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TRANSLATION FRAMED SURFACES GENERATED BY NON-NULL FRAMED CURVES IN MINKOWSKI 3-SPACE $\mathbb{E}_1^3$

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In this paper, first we obtain the conditions for the existence and uniqueness of non-null framed curves as well as non-null framed surfaces in Minkowski 3-space. Further, we study the timelike and spacelike translation framed surfaces generated by non-null framed curves and obtain the basic invariants of such surfaces in $\mathbb{E}_1^3$. We also find the curvatures of timelike and spacelike translation framed surfaces generated by non-null framed curves. Finally, we classify the translation framed surfaces generated by non-null framed curves lying in mutually perpendicular coordinate planes of $\mathbb{E}_1^3$ with $\mu_K \equiv 0$ and $\mu_H \equiv 0$.

Keywords: Framed curve, Framed surface, Translation framed surface, Curvature and invariants of a translation framed surface.

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1. Introduction

A translation surface is a special case of Darboux surface which is the union of 'equivalent' curves ('equivalent' in the sense that, the curves are images of one another by some isometries of the space), also known as generating curves of the surface. A Darboux surface is defined as the movement of curves by rigid motions of the space. Therefore, it can be parametrized as $X(u, v) = A(v).\alpha(u) + \beta(v)$, where α, β are two space curves and A is an orthogonal matrix. When the orthogonal matrix A is identity matrix the surface is called a translation

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surface. Thus, a generalized type of a translation surface is given by

$$X(u,v) = \alpha(u) + \beta(v).$$

Translation surface which is known as the double curve in differential geometry is base for roofing structures. The construction and design of free form glass roofing structures are generally created with the help of curved (formed) glass panes or planar triangular glass facets. Recently, classification of translation surfaces under some conditions on curvatures has been studied in Euclidean as well as Minkowski space ([1],[10],[11],[15]).

A framed curve in Minkowski 3-space is a curve with an assigned frame which moves along the curve. In [7], Honda and Takahashi defined the curvature functions of the framed curve in $\mathbb{E}^3$, similar to a regular curve. By using curvature functions, they obtained the existence and the uniqueness theorem for the framed curves. The curvature functions of a framed curve are used to investigate the curve along with its singularities. On the other hand, a framed surface is defined to be a surface with an assigned moving frame which is used to analyze properties and singularities of the surface. In [4], by using the moving frames in $\mathbb{E}^3$, the basic invariants and the curvatures of framed surfaces are introduced by Fukunaga and Takahashi. They studied the properties of framed surfaces using the basic invariants of the surfaces and gave some examples.

In [5], Fukunaga and Takahashi reviewed the theories for framed surfaces, framed curves and one-parameter families of framed curves in $\mathbb{E}^3$. They showed that up to congruence, the surface along with the moving frame can be determined by the basic invariants of the framed surface and the curvature of a one parameter family of framed curves. In [6], the authors studied the translation surfaces with assigned moving frame and discussed the various singularities that arise on such surfaces with help of the notion of framed curves and surfaces.

In this paper, we study the non-null translation framed surfaces generated by non-null framed curves in $\mathbb{E}^3_1$. The paper is arranged as follows. There are some basic results in section 2. In section 3, we study non-null framed curves in $\mathbb{E}^3_1$ and obtain the conditions for the existence and uniqueness of non-null framed curves. In section 4, first we study non-null framed surfaces in $\mathbb{E}^3_1$ and find their curvatures and existence and uniqueness conditions. Further, we study the timelike and spacelike translation framed surfaces generated by non-null framed curves and obtain the basic invariants of such surfaces in $\mathbb{E}^3_1$. We also find the curvatures of timelike and spacelike translation framed surfaces generated by non-null framed curves.

Finally, we classify the translation framed surfaces generated by non-null framed curves lie in the coordinate planes of $\mathbb{E}_1^3$ with $\mu_K \equiv 0$ and $\mu_H \equiv 0$.

2. Preliminaries

The Minkowski 3-space, denoted by $\mathbb{E}_1^3$, is a three dimensional real vector space $\mathbb{R}^3$ endowed with the metric tensor $\langle .,. \rangle = -dx^2 + dy^2 + dz^2$. The (Lorentzian) scalar and cross product are defined by:

$$\begin{cases} \langle x, y \rangle = -x_1 y_1 + x_2 y_2 + x_3 y_3, \\ x \times y = (-x_2 y_3 + x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1), \end{cases} \tag{2.1}$$

where $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3)$ belong to $\mathbb{E}_1^3$. This space is also known as Lorentz-Minkowski space. A vector $x \in \mathbb{E}_1^3$ is said to be spacelike when $\langle x, x \rangle > 0$ or $x = 0$, timelike when $\langle x, x \rangle < 0$ and lightlike(null) when $\langle x, x \rangle = 0$, $x \neq 0$. A curve in $\mathbb{E}_1^3$ is called spacelike, timelike or lightlike when the velocity vector of the curve is spacelike, timelike or lightlike, respectively. The norm of a vector $x \in \mathbb{E}_1^3$ is defined as $\|x\| = \sqrt{|\langle x, x \rangle|}$. The hyperbolic and Lorentzian unit spheres are defined as

$$H_0^2 = \{x \in \mathbb{E}_1^3 \| \langle x, x \rangle = -1\}$$

and

$$S_1^2 = \{x \in \mathbb{E}_1^3 \| \langle x, x \rangle = 1\},$$

respectively. Let $\gamma = \gamma(s) : I \to \mathbb{E}_1^3$ be an arbitrary curve. The curve γ is said to be an unit speed curve (or parameterized by the arc-length parameter s) if $\langle \gamma'(s), \gamma'(s) \rangle = \pm 1$ for any $s \in I$.

For a spacelike curve $\gamma : I \to \mathbb{E}_1^3$ parametrized with arclength parameter s, let $\{t, n, b\}$ be the moving Frenet frame along the curve, where $t(s) = \gamma'(s)$ is the unit tangent vector, n is the unit normal vector defined as the unit vector in the direction $t'(s)$ such that $t'(s) = \kappa(s)\, n(s)$, where $\kappa(s)$ is the curvature of the curve and $b(s) = t(s) \times n(s)$. The second curvature (torsion) of the curve is given by $\tau = \epsilon \langle b', n \rangle$, where $\epsilon = \langle n, n \rangle$.The Frenet-Serret equations of the spacelike curve are given as

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\epsilon\kappa & 0 & \tau \\ 0 & \tau & 0 \end{bmatrix} = \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t,t\rangle = 1$, $\langle n,n\rangle = \epsilon$, $\langle b,b\rangle = -\epsilon$, $\langle t,b\rangle = \langle t,n\rangle = \langle n,b\rangle = 0$. If $\epsilon = 1$, $\gamma(s)$ is a spacelike curve with the spacelike principal normal n and the timelike binormal b, while if $\epsilon =$ -1 then γ is a spacelike curve with the timelike principal normal n and the spacelike binormal b.
For a timelike curve γ, we define Frenet frame in similar way except for the torsion is given by $\tau = -\langle b', n\rangle$. The Frenet-Serret equations are given by

$$\begin{bmatrix} t' \\ n' \\ b' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} = \begin{bmatrix} t \\ n \\ b \end{bmatrix},$$

where $\langle t,t\rangle =$ -1, $\langle n,n\rangle = 1$, $\langle b,b\rangle = 1$, $\langle t,n\rangle = \langle t,b\rangle = \langle n,b\rangle = 0$.
A surface in $\mathbb{E}^3_1$ is said to be a spacelike, timelike or lightlike if the metric on the surface is positive definite, indefinite or degenerate, respectively. The type of a surface can also be expressed in terms of the causal character of the normal vector of the surface by the following lemma.

Lemma 2.1. [8] *A surface in Minkowski 3-space is spacelike, timelike or lightlike if and only if at every point of the surface there exists a normal which is timelike, spacelike or lightlike, respectively.*

Definition 2.1. [14] *Let v and w be two spacelike vectors. Then, there exists a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = \|v\|\|w\|\cos\theta$.*

Definition 2.2. [14] *Let v be a spacelike vector and w be a timelike vector in $\mathbb{E}^3_1$. Then, there exists a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = \|v\|\|w\|\sinh\theta$.*

Definition 2.3. [12] *Let v and w be two timelike vectors in the same time cone of $\mathbb{E}^3_1$, i.e. $\langle v,w\rangle < 0$. Then, there exists a unique non-negative real number $\theta \geq 0$, such that $\langle v,w\rangle = -\|v\|\|w\|\cosh\theta$.*

Lagrange's Identity: For any vectors $\eta, \xi \in \mathbb{E}^3_1$, we have $\langle \eta\times\xi, \eta\times\xi\rangle = -\langle \eta,\eta\rangle\langle\xi,\xi\rangle + \langle\eta,\xi\rangle^2$.

3. Framed curves in Minkowski 3-space

In this section we define the Frenet type formula for the framed curves and give existence and uniqueness theorem of the framed curves in $\mathbb{E}^3_1$.

