**International Journal of Maps in Mathematics**
*Volume 7, Issue 1, 2024, Pages:2-19*
E-ISSN: 2636-7467
**www.journalmim.com**

# A NEW PARAMETRIZATION OF CARTAN NULL BERTRAND CURVE IN MINKOWSKI 3-SPACE

STUTI TAMTA AND RAM SHANKAR GUPTA *

*Dedicated to the memory of the late Professor Krishan Lal Duggal(1929-2022)*

Abstract. We define and study a new parametrization of a Bertrand pair $\{\alpha, \alpha^*\}$, where $\alpha$ is a Cartan null Bertrand curve and $\alpha^*$ is a Bertrand partner curve of $\alpha$ in Minkowski 3-space by not taking the principal normal vector of the Cartan null Bertrand curve $\alpha$ parallel to $\overrightarrow{\alpha^* \alpha}$. We characterize both cases when the curve $\alpha^*$ is non-null and the null Bertrand partner of the curve $\alpha$. Further, we investigate this type of Bertrand pair curve as a helix and a slant helix. Also, we provide some examples.

**Keywords**: Bertrand curves, general helices, slant helices, Cartan null curve, non-null curve, Minkowski 3-space.

**2020 Mathematics Subject Classification**: 53B30.

## 1. Introduction

In 1802, Lancret [14] defined a helix as a curve whose tangent vector makes a constant angle with a fixed straight line called the directrix. Later in 1845, Saint Venant [16] obtained a necessary and sufficient condition for a curve to be a general helix if its ratio of curvature to torsion is constant. In 1995, Scofield studied closed-form arc-length parametrizations for curves of constant precession and slant helices with a constant speed of precession [17]. In 2004, Izumiya and Takeuchi introduced the concept of the slant helix in $\mathbb{E}^3$ saying that the principal normal lines make a constant angle with a fixed direction. They characterized a

*Received:2023.05.10* *Revised: 2023.09.25* *Accepted:2023.11.23*

* Corresponding author

Stuti Tamta ⋄ stutitams@gmail.com ⋄ https://orcid.org/0000-0002-9432-1037

Ram Shankar Gupta ⋄ ramshankar.gupta@gmail.com ⋄ https://orcid.org/0000-0003-0985-3280.

curve as a slant helix if and only if the principal image of the major normal indicatrix has a constant geodesic curvature [10].

In 2010, Kula et al. studied the relationship between slant helices and helices, and they characterized slant helices in $\mathbb{E}^3$ in terms of differential equations [13]. In 2011, Ali and Lopez [1] characterized a non-null spacelike and timelike curve with a spacelike principal normal vector to be a slant helix in $\mathbb{E}_1^3$ if and only if either one of the two functions

$$\left(\frac{\tau}{k}\right)' \frac{k^2}{(k^2-\tau^2)^{3/2}} \quad or \quad \left(\frac{\tau}{k}\right)' \frac{k^2}{(\tau^2-k^2)^{3/2}} \tag{1.1}$$

is a constant function and $\tau^2 - k^2 \neq 0$.

In 2019, Liu and Pei [15] characterized a null Cartan curve $\alpha$ to be a slant helix in $\mathbb{E}_1^3$ if and only if the principal image of the major normal indicatrix has a constant geodesic curvature $k_g$, i.e.,

$$k_g = \frac{\tau'(s)}{2\sqrt{2}|\tau(s)|^{3/2}} \tag{1.2}$$

is a constant function for a non-zero torsion $\tau(s)$ of the curve.

In the fields of computer-aided design and computer graphics, helices can be used for tool path description, the simulation of kinematic motion, the design of highways, etc. [21]. Also, helix and slant helix play an important role in curve theory with numerous applications in the biological sciences, physics, etc. For instance, in the biological sciences, curves are used in the analysis of Deoxyribonucleic Acid (DNA), and in physics, they are used in characterizing the motion of particles in a magnetic field.

In 1845, Saint Venant [16] posed the question of whether the principal normal of a curve is the principal normal of another curve on the surface generated by the principal normal of the given one. Bertrand [4] gave an answer to this question in 1850 and introduced curves with the property that the principal normal vector of a curve $\alpha$ coincides with the principal normal vector of another curve $\alpha^*$ at their corresponding points. Further, these curves were characterized in $\mathbb{E}^3$ with condition $ak + b\tau = 1$, where $a$ and $b$ are nonzero constants and $k$ and $\tau$ are the curvature and torsion of the curve, respectively [7]. Also, Bertrand curves and their characterizations were studied by many researchers in Minkowski 3-space (see [2, 3, 9, 11, 20]). In [3], Balgetir et al. studied the Cartan null Bertrand pair curve $\{\alpha, \alpha^*\}$ in $\mathbb{E}_1^3$. Later in 2021, Gokcek and Erdem [8] studied the Cartan null Bertrand curve $\alpha$ with the non-null Bertrand partner curve $\alpha^*$ in $\mathbb{E}_1^3$. In [6], Camci, et al. introduced a new relationship between a Bertrand pair $\alpha$ and $\alpha^*$ in $\mathbb{E}^3$ by not taking the vector $\overrightarrow{\alpha^*\alpha}$ parallel to a normal vector of Bertrand curve $\alpha$. Using this approach, the present authors studied a new parametrization of Bertrand partner curves and spherical indicatrices in Euclidean 3-space [18, 19].

In view of this, we define and study a new parametrization of a Bertrand pair $\{\alpha,\, \alpha^*\}$, where $\alpha$ is a Cartan null Bertrand curve and $\alpha^*$ is a Bertrand partner curve of $\alpha$ in Minkowski 3-space by not taking the vector $\overrightarrow{\alpha^*\,\alpha}$ parallel to $N$ of $\alpha$ in Minkowski 3-space.

