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# CONTRIBUTION TO NULL KILLING MAGNETIC TRAJECTORIES

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Abstract. We analyze null magnetic trajectories of a magnetic field on a timelike surface in Minkowski 3−space $\mathbb{E}_1^3$. We show that the Lorentz force can be written into the Darboux frame field of a null trajectory on the surface. We give the necessary and sufficient condition for writing a null curve as the magnetic trajectory of the magnetic field. After creating a variation, we derive the Killing magnetic flow equations with regard to the geodesic curvature, geodesic torsion and normal curvature of the curve $\gamma$ on the timelike surface. Finally we examine the geodesics of some timelike surfaces in $\mathbb{E}_1^3$.

## 1. Introduction

Any magnetic vector field is known divergence zero vector field in three- dimensional spaces. A magnetic trajectory of a magnetic flow created by magnetic vector field is a curve called as magnetic. Although the problem of investigating magnetic trajectories appears to be physical problem, recent studies show that the characterization of magnetic flow in a magnetic field have brought variational perspective in more geometrical manner. In particular, magnetic curves have been developed by techniques of differential geometry and methods of

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calculus of variation from basic spaces to manifolds because the Lorentz force equation is a minimizer of the functional $\mathcal{L} : \Gamma \to \mathbb{R}$ defined by

$$\mathcal{L}(\gamma) : \frac{1}{2} \int_{\gamma} \langle \gamma', \gamma \rangle' \, dt + \omega(\gamma') \, dt,$$

where $\Gamma$ is a family of smooth curves that connect two fixed point of $U$, $\gamma$ is a curve choosing from $\Gamma$ and $\omega$ is a potential $1-$form. The Euler-Lagrange equation of the functional $\mathcal{L}$ is derived as

$$\phi(\gamma') = \nabla_{\gamma'} \gamma', \tag{1.1}$$

where $\phi$ is the skew-symmetric operator. The critical point of the functional $\mathcal{L}$ corresponds to a solution of the Lorentz force equation. So the solutions of the equations could be interpreted with a more geometric point of view [ 1, 3, 4, 5, 7, 10, 13 ].

In this work we consider null Killing magnetic trajectories on a timelike surface $S$ in Minkowski $3-$space $\mathbb{E}_1^3$. Also, we get equation of the Lorentz force by using the Darboux frame field of a null magnetic curve on the such surface and give equations of the Killing magnetic flow by means of the structures of a magnetic vector field in $\mathbb{E}_1^3$. Then we apply this formulation to give results about magnetic curves on the pseudo-sphere and the pseudo-cylinder surfaces, so we show that geodesics of these surfaces are null magnetic curves.

## 2. Preliminaries

We consider that $\mathbb{E}_1^3$ denotes Minkowski $3-$space with the inner product

$$\langle u, w \rangle = -u_1 w_1 + u_2 w_2 + u_3 w_3$$

which is a non-degenerate, symmetric and bilinear form and the vector product

$$u \times w = (-u_2 w_3 + u_3 w_2, u_3 w_1 - u_1 w_3, u_1 w_2 - u_2 w_2),$$

where $u = (u_1, u_2, u_3)$, $w = (w_1, w_2, w_3) \in \mathbb{E}_1^3$. A vector $u$ in $\mathbb{E}_1^3$ is called a spacelike vector if $\langle u, u \rangle > 0$ or $u = 0$, a timelike vector if $\langle u, u \rangle < 0$, or null (lightlike) vector if $\langle u, u \rangle = 0$ and $u \neq 0$. A regular curve in $\mathbb{E}_1^3$ is called spacelike, timelike or null, if its velocity vector is spacelike, timelike or null, respectively. A non-degenerate surface is named in terms of the induced metric. If the induced metric is indefinite, a non-degenerate surface is called timelike [9 12 ].

