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EXPONENTIAL DECAY FOR A THERMO-VISCOELASTIC BRESSE SYSTEM WITH SECOND SOUND AND DELAY TERMS

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Abstract. In this paper, we consider a thermo-viscoelastic Bresse system with second sound and delay terms, where the heat flux is given by Cattaneo's law. Regardless of the speeds of wave propagation and the stable number, which is introduced in [14, 15], we prove an exponential stability result using energy method under suitable assumptions on the weights of the delays and the frictionals damping.

1. Introduction

In the present paper, we consider the following thermo-viscoelastic Bresse system with second sound and delay terms

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu_1\varphi_t + \mu_2\varphi_t\left(x, t - \tau_1\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \delta\int_0^t g\left(t - s\right)\psi_{xx}\left(x, s\right)ds + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) + \lambda_1\omega_t + \lambda_2\omega_t\left(x, t - \tau_2\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \alpha q_t + \beta q + \theta_x = 0, \end{cases} \tag{1.1}$$

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* Dedicated to Professor Sadık Keleş on the occasion of his retirement from Inonu University.

with the initial data and boundary conditions

$$\begin{aligned}
&\varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x),\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x),\\
&\omega(x,0)=\omega_0(x),\omega_t(x,0)=\omega_1(x),\theta(x,0)=\theta_0(x),\ \theta_t(x,0)=\theta_1(x),\\
&q(x,0)=q_0(x),\ q_t(x,0)=q_1(x).
\end{aligned}$$

$$\begin{aligned}
&\varphi(0,t)=\psi_x(0,t)=\omega_x(0,t)=\theta(0,t)=\omega(L,t)=\psi(L,t)=\varphi_x(L,t)=q(L,t)=0,\\
&\varphi_t(x,t-\tau_1)=f_0(x,t-\tau_1),\ \ (x,t)\in(0,L)\times(0,\tau_1),\\
&\omega_t(x,t-\tau_2)=\widetilde{f_0}(x,t-\tau_2),\ \ (x,t)\in(0,L)\times(0,\tau_2).
\end{aligned} \tag{1.2}$$

where $(x,t)\in(0,L)\times\mathbb{R}_+$, $\rho_1,\rho_2,\rho_3,\alpha,\beta,k,k_0,l,b,\delta,\gamma,\mu_1,\lambda_1$ are positive constants, μ_2 and λ_2 are real numbers, $\tau_1,\tau_2>0$ represent the time delays, θ is the difference temperature, q is the heat flux and g is a positive function satisfying some conditions to be determined later.

Originally, the Bresse system consists of three wave equations where the main variables describing the longitudinal, vertical and shear angle displacements, which can be represented as (see [6]):

$$\begin{cases}
\rho_1\varphi_{tt}=Q_x+lN+F_1,\\
\rho_2\psi_{tt}=M_x-Q+F_2,\\
\rho_1\omega_{tt}=N_x-lQ+F_3,
\end{cases} \tag{1.3}$$

where in our work

$$\begin{aligned}
M &= b\psi_x-\delta\int_0^t g(t-s)\,\psi_x(.,s)\,ds,\ \ N=k_0(\omega_x-l\varphi),\ \ Q=k(\varphi_x+\psi+l\omega),\\
F_1 &= -\mu_1\varphi_t-\mu_2\varphi_t(.,t-\tau_1),\ \ F_2=0,\ \text{and}\ F_3=-\lambda_1\omega_t-\lambda_2\omega_t(.,t-\tau_2).
\end{aligned}$$

N, Q and M denote the axial force, the shear force and the bending moment. By ω,φ, and ψ, we are denoting the longitudinal, vertical and shear angle displacements. Here $\rho_1=\rho A$, $\rho_2=\rho l$, $b=EI$, $k_0=EA$, $k=k_0GA$ and $l=R^{-1}$. For material properties, we use ρ for density, E for the modulus of elasticity, G for the shear modulus, k for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross-section and R for the radius of curvature and we assume that all this quantities are positives. Also by F_i we are denote external forces. The Bresse system (1.3), is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered ($l=0$).

