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YAMABE SOLITONS ON Sol_3 SPACE

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. The aim of this work is to find the existence/non-existence of Yamabe solitons and gradient Yamabe solitons of Sol_3 space with left-invariant Riemannian and Lorentzian metric. We show that there exists an expanding Yamabe soliton and a gradient Yamabe soliton with a constant potential function on Sol_3 space.

Keywords: Solvable Lie group, Yamabe soliton, gradient Yamabe soliton, Riemannian metric, Lorentzian metric, left-invariant metric.

2010 Mathematics Subject Classification: 53B30, 53C50, 53C30.

1. Introduction

Thurston [12] gave a classification of 3-dimensional homogeneous manifolds into eight model spaces, which are real space forms having groups of isometries of dimension 6, $\mathbb{S}^2 \times \mathbb{R}$, $\mathbb{H}^2 \times \mathbb{R}$, Nil_3 the Heisenberg group, the universal covering $\widetilde{SL_2}\mathbb{R}$ of $SL_2\mathbb{R}$ having a group of isometries of dimension 4, and Sol_3 space with a group of isometries of dimension 3. The Sol_3 space is a simply connected homogeneous 3-dimensional manifold having the smallest number of isometries. The Poincaré conjecture is a special case of the Thurston conjecture, which states that every compact orientable 3-manifold has a canonical decomposition into pieces that each have one of the eight types of geometric structures. In the last three decades, there have been extensive studies to understand this problem; however, the most important

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efforts are due to R. Hamilton [5]. G. Perelman gave a proof of the Thurston conjecture using Ricci flow. Henceforth, this technique attracted the attention of researchers to study problems in homogeneous spaces. Although most of the investigation has been done in the case of the Riemannian setting using the Ricci flow technique (see [9, 10, 8] and references therein). However, in the Lorentzian setting, it has been studied in the last decade (see [2, 7]). Yamabe flows are well-posed in the Riemannian setting, which may not be true in the Lorentzian case due to the non-existence of short-time solutions in general because of the lack of parabolicity. In this paper, we study Yamabe soliton and gradient Yamabe soliton on Sol_3 space with left-invariant Riemannian and Lorentzian metrics.

A Yamabe soliton on a complete Riemannian manifold satisfies [5]:

$$\frac{1}{2}\mathcal{L}_V g = (\nu - r)g, \tag{1.1}$$

where $\mathcal{L}_V$ is the Lie-derivative along the smooth potential field V, g is the Riemannian metric, ν a real scalar, and r is the scalar curvature of g. Also, Yamabe solitons serve as solutions of the Yamabe flow of Hamilton [5], which develops along the symmetries of the flow. The soliton is steady, shrinking or expanding if $\nu = 0$, > 0, or < 0, respectively. If $V = \operatorname{grad} F$ for some real-valued function $F \in C^\infty(M)$, then it is called the gradient Yamabe soliton.

On a smooth Riemannian manifold (M, g_0), the evolution of the metric g_0 in time t to $g = g(t)$ through the equation

$$\frac{\partial}{\partial t} g_t = -rg, \; g(0) = g_0,$$

is known as the Yamabe flow [5]. Yamabe flow is significant as it is a natural geometric deformation to metrics of constant scalar curvature. In mathematical physics, Yamabe flow corresponds to the fast diffusion case of the porous medium equation (the plasma equation). A Yamabe soliton is a special solution of the Yamabe flow. If V is Killing, then Yamabe soliton is called trivial Yamabe soliton.

In 2012, Calviño-Louzao et al. [3] gave a geometric characterization of Yamabe solitons on three-dimensional homogeneous Lorentzian manifolds. In 2013, Daskalopoulos and Sesum [4] classified the locally conformally flat gradient Yamabe solitons with positive sectional curvature. In 2017, Neto and Tenenblat [6] investigated gradient Yamabe solitons, conformal to an $n-$dimensional pseudo-Euclidean space. Recently, Shaikh et al. [11] examined a gradient Yamabe soliton with some additional conditions and proved that it must be of constant scalar curvature.

