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GENERALIZED SOLITONIC CHARACTERISTICS IN TRANS PARA SASAKIAN MANIFOLDS

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In the current research, we quantify the almost generalized Ricci soliton (AGRS) on the trans-para-Sasakian manifold (TPS-manifold) as well as the gradient almost generalized Ricci soliton (GAGRS). Trans-para Sasakian manifolds that meet certain criteria are also required to be Einstein manifolds. It is demonstrated that the almost generalized Ricci soliton equation is also satisfied by some manifolds, notably β-para-Kenmotsu manifolds, α-para-Sasakian. The fact that a compact trans-para-Sasakian admits both a convex Einstein potential with non-negative scalar curvature and a gradient almost generalized Ricci soliton with Hodge-de Rham potential has also been covered. Finally, we furnished an example which illustrates our finding.

Keywords: Almost generalized Ricci solitons, gradient almost generalized Ricci soliton, Trans-para Sasakian manifold, Einstein manifold.

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1. Introduction

The most important geometrical tool to explain the geometric structures in Riemannian geometry (semi-Riemannian) over the last two decades has been the theory of geometric flows. Since they arise as potential models of discontinuities, the study of discontinuities

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(singularities) of the flows involves a special class of solutions where the metric changes via dilations and diffeomorphisms. They are often called soliton solutions. In 1982, R. S. Hamilton [10] developed the idea of Ricci flow such that

$$\frac{\partial g}{\partial t} = -2S_{ric}(g). \tag{1.1}$$

On a Riemannian manifold (M, g), a Ricci soliton struture (g, V, λ) can be expressed by

$$S_{ric} + \frac{1}{2}\mathfrak{L}_\theta g + \Lambda g = 0, \tag{1.2}$$

here $\mathfrak{L}_\theta$ is the Lie derivative along the vector field θ, Λ is a scalar, and S_{ric} is the Ricci tensor. Ricci soliton is defined as $\Lambda < 0, \Lambda = 0$, and $\Lambda > 0$, respectively. It can also be described as expanding, stable, or shrinking.

Equation (1.2) takes on the form of a gradient Ricci soliton if the vector field $\theta = grad(\psi)$, where ψ is potential function on manifold.

$$Hess\psi = S_{ric} + \Lambda g. \tag{1.3}$$

Pigola et al. [21] argue that if we consider $\Lambda \in C^\infty(M)$, sometimes referred to as a soliton function, so we could assert that (M, g) is *almost generalized Ricci solitons* (AGRS).

Plenty of mathematicians are drawn to this idea. Therefore, how self-similar solutions are categorized to Ricci flows has received a lot of attention in recent years.This problem has significant practical implications in fields such as thermodynamics, control theory, optics, mechanics, phase space of dynamical systems, and many other departments of pure mathematics.

Ricci solitons are significant because they are both logical generalizations of Einstein metrics. A few generalizations, for example quasi-Einstein manifolds [4], generalized quasi-Einstein manifolds [5] and gradient Ricci solitons [3], are crucial in the solutions of some manifolds have their local structure derived from Ricci flows.

Overarching in reference [19], Nurowski and Randall initially defined Ricci soliton as a kind of over determined framework for equations.

$$\frac{1}{2}\mathfrak{L}_\theta g - bS_{ric} - \Lambda g + a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp = 0, \tag{1.4}$$

where $\mathcal{U}^\sharp$ denotes the canonical 1-from and a, b are real constants .

If $\mathcal{U} = \nabla\psi$, where $\psi \in C^\infty(M)$, (M, g) is referred to as a gradient almost generalized Ricci soliton (GAGRS) in that case. As a result, (1.4) becomes

$$\nabla^2\psi - 2bS_{ric} - 2\Lambda g + 2a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp = 0. \tag{1.5}$$

However, Kaneyuki and Konzai started researching an almost para-contact structure on semi-Riemannian manifolds [12]. Zamkovoy has done extensive research on para contact metric manifolds [35]. Furthermore, trans-para-Sasakian manifold geometry was given by Zamkovoy in 2019 [37]. Siddiqi also has investigated lightlike hypersurfaces [27] and null hypersurfaces of trans-para-Sasakian manifold [26].

Structures that are an almost contact manifold M are known as trans-Sasakian structures [20], if $M \times \mathbb{R}$, the product manifolds, are members of class W_4 [9]. Marrero and Chinea are fully characterized trans-Sasakian structures of type (α, β) in [16].

The trans-para-Sasakian manifolds are seen by Zamkovoy in [37] as an analogy of the trans-Sasakian manifolds. A trans-para-Sasakian structure of type (α, β), where α and β are smooth functions, is called a trans-para-Sasakian manifold [28]. The manifolds of type (α, β) that are trans-para-Sasakian are the para-Sasakian manifolds in the case of $\alpha = 1$, the para-Kenmostu manifolds in the case of $\beta = 1$ [37], and the para-cosympletic manifolds $(\alpha = \beta = 0)$ [13].

During last two decades, many geometers exclusively studied the Ricci solitons and an extension [24] of Ricci solitons namely, η-Ricci solitons on different manifolds such as Riemannian manifold [22], Kenmotsu manifold [18], K-contact manifolds and (k, μ)-contact manifolds [29] and trans-Sasakian manifolds [31]. Following Siddiqi [25], who also discussed generalized Ricci soliton. Mekki and Cherif studied another generic concept known as generalized Ricci soliton on Sasakian manifolds [17]. In this research note, we studied the almost generalized Ricci soliton and almost gradient generalized Ricci soliton in trans-para-Sasakian manifolds as a result of the aforementioned sources and comments.

