CHARACTERISTICS OF LIGHTLIKE HYPERSURFACES OF TRANS-PARA SASAKIAN MANIFOLDS

MOHD DANISH SIDDIQI ᴵᴰ *

ABSTRACT. In this research article, we study three lightlike hypersurfaces of trans-para Sasakian manifolds with a quarter-symmetric metric connection: (1) re-current, (2) Lie re-current and (3) Hopf-lightlike hypersurface. Also, we discuss some properties of a screen semi-invariant lightlike hypersurface of trans-para Sasakian manifolds with a quarter-symmetric metric connection. Furthermore, we show that a conformal hypersurface is screen totally geodesic lightlike hypersurfaces. Finally we prove the integrability conditions for the distributions of screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold with a quarter-symmetric metric connection.

1. INTRODUCTION

In 1975, S. Golab [11] initiated the study of a quarter-symmetric connection in a differentiable manifold.

A linear connection $\bar{\nabla}$ on an n-dimensional Riemannian manifold (M, g) is called a quarter-symmetric connection if its torsion tensor T of the connection $\bar{\nabla}$ [11]

$$T(X, Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X, Y] \tag{1.1}$$

satisfies

$$T(X,Y) = \eta(Y)\varphi X - \eta(X)\varphi Y, \tag{1.2}$$

where η is a 1-form and φ is a $(1,1)$ tensor field.

In particular, if $\varphi(X) = X$, then the quarter-symmetric connection is reduces to the semi-symmetric connection [9]. Thus the notion of a quarter-symmetric connection is the extension of the semi-symmetric connection. Moreover if, a quarter-symmetric connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0 \tag{1.3}$$

for all $X, Y, Z \in T(M)$, where $T(M)$ is the Lie algebra of vector fields of the manifold M, then $\bar{\nabla}$ is said to be a quarter-symmetric metric connection. Otherwise it is said to be a quarter-symmetric non-metric connection.

After S. Golab [11], numerous geometers (see [16], [24], [25], [2]) continued the systematic and specific study of a quarter-symmetric metric connection with various structures in several ways to a different extent .

The differential geometry of lightlike hypersurfaces is one of the most specific topic in the theory of lightlike submanifolds. Lightlike hypersurfaces have several significant applications in mathematical physics [4], electromagnetic [5], black hole theory [3], string theory and general relativity [10]. A submanifold of a semi-Riemannian manifold is called a lightlike submanifold if the induced metric is degenerate. In 1996, K. L. Duggal, A. Bejancu established the conception of lightlike submanifolds of almost contact metric manifolds [5].

Also K. L. Duggal with B. Sahin and others geometers have further developed this concept and studied many new classes of lightlike submanifolds (for more details see [1], [6], [7], [8], [13]).

Furthermore, K. L. Duggal and R. Sharma [9] also studied some properties of semi-Riemmnain manifold with a semi-symmetric metric connection. They proved that these geometric results have many physical applications in real world.

Inspired by the above studies others geometers like D. H. Jin have been exclusively studied on lightlike hypersufaces with respect to the different connections such as semi symmetric metric and quarter-symmetric metric connection (cf. [17], [18], [19], [20], [21], [22]).

On the other hand, in 1985, S. Kaneyuki and M. Konzai [23] initiated the study of a para-complex structure and almost para-contact structure on a semi-Riemannian manifold.

S. Zamkovoy [26] has extensively studied para contact metric manifolds after that there are many papers discussed the contribution of para-contact geometry of a semi-Riemannian manifolds ([27, 28]. In 2019, S. Zamkovoy [27] also introduced the geometry of trans-para-Sasakian manifolds. An almost contact structure on a manifold M is called a trans-Sasakian structure if the product manifolds $M \times \mathbb{R}$ belongs to the class W_4 [12]. In ([14], [15]), J. C. Marrero and D. Chinea are completely characterized trans-Sasakian structures of types (α, β). We note that the trans-Sasakian structures of type $(\alpha, 0)$, $(0, \beta)$ and $(0, 0)$ are α-Sasakian [12], β-Kenmotsu [14], and cosympletic [14], respectively. In [27], S. Zamkovoy consider the trans-para-Sasakian manifolds as an analogue of the trans-Sasakian manifolds. A trans-para-Sasakian manifolds is a trans-para-Sasakian structure of type (α, β), where α and β are smooth functions. The trans-para-Sasakian manifolds of type (α, β), are called para-Sasakian manifolds $(\alpha = 1)$, para-Kenmotsu manifolds $(\beta = 1)$ [14] and para-cosympletic manifolds $(\alpha = \beta = 0)$.

Motivated by the above research articles, we consider the three types of lightlike hypersurfaces of trans-para-Sasakian manifolds with respect to the quarter-symmetric metric connection in the present framework.

2. Preliminaries

A $(2n + 1)$-dimensional smooth manifold M has an almost paracontact structure (φ, ξ, η) if it admits a tensor field φ of type $(1, 1)$, a vector field ξ and a 1-form η satisfying the following compatibility conditions

$$\varphi^2 X = X - \eta(X)\xi, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1. \tag{2.4}$$

The distribution $\mathbb{D} : p \in M \longrightarrow \mathbb{D}_p \subset T_pM$: $\mathbb{D}_p = Ker\eta = \{X \in T_pM : \eta(X) = 0\}$ is called a paracontact distribution generated by η.