The article is organised as follows: In Section 2, we recall the group structure, connection, and curvature of the Sol_3 group. In Section 3, we investigate Yamabe and gradient Yamabe solitons on Sol_3 space with the Riemannian metric. In Section 4, we examine Yamabe and gradient Yamabe solitons on Sol_3 space with the Lorentzian metric.

2. Preliminaries

In this section we recall some basic facts on Sol_3 given in [1].

The Sol_3 space is defined as a group of 3×3 matrices

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^{z} & y \\ 0 & 0 & 1 \end{pmatrix},$$

with the group structure given by

$$(x', y', z') \star (x, y, z) = (e^{-z'}x + x', e^{z'}y + y', z + z'),$$

where $(x, y, z) \in \mathbb{R}^3$.

We denote by ∇ and R the Levi-Civita connection and the Riemann curvature tensor of (Sol_3, g), respectively, such that R is given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and by Ric the Ricci tensor of (Sol_3, g), which is defined by

$$Ric(X,Y) = \sum_{k=1}^{3} g(E_k, E_k) g(R(E_k, X)Y, E_k),$$

where $\{E_k\}_{k=1,\ldots,3}$ is an orthonormal basis.

3. Yamabe and gradient Yamabe solitons on Sol_3 space with Riemannian metric

In this section, we examine the existence of Yamabe and gradient Yamabe solitons on a three-dimensional solvable Lie group (Sol_3, g) with the Riemannian metric.

We consider Sol_3 space with a left-invariant Riemannian metric

$$g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2, \tag{3.2}$$

with a left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}, \tag{3.3}$$

where $(x, y, z) \in \mathbb{R}^3$.

The non-vanishing Lie brackets are

$$[E_1, E_3] = E_1,\ [E_2, E_3] = -E_2. \tag{3.4}$$

Using (3.2), (3.3), and (3.4), the Levi-Civita connection ∇ is given by

$$(\nabla_{E_i} E_j) = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.5}$$

where $i,\ j = 1, 2, 3$.

The non-vanishing components of Riemann curvature tensor and Ricci tensor are

$$\begin{cases} R(E_1, E_2)E_1 = -E_2 = R(E_2, E_3)E_3,\ R(E_j, E_3)E_j = E_3, \text{ for } j = 1, 2, \\ R(E_1, E_3)E_3 = -E_1,\ R(E_1, E_2)E_2 = E_1,\ S(E_3, E_3) = -2. \end{cases} \tag{3.6}$$

The scalar curvature r of the Riemannian Sol_3 Lie group is

$$r = \sum_{i=1}^{3} g(E_i, E_i)S(E_i, E_i) = -2. \tag{3.7}$$

Let

$$V = f_1E_1 + f_2E_2 + f_3E_3, \tag{3.8}$$

be an arbitrary potential vector field on (Sol_3, g), where $f_1,\ f_2$ and f_3 are smooth functions of $x,\ y$ and z. We denote the coordinate basis $\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\}$ by $\{\partial_x, \partial_y, \partial_z\}$.

Now, we have

Theorem 3.1. *The Sol_3 space with a left-invariant Riemannian metric given by (3.2) satisfies a Yamabe soliton equation*

$$\mathcal{L}_V g = 2(-2 - r)g,$$

with

$$V = (\alpha_1 - \delta_4 x)\partial_x + (\alpha_2 + \delta_4 y)\partial_y + \delta_4\partial_z,$$

where $\alpha_1,\ \alpha_2,\ \delta_4 \in \mathbb{R}$. Moreover, the left-invariant Riemannian metric (3.2) is an expanding Yamabe soliton.