The issue of existence and stability of Bresse system has attracted a great deal of attention in the last decades (e.g. [1, 2, 3, 6, 10, 11, 12, 16, 17, 18, 21, 22]). In the absence of viscoelastic damping ($g=0$), frictionals damping $\mu_1=\lambda_1=0$ and delay terms $\mu_2=\lambda_2=0$, Keddi et al. [14] studied the following one-dimensional thermoelastic Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0, \end{cases} \tag{1.4}$$

where the heat conduction is given by Cattaneo's law effective in the shear angle displacement. They established the well-posedness of the system and proved, under a condition on the parameters ζ, k and k_0, which is

$$\zeta := \left(1 - \frac{\tau k \rho_3}{\rho_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right) - \frac{\tau\gamma^2}{b} = 0 \text{ and } k = k_0,$$

that the system was exponentially stable depending on the stable number of the system, and showed that in general, the system was polynomially stable if $\zeta \neq 0$ and $k = k_0$. Li et al. [15] extended this last result to the following Bresse system with delay

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\omega\right)_x - k_0 l\left(\omega_x - l\varphi\right) + \mu\varphi_t\left(x, t - \tau_0\right) = 0, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\omega\right) + \gamma\theta_x = 0, \\ \rho_1\omega_{tt} - k_0\left(\omega_x - l\varphi\right)_x + kl\left(\varphi_x + \psi + l\omega\right) = 0, \\ \rho_3\theta_t + q_x + \gamma\psi_{tx} = 0, \\ \tau q_t + \beta q + \theta_x = 0. \end{cases} \tag{1.5}$$

They proved that the system is well-posed by using the semigroup method, and under a similar condition on the precedent parameters, that is

$$\zeta := \left(\tau - \frac{\rho_1}{k\rho_3}\right)\left(\frac{\rho_2}{b} - \frac{\rho_1}{k}\right) - \frac{\tau\gamma^2\rho_1}{bk\rho_3} = 0 \text{ and } k = k_0,$$

they showed that the dissipation induced by the heat is strong enough to exponentially stabilize the system in the presence of a "small" delay when the stable number is zero.

Motivated by the works mentioned above, we investigate system (1.1) under suitable assumptions and show that even in the presence of the viscoelastic term ($g \neq 0$), the frictionals damping ($\lambda_1, \mu_1 \neq 0$) and the second delay term ($\lambda_2 \neq 0$), we can establish an exponential decay result regardless of the stable number ζ. Introducing the viscoelastic term together with the frictionals damping in the internal feedback of thermoelastic Bresse system with second sound makes our problem different from those considered so far in the literature. We prove our result by using the energy method together with some hypotheses on the weights of the delays and the frictionals damping as well the relaxation function g.

This paper is organized as follows: In Section 2, we introduce some assumptions needed in our work. In section 3, we shall give some technical lemmas and state with proof our main result.

2. Preliminaries

In this section, we present some materials needed in the proof of our results. We also state, without proof, a local existence result for problem (1.1). The proof can be established by using Faedo–Galerkin method [7]. Throughout this paper, c or C represents a generic positive constant and is different in various occurrences.

We shall use the following assumptions:

$(\mathbf{A}_1)$ $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$g(0) > 0, \qquad b - \delta \int_0^\infty g(s)ds = b - \delta g_1 = l > 0, \tag{2.6}$$

$(\mathbf{A}_2)$ There exists a non-increasing differentiable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$g'(t) \le -\eta(t)g(t), \qquad t \ge 0 \quad \text{and} \quad \int_0^\infty \eta(t)dt = +\infty. \tag{2.7}$$

Remark 2.1. *Since g is positive and $g(0) > 0$ then for any $t_0 > 0$ we have*

$$\int_0^t g(s)\,ds \ \ge \int_0^{t_0} g(s)\,ds = g_0 > 0, \quad \forall t \ge t_0. \tag{2.8}$$

We introduce the new variable as in [20]

$$z_1(x,\rho,t) = \varphi_t(x, t - \tau_1\rho), \qquad x \in (0,L), \rho \in (0,1), t > 0, \tag{2.9}$$

$$z_2(x,\rho,t) = \omega_t(x, t - \tau_2\rho), \qquad x \in (0,L), \rho \in (0,1), t > 0. \tag{2.10}$$

Then, we have

$$\begin{aligned} \tau z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) &= 0, \qquad x \in (0,L), \rho \in (0,1), t > 0, \\ \tau z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) &= 0, \qquad x \in (0,L), \rho \in (0,1), t > 0. \end{aligned}$$