By the definition of an almost paracontact structure the endomorphism φ has rank $2n$.

If a $(2n+1)$-dimensional manifold M with (φ, ξ, η) structure admits a pseudo-Riemannian metric g such that

$$g(\varphi X, \varphi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.5}$$

where $X, Y \in T(M)$ then we say that M has an almost paracontact metric structure with compatible metric g. Any compatible metric g with a given almost paracontact structure with signature $(n + 1, n)$. Note that setting $Y = \xi$, we have $\eta(X) = g(X, \xi)$. Further, any

almost paracontact structure admits a compatible metric.

Definition 2.1. *An almost paracontact metric manifold $(M, \varphi, \eta, \xi, g)$ is said to be a para-contact metric manifold if $(g(X, \varphi Y) = d\eta(X, Y)$, where $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y])$ and η is a paracontact form.*

A paracontact structure on M naturally gives rise to an almost paracomplex structure on the product $M \times \mathbb{R}$. If this almsot paracomplex struture is integrable, then the given paracontact metric manifold is said to a para-Sasakian (see [20]). A paracontact metric manifold is a para-Sasakian if and only if

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X. \tag{2.6}$$

The manifold $(M, \varphi, \xi, \eta, g)$ of dimension $(2n + 1)$ is said to be trans-para-Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = \alpha(-g(X, Y)\xi + \eta(Y)X) + \beta(g(X, \varphi Y)\xi + \eta(Y)\varphi X). \tag{2.7}$$

From (2.7), we also have

$$\nabla_X \xi = -\alpha \varphi X - \beta(X - \eta(X)\xi). \tag{2.8}$$

Now, we have the following lemma [17]

Lemma 2.1. [17] *Let $(M, \varphi, \eta, \xi, g)$ be a trans-para-Sasakian manifold. Then we have*

$$R(X, Y)\xi = -(\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y], \tag{2.9}$$

$$R(\xi, Y)Z = -(\alpha^2 + \beta^2)[g(Y, Z)\xi - \eta(Z)X], \tag{2.10}$$

$$S(X, \xi) = -2n(\alpha^2 + \beta^2)\eta(X), \tag{2.11}$$

$$(\nabla_X \eta)Y = \alpha g(X, \varphi Y) - \beta(g(X, Y) - \eta(X)\eta(Y)), \tag{2.12}$$

for all $X, Y, Z \in T(M)$, where R is a Riemannian curvature tensor and S is a Ricci curvature tensor.

3. Quarter-Symmetric metric connection

In this section, we express few tensorial relations for a trans-para Sasakian manifold with quarter-symmetric metric connection.

Let $\bar{\nabla}$ be a linear connection and ∇ be the Levi-Civita connection of a almost paracontact metric manifold M such that

$$\bar{\nabla}_X Y = \nabla_X Y + H(X, Y), \tag{3.13}$$

where H is a $(1,1)$-tensor type. For $\bar{\nabla}$ to be a quarter-symmetric metric connection in M, we have [11]

$$H(X, Y) = \frac{1}{2}[T(X, Y) + T'(X, Y) + T'(Y, X)], \tag{3.14}$$

where

$$g(T'(X, Y), Z) = g(T(Z, X)Y). \tag{3.15}$$

From (1.1) and (3.15), we find

$$T'(X, Y) = \eta(X)\varphi Y - g(\varphi X, Y)\xi. \tag{3.16}$$

Using (1.1) and (3.16) in (3.14), we arrive at

$$H(X, Y) = \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{3.17}$$

Therefore, a quarter-symmetric metric connection $\bar{\nabla}$ in a trans-para Sasakian manifold is given by [16]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)\varphi X - g(\varphi X, Y)\xi. \tag{3.18}$$

Now, using (3.18), (2.7) and (2.8), we obtain the following results:

Theorem 3.1. *Let M be a trans-para Sasakian manifold with a quarter-symmetric metric connection. Then*

$$(\bar{\nabla}_X \varphi)Y = (1 - \alpha)\{g(X, Y)\xi - \eta(Y)X\} + \beta\{g(X, \varphi Y)\xi + \eta(Y)\varphi X\} \tag{3.19}$$

$$\bar{\nabla}_X \xi = (1 - \alpha)\varphi X - \beta(X - \eta(X)\xi). \tag{3.20}$$

4. Lightlike hypersurfaces

Let $\bar{M}$ be a semi-Riemannian manifold with index r, $0 < r < 2n + 1$ and M be a hypersurface of $\bar{M}$, with induced metric $g = \bar{g}$. M is a null hypersurface of $\bar{M}$ if the metric g is of rank $2n - 1$. The orthogonal complement $TM^\perp$ of the tangent space TM, given as

$$TM^\perp = \left\{ X_p \in T_p M^\perp : g_p(X_p, Y_p) = 0, \forall \quad Y_p \in \Gamma(T_p M) \right\}$$

is a distribution of rank 1 on M. If $TM^\perp \subset TM$ and then coincides with the *radical distribution* $Rad(TM)$ such that

$$Rad(TM) = TM \cap TM^\perp. \tag{4.21}$$

A complementary bundle of $TM^\perp$ in TM is a non-degenerate distribution of constant rank $2n - 1$ over M. It is known as a *screen distribution* and denoted by $S(TM)$.

