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POINTWISE HEMI-SLANT RIEMANNIAN MAPS INTO ALMOST HERMITIAN MANIFOLDS AND CASORATI INEQUALITIES

YILMAZ GÜNDÜZALP, MEHMET AKİF AKYOL, AND BAYRAM ŞAHİN *

Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In the present paper, we introduce a new class of Riemannian maps which are called *pointwise hemi-slant Riemannian maps* from Riemannian manifolds to almost Hermitian manifolds as a natural generalization of hemi-slant submanifolds, hemi-slant submersions and hemi-slant Riemannian maps in a very natural way. We mention some examples, present a characterization and obtain the geometry of foliations in terms of the distributions which are involved in the definition of such maps. We also find necessary and sufficient conditions for pointwise hemi-slant Riemannian maps to be totally geodesic. Finally, we obtain Casorati curvatures for pointwise hemi-slant Riemannian maps in complex space form.

Keywords: Kaehler manifold, Riemannian map, pointwise hemi-slant submanifold, hemi-slant function, pointwise hemi-slant Riemannian map.

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1. Introduction

A considerable flaw in Riemannian geometry is to compare some geometric properties of suitable types of maps between Riemannian manifolds. Such suitable maps between Riemannian manifolds are isometric immersions and Riemannian submersions. Many geometers

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* Corresponding author

Yılmaz Gündüzalp ⋄ ygunduzalp@dicle.edu.tr ⋄ https://orcid.org/0000-0002-0932-949X

Mehmet Akif Akyol ⋄ mehmetakifakyol@bingol.edu.tr ⋄ https://orcid.org/0000-0001-7413-8844

Bayram Şahin ⋄ bayram.sahin@ege.edu.tr ⋄ https://orcid.org/0000-0002-9372-1151.

have been studied these maps between manifolds in ([1, 2, 9, 10, 12, 13, 14, 16, 18, 19, 25, 27, 32, 34, 33, 36, 35, 40, 37, 45, 48, 49]).

As a natural generalization of isometric immersions and Riemannian submersions, Fischer [17] defined the concept of Riemannian maps between Riemannian manifolds as follows: Let (M, g) and $(N, \bar{g})$ be Riemannian manifolds and Ψ is a smooth map between them. Then the tangent bundle of M has the following decompostion

$$TM = ker\Psi_* \oplus (ker\Psi_*)^{\perp},$$

where $ker\Psi_*$ denotes the kernel space of Ψ_* and $(ker\Psi_*)^{\perp}$ is the orthogonal complementary space to $ker\Psi_*$. In a similar way, the tangent bundle of N has the following decompostion

$$TN = (range\Psi_*) \oplus (range\Psi_*)^{\perp}$$

where $range\Psi_*$ denotes the range of Ψ_* and $range\Psi_*)^{\perp}$ is the orthogonal complementary space to $range\Psi_*$. Now, if the horizontal restriction $\Psi^h_{*p_1} : (ker\Psi^{\perp}_*) \longrightarrow (range\Psi_{*p_1})$ is a linear isometry between the inner product spaces $((ker\Psi_{*p_1})^{\perp}, g(p_1) \mid_{(ker\Psi_{*p_1})^{\perp}})$ and $(range\Psi_{*p_1}, \bar{g}(p_2) \mid_{(range\Psi_*)_{p_1}}), p_2 = \Psi(p_1)$ then a smooth map $\Psi : (M, g) \longrightarrow (N, \bar{g})$ is called Riemannian map at $p_1 \in M$. One can see that Riemannian submersions and isometric immersions are particular Riemannian maps with $(range\Psi_*)^{\perp} = 0$ and $ker\Psi_* = 0$, respectively.

Inspried by Fischer's article, B. Şahin introduced anti invariant Riemannian maps, holomorphic Riemannian maps and semi-invariant Riemannian maps to almost Hermitian manifolds and studied the geomerty of total spaces and base spaces ([39, 41]). This notion has opened a new original and effective area in the theory of Riemannian maps. Since then many geometers have studied Riemannian maps in different kinds of structures in [3, 4, 5, 20, 29, 28, 31, 38, 44, 43]. Recent developments in the theory of Riemannian map can be found in the books [30, 42].

On the other hand, in [11], Casorati introduced Casorati curvature which is a very natural concept of regular surfaces in the three-dimensional Euclidean space. One can see some optimal inequalities involving Casorati curvatures in ([7, 6, 15, 22, 24, 46, 47, 23, 51, 52]).

Hemi-slant submanifolds were introduced by Carriazo (Bi-slant immersions. in: Proc. ICRAMS 2000, Kharagpur, India, 2000, 88–97.) and Şahin (Annales Polonici Mathematici 95 (2009), 207-226) as a generalization of slant submanifolds. Hemi-slant submersions were introduced by Taştan, Şahin and Yanan (Mediterr. J. Math. 13, 2171–2184 (2016)) as a natural generalization of slant submersions. On the other hand, hemi-slant Riemannian maps were defined by Şahin (Mediterr. J. Math. 14, 10 (2017)) as a natural generalization of hemi-slant submanifolds and hemi-slant submersions. In 2022, Gündüzalp and Akyol defined pointwise slant Riemannian maps as a generalization of pointwise slant submanifolds [14] and pointwise slant submersions [25] in a natural way in [21]. They obtained simple characterizations and geometrical properties of pointwise slant Riemannian maps. As far as we know, no author has studied pointwise hemi-slant Riemannian maps so far. In the present paper, we are motivated to fill a gap in the literature by giving the notion of pointwise hemi-slant Riemannian maps, in which the base space consist of an anti-invariant and a slant distribution, as a special case of slant submanifold, hemi-slant submanifold, pointwise slant submanifold, slant submersions, hemi-slant submersions and hemi-slant Riemannian map and investigate the geometry of these maps.

The paper is organized as follows. Section 2 includes the main properties of the Riemannian maps, the tensors introduced by B. O'Neill and the second fundamental form of a map. Section 3 contains the definition of pointwise hemi-slant Riemannian maps from Riemannian manifolds to almost Hermitian manifolds, many examples and investigate the geometry of foliations which are arisen from the definition of a pointwise hemi-slant Riemannian map and obtain decomposition theorems by using these maps. We also find necessary and sufficient conditions for pointwise hemi-slant Riemannian maps to be totally geodesic. Finally, we obtain Casorati curvatures for pointwise hemi-slant Riemannian maps in complex space form.

2. Preliminaries

Let (M_1, g_{M_1}, J_1) be an almost Hermitian manifold. This means that M_1 admits a tensor field J_1 of type $(1,1)$ on M_1 such that

$$J_1^2 = -I, \;\; g_{M_1}(J_1\xi_1, J_1\xi_2) = g_{M_1}(\xi_1, \xi_2), \;\; \xi_1, \xi_2 \in \Gamma(TM_1). \tag{2.1}$$

An almost Hermitian manifold M_1 is called Kaehler manifold [50] if

$$(\nabla_{\xi_1} J_1)\xi_2 = 0, \quad \xi_1, \xi_2 \in \Gamma(TM_1), \tag{2.2}$$

where ∇ denotes the Riemannian connection of the metric g_{M_1} on M_1.