Hence, problem (1.1)-(1.2) is equivalent to the following system, where $(x,\rho,t)\in(0,L)\times(0,1)\times\mathbb{R}_+$

$$\begin{cases} \rho_1\varphi_{tt}-k\left(\varphi_x+\psi+l\omega\right)_x-k_0l\left(\omega_x-l\varphi\right)+\mu_1\varphi_t+\mu_2z_1(x,1,t)=0,\\ \tau_1z_{1_t}(x,\rho,t)+z_{1_\rho}(x,\rho,t)=0,\\ \rho_2\psi_{tt}-b\psi_{xx}+k\left(\varphi_x+\psi+l\omega\right)+\delta\int_0^t g\left(t-s\right)\psi_{xx}\left(x,s\right)ds+\gamma\theta_x=0,\\ \rho_1\omega_{tt}-k_0\left(\omega_x-l\varphi\right)_x+kl\left(\varphi_x+\psi+l\omega\right)+\lambda_1\omega_t+\lambda_2z_2(x,1,t)=0,\\ \tau_2z_{2_t}(x,\rho,t)+z_{2_\rho}(x,\rho,t)=0,\\ \rho_3\theta_t+q_x+\gamma\psi_{tx}=0,\\ \alpha q+\beta q+\theta_x=0, \end{cases} \tag{2.11}$$

with the following initial data and boundary conditions

$$\begin{cases} \varphi\left(x,0\right)=\varphi_0\left(x\right),\ \varphi_t\left(x,0\right)=\varphi_1\left(x\right), & x\in(0,L),\\ \psi\left(x,0\right)=\psi_0\left(x\right),\psi_t\left(x,0\right)=\psi_1\left(x\right), & x\in(0,L),\\ \omega\left(x,0\right)=\omega_0\left(x\right),\ \omega_t\left(x,0\right)=\omega_1\left(x\right), & x\in(0,L),\\ q\left(x,0\right)=q_0\left(x\right),\ q_t\left(x,0\right)=q_1\left(x\right), & x\in(0,L),\\ \theta\left(x,0\right)=\theta_0\left(x\right),\ \theta_t\left(x,0\right)=\theta_1\left(x\right), & x\in(0,L),\\ z_1(x,\rho,0)=f_0(x,-\rho\tau_1),\ z_2(x,\rho,0)=\widetilde{f}_0(x,-\rho\tau_2) & (x,\rho)\in(0,L)\times(0,1)\\ z_1(x,0,t)=\varphi_t(x,t),\quad z_2(x,0,t)=\omega_t(x,t) & (x,t)\in(0,L)\times(0,+\infty),\\ \varphi\left(0,t\right)=\psi_x\left(0,t\right)=\omega_x\left(0,t\right)=\theta\left(0,t\right)=0, & t\in(0,+\infty),\\ \omega\left(L,t\right)=\psi\left(L,t\right)=\varphi_x\left(L,t\right)=q\left(L,t\right)=0, & t\in(0,+\infty). \end{cases} \tag{2.12}$$

Along this paper, we use the following notations

$$(f\diamond v)(t)=\textstyle\int_0^t f(t-s)\left(v(t)-v(s)\right)ds,\quad \forall v\in L^2(0,L),$$

$$(f\circ v)(t)=\textstyle\int_0^t f(t-s)(v(s)-v(t))^2ds.$$

The energy functional associated to (2.11)-(2.12), is

$$\begin{aligned} \mathcal{E}(t) &= \frac{1}{2}\int_0^L\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+\rho_3\theta^2+\alpha q^2+\left(b-\delta\int_0^t g(s)ds\right)\psi_x^2\right\}dx\\ &\quad+\frac{1}{2}\int_0^L\left\{\xi_1\int_0^1 z_1^2(x,\rho,t)d\rho+\xi_2\int_0^1 z_2^2(x,\rho,t)d\rho+k\left(\varphi_x+\psi+l\omega\right)^2\right\}dx\\ &\quad+\frac{1}{2}\int_0^L\left\{k_0\left(\omega_x-l\varphi\right)^2+\delta(g\circ\psi_x)\right\}dx \end{aligned} \tag{2.13}$$

we denote $\mathcal{E}(t)=\mathcal{E}\left(t,\varphi,\psi,\omega,\theta,q,z_1,z_2\right)$ and $\mathcal{E}(0)=\mathcal{E}\left(0,\varphi_0,\psi_0,\omega_0,\theta_0,q_0,f_0,\widetilde{f}_0\right)$ for simplicity of notations.