Let $(M, g, S(TM))$ be a lightlike hypersurface of a semi-Riemannian manifold $\bar{M}$. Then there exists a unique rank over subbundle $tr(TM)$ called the lightlike *transversal vector* bundle of of M with respect to $S(TM)$, such that for any null section ξ of $Rad(TM)$ on coordinate neighborhood U of M, there exists a unique section N of $tr(TM)$ on U satisfying

$$g(N, X) = 0, \quad g(N, N) = 0, \quad g(\mathcal{N}, \xi) = 1, \forall X \in \Gamma(S(TM)) \tag{4.22}$$

Then, we have the decomposition on the tangent bundle

$$TM = S(TM) \perp Rad(TM), \tag{4.23}$$

$$T\bar{M} = TM \oplus tr(TM) = S(TM) \perp \left\{ Rad(TM) \oplus tr(TM) \right\}. \tag{4.24}$$

Let $P : TM \longrightarrow S(TM)$ be the projection morphism. Then, we have the local Gauss-Weingarten formulas of M and $S(TM)$ as follows

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N, \tag{4.25}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{tr} N, \tag{4.26}$$

$$\nabla_X PT = \nabla_X^* PY + C(X, PY)\xi, \tag{4.27}$$

$$\bar{\nabla}_X \xi = -A_\xi^* X - \tau(X)\xi \tag{4.28}$$

for any $X, Y \in \Gamma(TM)$, where ∇ is a linear connection on M and ∇^* is a linear connection on $S(TM)$ and B, A_N and τ are called the local second fundamental form on $T(M)$ respectively.

It is well know that the induced connection ∇ is quarter-symmetric non-metric connection and we get

$$(\nabla_X g) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y), \tag{4.29}$$

$$T(X, Y) = \eta(X)Y - \eta(Y)X. \tag{4.30}$$

where T is the torsion tensor with respect to the induced connection ∇ on M, B is symmetric on $T(M)$ and $\eta(X) = g(X, N)$ is a differential 1-form on TM.

For the second fundamental form B, we have

$$B(X, \xi) = 0. \tag{4.31}$$

The local second fundamental forms are related to their shape operators by

$$B(X, PY) = g(A_\xi^* X, PY), \qquad g(A_\xi^* X, N) = 0, \tag{4.32}$$

$$C(X, PY) = g(A_N X, PY), \qquad g(A_N X, N) = 0. \tag{4.33}$$

From (4.32), A_ξ^* is a $S(TM)$-valued real self-adjoint operator and satisfies

$$A_\xi^* \xi = 0. \tag{4.34}$$

5. Screen Semi-invariant lightlike hypersurfaces

This segment deal with screen semi-invariant lightlike hypersurfaces of a trans-para Sasakian manifold equipped with a quarter-symmetric metric connection.

Let M be a lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with $\xi \in \Gamma(TM)$. If ξ is a local section of $\Gamma Rad(TM)$, then

$$g(\varphi\xi, \xi) = 0, \tag{5.35}$$

and $\varphi\xi$ is tangent to M. Therefore, we obtain a distribution $\varphi(Rad(TM))$ of dimension 1 on M.

If

$$\varphi((tr(TM)) = (tr(TM), \quad and \quad \varphi(Rad(TM)) = Rad(TM), \tag{5.36}$$

then lightlike hypersurface M is called a screen semi-invariant lightlike hypersurface of $\bar{M}$ [1].

Since M is a screen semi-invariant lightlike hypersurface

$$g(\varphi N, N) = 0 \tag{5.37}$$

$$g(\varphi N, \xi) = -g(N, \varphi \xi) = 0. \tag{5.38}$$

$$g(N, \xi) = 1 \tag{5.39}$$

from (2.5), we obtain

$$g(\varphi \xi, \varphi N) = -1. \tag{5.40}$$

Therefore, $\varphi(Rad(TM)) \oplus \varphi(tr(TM))$ is a non-degenerate vector subbundle of screen distributions $S(TM)$.

Now, since $S(TM)$ and $\varphi(Rad(TM)) \oplus \varphi(tr(TM))$ are non-degenerate distribution $\bar{D}_0$ such that

$$S(TM) = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\}. \tag{5.41}$$

Therefore, $\varphi(D_0) = D_0$ and $\xi \in D_0$. In view of (4.23), (4.25) and (5.41) we obtain the followings

$$TM = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\} \perp Rad(TM) \tag{5.42}$$

$$T\bar{M} = D_0 \perp \{\varphi(Rad(TM)) \oplus \varphi(tr(TM))\} \perp \{Rad(TM) \oplus tr(TM)\}. \tag{5.43}$$