Let (M_1, g_{M_1}) and (M_2, g_{M_2}) be Riemannian manifolds and Ψ is a differentiable map between them. Then the differential Ψ_* of Ψ can be viewed a section of the bundle $Hom(TM_1, \Psi^{-1}TM_2) \to M_1$, where $\Psi^{-1}TM_2$ is the pullback bundle which has fibres $(\Psi^{-1}TM_2)_q = T_{\Psi(q)}M_2$, $q \in M_1$. $Hom(TM_1, \Psi^{-1}TM_2)$ has a connection ∇ induced from the Levi-Civita connection ∇^{M_1} and the pullback connection. The second fundamental form of Ψ is given by [8]

$$(\nabla\Psi_*)(\xi_1,\xi_2) = \nabla^{\Psi}_{\xi_1}\Psi_*\xi_2 - \Psi_*(\nabla^{M_1}_{\xi_1}\xi_2) \tag{2.3}$$

for $\xi_1,\xi_2 \in \Gamma(TM_1)$, where ∇^{Ψ} is the pullback connection. On the other hand, it is shown in ([39]) that $(\nabla\Psi_*)(\xi_1,\xi_2)$ has no components in $Im\Psi_*$, provided that $\xi_1,\xi_2 \in \Gamma((ker\Psi_*)^{\perp})$. More exactly,

$$(\nabla\Psi_*)(\xi_1,\xi_2) \in \Gamma((Im\Psi_*)^{\perp}),\ \ \forall\xi_1,\xi_2 \in \Gamma((ker\Psi_*)^{\perp}), \tag{2.4}$$

here $(Im\Psi_*)^{\perp}$ is the subbundle of $\Psi^{-1}(TM_2)$ with fibre $\Gamma(\Psi_*(T_qM_1)^{\perp})$, $q \in M_1$.
Let Ψ be a Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Riemannian manifold (M_2, g_{M_2}). Then $\forall\xi_1,\xi_2,Y_3 \in \Gamma((ker\Psi_*)^{\perp})$, we have

$$g_{M_2}((\nabla\Psi_*)(\xi_1,\xi_2),\Psi_*(Y_3)) = 0. \tag{2.5}$$

O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ are defined by, respectively,

$$\mathcal{T}_{\xi_1}\xi_2 = h\nabla_{v\xi_1}v\xi_2 + v\nabla_{v\xi_1}h\xi_2 \tag{2.6}$$

and

$$\mathcal{A}_{\xi_1}\xi_2 = v\nabla_{h\xi_1}h\xi_2 + h\nabla_{h\xi_1}v\xi_2 \tag{2.7}$$

for every $\xi_1,\xi_2 \in \Gamma(TM_1)$, where ∇ is the Levi-Civita connection of g_{M_1}. Here h and v are the projections on horizontal and vertical distributions, respectively. It is known that the tensor fields $\mathcal{T}$ is symetric and $\mathcal{A}$ is anti-symetric tensors. By using (2.6) and (2.7), we obtain

$$\nabla_{\eta_1}\eta_2 = \mathcal{T}_{\eta_1}\eta_2 + \hat{\nabla}_{\eta_1}\eta_2; \tag{2.8}$$

$$\nabla_{\eta_1}\xi_1 = \mathcal{T}_{\eta_1}\xi_1 + h\nabla_{\eta_1}\xi_1; \tag{2.9}$$

$$\nabla_{\xi_1}\eta_1 = \mathcal{A}_{\xi_1}\eta_1 + v\nabla_{\xi_1}\eta_1; \tag{2.10}$$

$$\nabla_{\xi_1}\xi_2 = \mathcal{A}_{\xi_1}\xi_2 + h\nabla_{\xi_1}\xi_2, \tag{2.11}$$

for any $\xi_1,\xi_2 \in \Gamma((ker\Psi_*)^{\perp})$, $\eta_1,\eta_2 \in \Gamma(ker\Psi_*)$, here $\hat{\nabla}_{\eta_1}\eta_2 = v\nabla_{\eta_1}\eta_2$.

We denote by ∇^2 both the levi-Civita connection of (M_2, g_{M_2}) and its pullback along Ψ. Then according to [26], for any vector field ξ_1 on M_1 and any section η_1 of $(range\Psi_*)^{\perp}$, where $(range\Psi_*)^{\perp}$ is the subbundle of $\Psi^{-1}(TM_2)$ with fiber $(\Psi_*(T_qM_1))^{\perp}-$ orthogonal complement of $(\Psi_*(T_qM_1))$ for g_{M_2} over q, we have $\nabla^{\Psi\perp}_{\xi_1}\eta_1$ which is the orthogonal projection of $\nabla^2_{\xi_1}\eta_1$ on $(\Psi_*(T_qM_1))^{\perp}-$such that $\nabla^{\Psi\perp}g_{M_2} = 0$. We now define $\mathcal{S}_{\eta_1}$ as

$$\nabla^2_{\Psi_*\xi_1}\eta_1 = -\mathcal{S}_{\eta_1}\Psi_*\xi_1 + \nabla^{\Psi\perp}_{\xi_1}\eta_1 \tag{2.12}$$

where $\mathcal{S}_{\eta_1}\Psi_*\xi_1$ is tangential component of $\nabla^2_{\Psi_*\xi_1}\eta_1$. It is easy to see that $\mathcal{S}_{\eta_1}\Psi_*\xi_1$ is bilinear in η_1 and $\Psi_*\xi_1$ and $\mathcal{S}_{\eta_1}\Psi_*\xi_1$ at q depends only on U_{1q} and $\Psi_{*q}Y_{1q}$. Thus,for $\xi_1, \xi_2 \in \Gamma((ker\Psi_*^{\perp})$ and $\eta_1 \in \Gamma((range\Psi_*)^{\perp})$, we get

$$g_{M_2}(\mathcal{S}_{\eta_1}\Psi_*\xi_1, \Psi_*\xi_2) = g_{M_2}(\eta_1, (\nabla\Psi_*)(\xi_1, \xi_2)). \tag{2.13}$$

Since $(\nabla\Psi_*)$ is symmetric, it follows that $\mathcal{S}_{\eta_1}$ is a symmetric linear transformation of $range\Psi_*$.

3. Pointwise hemi-slant Riemannian maps to Kaehler manifolds

Let $\Psi : (M_1, g_{M_1}) \to (M_2, g_{M_2}, J_2)$ be a Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to an almost Hermitian manifold (M_2, g_{M_2}, J_2). If, at each given point $p \in M_2$, the Wirtinger angle $\phi(X)$ between $J_2\Psi_*(X)$ and the space $range\Psi_*$ is independent of the choice of the nonzero tangent vector $\Psi_*(X)$ in $range\Psi_*$, then we say that Ψ is a pointwise slant Riemannian map. In this case, the angle ϕ can be regarded as a function on M_2, which is called the slant function of the pointwise slant Riemannian map.

Let $\mathcal{D}$ be a differentiable distribution on M_2. Then $\mathcal{D}$ is pointwise slant if and only if there exists a function $\mu \in [-1, 0]$ such that $(\gamma Q_\phi)^2\eta = \mu\eta$ for $\eta \in \mathcal{D}$, where Q_ϕ denotes the orthogonal projection on $\mathcal{D}$. Moreover, in this case $\mu = -\cos^2\phi$.