Proof. In view of (3.2) and (3.8), we have

$$\begin{cases} (\mathcal{L}_V g)(E_1, E_1) = 2(f_3 + e^{-z}\partial_x f_1),\ (\mathcal{L}_V g)(E_1, E_2) = e^{-z}\partial_x f_2 + e^z \partial_y f_1, \\ (\mathcal{L}_V g)(E_1, E_3) = e^{-z}\partial_x f_3 - f_1 + \partial_z f_1,\ (\mathcal{L}_V g)(E_2, E_2) = -2(f_3 - e^z \partial_y f_2), \\ (\mathcal{L}_V g)(E_2, E_3) = e^z \partial_y f_3 + f_2 + \partial_z f_2,\ (\mathcal{L}_V g)(E_3, E_3) = 2\partial_z f_3. \end{cases} \tag{3.9}$$

Thus, by using (3.2), (3.7), and (3.9) in (1.1), we find that (Sol_3, g) is a Yamabe soliton if and only if the following system of equations holds:

$$f_3 + e^{-z}\partial_x f_1 = \nu + 2, \tag{3.10}$$

$$e^{-z}\partial_x f_2 + e^z \partial_y f_1 = 0, \tag{3.11}$$

$$e^{-z}\partial_x f_3 - f_1 + \partial_z f_1 = 0, \tag{3.12}$$

$$-f_3 + e^z \partial_y f_2 = \nu + 2, \tag{3.13}$$

$$e^z \partial_y f_3 + f_2 + \partial_z f_2 = 0, \tag{3.14}$$

$$\partial_z f_3 = \nu + 2. \tag{3.15}$$

From (3.15), we get

$$f_3 = (\nu + 2)z + F(x, y), \tag{3.16}$$

where $F = F(x, y)$ is a real-valued smooth function on $\mathbb{R}^2$.

From (3.10), we obtain

$$\partial_x f_1 = e^z\big((\nu + 2) - (\nu + 2)z - F\big). \tag{3.17}$$

Differentiating (3.17) with respect to z, we get

$$\partial_z \partial_x f_1 = -e^z\big((\nu + 2)z + F\big). \tag{3.18}$$

Differentiating (3.12) with respect to x, and using (3.16), (3.17), and (3.18) therein we find $\nu = -2$ and

$$\partial_x^2 F = 0. \tag{3.19}$$

Further, (3.13) gives

$$\partial_y f_2 = e^{-z}\big((\nu + 2) + (\nu + 2)z + F\big). \tag{3.20}$$

Differentiating (3.20) with respect to z, we obtain

$$\partial_z \partial_y f_2 = -e^{-z}\big((\nu + 2)z + F\big). \tag{3.21}$$

Next, differentiating (3.14) with respect to y and therein using (3.16), (3.20), and (3.21), we find $\nu = -2$ and

$$\partial_y^2 F = 0. \tag{3.22}$$

From (3.16), (3.19), and (3.22), we derive that

$$f_3 = F(x,y) = \delta_1 x + \delta_2 y + \delta_3 xy + \delta_4, \tag{3.23}$$

where $\delta_i \in \mathbb{R}$.

From (3.10) and (3.23), we get

$$f_1 = -e^z\Big(\frac{\delta_1 x^2}{2} + \delta_2 xy + \frac{\delta_3 yx^2}{2} + \delta_4 x\Big) + T(y,z), \tag{3.24}$$

where T is a smooth function.

From (3.13) and (3.23), we obtain

$$f_2 = e^{-z}\Big(\delta_1 xy + \frac{\delta_2 y^2}{2} + \frac{\delta_3 xy^2}{2} + \delta_4 y\Big) + I(x,z), \tag{3.25}$$

where I is a smooth function.

Using (3.24) and (3.25) in (3.11), we get

$$\Big(e^z\partial_y T + e^{-2z}\Big(\delta_1 y + \frac{\delta_3 y^2}{2}\Big)\Big) + \Big(e^{-z}\partial_x I - e^{2z}\Big(\delta_2 x + \frac{\delta_3 x^2}{2}\Big)\Big) = 0. \tag{3.26}$$

Since (3.26) holds for all values of z, therefore, it implies that

$$e^z\partial_y T + e^{-2z}\Big(\delta_1 y + \frac{\delta_3 y^2}{2}\Big) = 0, \quad e^{-z}\partial_x I - e^{2z}\Big(\delta_2 x + \frac{\delta_3 x^2}{2}\Big) = 0. \tag{3.27}$$