## 2. Preliminaries

The Lorentz-Minkowski $\mathbb{E}^3_1$ is a space with metric,

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2,$$

where $(x_1, x_2, x_3)$ is a rectangular coordinate system. With respect to this metric, an arbitrary vector $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ is said to be spacelike if $\langle \alpha, \alpha \rangle > 0$, timelike if $\langle \alpha, \alpha \rangle < 0$, and null if $\langle \alpha, \alpha \rangle = 0$. Similarly, if $\alpha = \alpha(s)$ denotes the position vector of an arbitrary non-null curve in $\mathbb{E}^3_1$, then it is called timelike and spacelike if all of its velocity vectors $\alpha'(s)$ are timelike and spacelike, respectively. The norm of the vector $\alpha$ is given by $||\alpha'|| = \sqrt{|\langle \alpha', \alpha' \rangle|}$. A non-null curve $\alpha(s)$ is parameterized by arc length $s$ if $\langle \alpha'(s), \alpha'(s) \rangle = \pm 1$. A null curve is parameterized by pseudo-arc $s$ if $\langle \alpha''(s),\, \alpha''(s) \rangle = 1$. If a null curve is parameterized by a pseudo-arc function, it is referred to as a Cartan null curve.

Let $\{T,\, N,\, B\}$ be the moving Frenet frame along a curve in $\mathbb{E}^3_1$, consisting of the tangent, the principal normal, and the binormal vector field, respectively. Depending on the causal character of $\alpha$, the Frenet equations have the following forms:

**Case I.** If $\alpha$ is a non-null curve, the Frenet formulas are [12]:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \epsilon_1 k_1 & 0 \\ -\epsilon_0 k_1 & 0 & \epsilon_2\, k_2 \\ 0 & -\epsilon_1 k_2 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.3}$$

where $\langle T, T \rangle = \epsilon_0$, $\langle N, N \rangle = \epsilon_1$, $\langle B, B \rangle = \epsilon_2$, and $\epsilon_0, \epsilon_1, \epsilon_2 \in \{-1, 1\}$, and $k(s)$, $\tau(s)$ are curvature and torsion of $\alpha$.

**Case II.** If $\alpha$ is a Cartan null curve, the Frenet formulas are [5]:

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k_1 & 0 \\ k_2 & 0 & -k_1 \\ 0 & -k_2 & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{2.4}$$

where $\langle T, B\rangle = \langle N, N\rangle = 1$, and $\langle T, T\rangle = \langle B, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0$, and $k_1(s)$, $k_2(s)$ are curvature and torsion of $\alpha$.

In [3, 8], the authors defined the Cartan null Bertrand curve $\alpha : I \to \mathbb{E}^3_1$ with Bertrand partner curve $\alpha^* : I^* \to \mathbb{E}^3_1$ as follows:

$$\alpha^*(s^*) = \alpha(s) + \lambda(s)\, N(s), \tag{2.5}$$

such that the principal normal vectors of $\alpha(s)$ and $\alpha^*(s^*)$ coincides at $s \in I$, $s^* \in I^*$, where $\lambda(s)$ is $C^\infty$-function on $I$.

Now, we define a new parametrization of a Bertrand pair $\{\alpha, \alpha^*\}$, where $\alpha$ is a Cartan null curve and $\alpha^*$ is a Bertrand partner curve of $\alpha$ in $\mathbb{E}^3_1$ such that the vector $\overrightarrow{\alpha\,\alpha^*}$ does not have to be parallel to $N$, which is given by

$$\alpha^*(s^*) = \alpha(s) + u(s)\, T(s) + v(s)\, N(s) + w(s)\, B(s), \tag{2.6}$$

where $u(s)$, $v(s)$ and $w(s)$ are differentiable functions and $\{T(s), N(s), B(s)\}$ is the Frenet-Serret frame of $\alpha(s)$. If we take $u = w = 0$ in (2.6), we obtain (2.5). Hence, (2.6) is the generalization of Cartan null Bertrand curves in $\mathbb{E}^3_1$.

## 3. New parametrization of Cartan null Bertrand curve in $\mathbb{E}^3_1$

In this section, we study a pair curve $\{\alpha, \alpha^*\}$ in $\mathbb{E}^3_1$ satisfying (2.6), where $\alpha$ is a Cartan null curve with curvature $k_1$ and torsion $k_2$, and $\alpha^*$ is a Bertrand partner curve of $\alpha$ with curvature $k_1^*$ and torsion $k_2^*$.

Now onwards, we denote the geodesic curvatures of the principal normal indicatrices (images) of a Cartan null Bertrand curve $\alpha$ by $\Gamma$, and that of a timelike and spacelike Bertrand partner curve $\alpha^*$ by $\Gamma_1^*$ and $\Gamma_2^*$, respectively.

Also, we set

$$\mu = \frac{(1 + u' + v\, k_2)}{\frac{ds^*}{ds}}, \quad \nu = \frac{(w' - v\, k_1)}{\frac{ds^*}{ds}}, \quad h = \frac{\mu}{\nu}, \tag{3.7}$$

and

$$\begin{cases} \beta = \dfrac{(\mu\, k_1 - \nu\, k_2)}{k_1^*\, \frac{ds^*}{ds}} = \pm 1, & \rho(s) = \dfrac{(w' - v\, k_1)\,(h^2\, k_1^2 - k_2^2)}{2\, k_1^* \left(\frac{ds^*}{ds}\right)^2}, \\ \eta(s) = -\dfrac{(w' - v\, k_1)\,(h^2\, k_1^2 - k_2^2)}{2\, h\, k_1^* \left(\frac{ds^*}{ds}\right)^2}, & \rho(s) = -h\, \eta(s). \end{cases} \tag{3.8}$$

Next, we have:

**Theorem 3.1.** *Let $\alpha : I \to \mathbb{E}_1^3$ be a Cartan null curve in $\mathbb{E}_1^3$ with curvatures $k_1(s) \neq 0$ and $k_2(s)$ satisfying (2.6).*

*(i) If $\alpha^*$ is a timelike curve with $k_1^* \neq 0$, then $\{\alpha, \alpha^*\}$ is a Bertrand pair in $\mathbb{E}_1^3$ if and only if there exist differentiable functions $u$, $v$, $w$, and a real number $h$ satisfying*

$$v' + u\,k_1 - w\,k_2 = 0, \quad h < 0, \quad \nu^2 = -\frac{1}{2\,h}, \quad h\,k_1 - k_2 \neq 0. \tag{3.9}$$

*(ii) If $\alpha^*$ is a spacelike curve with $k_1^* \neq 0$ and having a spacelike principal normal vector, then $\{\alpha, \alpha^*\}$ is a Bertrand pair in $\mathbb{E}_1^3$ if and only if there exist differentiable functions $u$, $v$, $w$, and a real number $h$ satisfying*

$$v' + u\,k_1 - w\,k_2 = 0, \quad h > 0, \quad \nu^2 = \frac{1}{2\,h}, \quad h\,k_1 - k_2 \neq 0. \tag{3.10}$$

