}_1.\end{aligned}$$

(3) By (S1), (S2) and (2), we have

$$\begin{aligned}(\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) &= (((\breve{a}_1|(\breve{a}_1)|(\breve{a}_1|\breve{a}_1))|(0|0))|((\breve{a}_1|(\breve{a}_1)|(\breve{a}_1|\breve{a}_1))|(0|0)))\\ &= ((0|0)|((\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_1)))|((0|0)|((\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_1)))\\ &= (\breve{a}_1|\breve{a}_1)|(\breve{a}_1|\breve{a}_1)\\ &= \breve{a}_1.\end{aligned}$$

(4) By (S1), (S2) and (2), it is implied that

$$\begin{aligned}\breve{a}_1|0 &= \breve{a}_1|((0|0)|(0|0))\\ &= ((0|0)|(\breve{a}_1|\breve{a}_1))|((0|0)|(0|0))\\ &= ((0|0)|(0|0))|((0|0)|(\breve{a}_1|\breve{a}_1))\\ &= (0|0).\end{aligned}$$

(5) By (S1) and (S3), we have

$$\begin{aligned}\breve{a}_1|((\breve{a}_2|(\breve{a}_3|\breve{a}_3))|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))) &= (((\breve{a}_1|\breve{a}_2)|(\breve{a}_1|\breve{a}_2))|(\breve{a}_3|\breve{a}_3))\\ &= (((\breve{a}_2|\breve{a}_1)|(\breve{a}_2|\breve{a}_1))|(\breve{a}_3|\breve{a}_3))\\ &= \breve{a}_2|((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3))).\end{aligned}$$

(6) It is obtained from (sBH.1) and (5).

(7) In (S3), by substituting $[\breve{a}_2 := \breve{a}_1|(\breve{a}_2|\breve{a}_2)]$ and $[\breve{a}_3 := \breve{a}_2|\breve{a}_2]$ and applying (S1), (S3) and (sBH.1), we obtain

$$\begin{aligned}
\breve{a}_1|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|\breve{a}_2)) &= \breve{a}_1|(((\breve{a}_2|\breve{a}_2)|\breve{a}_1|(\breve{a}_2|\breve{a}_2))| \\
&\quad ((\breve{a}_2|\breve{a}_2)|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))) \\
&= ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \\
&= (\breve{a}_1|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\
&= 0|0.
\end{aligned}$$

(8) By (S1), (S3), (sBH.1) and (4), we have

$$\begin{aligned}
((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|\breve{a}_1) &= (\breve{a}_1|\breve{a}_1)|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\
&= (((\breve{a}_1|\breve{a}_1)|\breve{a}_1)|((\breve{a}_1|\breve{a}_1)|\breve{a}_1))|(\breve{a}_2|\breve{a}_2) \\
&= ((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|\breve{a}_2) \\
&= 0|(\breve{a}_2|\breve{a}_2) \\
&= 0|0.
\end{aligned}$$

(9) By (S1), (S3) and (sBH.1), we get

$$\begin{aligned}
((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) &= (\breve{a}_2|\breve{a}_2)|((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))| \\
&\quad (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))) \\
&= (((\breve{a}_2|\breve{a}_2)|\breve{a}_1)|((\breve{a}_2|\breve{a}_2)|\breve{a}_1))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \\
&= (\breve{a}_1|(\breve{a}_2|\breve{a}_2))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))| \\
&\quad (\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \\
&= 0|0.
\end{aligned}$$

□

Theorem 3.1. *Let $(A, |, 0)$ be a Sheffer stroke BH-algebra. If we define*

$\breve{a}_1 * \breve{a}_2 := (\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)),$

then $(A, *, 0)$ *is a BH-algebra.*

Proof. By using (sBH.1), (sBH.2), Lemma 3.2 (3), we have

$(BH.1) : \breve{a}_1 * \breve{a}_1 = (\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = 0.$

$(BH.2)$: $\breve{a}_1 * \breve{a}_2 = (\breve{a}_1|(\breve{a}_2|(\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|(\breve{a}_2)) = 0$ and $\breve{a}_2 * \breve{a}_1 = (\breve{a}_2|(\breve{a}_1|(\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|(\breve{a}_1)) = 0$ imply $\breve{a}_1 = \breve{a}_2$.

$(BH.3)$: $\breve{a}_1 * 0 = (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) = \breve{a}_1$.

Then $(A, *, 0)$ is a BH-algebra. □

Example 3.2. *Consider the Sheffer stroke BH-algebra* $(A, |, 0)$ *in Example 3.1 and define the binary operation* $*$ *by Table 4.*

Table 4

$*$	0	x	y	1
0	0	0	0	0
x	x	0	x	0
y	y	y	0	0
1	1	y	x	0

Then $(A, *, 0)$ *is a BH-algebra.*

Theorem 3.2. *Let* $(A, *, 0, 1)$ *be a bounded BH-algebra. If we define* $\breve{a}_1|\breve{a}_2 = (\breve{a}_1 * \breve{a}_2^0)^0$ *and* $\breve{a}_1^0 = 1 * \breve{a}_1$, *where* $\breve{a}_1 * (1 * \breve{a}_1) = \breve{a}_1$ *and* $1 * (1 * \breve{a}_1) = \breve{a}_1$, *then* $(A, |, 0)$ *is a Sheffer stroke BH-algebra.*

Proof. $(sBH.1)$: By using (BH.1), we have

$$\begin{aligned}(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) &= (\breve{a}_1 * \breve{a}_1)^0|(\breve{a}_1 * \breve{a}_1)^0 \\ &= ((\breve{a}_1 * \breve{a}_1)^0)^0 \\ &= \breve{a}_1 * \breve{a}_1 \\ &= 0.\end{aligned}$$

$(sBH.2)$: By using (BH.2), we obtain

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = \breve{a}_1 * \breve{a}_2 = 0$$

and $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = \breve{a}_2 * \breve{a}_1 = 0$ imply $\breve{a}_1 = \breve{a}_2$.

