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# $g$-$\mathcal{H}$-REGULAR AND $g$-$\mathcal{H}$-NORMAL HEREDITARY SPACES

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Abstract. This paper presents the introduction of the concept of $g$-$\mathcal{H}$-regularity within the context of hereditary spaces. It delves into an exploration of various properties associated with $g$-$\mathcal{H}$-normality, offering proofs for some of these properties. Additionally, the paper investigates the characterization of $g$-$\mathcal{H}$-normality through the application of modified versions of Urysohn's lemma and the famous Tietze Extension Theorem.

**Keywords**: Generalized Topological spaces, Hereditary classes, $\mathcal{H}_g$-closed sets, $g$-$\mathcal{H}$-normal spaces, $g$-$\mathcal{H}$-regular spaces, Urysohn's Lemma, Tietze Extension Theorem.

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## 1. Introduction

The separation axioms play a pivotal role in the examination of topological spaces by enabling us to use topological methods to distinguish between disjoint sets and distinct points. In 2002, Á Császár [7] introduced the concept of generalized topology, which expands upon this framework. For a non-empty set $X$, a family $\mu$ of subsets of $X$ is designated as a generalized topology on $X$ if it satisfies two fundamental properties: it must include the empty set $\emptyset$ and remain closed under arbitrary unions [6]. The pair $(X, \mu)$ is referred to as a generalized topological space, where the elements of $\mu$ are known as $\mu$-open sets, and their complements are designated as $\mu$-closed sets. We define the closure of a set $A$ in this context as $cl_\mu(A)$, given by $\cap\{F \subset X : X - F \in \mu, A \subset F\}$, and the interior of $A$ as $int_\mu(A)$, defined as $\cup\{G \subset X : G \in \mu, G \subset A\}$.

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In 2004, Császár [8] introduced a modified framework for separation axioms ($\mu - T_0$, $\mu - T_1$, $\mu - T_2$, $\mu - S_1$, $\mu - S_2$) tailored specifically for generalized topologies, where the conventional open sets are substituted with $\mu$-open sets. In 2007, he introduced the concept of normality for generalized topological spaces and demonstrated several properties of normal spaces. These properties were characterized using an adapted version of Urysohn's lemma [10]. Sarsak [17] expanded the study of separation axioms by introducing $\mu - D_0$, $\mu - D_1$, $\mu - D_2$ generalized topological spaces. Xun et al. [18] conducted research on generalized topological spaces and provided characterizations for $\mu - T_i$ spaces, for $i = 0, 1, 2, 3, 4$, as well as $\mu - T_D$ spaces and $\mu - R_0$ spaces. Additionally, Min conducted a study on separation axioms within generalized topological spaces in [14].

Also hereditary classes, initially introduced by Császár [9], have been a subject of ongoing exploration by numerous researchers over time. A non-empty family $\mathcal{H}$ consisting of subsets of $X$ is termed a hereditary class on $X$ if, whenever $A$ is a subset of $B$ and $B$ is a member of $\mathcal{H}$, $A$ must also belong to $\mathcal{H}$. The triple $(X, \mu, \mathcal{H})$ is denoted as a hereditary generalized topological space, or simply a hereditary space. Császár [9] defined an operator $cl^*(A) = A \cup A^*$, where $A^* = \{x \in X : U \cap A \notin \mathcal{H} \text{ for each } U \in \mu, x \in U\}$ for $A \subset X$. This operator induces another generalized topology, denoted as $\mu^*$, which is finer than $\mu$, and it is referred to as the $*$-generalized topology. The constituents of $\mu^*$ are known as $*$-open sets and their complements are designated as $*$-closed sets. Additionally, $int^*(A) = \cup\{G \subset X : G \in \mu^*, G \subset A\}$. The exploration of hereditary spaces has been a subject of ongoing research by various authors [12, 1]. In a separate study, the author investigated some generalized separation axioms, such as Hausdorff modulo $\mathcal{H}$ and $\mathcal{H}$-regularity, as outlined in [5].

In 2009, Navaneethakrishnan et al. [16] introduced and examined the notions of $\mathcal{I}_g$-normal and $\mathcal{I}_g$-regular ideal topological spaces, utilizing the concepts of $\mathcal{I}_g$-open and $\mathcal{I}_g$-closed sets [15]. The author later extended these concepts to $\mathcal{H}_g$-normal and $\mathcal{H}_g$-regular hereditary spaces in [4]. Furthermore, the author also introduced and investigated the concept of $g$-$\mathcal{H}$-normal spaces in the same study [4].