We can assign a frame to any point of a null curve since we investigate the geometry of the curve. This frame is known as Cartan frame field along a null curve in $\mathbb{E}_1^3$. Let $\gamma = \gamma(s)$

be a null curve in $\mathbb{E}_1^3$. Let $T$ denote a null vector field along $\gamma$. So, there exists a null vector field $B$ along $\gamma$ satisfying $\langle T, B\rangle = 1$. If we write $N = B \times T$, then we can obtain a Cartan frame field $\mathcal{F} = \{T, N, B\}$ along $\gamma$. A Cartan framed null curve $(\gamma, \mathcal{F})$ is given by

$$T(s) = \gamma'(s), \quad N(s) = \gamma''(s), \quad B(s) = -\gamma'''(s) - \frac{1}{2} < \gamma'''(s), \gamma'''(s) > \gamma'(s)$$

at a point $\gamma(s)$, where

$$\langle T, T\rangle = \langle B, B\rangle = \langle T, N\rangle = \langle N, B\rangle = 0,$$
$$\langle N, N\rangle = \langle T, B\rangle = 1.$$

We have the following derivative equations of the Cartan frame (generally knows as Frenet equations)

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\kappa & 0 & -1 \\ 0 & \kappa & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix},$$

where

$$\kappa(s) = \frac{1}{2} < \gamma'''(s), \gamma'''(s) >,$$

[2, 8, 12].

In order to study the geometry of a null curve on a timelike surface, we can construct a suitable frame, which is known the Darboux frame field, to any point of the curve. Let $(\gamma, \mathcal{F})$ be a null curve with frame $\mathcal{F} = \{T, N, B\}$ and $S$ an oriented timelike surface in Minkowski $3-$space. The Darboux frame at $\gamma(s)$ of $\gamma$ is the orthonormal basis $\{T, Q, n\}$ of $\mathbb{E}_1^3$, where $Q$ is the unique vector obtained by

$$Q = \frac{1}{\langle V, T\rangle}\{V - \frac{\langle V, V\rangle}{2\langle V, T\rangle}T\}, \quad V \in T_{\gamma(s)}M, \quad \langle V, T\rangle \neq 0,$$

and $n$ is the spacelike unit normal of $S$ which is defined by $n = T \times Q$. So, we have

$$\langle T, T\rangle = \langle Q, Q\rangle = \langle Q, n\rangle = \langle T, n\rangle = 0,$$
$$\langle n, n\rangle = \langle T, Q\rangle = 1.$$

The first order variation of $\{T, Q, n\}$ is expressed as follow

$$\begin{bmatrix} T' \\ Q' \\ n' \end{bmatrix} = \begin{bmatrix} \kappa_g & 0 & \kappa_n \\ 0 & -\kappa_g & \tau_g \\ -\tau_g & -\kappa_n & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ n \end{bmatrix}, \tag{2.2}$$

where the functions $\kappa_g$, $\kappa_n$ and $\tau_g$ are called the geodesic curvature, the normal curvature and the geodesic torsion of the curve $\gamma$, respectively. From the comparison of Cartan and Darboux frames, we have

$$\kappa_n = \pm 1 \tag{2.3}$$

[6, 12].

## 3. Magnetic Vector Fields

The Lorentz force $\phi$ corresponding the magnetic field $V$ is given by

$$\phi\left(\gamma'\right) = V \times \gamma'.$$

A curve $\gamma$ in $\mathbb{E}_1^3$ is called magnetic curve of a magnetic field $V$ if its tangent vector field satisfies

$$\nabla_{\gamma'}\gamma' = \phi\left(\gamma'\right) = V \times \gamma'. \tag{3.4}$$

The Lorentz force $\phi$ of a magnetic field $F$ in $\mathbb{E}_1^3$ is defined to be skew symmetric operator given by

$$< \phi(X), Y >= F(X, Y)$$

for vector fields $X$ and $Y$. The mixed product of the vector fields $X$, $Y$ and $Z$ is given by

$$< X \times Y, Z >= \Omega(X, Y, Z),$$

where $\Omega$ a volume on $\mathbb{E}_1^3$. So, the Lorentz force of the corresponding Killing magnetic force is given as $\phi(X) = V \times X$, where $V$ is a Killing vector field [13].

Then we can give the following proposition for the Lorentz force.

