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SEMI-SYMMETRIC METRIC CONNECTION ON COSYMPLECTIC MANIFOLDS

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Abstract. In this paper we studied almost contact manifolds with semi-symmetric connection, especially Sasakian manifolds. Curvature, sectional curvature and ϕ-sectional curvature are calculated by semi-symmetric connection. Furthermore; geometric properties of integral submanifold of Sasakian manifolds are investigated.

1. Introduction

The idea of a semi symmetric connection on a smooth manifolds was first introduce by Friedmann and Schouten in 1924, [3]. The Sasakian manifolds were introduced in the 1960's by S. Sasaki as an odd-dimensional analogous of Kaehler manifolds. Kaehler manifolds are a classical object of differential geometry and well studied in literature. Compared to that Sasakian manifolds have only recently become subject of deeper research in mathematics and physics. Semi-symmetric connection studied by many authors from 1924 so far. In 1993, Benjancu and Duggal [2] introduced the concept of (ε)-Sasakian manifolds. Afterwards, in 2014, Ram Nawal Singh, Shravan Kumar Pandey, Giteshwari Pandey and Kiran Tiwari examined semi-symmetric connection in an (ε)-Kenmotsu manifold.

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In the present paper, in the first section, Sasakian manifold are examined, then in next section cosymplectic manifolds are studied using semi symmetric metric connection.

2. Preliminaries

Let M be an odd dimensional smooth manifold with a Riemannian metric g and Riemannian connection ∇. Denote by TM the Lie algebra of vector fields on M. Then M is said to be an almost contact metric manifold if there exist on M a tensor ϕ of type $(1,1)$, a vector field ξ called structure vector field and η, the dual 1-form of ξ satisfying the following

$$\phi^2 X = -X + \eta(X)\xi, \quad g(X,\xi) = \eta(X) \tag{2.1}$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \quad \eta \circ \phi = 0 \tag{2.2}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{2.3}$$

for any $X, Y \in TM$. In this case

$$g(\phi X, Y) = -g(X, \phi Y). \tag{2.4}$$

If $d\eta(X,Y) = g(X,\phi Y)$, for every $X, Y \in TM$, then we say that M is a contact metric manifold. If ξ is a killing vector field with respect to g, the contact metric structure is called a K-contact structure. It is easy to prove that a contact metric manifold is K-contact if and only if $\nabla_X \xi = -\phi X$, for any $X \in TM$, where ∇ denotes the Levi-Civita connection on M. We are thus led to define four tensors N^1 , N^2 , N^3 , N^4 by

$$\begin{aligned} N^{(1)}(X,Y) &= [\phi,\phi](X,Y) + 2d\eta(X,Y)\xi, \\ N^{(2)}(X,Y) &= (\mathcal{L}_{\phi X}\eta)(Y) - (\mathcal{L}_{\phi Y}\eta)(X), \\ N^{(3)} &= (\mathcal{L}_{\xi}\phi)X, \\ N^{(4)} &= (\mathcal{L}_{\xi}\eta)X. \end{aligned}$$

An almost contact structure (ϕ,ξ,η) is normal if and only if these four tensors are equal to zero. Now we give some useful theorems.

Theorem 2.1. [2] *An almost contact metric struture (ϕ,ξ,η,g) is Sasakian if and only if*

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X.$$

A Sasakian manifold is K-contact then ξ is Killing vector field and $\nabla_X \xi = -\phi X$.

Proposition 2.1. [2] *On a Sasakian manifold,*

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

Theorem 2.2. [2] *A contact metric manifold is K-contact if and only if the sectional curvature of all plane sections containing ξ are equal to 1. Moreover, on a K-contact manifold,*

$$R(X,\xi)\xi = X - \eta(X)\xi.$$

Let M be a submanifold of $\tilde{M}$ and TM and $T^{\perp}M$ be the Lie algebras of vector fields tangential and normal to $\tilde{M}$, respectively. Suppose $\tilde{\nabla}$ is the induced Levi-Civita connection on $\tilde{M}$. The Gauss and Weingarten formulas are given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.5}$$

$$\tilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{2.6}$$

for all $X, Y \in TM$ and $V \in T^{\perp}M$, where $\nabla^{\perp}$ is the connection on the normal bundle $T^{\perp}M$, h is the second fundamental form and A_V is the Weingarten map associated with V as

$$g(A_V X, Y) = g(h(X,Y), V), \tag{2.7}$$

for then using the standard formula namely Koszul formula for the Levi-Civita connection,

$$\begin{aligned} g(\nabla_X Y, Z) = \frac{1}{2}\{&Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) \\ &+ g([X,Y],Z) + g([Z,X],Y) + g([Z,Y],X)\}, \end{aligned} \tag{2.8}$$

for all $X, Y \in TM$.