2. Preliminaries

If a $(2n+1)$-dimensional smooth manifold Θ admits a vector field ζ, a 1-form γ, and a tensor field Φ of type $(1,1)$, and a pseudo-Riemannian metric g then it has an almost paracontact structure (Φ, ζ, γ, g) such that [2]

$$\Phi^2 p = p - \gamma(p)\zeta, \quad \Phi(\zeta) = 0, \quad \gamma \circ \Phi = 0, \quad \gamma(\zeta) = 1. \tag{2.6}$$

The definition of almost paracontact structure immediately leads to the rank $2n$ of the endomorphism Φ.

$$g(\Phi p, \Phi q) = -g(p, q) + \gamma(p)\gamma(q), \tag{2.7}$$

then g is said to be compatible with signature $(n+1, n)$ and Θ has an almost paracontact metric structure.

Observe that when $q = \zeta$ is set, $\gamma(p) = g(p, \zeta)$. Moreover, a compatible metric admits any almost paracontact structure. If

$$g(p, \Phi q) = d\gamma(p, q),$$

where $d\gamma(p, q) = \frac{1}{2}(p\gamma(q) - q\gamma(p) - \gamma([p, q]))$, then γ is a paracontact form and the almost paracontact metric manifold $(\Theta, \Phi, \gamma, \zeta, g)$ is defined as a paracontact metric manifold.

An almost paracomplex structure on the product $\Theta^{(2n+1)} \times \mathbb{R}$ easily arises from a paracontact structure on a $\Theta^{(2n+1)}$. The provided paracontact metric manifold is called para-Sasakian if this almost paracomplex structure is integrable. Comparably, a paracontact metric manifold is a para-Sasakian if and only if (see [36]).

$$(\nabla_p \Phi)q = -g(p, q)\zeta + \gamma(q)p, \tag{2.8}$$

the manifold $(\Theta, \Phi, \zeta, \gamma, g)$ of dimension $(2n+1)$ is said to be trans-para-Sasakian manifolds (TPS-manifolds) if and only if

$$(\nabla_p \Phi)Y = \alpha(-g(p, q)\zeta + \gamma(q)p) + \beta(g(p, \Phi q)\zeta + \gamma(q)\Phi p), \tag{2.9}$$

from (2.9), we also have

$$\nabla_p \zeta = -\alpha \Phi p - \beta(p - \gamma(p)\zeta). \tag{2.10}$$

The gradient of a smooth function ψ on Θ is defined as follows

$$g(grad\psi, p) = p(\psi). \tag{2.11}$$

The definition of ψ's *Hessian* is

$$(Hess\Psi)(p, q) = g(\nabla_p grad\Psi, q), \tag{2.12}$$

where $p, q \in \Gamma(T\Theta)$.

We defined $p \in \Gamma(T\Theta)$. $\mathcal{U}^\sharp \in \Gamma(\bar{T}\Theta)$ by

$$\mathcal{U}^\sharp(q) = g(p, q). \tag{2.13}$$

The AGRS equation in Riemannian manifold Θ is given by [19]

$$\mathfrak{L}_\theta g = -2a\mathcal{U}^\sharp \odot \mathcal{U}^\sharp + 2bS_{ric} + 2\Lambda g, \tag{2.14}$$

where $p \in \Gamma(T\Theta)$ and the Lie-derivative is defined as

$$(\mathfrak{L}_\theta g)(q, t) = g(\nabla_q \theta, t) + g(\nabla_t \theta, q) \tag{2.15}$$

where $q, t \in \Gamma(T\Theta)$. Equation (1.4), furthermore, is refers to an expansion of

(1) If $a = b = \Lambda = 0$, then Killing's equation.
(2) If $a = b = 0$, then equation for homotheties.
(3) If $a = 0, b = -1$,then Ricci soliton.
(4) If $a = 1, b = \frac{-1}{n-2}$, then Einstein-Weyl geometry.
(5) If $a = 1, b = \frac{-1}{n-2}, \lambda = 0$, then we have metric projective structures with skew-symmetric Ricci tensor in projective class.
(6) If $a = 1, b = \frac{1}{2}$, then we have Vacuum near-horzion geometry equation (for more details see [7], [8], [11], [14]).

A generalization of Einstein manifolds [5] is given by equation (1.4). Observe that the gradient AGRS equation is provided by: if $p = grad\psi$, where $\psi, \Lambda \in C^\infty(\Theta)$

$$Hess\psi + adf \odot df = bS_{ric} + \Lambda g. \tag{2.16}$$

3. Gradient almost generalized Ricci soliton on trans para Sasakian manifolds

The following relations hold in a $(2n+1)$-dimensional TPS manifold Θ [37]:

$$\Re(p, q)\zeta = -(\alpha^2 + \beta^2)[\gamma(q)p - \gamma(p)q] - 2\alpha\beta[\gamma(q)\Phi p - \gamma(p)\Phi q] \tag{3.17}$$
$$+[(q\alpha)\Phi p - (p\alpha)\Phi q + (q\beta)\Phi^2 p - (p\beta)\Phi^2 q].$$

$$S_{ric}(p, \zeta) = [(-2n(\alpha^2 + \beta^2) - (\zeta\beta)]\gamma(p) + ((\Phi p)\alpha) + (n-2)(p\beta), \tag{3.18}$$

$$Q\zeta = -2n(\alpha^2 + \beta^2) - (\zeta\beta))\zeta + \Phi(grad\alpha) - (n-2)(grad\beta), \tag{3.19}$$

where Q is the Ricci operator provided by $S_{ric}(p,q) = g(Qp,q)$, and $\Re$ is the curvature tensor.