Definition 3.1. *Let (M_1, g_{M_1}) be a Riemannian manifold and (M_2, g_{M_2}, J_2) be an almost Hermitian manifold. Then we say that a Riemannian map $\Psi : M_1 \to M_2$ is a pointwise hemi-slant Riemannian map if there exists a pair of orthogonal distributions $\mathcal{D}^\phi$ and $\mathcal{D}^\perp$ on $range\Psi_*$ such that*

(1) *The space $range\Psi_*$ admits the orthogonal direct decomposition $\mathcal{D}^\phi \oplus \mathcal{D}^\perp$.*
(2) *The distribution $\mathcal{D}^\perp$ is totally real.*
(3) *The distribution $\mathcal{D}^\phi$ is pointwise slant with slant function ϕ.*

In this case, the angle ϕ can be regarded as a function on M_2, which is called the hemi-slant function of the pointwise hemi-slant Riemannian map.

Now we say that the pointwise hemi-slant Riemannian map Ψ is proper if $\mathcal{D}^\perp \neq \{0\}$ and $\phi \neq 0, \frac{\pi}{2}$.

Then, for $\eta_1 \in \Gamma(range\Psi_*)$, we can write

$$J_2\eta_1 = \mathcal{N}_1\eta_1 + \mathcal{N}_2\eta_1, \tag{3.14}$$

here $\mathcal{N}_1\eta_1 \in \Gamma(\mathcal{D}^\phi)$ and $\mathcal{N}_2\eta_1 \in \Gamma(\mathcal{D}^\perp)$ and we can write

$$J_2\eta_1 = \gamma\eta_1 + \delta\eta_1, \tag{3.15}$$

here $\gamma\eta_1 \in \Gamma(range\Psi_*)$ and $\delta\eta_1 \in \Gamma((range\Psi_*)^\perp)$. Also, for any $Y_1 \in \Gamma((range\Psi_*)^\perp)$, we get

$$J_2 Y_1 = \bar{\gamma} Y_1 + \bar{\delta} Y_1, \tag{3.16}$$

here $\bar{\gamma} Y_1 \in \Gamma(range\Psi_*)$ and $\bar{\delta} Y_1 \in \Gamma((range\Psi_*)^\perp)$.

Theorem 3.1. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to an almost Hermitian manifold (M_2, g_{M_2}, J_2) with hemi-slant function ϕ.*

$$\gamma^2\eta_1 = -(\cos^2\phi)\eta_1 \tag{3.17}$$

for any $\eta_1 \in \Gamma(\mathcal{D}^\phi)$.

Proof. Since,

$$\cos\phi = \frac{g_{M_2}(J_2\eta_1, \gamma\eta_1)}{|J_2\eta_1||\gamma\eta_1|} = -\frac{g_{M_2}(\eta_1, \gamma^2\eta_1)}{|\eta_1||\gamma\eta_1|}$$

and $\cos\phi == \frac{|\gamma\eta_1|}{|J_2\eta_1|}$, for $\eta_1 \in \Gamma(\mathcal{D}^\phi)$ we obtain

$$\cos^2\phi = -\frac{g_{M_2}(\eta_1, \gamma^2\eta_1)}{|\eta_1|^2}.$$

Hence,

$$\gamma^2\eta_1 = -(\cos^2\phi)\eta_1.$$

Also converse of Theorem 3.1, it can be directly verified. □

Moreover, for any $\eta_1, U_2 \in \Gamma(\mathcal{D}^\phi)$ we have

$$g_{M_2}(\gamma\eta_1, \gamma U_2) = \cos^2\phi g_{M_2}(\eta_1, U_2) \tag{3.18}$$

$$g_{M_2}(\delta\eta_1, \delta U_2) = \sin^2\phi g_{M_2}(\eta_1, U_2). \tag{3.19}$$

Furthermore, for $\eta_1 \in \Gamma(\mathcal{D}^\phi)$ we obtain

$$\bar{\gamma}\delta\eta_1 = -\sin^2\phi\eta_1, \;\; \bar{\delta}\delta\eta_1 = -\delta\gamma\eta_1. \tag{3.20}$$

Example 3.1. *Let $(\mathbb{R}^8, g_{\mathbb{R}^8})$ be the Euclid space. Consider $\{J_1, J_2\}$ a pair of almost complex structures on R^8 satisfying $J_1J_2 = -J_2J_1$, here*

$$J_1(a_1, ..., a_8) = (-a_3, -a_4, a_1, a_2, -a_7, -a_8, a_5, a_6)$$

and

$$J_2(a_1, ..., a_8) = (-a_2, a_1, a_4, -a_3, -a_6, a_5, a_8, -a_7).$$

For any real-valued function $\lambda : \mathbb{R}^8 \to \mathbb{R}$, we define new almost complex structure J_λ on $\mathbb{R}^8$ by $J_\lambda = (\cos\lambda)J_1 + (\sin\lambda)J_2$.

Then $\mathbb{R}^8_\lambda = (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8})$ *is an almost Hermitian manifold.*
Consider a Riemannian map $\Psi : \mathbb{R}^8 \to \mathbb{R}^8_\lambda$ *by*

$$\Psi(y_1, ..., y_8) = (y_1, y_3, y_6, y_8, \pi, e, c_1, c_2).$$

Then the map Ψ *is a proper pointwise hemi-slant Riemannian map with the hemi-slant function* λ *such that*

$$\mathcal{D}^\phi = span\{\frac{\partial}{\partial z_6}, \frac{\partial}{\partial z_8}\}, \text{ and } \mathcal{D}^\perp = span\{\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}\}.$$

Also, we obtain

$$(range_*)^\perp = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_4}, \frac{\partial}{\partial z_5}, \frac{\partial}{\partial z_7}\},$$

here $z_1, ..., z_8$ *are the local coordinates on* $\mathbb{R}^8$.

Theorem 3.2. *Let* Ψ_1 *be a Riemannian submersion from a Riemannian manifold* (M_1, g_{M_1}) *onto an almost Hermitian manifold* (M_2, g_{M_2}, J_2) *and* Ψ_2 *a poitwise hemi-slant immersion from* (M_2, g_{M_2}, J_2) *to an almost Hermitian manifold* (M_3, g_{M_3}, J_2). *Then* $\Psi_2 \circ \Psi_1$ *is a pointwise hemi-slant Riemannian map.*

This theorem is obvious from ([38], Theorem 5.2), and therefore we omit its proof.

As an application of the above Theorem, we give the following example of proper pointwise hemi-slant Riemannian map.

Example 3.2. *Let* $(\mathbb{R}^8, g_{\mathbb{R}^8})$ *be the Euclid space. Consider* $\{J_1, J_2\}$ *a pair of almost complex structures on* R^8 *satisfying* $J_1J_2 = -J_2J_1$, *here*

$$J_1(a_1, ..., a_8) = (-a_2, a_1, -a_4, a_3, -a_6, a_5, -a_8, a_7)$$

and

$$J_2(a_1, ..., a_8) = (-a_3, a_4, a_1, -a_2, -a_7, a_8, a_5, -a_6).$$

For any real-valued function $\lambda : \mathbb{R}^8 \to \mathbb{R}$, *we define new almost complex structure* J_λ *on* $\mathbb{R}^8$ *by* $J_\lambda = (\cos\lambda)J_1 + (\sin\lambda)J_2$.
Then, $\mathbb{R}^8_\lambda = (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8})$ *is an almost Hermitian manifold. Consider the map*

$$\Psi : (\mathbb{R}^8, g) \to (\mathbb{R}^8_\lambda, J_\lambda, g_{\mathbb{R}^8}), \;\; \Psi(y_1, ..., y_8) = (y_1, 0, 0, y_4, 0, 0, y_8, y_7)$$

which is the the composition of the Riemannian submersion

$$\Psi_1 : (\mathbb{R}^8, g) \to \mathbb{E}^4, \;\; \Psi_1(y_1, ..., y_8) = (y_1, y_4, y_7, y_8)$$

followed by the pointwise hemi-slant immersion

$$\Psi_2 : \mathbb{E}^4 \to (\mathbb{R}^8, J_\lambda, g_{\mathbb{R}^8}),\ \ \Psi_2(u_1, ..., u_4) = (u_1, 0, 0, u_2, 0, 0, u_4, u_3).$$