Recently, in 2024, Mesfer H. Alqahtani et al. [2] introduced a category of $\aleph$-open sets in topological spaces and discussed its relationships with many different classes of open sets. Additionally, the concepts of continuity of functions and separation axioms have been investigated.

Many authors introduced modified separation axioms for generalized topologies in different set-ups, which motivates the author to investigate further on separation axioms and find the generalizations of well known results for the same. This paper builds upon the previous research to provide further characterizations and a modified version of the well-known Urysohn lemma and Tietze Extension Theorem specifically tailored for $g$-$\mathcal{H}$-normal spaces. Additionally, the concept of $g$-$\mathcal{H}$-regular space is introduced and various properties of this space are investigated.

## 2. Preliminaries

In our study, we will make reference to the following definitions and theorems:

**Definition 2.1.** [5] *The generalized topological space $(X, \mu)$ is said to be $\mu$-regular if, for every point $x$ within $X$ and for every $\mu$-closed set $F$ that does not include $x$, there exist two disjoint $\mu$-open sets denoted as $U$ and $V$ in $X$, satisfying the conditions that $x$ is an element of $U$ and $F$ is entirely contained within $V$.*

**Definition 2.2.** [3] *In the context of a generalized topological space $(X, \mu)$ and any subset $Y$ of $X$, the collection $\{Y \cap G : G \in \mu\}$ is a generalized topology on $Y$, which particularly is referred to as the subspace generalized topology or relative generalized topology and it is denoted by $\mu_Y$. Consequently, when we equip the set $Y$ with this generalized topology $\mu_Y$, it is described as a generalized subspace (or simply subspace) of $X$.*

**Definition 2.3.** [4] *A subset $A$ of a generalized topological space $X$ is said to be $\mathcal{H}_g$-closed when it satisfies the condition that if $U$ is a $\mu$-open set containing $A$, then $A^*$ must be entirely contained within $U$. $A$ is said to be $\mathcal{H}_g$-open if $X - A$ is $\mathcal{H}_g$-closed.*

**Remark 2.1.** [4] *Each $\mu$-open set is also $\mathcal{H}_g$-open and each $\mu$-closed set is also $\mathcal{H}_g$-closed.*

**Definition 2.4.** [4] *A subset $A$ of a generalized topological space $X$ is called $g$-$\mu$-closed when it satisfies the condition that if $U$ is $\mu$-open set cotaining $A$, then $cl_\mu(A)$, the $\mu$-closure of $A$, must be entirely contained within $U$. $A$ is said to be $g$-$\mu$-open if $X - A$ is $g$-$\mu$-closed.*

**Definition 2.5.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is considered to be $\mathcal{H}_g$-regular when, for any point $x$ and a $\mu$-closed set $B \subset X$, provided that $B$ does not contain $x$, there exist two disjoint $\mathcal{H}_g$-open sets $U$ and $V$, within $X$ satisfying $x \in U$ and $B \subset V$.*

**Definition 2.6.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is $\mathcal{H}_g$-normal if, for any two disjoint $\mu$-closed sets $A$ and $B$ in $X$, there exist two disjoint $\mathcal{H}_g$-open sets $U$ and $V$ within $X$, such that $A$ is entirely contained in $U$ and $B$ is entirely contained in $V$.*

**Definition 2.7.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-normal if, for any two disjoint $\mathcal{H}_g$-closed sets $A$ and $B$ in $X$, there exist two disjoint $\mu$-open sets $U$ and $V$ within $X$ such that $A$ is entirely contained in $U$ and $B$ is entirely contained in $V$.*

**Theorem 2.1.** [9] *For any two subsets $A$ and $B$ of a hereditary space $(X, \mu, \mathcal{H})$, the following properties hold.*

(1) $A^* \subset cl_\mu(A)$.
(2) $A^*$ *is $\mu$-closed set, and therefore* $A^* = cl_\mu(A^*) = cl_\mu(A)$.
(3) $cl^*(A) = A^* = cl_\mu(A) = cl_\mu(A^*)$, *whenever* $A \subset A^*$.
(4) *If* $\mathcal{H} = \{\emptyset\}$, *then* $A^* = cl_\mu(A) = cl^*(A)$.
(5) $(A \cap B)^* \subset A^* \cap B^*$.