**Proposition 3.1.** *Let $\gamma$ be a null magnetic curve on a timelike surface $S \subset \mathbb{E}_1^3$ and $\{T, Q, n\}$ is the Darboux frame field along $\gamma$. Then the Lorentz force in the Darboux frame $\{T, Q, n\}$ is written as follows*

$$\phi(T) = \kappa_g T + \kappa_n n, \tag{3.5}$$

$$\phi(Q) = -\kappa_g Q + \omega n \tag{3.6}$$

*and*

$$\phi(n) = -\omega T - \kappa_n Q, \tag{3.7}$$

*where the function $\omega(s) =< \phi(Q(s)), n(s) >$ associated with each magnetic curve is quasislope measured with respect to the magnetic vector field $V$.*

**Proof.** The unit tangent vector to $\gamma$ at a point $\gamma(s)$ of $\gamma$ is $T(s) = \gamma'(s)$. Then from (1.1), we have

$$\phi(T) = \nabla_T T = V \times T.$$

By using the Darboux formulas (2.2), we get

$$\phi(T) = \kappa_g T + \kappa_n n$$

and

$$< \phi(T), Q >= \kappa_g \quad \text{and} \quad < \phi(T), n >= \kappa_n.$$

Similarly, we can write the linear expansion of $\phi(Q), \phi(n) \in S$ as follows

$$\phi(Q) =< \phi(Q), Q > T+ < \phi(Q), T > Q+ < \phi(Q), n > n$$

and

$$\phi(n) =< \phi(n), Q > T+ < \phi(n), T > Q+ < \phi(n), n > n,$$

respectively. Taking into consideration Eqs. (3.4) and (3.5), we get

$$< \phi(Q), T >=< V \times Q, T >= - < V \times T, Q >= - < \phi(T), Q >= -\kappa_g$$

and

$$< \phi(n), T >=< V \times n, T >= - < V \times T, n >= - < \phi(T), n >= -\kappa_n.$$

Since $\phi$ is a skew-symmetric operator, we get $< \phi(Q), Q >=< \phi(n), n >= 0$.

Then by using Proposition 3.1 we can write the magnetic vector field according to Darboux frame on a timelike surface $S$ in the following.

**Proposition 3.2.** *A null curve $\gamma : I \subset \mathbb{R} \to S$ is a magnetic trajectory of a magnetic field $V$ if and only if $V$ can be written along $\gamma$ as*

$$V = \omega T - \kappa_n Q + \kappa_g n. \tag{3.8}$$

**Proof.** Suppose that $\gamma$ is a null magnetic curve along a magnetic field $V$ with the Darboux frame field $\{T, Q, n\}$. Then, $V$ can written as $V =< V, Q > T+ < V, T > Q+ < V, n > n$. To find coefficient of $V$, we use the Lorentz force in Darboux frame equations (3.5–3.7):

$$\begin{aligned} \omega &= < \phi(Q), n >=< V, Q \times n >=< V, Q >, \\ \kappa_n &= < \phi(T), n >= - < V, n \times T >= - < V, T > \end{aligned}$$

and

$$\kappa_g =< \phi\left(T\right), Q >=< V, T \times Q >=< V, n > .$$

## 4. Killing Magnetic Flow Equation for Null Magnetic Trajectories

Let $\gamma : I \to S$ be pseudo-parametrized null curve on a timelike surface in $\mathbb{E}_1^3$ and $V$ a magnetic vector field along that curve. One can take a variation of $\gamma$ in the direction of $V$, say a map

$$\begin{array}{rcl} \Gamma: \quad [0,1] \times (-\varepsilon, \varepsilon) & \to & S \\ (s,t) & \to & \Gamma(s,t) \end{array}$$

which satisfies

$$\Gamma\left(s,0\right) = \gamma\left(s\right), \ \left(\frac{\partial \Gamma(s,t)}{\partial t}\right)_{t=0} = V\left(s\right) \text{ and } \left(\frac{\partial \Gamma(s,t)}{\partial s}\right)_{t=0} = \gamma'\left(s\right).$$