Then $(A, |, 0)$ is a Sheffer stroke BH-algebra. □

Example 3.3. *Consider a set* $A = \{0, x, y, z, t, u, v, 1\}$*, and define the binary operation by Table 5 and the Sheffer stroke "*$|$*" by Table 6. Then* $(A, *, 0, 1)$ *is a a bounded BH-algebra and* $(A, |, 1)$ *is a Sheffer stroke BH-algebra.*

TABLE 5

$*$	0	x	y	z	t	u	v	1
0	0	0	0	0	0	0	0	0
x	x	0	x	x	0	0	x	0
y	y	y	0	y	0	y	0	0
z	z	z	z	0	z	0	0	0
t	t	y	x	t	0	y	x	0
u	u	z	u	x	z	0	x	0
v	v	v	z	y	z	y	0	0
1	1	v	u	t	z	y	x	0

TABLE 6

$\|$	0	x	y	z	t	u	v	1
0	1	1	1	1	1	1	1	1
x	1	v	1	1	v	v	1	v
y	1	1	u	1	u	1	u	u
z	1	1	1	t	1	t	t	t
t	1	v	u	1	z	v	u	z
u	1	v	1	t	v	y	t	y
v	1	1	u	t	u	t	x	x
1	1	v	u	t	z	y	x	0

Theorem 3.3. *Let* $(A, |_A, 0_A)$ *and* $(B, |_B, 0_B)$ *be Sheffer stroke BH-algebras. Then,* $(A \times B, |_{A\times B}, 0_{A\times B})$ *is a Sheffer stroke BH-algebra where the set* $A \times B$ *is the Cartesian product of* A *and* B*, the operation* $|_{A\times B}$ *is defined by* $(\breve{a}_1, b_1)|_{A\times B}(\breve{a}_2, b_2) = (\breve{a}_1|_A\breve{a}_2, b_1|_B b_2)$ *and* $0_{A\times B} = (0_A, 0_B)$.

Definition 3.2. *A Sheffer stroke BH-algebra* A *is called an associative Sheffer stroke BH-algebra if* $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|\breve{a}_3) = (\breve{a}_1|(\breve{a}_2|(\breve{a}_3|\breve{a}_3)))$ *holds for all* $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$.

Theorem 3.4. *Let A be an associative Sheffer stroke BH-algebra. Then the following properties are hold:*

(1) $(0|(\breve{a}_1|\breve{a}_1))|(0|(\breve{a}_1|\breve{a}_1)) = \breve{a}_1$,

(2) $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = \breve{a}_2$,

(3) $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))$,

(4) $((\breve{a}_3|(\breve{a}_1|\breve{a}_1))|(\breve{a}_3|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) = \breve{a}_1|(\breve{a}_2|\breve{a}_2)$,

(5) $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ *implies* $\breve{a}_1 = \breve{a}_2$,

(6) $((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) = 0|0$,

(7) $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|\breve{a}_3) = ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|\breve{a}_2)$,

(8) $((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(a_4|a_4)) = ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|(a_4|a_4))$.

Proof. (1) By Lemma 3.2 (3), (S2) and (sBH.1), we have

$$\begin{aligned}(0|(\breve{a}_1|\breve{a}_1))|(0|(\breve{a}_1|\breve{a}_1)) &= (((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|\breve{a}_1))|(((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|\\ &\quad (\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|\breve{a}_1))\\ &= (\breve{a}_1|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))\\ &= (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0))\\ &= \breve{a}_1.\end{aligned}$$

(2) By (sBH.1) and (1), we obtain

$$\begin{aligned}(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) &= ((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))(\breve{a}_2|\breve{a}_2)|\\ &\quad ((\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)))(\breve{a}_2|\breve{a}_2)\\ &= (0|(\breve{a}_2|\breve{a}_2))|(0|(\breve{a}_2|\breve{a}_2))\\ &= \breve{a}_2.\end{aligned}$$

(3) By (S2), (sBH.1) and (2),

$((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))$

$|((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))$

$= ((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))$

$= (\breve{a}_2|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))$

$= (\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))$

$= 0.$

Similarly, we get

$$((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) = 0.$$

Therefore, we obtain from (sBH.2) that $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))$.

(4) By (sBH.1), (1) and (3),

$$\begin{aligned}((\breve{a}_3|(\breve{a}_1|\breve{a}_1))|(\breve{a}_3|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) &= ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \\ &= \breve{a}_1|(\breve{a}_3|(\breve{a}_3|(\breve{a}_2|\breve{a}_2))) \\ &= \breve{a}_1|(((\breve{a}_3|(\breve{a}_3|\breve{a}_3))|(\breve{a}_3|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|\breve{a}_2)) \\ &= \breve{a}_1|(0|(\breve{a}_2|\breve{a}_2)) \\ &= \breve{a}_1|(\breve{a}_2|\breve{a}_2).\end{aligned}$$

(5) Let $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$. Then $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = 0$ from (3). Thus $\breve{a}_1 = \breve{a}_2$ from (sBH.2).

(6) $((\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(\breve{a}_2|\breve{a}_2) = (\breve{a}_2|(\breve{a}_2|\breve{a}_2)) = 0|0$ from (2) and (sBH.1).

(7) By (3), we have

$$\begin{aligned}((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|\breve{a}_3) &= (\breve{a}_1|(\breve{a}_2|(\breve{a}_3|\breve{a}_3))) \\ &= (\breve{a}_1|(\breve{a}_3|(\breve{a}_2|\breve{a}_2))) \\ &= ((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|\breve{a}_2).\end{aligned}$$

(8) By (3) and (7), we have

$$\begin{aligned}((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(a_4|a_4)) &= ((\breve{a}_1|(\breve{a}_2|(a_4|a_4)))|(\breve{a}_1|(\breve{a}_2|(a_4|a_4))))|(\breve{a}_3|\breve{a}_3) \\ &= (((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(a_4|a_4)| \\ &\quad ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(a_4|a_4))| \\ &\quad (\breve{a}_3|\breve{a}_3) \\ &= ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(a_4|(\breve{a}_3|\breve{a}_3)) \\ &= ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_3|(a_4|a_4)).\end{aligned}$$

□

Definition 3.3. *A non-empty subset S of a Sheffer stroke BH-algebra A is called a Sheffer stroke BH-subalgebra of A if $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in S$, for all $\breve{a}_1, \breve{a}_2 \in S$.*

Example 3.4. *In Example 3.1,* $S_1 = \{0, x\}$, $S_2 = \{0, y\}$ *and* $S_3 = \{0, x, y\}$ *are subalgebras of* A.