Definition 3.1. [9] *Let $\gamma : I \to \mathbb{E}^3_1$ be a curve in $\mathbb{E}^3_1$. Then the map $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}^3_1 \times \Theta$ is called a spacelike framed curve if*

$$\langle \gamma'(t), \vartheta_1(t)\rangle = 0,\ \langle \gamma'(t), \vartheta_2(t)\rangle = 0, \forall t \in I,$$

such that $\rho(t) = \vartheta_1(t) \times \vartheta_2(t)$ *is an arbitrary spacelike vector field, where*

$\Theta = \{(u, v) \in S_1^2 \times H_0^2 | \langle u, v\rangle = 0\}$ *or* $\Theta = \{(u, v) \in H_0^2 \times S_1^2 | \langle u, v\rangle = 0\}$.

Definition 3.2. [9] *Let* $\gamma : I \to \mathbb{E}_1^3$ *be an arbitrary curve in* $\mathbb{E}_1^3$. *Then the map* $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ *is called a timelike framed curve if*

$$\langle \gamma'(t), \vartheta_1(t)\rangle = 0,\ \langle \gamma'(t), \vartheta_2(t)\rangle = 0, \forall t \in I,$$

such that $\rho(t) = \vartheta_1(t) \times \vartheta_2(t)$ *is a timelike vector field, where*

$\Theta = \{(u, v) \in S_1^2 \times S_1^2 | \langle u, v\rangle = 0\}$.

Definition 3.3. *Let* $(\gamma, \vartheta_1, \vartheta_2)$ *and* $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2) : I \to \mathbb{E}_1^3 \times \Theta$ *are framed curves. We say that* γ *and* $\bar{\gamma}$ *have the same causal character of the moving frame if the vector triplets* $\{\vartheta_1, \vartheta_2, \rho\}$ *and* $\{\bar{\vartheta}_1, \bar{\vartheta}_2, \bar{\rho}\}$ *have the same causal characters, respectively.*

3.1. **Frenet-Serret type formula for framed curves.** Let $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ be an spacelike framed curve and $\rho(t) = \vartheta_1(t) \times \vartheta_2(t)$. The Frenet-Serret type formula is given by

$$\begin{bmatrix} \vartheta_1' \\ \vartheta_2' \\ \rho' \end{bmatrix} = \begin{bmatrix} 0 & -\delta\kappa_1 & \kappa_2 \\ -\delta\kappa_1 & 0 & \kappa_3 \\ -\delta\kappa_2 & \delta\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \rho \end{bmatrix}, \tag{3.2}$$

where $\delta = \langle \vartheta_1, \vartheta_1\rangle = -\langle \vartheta_2, \vartheta_2\rangle$. $\kappa_1 = \langle \vartheta_1', \vartheta_2\rangle, \kappa_2 = \langle \vartheta_1', \rho\rangle, \kappa_3 = \langle \vartheta_2', \rho\rangle$. Moreover we can find a smooth function $\tau(t)$ such that $\gamma'(t) = \tau(t)\rho(t)$. We call the functions $(\tau(t), \kappa_1(t), \kappa_2(t), \kappa_3(t))$ the curvature of the framed curve.

Similarly, the Frenet-Serret type formula for a timelike framed curve $(\gamma, \vartheta_1, \vartheta_2)$ can be given by

$$\begin{bmatrix} \vartheta_1' \\ \vartheta_2' \\ \rho' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & -\kappa_2 \\ -\kappa_1 & 0 & -\kappa_3 \\ -\kappa_2 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \vartheta_1 \\ \vartheta_2 \\ \rho \end{bmatrix}, \tag{3.3}$$

where $\kappa_1 = \langle \vartheta_1', \vartheta_2\rangle, \kappa_2 = \langle \vartheta_1', \rho\rangle, \kappa_3 = \langle \vartheta_2', \rho\rangle$.

3.2. **Existence and uniqueness of the framed curves in $\mathbb{E}_1^3$.**

Theorem 3.1. *Let* $(\tau(t), \kappa_1(t), \kappa_2(t), \kappa_3(t)) : I \to \mathbb{R}^4$ *be a smooth map. Then there exist framed curves* $(\gamma, \vartheta_1, \vartheta_2) : I \to \mathbb{E}_1^3 \times \Theta$ *with three different causality whose curvatures are* $(\tau(t), \kappa_1(t), \kappa_2(t), \kappa_3(t))$.

Proof. Let $t_0 \in I$ and let $\{e_1, e_2, e_3\}$ be an pseudo orthonormal basis for $\mathbb{E}^3_1$. First we suppose that e_3 is a timelike vector and the basis is positively oriented. We need to solve the following ODE system

$$\begin{aligned}\vartheta_1' &= \kappa_1\vartheta_2 + \kappa_2\rho, \\ \vartheta_2' &= -\kappa_1\vartheta_1 + \kappa_3\rho, \\ \rho' &= \kappa_2\vartheta_1 + \kappa_3\vartheta_2,\end{aligned} \tag{3.4}$$

with initial conditions, $\vartheta_1(t_0) = e_1, \vartheta_2(t_0) = e_2, \rho(t_0) = e_3$. Then by existence and uniqueness of the solution of a system of ODE, we get $\{\vartheta_1, \vartheta_2, \rho\}$ to be the unique solution and define

$$\gamma(t) = \int_{t_0}^{t} \tau(s)\rho(s)ds. \tag{3.5}$$

Then we have to prove that the framed curve $(\gamma(t), \vartheta_1(t), \vartheta_2(t))$ is a timelike curve with curvature functions $(\tau, \kappa_1, \kappa_2, \kappa_3)$. We first show that the moving frame $\{\vartheta_1(t), \vartheta_2(t), \rho(t)\}$ is an pseudo orthonormal basis of $\mathbb{E}^3_1$ with the same causal properties as of the initial basis $\{e_1, e_2, e_3\}$. Consider the ODE system,

$$\begin{aligned}\langle\vartheta_1, \vartheta_1\rangle' &= 2\kappa_1\langle\vartheta_1, \vartheta_2\rangle + 2\kappa_2\langle\rho, \vartheta_1\rangle, \\ \langle\vartheta_2, \vartheta_2\rangle' &= -2\kappa_1\langle\vartheta_1, \vartheta_2\rangle + 2\kappa_3\langle\rho, \vartheta_2\rangle, \\ \langle\rho, \rho\rangle' &= 2\kappa_2\langle\vartheta_1, \rho\rangle + 2\kappa_3\langle\rho, \vartheta_2\rangle, \\ \langle\vartheta_1, \vartheta_2\rangle' &= \kappa_1\langle\vartheta_2, \vartheta_2\rangle + \kappa_2\langle\rho, \vartheta_2\rangle - \kappa_1\langle\vartheta_1, \vartheta_1\rangle + \kappa_3\langle\rho, \vartheta_1\rangle, \\ \langle\vartheta_1, \rho\rangle' &= \kappa_1\langle\vartheta_2, \rho\rangle + \kappa_2\langle\rho, \rho\rangle + \kappa_2\langle\vartheta_1, \vartheta_1\rangle + \kappa_3\langle\vartheta_2, \vartheta_1\rangle, \\ \langle\vartheta_2, \rho\rangle' &= -\kappa_1\langle\vartheta_1, \rho\rangle + \kappa_3\langle\rho, \rho\rangle + \kappa_2\langle\vartheta_1, \vartheta_2\rangle + \kappa_3\langle\vartheta_2, \vartheta_2\rangle,\end{aligned}$$

with initial conditions $\langle\vartheta_1, \vartheta_1\rangle = 1, \langle\vartheta_2, \vartheta_2\rangle = 1, \langle\rho, \rho\rangle = -1, \langle\vartheta_1, \vartheta_2\rangle = 0, \langle\vartheta_1, \rho\rangle = 0, \langle\vartheta_2, \rho\rangle = 0$. On the other hand, the constant functions $f_1(t) = 1, f_2(t) = 1, f_3(t) = -1, f_4(t) = 0, f_5(t) = 0, f_6(t) = 0$ satisfy the same ODE system and initial conditions, so by uniqueness of the solution,

$$-\langle\rho, \rho\rangle = \langle\vartheta_1, \vartheta_1\rangle = \langle\vartheta_2, \vartheta_2\rangle = 1, \ \langle\vartheta_1, \vartheta_2\rangle = \langle\vartheta_1, \rho\rangle = \langle\vartheta_2, \rho\rangle = 0.$$

This implies that $\{\vartheta_1, \vartheta_2, \rho\}$ is a pseudo orthonormal basis of $\mathbb{E}^3_1$. From (3.4), $\gamma'(t) = \tau(t)\rho(t)$, and hence $\langle\gamma', \gamma'\rangle = \tau^2\langle\rho, \rho\rangle = -\tau^2 < 0$, considering $\tau \neq 0$, this implies that γ is a timelike framed curve with curvatures $(\tau, \kappa_1, \kappa_2, \kappa_3)$.