For state a local existence result, we introduce the vector function $\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z_1, z_2)^T$, where $u = \varphi_t$, $v = \psi_t$ and $w = \omega_t$, using the standard Lebesgue space $L^2(0,L)$ and the Sobolev space $H_0^1(0,L)$ with their usual scalar products and norms for define the space $\mathcal{H}$ as follows

$$\mathcal{H} := H_*^1(0,L) \times L^2(0,L) \times \left[\widetilde{H_*^1}(0,L) \times L^2(0,L)\right]^2 \times \left[L^2(0,L)\right]^2 \times \left[L^2((0,L)\times(0,1))\right]^2,$$

where

$$\begin{aligned} H_*^1(0,L) &= \left\{f \in H^1(0,L),\ f(0) = 0\right\}, \\ \widetilde{H_*^1}(0,L) &= \left\{f \in H^1(0,L),\ f(L) = 0\right\}, \\ H_*^2(0,L) &= H^2(0,L) \cap H_*^1(0,L), \\ \widetilde{H_*^2}(0,L) &= H^2(0,L) \cap \widetilde{H_*^1}(0,L). \end{aligned}$$

Proposition 2.1. *Let $\Phi_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, \theta_0, q_0, f_0, \widetilde{f_0})^T \in \mathcal{H}$ be given. Assume that $(A_1), (A_2)$, $\mu_1 > |\mu_2|$ and $\lambda_1 > |\lambda_2|$ are satisfied. Then Problem (2.11)-(2.12) possesses a unique global (weak) solution satisfying*

$$\Phi = (\varphi, u, \psi, v, \omega, w, \theta, q, z_1, z_2)^T \in C\left(\mathbb{R}_+; \mathcal{H}\right).$$

3. **Exponential stability**

In this section, we state and prove our exponential decay result for the energy of the solution of system (1.1)-(1.2), using the Lyapunov functional which is equivalent to the energy functional. To achieve our goal, we need the following technical lemmas.

The two inequalities in the following lemma are introduced in [8] and [13] respectively.

Lemma 3.1. *For any function $g \in C([0,+\infty), \mathbb{R}_+)$ and any $v \in L^2(0,L)$ we have*

$$[g \diamond v(t)]^2\, dx \leq \left(\int_0^t g(s)ds\right) g \circ v(t), \quad \forall t \geq 0, \tag{3.14}$$

$$\int_0^L \left(\int_0^t g(t-s) v_x(s) ds\right)^2 dx \leq 2g_1 \int_0^L g \circ v_x dx + 2g_1 \int_0^L v_x^2 dx. \tag{3.15}$$

Lemma 3.2. *(**Poincaré-type Scheeffer's inequality**, [19]): Let $h \in H_0^1(0,L)$. Then it holds*

$$\int_0^L |h|^2\, dx \leq\ c \int_0^L |h_x|^2\, dx, \quad c = \frac{L^2}{\pi^2}. \tag{3.16}$$

Lemma 3.3. [10] *There exists a positive constant c such that the following inequality holds for every $(\varphi, \psi, \omega) \in \left[H_0^1(0,L)\right]^3$*

$$\int_0^L (\varphi_x^2 + \psi_x^2 + \omega_x^2) dx \leq c \int_0^L \left[b\psi_x^2 + k(\varphi_x + \psi_x + \omega_x)^2 + k_0(\omega_x - l\varphi)^2\right] dx. \tag{3.17}$$

Lemma 3.4. *Let $(\varphi,\psi,\omega,\theta,q,z_1,z_2)$ be the solution of (2.11)-(2.12). Then the energy functional satisfies, for some n_0, $n_0^{'}>0$,*

$$\begin{aligned}\mathcal{E}^{'}(t) \quad \leq \quad & -\beta\int_0^L q^2dx+\frac{\delta}{2}\int_0^L (g^{'}\circ\psi_x)dx-\frac{\delta}{2}g(t)\int_0^L\psi_x^2dx-n_0\left(\int_0^L\varphi_t^2dx+\int_0^L z_1^2(x,1,t)dx\right)\\ & -n_0^{'}\left(\int_0^L\omega_t^2dx+\int_0^L z_2^2(x,1,t)dx\right)\leq 0\end{aligned}$$

where

$$\tau_1\left|\mu_2\right|<\xi_1<\tau_1(2\mu_1-\left|\mu_2\right|) \text{ and } \tau_2\left|\lambda_2\right|<\xi_2<\tau_2(2\lambda_1-\left|\lambda_2\right|). \tag{3.18}$$