We recall that a spacelike or timelike curve in $\mathbb{E}_1^3$ can be reparametrize by an arclength. However, there would be not sense reparametrize by the arclength for a null curve $\gamma$. However, it has pseudo arc-length parametrized $\alpha(s) = \gamma(\phi(s))$, such that $\|\alpha''(s)\| = 1$, where $\phi$ is the differential function in suitable interval. Thus, we have the following equations:

$$\begin{aligned} T(s,t) &= \left(\tfrac{\partial \Gamma(s,t)}{\partial s}\right)_{t=0} = \gamma'\left(s\right), \\ \beta(s,t) &= \left(< \left(\tfrac{\partial^2 \Gamma(s,t)}{\partial s^2}\right)_{t=0}, \left(\tfrac{\partial^2 \Gamma(s,t)}{\partial s^2}\right)_{t=0} >\right)^{1/4}, \end{aligned}$$

( see [9, 12 ]) .

By using above variational formulas, we have the following equalities (by similar method that of [3, 10] ).

**Lemma 4.1.** *We consider that $\gamma$ is a null curve on a timelike surface in $\mathbb{E}_1^3$ and a magnetic vector field $V$ is a variational vector field along the variation $\Gamma$. So we can give the following expressions;*

$$V\left(\beta\right) = \frac{1}{2\beta^3} < \nabla_T \nabla_T V, \nabla_T T >, \tag{4.9}$$

$$V\left(\kappa\right) = \frac{1}{2} V\left(< \nabla_T \nabla_T T, \nabla_T \nabla_T T >\right) =< \nabla_T^3 V, \nabla_T^2 T > . \tag{4.10}$$

**Proposition 4.1.** *(see [11]) . Let $V\left(s\right)$ be the restriction to $\gamma\left(s\right)$ of a Killing vector field, then*

$$V\left(\beta\right) = V\left(\kappa\right) = 0. \tag{4.11}$$

Thus, Killing magnetic flow equations can be given the following theorem.

**Theorem 4.1.** *Let $\gamma$ be a null curve on $S$ in $\mathbb{E}_1^3$. Suppose that $V = \omega T - \kappa_n Q + \kappa_g n$ is a Killing vector field along $\gamma$. Then the magnetic trajectories are curves on $S$ satisfying following differential equations*

$$b\kappa_g + c\kappa_n = 0 \tag{4.12}$$

*and*

$$\begin{aligned} -a' + 2c\tau_g + b'\kappa_g' - b\kappa_g\kappa_g' - c\kappa_n\kappa_g' + \kappa_g^2 b' - b\kappa_g^3 \\ - c\kappa_n\kappa_g^2 - \kappa_n\tau_g b' + 2b\kappa_g\kappa_n\tau_g + c'\kappa_g\kappa_n = 0, \end{aligned} \tag{4.13}$$

*where*

$$\begin{aligned} &a = \omega'' + 2\omega'\kappa_g + \omega\kappa_g' - 2\kappa_g'\tau_g - \kappa_g\tau_g' + \omega\kappa_g^2 - \kappa_g^2\tau_g \\ &-\omega\kappa_n\tau_g + \kappa_n\tau_g^2, \\ &b = -\omega + \tau_g - \kappa_g'\kappa_n, \\ &c = 2\omega'\kappa_n + \omega\kappa_g\kappa_n - \kappa_g\kappa_n\tau_g - \kappa_n\tau_g' + \kappa_g''. \end{aligned}$$

**Proof.** Assume that $V$ is a Killing vector field along $\gamma$ on $S$. Along any magnetic trajectory $\gamma$, we have $V = \omega T - \kappa_n Q + \kappa_g n$. Using (2.3), we get

$$\nabla_T V = \left(\omega' + \omega\kappa_g - \kappa_g\tau_g\right) T + \left(\omega\kappa_n - \kappa_n\tau_g + \kappa_g'\right) n. \tag{4.14}$$

