International Journal of Maps in Mathematics

*Volume **7**, Issue **1**, 2024, Pages:**33-44***

E-ISSN: 2636-7467

www.journalmim.com

A STUDY OF φ-RICCI SYMMETRIC LP-KENMOTSU MANIFOLDS

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Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)

Abstract. In the current article we characterize φ-Ricci symmetric (φ-RS) and weakly φ-Ricci symmetric (weakly φ-RS) LP-Kenmotsu m-manifolds ((LP-K)$_m$). We also examine the characteristic of an (LP-K)$_3$ of scalar curvature 6. Moreover, we study (LP-K)$_m$ admitting ω-parallel Ricci tensor. At last, we construct an example of φ-RS (LP-K)$_3$ to verify some of our results.

Keywords: Einstein manifold, φ-Ricci symmetric manifolds, LP-Kenmotsu manifolds, scalar curvature, Ricci tensor.

2010 Mathematics Subject Classification: 53C25, 53C50, 53C80.

1. Introduction

Approximately five decades ago, the notion of Kenmotsu manifold as a class of almost contact metric manifolds was introduced by Kenmotsu [19]. Kenmotsu has proved that a locally Kenmostu manifold is a warped product $\mathcal{I}\times_{\mathfrak{f}} \aleph$ of an interval $\mathcal{I}$ and a Kähler manifold $\aleph$ with warping function $\mathfrak{f}(\mathfrak{t}) = \rho e^{\mathfrak{t}}$, where ρ ($\neq 0$) is a constant. In 1976, the idea of almost para-contact Riemannian manifolds was proposed by Sato [20]. Then, as a class of almost contact Riemannian manifolds, para-Sasakian and Special para-Sasakian manifolds have been

Received:2023.10.02 *Revised:2024.01.09* *Accepted:2024.01.12*

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defined and studied in [1] by Adati and Matsumoto. In 1989, Matsumoto [14] defined and studied Lorentzian para-Sasakian manifolds. Later, Mihai and Rosca also contribured some remarks on this manifold [16]. The authors Sinha and Prasad [22] studied para-Kenmotsu manifolds. In 2018, the first and second authors proposed and investigated a new class of Lorentzian almost para-contact metric manifolds namely LP-Kenmotsu manifolds [11]. Recently, numerous geometers studied LP-Kenmotsu manifolds in many ways to different point of views such as [2, 17, 12, 9, 15] and many others. Several mathematicians have studied the notion of weakly local symmetric Riemannian manifolds with different approaches in various fields. In 1977, Takahashi [23] introduced the concept of locally φ-symmetric Sasakian manifolds. The φ-symmetric notion in contact geometry was initiated and studied by Vanhecke, Buecken and Boeckx [5]. About two decades ago, the authors De, Shaikh and Biswas have studied φ-recurrent Sasakian manifolds [6] by generalizing the idea of locally φ-symmetric manifolds. In [8], the author studied φ-symmetric Kenmotsu manifolds in which he had given a number of examples. In 2008, De and Sarkar [7] studied φ-RS Sasakian manifolds. Later in 2009, φ-RS Kenmotsu manifold was studied by Shukla and Shukla [21].

This paper is structured in the following manner: Section 2 contains preliminaries, where some basic results are mentioned. In section 3, we study φ-RS $(\text{LP-K})_m$ and prove that an $(\text{LP-K})_m$ is Einstein manifold, if it is φ-symmetric. In section 4, we study of φ-RS $(\text{LP-K})_3$, here we proved that an $(\text{LP-K})_3$ is locally φ-RS, if and only if $\underline{\text{r}}$ is constant. Section 5 is devoted to the study of weakly φ-RS $(\text{LP-K})_m$ and it is proven that a weakly φ-RS $(\text{LP-K})_m$ is an ω-Einstein manifold. Section 6 deals with the study of $(\text{LP-K})_m$ admitting ω-parallel Ricci tensor. At last an example of $(\text{LP-K})_3$ is modeled to inquire some of our findings.

2. Preliminaries

Let $\mathcal{M}^m$ $(\varphi, \zeta, \omega, g)$ be a Lorentzian metric manifold, where φ: $(1,1)$ tensor field, ζ: a characteristic vector field, ω: a 1-form and g: the Lorentz metric. We are well acquainted with the following results [3, 4, 18]:

$$\begin{cases} \varphi\zeta = 0, \\ \omega(\varphi\underline{\text{U}}) = 0, \\ \omega(\zeta) + 1 = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} \varphi^2 U - U - \omega(U)\zeta = 0, \\ g(U,\zeta) - \omega(U) = 0, \end{cases} \tag{2.2}$$

$$g(\varphi U, \varphi V) - g(U, V) = \omega(U)\omega(V), \tag{2.3}$$

$$(\bar{\nabla}_U \varphi)V = -g(\varphi U, V)\zeta - \omega(V)\varphi U, \tag{2.4}$$

$$\bar{\nabla}_U \zeta = -U - \omega(U)\zeta, \tag{2.5}$$

for all vector fields U, V on $\mathcal{M}^m$ and $\bar{\nabla}$ represents the Levi-Civita connection of g, then $\mathcal{M}^m$ $(\varphi, \zeta, \omega, g)$ is said to be an (LP-K)$_m$ [11, 10].