It is easy to verify that Ψ *is a pointwise hemi-slant Riemannian map with the slant function* $\phi = f$ *such that*

$$\mathcal{D}^\phi = span\{\frac{\partial}{\partial z_7}, \frac{\partial}{\partial z_8}\},\ \ and\ \ \mathcal{D}^\perp = span\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_4}\}.$$

Also, we obtain

$$(range_*)^\perp = span\{\frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, \frac{\partial}{\partial z_5}, \frac{\partial}{\partial z_6}\},$$

here $z_1, ..., z_8$ *are the local coordinates on* $\mathbb{R}^8_\lambda$.

First note that for $\Psi_*\xi_1 \in \mathcal{D}^\phi$ and $\Psi_*\xi_2 \in \mathcal{D}^\perp$, we get $g_{M_2}(\Psi_*\xi_1, \Psi_*\xi_2) = 0$. Then, Riemannian map Ψ implies that $g_{M_1}(\xi_1, \xi_2) = 0$. So we obtain two orthogonal distributions $\tilde{\mathcal{D}^\phi}$ and $\tilde{\mathcal{D}^\perp}$ such that

$$(ker\Psi_*)^\perp = \tilde{\mathcal{D}^\phi} \oplus \tilde{\mathcal{D}^\perp}.$$

Let Ψ be a $C^\infty-$map from a Riemannian manifold (M_1, g_{M_1}) to a Riemannian manifold (M_2, g_{M_2}). Then, the adjoint map ${}^*(\Psi_*)_{q_1}$ of the differential $(\Psi_*)_{q_1}$, $q_1 \in M_1$, is given by

$$g_{M_2}((\Psi_*)_{q_1}\eta_1, Y_1) = g_{M_1}(\eta_1,{}^* (\Psi_*)_{q_1} Y_1) \tag{3.21}$$

for any $\eta_1 \in T_{q_1}M_1$ and $Y_1 \in T_{\Psi(q_1)}M_2$. Furthermore if the map Ψ is a Riemannian map, then for $\eta_1 \in (range\Psi_*)_{\Psi(q_1)}$ and $Y_1 \in (ker(\Psi_*)_{q_1})^\perp$, we obtain

$$(\Psi_*)^*_{q_1}(\Psi_*)_{q_1}\eta_1 = \eta_1,\ \ {}^*(\Psi_*)_{q_1}(\Psi_*)_{q_1}Y_1 = Y_1,$$

thus the linear map ${}^*(\Psi_*)_{q_1} : (range\Psi_*)_{\Psi(q_1)} \to (ker(\Psi_*)_{q_1})^\perp$ is an isomorphism. Define $\mathrm{C} =^* (\Psi_*)_{q_1}\gamma(\Psi_*)$. From Theorem 3.1, we obtain:

Corollary 3.1. *Let* Ψ *be a pointwise hemi-slant Riemannian map from a Riemannian manifold* (M_1, g_{M_1}) *to an Hermitian manifold* (M_2, g_{M_2}, J_2) *with the hemi-slant function* ϕ*. Then,* $\eta_1 \in \Gamma(\mathrm{D}^\phi)$ *we have*

$$\mathrm{C}^2\eta_1 = -\cos^2\phi\eta_1. \tag{3.22}$$

For $Y_1, Y_2, \acute{Y_2} \in (ker(\Psi_*)_{q_1})^\perp$ with $\Psi_*\acute{Y_2} = \gamma\Psi_*Y_2$, we define

$$(\nabla^\Psi_{Y_1}\delta)\Psi_*Y_2 = \bar{\delta}(\nabla\Psi_*)(Y_1, Y_2) - (\nabla\Psi_*)(Y_1, \acute{Y_2}). \tag{3.23}$$

Proposition 3.1. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. If the tensor δ is parallel,then $\eta_1, U_2 \in \Gamma(\mathcal{D}^{\phi})$ we obtain*

$$(\nabla\Psi_*)(\mathrm{C}\eta_1, \mathrm{C}U_2) = -\cos^2\phi(\nabla\Psi_*)(\eta_1, U_2). \tag{3.24}$$

Proof. Assume that δ is parallel. Then, using (3.24), for $\eta_1, U_2 \in \Gamma(\mathcal{D}^{\phi})$ we get

$$\bar{\delta}(\nabla\Psi_*)(\eta_1, U_2) = (\nabla\Psi_*)(\eta_1, \mathrm{C}U_2).$$

By replacing η_1 and U_2 , we have

$$\bar{\delta}(\nabla\Psi_*)(U_2, \eta_1) = (\nabla\Psi_*)(U_2, \mathrm{C}\eta_1).$$

Since the tensor $(\nabla\Psi_*)$ is symmetric, we obtain

$$(\nabla\Psi_*)(\eta_1, \mathrm{C}U_2) = (\nabla\Psi_*)(U_2, \mathrm{C}\eta_1).$$

Thus we have

$$(\nabla\Psi_*)(\mathrm{C}\eta_1, \mathrm{C}U_2) = (\nabla\Psi_*)(\eta_1, \mathrm{C}^2U_2) = -\cos^2\phi(\nabla\Psi_*)(\eta_1, U_2).$$

□

Theorem 3.3. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. Then, the following assertions are equivalent:*

(a) distribution $\mathcal{D}^{\perp}$ defines a totally geodesic foliation on M_2,

(b)

$$g_{M_2}((\nabla\Psi_*)(\eta_1, {}^*\Psi_*((\gamma\Psi_*(U_3)))), J_2\Psi_*(U_2)) = g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2))$$

and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, {}^*\Psi_*(\bar{\gamma}Y_1)), J_2\Psi_*(U_2)) = g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}J_2\Psi_*(U_2), \bar{\delta}Y_1), \tag{3.25}$$

(c) Ψ satisfies (3.25) and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, U_2), \delta\gamma\Psi_*(U_3)) = g_{M_2}(\bar{\gamma}\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), \Psi_*(U_2))$$

for any $\Psi_*(\eta_1), \Psi_*(U_2) \in \Gamma(\mathcal{D}^{\perp})$, $\Psi_*(U_3) \in \Gamma(\mathcal{D}^{\phi})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$.