Now, we take $D_1 = Rad(TM) \perp \varphi(Rad(TM)) \perp D_0$ and $D_2 = \varphi(tr(TM))$ on M, we get

$$TM = D_1 \oplus D_2. \tag{5.44}$$

Let the local null vector fields $V = \varphi \xi$ and $U = \varphi N$ and denote the projection morphism of TM into D_1 and D_2 by P_1 and P_2, respectively. Therefore , for $X \in \Gamma(TM)$,we have

$$X = P_1 X + P_2 X, \qquad P_2 X = u(X)\bar{\mathcal{U}}, \tag{5.45}$$

where u is a differential 1-form locally defined by

$$u(X) = -g(\varphi \xi, X), \quad and \quad v(X) = -g(\varphi N, X). \tag{5.46}$$

Operating φ on X, we get

$$\varphi X = \varphi(P_1 X) + u(X)N. \tag{5.47}$$

If we put $\varphi X = \varphi(P_1 X)$ in above relation, we obtain the following:

$$\varphi X = \omega X + u(X)N, \tag{5.48}$$

where ω is a tensor field defined as $\omega = \varphi \circ P_1$ of type $(1,1)$.

Again operating ω to (5.48), we get

$$\omega^2 X = X - \eta(X)\xi - u(X)(U), \qquad u(U) = 1. \tag{5.49}$$

Replacing Y by ξ in (4.25) with (3.19) and (5.48), we have

$$\nabla_X \xi = (1-\alpha)\omega X + \beta(X - \eta(X)\xi), \tag{5.50}$$

$$B(X, \xi) = (1-\alpha)u(X). \tag{5.51}$$

From the covariant derivative of $g(\xi, N) = 0$ in terms of X with (3.19), (5.49) and (4.33), we obtained that

$$C(X, \xi) = (1-\alpha)v(X) + \beta\eta(X). \tag{5.52}$$

Now, from (4.23) comparing the different components, we get

$$(\nabla_X \omega)Y = (1-\alpha)[g(X,Y)\xi - \eta(Y)X] + \beta[g(X,\varphi Y)\xi + \eta(Y)\omega X] \tag{5.53}$$

$$+B(X,Y)\bar{U} + u(Y)A_N X,$$

$$(\nabla_X u)Y = u(Y)\tau(X) - B(X,\omega X) + \beta\eta(Y)u(X), \tag{5.54}$$

$$(\nabla_X v)Y = v(Y)\tau(X) + g(A_N X, \omega Y) + [(1-\alpha)\eta(X) + \beta v(X)]\eta(Y), \tag{5.55}$$

$$\nabla_X \bar{U} = \omega(A_N X - \tau(X)\bar{U} + [(1-\alpha)\eta(X) + \beta v(X)]\xi, \tag{5.56}$$

$$\nabla_X \bar{V} = \omega(A_E^* X) - \tau(X)U + \beta u(X)\xi, \tag{5.57}$$

$$B(X, \bar{U}) = C(X, \bar{V}). \tag{5.58}$$

where $\bar{U}$ and $\bar{V}$ are the structure tensor fields on M.

6. Re-current screen semi-invariant lightlike hypersurface

Now, we give the following definition

Definition 6.1. *Let M be a screen semi-invariant lightlike hypersurface of trans-para Sasakian manifold $\bar{M}$ and μ be a 1-form on M. If M admits a re-current tensor field ω such that*

$$(\nabla_X \omega)Y = \mu(X)\omega Y \tag{6.59}$$

then said to be recurrent [9].

Theorem 6.1. *Let M is a re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then*

(1) $\alpha = 1$, $\beta = 0$ *i.e., $\bar{M}$ is a para-Sasakian manifold,*

(2) ω *is parallel with respect to the induced connection ∇ on M,*

(3) $A_N X = -\mu(X)\bar{U} - v(X)\xi$

(4) $A_\xi^* X = -\mu(X)\bar{V} - u(X)\xi$.

for all $X, Y \in \Gamma T(M)$.

Proof. (1) From (5.53), we have

$$\mu(X)\omega Y = (1-\alpha)(g(X,Y)\xi - \eta(Y)X) + \beta(g(X,\varphi Y)\xi + \eta(Y)\omega X) \tag{6.60}$$

$$+ B(X,Y)\bar{U} + u(Y)A_N X.$$

Setting $Y = \xi$ in (6.60) and using (2.4), we obtained that

$$(1-\alpha)\{X - \eta(X)\xi + u(X)U\} + \beta\omega X = 0. \tag{6.61}$$

Putting $X = \xi$ in (6.61) and using the fact that $\omega\xi = V$, we have

$$(1-\alpha)\xi + \beta V = 0. \tag{6.62}$$

Taking the scalar product with N and $\bar{U}$ to the above equation, we get

$$\alpha = 1, \qquad \beta = 0. \tag{6.63}$$

Therefore, $\bar{M}$ is a para-Sasakian manifold with a quarter-symmetric metric connection and we arrive at (1).

(2) Taking $Y = \xi$ to (6.60) and in view (4.32) and (5.46), we get

$$\mu(X)V = -g(X,\xi)\xi. \tag{6.64}$$

Taking inner product of $\bar{U}$ it follows that $\mu = 0$. Thus, ω is parallel with respect to the connection ∇ and we arrive at (2).

(3) Now taking $Y = \bar{U}$ in (6.60) and using the fact that $\mu(X) = 0$, we obtain (3). Similarly taking inner product $\bar{V}$ to (6.60), we get (4).