In (LP-K)$_m$, the following results hold:

$$(\bar{\nabla}_U \omega)V = -\omega(U)\omega(V) - g(U, V), \tag{2.6}$$

$$\omega(R(U,V)Z) = g(V,Z)\omega(U) - g(U,Z)\omega(V), \tag{2.7}$$

$$R(U,V)\zeta = \omega(V)U - \omega(U)V, \tag{2.8}$$

$$R(\zeta, U)V = g(U,V)\zeta - \omega(V)U, \tag{2.9}$$

$$\mathcal{S}(U,\zeta) = (m-1)\omega(U), \qquad Q\zeta = (m-1)\zeta, \tag{2.10}$$

$$(\bar{\nabla}_Z R)(U,V)\zeta = g(U,Z)V g(V,Z)U + R(U,V)Z, \tag{2.11}$$

$$\mathcal{S}(\varphi U, \varphi V) = \mathcal{S}(U,V) + (m-1)\omega(U)\omega(V) \tag{2.12}$$

for all vector fields U, V, Z on (LP-K)$_m$, where R is the Riemannian curvature tensor, $\mathcal{S}$ is the Ricci tensor and Q indicates the Ricci operator such that $\mathcal{S}(U,V) = g(QU, V)$.

Remark 2.1. [13] *If an (LP-K)$_m$ possesses the constant scalar curvature, then* $r = m(m-1)$.

3. φ-RS (LP-K)$_m$

We start this section with the following definitions:

Definition 3.1. *An (LP-K)$_m$ is called*

(i) φ-RS if

$$\varphi^2((\bar{\nabla}_U Q)(V)) = 0, \tag{3.13}$$

(ii) φ-symmetric if

$$\varphi^2((\bar{\nabla}_K R)(U, V)Z) = 0 \tag{3.14}$$

for any vector fields U, V, Z, K on (LP-K)$_m$. In case, U, V are orthogonal to ζ, then φ-RS (LP-K)$_m$ is named locally φ-RS.

Definition 3.2. *An (LP-K)$_m$ is called Einstein manifold, if its $\mathcal{S}$ is of the form*

$$\mathcal{S}(\underline{U},\underline{V})=\lambda g(\underline{U},\underline{V}),$$

where λ is a constant.

Theorem 3.1. *An (LP-K)$_m$ is φ-RS, iff it is Einstein manifold.*

Proof. Let an (LP-K)$_m$ be φ-RS. Then we have

$$\varphi^2((\bar{\nabla}_{\underline{U}}Q)(\underline{V}))=0,$$

which by using (2.2) becomes

$$(\bar{\nabla}_{\underline{U}}Q)\underline{V}+\omega((\bar{\nabla}_{\underline{U}}Q)\underline{V})\zeta=0. \tag{3.15}$$

The inner product of (3.15) with $\underline{Z}$ lead to

$$g((\bar{\nabla}_{\underline{U}}Q)\underline{V},\underline{Z})+\omega((\bar{\nabla}_{\underline{U}}Q)\underline{V})\omega(\underline{Z})=0,$$

which after simplification takes the form

$$g(\bar{\nabla}_{\underline{U}}(Q\underline{V}),\underline{Z})-\mathcal{S}(\bar{\nabla}_{\underline{U}}\underline{V},\underline{Z})+\omega((\bar{\nabla}_{\underline{U}}Q)\underline{V})\omega(\underline{Z})=0. \tag{3.16}$$

By taking $\underline{V}=\zeta$ in (3.16), then using (2.5) and (2.10), we have

$$(m-1)g(\bar{\nabla}_{\underline{U}}\zeta,\underline{Z})+\mathcal{S}(\underline{U},\underline{Z})+\omega(\underline{U})\mathcal{S}(\zeta,\underline{Z})+\omega((\bar{\nabla}_{\underline{U}}Q)\zeta)\omega(\underline{Z})=0. \tag{3.17}$$

Now by virtue of (2.5) and (2.10), (3.17) turns to

$$S(\underline{U},\underline{Z})-(m-1)g(\underline{U},\underline{Z})+\omega((\bar{\nabla}_{\underline{U}}Q)\zeta)\omega(\underline{Z})=0. \tag{3.18}$$