Furthermore, we have a TPS manifold

$$\Phi(grad\alpha) = -(2n-1)(grad\beta), \tag{3.20}$$

$$2\alpha\beta - (\zeta\alpha) = 0. \tag{3.21}$$

Lemma [15] follows from combining (3.17) and (3.21) for constants α and β.

Lemma 3.1. [15] *Let* $(\Theta^{(2n+1)}, \Phi, \gamma, \zeta, g)$ *be a* TPS*-manifold. Then we have*

$$\Re(p,q)\zeta = -(\alpha^2+\beta^2)[\gamma(q)p - \gamma(p)q], \tag{3.22}$$

$$\Re(\zeta,q)t = -(\alpha^2+\beta^2)[g(q,t)\zeta - \gamma(t)q], \tag{3.23}$$

$$S_{ric}(p,\zeta) = -2n(\alpha^2+\beta^2)\gamma(p), \tag{3.24}$$

$$(\nabla_p\gamma)q = \alpha g(p,\Phi q) - \beta(g(p,q) - \gamma(p)\gamma(q)), \tag{3.25}$$

$$Q\zeta = -[2n(\alpha^2+\beta^2)]\zeta, \tag{3.26}$$

where for all $p, q, t \in T(\Theta)$.

Example 3.1. *Let* (x,y,z) *be the Cartesian coordinates in* $\mathbb{R}^3$*. Assume a 3-dimensional manifold* $\Theta = \{(x,y,z) \in \mathbb{R}^3 | z \neq 0\}$*. Let the linearly independent vector fields* $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ *are linearly independent at each point of* Θ *defined as*

$$\mathcal{E}_1 = e^z(\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}), \quad \mathcal{E}_2 = e^z\frac{\partial}{\partial y}, \quad \mathcal{E}_3 = \frac{\partial}{\partial z}.$$

Let g be the pseudo-Riemannian metric defined by

$$g(\mathcal{E}_1,\mathcal{E}_1) = -g(\mathcal{E}_2,\mathcal{E}_2) = g(\mathcal{E}_3,\mathcal{E}_3) = 1, \quad g(\mathcal{E}_1,\mathcal{E}_2) = g(\mathcal{E}_2,\mathcal{E}_3) = g(\mathcal{E}_3,\mathcal{E}_1) = 0.$$

Moreover, the 1-form γ is given by $\zeta = \mathcal{E}_3$ and $\gamma(p) = g(p,\mathcal{E}_3)$. Let Φ be the (1,1) tensor field defined by

$$\Phi(\mathcal{E}_1) = \mathcal{E}_2,\ \Phi(\mathcal{E}_2) = \mathcal{E}_1,\ \Phi(\mathcal{E}_3) = 0,$$

for any vector field p on Θ. Using the linearity of Φ and g, we then obtain $\gamma(\mathcal{E}_3) = 1$, $\phi^2 p = p - \gamma(p)\zeta$, with $\zeta = \mathcal{E}_3$.

Moreover, for all vector fields p and q on Θ, we have

$$g(\Phi p, \Phi q) = -g(p,q) + \gamma(p)\gamma(q).$$

Therefore, in $\mathbb{R}^3$, the structure (Φ, ζ, γ, g) defines a paracontact structure for $\mathcal{E}_3 = \zeta$ [36]. Let $\Re$ be the curvature tensor of g and ∇ be the Levi-Civita connection with respect to metric g. Next, we have

$$[\mathcal{E}_1, \mathcal{E}_2] = ye^z\mathcal{E}_2 - e^{2z}\mathcal{E}_3[\mathcal{E}_1, \mathcal{E}_3] = -\mathcal{E}_3 \qquad [\mathcal{E}_2, \mathcal{E}_3] = -\mathcal{E}_2.$$

Now, we have Koszul's formula

$$2g(\nabla_p q, t) = pg(q,t) + qg(t,p) - tg(p,q) - g(p,[q,t])$$

$$-g(q,[p,t]) + g(t,[p,q]).$$

Therefore, in light of above formula, we turn up

$$\nabla_{\mathcal{E}_1}\mathcal{E}_1 = \mathcal{E}_3, \quad \nabla_{\mathcal{E}_1}\mathcal{E}_2 = -\frac{1}{2}e^{2z}\mathcal{E}_3, \quad \nabla_{\mathcal{E}_1}\mathcal{E}_3 = -\mathcal{E}_1 - \frac{1}{2}e^{2z}\mathcal{E}_2, \tag{3.27}$$

$$\nabla_{\mathcal{E}_2}\mathcal{E}_2 = -ye^z\mathcal{E}_1 - \mathcal{E}_3, \quad \nabla_{\mathcal{E}_2}\mathcal{E}_1 = -ye^z\mathcal{E}_2 + \frac{1}{2}e^{2z}\mathcal{E}_3,$$

$$\nabla_{\mathcal{E}_3}\mathcal{E}_1 = -\frac{1}{2}e^{2z}\mathcal{E}_2, \quad \nabla_{\mathcal{E}_3}\mathcal{E}_2 = -\frac{1}{2}e^{2z}\mathcal{E}_1, \quad \nabla_{\mathcal{E}_3}\mathcal{E}_3 = 0.$$

The fact that (Φ, ζ, γ, g) is a TPS-structure on Θ is evident from the above. Thus, $\Theta^3(\Phi, \zeta, \gamma, g)$, with $\beta = 1$ and $\alpha = \frac{1}{2}e^{2z} \neq 0$, is a TPS- manifold.