Theorem 6.2. *Let M be a re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then D_1 and D_2 are parallel distributions on M.*

Proof. Taking inner product with $\bar{V}$ to (5.53) and in view of (6.59), we can write as

$$B(X,Y) = u(Y)u(A_N X). \tag{6.65}$$

Putting $Y = \bar{V}$ and $Y = \omega Z$ in (6.65), we get

$$B(X,Y) = 0, \quad and \quad B(X,\omega Z) = 0. \tag{6.66}$$

Now, from (5.48) and (5.57), we find for all $Z \in \Gamma(D_0)$,

$$g(\nabla_X \xi, \bar{V}) = B(X,\bar{V}), \tag{6.67}$$

$$g(\nabla_X Z, \bar{V}) = B(X,\omega Z), \quad g(\nabla_X \bar{V}, \bar{V}) = 0. \tag{6.68}$$

From these equations and (6.66), we see that

$$\nabla_X Y \in \Gamma(D_1), \quad \forall X \in \Gamma(TM), \quad \forall Y \in \Gamma(D_1).$$

and hence D_1 is a parallel distribution on M.

On the other hand, setting $Y = \bar{U}$ in (6.60, , we have

$$B(X,\bar{U})\bar{U} = A_N X. \tag{6.69}$$

Using $\omega \bar{U} = 0$ in (6.69), it is obtained that

$$\omega(A_N X) = 0. \tag{6.70}$$

Using this result and equation (5.56) reduced to

$$\nabla_X \bar{U} = \tau(X)\bar{U}. \tag{6.71}$$

It follows that

$$\nabla_X \bar{U} \in \Gamma(D_2), \quad \forall X \in \Gamma(TM),$$

and hence D_2 is a parallel distribution on M.

Therefore immediate consequence of the above theorem and from equation (5.44), we have the following theorem

Theorem 6.3. *Let M be a re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection . Then M is locally a product manifolds $C_{\bar{U}} \times M$ [23], where $C_{\bar{U}}$ is a null curve tangent to D_2 and M is a leaf of the distribution D_1.*

Now, we have following

Definition 6.2. [9] *A lightlike hypersurface of semi-Riemannian manifold is said to be screen conformal if there exists a non-zero smooth function λ such that*

$$A_N X = \lambda A_N^* X \quad or \quad C(X, PY) = \lambda B(X, Y). \tag{6.72}$$

Theorem 6.4. *Let M be a re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Consider that M is a screen conformal lightlike hypersurface. Then M is either geodesic or screen totally geodesic if and only if $X \in \Gamma(D_0)$.*

Proof.   Since M is screen conformal, from Theorem (6.1) using relations (3) and (4), we get

$$\mu(X)U + v(X)\xi = \lambda(\mu(X)\bar{V} + u(X)\xi). \tag{6.73}$$

Taking an inner product with $\bar{V}$ to (6.73), we have

$$\mu(X) = 0. \tag{6.74}$$

So, by using relation (3), (4) and Theorem (6.1), we get the required assertion.

7. Lie re-current screen semi-invariant lightlike hypersurface

This section starts with the following definition:

Definition 7.1. [9] *Let M be a screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection and ρ be a 1-form on M. Then M is said to be Lie re-current if it admits a Lie re-current tensor field ω such that*

$$(\mathcal{L}_X\omega)Y = \rho(X)\omega Y, \tag{7.75}$$

where $\mathcal{L}_X$ denotes the Lie derivative on M with respect to X that is

$$(\mathcal{L}_X\omega)Y = [X, \omega Y] - \omega[X, Y]. \tag{7.76}$$

If the structure tensor field ω satisfies the condition

$$\mathcal{L}_X\omega = 0, \tag{7.77}$$

then ω is said to be Lie parallel. A screen semi-invaraint lightlike hypersurface M of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection is called Lie re-current if its structure tensor field ω is Lie re-current.

Theorem 7.1. *Let M be a Lie re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection . Then the structure tensor field ω is Lie parallel.*

Proof. In view of (7.76), (7.77) and (5.53), we get

$$\rho(X)\omega Y = -\nabla_{\omega Y}X + \omega\nabla_Y X + u(Y)A_N X - B(X,Y)\bar{U} \tag{7.78}$$

$$+(1-\alpha)[g(X,Y)\xi - \eta(Y)X] + \beta g[X,\varphi Y)\xi + \beta\eta(Y)\omega X].$$

Putting $Y = E$ in (7.78) and by the use of (4.31), we have

$$\rho(X)\bar{V} = -\nabla_{\bar{V}}X + \omega\nabla_E X - \beta u(X)\xi. \tag{7.79}$$

Taking inner product with $\bar{V}$ to (7.79), we obtain

$$g(\nabla_{\bar{V}}X, \bar{V}) = u(\nabla_{\bar{V}}X) = 0, \qquad and \quad \eta(\nabla_{\bar{V}}X) = \beta u(X). \tag{7.80}$$

Replacing Y by $\bar{V}$ in (7.78) and using the fact that $\eta(Y) = 0$, we have

$$\rho(X)E = -\nabla_{\bar{E}}X + \omega\nabla_{\bar{V}}X + B(X,\bar{V}) + \bar{U} + (1-\alpha)u(X)\xi. \tag{7.81}$$

Applying ω to the above equation, using (5.49) with (7.80), it is obtained that

$$\rho(X)E = -\nabla_{\bar{E}}X + \omega\nabla_{\bar{V}}X + \bar{U} + \beta u(X)\xi. \tag{7.82}$$

Comparing the above equation with (7.79), we get $\rho = 0$. Therefore we arrive at ω is Lie-parallel.