*Further, in both the cases (i) and (ii), if* $\begin{cases} k_2^* \neq 0, & \textit{then} \quad h\,k_1 + k_2 \neq 0, \\ k_2^* = 0, & \textit{then} \quad h\,k_1 + k_2 = 0. \end{cases}$

*Proof.* (*i*) Let $\{\alpha, \alpha^*\}$ be a Bertrand pair in $\mathbb{E}_1^3$ satisfying (2.6) such that $\alpha$ is a Cartan null curve and $\alpha^*$ is a timelike curve. Differentiating (2.6) with respect to $s$ and then using (2.3) and (2.4), we obtain

$$T^*\,\frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (v' + u\,k_1 - w\,k_2)\,N + (w' - v\,k_1)\,B. \tag{3.11}$$

Taking the inner product of (3.11) with $N$, we get

$$v' + u\,k_1 - w\,k_2 = 0. \tag{3.12}$$

Using (3.12) in (3.11), we obtain

$$T^*\,\frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (w' - v\,k_1)\,B. \tag{3.13}$$

Using (3.7) in (3.13), we have

$$T^* = \mu\,T + \nu\,B. \tag{3.14}$$

If we take the inner product of the equation (3.14) first with $T$ and then with $N$, the following results are obtained

$$-1 = 2\,\mu\,\nu, \tag{3.15}$$

which gives $\mu \neq 0$ and $\nu \neq 0$. Consequently, from (3.7), we find that $1 + u' + v\,k_2 \neq 0$ and $w' - v\,k_1 \neq 0$.

Using the third relation of (3.7) in (3.15), we get

$$2\,h\,\nu^2 = -1, \tag{3.16}$$

which gives $h < 0$.

Now, differentiating (3.14) with respect to $s$ and then using (2.3) and (2.4), we obtain

$$k_1^*\,N^*\,\frac{ds^*}{ds} = \mu'\,T + (\mu\,k_1 - \nu\,k_2)\,N + \nu'\,B. \tag{3.17}$$

Taking the inner product of (3.17) with $T$ and $B$, we find

$$\nu' = 0, \quad \mu' = 0. \tag{3.18}$$

Using (3.18) in (3.17), we obtain

$$k_1^*\,N^*\,\frac{ds^*}{ds} = (\mu\,k_1 - \nu\,k_2)\,N. \tag{3.19}$$

Now, taking the inner product of (3.19) with itself and using (3.16) and the third relation of (3.7), we get

$$(k_1^*)^2\left(\frac{ds^*}{ds}\right)^2 = -\frac{1}{2\,h}\,(h\,k_1 - k_2)^2, \tag{3.20}$$

which gives $(h\,k_1 - k_2) \neq 0$. Now, using the first relation of (3.8), we have

$$N^* = \beta\,N. \tag{3.21}$$

Differentiating (3.21) with respect to $s$ and then using (2.3) and (2.4), we obtain

$$k_2^*\,B^*\,\frac{ds^*}{ds} = \beta\,(k_2\,T - k_1\,B) - \,k_1^*\,T^*\,\frac{ds^*}{ds}. \tag{3.22}$$

Using (3.13) in (3.22), we get

$$k_2^*\,B^*\,\frac{ds^*}{ds} = \big(\beta\,k_2 - k_1^*\,(1 + u' + v\,k_2)\big)\,T - \big(\beta\,k_1 + k_1^*\,(w' - v\,k_1)\big)\,B. \tag{3.23}$$

Using the first relation of (3.8) in (3.23), we have

$$\begin{aligned} k_2^*\,B^*\,\frac{ds^*}{ds} = {} & \Big(\frac{\nu\,k_2\,(h\,k_1 - k_2)}{k_1^*\,\frac{ds^*}{ds}} - k_1^*\,(1 + u' + v\,k_2)\Big)\,T - \Big(\frac{\nu\,k_1\,(h\,k_1 - k_2)}{k_1^*\,\frac{ds^*}{ds}} \\ & + k_1^*\,(w' - v\,k_1)\Big)\,B. \end{aligned} \tag{3.24}$$

Now, using (3.7) in (3.24), we find

$$\begin{aligned} k_2^*\,B^*\,\frac{ds^*}{ds} = (w' - v\,k_1)\,\Big(\Big(&\frac{k_2\,(h\,k_1 - k_2) - k_1^{*2}\,h\,(\frac{ds^*}{ds})^2}{k_1^*\,(\frac{ds^*}{ds})^2}\Big)\,T \\ &- \Big(\frac{k_1\,(h\,k_1 - k_2) + k_1^{*2}\,(\frac{ds^*}{ds})^2}{k_1^*\,(\frac{ds^*}{ds})^2}\Big)\,B\Big). \end{aligned} \tag{3.25}$$

Using (3.20), and the second and third relations of (3.8) in (3.25), we obtain

$$k_2^* \, B^* \, \frac{ds^*}{ds} = \rho(s) \, T(s) + \eta(s) \, B(s). \tag{3.26}$$

Taking the inner product of (3.26) with itself, we get

$$k_2^{*2} \left(\frac{ds^*}{ds}\right)^2 = 2 \, \rho(s) \, \eta(s) = -2 \, h \, \eta(s)^2. \tag{3.27}$$

From (3.27), depending upon $k_2^* = 0$ or $k_2^* \neq 0$, we find that $h \, k_1 + k_2 = 0$ or $h \, k_1 + k_2 \neq 0$.