Substituting $\underline{U}\to\varphi\underline{U}$ as well as $\underline{Z}\to\varphi\underline{Z}$ in (3.18), we find

$$\mathcal{S}(\varphi\underline{U},\varphi\underline{Z})=(m-1)g(\varphi\underline{U},\varphi\underline{Z}). \tag{3.19}$$

Keeping in mind (2.3) and (2.12), (3.19) leads to

$$\mathcal{S}(\underline{U},\underline{Z})=(m-1)g(\underline{U},\underline{Z}). \tag{3.20}$$

Conversely, we assume that (LP-K)$_m$ is an Einstein manifold. Therefore, by the Definition 3.2, we have $Q\underline{U}=\lambda\underline{U}$, from which we conclude

$$\varphi^2((\bar{\nabla}_{\underline{U}}Q)(\underline{V}))=0.$$

This completes the proof. □

Corollary 3.1. *An (LP-K)$_m$ is Einstein manifold, if it is φ-symmetric.*

Proof. Let an (LP-K)$_m$ be φ-symmetric manifold. Then we have

$$\varphi^2((\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\underline{Z}) = 0 \tag{3.21}$$

for any vector fields $\underline{U}$, $\underline{V}$, $\underline{Z}$, $\underline{K}$ on (LP-K)$_m$.

By using (2.2) in (3.21), it yields

$$(\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\underline{Z} - g((\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\zeta,\underline{Z})\zeta = 0. \tag{3.22}$$

Now in view of (2.11), (3.22) takes the form

$$\begin{aligned}(\bar{\nabla}_{\underline{K}}\underline{R})(\underline{U},\underline{V})\underline{Z} - g(\underline{U},\underline{K})g(\underline{V},\underline{Z})\zeta \\ +g(\underline{V},\underline{K})g(\underline{U},\underline{Z})\zeta - g(\underline{R}(\underline{U},\underline{V})\underline{K},\underline{Z})\zeta = 0.\end{aligned} \tag{3.23}$$

On contracting (3.23), we obtain

$$(\bar{\nabla}_{\underline{K}}\mathcal{S})(\underline{V},\underline{Z}) - g(\underline{V},\underline{Z})\omega(\underline{K}) + g(\underline{V},\underline{K})\omega(\underline{Z}) + \omega(\underline{R}(\underline{K},\underline{Z})\underline{V}) = 0. \tag{3.24}$$

By virtue of (2.7), equation (3.24) reduces to

$$(\bar{\nabla}_{\underline{K}}\mathcal{S})(\underline{V},\underline{Z}) = 0. \tag{3.25}$$

Consequenty, we obtain

$$\varphi^2((\bar{\nabla}_{\underline{K}}\mathcal{S})(\underline{V},\underline{Z})) = 0. \tag{3.26}$$

Thus φ-symmetric (LP-K)$_m$ is φ-RS. And hence Corollary 3.1 follows from Theorem 3.1. □

4. φ-RS (LP-K)$_3$

Theorem 4.1. *In case, the scalar curvature* $\underline{r}$ *of an (LP-K)$_3$ is* 6*, then (LP-K)$_3$ is φ-RS.*

Proof. In an (LP-K)$_3$, the curvature tensor $\underline{R}$ is given by [11, 24]

$$\begin{aligned}\underline{R}(\underline{U},\underline{V})\underline{Z} &= (\frac{\underline{r}}{2}-2)[g(\underline{V},\underline{Z})\underline{U} - g(\underline{U},\underline{Z})\underline{V}] \\ &+(\frac{\underline{r}}{2}-3)[g(\underline{V},\underline{Z})\omega(\underline{U})\zeta - g(\underline{U},\underline{Z})\omega(\underline{V})\zeta] \\ &+(\frac{\underline{r}}{2}-3)[\omega(\underline{V})\omega(\underline{Z})\underline{U} - \omega(\underline{U})\omega(\underline{Z})\underline{V}]\end{aligned} \tag{4.27}$$

for all vector fields $\underline{U}, \underline{V}, \underline{Z}$ on (LP-K)$_3$.