Similarly, we can show that $(\gamma, \vartheta_1, \vartheta_2)$ is a spacelike framed curve with the spacelike vector ϑ_1 if e_2 is timelike, and is a spacelike framed curve with the timelike vector ϑ_1 if e_1 is timelike. □

Proposition 3.1. [2] *For any vectors $a, b \in \mathbb{E}_1^3$ and an isometry $M \in SO_1(3)$, we have*

$$\langle a, b\rangle = \langle Ma, Mb\rangle, \tag{3.6}$$
$$a \times b = Ma \times Mb.$$

Definition 3.4. [9] *Let $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2) : I \to \mathbb{E}_1^3 \times \Theta$ be framed curves of same causal character. We say that $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2)$ are congruent as framed curves through a Lorentzian motion if there exists a matrix $M \in SO_1(3)$ and a constant vector $c \in \mathbb{E}_1^3$ such that*

$$\bar{\gamma}(t) = M(\gamma(t)) + c, \tag{3.7}$$
$$\bar{\vartheta}_i(t) = M(\vartheta_i(t)),$$

for all $t \in I$, where the matrix M satisfies $M^T G M = G$, $Det(M) = 1, G = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Lemma 3.1. [9] *Let the framed curves $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2)$ be congruent. Then their curvatures coincide, i.e. the curvatures $(\tau, \kappa_1, \kappa_2, \kappa_3)$ are invariant under a Lorentzian motion.*

Theorem 3.2. *Let $(\gamma, \vartheta_1, \vartheta_2)$ and $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2) : I \to \mathbb{E}_1^3 \times \Theta$ be framed curves that have the same causal character of the moving frames. If they have the same corresponding curvatures then they are congruent as framed curves through a Lorentzian motion.*

Proof. Let $t_0 \in I$ and consider the isometry $A \in SO_1(3)$ such that $\bar{\vartheta}_i(t_0) = A\vartheta_i(t_0),\ \bar{\rho}(t_0) = A\rho(t_0)$. If $c = \bar{\gamma}(t_0) - A \circ \gamma(t_0)$, define the rigid motion $Mx = Ax + c$. We know that by above lemma 3.6, that the framed curve $(M \circ \gamma, A\vartheta_1, A\vartheta_2)$ satisfies the same ODE system as $(\bar{\gamma}, \bar{\vartheta}_1, \bar{\vartheta}_2)$. As the initial conditions coincide, then by uniqueness of ODE system,

$$\bar{\gamma}(t) = M \circ \gamma(t),$$
$$\bar{\vartheta}_i(t) = A\vartheta_i(t),\ i = 1, 2,$$

which completes the proof. □

4. Translation Framed surfaces in $\mathbb{E}_1^3$

Definition 4.1. *A smooth map $(\sigma, \xi, \eta) : \Omega \subset \mathbb{E}^2 \to \mathbb{E}_1^3 \times \Theta$ is said to be a spacelike framed surface if the following conditions hold*

$$\sigma_s(s,t).\xi(s,t) = 0,\ \sigma_t(s,t).\xi(s,t) = 0, \forall (s,t) \in \Omega, \tag{4.8}$$

where $\Theta = \{(u,v) \in H_0^2 \times S_1^2 | u.v = 0\}$.
Also, we say that the map $(\sigma, \xi, \eta) : \Omega \subset \mathbb{E}^2 \to \mathbb{E}_1^3 \times \Theta$ is a timelike framed surface if condition (4.8) *holds with $\Theta = \{(u,v) \in S_1^2 \times S_1^2 | u.v = 0\}$ or $\Theta = \{(u,v) \in S_1^2 \times H_0^2 | u.v = 0\}$.*
For a framed surface (σ, ξ, η), the map $(\xi, \eta) : \Omega \to \Theta$, is a moving frame while $\sigma : \Omega \to \mathbb{E}_1^3$ is called the framed base surface.

4.1. **Basic invariants of a framed surface.** Let's define $\zeta(s,t) = \xi(s,t) \times \eta(s,t)$, then with respect to the moving frame $\{\xi(s,t), \eta(s,t), \zeta(s,t)\}$ along $\sigma(s,t)$, the basic invariants are defined as follows

Case(i):- For the spacelike surface, ξ is a timelike vector and η, ζ are spacelike vectors. Then

$$\begin{bmatrix} \sigma_s \\ \sigma_t \end{bmatrix} = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix}, \tag{4.9}$$

where $c_1 = \sigma_s.\eta$, $c_2 = \sigma_t.\eta$, $d_1 = \sigma_s.\zeta$, $d_2 = \sigma_t.\zeta$.

$$\begin{bmatrix} \xi_s \\ \eta_s \\ \zeta_s \end{bmatrix} = \begin{bmatrix} 0 & l_1 & m_1 \\ l_1 & 0 & n_1 \\ m_1 & -n_1 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{4.10}$$

$$\begin{bmatrix} \xi_t \\ \eta_t \\ \zeta_t \end{bmatrix} = \begin{bmatrix} 0 & l_2 & m_2 \\ l_2 & 0 & n_2 \\ m_2 & -n_2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix},$$

where $l_1 = \xi_s.\eta$, $m_1 = \xi_s.\zeta$, $n_1 = \eta_s.\zeta$ and $l_2 = \xi_t.\eta$, $m_2 = \xi_t.\zeta$, $n_2 = \eta_t.\zeta$.
We call the smooth functions $c_i, d_i, l_i, m_i, n_i : \Omega \to R$, i = 1, 2 the basic invariants of the framed surface. Let the above matrices be denoted by $\Lambda, \Delta_1, \Delta_2$, respectively, as follows

$$\Lambda = \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & l_1 & m_1 \\ l_1 & 0 & n_1 \\ m_1 & -n_1 & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0 & l_2 & m_2 \\ l_2 & 0 & n_2 \\ m_2 & -n_2 & 0 \end{bmatrix}.$$

Then, using the integrability condition $\sigma_{st} = \sigma_{ts}$ and $\Delta_{2,s} - \Delta_{1,t} = \Delta_1\Delta_2 - \Delta_2\Delta_1$, the basic invariants satisfy the following conditions:

$$\begin{cases} c_{1,t} - d_1 g_2 = c_{2,s} - d_2 n_1, \\ d_{1,t} - c_2 g_1 = d_{2,s} - c_1 n_2, \\ c_1 e_2 + d_1 f_2 = c_2 e_1 + d_2 m_1. \end{cases} \tag{4.11}$$

$$\begin{cases} l_{1,t} - m_1 n_2 = l_{2,s} - m_2 n_1, \\ m_{1,t} - l_2 n_1 = m_{2,s} - l_1 n_2, \\ n_{1,t} + l_1 m_2 = n_{2,s} + l_2 m_1. \end{cases} \tag{4.12}$$

Case(ii):- For the timelike surface, ξ is a spacelike vector and one of the vectors η or ζ is a timelike vector and other is spacelike. So let $\langle \eta, \eta \rangle = \delta = -\langle \zeta, \zeta \rangle$, where $\delta = \pm 1$, accordingly. Then

$$\begin{bmatrix} \sigma_s \\ \sigma_t \end{bmatrix} = \delta \begin{bmatrix} c_1 & -d_1 \\ c_2 & -d_2 \end{bmatrix} \begin{bmatrix} \eta \\ \zeta \end{bmatrix}, \tag{4.13}$$

where $c_1 = \sigma_s.\eta$, $c_2 = \sigma_t.\eta$, $d_1 = \sigma_s.\zeta$, $d_2 = \sigma_t.\zeta$.

$$\begin{bmatrix} \xi_s \\ \eta_s \\ \zeta_s \end{bmatrix} = \delta \begin{bmatrix} 0 & l_1 & -m_1 \\ -\delta l_1 & 0 & -n_1 \\ -\delta m_1 & -n_1 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}, \tag{4.14}$$

$$\begin{bmatrix} \xi_t \\ \eta_t \\ \zeta_t \end{bmatrix} = \delta \begin{bmatrix} 0 & l_2 & -m_2 \\ -\delta l_2 & 0 & -n_2 \\ -\delta m_2 & -n_2 & 0 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix},$$

where $l_1 = \xi_s.\eta$, $m_1 = \xi_s.\zeta$, $n_1 = \eta_s.\zeta$ and $l_2 = \xi_t.\eta$, $m_2 = \xi_t.\zeta$, $n_2 = \eta_t.\zeta$.