By integration (3.27) gives

$$T = -e^{-3z}\big(\frac{\delta_1 y^2}{2} + \frac{\delta_3 y^3}{6}\big) + \bar{T}(z), \quad I = e^{3z}\big(\frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}\big) + \bar{I}(z), \tag{3.28}$$

where $\bar{T}$ and $\bar{I}$ are smooth functions. So,

$$\begin{cases} f_1 = -e^z\Big(\frac{\delta_1 x^2}{2} + \delta_2 xy + \frac{\delta_3 yx^2}{2} + \delta_4 x\Big) - e^{-3z}\Big(\frac{\delta_1 y^2}{2} + \frac{\delta_3 y^3}{6}\Big) + \bar{T}(z), \\ f_2 = e^{-z}\Big(\delta_1 xy + \frac{\delta_2 y^2}{2} + \frac{\delta_3 xy^2}{2} + \delta_4 y\Big) + e^{3z}\big(\frac{\delta_2 x^2}{2} + \frac{\delta_3 x^3}{6}\big) + \bar{I}(z). \end{cases} \tag{3.29}$$

Now putting the values of f_1 and f_3 in (3.12), we obtain

$$e^{-z}\big(\delta_1 + \delta_3 y\big) + 4e^{-3z}\Big(\frac{\delta_1 y^2}{2} + \frac{\delta_3 y^3}{6}\Big) - \bar{T}(z) + \bar{T}'(z) = 0. \tag{3.30}$$

Since (3.30) holds for all z, therefore, it gives that $\delta_1 = \delta_3 = 0$, and

$$\bar{T} = \alpha_1 e^z,$$

where $\alpha_1 \in \mathbb{R}$.

Putting the values of f_2 and f_3 in (3.14), we find

$$e^z\big(\delta_2+\delta_3 x\big)+4e^{3z}\Big(\frac{\delta_2 x^2}{2}+\frac{\delta_3 x^3}{6}\Big)+\bar{I}(z)+\bar{I}'(z)=0. \tag{3.31}$$

Since (3.31) holds for all z, therefore, it implies that $\delta_2=\delta_3=0$, and

$$\bar{I}=\alpha_2 e^{-z},$$

where $\alpha_2\in\mathbb{R}$.

Hence, the solution of the system of equations (3.10)$\sim$(3.15) is given by

$$f_1=(\alpha_1-\delta_4 x)e^z,\quad f_2=(\alpha_2+\delta_4 y)e^{-z},\quad f_3=\delta_4, \tag{3.32}$$

where α_1, α_2, $\delta_4\in\mathbb{R}$.

Hence, the Riemannian three-dimensional Lie group Sol_3 admits an expanding Yamabe soliton for appropriate vector fields given by (3.32). □

Theorem 3.2. *The Sol_3 space with a left-invariant Riemannian metric given by (3.2) satisfies a gradient Yamabe soliton equation*

$$\mathcal{L}_{\mathrm{grad}\,F}\,g=2(-2-r)g,$$

where the potential function F is constant.

Proof. Let $V=\mathrm{grad}\,F$ be an arbitrary gradient vector field on (Sol_3,g) with potential function F. Then V is given by

$$\mathrm{grad}\,F=e^{-2z}\partial_x F\,\partial_x+e^{2z}\,\partial_y F\,\partial_y+\partial_z F\,\partial_z.$$

From (3.32), we see that (Sol_3,g) is a gradient Yamabe soliton if and only if the potential function F satisfies the following systems:

$$\partial_x F = e^{2z}(\alpha_1-\delta_4 x), \tag{3.33}$$

$$\partial_y F = e^{-2z}(\alpha_2+\delta_4 y), \tag{3.34}$$

$$\partial_z F = \delta_4. \tag{3.35}$$

Differentiating (3.33) with respect to z and (3.35) with respect to x, and equating them we obtain

$$2e^{2z}(\alpha_1-\delta_4 x)=0, \tag{3.36}$$

which gives $\alpha_1 = 0$ and $\delta_4 = 0$. Further, taking the derivative of (3.34) with respect to z and (3.35) with respect to y and equating them we get

$$2e^{-2z}(\alpha_2 + \delta_4 y) = 0, \tag{3.37}$$

which gives $\alpha_2 = 0$ and $\delta_4 = 0$. So, $F =$ constant. Hence the result. □