Conversely, let $\alpha$ be a Cartan null curve with curvatures $k_1 \neq 0$ and $k_2$ in $\mathbb{E}_1^3$ satisfying (3.10). Then, we can define the curve $\alpha^*$ as (2.6). Differentiating (2.6) with respect to $s$ and then using (2.4), we obtain

$$T^* = \mu \, T + \nu \, B. \tag{3.28}$$

Using the third relation of (3.7) and the third relation of (3.9) in (3.28), we get

$$T^* = \frac{1}{\sqrt{-2h}} \, (h \, T + B), \quad \langle T^*, \, T^* \rangle = -1. \tag{3.29}$$

Now, differentiating (3.29) with respect to $s$ and then using (2.4), we get

$$\frac{dT^*}{ds} = \frac{1}{\sqrt{-2 \, h}} \, (h \, k_1 - k_2) \, N, \tag{3.30}$$

which gives

$$k_1^* = || \frac{dT^*}{ds^*} || = \frac{\xi_1 \, (h \, k_1 - k_2)}{\sqrt{-2h} \, \frac{ds^*}{ds}}, \tag{3.31}$$

where $\xi_1 = \pm 1$. Now, $N^*$ can be obtained as

$$N^* = \xi_1 \, N, \quad \langle N^*, \, N^* \rangle = 1. \tag{3.32}$$

Differentiating (3.32) with respect to $s$ and using $\epsilon_0 = -1$, $\epsilon_2 = 1$, (2.3) and (2.4), we obtain

$$(k_1^* \, T^* + k_2^* \, B^*) \, \frac{ds^*}{ds} = \xi_1 \, (k_2 \, T - k_1 \, B). \tag{3.33}$$

Taking the inner product of (3.33) with itself, we get

$$(-k_1^{*2} + k_2^{*2}) \left(\frac{ds^*}{ds}\right)^2 = -2k_1 \, k_2. \tag{3.34}$$

Using (3.31) in (3.34), we get

$$k_2^* = \frac{\xi_2 \, (h \, k_1 + k_2)}{\sqrt{-2h} \, \frac{ds^*}{ds}}, \tag{3.35}$$

where $\xi_2 = \pm 1$.

Using (3.29), (3.31), and (3.35) in (3.33), we find

$$B^* = \frac{\xi_1\,\xi_2}{\sqrt{-2\,h}}\,(-h\,T + B), \quad \langle B^*, B^*\rangle = 1. \tag{3.36}$$

Then, $\alpha^*$ is a timelike curve, and the Bertrand partner curve of the null Cartan curve $\alpha$. Thus, $\alpha$ is a Bertrand curve.

*(ii)* Let $\{\alpha,\, \alpha^*\}$ be a Bertrand pair in $\mathbb{E}^3_1$ satisfying (2.6) such that $\alpha$ is a Cartan null curve and $\alpha^*$ is a spacelike curve. Differentiating (2.6) with respect to $s$, and using (2.3) and (2.4), we get

$$T^*\frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (v' + u\,k_1 - w\,k_2)\,N + (w' - v\,k_1)\,B. \tag{3.37}$$

Taking the inner product of (3.37) with $N$, we obtain

$$v' + u\,k_1 - w\,k_2 = 0. \tag{3.38}$$

Using (3.38) in (3.37), we obtain

$$T^*\,\frac{ds^*}{ds} = (1 + u' + v\,k_2)\,T + (w' - v\,k_1)\,B. \tag{3.39}$$

Using (3.7) in (3.39), we have

$$T^* = \mu\,T + \nu\,B. \tag{3.40}$$

Taking the inner product of (3.40) with itself, we find

$$1 = 2\,\mu\,\nu, \tag{3.41}$$

which gives $\nu \neq 0$ and $\mu \neq 0$. Consequently, from (3.7), we find that $1 + u' + v\,k_2 \neq 0$ and $w' - v\,k_1 \neq 0$.

Using the third relation of (3.7) in (3.41), we obtain

$$2\,h\,\nu^2 = 1, \tag{3.42}$$

which gives $h > 0$.

Now, differentiating (3.40) with respect to $s$ and then using (2.3) and (2.4), we get

$$k_1^*\,N^*\,\frac{ds^*}{ds} = \mu'\,T + (\mu\,k_1 - \nu\,k_2)\,N + \nu'\,B. \tag{3.43}$$

If we take the inner product of the equation (3.43) first with $T$ and then with $B$, the following results are obtained

$$\nu' = 0, \quad \mu' = 0. \tag{3.44}$$

Using (3.44) in (3.43), we obtain

$$k_1^*\, N^*\, \frac{ds^*}{ds} = (\mu\, k_1 - \nu\, k_2)\, N. \tag{3.45}$$

Now, taking the inner product of (3.45) with itself and using (3.42) and the third relation of (3.7), we get

$$(k_1^*)^2 \left(\frac{ds^*}{ds}\right)^2 = \frac{1}{2\,h}\,(h\,k_1 - k_2)^2, \tag{3.46}$$

which gives $(h\,k_1 - k_2) \neq 0$. Now, using the first relation of (3.8), we have

$$N^* = \beta\, N. \tag{3.47}$$

Differentiating (3.47) with respect to $s$ and then using (2.3) and (2.4), we get

$$-k_2^*\, B^*\, \frac{ds^*}{ds} = \beta\,(k_2\, T - k_1\, B) + \, k_1^*\, T^*\, \frac{ds^*}{ds}. \tag{3.48}$$

Using (3.39) in (3.48), we get

$$-k_2^*\, B^*\, \frac{ds^*}{ds} = \beta\,(k_2\, T - k_1\, B) + \, k_1^*\, \Big((1 + u' + v\, k_2)\, T + (w' - v\, k_1)\, B\Big). \tag{3.49}$$

Using (3.7), (3.8) and (3.46) in (3.49), we get

$$-k_2^*\, B^*\, \frac{ds^*}{ds} = \rho(s)\, T(s) - \eta(s)\, B(s). \tag{3.50}$$

Taking the inner product of (3.50) with itself, we get

$$k_2^{*2} \left(\frac{ds^*}{ds}\right)^2 = 2\,\rho(s)\,\eta(s) = -2\,h\,\eta(s)^2. \tag{3.51}$$

From (3.51), depending upon $k_2^* = 0$ or $k_2^* \neq 0$, we find that $h\,k_1 + k_2 = 0$ or $h\,k_1 + k_2 \neq 0$.