We calculate derivative of (4.14) as follows

$$\begin{aligned} \nabla_T^2 V = &\left(\omega'' + 2\omega'\kappa_g + \omega\kappa_g' - 2\kappa_g'\tau_g - \kappa_g\tau_g' + \omega\kappa_g^2 \right. \\ &\left. -\kappa_g^2\tau_g - \omega\kappa_n\tau_g + \kappa_n\tau_g^2\right) T + \left(-\omega + \tau_g - \kappa_g'\kappa_n\right) Q \\ &\left(2\omega'\kappa_n + \omega\kappa_g\kappa_n - \kappa_g\kappa_n\tau_g - \kappa_n\tau_g' + \kappa_g''\right) n \\ &= aT + bQ + cn. \end{aligned} \tag{4.15}$$

Substituting (4.15) into (4.9), we derive

$$V(\beta) = b\kappa_g + c\kappa_n = 0.$$

For variation of $\kappa$, taking derivative of (4.15), we have,

$$\begin{aligned} \nabla_T^3 V = (a' + a\kappa_g - c\tau_g)T + (b' - b\kappa_g - c\kappa_n)Q \\ + (a\kappa_n + b\tau_g + c')n. \end{aligned} \tag{4.16}$$

Substituting (4.16), (2.2) and (2.3) into (4.10), we obtain

$$\begin{aligned} V(\kappa) = -a' + 2c\tau_g + b'\kappa_g' - b\kappa_g\kappa_g' - c\kappa_n\kappa_g' + \kappa_g^2 b' - b\kappa_g^3 \\ - c\kappa_n\kappa_g^2 - \kappa_n\tau_g b' + 2b\kappa_g\kappa_n\tau_g + c'\kappa_g\kappa_n = 0. \end{aligned}$$

**Definition 4.1.** *Any null curve on a timelike surface $S$ is called the null magnetic trajectory of a magnetic field $V$ if it satisfies the differential equation system (4.12) and (4.13).*

## 5. Applications

**Magnetic trajectories on a timelike pseudo-sphere:** We consider the timelike pseudo-sphere with radius $r$,

$$\mathbb{S}_1^2(r) = \left\{(x_1, x_2, x_3) \in E_1^3 : x_1^2 + x_2^2 + x_3^2 = r^2\right\}.$$

The geodesic torsion $\tau_g$ vanishes for all curves on $\mathbb{S}_1^2(r)$ and the normal curvature $\kappa_n^2 = 1$ [12]. Then any null geodesic curve $\gamma$ on $\mathbb{S}_1^2(r)$ is a magnetic trajectory of a magnetic field $V$ if and only if $V$ can be written along $\gamma$ as

$$V = \omega T \pm Q,$$

where $\omega$ is a constant.

**Magnetic trajectories on a pseudo-cylinder:** The pseudo-cylinder

$$\mathbb{C}_1^2(1) = \left\{ (x, y, z) \in \mathbb{E}_1^3 \middle| -x^2 + y^2 = 1,\ z \in \mathbb{R}\right\}$$

is a timelike surface and parametrized by

$$X(u, v) = (\sinh s, \cosh s, s),$$

where $r$ is radius of the circle. Then for a null geodesic

$$\gamma(s) = (\sinh s, \cosh s, s)$$

on $\mathbb{C}_1^2(1)$, we have

$$\kappa_g = 0,\ \kappa_n = 1 \text{ and } \tau_g = -\frac{1}{2},$$

(see [6, 12]). So, the null geodesic $\gamma$ on a pseudo-cylinder are magnetic trajectories of the magnetic field

$$V = \omega T - Q$$

where $\omega$ is a constant (see Fig (5.1)).

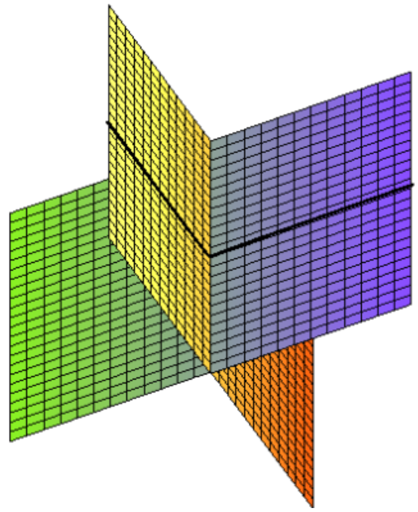

FIGURE 1. A null magnetic trajectory on the pseudo-cylinder

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