Theorem 7.2. *Let M be a Lie re-current screen semi-invariant lightlike hypersurface of a tans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then $\alpha = 1$, $\beta = 0$ and $\bar{M}$ is a para Sasakian manifold.*

Proof.   Replacing X by $\bar{U}$ in (7.79) and using (4.32), (4.33), (5.46), (5.53)-(5.56) and $\omega\bar{U} = 0$ and $\omega\xi = 0$, it is obtained that

$$u(Y)A_N\bar{U} - \omega(A_N\omega Y) - A_NY - \tau(\omega Y)\bar{U} \tag{7.83}$$

$$-\alpha v(Y)\xi + \beta\eta(Y)\xi - \alpha\eta(Y)\bar{U} = 0.$$

Taking an inner product with ξ into (7.83) and using the fact that

$$C(X,\xi) = (1-\alpha)v(X) + \beta\eta(X),$$

it is obtained that $(1-\alpha)v(Y) = 0$ and $\beta\eta(Y) = 0$, and hence $\alpha = 1$, $\beta = 0$. That is, $\bar{M}$ is a para Sasakian manifold.

Theorem 7.3. *Let M be a Lie re-current screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection . Then the following statements are holds:*

(1) $\tau = \beta\eta$ *on TM, and*

(2) $A^*_\xi\bar{U} = 0,$ *and* $A^*_\xi\bar{V} = 0.$

for all $X, Y \in T(M)$.

Proof.   Taking inner product with N to (7.79) and using, (4.33), we have

$$-g(\nabla_Y X, N) + g(\nabla_Y X, \bar{U}) = \beta\eta(Y)u(X), \tag{7.84}$$

since $\alpha = 1$ in (7.84). Replacing X by ξ in (7.84) and using (4.28) and (4.32), we get

$$B(X,\bar{U}) = \tau(\omega X). \tag{7.85}$$

Taking $X = \bar{U}$ and using (5.58) and $\omega\bar{U} = 0$, we have

$$C(\bar{U}, \bar{V}) = B(\bar{U}, \bar{U}) = 0. \tag{7.86}$$

taking the inner product with $\bar{V}$ in (7.82) and using (4.32), (5.58), (7.86), and $\alpha = 0$, it is obtained that

$$B(X, \bar{U}) = -\tau(\omega X). \tag{7.87}$$

Comparing the above equation with (7.81), it is obtained that $\tau(\omega X) = 0$.

Replacing X by $\bar{V}$ in (7.83) and using (5.57), we have

$$B(\omega Y, \bar{U}) + \beta\eta(Y) = \tau(Y). \tag{7.88}$$

Taking $Y = \bar{U}$ and $Y = \xi$ and using $\omega\bar{U} = \omega\xi = 0$, it is obtained that

$$\tau(\bar{U}) = 0, \qquad \tau(\xi) = -\beta. \tag{7.89}$$

Setting $X = \omega Y$ to $\tau\omega X) = 0$ and using (5.49) and (7.89), we get $\tau(X) = -\beta\eta(X)$. Thus we have (1).

As $\tau(\omega X) = 0$, from (4.32) and (7.84), we have $g(A^*_\xi \bar{U}, X) = 0$. The non-degeneracy of $S(TM)$ implies that $A^*_E \bar{U} = 0$. Putting X by ξ to (7.80) and using (4.34) and $\tau(\omega X) = 0$, then we obtained $A^*_\xi \bar{V} = 0$, thus we arrive at (2).

8. SCREEN SEMI-INVARIANT HOPF LIGHTLIKE HYPERSURFACE

Definition 8.1. *Let M be a screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ and $\bar{U}$ be a structure tensor field on M. The structure tensor field $\bar{U}$ is called principal if there exists a smooth function σ and $X \in (TM)$ such that*

$$A^*_\xi X = \sigma\bar{U}. \tag{8.90}$$

A screen semi-invariant lightlike hypersurface M of a trans-para Sasakian manifold $\bar{M}$ is called a Hopf lightlike hypersurface if it admits principal vector field $\bar{U} \in (M)$ [9].

If we consider (8.90), from (4.32) and (5.46), we obtain

$$B(X, \bar{U}) = -\sigma v(X), \quad and \quad C(X, \bar{V}) = -\sigma u(X). \tag{8.91}$$

Now, we have the following theorems

Theorem 8.1. *Let M be a screen semi-invariant Hopf lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If M is screen totally umbilical then $\kappa = 0$ and M is a screen totally geodesic null hypersurface for $X, Y \in \Gamma T(M)$.*

Proof. We know that M is screen totally umbilical lightlike hypersurface if there exists a smooth function f such that $A_N X = fg(X, Y)$ or

$$C(X, PY) = fg(X, Y), \tag{8.92}$$

and $f = 0$, we say that M is a screen totally geodesic lightlike hypersurface.