**Lemma 2.1.** [10] *Let $\beta$ be any family of subsets of the space $X$. The family $\nu$ of subsets of $X$ consists of the $\emptyset$ and all sets $N$ that can be expressed as the union of sets $\bigcup_{i\in I} B_i$, where $B_i \in \beta$ and $I$ is a non-empty index set, is a generalized topology on $X$, which is referred to as the generalized topology generated by the base $\beta$.*

**Example 2.1.** [10] *Consider $X = \mathbf{R}$ and the family of subsets $\beta = \{(-\infty, t) : t \in \mathbf{R}\} \bigcup \{(t, +\infty) : t \in \mathbf{R}\}$. Then the generalized topology on $\mathbf{R}$ generated by $\beta$, denoted by $\nu$, is known as the usual generalized topology.*

**Lemma 2.2.** [10] *Suppose $\mu$ is a generalized topology on the space $X$ and the generalized topology $\nu$ on another space $Y$ is generated by the base $\beta$. Then a mapping $f : X \to Y$ is considered $(\mu, \nu)$-continuous if and only if the inverse image of each set $B \in \beta$ under the map $f$, denoted as $f^{-1}(B)$, belongs to the generalized topology $\mu$.*

**Theorem 2.2.** [4] *A hereditary space $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-normal if and only if, for every $\mathcal{H}_g$-closed set $A$ within $X$ and an $\mathcal{H}_g$-open set $B$ that contains $A$, there exists a $\mu$-open set $V$ satisfying $A \subset V \subset cl_\mu(V) \subset B$.*

## 3. $g$-$\mathcal{H}$-regular and $g$-$\mathcal{H}$-normal Spaces

**$g$-$\mathcal{H}$-regular Spaces.** This section will provide an introduction to the concept of $g$-$\mathcal{H}$-regular hereditary spaces and delve into an exploration of the different properties related to these spaces.

**Definition 3.1.** *A hereditary space $(X, \mu, \mathcal{H})$ is defined to be g-$\mathcal{H}$-regular if when, for a point $x$ in $X$ and an $\mathcal{H}_g$-closed set $A$ that does not contain $x$, there exist two disjoint $\mu$-open sets $U$ and $V$ such that $x$ is an element of $U$ and $A$ is entirely contained within $V$.*

**Definition 3.2.** *A generalized topological space $(X, \mu)$ is termed g-$\mu$-regular if, for a point $x$ within $X$ and a g-$\mu$-closed set $A$ that does not include $x$, there exist two disjoint $\mu$-open sets $U$ and $V$ such that $x$ is a member of $U$ and $A$ is entirely contained within $V$.*

**Remark 3.1.** *Every hereditary space that is g-$\mathcal{H}$-regular is also $\mu$-regular because each set that is $\mu$-closed is also $\mathcal{H}_g$-closed, however the converse does not necessarily hold, as illustrated in Example 3.1.*

**Example 3.1.** *Let $X = \{p, q, r\}$, $\mu = \{\emptyset, \{p\}, \{q, r\}, X\}$ and $\mathcal{H} = \{\emptyset\}$. This space is $\mu$-regular, but not g-$\mathcal{H}$-regular, since $\{r\}$ is $\mathcal{H}_g$-closed set that does not contain $q$ and there are no disjoint $\mu$-open sets that contain $q$ and $\{r\}$.*

The following Theorems 3.1 and 3.2 give characterizations of $g$-$\mathcal{H}$-regular spaces.

**Theorem 3.1.** *A hereditary space $(X, \mu, \mathcal{H})$ is g-$\mathcal{H}$-regular if, and only if, for every point $x \in X$ and each $\mathcal{H}_g$-open set $A$ in $X$ that includes $x$, there is a $\mu$-open set $V$ satisfying that $x \in V \subset cl_\mu(V) \subset A$.*