Theorem 3.5. *Let* $(A, |, 0)$ *be a Sheffer stroke BH-algebra and* $\emptyset \neq S \subseteq A$. *Then the following are equivalent:*

(a) *S is a subalgebra of* A,

(b) $(\breve{a}_1|(\breve{a}_2|(0|0)))|(\breve{a}_1|(\breve{a}_2|(0|0))) \in S$ *for any* $\breve{a}_1, \breve{a}_2 \in S$.

Proof. $(a) \Rightarrow (b)$: Since $S \neq \emptyset$, there exists an element $\breve{a}_1 \in S$ and

$$0 = (\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) \in S.$$

Since S is closed under $|$, $(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0)) \in S$ and thus

$(\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))))|(\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))))$
$= (\breve{a}_1|(\breve{a}_2|(0|0)))|(\breve{a}_1|(\breve{a}_2|(0|0))) \in S.$

$(b) \Rightarrow (a)$: By using Lemma 3.2 (3), we get $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))))|(\breve{a}_1|((\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0))|(\breve{a}_2|(0|0)))) = (\breve{a}_1|(\breve{a}_2|(0|0)))|(\breve{a}_1|(\breve{a}_2|(0|0))) \in S$ for any $\breve{a}_1, \breve{a}_2 \in S$. □

Definition 3.4. *The set* $A_+ = \{\breve{a}_1 \in A | (0|(\breve{a}_1|\breve{a}_1))|(0|(\breve{a}_1|\breve{a}_1)) = 0\}$ *is called a BCA-part of* A.

Example 3.5. *Given the Sheffer stroke BH-algebra in Example 3.1. Then it is obvious that the set* $A_+ = \{0, x, y, 1\}$ *is a BCA-part of* A.

Theorem 3.6. *Let* A *be a Sheffer stroke BH-algebra. Then* A_+ *is a subalgebra of* A.

Proof. Clearly, $0 \in A_+$ and so A_+ is nonempty. Let $\breve{a}_1, \breve{a}_2 \in A_+$. By (S2) and Lemma 3.2 (4), we have

$(0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))))$
$= (0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))))|(0|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))))$
$= (0|0)|(0|0)$
$= 0.$

Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in A_+$ and so A_+ is a subalgebra of A. □

Definition 3.5. *A non-empty subset N of A is said to be normal subset of A if*

$$(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b)))|(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b))) \in N,$$

for any $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)), (a|(b|b))|(a|(b|b)) \in N$.

Example 3.6. *In Example 3.1, $N = \{0, x\}$ is a normal subset of A when $[\breve{a}_1 := 0], [\breve{a}_2 := 1], [a := x]$, and $[b := y]$. Since $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (0|(1|1))|(0|(1|1)) = 0 \in N$ and $(a|(b|b))|(a|(b|b)) = (x|(y|y))|(x|(y|y)) = x \in N$, we have*

$$(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b)))|(((\breve{a}_1|(a|a))|(\breve{a}_1|(a|a)))|(\breve{a}_2|(b|b)))$$
$$= (((0|(x|x))|(0|(x|x)))|(1|(y|y)))|(((0|(x|x))|(0|(x|x)))|(1|(y|y))) = 1|1 = 0 \in N.$$

Theorem 3.7. *Every normal subset N of a Sheffer stroke BH-algebra A is a Sheffer stroke subalgebra of A.*

Proof. If $\breve{a}_1, \breve{a}_2 \in N$ then $(\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)), (\breve{a}_2|(0|0))|(\breve{a}_2|(0|0)) \in N$. Since N is normal , then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = ((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(0|(0|0))|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1| (\breve{a}_2|\breve{a}_2)))|(0|(0|0)) \in N$. Therefore, N is a Sheffer stroke subalgebra of A. □

Remark 3.1. *The converse of Theorem 3.7 does not hold. In Example 3.1, $N = \{0, x, y\}$ is a subalgebra of A, but it is not normal, since $(1|(y|y))|(1|(y|y)) = x \in N$, $(0|(1|1))|(0|(1|1)) = 0 \in N$, while $(((1|(0|0))|(1|(0|0)))|(y|(1|1)))|(((1|(0|0))|(1|(0|0)))|(y|(1|1))) = 0|0 = 1 \notin N$.*

Proposition 3.1. *Let N be a Sheffer stroke normal subalgebra of A. If $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ for $\breve{a}_1, \breve{a}_2 \in N$, then $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) \in N$.*

Proof. Let $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$. Since $(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2)) = 0 \in N$ and N is a normal subalgebra,

$(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) = ((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))| (\breve{a}_2|(\breve{a}_2|\breve{a}_2)))|((\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|((\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))) \in N$ from (sBH.1) and Lemma 3.2 (3). □

4. On Filters of Sheffer stroke BH-algebras

We introduce the notion of filter in a Sheffer stroke BH-algebra in this section.