Theorem 3.1. *If $\Theta^{(2n+1)}$ be a TPS-manifolds, and satisfies the $AGRS$ (1.4) with restriction $a[\lambda - 2nb(\alpha^2 + \beta^2)] \neq -1$. Then ψ is a constant function. In addition, if $b \neq 0$, then Θ is an Einstein.*

Lemma 3.1 gives us the following observations:

Corollary 3.1. *If $\Theta^{(2n+1)}$ be a TPS-manifolds, and satisfies the $AGRS$ $Hess\psi + S_{ric} = \Lambda g$, then ψ is a constant function and Θ is an Einstein.*

Corollary 3.2. *In a TPS-manifolds Θ, there is no non-constant smooth function ψ, such that $Hess\psi = \Lambda g$, for some constant Λ.*

We must first show the following lemmas in order to proceed with the proof of the Theorem (3.1).

Lemma 3.2. *Let Θ be a TPS-manifold. Then we have*

$$(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) = -(\alpha^2+\beta^2)g(p,q) + g(\nabla_\zeta\nabla_\zeta p, q) + qg(\nabla_\zeta p,\zeta), \tag{3.28}$$

where $p,q \in \Gamma(T\Theta)$ and q is orthogonal to ζ.

Proof. Based on the Lie-derivative property, we may observe that

$$(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) = \zeta((\mathfrak{L}_p g)(q,\zeta)) - (\mathfrak{L}_p g)(\mathfrak{L}_\zeta q,\zeta) - (\mathfrak{L}_p g)(q,\mathfrak{L}_\zeta\zeta). \tag{3.29}$$

Since $\mathfrak{L}_\zeta q = [\zeta,q]$, $\mathfrak{L}_\zeta\zeta = [\zeta,\zeta]$, by adopting (2.16) and (4.51), we have

$$\begin{aligned}(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) &= \zeta g(\nabla_q p,\zeta) + \zeta g(\nabla_\zeta p, q) - g(\nabla_{[\zeta,q]}p,\zeta) \\ &\quad -g(\nabla_\zeta p,[\zeta,q]) \\ &= g(\nabla_\zeta\nabla_q p,\zeta) + g(\nabla_q p,\nabla_\zeta\zeta) + g(\nabla_\zeta\nabla_\zeta p,q) \\ &\quad +g(\nabla_\zeta p,\nabla_\zeta q) - g(\nabla_\zeta p,\nabla_\zeta q) - g(\nabla_{[\zeta,q]}p,\zeta) + g(\nabla_\zeta p,\nabla_q\zeta).\end{aligned} \tag{3.30}$$

By (1.4), we turn up $\nabla_\zeta\zeta = \Phi\zeta = 0$, therefore we gain

$$\begin{aligned}(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) &= g(\nabla_\zeta\nabla_q p,\zeta) + g(\nabla_\zeta\nabla_\zeta p,q) - g(\nabla_{[\zeta,q]}p,\zeta) \\ &\quad +qg(\nabla_\zeta p,\zeta) - g(\nabla_q\nabla_\zeta p,\zeta).\end{aligned} \tag{3.31}$$

Utilizing (4.51) and (3.29), we turn up

$$(\mathfrak{L}_\zeta(\mathfrak{L}_p g))(q,\zeta) = g(\Re(\zeta,q)p,\zeta) + g(\nabla_\zeta\nabla_\zeta p,q) + qg(\nabla_\zeta p,\zeta). \tag{3.32}$$

When $g(q,\zeta)=0$ is taken from (4.51), we find

$$g(\Re(\zeta,q)p,\zeta) = g(R(q,\zeta)\zeta,p) = (\alpha^2+\beta^2)g(p,q). \tag{3.33}$$

(3.29) and (4.52) provide the Lemma. □

We now have another helpful Lemma.

Lemma 3.3. *If Θ be a Riemannian manifold, and let $\psi \in C^\infty(\Theta)$. Then we have*

$$(\mathfrak{L}_\zeta(df\odot df))(q,\zeta) = q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\xi(\psi)), \tag{3.34}$$

where $\zeta,q \in \Gamma(T\Theta)$.

Proof. We compute

$$(\mathfrak{L}_\zeta(df \odot df))(q,\zeta) = \zeta(q(\psi)\zeta(\psi) - [\zeta,q](\psi)\zeta(\psi) - q(\psi)[\zeta,\zeta](\psi)$$

$$= \zeta(q(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi)) - [\zeta,q](\psi)\zeta(\psi).$$

Since $[\zeta,q](\psi) = \zeta(q(\psi)) - q(\zeta(\psi))$, we gain

$$(\mathfrak{L}_\zeta(df \odot df))(q,\zeta) = [\zeta,q](\psi)\zeta(\psi) + q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi)) - [\zeta,q](\psi)\zeta(\psi)$$

$$= q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi)).$$

□

Lemma 3.4. *If Θ^{2n+1} be a TPS-manifold and satisfies the AGRS equation (2.16). Then we have*

$$\nabla_\zeta\ grad\psi = [\Lambda - 2nb(\alpha^2+\beta^2)]\zeta - a\zeta(\psi)grad\psi. \tag{3.35}$$