*Proof.* In a $g$-$\mathcal{H}$-regular space $X$, consider a point $x$ and an $\mathcal{H}_g$-open set $A$ containing $x$. Then $X - A$ is $\mathcal{H}_g$-closed set that does not contain $x$. Since $X$ is $g$-$\mathcal{H}$-regular, there exist two disjoint $\mu$-open sets, $V$ and $W$, such that $x$ belongs to $V$ and $(X - A)$ is a subset of $W$. The fact $V \cap W = \emptyset$ implies that $cl_\mu(V) \subset X - W$. Consequently, $x \in V \subset cl_\mu(V) \subset X - W \subset A$. Conversely, suppose $x$ is an element of $X$ and $A$ is any $\mathcal{H}_g$-closed sets in $X$ that does not contain $x$. In this case, $X - A$ is $\mathcal{H}_g$-open set containing $x$. Then there exists a $\mu$-open set $V$ such that $x \in V \subset cl_\mu(V) \subset X - A$. By defining $W = X - cl_\mu(V)$, there will be two disjoint $\mu$-open sets $V$ and $W$ with the properties that $x \in V$ and $A \subset W$. Therefore $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-regular. □

By setting $\mathcal{H} = \{\emptyset\}$ in the above Theorem 3.1, we can derive the following characterization of $g$-$\mu$-regular generalized topological spaces.

**Corollary 3.1.** *A generalized topological space $(X, \mu)$ is g-$\mu$-regular if and only if, for every point $x \in X$ and each g-$\mu$-open set $A$ that contains $x$, there exists a $\mu$-open set $U$ satisfying $x \in U \subset cl_\mu(U) \subset A$.*

**Theorem 3.2.** *A hereditary space $(X,\mu,\mathcal{H})$ is $g$-$\mathcal{H}$-regular if and only if, for every $x \in X$ and any $\mathcal{H}_g$-closed set $A$ that does not contain $x$, there exists a $\mu$-open set $V$ that contains $x$ such that $cl_\mu(V)$ is disjoint from $A$.*

*Proof.* The proof of the theorem is straightforward and follows directly from Theorem 3.1. □

The following Corollary 3.2 provides a way to characterize $g$-$\mu$-regular spaces when we take $\mathcal{H} = \{\emptyset\}$ in the Theorem 3.2.

**Corollary 3.2.** *A generalized topological space $(X,\mu)$ is $g$-$\mu$-regular if and only if, for every point $x \in X$ and for any $g$-$\mu$-closed set $A$ that does not include $x$, there exists a $\mu$-open set $V$ containing $x$ such that $cl_\mu(V)$ does not intersect with $A$.*

We have defined $\mathcal{H}_g$-regular hereditary spaces in [4]. Now we establish a relationship between $g$-$\mathcal{H}$-regularity and $\mathcal{H}_g$-regularity of hereditary spaces in the Theorem 3.3.

**Theorem 3.3.** *A hereditary space $(X,\mu,\mathcal{H})$, which is $g$-$\mathcal{H}$-regular, is also $\mathcal{H}_g$-regular.*

*Proof.* The straightforward proof lies in the fact that every $\mu$-open set is $\mathcal{H}_g$-open and every $\mu$-closed set is $\mathcal{H}_g$-closed. □

**Remark 3.2.** *Every $g$-$\mathcal{H}$-regular hereditary space is $\mathcal{H}_g$-regular, as shown in the Theorem 3.3, however the converse does not necessarily hold, as illustrated in Example 3.2.*

**Example 3.2.** *Let $X = \{p,q,r\}$, $\mu = \{\emptyset,\{p\},\{p,q\},\{p,r\},X\}$ and $\mathcal{H} = \{\emptyset,\{p\}\}$. Every $\mu$-open subset of $X$ is $*$-closed, therefore every subset of $X$ is $\mathcal{H}_g$-open, which makes the space $(X,\mu,\mathcal{H})$, $\mathcal{H}_g$-regular. $\{r\}$ is $\mathcal{H}_g$-closed set that does not contain $q$ and there are no disjoint $\mu$-open sets that contain $q$ and $\{r\}$. Therefore $(X,\mu,\mathcal{H})$ is not $g$-$\mathcal{H}$-regular.*

**$g$-$\mathcal{H}$-normal Spaces.** The notion of $g$-$\mathcal{H}$-normal hereditary spaces was originally introduced in the reference [4]. In this context, we will explore a range of properties and characterizations of these $g$-$\mathcal{H}$-normal hereditary spaces.