In particular, if we assume that the vector field η is timelike, then the basic invariants are given by

$$\Lambda = \begin{bmatrix} -c_1 & d_1 \\ -c_2 & d_2 \end{bmatrix}, \Delta_1 = \begin{bmatrix} 0 & -l_1 & m_1 \\ -l_1 & 0 & n_1 \\ -m_1 & n_1 & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0 & -l_2 & m_2 \\ -l_2 & 0 & n_2 \\ -m_2 & n_2 & 0 \end{bmatrix}.$$

Again using the integrability conditions, the basic invariants satisfy the following conditions:

$$\begin{cases} c_{1,t} - d_1 n_2 = c_{2,s} - d_2 n_1, \\ d_{1,t} + c_2 n_1 = d_{2,s} + c_1 n_2, \\ c_1 l_2 - b_1 m_2 = c_2 l_1 - d_2 m_1. \end{cases} \tag{4.15}$$

$$\begin{cases} l_{1,t} - m_1 n_2 = l_{2,s} - m_2 n_1, \\ m_{1,t} + l_2 n_1 = m_{2,s} + l_1 n_2, \\ n_{1,t} - l_1 m_2 = n_{2,s} - l_2 m_1. \end{cases} \tag{4.16}$$

4.2. **Existence and Uniqueness of framed surfaces in $\mathbb{E}_1^3$.**

Theorem 4.1. *For arbitrary given smooth functions $c_i, d_i, l_i, m_i, n_i : \Omega \to \mathbb{R}, i = 1, 2,$ defined on a simply connected domain Ω, satisfying the integrability conditions* (4.11) *and* (4.12) *(respectively,* (4.15) *and* (4.16)*), there exists a spacelike (respectively, timelike) framed surface $(\sigma, \xi, \eta) : \Omega \to \mathbb{E}_1^3 \times \Theta$ such that c_i, d_i, l_i, m_i, n_i are the basic invariants of the surface.*

Proof. By the integrability condition (4.12) (respectively, (4.16)), there exists a pseudo orthonormal frame $\{\xi, \eta, \zeta\}$ such that it satisfy ODE system (4.10) (respectively, (4.14)). Further, by the integrability condition (4.11) and (4.15), there exists a smooth map $\sigma : \Omega \to \mathbb{E}_1^3$ which satisfies the condition (4.9) and (4.13). Thus, we get a spacelike (respectively, timelike) framed surface (σ, ξ, η) with basic invariants $(\Lambda, \Delta_1, \Delta_2)$. □

Theorem 4.2. *Let (σ, ξ, η) and $(\bar{\sigma}, \bar{\xi}, \bar{\eta}) : \Omega \to \mathbb{E}_1^3 \times \Theta$ be framed surfaces of same causal character with basic invariants $(\Lambda, \Delta_1, \Delta_2)$ and $(\bar{\Lambda}, \bar{\Delta}_1, \bar{\Delta}_2)$, respectively. Then (σ, ξ, η) and $(\bar{\sigma}, \bar{\xi}, \bar{\eta})$ are congruent as framed surfaces if and only if the basic invariants coincide.*

Proof. Let $(s_0, t_0) \in U_0$ and consider the isometry $A \in O_1(3)$, such that $\bar{\xi}(s_0, t_0) = A \circ \xi(s_0, t_0), \bar{\eta}(s_0, t_0) = A \circ \eta(s_0, t_0)$ and $\bar{\zeta}(s_0, t_0) = A \circ \zeta(s_0, t_0)$. If $c = \bar{\sigma}(s_0, t_0) - A \circ \sigma(s_0, t_0)$, define the rigid motion $Mx = Ax + c$. Using the proposition 3.1, we see that the framed surface $(M \circ \sigma, A \circ \xi, A \circ \eta)$ and $(\bar{\sigma}, \bar{\xi}, \bar{\eta})$ both satisfy the same linear system of differential equations (4.13) and (4.14), i.e., basic invariants coincide. Now since initial conditions are same, by uniqueness theorem of system of ordinary differential equations, we find that $M \circ \sigma = \bar{\sigma}, A \circ \xi = \bar{\xi}, A \circ \eta = \bar{\eta}, A \circ \zeta = \bar{\zeta}$. Conversely, If (σ, ξ, η) and $(\bar{\sigma}, \bar{\xi}, \bar{\eta})$ are congruent then $M \circ \sigma = \bar{\sigma}, A \circ \xi = \bar{\xi}, A \circ \eta = \bar{\eta}, A \circ \zeta = \bar{\zeta}$, then again using proposition 3.1, we find that both framed surfaces have common basic invariants. □

4.3. **Curvatures of a Framed surface in $\mathbb{E}_1^3$.** We define curvatures of a framed surface $(\sigma, \xi, \eta) : \Omega \to \mathbb{E}_1^3 \times \Theta$ using the moving frame $\{\xi, \eta, \zeta = \xi \times \eta\}$ instead of $\{\sigma_s, \sigma_t, \xi\}$ as at singular points it may not be well defined. So first we obtain the matrix associated with the Weingarten map $W : TM \to TM$ with respect to the frame $\{\xi, \eta, \zeta = \xi \times \eta\}$ and then define the curvatures as determinant and trace of the map, where $TM = span\{\eta, \zeta\}$. Thus,

$$W(\eta) = -\eta\xi,\ W(\zeta) = -\zeta\xi, \tag{4.17}$$

where $\eta\xi$ and $\zeta\xi$ are the derivatives of the unit normal ξ with respect to the vector fields η and ζ, respectively. By using equation (4.9), we get

$$\begin{bmatrix} \eta \\ \zeta \end{bmatrix} = \frac{1}{\lambda} \begin{bmatrix} d_2 & -d_1 \\ -c_2 & c_1 \end{bmatrix} \begin{bmatrix} \sigma_s \\ \sigma_t \end{bmatrix},$$

where $\lambda = \det \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}$.

$$\begin{aligned} W(\eta) &= -\eta\xi = -\frac{1}{\lambda}(d_2\sigma_s - d_1\sigma_t)\xi = -\frac{1}{\lambda}(d_2\xi_s - d_1\xi_t), \\ W(\zeta) &= -\zeta\xi = -\frac{1}{\lambda}(-c_2\sigma_s + c_1\sigma_t)\xi = -\frac{1}{\lambda}(-c_2\xi_s + c_1\xi_t). \end{aligned}$$

Also using (4.10), we get

$$\begin{aligned} W(\eta) &= -\frac{1}{\lambda}((d_2 l_1 - d_1 l_2)\eta + (m_1 d_2 - m_2 d_1)\zeta), \\ W(\zeta) &= -\frac{1}{\lambda}((c_1 l_2 - c_2 l_1)\eta + (c_1 m_2 - c_2 m_1)\zeta). \end{aligned}$$

Thus, we get the Weingarten matrix as follows

$$W = -\frac{1}{\lambda} \begin{bmatrix} l_1 d_2 - l_2 d_1 & c_1 l_2 - c_2 l_1 \\ m_1 d_2 - m_2 d_1 & c_1 m_2 - c_2 m_1 \end{bmatrix}.$$

Now, we define $\mu_K = \lambda.\det W$ and $\mu_H = \lambda.\frac{1}{2}\mathrm{trace}(W)$. By direct calculation we obtain

$$\lambda = det \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix},\ \mu_K = det \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}, \tag{4.18}$$

$$\mu_H = -\frac{1}{2}\{det \begin{bmatrix} c_1 & m_1 \\ c_2 & m_2 \end{bmatrix} - det \begin{bmatrix} d_1 & l_1 \\ d_2 & l_2 \end{bmatrix}\}. \tag{4.19}$$

Where $\kappa_f = (\lambda, \mu_K, \mu_H)$ is the curvature of a spacelike framed surface. Similarly we find the curvature of a timelike framed surface as follows

$$\lambda = -det \begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix}, \; \mu_K = -det \begin{bmatrix} l_1 & m_1 \\ l_2 & m_2 \end{bmatrix}, \tag{4.20}$$

$$\mu_H = -\frac{\delta}{2}\{det \begin{bmatrix} c_1 & -m_1 \\ c_2 & m_2 \end{bmatrix} - det \begin{bmatrix} d_1 & -l_1 \\ d_2 & l_2 \end{bmatrix}\}, \tag{4.21}$$

where $\delta = \langle \eta, \eta \rangle$.

4.4. **Translation framed surface generated by framed curves in $\mathbb{E}_1^3$.** Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be framed curves with the curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$ in $\mathbb{E}_1^3$. Let $\sigma : I \times \bar{I} :\to \mathbb{E}_1^3$ be the translation surface parametrized as $\sigma(s,t) = \gamma(s) + \bar{\gamma}(t)$.

Proposition 4.1. [5] *Let $(\sigma, \nu_1^s, \nu_2^s) : \Omega \to \mathbb{E}_1^3 \times \Theta$ be a one parameter family of curves with respect to s and $(\sigma, \nu_1^t, \nu_2^t) : \Omega \to \mathbb{E}_1^3 \times \Theta$ be a one parameter family of curves with respect to t. If $\rho^s = \nu_1^s \times \nu_2^s$ and $\rho^t = \nu_1^t \times \nu_2^t$ are linearly independent for each $(s,t) \in \Omega$, then (σ, ξ, η) is a framed surface for some smooth mapping $(\xi, \eta) : \Omega \to \Theta$.*

For a translation surface $\sigma(s,t) = \gamma(s) + \bar{\gamma}(t)$ defined as above, we have (σ, ν_1, ν_2) and $(\sigma, \bar{\nu}_1, \bar{\nu}_2)$ as one parameter family of curves on the translation surface with respect to s and t, respectively. We consider a smooth map $(\xi, \eta) : \Omega \to \Theta$ defined by $\xi(s,t) = \frac{\rho(s)\times\bar{\rho}(t)}{\|\rho(s)\times\bar{\rho}(t)\|}$ and $\eta(s,t) = \rho(s)$, where $\rho = \nu_1 \times \nu_2$ and $\bar{\rho} = \bar{\nu}_1 \times \bar{\nu}_2$ such that the map $(\sigma, \xi, \eta) : \Omega \to E_1^3 \times \Theta$ is a framed surface and σ is a framed base surface by the Proposition 4.1. Considering the above construction we have the following corollary.