The inner product of (4.27) with $\underline{K}$ leads to

$$\begin{aligned}g(\underline{R}(\underline{U},\underline{V})\underline{Z},\underline{K}) &= (\frac{\underline{r}}{2}-2)[g(\underline{V},\underline{Z})g(\underline{U},\underline{K}) - g(\underline{U},\underline{Z})g(\underline{V},\underline{K})] \\ &+(\frac{\underline{r}}{2}-3)[g(\underline{V},\underline{Z})\omega(\underline{U})\omega(\underline{K}) - g(\underline{U},\underline{Z})\omega(\underline{V})\omega(\underline{K})] \\ &+(\frac{\underline{r}}{2}-3)[\omega(\underline{V})\omega(\underline{Z})g(\underline{U},\underline{K}) - \omega(\underline{U})\omega(\underline{Z})g(\underline{V},\underline{K})].\end{aligned} \tag{4.28}$$

Let $\{\underline{l}_1, \underline{l}_2, \underline{l}_3\}$ be the orthonormal basis of the tangent space at every point of $(\text{LP-K})_3$. Now setting $\underline{\mathrm{U}} = \underline{\mathrm{K}} = \underline{l}_i$ as well as proceeding for sum from $i = 1$ to 3 in equation (4.28), it provides

$$\mathcal{S}(\underline{\mathrm{V}}, \underline{\mathrm{Z}}) = (\frac{\underline{\mathrm{r}}}{2} - 1)g(\underline{\mathrm{V}}, \underline{\mathrm{Z}}) + (\frac{\underline{\mathrm{r}}}{2} - 3)\omega(\underline{\mathrm{V}})\omega(\underline{\mathrm{Z}}). \tag{4.29}$$

From (4.29) it follows that

$$\underline{\mathrm{Q}}\underline{\mathrm{V}} = (\frac{\underline{\mathrm{r}}}{2} - 1)\underline{\mathrm{V}} + (\frac{\underline{\mathrm{r}}}{2} - 3)\omega(\underline{\mathrm{V}})\zeta. \tag{4.30}$$

Differentiating (4.30) covariantly along $\underline{\mathrm{K}}$, we have

$$\begin{aligned}(\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{Q}})\underline{\mathrm{V}} + \underline{\mathrm{Q}}(\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{V}}) \quad &= \quad (\frac{\underline{\mathrm{r}}}{2} - 1)\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{V}} + \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\underline{\mathrm{V}} + \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\omega(\underline{\mathrm{V}})\zeta + (\frac{\underline{\mathrm{r}}}{2} - 3)(\bar{\nabla}_{\underline{\mathrm{K}}}\omega)(\underline{\mathrm{V}})\zeta \\ &+ (\frac{\underline{\mathrm{r}}}{2} - 3)\omega(\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{V}})\zeta + (\frac{\underline{\mathrm{r}}}{2} - 3)\omega(\underline{\mathrm{V}})\bar{\nabla}_{\underline{\mathrm{K}}}\zeta. \end{aligned} \tag{4.31}$$

By virtue of (4.30), (4.31) takes the form

$$\begin{aligned}(\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{Q}})\underline{\mathrm{V}} \quad &= \quad \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\underline{\mathrm{V}} + \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\omega(\underline{\mathrm{V}})\zeta + (\frac{\underline{\mathrm{r}}}{2} - 3)(\bar{\nabla}_{\underline{\mathrm{K}}}\omega)(\underline{\mathrm{V}})\zeta \\ &+ (\frac{\underline{\mathrm{r}}}{2} - 3)\omega(\underline{\mathrm{V}})\bar{\nabla}_{\underline{\mathrm{K}}}\zeta. \end{aligned} \tag{4.32}$$

By using (2.5) and (2.6) in (4.32), we have

$$\begin{aligned}(\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{Q}})\underline{\mathrm{V}} \quad &= \quad \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\underline{\mathrm{V}} + \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\omega(\underline{\mathrm{V}})\zeta - (\frac{\underline{\mathrm{r}}}{2} - 3)g(\underline{\mathrm{V}}, \underline{\mathrm{K}})\zeta \\ &- \quad (\frac{\underline{\mathrm{r}}}{2} - 3)\omega(\underline{\mathrm{V}})\omega(\underline{\mathrm{K}})\zeta - (\frac{\underline{\mathrm{r}}}{2} - 3)[\omega(\underline{\mathrm{V}})\underline{\mathrm{K}} + \omega(\underline{\mathrm{V}})\omega(\underline{\mathrm{K}})\zeta]. \end{aligned} \tag{4.33}$$

By operating φ^2 on both the sides of (4.33), then using (2.1) and (2.2), we arrive at

$$\varphi^2((\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{Q}})\underline{\mathrm{V}}) = \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}[\underline{\mathrm{V}} + \omega(\underline{\mathrm{V}})\zeta] - (\frac{\underline{\mathrm{r}}}{2} - 3)[\omega(\underline{\mathrm{V}})(\underline{\mathrm{K}} + \omega(\underline{\mathrm{K}})\zeta)]. \tag{4.34}$$

Since $\underline{\mathrm{r}} = 6$, therefore, from (4.34) it follows that

$$\varphi^2((\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{Q}})\underline{\mathrm{V}}) = 0. \tag{4.35}$$