**Theorem 3.4.** *Let $X$ be $g$-$\mathcal{H}$-normal space. Then a $\mu$-closed subspace of $X$ is $g$-$\mathcal{H}$-normal.*

*Proof.* In a $\mu$-closed subspace $Y$ of $X$, if $A$ and $B$ be two disjoint $\mathcal{H}_g$-closed sets, then, by Theorem 3.6, $A$ and $B$ are disjoint $\mathcal{H}_g$-closed subsets of the space $X$. Since $X$ is $g$-$\mathcal{H}$-normal, there exist two disjoint $\mu$-open sets $U$ and $V$ in $X$ such that $A$ is contained in $U$ and $B$ is contained in $V$. Then $U \cap Y$ and $V \cap Y$ are two disjoint $\mu_Y$-open sets in $Y$ such that $A = (A \cap Y) \subset (U \cap Y)$ and $B = (B \cap Y) \subset (V \cap Y)$. Hence $Y$ is $g$-$\mathcal{H}$-normal space. □

The Theorem 3.5 discussed below establishes a relationship between the notions of $g$-$\mathcal{H}$-normality and $\mathcal{H}_g$-normality within the context of hereditary spaces.

**Theorem 3.5.** *If a hereditary space $(X, \mu, \mathcal{H})$ is $g$-$\mathcal{H}$-normal, then it is $\mathcal{H}_g$-normal.*

*Proof.* The proof can be immediately established by the fact that every $\mu$-open set is $\mathcal{H}_g$-open and every $\mu$-closed set is $\mathcal{H}_g$-closed. □

**Remark 3.3.** *Every $g$-$\mathcal{H}$-normal hereditary space is $\mathcal{H}_g$-normal, as shown in the Theorem 3.5, however the converse does not necessarily hold, as illustrated in Example 3.3.*

**Example 3.3.** *Consider the hereditary space $X = \{p, q, r\}$, $\mu = \{\emptyset, \{p\}, \{p, q\}, \{p, r\}, X\}$ and $\mathcal{H} = \{\emptyset, \{p\}\}$. In this space, every $\mu$-open set is essentially $*$-closed, making every subset of $X$, $\mathcal{H}g$-open. Consequently, $(X, \mu, \mathcal{H})$ is $\mathcal{H}_g$-normal. However, $\{q\}$ and $\{r\}$ are two disjoint $\mathcal{H}_g$-closed sets which can not be separated by disjoint $\mu$-open sets and therefore $(X, \mu, \mathcal{H})$ is not $g$-$\mathcal{H}$-normal.*

**Theorem 3.6.** *Consider a generalized subspace $Y$ of the space $X$. If a subset $A \subset Y$ is $\mathcal{H}_g$-closed within $Y$, then $A$ is $\mathcal{H}_g$-closed in $X$.*

*Proof.* Let $U$ be a $\mu$-open set containing $A$. Then $(U \cap Y) \in \mu_Y$ and $A \subset (U \cap Y)$. Since $A$ is $\mathcal{H}_g$-closed in $Y$, $A^* \subset (U \cap Y) \subset U$. Therefore $A$ is $\mathcal{H}_g$-closed in $X$. □

**Corollary 3.3.** *Consider a generalized subspace $Y$ of the space $X$. If a subset $A \subset Y$ is $\mu_Y$-closed within $Y$ then $A$ is $\mathcal{H}_g$-closed in $X$.*

**Theorem 3.7.** *Let $Y$ be a generalized subspace of $X$. If a set $A$ is $\mathcal{H}_g$-closed within the space $X$ and $Y$ is $\mu$-closed within $X$, then the intersection $A \cap Y$ is $\mathcal{H}_g$-closed within $Y$.*

*Proof.* Consider $(A \cap Y) \subset U$ with $U \in \mu_Y$. Then $U$ can be expressed as $U = (G \cap Y)$ for some $G \in \mu$. Then $A = (A \cap Y) \cup (A \cap (X - Y)) \subset (U \cup (X - Y)) = (G \cap Y) \cup (X - Y) = (G \cup (X - Y)) \in \mu$, since $Y$ is $\mu$-closed in $X$. Also, $A$ is $\mathcal{H}_g$-closed set within $X$, $A^* \subset (G \cup (X - Y))$. Then $(A \cap Y)^* \subset (A^* \cap Y^*) \subset (A^* \cap Y) \subset ((G \cup (X - Y)) \cap Y) = G \cap Y = U$. Therefore $A$ is $\mathcal{H}_g$-closed in $Y$. □