Definition 4.1. *A filter of A is a nonempty subset $F \subseteq A$ satisfying*

$(SF.1)$ *If $\breve{a}_1, \breve{a}_2 \in F$, then*

$$(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) \in F$$

and

$$(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in F.$$

$(SF.2)$ *If* $\breve{a}_1 \in F$ *and* $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$*, then* $\breve{a}_2 \in F$*.*

Example 4.1. *Consider the Sheffer stroke BH-algebra in Example 3.3. Then it is obvious that* $\{t, 1\}$ *is a filter of* A*.*

Theorem 4.1. *The family* K_A *of all filters in* A *forms a complete lattice.*

Proof. Let $\{F_i\}_{i\in I}$ be a family of filters of A. If $\breve{a}_1, \breve{a}_2 \in \bigcap_{i\in I} F_i$, then $\breve{a}_1, \breve{a}_2 \in F_i$, for all $i \in I$. Since F_i is a filter of A, then $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))$, $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in F_i$. Thus,

$$(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))), (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in \bigcap_{i\in I} F_i.$$

- (i) Suppose that $\breve{a}_1 \in \bigcap_{i\in I} F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in \bigcap_{i\in I} F_i$ hold for $\breve{a}_1, \breve{a}_2 \in A$, that is $\breve{a}_1 \in F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ hold for all $i \in I$. Then it is obtained from (SF.2) that $\breve{a}_2 \in F_i$ for all $i \in I$. Then $\breve{a}_2 \in \bigcap_{i\in I} F_i$.

- (ii) Let η be the family of all filters of A containing the union $\bigcup_{i\in I} F_i$. Then $\bigcap \eta$ is a filter of A from (i). If $\bigwedge_{i\in I} F_i = \bigcap_{i\in I} F_i$ and $\bigvee_{i\in I} F_i = \bigcap \eta$, then $(K_A, \bigwedge, \bigvee)$ is a complete lattice.

□

Corollary 4.1. *Let* B *be a subset of* A*. Then there is the minimal filter* $\langle B\rangle$ *containing the subset* B*.*

Proof. Let $\varepsilon = \{F : F \textit{ is a filter of } A \textit{ containing } B\}$. Then $\langle B\rangle = \{x \in A : x \in \bigcap_{F\in\varepsilon} F\}$ is the minimal filter of A containing B. □

Theorem 4.2. *Let* S *be a subalgebra of* A*. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

holds for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$*, then* S *is a filter of* A*.*

Proof. $(SF.1)$ Let S be a subalgebra of A and $\breve{a}_1, \breve{a}_2 \in S$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in S$ and $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) \in S$. So $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in S$ and $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) \in S$.

$(SF.2)$ Let $\breve{a}_1 \in S$, $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))$. We obtain $\breve{a}_1 = \breve{a}_2$. Thus, $\breve{a}_2 \in S$. Therefore, S is a filter of A. □

Corollary 4.2. *Let S be a normal subalgebra of A. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \text{ implies } \breve{a}_1 = \breve{a}_3$$

holds for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then S is a filter of A.

Proof. It is obtained from Theorem 3.7 and Theorem 4.2. □

Theorem 4.3. *Let A be a Sheffer stroke BH-algebra. If*

$$(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = \breve{a}_2$$

holds for all $\breve{a}_1, \breve{a}_2 \in A$, then every non-empty subset S of A is a filter of A.

Proof. Let S be a non-empty subset of A.
$(SF.1)$ Let $\breve{a}_1, \breve{a}_2 \in S$. Then $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))) = \breve{a}_1 \in S$. Similarly, $(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = \breve{a}_2 \in S$.
$(SF.2)$ Let $\breve{a}_1 \in S$, $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$. Then $\breve{a}_2 = (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) = (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) = \breve{a}_1$ from Lemma 3.2 (3). Thus, $\breve{a}_2 \in S$. Therefore, S is a filter of A. □

Proposition 4.1. *Let $\{Fi, i \in \lambda\}$ be a family of Sheffer stroke BH-filters of A. Then $\bigcap_{i\in\lambda} F_i$ is a Sheffer stroke BH-filter of A.*

Proof. Let $\{F_i, i \in \lambda\}$ be a family of Sheffer stroke BH-filters of A.
$(SF.1)$ If $\breve{a}_1, \breve{a}_2 \in \bigcap_{i\in\lambda} F_i$, then $\breve{a}_1, \breve{a}_2 \in F_i$, for all $i \in \lambda$. Since F_i is a filter of A, we have

$$(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))), (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in F_i.$$

Then $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1))), (\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in \bigcap_{i\in\lambda} F_i$.
$(SF.2)$ Suppose that $\breve{a}_1 \in \bigcap_{i\in\lambda} F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in \bigcap_{i\in\lambda} F_i$ hold for $\breve{a}_1, \breve{a}_2 \in A$, that is $\breve{a}_1 \in F_i$ and $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0$ hold for all $i \in \lambda$. Then $\breve{a}_2 \in F_i$ for all $i \in \lambda$. Then $\breve{a}_2 \in \bigcap_{i\in\lambda} F_i$. Therefore $\bigcap_{i\in\lambda} F_i$ is a Sheffer stroke BH-filter of A. □

The union of Sheffer stroke BH-filters of Sheffer stroke BH-algebras may not be a Sheffer stroke BH-filter as in the following example.

Example 4.2. *Consider the Sheffer stroke BH-algebra in Example 3.3. Then it is obvious that $F_1 = \{t, 1\}$ and $F_2 = \{u, 1\}$ are two filters of A. The union of that filters is not a Sheffer stroke BH-filter of A. Since $u, t \in F_1 \bigcup F_2$, but $(u|(u|(t|t)))|(u|(u|(t|t))) = x \notin F_1 \bigcup F_2$.*

Proposition 4.2. *Let $\{F_i, i \in \lambda\}$ be a chain of Sheffer stroke BH-filters of A. Then $\bigcup_{i\in\lambda} F_i$ is a Sheffer stroke BH-filter of A.*

5. Homomorphisms on Sheffer stroke BH-algebras

In this section, we present some definitions and concepts about homomorphism between Sheffer stroke BH-algebras.

Definition 5.1. *Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras. A mapping $f : A \longrightarrow B$ is called a homomorphism if*

$$f(\breve{a}_1|_A\breve{a}_2) = f(\breve{a}_1)|_B f(\breve{a}_2),$$

for all $\breve{a}_1, \breve{a}_2 \in A$.

A Sheffer stroke BH-homomorphism f is called a Sheffer stroke BH-monomorphism if it is injective.