Proof. Let $q \in \Gamma(T\Theta)$, adopting the definition of Ricci curvature S_{ric} (1.4) , and the curvature restriction (4.51), we gain

$$\begin{aligned} S_{ric}(p,q) &= g(\Re(\zeta,\mathcal{E}_i)\mathcal{E}_i,q) \\ &= g(\Re(\mathcal{E}_i,q)\xi,\mathcal{E}_i) \\ &= -(\alpha^2+\beta^2)[\gamma(q)g(\mathcal{E}_i,\mathcal{E}_i) - \gamma(\mathcal{E}_i)g(p,\mathcal{E}_i) \\ &= (\alpha^2+\beta^2)[(2n+1)\gamma(q) - \gamma(q)] \\ &= -2n(\alpha^2+\beta^2)\gamma(q) \\ &= -2n(\alpha^2+\beta^2)g(\zeta,q), \end{aligned}$$

where $\{\mathcal{E}_1,\mathcal{E}_2,\cdots,\mathcal{E}_i\}$, and $1 \le i \le n$ is an orthonormal frame of Θ, indicates that

$$\Lambda g(\zeta,q) + bS_{ric}(\zeta,q) = \Lambda g(\zeta,q) - 2nb(\alpha^2+\beta^2)g(\zeta,q) \tag{3.36}$$

$$= [\Lambda - 2nb(\alpha^2+\beta^2)]g(\zeta,q).$$

In light of (1.4) and (3.35), we turn up

$$(Hess\psi)(\zeta,q) = -a\zeta(\psi)(q)(\psi) + [\Lambda - 2nb(\alpha^2+\beta^2)]g(\zeta,q) \tag{3.37}$$

$$= -a\zeta(\psi)g(grad\psi,q) + [\Lambda - 2nb(\alpha^2+\beta^2)]g(\zeta,q).$$

Accordingly, Lemma is inferred from both equation (3.35) and *Hessian* Definition (1.5). □

We can now establish Theorem 3.1 with the aid of Lemma 3.2, Lemma 3.3, and Lemma 3.4.

Proof. (Proof of Theorem 3.1) Consider $q \in \Gamma(T\Theta)$, such that $g(\zeta, q) = 0$. Lemma 3.1 gives us that, given $X = grad\ \psi$,

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) + g(\nabla_\zeta\nabla_\zeta\ grad\psi, q) + qg(\nabla_\zeta\ grad\psi, \zeta). \tag{3.38}$$

Using equation (3.37) and Lemma 3.1, we obtain

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) + [\Lambda + b(n-1)(\alpha^2+\beta^2)]g(\nabla_\zeta, q) - ag(\nabla_\zeta(\zeta(\psi)grad\ \psi), q)$$
$$+[\Lambda + b(n-1)(\alpha^2+\beta^2)]qg(\zeta,\zeta) - aq(\zeta(\psi)^2). \tag{3.39}$$

Since $\nabla_\zeta\zeta = 0$ and $g(\zeta,\zeta) = 1$, in view of equation (3.38), we get

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) - a\zeta(\zeta(\psi))q(\psi) - a\zeta(\psi)g(\nabla_\zeta\ grad\psi, q) \tag{3.40}$$
$$-2a\zeta(\psi)q(\zeta(\psi)).$$

Given $g(\xi, Y) = 0$ and Lemma 3.1 and equation (3.39), we have

$$2(\mathfrak{L}_\zeta(Hess\psi))(q,\zeta) = q(\psi) - a\zeta(\zeta(\psi))q(\psi) + a^2\zeta(\psi)^2q(\psi) \tag{3.41}$$
$$-2a\zeta(\psi)q(\zeta(\psi)).$$

Observe that $\mathfrak{L}_\zeta g = 0$, a Killing vector filed, follows from (1.4) and (1.5). This suggests that $\mathfrak{L}_\zeta S = 0$, which is what the Lie derivative to the GRS equation (2.16) delivers.

$$q(\psi) - a\zeta(\zeta(\psi))q(\psi) + a^2\zeta(\psi)^2q(\psi) - 2a\zeta(\psi)q(\zeta(\psi)) \tag{3.42}$$
$$= -2aq(\zeta(\psi))\zeta(\psi) - 2aq(\psi)\zeta(\zeta(\psi)),$$

is equivalent to

$$q(\psi)[1 + a\zeta(\zeta(\psi)) + a^2\zeta(\psi)^2] = 0. \tag{3.43}$$

Lemma 3.1 states that we have

$$a\zeta(\zeta(\psi)) = a\zeta g(\zeta, grad\ \psi) \tag{3.44}$$
$$= ag(\zeta, \nabla_\zeta\ grad\psi)$$
$$= a[\Lambda - 2nb(\alpha^2+\beta^2)] - a^2\zeta(\psi)^2.$$

In view of equations (3.42) and (3.43), we gain

$$q(\psi)[\Lambda - 2nb(\alpha^2+\beta^2)] = 0. \tag{3.45}$$

$$[\Lambda - 2nb(\alpha^2+\beta^2)] \neq -1,$$

which indicates that $grad\psi$ is parallel to ζ, and so $q(\psi) = 0$. Since $D = ker\gamma$ is not integrable anywhere, $grad\ \psi = 0$, indicating that ψ is a constant function. □

Now, the following scenarios exist for specific values of α and β:

Case 1.: For $\alpha = 0$ or $(\beta = 1)$, we can state:

Corollary 3.3. *If $\Theta^{(2n+1)}$ be a β-para Kenmotsu (or para Kenmotsu) manifold and satisfies the AGRS (1.5) with condition $a[\Lambda - 2nb\beta^2)] \neq -1$, then ψ is a constant function. In addition, if $b \neq 0$, then $\Theta^{(2n+1)}$ is Einstein .*

Case 2.: For $\beta = 0$, or $(\alpha = 1)$ we can state:

Corollary 3.4. *If $\Theta^{(2n+1)}$ be a α-para Sasakian (or para Sasakian) manifold and satisfies the AGRS (1.5) with condition $a[\Lambda - 2nb\alpha^2)] \neq -1$, then ψ is a constant function. Moreover, if $b \neq 0$, then $\Theta^{(2n+1)}$ is Einstein.*

4. Almost generalized Ricci solitons on compact trans para Sasakian manifolds

de Rham-Hodge's classical theorem states that harmonic forms can express the cohomology of an oriented closed Riemannian manifold. For an orientated compact Riemannian manifold with boundary, the analogous one still holds by imposing certain boundary requirements, including relative and absolute ones. However, these examples come from fully Riemannian manifolds. The following are some helpful definitions.

Definition 4.1. [33] *A C^2-function $\omega : \Theta \longrightarrow \mathbb{R}$ is considered to be harmonic if $\Delta\omega = 0$. The function ω is named subharmonic (resp. superharmonic) if $\Delta \geq 0$ (resp. $\Delta\omega \leq 0$), where Δ is the Laplacian operator in Θ.*

Definition 4.2. [35] *A function $\omega : \Theta \longrightarrow \mathbb{R}$ is called convex if the following inequality holds*

$$\omega \circ \delta(T) \leq (1 - T)\omega \circ \delta(0) + T\omega \circ \delta(1), \qquad \forall T \in [0, 1],$$

for any geodesic $\delta : [0, 1] \longrightarrow \Theta$. Therefore in this case ω is differentiable, then ω is convex if and only if ω satisfies

$$g(\nabla\omega, p) \leq \omega(e^x\nabla\omega) - \omega(x), \qquad \forall p \in T_x\Theta.$$

Let $p \in \chi(\Theta)$ and Θ be compact orientable TPS- manifolds. Then, according to the [1] *Hodge-de Rham* decomposition theorem, p can be stated as

$$p = \nabla \hbar + q, \tag{4.46}$$

where $\hbar \in C^{\infty}(\Theta)$ and $div(q) = 0$. The *Hodge-de Rham* potential is the name given to the function h [22].

Let (g, p, λ) be a compact AGRS on compact TPS-manifold Θ, we turn up

$$div(p) + 2n(2n+1)b(\alpha^2 + \beta^2) = n\Lambda - tr(a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp). \tag{4.47}$$

$div(X) = \Delta\hbar$ is implied by the Hodge-de Rham decomposition, so, using equation(4.47), we obtain

$$2n(2n+1)b(\alpha^2 + \beta^2) = n\Lambda - \Delta\hbar - tr(a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp). \tag{4.48}$$

Since Θ is $GAGRS$ with potential function, we obtain

$$2n(2n+1)b(\alpha^2 + \beta^2) = n\Lambda - \Delta f - tr(a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp). \tag{4.49}$$

Now, on equating (4.48) and (4.49), we turn

$$\Delta(f - \hbar) = 0.$$

Consequently, although Θ is compact, f is a harmonic function in Θ. Thus, for some constant c, $f = \hbar + c$. As so, we possess the following outcome.

Theorem 4.1. *If (g, p, Λ) is a compact $GAGRS$. If TPS- manifold Θ is also a $GAGRS$ with potential function f, then, up to constant, f equals to the Hodge-de Rham potential.*

Theorem 4.2. *Let $(\Theta, \zeta, \gamma, \Phi, g)$ be a complete TPS-manifold satisfying*

$$\frac{1}{2}\mathfrak{L}_p g - bS_{ric} \geq \Lambda g - a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp, \tag{4.50}$$

where $\mathcal{U}^\sharp$ is a canonical 1-from associated with p, a, b, and Λ are smooth functions, and p is a smooth vector field. If one of the following requirements is fulfilled and $\|p\|$ is bounded, then the TPS-manifold Θ is compact:

(1) $\Lambda \geq 0$ *and* $a > 0$, $c > 0$,

(2) $\Lambda > c > 0$ *and* $a \geq 0$,

for a constant $c > 0$.

Proof. If $\pi \in \Theta$ be a fixed point and $\delta : (0, \infty] \longrightarrow \Theta$ be a geodesic such that $\delta(0) = p$. Then along δ we compute

$$\mathfrak{L}_p g(\delta_1, \delta_1) = 2g(\nabla_{\delta_1} p, \delta_1) = 2\frac{d}{dt}[g(p, \delta_1)]. \tag{4.51}$$

Now, from (4.50) and (4.51), we have

$$-\int_0^T bS_{ric}(\delta_1, \delta_1)dt \geq \int_0^T \Lambda(\delta(t))g(\delta_1, \delta_1)dt - \int_0^T \frac{d}{dt}[g(p, \delta_1)dt - \int_0^t a(\delta(T))(\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp)(\delta_1, \delta_1)dt$$

$$= -\frac{1}{b}\left[\int_0^T \Lambda(\delta(t))dt + g(p_\pi, \delta_1(0)) - g(p_{\delta(T)}, \delta_1(T)) + \int_0^T a(\delta(T))\mathcal{U}^{\sharp^2}(\delta_1)dt\right]$$

$$\geq -\frac{1}{b}\left[\int_0^T \Lambda(\delta(t))dt + g(p_\pi, \delta_1(0)) - \|X_{\delta(T)}\| + \int_0^T a(\delta(T))\mathcal{U}^{\sharp^2}(\delta_1)dt\right].$$