**Theorem 3.8.** *If a set $A$ is $\mathcal{H}_g$-closed and set $B$ is $\mu$-closed, then their intersection $A \cap B$ is $\mathcal{H}_g$-closed.*

*Proof.* The proof can be deduced from the Theorems 3.6 and 3.7. □

*Urysohn's Lemma.* We will now provide a proof for the following variation of Urysohn's Lemma adapted for $g$-$\mathcal{H}$-normal hereditary spaces:

**Theorem 3.9.** *Necessary Condition for $g$-$\mathcal{H}$-Normality in Hereditary Space: Let $(X, \mu, \mathcal{H})$ be a $g$-$\mathcal{H}$-normal hereditary space and let $A$, $B$ be disjoint $\mathcal{H}_g$-closed subsets of $X$. Then there exist a function $f : X \to [0, 1]$ that is $(\mu, \nu)$-continuous where $\nu$ is the standard generalized topology on the interval $[0, 1]$, such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$.*

*Proof.* Consider the collection $D$ of dyadic fractions in the interval [0,1], defined as $\frac{m}{2^n}$, where $m = 0, 1, 2, .....2^n$ and $n = 0, 1, 2, 3, .....$ For each $r \in D$, we construct $\mu$-open sets $U_r$ and $\mu$-closed sets $F_r$ in such a way that:

(1) $U_r \subset F_r$ for each $r \in D$.
(2) If $r$ and $s$ are in $D$ and $r < s$, then $F_r \subset U_s$.

Start by setting $F_0 = A$ and $U_1 = X - B$. As $A$ and $B$ are disjoint, $A \subset (X - B)$. Using Theorem 2.2, since $A$ is $\mathcal{H}_g$-closed and $X - B$ is $\mathcal{H}_g$-open, there exists a $\mu$-open set and therefore $\mathcal{H}_g$-open $U_{\frac{1}{2}}$ such that $A \subset U_{\frac{1}{2}} \subset cl_\mu(U_{\frac{1}{2}}) \subset X - B$. Continuing this construction, we can obtain $U_r$ and $F_r$ for each $r \in D$, ensuring that $U_r \subset F_r$ and $F_r \subset U_s$ for $r < s$. Define a function $f : X \to [0, 1]$ as $f(x) = inf\{r \in D : x \in F_r\}$. Then $f(x) = 0$ for $x \in F_0 = A$ and $f(x) = 1$ for $x \in B$. To show that $f$ is $(\mu, \nu)$-continuous, it is sufficient to prove that $f^{-1}([0, a))$ and $f^{-1}((b, 1])$ are $\mu$-open sets in $X$. $f^{-1}([0, a)) = \cup\{U_r : r < a\}$ and $f^{-1}((b, 1]) = \cup\{X - F_r : r > b\}$, ensuring that both sets are $\mu$-open, making $f$, $(\mu, \nu)$-continuous. □

The following Theorem 3.10 provides a sufficient condition for $g$-$\mathcal{H}$-normality in hereditary spaces.

**Theorem 3.10.** *Sufficient Condition for $g$-$\mathcal{H}$-Normality in Hereditary Space: If $(X, \mu, \mathcal{H})$ is a hereditary space with the property that for any two disjoint $\mathcal{H}_g$-closed subsets $A$ and $B$ of $X$, there exist a function $f : X \to [0, 1]$ that is $(\mu, \nu)$-continuous, where $\nu$ is the standard generalized topology on the interval $[0, 1]$, such that $f(x) = 0$ for $x \in A$ and $f(x) = 1$ for $x \in B$, then $X$ is $g$-$\mathcal{H}$-normal.*

*Proof.* Consider the sets $f^{-1}([0, 1/2))$ and $f^{-1}((1/2, 1])$. These sets are disjoint $\mu$-open sets in $X$ containing $A$ and $B$, respectively. Consequently, $X$ is $g$-$\mathcal{H}$-normal. □

*Tietze Extension Theorem.* A modified version of the Tietze Extension Theorem has been established for $g$-$\mathcal{H}$-normal hereditary spaces, as outlined in Theorem 3.11.