Conversely, let $\alpha$ be a Cartan null curve in $E_1^3$ with curvatures $k_1 \neq 0$ and $k_2$ satisfying (3.10). Then, we can define the curve $\alpha^*$ as (2.6). Now, differentiating (2.6) with respect to $s$ and then using (2.4), we get

$$T^* = \mu\, T + \nu\, B. \tag{3.52}$$

Using the third relation of (3.7) and the third relation of (3.10) in (3.52), we obtain

$$T^* = \frac{1}{\sqrt{2h}}\,(h\, T + B), \quad \langle T^*, T^* \rangle = 1. \tag{3.53}$$

Now, differentiating (3.53) with respect to $s$ and then using (2.4), we get

$$\frac{dT^*}{ds} = \frac{1}{\sqrt{2\,h}}\,(h\,k_1 - k_2)\, N, \tag{3.54}$$

which gives

$$k_1^* = ||\frac{dT^*}{ds^*}|| = \frac{\xi_3\,(h\,k_1 - k_2)}{\sqrt{2h}\,\frac{ds^*}{ds}}, \tag{3.55}$$

where $\xi_3 = \pm 1$. Now, $N^*$ can be obtained as

$$N^* = \xi_3\,N, \quad \langle N^*,\, N^* \rangle = 1. \tag{3.56}$$

Differentiating (3.56) with respect to $s$ and using $\epsilon_0 = 1$, $\epsilon_2 = -1$, (2.3) and (2.4), we obtain

$$(-k_1^*\,T^* - k_2^*\,B^*)\,\frac{ds^*}{ds} = \xi_3\,(k_2\,T - k_1\,B). \tag{3.57}$$

Taking the inner product of (3.57) with itself, we get

$$(k_1^{*2} - k_2^{*2})\Big(\frac{ds^*}{ds}\Big)^2 = -2k_1\,k_2. \tag{3.58}$$

Using (3.55) in (3.58), we obtain

$$k_2^* = \frac{\xi_4\,(h\,k_1 + k_2)}{\sqrt{2h}\,\frac{ds^*}{ds}}, \tag{3.59}$$

where $\xi_4 = \pm 1$.

Using (3.53), (3.55), and (3.59) in (3.57), we find

$$B^* = \frac{-\xi_3\,\xi_4}{\sqrt{2\,h}}\,(h\,T - B), \quad \langle B^*,\, B^* \rangle = -1. \tag{3.60}$$

Then, $\alpha^*$ is a spacelike curve with a spacelike principal normal vector and the Bertrand partner curve of $\alpha$. As a result, $\alpha$ is a Bertrand curve, and the proof of the Theorem is complete. □

Now, from Theorem 3.1, we have:

**Corollary 3.1.** *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with non-zero curvature $k_1 \neq 0$, $k_2$, and the curve $\alpha^*$ given in (2.6) be a non-null Bertrand partner curve of $\alpha$ with the non zero curvatures $k_1^*$, $k_2^*$. Then $\alpha^*$ is a general helix if and only if $\alpha$ is a general helix.*

*Proof.* Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with the curvatures $k_1$, $k_2$ and the curve $\alpha^*$ is a Bertrand partner curve of $\alpha$.

$(i)$ If $\alpha^*$ is a timelike curve, then from (3.31) and (3.35), we have

$$\frac{k_1^*}{k_2^*} = \xi_1\,\xi_2\,\frac{h\,\frac{k_1}{k_2} - 1}{h\,\frac{k_1}{k_2} + 1}. \tag{3.61}$$

$(ii)$ If $\alpha^*$ is a spacelike curve, then from (3.55) and (3.59), we have

$$\frac{k_1^*}{k_2^*} = \xi_3\,\xi_4\,\frac{h\,\frac{k_1}{k_2} - 1}{h\,\frac{k_1}{k_2} + 1}. \tag{3.62}$$

Combining (3.61) and (3.62), the proof is complete. □

**Corollary 3.2.** *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with non-zero curvatures $k_1 = 1$, $k_2$, and the curve $\alpha^*$ be a non-null Bertrand partner curve of $\alpha$ with the non zero curvatures $k_1^*$, $k_2^*$ satisfying (2.6). Then $\alpha^*$ is a slant helix if and only if $\alpha$ is a slant helix. Moreover, we have*

$$\Gamma_1^* = -\xi_1\,\xi_2\,\Gamma, \quad \Gamma_2^* = \xi_3\,\xi_4\,\Gamma. \tag{3.63}$$

*Proof.* Assume that $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ is a Cartan null Bertrand curve in $\mathbb{E}_1^3$ with curvature $k_1 \neq 0$, $k_2$ and the curve $\alpha^*$ is a Bertrand partner curve of $\alpha$ with the non-zero curvatures $k_1^*$, $k_2^*$ satisfying (2.6). Now, if the curve $\alpha$ is a slant helix, and then for the principal normal vector $N$ of $\alpha$ and a constant vector field $U$, we have

$$\langle N,\, U\rangle = constant. \tag{3.64}$$

Since $N$ is parallel to $N^*$ from (3.64) we find

$$\langle N^*,\, U\rangle = constant, \tag{3.65}$$

which implies $\alpha^*$ is also a slant helix and converse is easy to prove. Further, we have:

$(i)$ If $\alpha^*$ is a timelike curve, then using $k_1^*$ and $k_2^*$ from (3.31) and (3.35) in (1.1), we have

$$\Gamma_1^* = -\xi_1\,\xi_2\frac{k_2'}{2\sqrt{2}\,k_2^{3/2}}. \tag{3.66}$$

$(ii)$ If $\alpha^*$ is a spacelike curve, then using $k_1^*$ and $k_2^*$ from (3.55) and (3.59) in (1.1), we have

$$\Gamma_2^* = \xi_3\,\xi_4\frac{k_2'}{2\sqrt{2}\,k_2^{3/2}}. \tag{3.67}$$

Then, using (1.2), (3.66), and (3.67), we have (3.63). Thus, the proof is complete. □

Now, we have:

**Theorem 3.2.** *Let $\alpha$ and $\alpha^*$ be a Cartan null curves in $\mathbb{E}_1^3$. Then, $\alpha^*$ is a Bertrand partner curve of the Bertrand curve $\alpha$ if*

$(i)$ *there exist differentiable functions $u$, $v$, and $w$ satisfying*

$$v' + u\,k_1 - w\,k_2 = 0, \quad w' - v\,k_1 = 0, \tag{3.68}$$

*and its Cartan null frames are related by*

$$T^* = \mu\, T, \quad N^* = \xi_5\, N, \quad B^* = \frac{1}{\mu}\, B, \tag{3.69}$$

*or*

(*ii*) *there exist differentiable functions* $u$, $v$, *and* $w$ *satisfying*

$$v' + u\, k_1 - w\, k_2 = 0, \quad 1 + u' + v\, k_2 = 0, \tag{3.70}$$

*and its Cartan null frames are related by*

$$T^* = \nu\, B, \quad N^* = -\xi_6\, N, \quad B^* = \frac{1}{\nu}\, T, \tag{3.71}$$

*where* $\xi_5 = \pm 1$, $\xi_6 = \pm 1$.