Corollary 4.1. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be framed curves in Minkowski 3-space such that $\rho(s)$ and $\bar{\rho}(t)$ are linearly independent for all $(s,t) \in I \times \bar{I}$, then $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$, defined by $\sigma(s,t) = \gamma(s) + \bar{\gamma}(t)$, $\xi(s,t) = \frac{\rho(s)\times\bar{\rho}(t)}{\|\rho(s)\times\bar{\rho}(t)\|}$ and $\eta(s,t) = \rho(s)$, is a translation framed surface.*

Theorem 4.3. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be timelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively in $\mathbb{E}_1^3$. Then the basic invariants of the timelike translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$ are obtained* *as follows*

$$
\begin{aligned}
c_1(s,t) &= -\tau(s),\\
d_1(s,t) &= 0,\\
c_2(s,t) &= \bar{\tau}(t)\rho(s).\bar{\rho}(t),\\
d_2(s,t) &= \bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2-1},\\
l_1(s,t) &= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t)-\kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
m_1(s,t) &= \frac{\rho(s).\bar{\rho}(t)}{(\rho(s).\bar{\rho}(t))^2-1}(\kappa_2(s)\nu_2(s).\bar{\rho}(t)-\kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
n_1(s,t) &= \frac{-1}{\sqrt{(\rho(s).\bar{\rho}(t))^2-1}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t)+\kappa_3(s)\nu_2(s).\bar{\rho}(t)),\\
l_2(s,t) &= 0,\\
m_2(s,t) &= \frac{-1}{(\rho(s).\bar{\rho}(t))^2-1}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s)-\bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)),\\
n_2(s,t) &= 0,
\end{aligned}
$$

where . denotes semi-Euclidean or Lorentzian scalar product.

Proof. Since the framed curves $(\gamma(s),\nu_1(s),\nu_2(s))$ and $(\bar{\gamma}(s),\bar{\nu}_1(s),\bar{\nu}_2(s))$ are timelike, by construction $\eta(s,t)=\rho(s)$ is a timelike vector field which belongs to the tangent space of the surface σ, therefore it is a timelike surface and furthermore ξ and ζ are spacelike vector. Now by using the Lagrange's identity $\langle u\times v,u\times v\rangle=-\langle u,u\rangle\langle v,v\rangle+\langle u,v\rangle^2$ and vector triple product $(u\times v)\times w=\langle v,w\rangle u-\langle u,w\rangle v$ for Minkowski space, we have

$$\|\rho(s)\times\bar{\rho}(t)\|=\sqrt{\epsilon(-\langle\rho(s),\rho(s)\rangle\langle\bar{\rho}(t),\bar{\rho}(t)\rangle+\langle\rho(s),\bar{\rho}(t)\rangle^2)}=\sqrt{\epsilon(-1+\langle\rho(s),\bar{\rho}(t)\rangle^2)},$$

where $\epsilon=\langle\xi,\xi\rangle=1$. Since by definition (of angle), $\langle\rho(s),\bar{\rho}(t)\rangle=-\cosh\theta$, therefore $-1+\langle\rho(s),\bar{\rho}(t)\rangle^2=-1+\cosh^2\theta=\sinh^2\theta\geq 0$. Also since ρ and $\bar{\rho}$ are linearly independent, $\rho.\bar{\rho}\neq 1$ therefore $\sinh^2\theta>0$. Thus, we have

$$
\begin{aligned}
c_1(s,t) &= \sigma_s(s,t).\eta(s,t)=\tau(s)\rho(s).\rho(s)=-\tau(s),\\
d_1(s,t) &= \sigma_s(s,t).\zeta(s,t)=\tau(s)\eta(s,t).\zeta(s,t)=0,\\
c_2(s,t) &= \sigma_t(s,t).\eta(s,t)=\bar{\tau}(t)\bar{\eta}(t).\eta(s)=\bar{\tau}(t)\rho(s).\bar{\rho}(t),
\end{aligned}
$$

$$\begin{aligned} d_2(s,t) &= \sigma_t(s,t).\zeta(s,t) = \bar{\tau}(t)\bar{\rho}(t).(\xi(s,t) \times \rho(s)) \\ &= \bar{\tau}(t)(\rho(s) \times \bar{\rho}(t)).\frac{\rho(s) \times \bar{\rho}(t)}{\|\rho(s) \times \bar{\rho}(t)\|} \\ &= \bar{\tau}(t)\|\rho(s) \times \bar{\rho}(t)\| = \bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}, \end{aligned}$$

Now, by using equation (3.3), we have $\rho_s(s) = \kappa_2(s)\nu_2(s) - \kappa_3(s)\nu_1(s)$. Thus

$$\begin{aligned} l_1(s,t) &= \xi_s(s,t).\eta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}((\rho(s) \times \rho_s(s)).\bar{\rho}(t)) \\ &= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}(\rho(s) \times (-\kappa_2(s)\nu_1(s) - \kappa_3(s)\nu_2(s)).\bar{\rho}(t)) \\ &= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)), \\ m_1(s,t) &= \xi_s(s,t).\zeta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|^2}(\rho_s(s) \times \bar{\rho}(t)).(\rho(s) \times \bar{\rho}(t) \times \rho(s)) \\ &= \frac{1}{(\rho(s).\bar{\rho}(t))^2 - 1}(\rho_s(s) \times \bar{\rho}(t)).((\rho(s).\bar{\rho}(t))\rho(s) + \bar{\rho}(t)) \\ &= \frac{1}{(\rho(s).\bar{\rho}(t))^2 - 1}(\rho(s).\bar{\rho}(t))(\rho(s) \times \rho_s(s)).\bar{\rho}(t) \\ &= \frac{\rho(s).\bar{\rho}(t)}{(\rho(s).\bar{\rho}(t))^2 - 1}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)), \\ n_1(s,t) &= \eta_s(s,t).\zeta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}\rho_s(s).((\rho(s) \times \bar{\rho}(t)) \times \rho(s)) \\ &= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}\rho_s(s).((\rho(s).\bar{\rho}(t))\rho(s) + \bar{\rho}(t)) \\ &= \frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}(\rho_s(s).\bar{\rho}(t)) \\ &= \frac{-1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t) + \kappa_3(s)\nu_2(s).\bar{\rho}(t)). \\ l_2(s,t) &= \xi_t(s,t).\eta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}(\rho(s) \times \bar{\rho}_t(t).\rho(s)) = 0, \\ m_2(s,t) &= \xi_t(s,t).\zeta(s,t) = \frac{1}{(\rho(s).\bar{\rho}(t))^2 - 1}(\rho(s) \times \bar{\rho}_t(t)).(\rho(s) \times \bar{\rho}(t) \times \rho(s)) \\ &= \frac{1}{(\rho(s).\bar{\rho}(t))^2 - 1}(\bar{\rho}_t(t) \times \bar{\rho}(t)).\rho(s) \\ &= \frac{1}{(\rho(s).\bar{\rho}(t))^2 - 1}(-\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) + \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)), \\ n_2(s,t) &= \eta_t(s,t).\zeta(s,t) = \rho_t(s,t).\zeta(s,t) = 0.\zeta(s,t) = 0. \end{aligned}$$

□

Corollary 4.2. *The curvature $\kappa_f = (\lambda, \mu_K, \mu_H)$ of the timelike translation framed surface in Theorem 4.4 is given as follows*

$$\lambda = \tau(s)\bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1},$$
$$\mu_K = \frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{((\rho(s).\bar{\rho}(t))^2 - 1)^{3/2}},$$
$$\mu_H = \frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) + \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2((\rho(s).\bar{\rho}(t))^2 - 1)}.$$

Proof. Using (4.20) and (4.21) and Theorem 4.4, we have

$$\lambda(s,t) = \tau(s)\bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1},$$
$$\begin{aligned}\mu_K(s,t) &= -l_1(s,t)m_2(s,t)\\ &= \frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{((\rho(s).\bar{\rho}(t))^2 - 1)^{3/2}}.\end{aligned}$$

$\delta = \langle \eta, \eta \rangle = -1$, so

$$\begin{aligned}\mu_H(s,t) =& \frac{1}{2}\{c_1 m_2 + c_2 m_1 - d_2 l_1\}\\ =& \frac{1}{2}\{\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(\rho(s).\bar{\rho}(t))^2 - 1}\\ &+ \frac{\bar{\tau}(t)(\rho(s).\bar{\rho}(t))^2}{(\rho(s).\bar{\rho}(t))^2 - 1}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\\ &- \bar{\tau}(t)\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}\frac{1}{\sqrt{(\rho(s).\bar{\rho}(t))^2 - 1}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\}\\ =& \frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) + \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2((\rho(s).\bar{\rho}(t))^2 - 1)}.\end{aligned}$$