**Theorem 3.11.** *Let $(X, \mu, \mathcal{H})$ be a $g$-$\mathcal{H}$-normal hereditary space and let $f : F \to \mathbf{R}$ be a $(\mu_F, \nu)$-continuous mapping, where $F$ is an $\mathcal{H}_g$-closed subset of $X$. Then there exist a $(\mu, \nu)$-continuous mapping $g : X \to \mathbf{R}$ such that $g(x) = f(x)$ for all $x \in F$, where $\nu$ is usual generalized topology on $\mathbf{R}$.*

*Proof.* We first assume that $f$ is bounded function with $c = sup\{|f(y)| : y \in F\}$. We define sets $A_0 = \{y \in F : f(y) \leq -c/3\}$ and $B_0 = \{y \in F : f(y) \geq c/3\}$. These sets are disjoint $\nu$-closed sets in the interval $[-c, c]$. Since $f$ is $(\mu_F, \nu)$-continuous mapping, $f^{-1}(A_0)$ and $f^{-1}(B_0)$ are disjoint $\mu_F$-closed sets and consequently $\mathcal{H}_g$-closed sets in $X$. By Theorem 3.9, there exists a $(\mu, \nu)$-continuous function $g_0 : X \to [-c/3, c/3]$ such that $g_0(A_0) = -c/3$ and $g_0(B_0) = c/3$. This function satisfies $|g_0| \leq c/3$ and $|f - g_0| \leq 2c/3$ on $F$. We then define sets $A_1 = \{y \in F : (f - g_0)(y) \leq -2c/9\}$ and $B_1 = \{y \in F : (f - g_0)(y) \geq 2c/9\}$. These sets are again disjoint $\nu$-closed sets in $[-c, c]$ and therefore $(f - g_0)^{-1}(A_1)$ and $(f - g_0)^{-1}(B_1)$ are disjoint $\mu$-closed sets in $X$, making them $\mathcal{H}_g$-closed sets in $X$. By applying Theorem 3.9, we obtain a $(\mu, \nu)$-continuous function $g_1 : X \to [-2c/9, 2c/9]$ such that $g_1(A_1) = -2c/9$ and $g_1(B_1) = 2c/9$ and $|g_1| \leq 2c/9$, $|f - g_0 - g_1| \leq 4c/9$ on $F$. This process is continued, producing a sequence $\{g_n\}$ of $(\mu, \nu)$-continuous functions defined on $X$ such that $|g_n| \leq \frac{2^n c}{3^{n+1}}$ and $|f - g_0 - g_1 ...... - g_n| \leq \frac{2^{n+1} c}{3^{n+1}}$ on $F$.

We define $h_n = g_0 + g_1 + ...... + g_n$ for $n \geq 1$. This is a sequence of $(\mu, \nu)$-continuous functions on $X$. For $n \geq m$, $|h_n - h_m|$ is bounded by $(\frac{2}{3})^{m+1} c$. Therefore, $\{h_n\}$ is a Cauchy sequence and converges uniformly to a real valued function $h$ on $X$. This limit function $h = \lim_{n\to\infty} h_n = \lim(g_0 + g_1 + ...... + g_n) = \sum_{n=0}^{\infty} g_n$ and therefore $h(x) = f(x)$ on $F$.

To complete the proof, we prove that $h$ is $(\mu, \nu)$-continuous function. Let $x \in X$ and $V$ be a $\nu$-open set in $\mathbf{R}$ containing $h(x)$. Since $h_n(x)$ converges uniformly to $h$, for any given $\epsilon > 0$, there exists an integer $N$ such that $h_n(x) \in V$ for all $n \geq N$. Since $h_n$ is $(\mu, \nu)$-continuous, there exists a $\mu$-open set $U$ in $X$ containing $x$ such that $h_n(U) \subset V$. Therefore, $h(U) = \lim_{n\to\infty} h_n(U) \subset V$. Thus, we have established that $h$ is $(\mu, \nu)$-continuous, concluding the proof. $\square$

## 4. Conclusion

This research primarily delves into two distinct areas within the realm of hereditary generalized topological spaces. The first area explores the concept of $g$-$\mathcal{H}$-regularity in hereditary spaces, which provides generalized versions of fundamental properties typically associated with regular topological spaces. In the second area, the focus shifts to the generalization of renowned results such as Urysohn's lemma and the Tietze Extension Theorem, specifically within the context of $g$-$\mathcal{H}$-normal hereditary spaces.

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