Cauchy-Schwarz inequality leads to the final inequality. If either of the two conditions (1) or (2) is true, the inequality above suggests that

$$\int_0^\infty bS_{ric}(\delta_1, \delta_1)dt = \infty. \tag{4.52}$$

Hence by Ambrose's Compactness Theorem [1] implies that TPS-manifold Θ is compact. □

5. Gradient almost generalized Ricci soliton on compact trans para Sasakian manifolds

In this segment, we discuss some results based on gradient almost generalized Ricci solitonon compact trans-para Sasakian manifold $n \geq 2$. Next, we articulate the following.

Theorem 5.1. [32] *IF $(\Theta, \Phi, \gamma, \zeta, g)$ be a compact TPS-manifold with constant scalar curvature and Θ admits a non-trivial conformal vector field p. If $\mathfrak{L}_p S_{ric} = \rho g$ for some $\rho \in C^\infty(\Theta)$, then Θ is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Hence, from Theorem (5.1) we can also state the next theorem:

Theorem 5.2. *Let $(\Theta, \Phi, \zeta, \gamma, g)$ be a compact GAGRS with Einstein potential f. If ∇f is non-trivial conformal vector field, then TPS-manifold Θ is isometric to the Euclidean sphere $\mathbb{S}^n$.*

Proof. Let (Θ, g) be a $GAGRS$. Then from (1.4) we deduce

$$\nabla^2 f - bS_{ric} = \Lambda g - a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp.$$

For each $\psi \in C^\infty(\Theta)$, $\nabla^2 f - \psi g$, if ∇f is a conformal vector field. The equation above now takes the form

$$bS_{ric} = (\psi - \Lambda)g - a\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp. \tag{5.53}$$

As a result, S_{ric} is only dependent on Θ points. Schur's lemma thus implies that R is constant. Once more, using $p = \nabla f$, we get

$$a\mathfrak{L}_p S_{ric} = (\psi - \Lambda)\mathfrak{L}_p g - a\mathfrak{L}_p(\mathcal{U}^\sharp \otimes \mathcal{U}^\sharp) \tag{5.54}$$

$$a\mathfrak{L}_p S_{ric} = (\psi - \Lambda)\psi g - a[q(\zeta(\psi))\zeta(\psi) + q(\psi)\zeta(\zeta(\psi))]. \tag{5.55}$$

This completes the proof. □

In [32] Yano already proved a following results.

Theorem 5.3. [32] *A compact manifold Θ with constant scalar curvature admits a non-trivial conformal vector field p such that $\mathfrak{L}_p g = 2\psi g, \quad \psi \neq 0$, then*

$$\int_\Theta \psi dV = 0. \tag{5.56}$$

Therefore in light of Theorem 5.3 we can state.

Theorem 5.4. *Let $(\Theta, \Phi, \zeta, \gamma, g)$ be a compact $GAGRS$ with Einstein potential f and $(\alpha^2 + \beta^2) \leq 0$. If ∇f is conformal vector field then TPS manifold Θ is shrinking or steady $GAGRS$.*

Proof. Taking the trace in (5.53)

$$2n(2n+1)b(\alpha^2 + \beta^2) = (2n+1)(\psi - \Lambda) - a\,|\zeta|^2 \tag{5.57}$$

which implies

$$\int_\Theta 2nb(\alpha^2 + \beta^2) + \frac{a}{(2n+1)}\,|\zeta|^2 = \int_\Theta (\psi - \Lambda). \tag{5.58}$$

If p is conformal vector field and the scalar curvature of Θ is constant $2n(2n+1)(\alpha^2 + \beta^2)$, then applying Theorem (5.3) we get

$$2n(2n+1)(\alpha^2 + \beta^2)\int_\Theta \left[b + \frac{a}{2n(2n+1)(\alpha^2 + \beta^2)}\,|\zeta|^2\right] = -(2n+1)\int_\Theta \Lambda. \tag{5.59}$$

Now, if $\Lambda < 0$, then above equation reduced

$$2n(2n+1)(\alpha^2 + \beta^2)\int_\Theta \left[b + \frac{a}{2n(2n+1)(\alpha^2 + \beta^2)}\,|\zeta|^2\right] < 0. \tag{5.60}$$

If M is a compact TPS-manifold, then Theorem (5.2) implies that Θ is isometric to $\mathbb{S}^n$. Because scalar curvature is preserved via isometry so $2n(2n+1)(\alpha^2+\beta^2) > 0$. Hence the above equation entails that

$$Vol(M) < \frac{1}{2n(2n+1)} \int_{\Theta} \left[2n(2n+1)b + \frac{a}{(\alpha^2+\beta^2)} |\zeta|^2 \right]. \tag{5.61}$$

□

Lemma 5.1. [5] *If $(\Theta, \Phi, \zeta, \gamma, g)$ be a GAGRS with Einstein potential f. Then we have*

$$\Delta f = 2n(2n+1)b(\alpha^2+\beta)^2) + (2n+1)\Lambda - a|\zeta|^2. \tag{5.62}$$