Proof. *Multiplying Equation $(2.11)_1$ by φ_t , $(2.11)_3$ by ψ_t, $(2.11)_4$ by ω_t, $(2.11)_6$ by θ_t and $(2.11)_7$ by q, then integrating over $(0,L)$. Next, multiplying $(2.11)_2$ by $(\xi_1/\tau_1)z_1$ and $(2.11)_5$ by $(\xi_2/\tau_2)z_2$ and integrating over $(0,L)\times(0,1)$ with respect to ρ and x, we get*

$$\begin{aligned} & \frac{1}{2}\frac{d}{dt}\int_0^L\left\{\rho_1\varphi_t^2+\rho_2\psi_t^2+\rho_1\omega_t^2+\rho_3\theta^2+b\psi_x^2\right\}dx \qquad (3.19)\\ & +\frac{1}{2}\frac{d}{dt}\int_0^L\left\{k\left(\varphi_x+\psi+l\omega\right)^2+k_0\left(\omega_x-l\varphi\right)^2+\alpha q^2\right\}dx\\ = \quad & -\mu_1\int_0^L\varphi_t^2-\lambda_1\int_0^L\omega_t^2-\mu_2\int_0^L z_1(x,1,t)\varphi_tdx-\beta\int_0^L q^2dx\\ & -\lambda_2\int_0^L z_2(x,1,t)\omega_tdx-\delta\int_0^L\psi_t\int_0^t g(t-s)\psi_{xx}(s)dsdx,\end{aligned}$$

and

$$\begin{aligned}\frac{\xi_1}{\tau_1}\int_0^L\int_0^1 z_1z_{1\rho}(x,\rho,t)d\rho dx &= \frac{\xi_1}{\tau_1}\int_0^L\int_0^1\frac{d}{2d\rho}z_1^2(x,\rho,t)d\rho dx \qquad (3.20)\\ &= \frac{\xi_1}{2\tau_1}\int_0^L[z_1^2(x,1,t)-z_1^2(x,0,t)]dx\\ &= \frac{\xi_1}{2\tau_1}\int_0^L z_1^2(x,1,t)dx-\frac{\xi_1}{2\tau_1}\int_0^L\varphi_t^2dx,\\ \frac{\xi_2}{\tau_2}\int_0^L\int_0^1 z_2z_{2\rho}(x,\rho,t)d\rho dx &= \frac{\xi_2}{\tau_2}\int_0^L\int_0^1\frac{d}{2d\rho}z_2^2(x,\rho,t)d\rho dx \qquad (3.21)\\ &= \frac{\xi_2}{2\tau_2}\int_0^L[z_2^2(x,1,t)-z_2^2(x,0,t)]dx \qquad (3.22)\\ &= \frac{\xi_2}{2\tau_2}\int_0^L z_2^2(x,1,t)dx-\frac{\xi_2}{2\tau_2}\int_0^L\omega_t^2dx. \qquad (3.23)\end{aligned}$$

Now, we estimate the last term on the left-hand side of (3.19).

$$\begin{aligned}\delta\int_0^L\psi_t(t)\int_0^t g(t-s)\psi_{xx}(s)dsdx &= \frac{\delta}{2}\frac{d}{dt}\int_0^L(g\circ\psi_x)dx+\frac{\delta}{2}g(t)\int_0^L\psi_x^2(t)dx \qquad (3.24)\\ & -\frac{\delta}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\int_0^1\psi_x^2(t)dx\right)-\frac{\delta}{2}\int_0^L(g^{'}\circ\psi_x)dx.\end{aligned}$$

We have also

$$\begin{aligned} -\mu_2 \int_0^L z_1(x,1,t)\varphi_t dx &\leq \frac{|\mu_2|}{2}\left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx\right), \\ -\lambda_2 \int_0^L z_2(x,1,t)\omega_t dx &\leq \frac{|\lambda_2|}{2}\left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx\right). \end{aligned}$$