Proof. For any $\Psi_*(\eta_1), \Psi_*(U_2) \in \Gamma(\mathcal{D}^\perp)$ and $\Psi_*(U_3) \in \Gamma(\mathcal{D}^\phi)$, using (2.2),(3.15) and (2.12) we obtain

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = -g_{M_2}(S_{J_2\Psi_*(U_2)}\Psi_*(\eta_1), \gamma\Psi_*(U_3)) + g_{M_2}((\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \delta\Psi_*(U_3)).$$

From (2.13), we arrive at

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = -g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*(\gamma\Psi_*(U_3))), J_2\Psi_*(U_2)) + g_{M_2}(\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \delta\Psi_*(U_3)). \tag{3.26}$$

On the other hand, using (2.2),(3.16) and (2.12), for $Y_1 \in \Gamma((range\Psi_*)^\perp)$ we have

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), Y_1) = -g_{M_2}(S_{J_2\Psi_*(U_2)}\Psi_*(\eta_1), \bar{\gamma}Y_1) + g_{M_2}(\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \bar{\delta}Y_1).$$

From (2.13), we get

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), Y_1) = -g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*(\bar{\gamma}Y_1)), J_2\Psi_*(U_2)) + g_{M_2}(\nabla^{\Psi\perp}_{\eta_1} J_2\Psi_*(U_2), \bar{\delta}Y_1). \tag{3.27}$$

(3.26) and (3.27) gives $(a) \Leftrightarrow (b)$. For any $\Psi_*(\eta_1), \Psi_*(U_2) \in \Gamma(\mathcal{D}^\perp)$ and $\Psi_*(U_3) \in \Gamma(\mathcal{D}^\phi)$, from (2.2) and (3.15) we get

$$g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = g_{M_2}(\nabla^\Psi_{\eta_1}\gamma^2\Psi_*(U_3), \Psi_*(U_2)) + g_{M_2}(\nabla^\Psi_{\eta_1}\delta\gamma\Psi_*(U_3), \Psi_*(U_2)) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2)).$$

Using (2.12)and (3.17), we obtain

$$\sin^2\phi g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = \sin 2\phi\Psi_*\eta_1(\phi) g_{M_2}(\Psi_*(U_3), \Psi_*(U_2) - g_{M_2}(S_{\delta\gamma\Psi_*(U_3)}\Psi_*(\eta_1), \Psi_*(U_2)) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2)).$$

So, from (2.13) we arrive at

$$\sin^2\phi g_{M_2}(\nabla^\Psi_{\eta_1}\Psi_*(U_2), \Psi_*(U_3)) = -g_{M_2}((\nabla\Psi_*)(\eta_1, U_2), \delta\gamma\Psi_*(U_3)) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), J_2\Psi_*(U_2)). \tag{3.28}$$

(3.27) and (3.28) gives $(a) \Leftrightarrow (c)$. $\square$

In a similar way we obtain:

Theorem 3.4. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. Then, the following assertions are equivalent:*
(a) distribution $\mathcal{D}^{\phi}$ defines a totally geodesic foliation on M_2,
(b)

$$g_{M_2}((\nabla\Psi_*)(U_2,^* \Psi_*((\gamma\Psi_*(U_3)))), J_2\Psi_*(\eta_1)) = g_{M_2}(\nabla^{\Psi\perp}_{U_2} J_2\Psi_*(\eta_1), \delta\Psi_*(U_3))$$

and

$$\begin{aligned} g_{M_2}((\nabla\Psi_*)(U_2,^* \Psi_*(\bar{\gamma}Y_1)), \delta\Psi_*(U_3)) &= g_{M_2}(\nabla^{\Psi\perp}_{U_2}\delta\Psi_*(U_3), \bar{\delta}Y_1) \\ &- g_{M_2}(\nabla^{\Psi\perp}_{U_2}\delta\gamma\Psi_*(U_3), Y_1), \end{aligned} \tag{3.29}$$

(c) Ψ satisfies (3.29) and

$$g_{M_2}((\nabla\Psi_*)(U_2, \eta_1), \delta\gamma\Psi_*(U_3)) = g_{M_2}(\bar{\gamma}\nabla^{\Psi\perp}_{U_2}\delta\Psi_*(U_3), \Psi_*(\eta_1))$$

for any $\Psi_*(\eta_1) \in \Gamma(\mathcal{D}^{\perp})$, $\Psi_*(U_2), \Psi_*(U_3) \in \Gamma(\mathcal{D}^{\phi})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$.

Using Theorems 3.3 and 3.4, we obtain:

Theorem 3.5. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with integrable distribution and the hemi-slant function ϕ. Then, the leaf of $(range\Psi_*)$ is a locally product Riemannian manifold $M_1^{\perp} \times M_2^{\phi}$ if and only if*

$$\begin{aligned} g_{M_2}((\nabla\Psi_*)(\eta_1,^* \Psi_*\bar{\gamma}(Y_1)), \delta\Psi_*(U_2)) &= g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2), \bar{\delta}Y_1) \\ &- g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\gamma\Psi_*(U_2), Y_1) \end{aligned}$$

and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, U_2), \delta\Psi_*(U_3)) = g_{M_2}(\bar{\gamma}\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_3), \Psi_*(U_2))$$

for any $\eta_1, U_2 \in \Gamma((ker\Psi^*)^{\perp})$, $\Psi_*(U_3) \in \Gamma(\mathcal{D}^{\phi})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$, here $M_1^{\perp}$ and M_2^{ϕ} denotes the leaves of $\mathcal{D}^{\perp}$ and $\mathcal{D}^{\phi}$, respectively.

Theorem 3.6. *Let Ψ be a pointwise hemi-slant Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Kaehler manifold (M_2, g_{M_2}, J_2) with the hemi-slant function ϕ. Then,*

Ψ is totally geodesic if and only if the following conditions are satisfied:

(a)

$$g_{M_2}((\nabla\Psi_*)(\eta_1,{}^*\Psi_*\bar{\gamma}(Y_1)),\delta\Psi_*(U_2)) = g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2),\bar{\delta}Y_1) - g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\gamma\Psi_*(U_2),Y_1)$$

for any $\eta_1, U_2 \in \Gamma((ker\Psi^*)^{\perp})$ and $Y_1 \in \Gamma((range\Psi_*)^{\perp})$,

(b)

$$\begin{aligned}\sin 2\phi\eta_1(\phi)g_{M_2}(\Psi_*(U_2),\Psi_*(U_3)) &= g_{M_2}((\nabla\Psi_*)(\eta_1,U_3),\delta\gamma\Psi_*(U_2))\\ &- g_{M_2}(S_{\delta\Psi_*(U_2)}\eta_1,\gamma\Psi_*(U_3)\\ &+ g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2),\delta\Psi_*(U_3))\\ &- \sin^2\phi g_{M_1}(h\nabla_{\eta_1}U_2,U_3)\end{aligned}$$

for any $\eta_1, U_2, U_3 \in \Gamma((ker\Psi_*)^{\perp})$,

(c) the distribution $ker\Psi_*$ is totally geodesic,

(d) the distribution $(ker\Psi_*)^{\perp}$ is integrable.