*Proof.* Let $\alpha$ is a Cartan null Bertrand curve in $\mathbb{E}^3_1$ with $k_1 \neq 0$, $k_2$ and the curve $\alpha^*$ is the Cartan null Bertrand partner curve of the curve $\alpha$ satisfying (2.6). Now, differentiating (2.6) with respect to $s$ and then using (2.4), we get

$$T^* \frac{ds^*}{ds} = (1 + u' + v\, k_2)\, T + (v' + u\, k_1 - w\, k_2)\, N + (w' - v\, k_1)\, B. \tag{3.72}$$

Taking the inner product of (3.72) with $N$, we obtain

$$v' + u\, k_1 - w\, k_2 = 0. \tag{3.73}$$

Using (3.73) in (3.72), we get

$$T^* \frac{ds^*}{ds} = (1 + u' + v\, k_2)\, T + (w' - v\, k_1)\, B. \tag{3.74}$$

Using (3.7) in (3.74), we have

$$T^* = \mu\, T + \nu\, B. \tag{3.75}$$

Taking the inner product of (3.75) with itself, we find

$$0 = 2\, \mu\, \nu. \tag{3.76}$$

Now, we have two cases:

**Case** (*i*) If $\nu = 0$, then we have

$$T^* = \mu T. \tag{3.77}$$

Now, differentiating (3.77) with respect to $s$ and then using (2.4), we get

$$k_1^*\, N^* \frac{ds^*}{ds} = \mu'\, T + \mu\, k_1\, N. \tag{3.78}$$

Taking the inner product of (3.78) with $B$, we find

$$\mu' = 0. \tag{3.79}$$

Using (3.79) in (3.78), we obtain

$$k_1^* \, N^* \, \frac{ds^*}{ds} = \mu \, k_1 \, N. \tag{3.80}$$

Taking the inner product of (3.80) with itself, we get

$$(k_1^*)^2 \left(\frac{ds^*}{ds}\right)^2 = \mu^2 \, k_1^2, \tag{3.81}$$

which gives

$$k_1^* = \frac{\xi_5 \, \mu \, k_1}{\frac{ds^*}{ds}}. \tag{3.82}$$

Using (3.82) in (3.80), we obtain

$$N^* = \xi_5 \, N. \tag{3.83}$$

Differentiating (3.83) with respect to $s$ and then using (2.4), we get

$$(k_2^* \, T^* - k_1^* \, B^*) \, \frac{ds^*}{ds} = \xi_5 \, (k_2 \, T - k_1 \, B). \tag{3.84}$$

Taking the inner product of (3.84) with itself, we get

$$k_1^* \, k_2^* \left(\frac{ds^*}{ds}\right)^2 = k_1 \, k_2. \tag{3.85}$$

Using (3.82) in (3.85), we obtain

$$k_2^* = \frac{\xi_5 \, k_2}{\mu \, \frac{ds^*}{ds}}. \tag{3.86}$$

Using (3.77), (3.82), and (3.86) in (3.84), we get

$$B^* = \frac{1}{\mu} \, B. \tag{3.87}$$

**Case** $(ii)$ If $\mu = 0$, then we have

$$T^* = \nu \, B. \tag{3.88}$$

Now, differentiating (3.88) with respect to $s$ and then using (2.4), we get

$$k_1^* \, N^* \, \frac{ds^*}{ds} = \nu' \, B - \nu \, k_2 \, N. \tag{3.89}$$

Taking the inner product of (3.89) with $T$, we find

$$\nu' = 0. \tag{3.90}$$

Using (3.90) in (3.89), we obtain

$$k_1^* \, N^* \, \frac{ds^*}{ds} = -\nu \, k_2 \, N. \tag{3.91}$$

Now, taking the inner product of (3.91) with itself, we get

$$(k_1^*)^2 \left(\frac{ds^*}{ds}\right)^2 = \nu^2 \, k_2^2, \tag{3.92}$$

which gives

$$k_1^* = \frac{\xi_6 \, \nu \, k_2}{\frac{ds^*}{ds}}. \tag{3.93}$$

Using (3.93) in (3.91), we obtain

$$N^* = -\xi_6 \, N. \tag{3.94}$$

Differentiating (3.94) with respect to $s$ and then using (2.4), we get

$$(k_2^* \, T^* - k_1^* \, B^*) \, \frac{ds^*}{ds} = -\xi_6 \, (k_2 \, T - k_1 \, B). \tag{3.95}$$

Taking the inner product of (3.95) with itself, we get

$$k_1^* \, k_2^* \left(\frac{ds^*}{ds}\right)^2 = k_1 \, k_2. \tag{3.96}$$

Using (3.93) in (3.96), we obtain

$$k_2^* = \frac{\xi_6 \, k_1}{\nu \, \frac{ds^*}{ds}}. \tag{3.97}$$

Using (3.88), (3.93), and (3.97) in (3.95), we get

$$B^* = \frac{1}{\nu} \, T. \tag{3.98}$$

Thus, the proof is complete. □

## 4. Examples

**Example 4.1.** *Let $\alpha(s)$ be a Cartan null curve in $\mathbb{E}_1^3$ given by*

$$\alpha(s) = \Big(\frac{1}{\sqrt{2}} \, \sinh(\sqrt{2}\, s) + \frac{1}{2} \, \cosh(\sqrt{2}\, s), \frac{1}{\sqrt{2}} \, \cosh(\sqrt{2}\, s) + \frac{1}{2} \, \sinh(\sqrt{2}\, s), \, \frac{1}{\sqrt{2}} \, s\Big),$$