So, we conclude

$$\begin{aligned} \mathcal{E}'(t) \leq& \frac{\delta}{2}\int_0^L (g' \circ \psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2 dx - \left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\int_0^L \varphi_t^2 dx \\ & - \left(\lambda_1 - \frac{\xi_2}{2\tau_2} - \frac{|\lambda_2|}{2}\right)\int_0^L \omega_t^2 dx - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\int_0^L z_1^2(x,1,t)dx \\ & - \left(\frac{\xi_2}{2\tau_2} - \frac{|\lambda_2|}{2}\right)\int_0^L z_2^2(x,1,t)dx. \end{aligned}$$

Using (3.18), we have, for some n_0, $n_0' > 0$,

$$\begin{aligned} \mathcal{E}'(t) \leq& \frac{\delta}{2}\int_0^L (g' \circ \psi_x)dx - \frac{\delta}{2}g(t)\int_0^L \psi_x^2 dx - n_0\left(\int_0^L \varphi_t^2 dx + \int_0^L z_1^2(x,1,t)dx\right) \\ & -n_0'\left(\int_0^L \omega_t^2 dx + \int_0^L z_2^2(x,1,t)dx\right) \leq 0. \end{aligned}$$

Lemma 3.5. *Let* $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ *be a solution of (2.11)-(2.12). Then the functional*

$$\mathcal{I}_1(t) = -\rho_2 \int_0^L \psi_t \left(\int_0^t g(t-s)(\psi(t) - \psi(s)ds\right)dx \tag{3.25}$$

satisfies for any $\delta' > 0$

$$\begin{aligned} \mathcal{I}_1'(t) \leq& -\rho_2\left(g_0 - \delta'\right)\int_0^L \psi_t^2 dx + \left(b^2 + \delta^2 g_1^2 - 2b\delta g_0\right)\delta'\int_0^L \psi_x^2 dx \\ & +k\delta'\int_0^L (\varphi_x + \psi + l\omega)^2 dx - \frac{\rho_2 g(0)}{4\delta'}\int_0^L (g' \circ \psi_x)dx \\ & +C\left(\delta'\right)\int_0^L g \circ \psi_x dx + \frac{1}{2}\int_0^L \theta^2 dx. \end{aligned} \tag{3.26}$$

Proof. *Taking the derivative of* $\mathcal{I}_1$, *using the third equation in (2.11), we obtain*

$$\begin{aligned} \mathcal{I}_1'(t) =& -\rho_2\int_0^L \psi_t\left(g' \diamond \psi\right)dx - \rho_2\left(\int_0^t g(s)\,ds\right)\int_0^L \psi_t^2 dx \qquad (3.27)\\ & + \left(b - \delta\int_0^t g(s)\,ds\right)\int_0^L (g \diamond \psi_x)\psi_x dx + k\int_0^L (\varphi_x + \psi + lw)(g \diamond \psi)dx \\ & +\delta\int_0^L (g \diamond \psi_x)^2 dx - \int_0^L \theta\,(g \diamond \psi_x)dx. \end{aligned}$$

By using Young's inequality, and (3.14), we get, for any $\delta' > 0$

$$\delta\int_0^L (g \diamond \psi_x)^2 dx \leq \delta g_1 \int_0^L (g \circ \psi_x)dx \tag{3.28}$$

$$-\int_0^L \psi_t \left(g' \diamond \psi\right) dx \leq \delta' \int_0^L \psi_t^2 dx - \frac{\rho_2 g(0)}{4\delta'} \int_0^L \left(g' \circ \psi_x\right) dx \tag{3.29}$$

$$k\int_0^L (\varphi_x + \psi + lw)(g \diamond \psi)\, dx \ \leq k\delta' \int_0^L (\varphi_x + \psi + lw)^2 dx + \frac{g_1 k}{4\delta'} \int_0^L (g \circ \psi_x)\, dx \tag{3.30}$$

$$\begin{aligned}\left(b - \delta \int_0^t g(s)\, ds\right) \int_0^L (g \diamond \psi_x)\, \psi_x dx \quad &\leq \quad \left(b^2 + \delta^2 g_1^2 - 2b\delta g_0\right) \delta' \int_0^L \psi_x^2 dx \\ &\quad + \frac{g_1}{4\delta'} \int_0^L (g \circ \psi_x)\, dx \end{aligned} \tag{3.31}$$

$$-\int_0^L \theta\, (g \diamond \psi_x)\, dx \leq \frac{1}{2} \int_0^L \theta^2 dx + \frac{g_1}{2} \int_0^L (g \circ \psi_x)\, dx. \tag{3.32}$$

Combining (3.27)-(3.32), the result follows.