Lemma 5.1. *Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a monomorphism. Then if F is a filter of A, $f(F)$ is a filter of B.*

Proof. Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a monomorphism.

- (i) Let F be a Sheffer stroke BH-filter of A and $\breve{a}_1, \breve{a}_2 \in f(F)$. Then there exist $x, y \in F$ such that $\breve{a}_1 = f(x), \breve{a}_2 = f(y)$. Since F is a filter, then $(\breve{a}_2|_B(\breve{a}_2|_B(\breve{a}_1|_B\breve{a}_1)))|_B(\breve{a}_2|_B (\breve{a}_2|_B(\breve{a}_1|_B\breve{a}_1))) = (f(y)|_B(f(y)|_B(f(x)|_B f(x))))|_B(f(y)|_B(f(y)|_B(f(x)|_B f(x)))) = f((y|_A(y|_A(x|_A x)))|_A(y|_A(y|_A(x|_A x)))) \in f(F)$. Hence $(\breve{a}_2|_B(\breve{a}_2|_B(\breve{a}_1|_B\breve{a}_1)))|_B(\breve{a}_2|_B(\breve{a}_2 |_B(\breve{a}_1|_B\breve{a}_1))) \in f(F)$. Similarly, $(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2)))|_B(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))) \in f(F)$.
- (ii) Let $\breve{a}_1 \in f(F)$ such that $(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2)) = 0_B$. Then there exist $x, y \in F$ such that $\breve{a}_1 = f(x)$ and $\breve{a}_2 = f(y)$. In this case, we have $(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))) |_B(\breve{a}_1|_B(\breve{a}_1|_B(\breve{a}_2|_B\breve{a}_2))) = (f(x)|_B(f(y)|_B f(y)))|_B(f(x)|_B(f(y)|_B f(y))) = f((x|_A(y|_A y))|_A(x|_A(y|_A y))) = 0_B = f(0_A)$. Since f is an injective, then $((x|_A(y|_A y))|_A(x|_A(y|_A y))) = 0_A$. Thus $y \in F$. So, $\breve{a}_2 = f(y) \in f(F)$. Therefore, $f(F)$ is a filter of B.

□

Theorem 5.1. *Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a homomorphism. If F is a filter of B, then $f^{-1}(F)$ is a filter of A.*

Proof. Let $(A, |_A, 0_A, 1_A)$ and $(B, |_B, 0_B, 1_B)$ be Sheffer stroke BH-algebras and $f : A \longrightarrow B$ be a homomorphism. Suppose that F is a filter of A.

• Let $\breve{a}_1, \breve{a}_2 \in f^{-1}(F)$. Then $f(\breve{a}_1), f(\breve{a}_2) \in F$. Since F is a filter, then $(f(\breve{a}_2)|_B((f(\breve{a}_2)|_B(f(\breve{a}_1) |_B f(\breve{a}_1)))))|_B(f(\breve{a}_2)|_B((f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1))) = f((\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))) \in F$. Therefore, $(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1))) \in f^{-1}(F)$. Similarly, $(\breve{a}_1|_A(\breve{a}_1 |_A(\breve{a}_2|_A\breve{a}_2)))|_A(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) \in f^{-1}(F)$.

• Let $\breve{a}_1 \in f^{-1}(F)$ such that $(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)) = 0_A$. Then $f(\breve{a}_1) \in F$ and $f((\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) = (f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2)))|_B(f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2))) = f(0_A) = 0_B$. Hence $f(\breve{a}_2) \in F$. Thus $\breve{a}_2 \in f^{-1}(F)$. Therefore, $f^{-1}(F)$ is a filter of A. □

Proposition 5.1. *Let A and B be Sheffer stroke BH-algebras and $f : (A, |_A, 0_A) \to (B, |_B, 0_B)$ be a Sheffer stroke BH-homomorphism. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \text{ implies } \breve{a}_1 = \breve{a}_3$$

holds for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then $Ker(f)$ is a Sheffer stroke BH-filter of A.

Proof. $(SF.1)$ Let $\breve{a}_1, \breve{a}_2 \in Ker(f)$. Then since $f(\breve{a}_1) = 0_B$ and $f(\breve{a}_2) = 0_B$, we have $f((\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))) = (f(\breve{a}_2)|_B(f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1)))|_B(f(\breve{a}_2)|_B (f(\breve{a}_2)|_B(f(\breve{a}_1)|_B f(\breve{a}_1))) = 0_B$. Hence $(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1)))|_A(\breve{a}_2|_A(\breve{a}_2|_A(\breve{a}_1|_A\breve{a}_1))) \in Ker(f)$. Similarly, $(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)))|_A(\breve{a}_1|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) \in Ker(f)$.

$(SF.2)$ Let $\breve{a}_1 \in Ker(f)$ and $\breve{a}_2 \in A$ such that $(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2)) = 0_A$. Then $f(\breve{a}_1) = 0_B$ and $f((\breve{a}_1)|_A(\breve{a}_2|_A\breve{a}_2))|_A(\breve{a}_1|_A(\breve{a}_2|_A\breve{a}_2))) = f(\breve{a}_1)|_B(f(\breve{a}_2)|_B f(\breve{a}_2)))|_B f(\breve{a}_1)|_B(f(\breve{a}_2) |_B f(\breve{a}_2))) = f(0_A)$. We get $(0_B|_B(f(\breve{a}_2)|_B(f(\breve{a}_2))))|_B(0_B|_B(f(\breve{a}_2)|_B(f(\breve{a}_2)))) = (f(\breve{a}_2)|_B(f(\breve{a}_2) |_B(f(\breve{a}_2))))|_B(f(\breve{a}_2)|_B(f(\breve{a}_2)|_B(f(\breve{a}_2))))$. We obtain $f(\breve{a}_2) = 0_B$. Thus $\breve{a}_2 \in Ker(f)$. Therefore, $Ker(f)$ is a filter of A. □

Lemma 5.2. *Let N be a normal subalgebra of A. Define a relation $\sim_N$ on A by $\breve{a}_1 \sim_N \breve{a}_2$ if and only if $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$, where $\breve{a}_1, \breve{a}_2 \in A$. Then $\sim_N$ is an equivalence relation on A.*

Proof. ▪ Reflexive: Since $0 \in A$, we have $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)) = 0 \in A$ i.e., $\breve{a}_1 \sim_N \breve{a}_1$ for any $\breve{a}_1 \in A$. This means that $\sim_N$ is reflexive.