Currently, function f convexity suggests that it is harmonic, or that $\Delta f = 0$, [32]. Therefore, (5.62) implies

$$2n(2n+1)b(\alpha^2+\beta)^2) + (2n+1)\Lambda - a|\zeta|^2 = 0. \tag{5.63}$$

$$\Lambda = \frac{a|\zeta|^2}{(2n+1)} - 2nb(\alpha^2+\beta^2). \tag{5.64}$$

Therefore, this leads the following result:

Theorem 5.5. *If f is a convex harmonic function on TPS-manifold $(\Theta, \Phi, \zeta, \gamma, g)$ and has non negative scalar curvature, then admitting a GAGRS with Einstein potential f is expanding, stable, or shrinking according as*

(1) $\frac{a|\zeta|^2}{(2n+1)} > 2nb(\alpha^2+\beta^2)$,

(2) $\frac{a|\zeta|^2}{(2n+1)} = 2nb(\alpha^2+\beta^2)$ *and*

(3) $\frac{a|\zeta|^2}{(2n+1)} < 2nb(\alpha^2+\beta^2)$, *respectively.*

Moreover, Lemma 5.1 entails the following:

Corollary 5.1. *If $(\Theta, \Phi, \zeta, \gamma, g)$ be a TPS-manifold admitting a GAGRS with Einstein potential f, then the Poisson equation satisfied by f becomes*

$$\Delta f = 2n(2n+1)b(\alpha^2+\beta)^2) + (2n+1)\Lambda - a|\zeta|^2. \tag{5.65}$$

Example 5.1. *Let $(\Theta, \Phi, \zeta, \gamma, g)$ be the 3-dimensional TPS-manifold considered in example 3.1.*

Let ∇ be a Levi-Civita connection. From (3.27), we obtain the following components of Riemannina curvature tensor and Ricci tensor:

$$\Re(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_2=\left(-\frac{3}{4}e^{4z}+1\right)\mathcal{E}_1,\quad \Re(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_3=-\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_1,\quad \Re(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_2=-e^{3z}y\mathcal{E}_1, \tag{5.66}$$

$$\Re(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_3=-\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_2,\quad \Re(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_2=-\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_3,$$

$$\Re(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_1=-e^{2z}\mathcal{E}_1-\left(\frac{1}{2}e^{4z}+1\right)\mathcal{E}_2+e^{3z}y\mathcal{E}_3,\quad \Re(\mathcal{E}_2,\mathcal{E}_3)\mathcal{E}_1=0,$$

$$\Re(\mathcal{E}_1,\mathcal{E}_3)\mathcal{E}_1=e^{3z}ye_2+\left(\frac{1}{4}e^{4z}+1\right)\mathcal{E}_3,\quad \Re(\mathcal{E}_1,\mathcal{E}_2)\mathcal{E}_3=-e^{3z}y\mathcal{E}_1.$$

$$S_{ric}(\mathcal{E}_1,\mathcal{E}_1)=-\frac{3}{4}e^{4z}-2,\quad S_{ric}(\mathcal{E}_2,\mathcal{E}_2)=-\frac{1}{2}e^{4z}+2\mathcal{E}_2,\quad S_{ric}(\mathcal{E}_3,\mathcal{E}_{3},)=-\frac{1}{2}e^{4z}-2. \tag{5.67}$$

From (2.14), we have

$$bS_{ric}(\mathcal{E}_i,\mathcal{E}_i)=-(\beta+\Lambda)g(\mathcal{E}_i,\mathcal{E}_i)+(a-\beta)\delta^i_j,\quad \{i=1,2,3\} \tag{5.68}$$

Now, we find the following cases corresponding to the different values of a and b in equation (2.14):

Case(1). For Killing vector field i.e., $a=b=0$, from (5.68) we find $\Lambda=-2\beta$, which is shrinking.

Case(2). In case of Ricci soliton $a=0,b=-1$, from (5.68), $\Lambda=-\left(\frac{3}{4}e^{4z}+2\right)-\beta$. Therefore, the data (g,ζ,Λ,a,b) is an AGRS on TPS-manifold $(\Theta,\Phi,\zeta,\gamma,g)$, is steady and shrinking according as $\frac{3}{4}e^{4z}+2<-\beta$, $\frac{3}{4}e^{4z}+2=\beta$, respectively.

Case(3). For Einstein-Weyl geometry case $a=1,b=\frac{-1}{n-2)}$, from (5.68), $\Lambda=(2\beta+1)-\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$. Now, the data (g,ζ,Λ,a,b) is an AGRS on TPS-manifold $(\Theta,\Phi,\zeta,\gamma,g)$ is steady, shrinking or expanding according as $(2\beta+1)=\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$, $(2\beta+1)<\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$ or $(2\beta+1)>\frac{1}{(n-2)}\left(\frac{1}{2}e^{4z}+2\right)$, respectively.

Case(4). For the geometry of Vacuum near horizon equation $a=1,b=\frac{1}{2}$, from (5.68), $\Lambda=(2\beta-2)-\left(\frac{1}{4}e^{4z}\right)$. The data (g,ζ,Λ,a,b) is an AGRS on TPS-manifold $(\Theta,\Phi,\zeta,\gamma,g)$, is steady, shrinking or expanding according as $(2\beta-2)=\left(\frac{1}{4}e^{4z}\right)$, $(2\beta-2)<\left(\frac{1}{4}e^{4z}\right)$ or $(2\beta-2)>\left(\frac{1}{4}e^{4z}\right)$, respectively.

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