4. Yamabe and gradient Yamabe solitons on Sol_3 space with Lorentzian metric

In this section, we examine the existence of Yamabe and gradient Yamabe solitons on a three-dimensional solvable Lie group with the Lorentzian metric.

We consider Sol_3 space with a left-invariant Lorentzian metric

$$g = e^{2z}dx^2 - e^{-2z}dy^2 + dz^2, \tag{4.38}$$

with a left-invariant orthonormal frame $\{E_1, E_2, E_3\}$ given by

$$E_1 = e^{-z}\frac{\partial}{\partial x},\ E_2 = e^{z}\frac{\partial}{\partial y},\ E_3 = \frac{\partial}{\partial z}, \tag{4.39}$$

where $(x, y, z) \in \mathbb{R}^3$.

The non-vanishing Lie brackets are

$$[E_1, E_3] = E_1,\ [E_2, E_3] = -E_2. \tag{4.40}$$

Using (4.38), (4.39), and (4.40) the Levi-Civita connection is given by

$$(\nabla_{E_i} E_j) = \begin{pmatrix} -E_3 & 0 & E_1 \\ 0 & -E_3 & -E_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{4.41}$$

where $i,\ j = 1, 2, 3$.

The non-vanishing components of Riemann curvature tensor and Ricci tensor are

$$\begin{cases} R(E_1, E_j)E_j = -E_1, \text{for } j = 2, 3,\ R(E_1, E_2)E_1 = -E_2 = R(E_2, E_3)E_3, \\ R(E_1, E_3)E_1 = E_3 = -R(E_2, E_3)E_2,\ S(E_3, E_3) = -2. \end{cases} \tag{4.42}$$

The scalar curvature r of the Lorentzian Sol_3 Lie group is

$$r = \sum_{i=1}^{3} g(E_i, E_i)S(E_i, E_i) = -2. \tag{4.43}$$

Now, we have

Theorem 4.1. *The Sol_3 space with a left-invariant Lorentzian metric given by (4.38) satisfies a Yamabe soliton equation*

$$\mathcal{L}_V g = 2(-2-r)g,$$

with

$$V = (\beta_1 - \gamma_4 x)\partial_x + (\beta_2 + \gamma_4 y)\partial_y + \gamma_4\partial_z,$$

where β_1, β_2, $\gamma_4 \in \mathbb{R}$. Moreover, the left-invariant Lorentzian metric (4.38) is an expanding Yamabe soliton.

Proof. In view of (3.8) and (4.38), we have

$$\begin{cases} (\mathcal{L}_V g)(E_1,E_1) = 2(f_3 + e^{-z}\partial_x f_1),\ (\mathcal{L}_V g)(E_1,E_2) = -e^{-z}\partial_x f_2 + e^{z}\partial_y f_1, \\ (\mathcal{L}_V g)(E_1,E_3) = e^{-z}\partial_x f_3 - f_1 + \partial_z f_1,\ (\mathcal{L}_V g)(E_2,E_2) = 2(f_3 - e^{z}\partial_y f_2), \\ (\mathcal{L}_V g)(E_2,E_3) = e^{z}\partial_y f_3 - f_2 - \partial_z f_2,\ (\mathcal{L}_V g)(E_3,E_3) = 2\partial_z f_3. \end{cases} \tag{4.44}$$