Therefore, in (8.92) replacing PY with $\bar{V}$ and use of (5.46) and (8.91), we find

$$fv(X) = fu(X). \tag{8.93}$$

Putting $X = \bar{U}$ in (8.93) we obtain $f = 0$. So, we get $A_N = 0 = C$ and $\kappa = 0 = g(A_N X, \bar{V})$. Therefore $\kappa = 0$ and M is a screen totally geodesic lightlike hypersurface.

Theorem 8.2. *Let M be a screen semi-invariant Hopf lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. If $\bar{V}$ is a parallel null vector field then M is a Hopf lightlike hypersurface such that $\kappa = 0$.*

Proof. Let us consider $\bar{V}$ is parallel null vector field, from (5.47) and (5.57), we find

$$\varphi(A_E^* X) - \beta u(A_E^* X) N + \tau(X) \bar{V}. \tag{8.94}$$

Applying φ to (8.94) and in view of (2.4), we have

$$A_E^* X - \beta u(A_E^* X) \bar{U} + \tau(X) E = 0. \tag{8.95}$$

Taking inner product with N to (8.95), we get at $\tau = 0$, which yields

$$A_E^* X = \beta u(A_E^* X) \bar{U}. \tag{8.96}$$

Therefore, we can say that M is a Hopf lightlike hypersurface. If we take inner product with $\bar{U}$ to (8.96), we find $\kappa(X) = 0 = B(X, \bar{U})$.

9. Integrability of screen semi-invariant lightlike hypersurface

This section explores the integrability conditions for the distributions engage with the screen semi-invariant hypersurface of a trans-para Sasakian manifold with a quarter-symmetric metric connection :

We note that $X \in D_1$ if and only if $u(X) = 0$. Now from (5.54), we have for all $X, Y \in \Gamma(TM)$.

$$u(\nabla_Y X) = \nabla_X u(Y) + u(Y)\tau(X) - B(X, \omega Y) + \beta\eta(Y)u(X) \tag{9.97}$$

from which we get

$$u([X, Y]) = B(X, \omega Y) - B(\omega X, Y) + \nabla_X u(Y) - \nabla_Y u(X) \tag{9.98}$$

$$+ u(Y)\tau(X) - u(X)\tau(Y) + \beta\eta(Y)u(X) - \beta\eta(X)u(Y).$$

Let $X, Y \in D_1$. Then $u(X) = 0 = u(Y)$, and from the equation (9.98) we get

$$u([X, Y]) = B(X, \omega Y) - B(\omega X, Y),$$

for all $X, Y \in D_1$. Thus we obtain a necessary and sufficient condition for the integrability of the distribution D_1 in the following:

Theorem 9.1. *Let M be a screen semi-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection . Then the distribution D_1 is integrable if and only if*

$$B(X, \omega Y) = B(\omega X, Y), \qquad X, Y \in \Gamma(D_1). \tag{9.99}$$

As a consequence of the theorem (9.1), we obtain a results based on radical anti-invariant lightlike hypersurface of trans-para Sasakian manifolds:

Theorem 9.2. *Let M be a radical anti-invariant lightlike hypersurface of a trans-para Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection. Then the screen distribution $S(TM)$ of M is an integrable distribution if and only if*

$$B(X, \omega Y) = B(\omega X, Y). \tag{9.100}$$

Now, we find a necessary and sufficient condition for the distribution D_2 to be integrable.

Theorem 9.3. *Let M be a screen semi-invariant lightlike hypersurface of a trans-a Sasakian manifold $\bar{M}$ with a quarter-symmetric metric connection . Then the distribution D_2 is integrable if and only if*

$$A_N \xi + (1 - \alpha)\bar{U} + \beta\omega\bar{U} = 0 \tag{9.101}$$

Proof.   It is Noted here that $X \in D_2$ if if and only if $\varphi X = \omega X = 0$. Now for all $X, Y \in \Gamma(TM)$, in view of (5.53), we arrive at

$$\omega(\nabla_X Y) = \nabla_X \omega(Y) - u(Y) A_N X - B(X,Y)\bar{U} \tag{9.102}$$

$$(1 - \alpha)(g(X,Y)\xi - \eta(Y)X) + \beta(g(X,\varphi Y)\xi + \eta(Y)\omega X).$$

From (9.102), we get

$$\omega([X,Y]) = \nabla_X \omega(Y) - \nabla_Y \omega(X) + u(X) A_N Y - u(Y) A_N X \tag{9.103}$$

$$+(1 - \alpha)(\eta(Y)X - \eta(X)Y) + \beta(\eta(Y)\omega X - \eta(X)\omega Y).$$

In particular for $X, Y \in D_2$, we get

$$\omega([X,Y]) = +u(X) A_N Y - u(Y) A_N X + (1 - \alpha)(\eta(Y)X - \eta(X)Y) \tag{9.104}$$

$$+\beta(\eta(Y)\omega X - \eta(X)\omega Y).$$

Setting $X = \bar{U}$ and $Y = \xi$ and hence, D_2 is integrable if and only if

$$\omega[\bar{U}, \xi] = 0 \tag{9.105}$$

which, in view of (9.105), is equivalent to (9.101).