3. Semi-symmetric metric connections

A linear connection $\bar{\nabla}$ defined on contact metric manifold M is said to be semi-symmetric connection[3], if its torsion tensor

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

satisfies

$$\bar{T}(X,Y) = \eta(Y)X - \eta(X)Y.$$

Further, a connection is called a semi-symmetric metric connection[5] if

$$(\bar{\nabla}_X g)(Y,Z) = 0.$$

The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection is given by[4]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{3.9}$$

Let M be a Sasakian manifold and ∇ be a Levi-Civita connection defined on M. Using 3.9 we obtain

$$(\bar{\nabla}_X \eta)Y = \bar{\nabla}_X g(Y,\xi) - \eta(\bar{\nabla}_X Y) = -\eta(\nabla_X Y) - \eta(Y)\eta(X) + g(X,Y). \tag{3.10}$$

From the definition immediately we obtain the following useful facts.

1)$\bar{\nabla}_X \phi Y = \nabla_X \phi Y - g(X,\phi Y)\xi$,

2)$\bar{\nabla}_{\phi X} Y = \nabla_{\phi X} Y + \eta(Y)\phi X - g(\phi X, Y)\xi$,

3)$\bar{\nabla}_{\phi X} \phi Y = \nabla_{\phi X} \phi Y + \eta(X)\eta(Y)\xi - g(X,Y)\xi$,

4)$\bar{\nabla}_{\phi X}\xi = 0$,

5)$\bar{\nabla}_\xi X = \nabla_\xi X$,

for all $X, Y \in TM$.

Lemma 3.1. *On Sasakian manifold,*

$$(\bar{\nabla}_{\phi X}\phi)Y = g(\phi X, Y)\xi - g(X,Y)\xi - \eta(Y)\phi X + \eta(Y)X.$$

Proof.

$$\begin{aligned}(\bar{\nabla}_{\phi X}\phi)Y = \bar{\nabla}_{\phi X}\phi Y - \phi\bar{\nabla}_{\phi X}Y &= \nabla_{\phi X}\phi Y - g(\phi X, \phi Y)\xi - \phi\nabla_{\phi X}Y - \eta(Y)\phi^2 X,\\ &= \nabla_{\phi X}\phi Y - \phi\nabla_{\phi X}Y + \eta(Y)\phi X - g(X,Y)\xi,\\ &= g(\phi X, Y)\xi - g(X,Y)\xi - \eta(Y)\phi X + \eta(Y)X.\end{aligned}$$

The proof is completed.

Let the curvature tensor$\bar{R}$ given by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z,$$

where $\bar{\nabla}$ is semi-symmetric connection. Using 3.9, we obtain routinely

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z &+ \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y,\\ &- g(Y,Z)X + g(X,Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi,\\ &+ g(\nabla_X\xi, Z)Y - g(Y,Z)\nabla_X\xi - g(Z,\nabla_Y\xi)X + g(X,Z)\nabla_Y\xi.\end{aligned} \tag{3.11}$$

For Sasakian manifolda the equation 3.9 reduces to

$$\begin{aligned}\bar{R}(X,Y)Z &= R(X,Y)Z + \eta(Z)\eta(Y)X - \eta(Z)\eta(X)Y, \\ &- g(Y,Z)X + g(X,Z)Y + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi, \\ &- g(\phi X,Z)Y + g(Y,Z)\phi X + g(Z,\phi Y)X - g(X,Z)\phi Y.\end{aligned} \tag{3.12}$$

To calculate the sectional curvature, first we have

$$\begin{aligned}\bar{R}(X,Y,Y,X) &= R(X,Y,Y,X) + \eta(Y)\eta(Y)g(X,X), \\ &- g(Y,Y)g(X,X) + g(X,Y)g(X,Y) + \eta(X)\eta(X)g(Y,Y).\end{aligned} \tag{3.13}$$