Hence, this completes the proof. □

Corollary 4.1. *An (LP-K)$_3$ is locally φ-RS, if and only if $\underline{\mathrm{r}}$ is constant.*

Proof. By taking $\underline{\mathrm{V}}$ as orthogonal to ζ, then (4.34) provides

$$\varphi^2((\bar{\nabla}_{\underline{\mathrm{K}}}\underline{\mathrm{Q}})\underline{\mathrm{V}}) = \frac{d\underline{\mathrm{r}}(\underline{\mathrm{K}})}{2}\underline{\mathrm{V}}. \tag{4.36}$$

The result follows from (4.36) and Theorem 4.1. □

5. Weakly φ-RS $(\text{LP-K})_m$

Definition 5.1. *An $(LP\text{-}K)_m$ is called weakly φ-RS if its Ricci operator Q satisfies*

$$\varphi^2((\bar{\nabla}_{\underline{U}}Q)(\underline{V})) = A(\underline{U})\varphi^2(Q(\underline{V})) + B(\underline{V})\varphi^2(Q(\underline{U})) + \mathcal{S}(\underline{V}, \underline{U})\varphi^2(\rho), \tag{5.37}$$

where $\underline{U}, \underline{V} \in (LP\text{-}K)_m$. A, B, D are 1-forms and ρ is a vector field associated with 1-form D, i.e., $g(\rho, \underline{Z}) = D(\underline{Z})$.

If the 1-forms $A = B = D = 0$, then the relation (5.37) reduces to the concept of φ-RS given by

$$\varphi^2((\nabla_{\underline{U}}Q)(\underline{V})) = 0. \tag{5.38}$$

This concept was initiated by Shukla and Shukla [21].

Now, we consider an $(\text{LP-K})_m$, which is weakly φ Ricci symmetric. Consequently, the relation (5.37) together with (2.2) gives

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}Q)(\underline{V}) + \omega((\bar{\nabla}_{\underline{U}}Q)(\underline{V}))\zeta &= A(\underline{U})[Q\underline{V} + \omega(Q\underline{V})\zeta] + B(\underline{V})[Q\underline{U} + \omega(Q\underline{U})\zeta] \\ &\quad + \mathcal{S}(\underline{V}, \underline{U})[\rho + \omega(\rho)\zeta],\end{aligned}$$

which can be written as

$$\begin{aligned}&\bar{\nabla}_{\underline{U}}(Q\underline{V}) - Q(\bar{\nabla}_{\underline{U}}\underline{V}) + \omega(\bar{\nabla}_{\underline{U}}(Q\underline{V}) - Q(\nabla_{\underline{U}}\underline{V}))\zeta = A(\underline{U})Q\underline{V} \\ &\quad + A(\underline{U})\omega(Q\underline{V})\zeta + B(\underline{V})[Q\underline{U} + \omega(Q\underline{U})\zeta] + \mathcal{S}(\underline{V}, \underline{U})\rho + \mathcal{S}(\underline{V}, \underline{U})\omega(\rho)\zeta.\end{aligned} \tag{5.39}$$

Taking the inner product of (5.39) with $\underline{Z}$ and using (2.2), we have

$$\begin{aligned}&g(\bar{\nabla}_{\underline{U}}(Q\underline{V}), \underline{Z}) - g(Q(\nabla_{\underline{U}}\underline{V}), \underline{Z}) + \omega(\bar{\nabla}_{\underline{U}}(Q\underline{V}) - Q(\nabla_{\underline{U}}\underline{V}))\omega(\underline{Z}) \\ &= A(\underline{U})g(Q\underline{V}, \underline{Z}) + A(\underline{U})\omega(Q\underline{V})\omega(\underline{Z}) + B(\underline{V})[g(Q\underline{U}, \underline{Z}) \\ &+ \omega(Q\underline{U})\omega(\underline{Z})] + \mathcal{S}(\underline{V}, \underline{U})D(\underline{Z}) + \mathcal{S}(\underline{V}, \underline{U})\omega(\rho)\omega(\underline{Z}),\end{aligned} \tag{5.40}$$

where $g(\rho, \underline{Z}) = D(\underline{Z})$.