*with curvature $k_1 = 1$ and torsion $k_2 = 1$.*

*The Frenet frame of $\alpha(s)$ is given by*

$$\begin{cases} T &= \Big(\cosh(\sqrt{2}\,s) + \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,s),\, \sinh(\sqrt{2}\,s) + \frac{1}{\sqrt{2}}\cosh(\sqrt{2}\,s),\, \frac{1}{\sqrt{2}}\Big), \\ N &= \big(\sqrt{2}\sinh(\sqrt{2}s) + \cosh(\sqrt{2}s),\, \sqrt{2}\cosh(\sqrt{2}s) + \sinh(\sqrt{2}s),\, 0\big), \\ B &= -\Big(\cosh(\sqrt{2}\,s) + \frac{1}{\sqrt{2}}\sinh(\sqrt{2}\,s),\, \sinh(\sqrt{2}\,s) + \frac{1}{\sqrt{2}}\cosh(\sqrt{2}\,s),\, -\frac{1}{\sqrt{2}}\Big). \end{cases}$$

*If we take $u = 2\,s$, $v = \frac{5\,s^2}{2}$, $w = -3\,s$ in (2.6), we find the Bertrand partner curve $\alpha^*(s^*)$ as:*

$$\alpha^* = \Big(\sinh(\sqrt{2}\,s)\,A(s) + \cosh(\sqrt{2}\,s)\,B(s),\, \cosh(\sqrt{2}\,s)\,A(s) + \sinh(\sqrt{2}\,s)\,B(s),\, 0\Big),$$

*where $A(s) = \frac{1+5\,s-5\,s^2}{\sqrt{2}}$, $B(s) = \frac{1+10\,s-5\,s^2}{2}$.*

*By computing the curvature and torsion of $\alpha^*$, we get*

$$k_1^* = \frac{2}{\sqrt{6-5\,s^2}}, \quad k_2^* = 0.$$

*Further, the Frenet frame of $\alpha^*$ is given by*

$$\begin{cases} T^* = \big(\sinh(\sqrt{2}\,s) + \sqrt{2}\cosh(\sqrt{2}\,s),\, \cosh(\sqrt{2}\,s) + \sqrt{2}\sinh(\sqrt{2}\,s),\, 0\big), \\ N^* = \big(\sqrt{2}\sinh(\sqrt{2}s) + \cosh(\sqrt{2}s),\, \sqrt{2}\cosh(\sqrt{2}s) + \sinh(\sqrt{2}s),\, 0\big), \\ B^* = \big(0,\, 0,\, 1\big). \end{cases}$$

*Thus, $\alpha^*(s^*)$ is a timelike Bertrand partner curve of the curve $\alpha(s)$.*

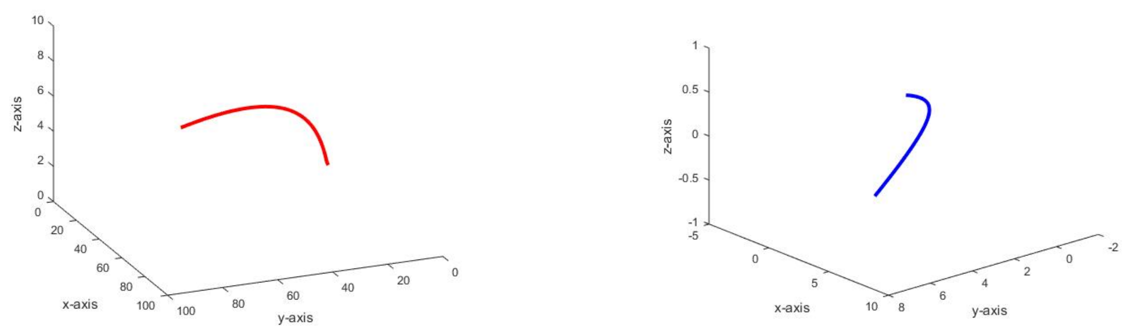



FIGURE 1. Curve $\alpha$ (red) and $\alpha^*$ (blue) in $\mathbb{E}_1^3$

**Example 4.2.** *Let $\alpha_1(s)$ be a Cartan null curve in $\mathbb{E}_1^3$ given by*

$$\alpha_1(s) = \Big(\sinh(s),\, \cosh(s),\, s\Big),$$

*with curvature $k_1 = 1$ and torsion $k_2 = 1/2$.*

*The Frenet frame of $\alpha_1(s)$ is given by*

$$\begin{cases} T_1 = \big(\cosh(s),\, \sinh(s),\, 1\big), \\ N_1 = \big(\sinh(s),\, \cosh(s),\, 0\big), \\ B_1 = \Big(\dfrac{-\cosh(s)}{2},\, \dfrac{-\sinh(s)}{2},\, \dfrac{1}{2}\Big). \end{cases}$$

*If we take $u = \frac{s}{2}$, $v = -\frac{1}{3}$, $w = s$ in (2.6), we find the Bertrand partner curve $\alpha_1^*(s^*)$ as:*

$$\alpha_1^* = \Big(\frac{2}{3}\sinh(s),\, \frac{2}{3}\cosh(s),\, 2\,s\Big).$$

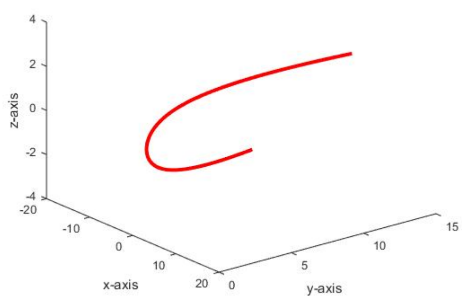
*By computing the curvature and torsion of $\alpha_1^*$, we get*

$$k_1^* = \frac{3}{16}, \quad k_2^* = \frac{9}{16}.$$

*Further, the Frenet frame of $\alpha_1^*$ is given by*

$$\begin{cases} T_1^* &= \frac{1}{2\sqrt{2}}\Big(\cosh(s),\, \sinh(s),\, 3\Big), \\ N_1^* &= \big(\sinh(s),\, \cosh(s),\, 0\big), \\ B_1^* &= -\frac{1}{2\sqrt{2}}\Big(3\cosh(s),\, 3\sinh(s),\, 1\Big). \end{cases}$$

*Thus, $\alpha_1^*(s^*)$ is a spacelike Bertrand partner curve of the curve $\alpha_1(s)$.*