Assume $\{X,Y\}$ are orthonormal, then

$$\bar{R}(X,Y,Y,X) = R(X,Y,Y,X) + \eta(Y)\eta(Y) + \eta(X)\eta(X) - 1, \tag{3.14}$$

therefore

$$\bar{K}(X,Y) = K(X,Y) + \eta(Y)\eta(Y) + \eta(X)\eta(X) - 1. \tag{3.15}$$

For Sasakian manifolds we have $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$, then from 3.9, we obtain

$$\bar{R}(X,Y)\xi = \eta(Y)X - \eta(X)Y - \eta(X)\phi Y + \eta(Y)\phi X.$$

3.1. **Integral submanifolds.**

Definition 3.1. *A submanifold N of M is an integral submanifold, if $\eta(X) = 0$ for every $X \in TN$.* [1]

Lemma 3.2. *Let M be a Sasakian manifold with a semi-symmetric metric connection. Assume N be an integral submanifold, then*

$$(\bar{\nabla}_X \phi)Y = g(X,Y)\xi,$$

for any $X, Y \in TN$.

Proof. If N be an integral submanifold, then ξ is normal to N, hence

$$(\bar{\nabla}_X\phi)Y = \bar{\nabla}_X\phi Y - \phi\bar{\nabla}_X Y = \nabla_X\phi Y - g(X,\phi Y)\xi - \phi\nabla_X Y + g(Y,\xi)\phi X = (\nabla_X\phi)Y.$$

Using theorem 2.1 the proof is trivial.

For integral submanifolds the equation 3.9 become to

$$\bar{R}(X,Y)Z = R(X,Y)Z + g(X,Z)Y - g(Y,Z)X - g(X,Z)\phi Y + g(Y,Z)\phi X, \tag{3.16}$$

which present the relation between curvature tensors of connections $\bar{\nabla}$ and ∇ in integral submanifolds of Sasakian manifolds. From 3.16, we get

$$g(\bar{R}(X,Y)Z,V) = g(R(X,Y)Z,V) + g(X,Z)g(Y,V) - g(Y,Z)g(X,V). \tag{3.17}$$

Suppose$\bar{R}(X,Y)Z = 0$, which by virtue of the equation 3.17 yields

$$g(R(X,Y)Z,V) = g(Y,Z)g(X,V) - g(X,Z)g(Y,V). \tag{3.18}$$

We know $R(X,\xi)\xi = X$, and we can caculate easily $R(\xi,X)\xi = X$, hence

$$\bar{R}(\xi,X)\xi = 2X - \phi X,$$

it's trivial $\bar{R}(X,Y)\xi = R(X,Y)\xi = 0$ and $\bar{R}(X,\xi)\xi = X$. Also, $\phi-$sectional curvature is defined by

$$K(\sqcap) = K(X,\phi X) = R(X,\phi X;\phi X,X).$$

Assume $X \in N$ be an unit vector field, then

$$\begin{aligned}\bar{R}(X,\phi X;\phi X,X) &= R(X,\phi X;\phi X,X) - g(X,X)g(\phi X,\phi X)\\ &= R(X,\phi X;\phi X,X) - 1,\end{aligned}$$

and we conclude $\bar{K} = K - 1$.

Lemma 3.3. *Let N be an integral submanifold of Sasakian maniold M, then*

$$\nabla_\xi Y = [\xi,Y],$$

for all $X,Y \in N$.

Proof. By 2.8, following equations are obtained

$$2g(\nabla_\xi X,Y) = \xi g(X,Y) + g([\xi,X],Y) + g([Y,\xi],X),$$

using $\xi g(X,Y) = g(\nabla_\xi X,Y) + g(X,\nabla_\xi Y)$ leads to

$$g(\nabla_\xi X,Y) = g(X,\nabla_\xi Y) + g([\xi,X],Y) + g([Y,\xi],X). \tag{3.19}$$

Also,

$$2g(\nabla_X\xi,Y) = \xi g(X,Y) + g([X,\xi],Y) + g([Y,\xi],X) = 0. \tag{3.20}$$

Comparing 3.19 and 3.20 complete the proof.