▪ Symmetric: Let $\breve{a}_1 \sim_N \breve{a}_2$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ and $(\breve{a}_2|(\breve{a}_1|\breve{a}_1))|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)) \in N$ by Proposition 3.1. We obtain $\breve{a}_2 \sim_N \breve{a}_1$ for any $\breve{a}_1, \breve{a}_2 \in A$.

▪ Transitive: Let $\breve{a}_1 \sim_N \breve{a}_2$ and $\breve{a}_2 \sim_N \breve{a}_3$. Then $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ and $(\breve{a}_2|(\breve{a}_3|\breve{a}_3))|$

$(\breve{a}_2|(\breve{a}_3|\breve{a}_3)) \in N$. By Proposition 3.1, we have $(\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \in N$. Since N is a normal subalgebra, then $(((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2)))|(((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))) = (((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(0|0))|(((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)))|(0|0))=((\breve{a}_1|(\breve{a}_3|\breve{a}_3))|(\breve{a}_1|(\breve{a}_3|\breve{a}_3)) \in N$. We obtain $\breve{a}_1 \sim_N \breve{a}_3$ for any $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$. □

Lemma 5.3. *An equivalence relation $\sim_*$ is a congruence relation if and only if $\breve{a}_1 \sim_* \breve{a}_2$ and $x \sim_* y$ imply $(\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)) \sim_* (\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))$.*

Lemma 5.4. *Let N be a normal subalgebra of A and the binary relation defined as Lemma 5.2. Then $\sim_N$ is a congruence relation on A.*

Proof. Let $x, y, \breve{a}_1, \breve{a}_2$ be any elements in A such that $\breve{a}_1 \sim_N \breve{a}_2$ and $x \sim_N y$, i.e., $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in N$ and $(x|(y|y))|(x|(y|y)) \in N$. Since N is a normal subalgebra, we get $(((\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y)))|(((\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y))) \in N$. Then $(\breve{a}_1|(x|x))|(\breve{a}_1|(x|x)) \sim_* (\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))$. Therefore, $\sim_N$ is a congruence relation on A. □

Denote the equivalence class containing $\breve{a}_1$ by $[\breve{a}_1]_N$, i.e., $[\breve{a}_1]_N = \{\breve{a}_2 \in N \mid \breve{a}_1 \sim_N \breve{a}_2\}$ and $A/N = \{[\breve{a}_1]_N \mid \breve{a}_1 \in A\}$.

Theorem 5.2. *Let N be a normal subalgebra of A. Then $(A/N, |, [0]_N)$ is a Sheffer stroke BH-algebra.*

Proof. If we define $[\breve{a}_1]_N|[\breve{a}_2]_N := [\breve{a}_1|\breve{a}_2]_N$, then the operation $|$ is well-defined, since if $\breve{a}_1 \sim_N p$ and $\breve{a}_2 \sim_N q$, then $(\breve{a}_1|(p|p))|(\breve{a}_1|(p|p)) \in N$ and $(\breve{a}_2|(q|q))|(\breve{a}_2|(q|q)) \in N$ implies $(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(p|(q|q)))|(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(p|(q|q))) \in N$ by normality of N. Then we have $(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \sim_N (((p|(q|q))|(p|(q|q)))$ and so $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))_N = (((p|(q|q))|(p|(q|q))))_N$. Note that $[0]_N = \{\breve{a}_1 \in A \mid \breve{a}_1 \sim_N 0\} = \{\breve{a}_1 \in A \mid (\breve{a}_1|(0|0))|(\breve{a}_1|(0|0)) \in N\} = \{\breve{a}_1 \in A \mid \breve{a}_1 \in N\} = N$. □

$(sBH.1)$ $([\breve{a}_1]_N|([\breve{a}_1]_N|[\breve{a}_1]_N)|([\breve{a}_1]_N|([\breve{a}_1]_N|[\breve{a}_1]_N) = [0]_N$,
$(sBH.2)$ $([\breve{a}_1]_N|([\breve{a}_2]_N|[\breve{a}_2]_N)|([\breve{a}_1]_N|([\breve{a}_2]_N|[\breve{a}_2]_N) = [0]_N$ and $([\breve{a}_2]_N|([\breve{a}_1]_N|[\breve{a}_1]_N)|([\breve{a}_2]_N|([\breve{a}_1]_N|[\breve{a}_1]_N) = [0]_N$ imply $[\breve{a}_1]_N = [\breve{a}_2]_N$.

The Sheffer stroke BH-algebra A/N discussed in Theorem 5.2 is called the quotient Sheffer stroke BH-algebra of A by N.

Example 5.1. *Consider the Sheffer stroke BH-algebra in Example 3.3. For the normal subalgebra $F = \{0, t\}$ of A, $\beta_F = \{(0,0), (x,x), (y,y), (z,z), (t,t), (u,u), (v,v), (1,1), (0,t), (t,0), (z,$*

$1),(1,z)\}$ is a congruence on A defined by F. Then $(A/F,|_{\beta_F},[0]_{\beta_F})$ is a Sheffer stroke BH-algebra with the following Hasse diagram in which the quotient set is $A/F=\{[0]_{\beta_F}$, $[x]_{\beta_F}$, $[y]_{\beta_F}$,

$[1]_{\beta_F}\}$:

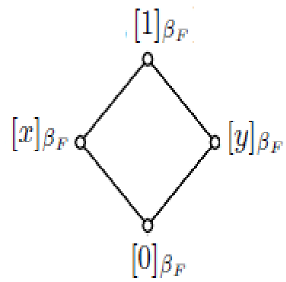

FIGURE 2. Hasse diagram

The binary operation $|_{\beta_F}$ on A/F has Cayley table in Table 7.