Setting $\underline{V} = \zeta$ in (5.40), it yields

$$\begin{aligned}&g(\bar{\nabla}_{\underline{U}}(Q\zeta), \underline{Z}) - g(Q(\bar{\nabla}_{\underline{U}}\zeta), \underline{Z}) + \omega(\bar{\nabla}_{\underline{U}}(Q\zeta) - (Q\bar{\nabla}_{\underline{U}}\zeta))\omega(\underline{Z}) \\ &= A(\underline{U})g(Q\zeta, \underline{Z}) + A(\underline{U})\omega(Q\zeta)\omega(\underline{Z}) + B(\zeta)[g(Q\underline{U}, \underline{Z}) \\ &+ \omega(Q\underline{U})\omega(\underline{Z})] + \mathcal{S}(\zeta, \underline{U})D(\underline{Z}) + \mathcal{S}(\zeta, \underline{U})\omega(\rho)\omega(\underline{Z}).\end{aligned} \tag{5.41}$$

By using (2.5) and (2.10) in (5.41), it gives

$$\mathcal{S}(\underline{U},\underline{Z})[1-B(\zeta)] = (m-1)[g(\underline{U},\underline{Z})+\omega(\underline{U})D(\underline{Z})] + (m-1)[B(\zeta)+\omega(\rho)]\omega(\underline{U})\omega(\underline{Z}). \tag{5.42}$$

Applying $\underline{U}\longrightarrow\varphi\underline{U}$ and $\underline{Z}\longrightarrow\varphi\underline{Z}$ in (5.42), then using relation (2.1), (2.3) and (2.12), we lead to

$$[1-B(\zeta)]\mathcal{S}(\underline{U},\underline{Z})+(m-1)[1-B(\zeta)]\omega(\underline{U})\omega(\underline{Z}) = (m-1)[g(\underline{U},\underline{Z})+\omega(\underline{U})\omega(\underline{Z})],$$

which is of the form

$$\mathcal{S}(\underline{U},\underline{Z}) = \mu g(\underline{U},\underline{Z})+\nu\omega(\underline{U})\omega(\underline{Z}), \tag{5.43}$$

where $\mu=\dfrac{(m-1)}{1-B(\zeta)}$ and $\nu=\dfrac{(m-1)B(\zeta)}{1-B(\zeta)}$, provided, $1-B(\zeta)\neq 0$. Thus, we state the following theorem:

Theorem 5.1. *A weakly φ-RS $(LP\text{-}K)_m$ is an ω-Einstein manifold.*

6. $(\text{LP-K})_m$ admitting ω-Parallel Ricci tensor

Definition 6.1. *The Ricci tensor of an $(LP\text{-}K)_m$ is said to be ω-parallel if it satisfies*

$$(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) = 0, \tag{6.44}$$

for all vector fields $\underline{U}$, $\underline{V}$, $\underline{Z}$ on $(LP\text{-}K)_m$.

Let the Ricci tensor of an $(\text{LP-K})_m$ be ω-parallel, therefore (6.44) holds. By the covariant differentiation of $\mathcal{S}(\varphi\underline{V},\varphi\underline{Z})$ along $\underline{U}$, we have

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) &= \bar{\nabla}_{\underline{U}}(\mathcal{S}(\varphi\underline{V},\varphi\underline{Z}))-\mathcal{S}((\bar{\nabla}_{\underline{U}}\varphi)\underline{V},\varphi\underline{Z})\\ &- \mathcal{S}(\varphi(\bar{\nabla}_{\underline{U}}\underline{V}),\varphi\underline{Z})-\mathcal{S}(\varphi\underline{V},(\bar{\nabla}_{\underline{U}}\varphi)\underline{Z})-\mathcal{S}(\varphi\underline{V},\varphi(\bar{\nabla}_{\underline{U}}\underline{Z})),\end{aligned}$$

which by virtue of (2.12) takes the form

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) &= (\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{V},\underline{Z})+\mathcal{S}(\bar{\nabla}_{\underline{U}}\underline{V},\underline{Z})+\mathcal{S}(\underline{V},\bar{\nabla}_{\underline{U}}\underline{Z})\\ &+ (n-1)[(\bar{\nabla}_{\underline{U}}\omega)(\underline{V})\omega(\underline{Z})+\omega(\bar{\nabla}_{\underline{U}}\underline{V})\omega(\underline{Z})\\ &+ \omega(\underline{V})(\bar{\nabla}_{\underline{U}}\omega)(\underline{Z})+\omega(\underline{V})\omega(\bar{\nabla}_{\underline{U}}\underline{Z})]-\mathcal{S}((\bar{\nabla}_{\underline{U}}\varphi)\underline{V},\varphi\underline{Z})\\ &- \mathcal{S}(\varphi(\bar{\nabla}_{\underline{U}}\underline{V}),\varphi\underline{Z})-\mathcal{S}(\varphi\underline{V},(\bar{\nabla}_{\underline{U}}\varphi)\underline{Z})-\mathcal{S}(\varphi\underline{V},\varphi(\bar{\nabla}_{\underline{U}}\underline{Z})).\end{aligned}$$