4. **Cosymplectic manifolds**

A normal almost contact metric manifold M is called a cosymplectic manifol if

$$(\nabla_X \phi)Y = 0, \quad \nabla_X \xi = 0, \tag{4.21}$$

where ∇ denotes Levi-Civita connection. From we have

$$(\bar{\nabla}_X \phi)Y = -\eta(Y)\phi X - g(X, \phi Y)\xi. \tag{4.22}$$

Following facts easily can be obtained

(1) $(\bar{\nabla}_X \phi Y) = \phi \nabla_X Y - g(X, \phi Y)\xi$,
(2) $\bar{\nabla}_\xi \phi X = \nabla_\xi \phi X$,
(3) $\bar{\nabla}_\xi Y = \nabla_\xi Y$.

Using obtained facts we obtain

$$g(\bar{\nabla}_X \phi Y, \xi) = g(\phi X, Y), \tag{4.23}$$

$$g(\nabla_\xi \phi X, Y) = g(\nabla_\xi \phi Y, X). \tag{4.24}$$

From 2.4 and 3.9 we get

$$\bar{\nabla}_X \xi = -\phi^2 X, \quad (\bar{\nabla}_X \phi)\xi = -\phi X. \tag{4.25}$$

Lemma 4.1. *Let M be a cosymplectic manifold, then*

$$\eta((\bar{\nabla}_X \phi)Y) = \eta(\bar{\nabla}_X \phi Y),$$

for all $X, Y \in TM$.

Proof. Using 4.22 and other obtained facts for Cosymplectic manifolds we get

$$\eta((\bar{\nabla}_X \phi)Y) = \eta(g(\phi X, Y)\xi) = g(\phi X, Y) = \eta(\bar{\nabla}_X \phi Y),$$

the proof is complete.

Based on theorem 6.8 [1] it can be seen $d\eta = 0$, then

$$2d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]) = 0. \tag{4.26}$$

Assume $X, Y \in TM$ are orthogonal elements. Using 2.8, 4.25 and 4.26 we find out

$$2g(\nabla_X \xi, Y) = X\eta(Y) - Y\eta(X) + g([X, \xi], Y) + \eta([X, Y]) + g([Y, \xi], X).$$

Therefore $g([X,\xi],Y)+g([Y,\xi],X)=0$. Using 2.8 we have

$$\eta(\nabla_X Y)=X\eta(Y),\quad \eta(\nabla_Y X)=Y\eta(X). \tag{4.27}$$

Since M is an almost cosymplectic manifold, from 4.26 following statement is valid

$$(\nabla_X \eta)(Y)-(\nabla_Y \eta)(X)=0.$$

Also, we have

$$(\bar{\nabla}_X \eta)(Y)=\eta(\bar{\nabla}_X \xi, Y)=g(\phi X,\phi Y).$$

Thus

$$(\bar{\nabla}_X \eta)Y+(\bar{\nabla}_Y \eta)X,$$

for all $X,Y\in TM$. Assume $\bar{\nabla}_X \phi Y=0$, for all $X,Y\in TM$, from 3.9 we obtain

$$\nabla_X \phi Y=g(X,\phi Y)\xi. \tag{4.28}$$

For cosymplectic manifold we have

$$g(\nabla_X \phi Y,\xi)=X\eta(\phi Y)-g(\phi Y,\nabla_X \xi)=0.$$

On the other side we know $g(\nabla_X \phi Y,\xi)=g(X,\phi Y)$ we realized that X is orthogonal to $Im\phi$. From 4.28 we have $\phi^2\nabla_X Y=0$, using 2.1 leads to

$$\nabla_X Y=\eta(\nabla_X Y)\xi.$$

Furthermore we have $\nabla_Y X=\eta(\nabla_Y X)\xi$, comparing last two equations we have

$$[X,Y]=(\nabla_X Y-\nabla_Y X)\xi. \tag{4.29}$$

Now we have proved

Theorem 4.1. *Let M be a cosymplectic manifold with semi symmetric metric connection $\bar{\nabla}$. If there is vector fields $X,Y\in TM$, such that $\bar{\nabla}_X Y=0$, then*

$$\phi([X,Y])=0.$$

Proof. From 4.29 the proof is trivial.

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