Thus, by using (4.38), (4.43), and (4.44) in (1.1), we find that (Sol_3, g) is a Yamabe soliton if and only if the following system of equations holds,

$$f_3 + e^{-z}\partial_x f_1 = \nu + 2, \tag{4.45}$$

$$-e^{-z}\partial_x f_2 + e^{z}\partial_y f_1 = 0, \tag{4.46}$$

$$e^{-z}\partial_x f_3 - f_1 + \partial_z f_1 = 0, \tag{4.47}$$

$$f_3 - e^{z}\partial_y f_2 = -\nu - 2, \tag{4.48}$$

$$e^{z}\partial_y f_3 - f_2 - \partial_z f_2 = 0, \tag{4.49}$$

$$\partial_z f_3 = \nu + 2. \tag{4.50}$$

From (4.50), we get

$$f_3 = (\nu+2)z + H(x,y), \tag{4.51}$$

where $H = H(x,y)$ is a real-valued smooth function on $\mathbb{R}^2$.

Using (4.51) in (4.48), we obtain

$$\partial_y f_2 = e^{-z}\big((\nu+2) + (\nu+2)z + H\big). \tag{4.52}$$

Differentiating (4.52) with respect to z, we get

$$\partial_z\partial_y f_2 = -e^{-z}\big((\nu+2)z + H\big). \tag{4.53}$$

Further, diferentiating (4.49) with respect to y and using (4.51)∼(4.53), we find that $\nu = -2$ and

$$\partial_y^2 H = 0. \tag{4.54}$$

On the other hand, using (4.51) in (4.45), we obtain

$$\partial_x f_1 = e^z\big((\nu+2) - (\nu+2)z - H\big). \tag{4.55}$$

Differentiating (4.55) with respect to z, we get

$$\partial_z \partial_x f_1 = -e^z\big((\nu+2)z + H\big). \tag{4.56}$$

Further, differentiating (4.47) with respect to x and using (4.51), (4.55), and (4.56), we find that $\nu = -2$ and

$$\partial_x^2 H = 0. \tag{4.57}$$

From (4.51), (4.54), and (4.57), we derive that

$$f_3 = H(x,y) = \gamma_1 x + \gamma_2 y + \gamma_3 xy + \gamma_4, \tag{4.58}$$

where $\gamma_i \in \mathbb{R}$.

From (4.45) and (4.58), we get

$$f_1 = -e^z\Big(\frac{\gamma_1 x^2}{2} + \gamma_2 xy + \frac{\gamma_3 y x^2}{2} + \gamma_4 x\Big) + K(y,z), \tag{4.59}$$

where K is a smooth function.

From (4.48) and (4.58), we obtain

$$f_2 = e^{-z}\Big(\gamma_1 xy + \frac{\gamma_2 y^2}{2} + \frac{\gamma_3 x y^2}{2} + \gamma_4 y\Big) + L(x,z), \tag{4.60}$$

where L is a smooth function.

Using (4.59) and (4.60) in (4.46), we get

$$\Big(e^z \partial_y K - e^{-2z}\Big(\gamma_1 y + \frac{\gamma_3 y^2}{2}\Big)\Big) - \Big(e^{-z}\partial_x L + e^{2z}\Big(\gamma_2 x + \frac{\gamma_3 x^2}{2}\Big)\Big) = 0. \tag{4.61}$$

Since (4.61) holds for all values of z, therefore, it implies that

$$e^z \partial_y K - e^{-2z}\big(\gamma_1 y + \frac{\gamma_3 y^2}{2}\big) = 0, \quad e^{-z}\partial_x L + e^{2z}\big(\gamma_2 x + \frac{\gamma_3 x^2}{2}\big) = 0. \tag{4.62}$$