In view of (2.4), (2.6), (2.10) and (2.12) the foregoing equation turns to

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\varphi\underline{V},\varphi\underline{Z}) &= (\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{V},\underline{Z}) - (n-1)g(\underline{U},\underline{V})\omega(\underline{Z}) \\ &- (n-1)g(\underline{U},\underline{Z})\omega(\underline{V}) + \mathcal{S}(\underline{U},\underline{Z})\omega(\underline{V}) + \mathcal{S}(\underline{U},\underline{V})\omega(\underline{Z}),\end{aligned}$$

which by virtue of (6.44) gives

$$\begin{aligned}(\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{V},\underline{Z}) &= (n-1)[g(\underline{U},\underline{V})\omega(\underline{Z}) + g(\underline{U},\underline{Z})\omega(\underline{V})] \\ &\quad -[\mathcal{S}(\underline{U},\underline{Z})\omega(\underline{V}) + \mathcal{S}(\underline{U},\underline{V})\omega(\underline{Z})].\end{aligned} \tag{6.45}$$

Let $\{\underline{l}_1, \underline{l}_2, \underline{l}_3......., \underline{l}_m\}$ be the orthonormal basis of the tangent space at every point of (LP-K)$_m$. Now setting $\underline{V} = \underline{Z} = \underline{l}_i$ as well as proceeding for sum from $i = 1$ to m in equation (6.45), it provides

$$\begin{aligned}\sum_{i=1}^{m}\epsilon_i(\bar{\nabla}_{\underline{U}}\mathcal{S})(\underline{l}_i,\underline{l}_i) &= (n-1)\sum_{i=1}^{m}\epsilon_i[g(\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta) + g(\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta)] \\ &- \sum_{i=1}^{m}\epsilon_i[g(Q\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta) + g(Q\underline{U},\underline{l}_i)g(\underline{l}_i,\zeta)],\end{aligned} \tag{6.46}$$

where $\epsilon_i = g(e_1, e_i)$. From (6.46) it follows that

$$dr(\underline{U}) = 0. \tag{6.47}$$

Thus, we conclude that $dr = 0$, i.e., r is constant and it is given by $r = m(m-1)$. Moreover, since $\mathcal{S}(\underline{U},\underline{V}) = g(Q\underline{U},\underline{V})$, then we obtain

$$\nabla_U|Q|^2 = 2\sum_{i=1}^{n}\epsilon_i g((\bar{\nabla}_{\underline{U}}Q)e_i, Qe_i). \tag{6.48}$$

By using (6.45) in above equation, we find

$$\nabla_{\underline{U}}|Q|^2 = 2\sum_{i=1}^{n}\epsilon_i g((\bar{\nabla}_{\underline{U}}Q)e_i, Qe_i) = 0. \tag{6.49}$$

This implies that

$$|Q|^2 = constant, \tag{6.50}$$

where Q is the Ricci operator. Hence, the relations (6.47) and (6.50) lead to the following result:

Theorem 6.1. *The scalar curvature of an (LP-K)$_{m>3}$ with the ω-parallel Ricci tensor is constant. Moreover, the norm of the Ricci operator is also constant.*

7. Illustration

We take a 3-dimensional smooth manifold $\mathcal{M}^3 = \{(\underline{u}, \underline{v}, \underline{w}) \in \mathbb{R}^3 : \underline{w} > 0)\}$, where $(\underline{u}, \underline{v}, \underline{w})$ denotes the basic coordinates on a 3-dimensional real space $\mathbb{R}^3$. Consider the vector fields $\{\underline{l}_1, \underline{l}_2, \underline{l}_3\}$, which is linearly independent on $\mathcal{M}^3$ and defined as

$$\underline{l}_1 = (sinh\underline{w} + cosh\underline{w})\frac{\partial}{\partial \underline{u}}, \quad \underline{l}_2 = (sinh\underline{w} + cosh\underline{w})\frac{\partial}{\partial \underline{v}}, \quad \underline{l}_3 = \frac{\partial}{\partial \underline{w}} = \zeta.$$

We define the Lorentz metric g on $\mathcal{M}^3$ as:

$$g_{pq} = g(\underline{l}_p, \underline{l}_q) = \begin{cases} -1 & \text{for } p = q = 3, \\ 0 & \text{for } p \neq q, \\ 1 & p = q = 1, 2. \end{cases}$$

Assume ω be a 1-form corresponding to the Lorentz metric g such that

$$\omega(\underline{U}) = g(\underline{U}, \underline{l}_3)$$

for any $\underline{U} \in \mathfrak{X}(\mathcal{M}^3)$, where $\mathfrak{X}(\mathcal{M}^3)$, denotes the collection of all smooth vector fields on $\mathcal{M}^3$. We define φ as follows

$$\varphi(\underline{l}_1) = \underline{l}_2, \quad \varphi(\underline{l}_2) = \underline{l}_1, \quad \varphi(\underline{l}_3) = 0.$$