TABLE 7

$\vert_{\beta_F}$	$[0]_{\beta_F}$	$[x]_{\beta_F}$	$[y]_{\beta_F}$	$[1]_{\beta_F}$
$[0]_{\beta_F}$	$[1]_{\beta_F}$	$[1]_{\beta_F}$	$[1]_{\beta_F}$	$[1]_{\beta_F}$
$[x]_{\beta_F}$	$[1]_{\beta_F}$	$[y]_{\beta_F}$	$[1]_{\beta_F}$	$[y]_{\beta_F}$
$[y]_{\beta_F}$	$[1]_{\beta_F}$	$[1]_{\beta_F}$	$[x]_{\beta_F}$	$[x]_{\beta_F}$
$[1]_{\beta_F}$	$[1]_{\beta_F}$	$[y]_{\beta_F}$	$[x]_{\beta_F}$	$[0]_{\beta_F}$

Theorem 5.3. *Let N be a normal subalgebra of A. Then $[0]_N$ is a normal subalgebra of A.*

Proof. Since $0_A\in[0]_N$, $[0]_N$ is non-empty. Let $(\breve{a}_1|(\breve{a}_1|\breve{a}_1))|(\breve{a}_1|(\breve{a}_1|\breve{a}_1)),(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))\in[0]_N$. Then $(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))=0\in[0]_N$. By Theorem 3.7, $[0]_N$ is a normal subalgebra of A. □

Theorem 5.4. *Let N be a filter of A and $(A/N,|',[0]_N)$ be a Sheffer stroke BH-algebra. If F is a filter of A such that $N\subseteq F$, then F/N is a Sheffer stroke BH-filter.*

Proof. Let F be a Sheffer stroke BH-filter of A.

- Let $[\breve{a}_1]_N,[\breve{a}_2]_N\in F/N$, then $([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N)))|'([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N)))=[(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))]_N$. Hence $([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N)))|'([\breve{a}_2]_N|'([\breve{a}_2]_N|'([\breve{a}_1]_N|'[\breve{a}_1]_N)))\in F/N$. (Since $(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))|(\breve{a}_2|(\breve{a}_2|(\breve{a}_1|\breve{a}_1)))\in F$, F is a filter of A).Similarly,

$([\breve{a}_1]_N|'([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N)))|'([\breve{a}_1]_N|'([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N))) \in F/N$.

• Let $\breve{a}_1 \in F/N$ such that $([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N))|'([\breve{a}_1]_N|'([\breve{a}_2]_N|'[\breve{a}_2]_N)) = [0]_N$. Then $[(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))]_N = [0]_N$. Hence $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(0|0)|((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)))|(0|0) \in N$. Since $((\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2))) \in N$, we have $\breve{a}_2 \in F/N$. We obtain $[\breve{a}_2]_N = [\breve{a}_1]_N$, then $[\breve{a}_2]_N \in F/N$. Therefore, F/N is a filter of A. □

Theorem 5.5. *Let N be a normal subalgebra of A. Then the mapping $\gamma : A \to A/N$ given by $\gamma(\breve{a}_1) := [\breve{a}_1]_N$ is a surjective Sheffer stroke BH-homomorphism and $Ker\gamma = N$.*

The mapping γ discussed in above theorem is called the natural(or canonical) homomorphism of A onto A/N.

Theorem 5.6. *Let $\varphi : A \to B$ be a Sheffer stroke BH-homomorphism. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

hold for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then φ is injective if and only if $Ker\varphi = \{0_A\}$.

Proof. Let $\breve{a}_1, \breve{a}_2 \in A$ with $\varphi(\breve{a}_1) = \varphi(\breve{a}_2)$. Then from (sBH.1), we obtain $(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2)))|(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2))) = 0_B$. So $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) \in Ker\varphi$. Since $Ker\varphi = \{0_A\}$, $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = 0_A$=$(\breve{a}_2|(\breve{a}_2|\breve{a}_2))|(\breve{a}_2|(\breve{a}_2|\breve{a}_2))$. Then $\breve{a}_1 = \breve{a}_2$. Hence φ is injective. The converse is trivial. □

Theorem 5.7. *Let $\varphi : A \to B$ be a Sheffer stroke BH-homomorphism. If*

$$(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)) = (\breve{a}_3|(\breve{a}_2|\breve{a}_2))|(\breve{a}_3|(\breve{a}_2|\breve{a}_2)) \textit{ implies } \breve{a}_1 = \breve{a}_3$$

hold for all $\breve{a}_1, \breve{a}_2, \breve{a}_3 \in A$, then $Ker\varphi$ is a normal subalgebra of A.

Proof. Since $0_A \in Ker\varphi$, Ker $\varphi \neq \emptyset$. Let $(\breve{a}_1|(\breve{a}_2|\breve{a}_2))|(\breve{a}_1|(\breve{a}_2|\breve{a}_2)), (x|(y|y))|(x|(y|y)) \in Ker\varphi$. Then $(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2)))|(\varphi(\breve{a}_1)|(\varphi(\breve{a}_2)|\varphi(\breve{a}_2))) = 0 = (\varphi(x)|(\varphi(y)|\varphi(y)))|(\varphi(x)|(\varphi(y)|\varphi(y)))$. Since $\varphi(\breve{a}_1) = \varphi(\breve{a}_2)$ and $\varphi(x) = \varphi(y)$, we obtain $\varphi((((\breve{a}_1)|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y)))|((((\breve{a}_1)|(x|x))|(\breve{a}_1|(x|x)))|(\breve{a}_2|(y|y)))$=$(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(\varphi(\breve{a}_2)|(\varphi(y)|\varphi(y))))|(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(\varphi(\breve{a}_2)|(\varphi(y)|\varphi(y))))$=$(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))))|(((\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))|(\varphi(\breve{a}_1)|(\varphi(x)|\varphi(x)))) = 0$. Then we have $((((\breve{a}_1)|(x|x))|(\breve{a}_1|(x|x)))|((\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))|(\breve{a}_2|(y|y))|(\breve{a}_2|(y|y)))) \in Ker\varphi$. Hence $Ker\varphi$ is a normal subalgebra of A. □

By Theorem 5.5 and 5.7, if $\varphi : A \to B$ is a Sheffer stroke BH-homomorphism, then $A/Ker\varphi$ is a Sheffer stroke BH-algebra.