□

Proposition 4.2. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be timelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the xz-plane and $\bar{\gamma}$ is contained in the xy-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$ obtained by the above curves, $\mu_K \equiv 0$ if and only if σ is a generalized cylinder.*

Proof. Let the curve γ be contained in the xz-plane and $\bar{\gamma}$ be contained in the xy-plane. Then we take $\nu_1(s) = (0, 1, 0)$, $\bar{\nu}_1(t) = (0, 0, 1)$ and $\rho(s) = (\rho_1(s), 0, \rho_3(s))$, $\bar{\rho}(t) = (\bar{\rho}_1(t), \bar{\rho}_2(t), 0)$ for some real smooth functions ρ_1, ρ_3, $\bar{\rho}_1$ and $\bar{\rho}_2$, which further gives $\nu_2(s) = \nu_1(s) \times \rho(s) =$

$(-\rho_3(s), 0, -\rho_1(s))$ and $\bar{\nu}_2(t) = \bar{\nu}_1(t) \times \bar{\rho}(t) = (-\bar{\rho}_2(t), -\bar{\rho}_1(t), 0)$. Also since ν_1 and $\bar{\nu}_1$ are fixed vectors, $\nu_1' = 0$ and $\bar{\nu}_1' = 0$ therefore from (3.3), $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Hence

$$\mu_K(s,t) = \frac{\kappa_3(s)\bar{\kappa}_3(t)\rho_3(s)\bar{\rho}_2(t)}{(\rho_1^2(s)\bar{\rho}_1^2(t) - 1)^{3/2}}.$$

Thus $\mu_K \equiv 0$ if and only if one of the functions $\bar{\kappa}_3, \kappa_3, \rho_3, \bar{\rho}_2$ is identically zero on an open interval in I or $\bar{I}$. So, if $\kappa_3 = 0$ or $\rho_3 = 0$ then γ is a part of a timelike straight line, while $\bar{\kappa}_3 = 0$ or $\bar{\rho}_2 = 0$ implies $\bar{\gamma}$ is a part of a timelike straight line. In either case σ is a generalized cylinder. □

Proposition 4.3. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be timelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the xz-plane and $\bar{\gamma}$ is contained in the xy-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$ generated by the framed curves, $\mu_H \equiv 0$ if and only if σ is a point or is a part of the following surface*

$$\sigma(s,t) = \Big(\frac{1}{c}\log\Big|\frac{\cosh(cu(s))}{\sinh(cv(t))}\Big| + B, v(t), u(s)\Big),$$

where B, c are some constants.

Proof. Using the similar constructions $\{\nu_1, \nu_2, \rho\}$ and $\{\bar{\nu}_1, \bar{\nu}_2, \bar{\rho}\}$ as in Proposition 4.2, we get $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Hence

$$\mu_H(s,t) = \frac{-\tau(s)\bar{\kappa}_3(t)\rho_3(s) - \bar{\tau}(t)\kappa_3(s)\bar{\rho}_2(t)}{2(\rho_1^2(s)\bar{\rho}_1^2(t) - 1)}.$$

Now $\mu_H \equiv 0$ if and only if $\tau(s)\bar{\kappa}_3(t)\rho_3(s) + \bar{\tau}(t)\kappa_3(s)\bar{\rho}_2(t) = 0$, or

$$\frac{\tau(s)\rho_3(s)}{\kappa_3(s)} = -\frac{\bar{\tau}(t)\bar{\rho}_2(t)}{\bar{\kappa}_3(t)} = C(constant). \tag{4.22}$$

By definition $\kappa_3(s) = \nu_2'(s).\rho(s) = \rho_{3,s}\rho_1 - \rho_{1,s}\rho_3$ and $\bar{\kappa}_3(t) = \bar{\rho}_{2,t}\bar{\rho}_1 - \bar{\rho}_{1,t}\bar{\rho}_2$, substituting into (4.22) we get,

$$C(\rho_{3,s}\rho_1 - \rho_{1,s}\rho_3) = \tau(s)\rho_3(s),$$

$$C(\bar{\rho}_{2,t}\bar{\rho}_1 - \bar{\rho}_{1,t}\bar{\rho}_2) = -\bar{\tau}(t)\bar{\rho}_2(t).$$

In the case $C = 0$, we have $\tau = 0$ or $\rho_3 = 0$ and $\bar{\tau} = 0$ or $\bar{\rho}_2 = 0$. If $\rho_3 = 0$ and $\bar{\rho}_2 = 0$ then $\kappa_3 = 0 = \bar{\kappa}_3$ which contradicts to the equation (4.22). Thus $\tau = 0$ and $\bar{\tau} = 0$ which implies that σ is a point.

Now in the case $C \neq 0$, replacing $c = \frac{1}{C}$ in the above equations we get,

$$\rho_{3,s}\rho_1 - \rho_{1,s}\rho_3 = c\tau(s)\rho_3(s), \tag{4.23}$$

$$\bar{\rho}_{2,t}\bar{\rho}_1 - \bar{\rho}_{1,t}\bar{\rho}_2 = -c\bar{\tau}(t)\bar{\rho}_2(t). \tag{4.24}$$

Since ρ is a timelike unit vector we take $\rho(s) = (\cosh(\theta(s)), 0, \sinh(\theta(s)))$, therefore $\rho_{1,s} = \theta_s \sinh(\theta)$ and $\rho_{3,s} = \theta_s \cosh(\theta)$. Using equation (4.23), we get

$$\theta_s = c\tau(s)\sinh(\theta(s)),$$
$$\int \frac{1}{\sinh(\theta)} d\theta = c\int \tau(s)ds + b,$$

which gives $e^\theta = \frac{1+Ae^{c\int\tau(s)ds}}{1-Ae^{c\int\tau(s)ds}}$, thus we get $\rho(s) = \big(\frac{1+A^2e^{2c\int\tau(s)ds}}{1-A^2e^{2c\int\tau(s)ds}}, 0, \frac{2Ae^{c\int\tau(s)ds}}{1-Ae^{2c\int\tau(s)ds}}\big)$. Now we calculate $\gamma(s) = \int \tau(s)\rho(s)ds$. Let $\gamma(s) = (\gamma_1(s), 0, \gamma_2(s))$, then we get $\gamma_1(s) = \int \tau(s)\rho_1(s)ds = -\frac{1}{c}\log\left(\frac{1-A^2e^{2c\int\tau(s)ds}}{e^{c\int\tau(s)ds}}\right)$ and $\gamma_3(s) = \int\tau(s)\rho_3(s)ds = \frac{1}{c}\log\left(\frac{1+Ae^{c\int\tau(s)ds}}{1-Ae^{c\int\tau(s)ds}}\right)$.
Let $u(s) = \frac{1}{c}\log\left(\frac{1+Ae^{c\int\tau(s)ds}}{1-Ae^{c\int\tau(s)ds}}\right)$, then γ is given by

$$\gamma(s) = \big(\frac{1}{c}\log\cosh(cu(s)) - \frac{1}{c}\log(2A), 0, u(s)\big).$$

Similarly, by equation (4.24), we obtain

$$\bar{\gamma}(t) = \big(-\frac{1}{c}\log\big|\sinh(cv(t))\big| + \frac{1}{c}\log(2\bar{A}), v(t), 0\big),$$

where $v(t) = -\frac{1}{c}\log\frac{1+\bar{A}e^{-c\int\bar{\tau}(t)dt}}{1-\bar{A}e^{-c\int\bar{\tau}(t)dt}}$. Thus

$$\begin{aligned}\sigma(s,t) &= \gamma(s) + \bar{\gamma}(t)\\ &= \Big(\frac{1}{c}\log\Big|\frac{\cosh(cu(s))}{\sinh(cv(t))}\Big| + B, v(t), u(s)\Big),\end{aligned}$$

where B is a constant. In fig. 1 we have diagram of the surface when $c = 1, B = 0$. $\square$

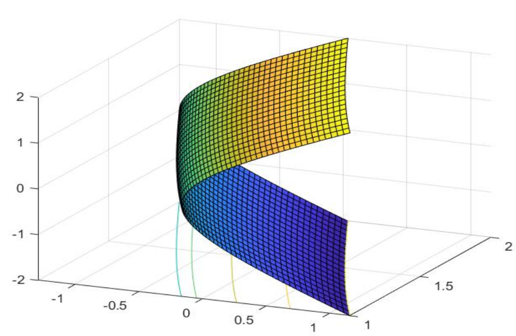

Figure 1.