Proof. For any $\eta_1, U_2, U_3 \in \Gamma((ker\Psi_*)^{\perp})$, from (2.2),(2.3),(2.11) and (3.15) we have

$$\begin{aligned}g_{M_2}((\nabla\Psi_*)(\eta_1,U_2),\Psi_*(U_3)) &= -g_{M_2}(\nabla^{\Psi}_{\eta_1}\gamma^2\Psi_*(U_2),\Psi_*(U_3))\\ &- g_{M_2}(\nabla^{\Psi}_{\eta_1}\delta\gamma\Psi_*(U_2),\Psi_*(U_3))\\ &+ g_{M_2}(\nabla^{\Psi}_{\eta_1}\delta\Psi_*(U_2),\gamma\Psi_*(U_3))\\ &+ g_{M_2}(\nabla^{\Psi}_{\eta_1}\delta\Psi_*(U_2),\delta\Psi_*(U_3)) - g_{M_1}(h\nabla_{\eta_1}U_2,U_3).\end{aligned}$$

Then, using (2.12),(2.13) and (3.17) we obtain

$$\begin{aligned}\sin^2\phi g_{M_2}((\nabla\Psi_*)(\eta_1,U_2),\Psi_*(U_3)) &= -\sin 2\phi\eta_1(\phi)g_{M_2}(\Psi_*(U_2),\Psi_*(U_3))\\ &+ g_{M_2}((\nabla\Psi_*)(\eta_1,U_3),\delta\gamma\Psi_*(U_2))\\ &- g_{M_2}(S_{\delta\Psi_*(U_2)}\eta_1,\gamma\Psi_*(U_3)\\ &+ g_{M_2}(\nabla^{\Psi\perp}_{\eta_1}\delta\Psi_*(U_2),\delta\Psi_*(U_3))\\ &- \sin^2\phi g_{M_1}(h\nabla_{\eta_1}U_2,U_3). \qquad (3.30)\end{aligned}$$

On the other hand, for $\eta_1, U_2 \in \Gamma((ker\Psi_*)^{\perp})$ and $V_1, V_2 \in \Gamma(ker\Psi_*)$, using (2.3), (2.8) and (2.11) we get

$$g_{M_2}((\nabla\Psi_*)(V_1,V_2),\Psi_*(\eta_1)) = -g_{M_1}(\mathrm{T}_{V_1}V_2,\eta_1) \qquad (3.31)$$

and

$$g_{M_2}((\nabla\Psi_*)(\eta_1, V_1), \Psi_*(U_2)) = -g_{M_1}(\mathrm{A}_{\eta_1}U_2, V_1). \tag{3.32}$$

Now, by using (3.30),(3.31), (3.32) and Theorem 3.3, the proof is completed. □

4. Casorati Inequalities along hemi-slant Riemannian maps to complex space forms

Lemma 4.1. [46] *Let* $W = \{(y_1, y_2, ..., y_m) \in \mathbb{R}^m : y_1 + y_2 + ... + y_m = z\}$ *be a hyperplane of* $\mathbb{R}^m$*, and* $g : \mathbb{R}^m \to \mathbb{R}$ *a quadratic form given by*

$$g(y_1, y_2, ..., y_m) = c\Sigma_{k=1}^{m-1}(y_k)^2 + d(y_m)^2 - 2\Sigma_{1\leq k<s\leq m}y_k y_s,\ c > 0,\ d > 0.$$

Then the constrained extremum problem $min_{(y_1,y_2,...,y_m)\in W}\, g$ *has the following solution:*

$$y_1 = y_2 = ... = y_{m-1} = \frac{z}{c+1},\ y_m = \frac{z}{d+1} = \frac{z(m-1)}{(c+1)d} = (c-m+2)\frac{z}{c+1},$$

provided that $d = \frac{m-1}{c-m+2}$.

Let (M_2, g_{M_2}, J_2) be a Kaehler manifold. The Riemannian-Christoffel curvature tensor of a complex space form $M_2(\nu)$ of constant holomorphic sectional curvature ν satisfies

$$\begin{aligned} R_{\mathcal{B}_2}(Y_1, Y_2, \mathcal{Y}_3, \mathcal{Y}_4) = \frac{\nu}{4}\{&g_{\mathcal{B}_2}(Y_1, \mathcal{Y}_4)g_{\mathcal{B}_2}(Y_2, \mathcal{Y}_3) - g_{\mathcal{B}_2}(Y_1, \mathcal{Y}_3)g_{\mathcal{B}_2}(Y_2, \mathcal{Y}_4) \\ &+ g_{\mathcal{B}_2}(Y_1, J_2\mathcal{Y}_3)g_{\mathcal{B}_2}(J_2Y_2, \mathcal{Y}_4) - g_{\mathcal{B}_2}(Y_2, J_2\mathcal{Y}_3)g_{\mathcal{B}_2}(J_2Y_1, \mathcal{Y}_4) \\ &+ 2g_{\mathcal{B}_2}(Y_1, J_2Y_2)g_{\mathcal{B}_2}(J_2\mathcal{Y}_3, \mathcal{Y}_4)\} \end{aligned} \tag{4.33}$$

for all vector fields $Y_1, Y_2, Y_3, Y_4 \in \Gamma(TM_2)$ ([50]).

Let Ψ be a Riemannian map from a Riemannian manifold (M_1, g_{M_1}) to a Riemannian manifold (M_2, g_{M_2}). Let R_{M_1} and R_{M_2} be the curvature tensor fields of ∇^{M_1} and ∇^{M_2}, respectively. Then, for all $Y_1, Y_2, Y_3, Y_4 \in \Gamma((ker\Psi_*)^\perp$, we obtain the Gauss formula given by ([40])

$$\begin{aligned} g_{M_2}(R_{\mathcal{B}_2}(\Psi_*Y_1, \Psi_*Y_2)\Psi_*Y_3, \Psi_*Y_4) = {}& g_{M_1}(R_{\mathcal{B}_1}(Y_1, Y_2)Y_3, Y_4) \\ &+ g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1, Y_3), (\nabla\Psi_*)(Y_2, Y_4)) \\ &- g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1, Y_4), (\nabla\Psi_*)(Y_2, Y_3)). \end{aligned} \tag{4.34}$$

Now, we suppose that Ψ is a pointwise hemi-slant Riemannian map from a Riemannian manifold $(M_1^{b_1}, g_{M_1})$ to the complex space form $(M_2^{b_2}(\nu), J_2, g_{M_2})$ such that $3 \leq p = rank\Psi <$

$min\{b_1, b_2\}$. Using (4.33) and (4.34) , for all $Y_1, Y_2, Y_3, Y_4 \in \Gamma((ker\Psi_*)^\perp$, we obtain

$$\begin{aligned} g_{M_1}(R_{\mathcal{B}_1}(Y_1,Y_2)Y_3,Y_4) = & \frac{\nu}{4}\{g_{\mathcal{B}_2}(Y_1,Y_4)g_{\mathcal{B}_2}(Y_2,Y_3) - g_{\mathcal{B}_2}(Y_1,Y_3)g_{\mathcal{B}_2}(Y_2,Y_4) \\ & + g_{\mathcal{B}_2}(\Psi_*Y_1, J_2\Psi_*Y_3)g_{\mathcal{B}_2}(J_2\Psi_*Y_2,\Psi_*Y_4) \\ & - g_{\mathcal{B}_2}(\Psi_*Y_2, J_2\Psi_*Y_3)g_{\mathcal{B}_2}(J_2\Psi_*Y_1,\Psi_*Y_4) \\ & + 2g_{\mathcal{B}_2}(\Psi_*Y_1, J_2\Psi_*Y_2)g_{\mathcal{B}_2}(J_2\Psi_*Y_3,\Psi_*Y_4)\} \\ & - g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1,Y_3),(\nabla\Psi_*)(Y_2,Y_4)) \\ & + g_{\mathcal{B}_2}((\nabla\Psi_*)(Y_1,Y_4),(\nabla\Psi_*)(Y_2,Y_3)). \end{aligned} \tag{4.35}$$