By integration (4.62) gives

$$K = e^{-3z}\big(\frac{\gamma_1 y^2}{2} + \frac{\gamma_3 y^3}{6}\big) + \bar{K}(z), \quad L = -e^{3z}\big(\frac{\gamma_2 x^2}{2} + \frac{\gamma_3 x^3}{6}\big) + \bar{L}(z), \tag{4.63}$$

where $\bar{K}$ and $\bar{L}$ are smooth functions. So, from (4.59) and (4.60), we get

$$\begin{cases} f_1 = -e^{z}\Big(\frac{\gamma_1 x^2}{2} + \gamma_2 xy + \frac{\gamma_3 y x^2}{2} + \gamma_4 x\Big) + e^{-3z}\Big(\frac{\gamma_1 y^2}{2} + \frac{\gamma_3 y^3}{6}\Big) + \bar{K}(z), \\ f_2 = e^{-z}\Big(\gamma_1 xy + \frac{\gamma_2 y^2}{2} + \frac{\gamma_3 x y^2}{2} + \gamma_4 y\Big) - e^{3z}\big(\frac{\gamma_2 x^2}{2} + \frac{\gamma_3 x^3}{6}\big) + \bar{L}(z). \end{cases} \tag{4.64}$$

Now, putting the values of f_1 and f_3 in (4.47), we obtain

$$e^{-z}\big(\gamma_1 + \gamma_3 y\big) - 4e^{-3z}\Big(\frac{\gamma_1 y^2}{2} + \frac{\gamma_3 y^3}{6}\Big) - \bar{K}(z) + \bar{K}'(z) = 0. \tag{4.65}$$

Since (4.65) holds for all z, therefore, it gives that $\gamma_1 = \gamma_3 = 0$, and

$$\bar{K} = \beta_1 e^{z},$$

where $\beta_1 \in \mathbb{R}$.

Putting the values of f_2 and f_3 in (4.49), we find

$$e^{z}\big(\gamma_2 + \gamma_3 x\big) + 4e^{3z}\Big(\frac{\gamma_2 x^2}{2} + \frac{\gamma_3 x^3}{6}\Big) - \bar{L}(z) - \bar{L}'(z) = 0. \tag{4.66}$$

Since (4.66) holds for all z, therefore, it implies that $\gamma_2 = \gamma_3 = 0$, and

$$\bar{L} = \beta_2 e^{-z},$$

where $\beta_2 \in \mathbb{R}$.

Hence, the solution of the system of equations (4.45)$\sim$(4.50) is given by

$$f_1 = (\beta_1 - \gamma_4 x)e^{z}, \quad f_2 = (\beta_2 + \gamma_4 y)e^{-z}, \quad f_3 = \gamma_4, \tag{4.67}$$

where β_1, β_2, $\gamma_4 \in \mathbb{R}$.

Hence Lorentzian three-dimensional Lie group Sol_3 admits an expanding Yamabe soliton for appropriate vector fields given by (4.67). □

Theorem 4.2. *The Sol_3 space with a left-invariant Lorentzian metric given by (4.38) satisfies a gradient Yamabe soliton equation*

$$\mathcal{L}_{\operatorname{grad} F}\, g = 2(-2 - r)g,$$

where the potential function F is constant.

Proof. Let $V = \operatorname{grad} F$ be an arbitrary gradient vector field on (Sol_3, g) with potential function F. Then V is given by

$$\operatorname{grad} F = e^{-2z}\partial_x F\, \partial_x + e^{2z}\, \partial_y F\, \partial_y + \partial_z F\, \partial_z.$$

From (4.67), we see that (Sol_3, g) is a gradient Yamabe soliton if and only if the potential function F satisfies the following systems:

$$\partial_x F = e^{2z}(\beta_1 - \gamma_4 x), \tag{4.68}$$

$$\partial_y F = e^{-2z}(\beta_2 + \gamma_4 y), \tag{4.69}$$

$$\partial_z F = \gamma_4. \tag{4.70}$$

Differentiating (4.68) with respect to z and (4.70) with respect to x, and equating them we obtain

$$2e^{2z}(\beta_1 - \gamma_4 x) = 0, \tag{4.71}$$

which gives $\beta_1 = 0$ and $\gamma_4 = 0$. Further, taking the derivative of (4.69) with respect to z and (4.70) with respect to y and equating them we get

$$2e^{-2z}(\beta_2 + \gamma_4 y) = 0, \tag{4.72}$$

which gives $\beta_2 = 0$ and $\gamma_4 = 0$. So, $F =$ constant. Hence the result. $\square$

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