References

[1] Acet, B. E., Yüksel, Perktas. S and Kilic, E., Symmetric lightlike hypersurfaces of a para-Sasakian space form, Analele Stiintifice Ale Universitatii Al. I. Cuza Iasi (N.S.), 2016; 2: 915-926.

[2] Ahmad, M., Siddiqi, M. D., and Ojha, J. P., Semi-invariant submanifolds of a Kenmotsu manifold immersed in an almost r-contact structure admitting a quarter symmetric non-metric connection, J. Math. Comput. Sci., 2012: 2(4): 982-998.

[3] Duggal K. L. Evolving null horizons near an isolated black hole. Applied Physics Research, 2016; 8(3): 90-93.

[4] Duggal K. L. Foliations of lightlike hypersurfaces and their physical interpretation. Central European Journal of Mathematics. 2012; 10(5): 1789-1800.

[5] Duggal, K. L and Bejancu, A., Lightlike submanifolds of semi-Riemannian manifolds and applications, Mathematics and Its Applications, Kluwer Publisher, 1996.

[6] Duggal, K. L and Sahin, B., Generalized Cauchy Riemann lightlike submanifolds of Kaehler manifolds, Acta Math. Hungarica, 2006; 112: 107-130.

[7] Duggal, K. L. and Sahin, B, Lightlike submanifolds of indefinite Sasakian manifolds, Int. J. Math. and Math. Sci., Article ID 57585, 2007.

[8] Duggal, K.L and Sahin, B., Differential geometry of lightlike submanifolds, Frontiers in Mathematics, 2010.

[9] Duggal, K. L., Sharma, R., Semi-symmetric metric connection in a semi-Riuemannian manifood, Indian J. Pure Appl. Math. 1986; 17: 1276-1283.

[10] Galloway, G. J., Lecture notes on spacetime geometry, Beijing Int. Math. Research Center, 207; 1-55.

[11] Golab, S., On semi-symmetric and quarter-symmetric linear connections, Tensor(N.S.), 1975; 29: 249-254.

[12] Gray, A and Hervella, L. M., The sixteen classes of almost Hermitian manifolds and their linear invariants, Ann. Mat. Pura. Appl. 1980; 123(4): 35-58.

[13] Haseeb, A, Khan, M. A., Siddiqi, M. D., Some more results on an (ε)-Kenmotsu manifold with a semi-symmetric semi-metric connection, Acta Mathematica Univ. Comenianae, 2016; 85(1): 9-20.

[14] Oubina, J. A., New classes of almost contact metric structures, Publ. Math. Debrecen, 1985; 32: 187-193.

[15] Marrero, J. C. and Chinea, D., On trans-Sasakian manifolds, Ann. Mat. Pura Appl., 1992; 162(4): 77-86.

[16] Mandal, K., De. U. c., Quarter-symmetric metric connection in a P-Sasakian manifold, Anal. Univ. Vest. Timis. Ser. Math. Inform., LIII, 2015; 137-150.

[17] Kumar, R, and Nagaich, R. K., Characteriztion of GCR-lightlike warped product of indefinite cosympletic manifolds, Novi Sad J. Math., 2012; 42: 111-126.

[18] Jin, D. H., Special lightlike hypersurfaces of indefinite Kaehler manifolds, Filomat, 2016; 30: 1919-1930.

[19] Jin, D. H., Lightlike hypersurfaces of indefinite Kaehler manifolds with a semi-symmetric non-metric connection, J. Korean Math. Soc., 2017; 54: 101-119.

[20] Jin, D. H., Geometry of lightlike hypersurface of an indefinite Sasakian manifolds, Ind. J. Pure and Appl. Math., 2010; 41(4): 569-581.

[21] Jin, D. H., Lightlike hypersurafce of a trans Sasakian manifold with quarter-symmetric metric connection, Applied. Math. Sci., 2015; 9(28): 1393-1406.

[22] Jin, D. H., Geometry of lightlike hypersurface of an indefnite Kaehler manifold with quarter-symmetric metric connection, Applied Math. Sci., 2016; 10(6): 289-299.

[23] Kaneyuki, S. and Konzai, M., Paracomplex structure and affine symmetric spaces, Tokyo J. Math., 1985; 8: 301-308.

[24] Rastogi, S. C., On quarter-symmetric metric connection, C. R. Acad. Sci. Bulgar, 1978; 31: 811-814.

[25] Siddiqi, M. D., Ahmed, M and Ojha, J. P., CR-submanifolds of nearly-trans hyperbolic sasakian manifolds admitting semi-symmetric-non-metric connection. African Diaspora J. Math. N.S., 2014; 17(10): 93-105.

[26] Zamkovoy, S., Canonical connection on paracontact manifolds, Ann. Glob. Anal. Geo., 2009; 36: 37-60.

[27] Zamkovoy, S., On the geometry of trans-para Sasakian manifolds, Filomat, 2019; 33(18): 6015-6024.

[28] Zamkovoy, S., On para-Kenmotsu manifolds, Filomat, 2018; 32(14): 4971-4980.

DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, JAZAN UNIVERSITY, JAZAN, KINGDOM OF SAUDI ARABIA

Email address: `anallintegral@gmail.com, msiddiqi@jazanu.edu.sa`