Since φ and g have linear nature, so it can be easily proved the following results:

$$\omega(\underline{l}_3) + 1 = 0, \qquad \varphi^2(\underline{U}) - \underline{U} - \omega(\underline{U})\underline{l}_3 = 0, \qquad g(\varphi\underline{U}, \varphi\underline{V}) - g(\underline{U}, \underline{V}) - \omega(\underline{U})\omega(\underline{V}) = 0$$

for all $\underline{U}, \underline{V} \in \mathfrak{X}(\mathcal{M}^3)$. This implies that for $\underline{l}_3 = \zeta$, the structure $(\varphi, \zeta, \omega, g)$ defines a Lorentzian paracontact structure and $(\mathcal{M}^3, \varphi, \zeta, \omega, g)$ is a Lorentzian paracontact manifold of dimension 3. The non-zero constituents of the Lie bracket are given as

$$[\underline{l}_3, \underline{l}_p] = \begin{cases} \underline{l}_p, p = 1, 2, \\ 0, \text{otherwise}. \end{cases}$$

The well-known Koszul's formula provides

$$\bar{\nabla}_{\underline{l}_p}\underline{l}_q = \begin{cases} -\underline{l}_3, p = q = 1, 2, \\ -\underline{l}_p, p = 1, 2, q = 3, \\ 0, \text{otherwise}. \end{cases}$$

From the above equations, it can be easily verified that $\bar{\nabla}_{\underline{U}}\underline{l}_3 = -\{\underline{U} + \omega(\underline{U})\underline{l}_3\}$ and $(\bar{\nabla}_{\underline{U}}\varphi)\underline{V} = -g(\varphi\underline{U}, \underline{V})\zeta - \omega(\underline{V})\varphi\underline{U}$ holds for each $\underline{U}, \underline{V} \in \mathfrak{X}(\mathcal{M}^3)$. Hence the Lorentzian paracontact manifold is an $(\text{LP-K})_3$. From the above equations, the non-zero constituents of $\underline{\mathrm{R}}$ are evaluated as follows

$$\underline{\mathrm{R}}(\underline{\mathrm{l}}_2,\underline{\mathrm{l}}_1)\underline{\mathrm{l}}_2 = -\underline{\mathrm{l}}_1, \quad \underline{\mathrm{R}}(\underline{\mathrm{l}}_2,\underline{\mathrm{l}}_3)\underline{\mathrm{l}}_2 = -\underline{\mathrm{l}}_3, \quad \underline{\mathrm{R}}(\underline{\mathrm{l}}_3,\underline{\mathrm{l}}_1)\underline{\mathrm{l}}_3 = \underline{\mathrm{l}}_1,$$

$$\underline{\mathrm{R}}(\underline{\mathrm{l}}_2,\underline{\mathrm{l}}_3)\underline{\mathrm{l}}_3 = -\underline{\mathrm{l}}_2, \quad \underline{\mathrm{R}}(\underline{\mathrm{l}}_2,\underline{\mathrm{l}}_1)\underline{\mathrm{l}}_1 = \underline{\mathrm{l}}_2, \quad \underline{\mathrm{R}}(\underline{\mathrm{l}}_1,\underline{\mathrm{l}}_3)\underline{\mathrm{l}}_1 = -\underline{\mathrm{l}}_3.$$

Thus we have

$$\underline{\mathrm{R}}(\underline{\mathrm{U}},\underline{\mathrm{V}})\underline{\mathrm{Z}} = -g(\underline{\mathrm{U}},\underline{\mathrm{Z}})\underline{\mathrm{V}} + g(\underline{\mathrm{V}},\underline{\mathrm{Z}})\underline{\mathrm{U}}, \tag{7.51}$$

which is a space of constant curvature 1.

The matrix representation of $\mathcal{S}$ is given by

$$\mathcal{S} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Thus we find $\underline{\mathrm{r}} = 6$. From (7.51) it follows that $\mathcal{S}(\underline{\mathrm{U}},\underline{\mathrm{V}}) = 2g(\underline{\mathrm{U}},\underline{\mathrm{V}}) \implies \mathrm{Q}\underline{\mathrm{U}} = 2\underline{\mathrm{U}}$, which implies that $\varphi^2((\bar{\nabla}_{\underline{\mathrm{W}}}\mathrm{Q})\underline{\mathrm{U}}) = 0$. As we see that $\mathcal{M}^3$ is φ-RS with the scalar curvature 6. Thus this illustration proves Theorem 4.1. Since $\mathcal{M}^3$ is φ-RS and Einstein, this illustration also admits Theorem 3.4 for three dimensional case.

Acknowledgments. The authors would like to thank the referees for some useful comments and their helpful suggestions that have improved the quality of this paper.

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