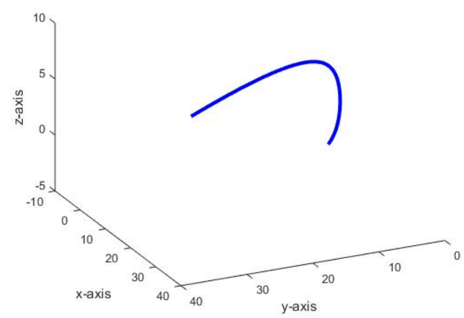




FIGURE 2. Curve $\alpha_1$ (red) and $\alpha_1^*$ (blue) in $\mathbb{E}_1^3$

**Example 4.3.** *If we take $u = \frac{s}{4}$, $v = \frac{1}{2}$, $w = \frac{s}{2}$ in (2.6) for the Cartan null curve $\alpha_1(s)$ in Example 4.2, we find the Bertrand partner curve $\alpha_2^*(s^*)$ as:*

$$\alpha_2^* = \frac{1}{2}\Big(3\sinh(s),\, 3\cosh(s),\, 3\,s\Big).$$

*By computing the curvature and torsion of $\alpha_2^*$, we get*

$$k_1^* = 1, \quad k_2^* = \frac{1}{3}.$$

*Further, the Frenet frame of $\alpha_2^*$ is given by*

$$\begin{cases} T_2^* &= \sqrt{\frac{3}{2}}\Big(\cosh(s),\, \sinh(s),\, 1\Big), \\ N_2^* &= \big(\sinh(s),\, \cosh(s),\, 0\big), \\ B_2^* &= \frac{1}{\sqrt{6}}\Big(-\cosh(s),\, -\sinh(s),\, 1\Big). \end{cases}$$

*Thus, $\alpha_2^*(s^*)$ is a Cartan null Bertrand partner curve of the curve $\alpha_1(s)$.*

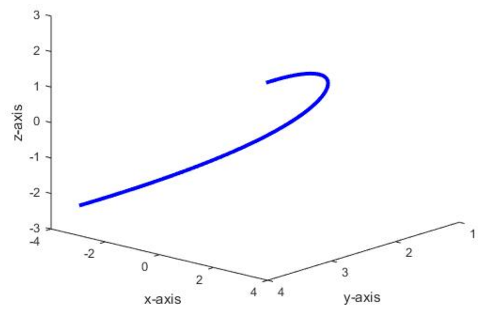



FIGURE 3. Cartan null Bertrand partner curve $\alpha_2^*$ of a null Cartan curve $\alpha_1$ in $\mathbb{E}_1^3$

## Acknowledgment

The first author is grateful to the University Grant Commission (UGC), Govt. of India, for awarding her a Senior Research Fellowship (UGC-SRF) with ID: JUNE18-413857. The authors are thankful to the reviewers for valuable suggestions which improved the manuscript.

## References

[1] Ali, A. T. & López, R. (2011). Slant helices in Minkowski space $\mathbb{E}_1^3$. J. Korean Math. Soc., 48(1), 159-167.

[2] Balgetir, H., Bektas, M. & Ergut, M. (2004). Bertrand curve for non-null curves in 3-dimensional Lorentzian space. Hadronic J., 27(2), 229-236.

[3] Balgetir, H.,Bektas, M. & Inoguchi, J. (2004/05). Null Bertrand curves in Minkowski 3-space and their characterizations. Note Mat., 23(1), 7-13.

[4] Bertrand, J.M. (1850). Mémoire sur la théorie des courbes á double courbure. Comptes Rendus, 15, 332-350.

[5] Bonnor, W. B. (1969). Null curves in a Minkowski space-time. Tensor 20, 229-242.

[6] Camci, C., Ucum, A. & Ilarslan, K. (2020). A new approach to Bertrand curves in Euclidean 3-space. J. Geom., 111(3) (15 pages).

[7] Do Carmo, M. P. (1976). Differential Geometry of Curves and Surfaces. Prentice Hall, Englewood Cliffs, NJ.

[8] Gokcek, F. & Erdem, H.A. (2021). On Cartan null Bertrand curves in Minkowski 3-space. Facta Univ., Math. Inform., 36(5), 1079-1088.

[9] Ilarslan, K. & Aslan, A. (2017). On spacelike Bertrand curve in Minkowski 3-space. Konuralp J. Math., 5(1), 214-222.

[10] Izumiya, S.& Takeuchi, N. (2004). New special curves and developable surfaces. Turk. J. Math., 28, 531-537.

[11] Jin, D. H. (2008). Null Bertrand curves in a Lorentz manifold. J. Korea Soc. Math. Educ. Ser. B Pure Appl. Math.,15 (3), 209-215.

[12] Kuhnel, W. (1999). Differential geometry: Curves - Surfaces - Manifolds. Braunschweig, Wiesbaden.

[13] Kula, L., Ekmekci, N., Yaylı, Y. & Ilarslan, K. (2010). Characterizations of slant helices in Euclidean 3-space. Turk. J. Math., 34(2), 261–273.

[14] Lancret, M.A. (1806). M*á*moire sur les courbes *à* double courbure. M*é*moires pr*é*sent*é*s *à* l'Institut1., 416-454.

[15] Liu, T. & Pei, D. (2019). Null helices and Cartan slant helices in Lorentz–Minkowski 3-space. Int. J. Geom. Methods Mod., 16(11), 1950179 (16 pages).

[16] Saint, V. B. (1845). M*é*moire sur les lignes courbes non planes. Journal de l'Ecole Polytechnique, 18, 1-76.

[17] Scofield, P. D. (1995). Curves of constant precession. Amer. Math. Monthly, 102, 531–537.

[18] Tamta, S. & Gupta, R. S. New parametrization of Bertrand partner D-curves in $\mathbb{E}^3$. Bol. Soc. Paran. Mat., (in press) doi:10.5269/bspm.63309.

[19] Tamta, S. & Gupta, R. S. (2023). Spherical Indicatrix of a new approach to Bertrand curves in Euclidean 3-space. Kyungpook Math. J., 63(2), 263-285.

[20] Ucum, A. & Ilarslan, K. (2016). On timelike Bertrand curve in Minkowski 3-space. Honam Math. J., 38(3), 467-477.

[21] Yang, X. (2003). High accuracy approximation of helices by quintic curves. Comput. Aided Geom. Des., 20, 303-317.

University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University Sector-16C, Dwarka, New Delhi-110078 India.

University School of Basic and Applied Sciences, Guru Gobind Singh Indraprastha University Sector-16C, Dwarka, New Delhi-110078 India.