Theorem 5.8. *Let $\varphi : A \rightarrow B$ be a Sheffer stroke BH-homomorphism. Then $A/Ker\varphi \cong Im\varphi$. In particular, if φ is surjective, then $A/Ker \cong B$.*

Theorem 5.9. *Let N and K be normal subalgebra of A, and $K \subseteq N$. Then $A/N \cong (A/K)/(N/K)$.*

Theorem 5.10. *Let A, B and C be Sheffer stroke BH-algebras, and $h : A \rightarrow B$ be a Sheffer stroke BH-epimorphism and $g : A \rightarrow C$ be a Sheffer stroke BH-homomorphism. If $Ker(h) \subseteq Ker(g)$, then there exists a unique Sheffer stroke BH-homomorphism $f : A \rightarrow B$ satisfying $f \circ h = g$.*

Theorem 5.11. *Let A, B and C be Sheffer stroke BH-algebras, and $h : B \rightarrow C$ be a Sheffer stroke BH-homomorphism and $g : A \rightarrow C$ be a Sheffer stroke BH-monomorphism. If $Im(g) \subseteq Im(h)$, then there exists a unique Sheffer stroke BH-homomorphism $f : A \rightarrow B$ satisfying $h \circ f = g$.*

Proof. For each $\breve{a}_1 \in A, g(\breve{a}_1) \in Im(g) \subseteq Im(h)$. Since h is a Sheffer stroke BH-monomorphism, there exists a unique $\breve{a}_2 \in B$ such that $h(\breve{a}_2) = g(\breve{a}_1)$. Define a map $f : A \rightarrow B$ by $f(\breve{a}_1) = \breve{a}_2$. Then $h \circ f = g$. Let $\breve{a}_3, a_4 \in A$, then $g((\breve{a}_3|(a_4|a_4))|(\breve{a}_3|(a_4|a_4))) = h(f((\breve{a}_3|(a_4|a_4))|(\breve{a}_3|(a_4|a_4))))$. Since h is injective, $f((\breve{a}_3|(a_4|a_4))|(\breve{a}_3|(a_4|a_4))) = f(\breve{a}_3|(a_4|a_4))$ $|f((\breve{a}_3|(a_4|a_4)))=f(\breve{a}_3|f(a_4)|f(a_4))|f((\breve{a}_3)|f(a_4)|f(a_4))$. Therefore, f is a Sheffer stroke BH-homomorphism. The uniqueness of f follows from the fact that h is a Sheffer stroke BH-monomorphism. □

Theorem 5.12. *Let A and B be Sheffer stroke BH-algebras and $f : A \rightarrow B$ be a Sheffer stroke BH-homomorphism. If N is a normal subalgebra of A such that $N \subseteq Ker(f)$, then $\bar{f} : A/N \rightarrow B$ defined by $\bar{f}([\breve{a}_1]_N) := f(\breve{a}_1)$ for all $\breve{a}_1 \in A$ is a unique Sheffer stroke BH-homomorphism such that $\bar{f} \circ \gamma = f$ where $\gamma : A \rightarrow A/N$ is natural Sheffer stroke BH-homomorphism.*

Corollary 5.1. *Let A and B be Sheffer stroke BH-algebras and $f : A \rightarrow B$ be a Sheffer stroke BH-homomorphism. If N is a normal subalgebra of A such that $N \subseteq Ker(f)$, then the following are equivalent:*

(i) there exists a unique Sheffer stroke BH-homomorphism $\bar{f} : A/N \rightarrow B$ such that $\bar{f} \circ \gamma = f$ where $\gamma : A \rightarrow A/N$ is the natural Sheffer stroke BH-homomorphism;

(ii) $N \subseteq Ker(f)$.

Furthermore, $\bar{f}$ is a Sheffer stroke BH-monomorphism if and only if $N = Ker(f)$.

Proof. $(ii) \Rightarrow (i)$: It is obtained from Theorem 5.11.

$(i) \Rightarrow (ii)$: If $\breve{a}_1 \in N$, then $f(\breve{a}_1) = (\bar{f} \circ \gamma)(\breve{a}_1) = \bar{f}([\breve{a}_1]_N) = \bar{f}([0]_N) = f(0) = 0$. Thus, $\breve{a}_1 \in Ker(f)$.

Furthermore, $\bar{f}$ is a monomorphism if and only if $Ker\bar{f} = \{N\}$ if and only if $f(\breve{a}_1) = 0$ implies $[\breve{a}_1]_N = [0]_N = N$ if and only if $Ker(f) \subseteq N$. □

6. Conclusion

In this study, we have given a Sheffer Stroke BH-algebra, and study a Cartesian product, a filter, a homomorphism between Sheffer stroke BH-algebras, kernel and many features in Sheffer stroke BH-algebras. After giving basic definitions and concepts about Sheffer stroke operation and a BH-algebra, we describe a Sheffer stroke BH-algebra and present basic notions about this algebraic structure. We show that a Sheffer stroke BH-algebra is a BH-algebra and that a Cartesian product of two Sheffer stroke BH-algebras is a Sheffer Stroke BH-algebra. After defining a subalgebra and a normal subset, we introduce the relation between a subalgebra and a normal subset on Sheffer stroke BH-algebra. We define a filter of a Sheffer stroke BH-algebra. Finally, a homomorphism between two Sheffer stroke BH-algebras is described and it is stated that mentioned notions are preserved under this homomorphism. It is shown that a kernel of a homomorphism is a filter of Sheffer stroke BH-algebra under one condition.

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