Let $q \in M_1$ and consider

$\{\Psi_*E_1, \Psi_*E_2 = \sec\phi\gamma\Psi_*E_1, ..., \Psi_*E_{2n-1}, \Psi_*E_{2n} = \sec\phi\gamma\Psi_*E_{2n-1}, \Psi_*E_{2n+1}, ..., \Psi_*E_p\}$ and $\{E_{p+1}, E_{p+2}, ..., E_{b_2}\}$ two orthonormal bases of $(ker\Psi_*)^\perp$ and $(range\Psi_*)^\perp$, respectively. Then, it follows that the dimension of $range\Psi_*$ is p. We defined the scalar curvature $\tau^{(ker\Psi_*)^\perp}$ on the horizontal space $(ker\Psi_{*q})^\perp$ by

$$\tau^{(ker\Psi_*)^\perp} = \Sigma_{k,s=1}^p g_{M_1}(R_{M_1}(E_k,E_s)E_s,E_k) \tag{4.36}$$

and the normalized scalar curvature $\kappa^{(ker\Psi_*)^\perp}$ of $(ker\Psi_{*q})^\perp$ as

$$\kappa^{(ker\Psi_*)^\perp} = \frac{2\tau^{(ker\Psi_*)^\perp}}{p(p-1)}. \tag{4.37}$$

Then, we can write

$$\psi_{ks}^\beta = g_{\mathcal{B}_2}((\nabla\Psi_*)(E_k,E_s),E_\beta),\ k,s = 1,...,p,\ \beta = p+1,...,b_2, \tag{4.38}$$

$$\|\psi\|^2 = \Sigma_{k,s=1}^p g_{\mathcal{B}_2}((\nabla\Psi_*)(E_k,E_s),(\nabla\Psi_*)(E_k,E_s)) \tag{4.39}$$

$$trace\psi = \Sigma_{k=1}^p(\nabla\Psi_*)(E_k,E_k),\ \|trace\psi\|^2 = g_{\mathcal{B}_2}(trace\psi, trace\psi). \tag{4.40}$$

The squared norm of ψ, the second fundamental form of the horizontal space $(ker\Psi_*)^\perp$ over the manifold $(M_2^{b_2}, J_2, g_{M_2})$, is denoted by $\mathcal{C}$ and is called the Casorati curvature of the horizontal space $(ker\Psi_*)^\perp$. Thus, we obtain

$$\mathcal{C} = \frac{1}{p}\|\psi\|^2 = \frac{1}{p}\Sigma_{\beta=p+1}^{b_2}\Sigma_{k,s=1}^p(\psi_{ks}^\beta)^2. \tag{4.41}$$

Now, assume that $\mathrm{L}^{(ker\Psi_*)^\perp}$ is a $t-$dimensional subspace $(ker\Psi_*)_q^\perp$, $2 \leq t$ and let $\{E_1, E_2, ..., E_t\}$ be an orthonormal basis of $\mathrm{L}^{(ker\Psi_*)^\perp}$. Then the Casorati curvature $\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}$ of $\mathrm{L}^{(ker\Psi_*)^\perp}$ defined as

$$\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) = \frac{1}{t}\|T\|^2 = \frac{1}{t}\Sigma_{\beta=p+1}^{b_2}\Sigma_{k,s=1}^t(T_{ks}^\beta)^2.$$

The normalized $\sigma^{(ker\Psi_*)^\perp}-$ Casorati curvatures $\sigma_{\mathcal{C}}^{(ker\Psi_*)\perp}(p-1)$ and $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1)$ of $(ker\Psi_*)_q)^\perp$ are given by

$[\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1)]_q = \frac{1}{2}\mathcal{C}_q^{(ker\Psi_*)^\perp} + \frac{p+1}{2p} inf\{\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) : \mathrm{L}^{(ker\Psi_*)^\perp}$ a hyperplane of $(ker\Psi_*)_q)^\perp\}$, and

$[\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1)]_q = 2\mathcal{C}_q^{(ker\Psi_*)^\perp} - \frac{2p-1}{2p} inf\{\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) : \mathrm{L}^{(ker\Psi_*)^\perp}$ a hyperplane of $(ker\Psi_*)_q^\perp\}$.

Using (4.35), (4.36) and (4.41) we arrive at

$$\frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi = 2\tau^{(ker\Psi_*)^\perp}(q) + p\mathcal{C}^{(ker\Psi_*)^\perp} - \|trace\psi\|^2, \tag{4.42}$$

here $\tau^{(ker\Psi_*)^\perp}$ is the scalar curvature of $(ker\Psi_*)^\perp$.

Now we define a function $\mathcal{Q}^{(ker\Psi_*)^\perp}$ associated with the following quadratic polynomial with respect to the components of ψ :

$$\begin{aligned}\mathcal{Q}^{(ker\Psi_*)^\perp} &= \frac{1}{2}[(p^2-p)\mathcal{C}^{(ker\Psi_*)^\perp} + (p^2-1)\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ &\quad - 2\tau^{(ker\Psi_*)^\perp} + \frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi.\end{aligned}$$

Without loos of generality, by supposing that the hyperplane $\mathrm{L}^{(ker\Psi_*)^\perp}$ is spanned by $\{E_1, ..., E_{p-1}\}$, using (4.42) one can produce

$$\begin{aligned}\mathcal{Q}^{(ker\Psi_*)^\perp} &= \Sigma_{\beta=p+1}^{b_2}\Sigma_{k=1}^{p-1}[p(\psi_{kk}^\beta)^2 + (p+1)(\psi_{kp}^\beta)^2] \\ &\quad + \Sigma_{\beta=p+1}^{b_2}[2(p+1)\Sigma_{1=k<s}^{p-1}(\psi_{ks}^\beta)^2 \\ &\quad - 2\Sigma_{1=k<s}^{p}\psi_{kk}^\beta\psi_{ss}^\beta + \frac{p-1}{2}(\psi_{pp}^\beta)^2] \\ &\geq \Sigma_{\beta=p+1}^{b_2}[\Sigma_{k=1}^{p-1}p(\psi_{kk}^\beta)^2 + \frac{p-1}{2}(\psi_{pp}^\beta)^2 \\ &\quad - 2\Sigma_{1=k<s}^{p}\psi_{kk}^\beta\psi_{ss}^\beta].\end{aligned} \tag{4.43}$$

For $\beta = p+1, ..., b_2$, let us consider the quadratic form $g_\beta : R^{b_2} \to R$ defined by

$$g_\beta(\psi_{11}^\beta, ..., \psi_{pp}^\beta) = \Sigma_{k=1}^{p-1}p(\psi_{kk}^\beta)^2 + \frac{p-1}{2}(\psi_{pp}^\beta)^2 - 2\Sigma_{k<s=1}^{p}\psi_{kk}^\beta\psi_{ss}^\beta, \tag{4.44}$$

and the constrained extremum problem, $ming_\beta$, subject to

$$\Phi^\beta : \psi_{11}^\beta + ... + \psi_{pp}^\beta = z^\beta,$$

here z^β is a real constant. From Lemma 4.1, we obtain $c = p,\ \ d = \frac{p-1}{2}$.