Lemma 3.6. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\epsilon_1, \epsilon_2, \epsilon_3 > 0$, the functional*

$$\mathcal{I}_2(t) = -\frac{\rho_2\rho_3}{\gamma} \int_0^L \theta \int_0^x \psi_t(y) dy dx \tag{3.33}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_2'(t) \quad &\leq \quad -\frac{\rho_2}{\gamma} \int_0^L \psi_t^2 dx + \epsilon_1 \int_0^L (\varphi_x + \psi + l\omega)^2 dx + c\left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + 1\right) \int_0^L \theta^2 dx \\ &\quad + (\epsilon_2 + 2g_1\epsilon_3) \int_0^L \psi_x^2 dx + c \int_0^L q^2 dx + 2g_1\epsilon_3 \int_0^L g \circ \psi_x dx. \end{aligned} \tag{3.34}$$

Proof. A simple differentiation of $\mathcal{I}_2$, then exploiting the third and sixth equations in (2.11), leads to

$$\begin{aligned}\mathcal{I}_2'(t) \quad &= \quad -\rho_2 \int_0^L \psi_t^2 dx + \rho_3 \int_0^L \theta^2 dx - \frac{\rho_2}{\gamma} \int_0^L q\psi_t dx - \frac{b\rho_3}{\gamma} \int_0^L \theta \psi_x dx \\ &\quad - \frac{k\rho_3}{\gamma} \int_0^L (\varphi_x + \psi + l\omega) \int_0^x \theta(y) dy dx + \frac{\delta\rho_3}{\gamma} \int_0^L \theta \int_0^t g(t-s)\psi_x ds dx. \end{aligned}$$

Estimate (3.34) follows by using Cauchy–Schwarz and Young's inequalities.

Lemma 3.7. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\epsilon_4 > 0$, the functional*

$$\mathcal{I}_3(t) = \alpha\rho_3 \int_0^L \theta \int_0^x q(y) dy dx \tag{3.35}$$

satisfies the estimate

$$\mathcal{I}_3'(t) \leq -\frac{\rho_3}{2} \int_0^L \theta^2 dx + \delta' \int_0^L \psi_t^2 dx + c\left(1 + \frac{1}{4\delta'}\right) \int_0^L q^2 dx. \tag{3.36}$$

Proof. A simple differentiation of $\mathcal{I}_3$, then exploiting the last two equations in (2.11), leads to

$$\mathcal{I}_3'(t) = -\rho_3 \int_0^L \theta^2 dx + \alpha \int_0^L q^2 dx + \alpha\gamma \int_0^L q\psi_t dx - \beta\rho_3 \int_0^L \theta \int_0^x q(y)dydx.$$

Estimate (3.36) follows by using Cauchy–Schwarz and Young's inequalities.

Lemma 3.8. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\delta' > 0$, the functional*

$$\mathcal{I}_4(t) = \rho_1 \int_0^L \varphi_t \left(\varphi + \int_0^x \psi(y)dy\right) dx \tag{3.37}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_4'(t) \leq & -\frac{k}{2}\int_0^L (\varphi_x + \psi + l\omega)^2 dx - \frac{lk_0}{2}\int_0^L (\omega_x - l\varphi)^2 dx + \delta' c \int_0^L \psi_t^2 dx \\ & + \left(c + \frac{1}{4\delta'}\right)\int_0^L \varphi_t^2 dx + c\int_0^L z_1^2(x,1,t)dx. \end{aligned}\tag{3.38}$$

Proof. A simple differentiation of $\mathcal{I}_4$, then exploiting the first equation in (2.11), leads to

$$\begin{aligned}\mathcal{I}_4'(t) = & \rho_1 \int_0^L \varphi_t \int_0^x \psi_t(y)dydx - \mu_2 \int_0^L \left(\varphi + \int_0^x \psi(y)dy\right) z_1(x,1,t)dx \\ & -k\int_0^L (\varphi_x + \psi + l\omega)^2 dx + \rho_1 \int_0^L \varphi_t^2 dx - lk_0 \int_0^L (\omega_x - l\varphi)^2 dx \\ & -\mu_1 \int_0^L \varphi_t \left(\varphi + \int_0^x \psi(y)dy\right) dx.\end{aligned}$$

Using Cauchy–Schwarz, Poincaré and Young's inequalities gives (3.38).