Theorem 4.4. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ a spacelike and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be a timelike framed curve with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively in $\mathbb{E}_1^3$. Then the basic invariants of the timelike translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$, are obtained as follows*

$$
\begin{aligned}
c_1(s,t) &= \tau(s), \\
d_1(s,t) &= 0, \\
c_2(s,t) &= \bar{\tau}(t)\rho(s).\bar{\rho}(t), \\
d_2(s,t) &= \bar{\tau}(t)\sqrt{1 + (\rho(s).\bar{\rho}(t))^2}, \\
l_1(s,t) &= \frac{1}{\sqrt{1 + (\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)), \\
m_1(s,t) &= \frac{\rho(s).\bar{\rho}(t)}{1 + (\rho(s).\bar{\rho}(t))^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)), \\
n_1(s,t) &= \frac{\delta}{\sqrt{1 + (\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t) - \kappa_3(s)\nu_2(s).\bar{\rho}(t)), \\
l_2(s,t) &= 0, \\
m_2(s,t) &= \frac{1}{1 + (\rho(s).\bar{\rho}(t))^2}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)), \\
n_2(s,t) &= 0,
\end{aligned}
$$

where $\delta = \langle \nu_1, \nu_1 \rangle = \pm 1$.

Proof. Since the framed curves $(\gamma(s), \nu_1(s), \nu_2(s))$ is spacelike and $(\bar{\gamma}(s), \bar{\nu}_1(s), \bar{\nu}_2(s))$ is timelike, by construction $\sigma_t(s,t) = \bar{\gamma}'(t)$ is a timelike vector field which belongs to the tangent space of the surface σ, hence σ is a timelike surface and ξ and η are spacelike vectors, ζ is a timelike vector. Thus, we have

$$\|\rho(s) \times \bar{\rho}(t)\| = \sqrt{\epsilon(\langle \rho(s), \rho(s)\rangle\langle \bar{\rho}(t), \bar{\rho}(t)\rangle + \langle \rho(s), \bar{\rho}(t)\rangle^2)} = \sqrt{\epsilon(1 + \langle \rho(s), \bar{\rho}(t)\rangle^2)},$$

we have $\langle \rho(s), \bar{\rho}(t)\rangle = \sinh\theta$, $1+\langle \rho(s), \bar{\rho}(t)\rangle^2 = 1+\sinh^2\theta = \cosh^2\theta > 0$, hence $\epsilon = \langle \xi, \xi\rangle = 1$, and $\|\rho(s) \times \bar{\rho}(t)\| = \sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}$, we have

$$
\begin{aligned}
c_1(s,t) &= \sigma_s(s,t).\eta(s,t) = \tau(s)\rho(s).\rho(s) = \tau(s), \\
d_1(s,t) &= \sigma_s(s,t).\zeta(s,t) = \tau(s)\eta(s,t).\zeta(s,t) = 0, \\
c_2(s,t) &= \sigma_t(s,t).\eta(s,t) = \bar{\tau}(t)\bar{\rho}(t).\rho(s) = \bar{\tau}(t)\rho(s).\bar{\rho}(t), \\
d_2(s,t) &= \sigma_t(s,t).\zeta(s,t) = \bar{\tau}(t)\bar{\rho}(t).(\xi(s,t) \times \rho(s))
\end{aligned}
$$

$$= \bar{\tau}(t)(\rho(s) \times \bar{\rho}(t)).\frac{\rho(s) \times \bar{\rho}(t)}{\|\rho(s) \times \bar{\rho}(t)\|}$$

$$= \bar{\tau}(t)\|\rho(s) \times \bar{\rho}(t)\| = \bar{\tau}(t)\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2},$$

By using (3.2), we have $\rho_s(s) = -\delta\kappa_2(s)\nu_1(s) + \delta\kappa_3(s)\nu_2(s)$, hence

$$l_1(s,t) = \xi_s(s,t).\eta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}((\rho(s) \times \rho_s(s)).\bar{\rho}(t))$$

$$= \frac{1}{\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}}((\rho(s) \times (-\delta\kappa_2(s)\nu_1(s) + \delta\kappa_3(s)\nu_2(s)).\bar{\rho}(t))$$

$$= \frac{\delta}{\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}}(-\kappa_2(s)(-\delta\nu_2(s)) + \kappa_3(s)(-\delta\nu_1(s))).\bar{\rho}(t)$$

$$= \frac{1}{\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)),$$

$$m_1(s,t) = \xi_s(s,t).\zeta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|^2}(\rho_s(s) \times \bar{\rho}(t)).(\rho(s) \times \bar{\rho}(t) \times \rho(s))$$

$$= \frac{1}{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}(\rho_s(s) \times \bar{\rho}(t)).((\rho(s).\bar{\rho}(t))\rho(s) + \bar{\rho}(t))$$

$$= \frac{1}{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}(\rho(s).\bar{\rho}(t))(\rho(s) \times \rho_s(s)).\bar{\rho}(t)$$

$$= \frac{\rho(s).\bar{\rho}(t)}{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)),$$

$$n_1(s,t) = \eta_s(s,t).\zeta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}\rho_s(s).((\rho(s) \times \bar{\rho}(t)) \times \rho(s))$$

$$= \frac{1}{\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}}\rho_s(s).((\rho(s).\bar{\rho}(t))\rho(s) - \bar{\rho}(t))$$

$$= \frac{-1}{\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}}(\rho_s(s).\bar{\rho}(t))$$

$$= \frac{\delta}{\sqrt{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t) - \kappa_3(s)\nu_2(s).\bar{\rho}(t)),$$

$$l_2(s,t) = \xi_t(s,t).\eta(s,t) = \frac{1}{\|\rho(s) \times \bar{\rho}(t)\|}(\rho(s) \times \bar{\rho}_t(t).\rho(s)) = 0,$$

$$m_2(s,t) = \xi_t(s,t).\zeta(s,t) = \frac{1}{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}(\rho(s) \times \bar{\rho}_t(t)).(\rho(s) \times \bar{\rho}(t) \times \rho(s))$$

$$= \frac{-1}{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}(\bar{\rho}_t(t) \times \bar{\rho}(t)).\rho(s)$$

$$= \frac{1}{1 + \langle \rho(s), \bar{\rho}(t)\rangle^2}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)),$$

$$n_2(s,t) = \eta_t(s,t).\zeta(s,t) = \rho_t(s,t).\zeta(s,t) = 0.\zeta(s,t) = 0.$$

□

Corollary 4.3. *The curvature $\kappa_f = (\lambda, \mu_K, \mu_H)$ of the timelike translation framed surface given in Theorem 4.5 is given as follows*

$$\lambda = -\tau(s)\bar{\tau}(t)\sqrt{1+(\rho(s).\bar{\rho}(t))^2},$$

$$\mu_K = -\frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(1+(\rho(s).\bar{\rho}(t))^2)^{3/2}},$$

$$\mu_H = -\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) - \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2(1+(\rho(s).\bar{\rho}(t))^2)}.$$

Proof. Using (4.20) and (4.21) and Theorem 4.5, we have

$$\lambda(s,t) = \tau(s)\bar{\tau}(t)\sqrt{1+(\rho(s).\bar{\rho}(t))^2},$$

$$\begin{aligned}\mu_K(s,t) &= -l_1(s,t)m_2(s,t)\\ &= -\frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(1+(\rho(s).\bar{\rho}(t))^2)^{3/2}}.\end{aligned}$$

$\delta = \langle \eta, \eta \rangle = 1$, so

$$\begin{aligned}\mu_H(s,t) &= -\frac{1}{2}\{c_1m_2 + c_2m_1 - d_2l_1\}\\ &= -\frac{1}{2}\{\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{1+(\rho(s).\bar{\rho}(t))^2}\\ &\quad + \frac{\bar{\tau}(t)(\rho(s).\bar{\rho}(t))^2}{1+(\rho(s).\bar{\rho}(t))^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\\ &\quad - \bar{\tau}(t)\sqrt{1+(\rho(s).\bar{\rho}(t))^2}\frac{1}{\sqrt{1+(\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))\}\\ &= -\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) - \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2(1+(\rho(s).\bar{\rho}(t))^2)}.\end{aligned}$$

□

Proposition 4.4. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ be an spacelike and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be a timelike framed curve with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the yz-plane and $\bar{\gamma}$ is contained in the xz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}^3_1 \times \Theta$, $\mu_K \equiv 0$ if and only if σ is a generalized cylinder.*

Proof. We take $\nu_1(s) = (1,0,0)$, $\bar{\nu}_1(t) = (0,1,0)$ and $\rho(s) = (0, \rho_2(s), \rho_3(s))$, $\bar{\rho}(t) = (\bar{\rho}_1(t), 0, \bar{\rho}_3(t))$ for some real smooth functions ρ_2, ρ_3, $\bar{\rho}_1$ and $\bar{\rho}_3$. Then we get $\nu_2(s) = \rho(s) \times \nu_1(s) = (0, \rho_3(s), \rho_2(s))$ and $\bar{\nu}_2(t) = \bar{\nu}_1(t) \times \bar{\rho}(t) = (-\bar{\rho}_3(t), 0, -\bar{\rho}_1(t))$. Since ν_1 and $\bar{\nu}_1$ are fixed vectors, $\nu_1' = 0$ and $\bar{\nu}_1' = 0$ therefore $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Now by following the similar steps to the Proposition 4.2 we get the result. □

Proposition 4.5. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ be an spacelike and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be a timelike framed curve with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the yz-plane and $\bar{\gamma}$ is contained in the xz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$, $\mu_H \equiv 0$ if and only if σ is a point or is a part of the following surface*

$$\sigma(s,t) = \Big(v(t), u(s), \frac{1}{c}\log\Big|\frac{2\csc(cu(s))}{\cosh(cv(t))}\Big|\Big),$$

where c is some constant.