Thus, by Lemma 4.1 we get the critical point $(\psi_{11}^\beta, ..., \psi_{pp}^\beta)$, given by

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1\,p-1}^\beta = \frac{z^\beta}{p+1},\ \psi_{pp}^\beta = \frac{2z^\beta}{p+1},$$

is a global minimum point. Also, $g_\beta(\psi_{11}^\beta, ..., \psi_{pp}^\beta) = 0$. Moreover we obtain

$$\mathcal{Q}^{(ker\Psi_*)^\perp} \geq 0, \tag{4.45}$$

which implies

$$\begin{aligned} 2\tau^{(ker\Psi_*)^\perp} &\leq \frac{1}{2}[(p^2-p)\mathcal{C}^{(ker\Psi_*)^\perp} + (p^2-1)\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ &+ \frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi \end{aligned} \tag{4.46}$$

and using (4.46) we obtain

$$\begin{aligned} \kappa^{(ker\Psi_*)^\perp} &\leq [\frac{1}{2}\mathcal{C}^{(ker\Psi_*)^\perp} + \frac{p+1}{2p}\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ &+ \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi \end{aligned} \tag{4.47}$$

for all hyperplane $\mathrm{L}^{(ker\Psi_*)^\perp}$ of $(ker\Psi_*)^\perp$.

Similarly, we can write

$$\begin{aligned} \mathcal{Z}^{(ker\Psi_*)^\perp} = 2(p^2-p)\mathcal{C}^{(ker\Psi_*)^\perp} - \frac{1}{2}(2p^2-3p+1)\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp}) \\ - 2\tau^{(ker\Psi_*)^\perp} + \frac{\nu}{4}(p^2-p) + \frac{3n\nu}{2}\cos^2\phi, \end{aligned}$$

here hyperplane $\mathrm{L}^{(ker\Psi_*)^\perp}$ is a hyperplane of $(ker\Psi_*)^\perp$. From here,

$$\mathcal{Z}^{(ker\Psi_*)^\perp} \geq 0, \tag{4.48}$$

which implies

$$\begin{aligned} \kappa^{(ker\Psi_*)^\perp} \leq 2\mathcal{C}^{(ker\Psi_*)^\perp} - \frac{2p-1}{2p}\mathcal{C}^{(ker\Psi_*)^\perp}(\mathrm{L}^{(ker\Psi_*)^\perp})] \\ + \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi. \end{aligned} \tag{4.49}$$

Now, taking the infimum in (4.47) and the supremum in (4.49) over all hyperplanes $\mathrm{L}^{(ker\Psi_*)^\perp}$ of $(ker\Psi_*)^\perp$ and analyzing the equality case in (4.45) and (4.48), respectively, we get:

Theorem 4.1. *Let Ψ be a pointwise hemi-slant Riemannian map a Riemannian manifold $(M_1^{b_1}, g_{M_1})$ to a complex space form $(M_2^{b_2}(\nu), J_2, g_{M_2})$ with hemi-slant function ϕ, $3 \leq p = rank\Psi < min\{b_1, b_2\}$. Then the normalized $\sigma-$ Casorati curvatures $\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ and $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ on $(ker\Psi_*)_q^\perp$ satisfy*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi, \tag{4.50}$$

$$(ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4} + \frac{3n\nu}{2p(p-1)}\cos^2\phi. \tag{4.51}$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ *if and only if with respect to suitable orthonormal basis* $\{E_1, ..., E_p\}$ *on* $(ker\Psi_*)_q^\perp$ *and* $\{E_{p+1}, ..., E_{b_2}\}$ *on* $((range\Psi_*)_q)^\perp$*, the components of* ψ *satisfy*

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1p-1}^\beta = \frac{1}{2}\psi_{pp}^\beta,\ \ \beta \in \{p+1, p+2, ..., b_2\},$$

$$\psi_{ks}^\beta = 0,\ \ k, s \in \{1, , ..., p\}(k \neq s),\ \ \beta \in \{p+1, p+2, ..., b_2\}.$$

Using the Theorem 4.1, we obtain the following results.

Corollary 4.1. *Let* Ψ *be a pointwise hemi-slant Riemannian map a Riemannian manifold* $(M_1^{b_1}, g_{M_1})$ *to a complex space form* $(M_2^{b_2}(\nu), J_2, g_{M_2})$ *with hemi-slant function* $\phi = \frac{\pi}{2}$, $3 \leq p = rank\Psi < min\{b_1, b_2\}$. *Then the normalized* $\sigma-$ *Casorati curvatures* $\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *and* $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *on* $(ker\Psi_*)_q^\perp$ *satisfy*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4}$$

$$(ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{4}.$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ *if and only if with respect to suitable orthonormal basis* $\{E_1, ..., E_p\}$ *on* $(ker\Psi_*)_q^\perp$ *and* $\{E_{p+1}, ..., E_{b_2}\}$ *on* $((range\Psi_*)_q)^\perp$*, the components of* ψ *satisfy*

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1p-1}^\beta = \frac{1}{2}\psi_{pp}^\beta,\ \ \beta \in \{p+1, p+2, ..., b_2\},$$

$$\psi_{ks}^\beta = 0,\ \ k, s \in \{1, , ..., p\}(k \neq s),\ \ \beta \in \{p+1, p+2, ..., b_2\}.$$

Corollary 4.2. *Let* Ψ *be a pointwise hemi-slant Riemannian map a Riemannian manifold* $(M_1^{b_1}, g_{M_1})$ *to a complex space form* $(M_2^{b_2}(\nu), J_2, g_{M_2})$ *with hemi-slant function* $\phi = 0$, $3 \leq p = rank\Psi < min\{b_1, b_2\}$. *Then the normalized* $\sigma-$ *Casorati curvatures* $\sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *and* $\bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}$ *on* $(ker\Psi_*)_q^\perp$ *satisfy*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{2}(\frac{1}{2} + \frac{3n}{p(p-1)})$$

$$(ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1) + \frac{\nu}{2}(\frac{1}{2} + \frac{3n}{p(p-1)}).$$

Furthermore, the equality case holds in any inequalities at a point $q \in M_1$ *if and only if with respect to suitable orthonormal basis* $\{E_1, ..., E_p\}$ *on* $(ker\Psi_*)_q^\perp$ *and* $\{E_{p+1}, ..., E_{b_2}\}$ *on* $((range\Psi_*)_q)^\perp$*, the components of* ψ *satisfy*

$$\psi_{11}^\beta = \psi_{22}^\beta = ... = \psi_{p-1p-1}^\beta = \frac{1}{2}\psi_{pp}^\beta, \ \ \beta \in \{p+1, p+2, ..., b_2\},$$

$$\psi_{ks}^\beta = 0, \ \ k, s \in \{1, , ..., p\}(k \neq s), \ \ \beta \in \{p+1, p+2, ..., b_2\}.$$

Corollary 4.3. *Let* Ψ *be a pointwise hemi-slant Riemannian map a Riemannian manifold* $(M_1^{b_1}, g_{M_1})$ *to the complex Euclidean space* $\mathbb{C}^{\frac{b_2}{2}}$ *with hemi-slant function* ϕ, $3 \leq p = rank\Psi < min\{b_1, b_2\}$. *Then we get*

$$(i)\ \kappa^{(ker\Psi_*)^\perp} \leq \sigma_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1), \ (ii)\ \kappa^{(ker\Psi_*)^\perp} \leq \bar{\sigma}_{\mathcal{C}}^{(ker\Psi_*)^\perp}(p-1).$$

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Dicle University, Faculty of Science, Department of Mathematics, 21280, Diyarbakir, Türkiye

Bingol University, Faculty of Arts and Sciences, Department of Mathematics, 12000, Bingöl, Türkiye

Ege University, Department of Mathematics, 35100, İzmir, Türkiye