Proof. Working with the same frames $\{\nu_1, \nu_2, \rho\}$ and $\{\bar{\nu}_1, \bar{\nu}_2, \bar{\rho}\}$ as defined in the Proposition 4.4, we get $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Since ρ is an spacelike unit vector and $\bar{\rho}$ is a timelike unit vector so we take $\rho(s) = (0, \cos\theta(s), \sin\theta(s))$ and $\bar{\rho} = (\cosh\theta(t), 0, \sinh\theta(t))$ and by following the similar steps to the Proposition 4.3 we obtain

$$\gamma(s) = \big(0, u(s), \frac{1}{c}\log\left(2\csc\left(cu(s)\right)\right)\big),$$
$$\bar{\gamma}(t) = \big(v(t), 0, -\frac{1}{c}\log\cosh\left(cv(t)\right)\big),$$

where $u(s) = -\frac{1}{c}\log(\tan(\frac{c}{2}\int\bar{\tau}(t)dt + b))$ and $v(t) = \frac{2}{c}\arctan(Ae^{c\int\bar{\tau}(t)dt})$. Thus

$$\begin{aligned} X(s,t) &= \gamma(s) + \bar{\gamma}(t) \\ &= \Big(v(t), u(s), \frac{1}{c}\log\Big|\frac{2\csc(cu(s))}{\cosh(cv(t))}\Big|\Big), \end{aligned}$$

where c is a constant. In fig. 2 we have diagram of the surface when $c = 1$. $\square$

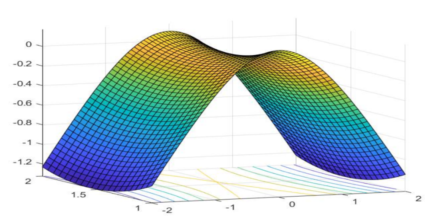

FIGURE 2.

Theorem 4.5. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be spacelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively in $\mathbb{E}^3_1$. Then the basic invariants of the spacelike translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}^3_1 \times \Theta$, are obtained as follows*

$$
\begin{aligned}
c_1(s,t) &= \tau(s),\\
d_1(s,t) &= 0,\\
c_2(s,t) &= \bar{\tau}(t)\rho(s).\bar{\rho}(t),\\
d_2(s,t) &= -\bar{\tau}(t)\sqrt{1-(\rho(s).\bar{\rho}(t))^2},\\
l_1(s,t) &= \frac{1}{\sqrt{1-(\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
m_1(s,t) &= \frac{\rho(s).\bar{\rho}(t)}{1-(\rho(s).\bar{\rho}(t))^2}(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t)),\\
n_1(s,t) &= \frac{\delta}{\sqrt{1-(\rho(s).\bar{\rho}(t))^2}}(\kappa_2(s)\nu_1(s).\bar{\rho}(t) - \kappa_3(s)\nu_2(s).\bar{\rho}(t)),\\
l_2(s,t) &= 0,\\
m_2(s,t) &= \frac{1}{1-(\rho(s).\bar{\rho}(t))^2}(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)),\\
n_2(s,t) &= 0,
\end{aligned}
$$

where $\delta = \langle \nu_1, \nu_1 \rangle = \pm 1$.

Proof. We can prove this theorem using similar steps as the Theorems 4.4, 4.5. □

Corollary 4.4. *The curvature $\kappa_f = (\lambda, \mu_K, \mu_H)$ of the spacelike translation framed surface given in Theorem 4.7 is given as follows*

$$
\begin{aligned}
\lambda &= -\tau(s)\bar{\tau}(t)\sqrt{1-(\rho(s).\bar{\rho}(t))^2},\\
\mu_K &= \frac{(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s))}{(1-(\rho(s).\bar{\rho}(t))^2)^{3/2}},\\
\mu_H &= -\frac{\tau(s)(\bar{\kappa}_2(t)\bar{\nu}_2(t).\rho(s) - \bar{\kappa}_3(t)\bar{\nu}_1(t).\rho(s)) - \bar{\tau}(t)(\kappa_2(s)\nu_2(s).\bar{\rho}(t) - \kappa_3(s)\nu_1(s).\bar{\rho}(t))}{2(1-(\rho(s).\bar{\rho}(t))^2)}.
\end{aligned}
$$

Proof. Proof is similar to the corollaries 4.2, 4.3. □

Proposition 4.6. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}^3_1 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}^3_1 \times \Theta$ be spacelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the yz-plane and $\bar{\gamma}$ is contained in the xz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}^3_1 \times \Theta$, $\mu_K \equiv 0$ if and only if σ is a generalized cylinder.*

Proof. We take $\nu_1(s) = (1, 0, 0)$ and $\bar{\nu}_1(t) = (0, 1, 0)$ then there exist real smooth functions ρ_2, ρ_3, $\bar{\rho}_1$ and $\bar{\rho}_3$ such that $\rho(s) = (0, \rho_2(s), \rho_3(s))$ and $\bar{\rho}(t) = (\bar{\rho}_1(t), 0, \bar{\rho}_3(t))$. Now by definition $\nu_2(s) = \rho(s) \times \nu_1(s) = (0, \rho_3(s), \rho_2(s))$ and $\bar{\nu}_2(t) = \bar{\nu}_1(t) \times \bar{\rho}(t) = (-\bar{\rho}_3(t), 0, -\bar{\rho}_1(t))$ and since ν_1 and $\bar{\nu}_1$ are fixed vectors, $\nu_1' = 0$ and $\bar{\nu}_1' = 0$ therefore $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Now by following the similar steps to the Proposition 4.2, we get the desired result. □

Proposition 4.7. *Let $(\gamma, \nu_1, \nu_2) : I \to \mathbb{E}_1^3 \times \Theta$ and $(\bar{\gamma}, \bar{\nu}_1, \bar{\nu}_2) : \bar{I} \to \mathbb{E}_1^3 \times \Theta$ be spacelike framed curves with curvatures $(\kappa_1, \kappa_2, \kappa_3, \tau)$ and $(\bar{\kappa}_1, \bar{\kappa}_2, \bar{\kappa}_3, \bar{\tau})$, respectively. Assume that γ is contained in the xz-plane and $\bar{\gamma}$ is contained in the yz-plane. Then for the translation framed surface $(\sigma, \xi, \eta) : I \times \bar{I} \to \mathbb{E}_1^3 \times \Theta$, $\mu_H \equiv 0$ if and only if σ is a point or is a part of the following surface*

$$\sigma(s,t) = \Big(v(t), u(s), \frac{1}{c}\log\Big|\frac{\sinh(cv(t))}{\sin(cu(s))}\Big|\Big),$$

where c is some constant.

Proof. Working with the frames $\{\nu_1, \nu_2, \rho\}$ and $\{\bar{\nu}_1, \bar{\nu}_2, \bar{\rho}\}$ as defined in the Proposition 4.6, we have $\kappa_1 = \kappa_2 = \bar{\kappa}_1 = \bar{\kappa}_2 = 0$. Since ρ and $\bar{\rho}$ are spacelike unit vectors so we take $\rho(s) = (0, \cos\theta(s), \sin\theta(s))$ and $\bar{\rho}(t) = (\sinh\theta(t), 0, \cosh\theta(t))$ and by following the similar steps to the Proposition 4.3 we obtain

$$\gamma(s) = \big(0, u(s), -\frac{1}{c}\log\left(2\sin\left(cu(s)\right)\right)\big),$$
$$\bar{\gamma}(t) = \big(v(t), 0, \frac{1}{c}\log\left(2\sinh\left(cv(t)\right)\right)\big),$$

where $u(s) = -\frac{1}{c}\log(\tan(\frac{c}{2}\int\bar{\tau}(t)dt + b))$ and $v(t) = \frac{1}{c}\log(\frac{1+\bar{A}e^{c\int\bar{\tau}(t)dt}}{1-\bar{A}e^{c\int\bar{\tau}(t)dt}})$. Thus,

$$\begin{aligned}\sigma(s,t) &= \gamma(s) + \bar{\gamma}(t)\\ &= \Big(v(t), u(s), \frac{1}{c}\log\Big|\frac{\sinh(cv(t))}{\sin(cu(s))}\Big|\Big),\end{aligned}$$

where c is a constant. In fig. 3 we have diagram of the surface when $c = 1$. □

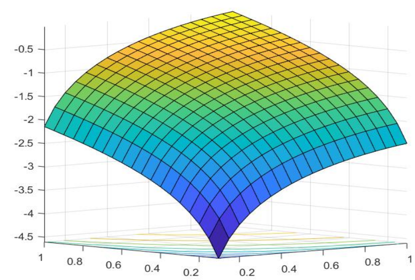

FIGURE 3.

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