Lemma 3.9. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12), then for $\delta', \epsilon_4 > 0$, the functional*

$$\mathcal{I}_5(t) = \rho_2 \int_0^L \psi\psi_t dx \tag{3.39}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_5'(t) \leq & \left(-\frac{b}{2} + \frac{\delta^2}{4\delta'} + \frac{\gamma^2}{\epsilon_4} + 2g_1\delta'\right)\int_0^L \psi_x^2 dx + 2g_1\delta' \int_0^L (g \circ \psi_x)\, dx \\ & +\rho_2 \int_0^L \psi_t^2 dx + \frac{k^2}{b}\int_0^L (\varphi_x + \psi + l\omega)^2 dx + \epsilon_4 \int_0^L \theta^2 dx. \end{aligned}\tag{3.40}$$

Proof. A simple differentiation of $\mathcal{I}_5$, then exploiting the first equation in (2.11), leads to

$$\begin{aligned}\mathcal{I}_5'(t) &= -\frac{b}{2}\int_0^L \psi_x^2 dx + \rho_2 \int_0^L \psi_t^2 dx + \gamma \int_0^L \theta\psi_x dx \\ &-k\int_0^L (\varphi_x+\psi+l\omega)\,\psi dx + \delta\int_0^L \psi_x \int_0^t g(t-s)\psi_x ds dx.\end{aligned}$$

Using (3.14), (3.15), Cauchy–Schwarz, Poincaré and Young's inequalities gives (3.40).

Lemma 3.10. *Let $(\varphi,\psi,\omega,\theta,q,z_1,z_2)$ be the solution of (2.11)-(2.12) and for $k=k_0$ and $\delta'>0$, the functional*

$$\mathcal{I}_6(t) = -\rho_1\int_0^L \varphi_t\,(\omega_x - l\varphi)\,dx - \rho_1\int_0^L \omega_t\,(\varphi_x+\psi+l\omega)\,dx \tag{3.41}$$

satisfies the estimate

$$\begin{aligned}\mathcal{I}_6'(t) &\le \left(2\delta' - k_0 l\right)\int_0^L (\omega_x - l\varphi)^2 dx + \left(\rho_1 l + \frac{\mu_1^2}{4\delta'}\right)\int_0^L \varphi_t^2 dx \\ &+\left(kl+2\delta'\right)\int_0^L (\varphi_x+\psi+l\omega)^2 dx + \left(\frac{\rho_1^2}{4\delta'}+\frac{\lambda_1^2}{4\delta'}-\rho_1 l\right)\int_0^L \omega_t^2 dx \\ &+\delta'\int_0^L \psi_t^2 dx + \frac{\mu_2^2}{4\delta'}\int_0^L z_1^2(x,1,t)dx + \frac{\lambda_2^2}{4\delta'}\int_0^L z_2^2(x,1,t)dx.\end{aligned} \tag{3.42}$$

Proof. A simple differentiation of $\mathcal{I}_6$, using the first and fourth equations in (2.11), leads to

$$\begin{aligned}\mathcal{I}_6'(t) &= -k_0 l\int_0^L (\omega_x - l\varphi)^2 dx + \rho_1 l\int_0^L \varphi_t^2 dx + kl\int_0^L (\varphi_x+\psi+l\omega)^2 dx \\ &-\rho_1 l\int_0^L \omega_t^2 dx - \rho_1\int_0^L \omega_t\psi_t dx + \mu_1\int_0^L \varphi_t\,(\omega_x - l\varphi)\,dx + \lambda_1\int_0^L \omega_t\,(\varphi_x+\psi+l\omega)\,dx \\ &+\mu_2\int_0^L z_1(x,1,t)\,(\omega_x - l\varphi)\,dx + \lambda_2\int_0^L z_2(x,1,t)\,(\varphi_x+\psi+l\omega)\,dx.\end{aligned}$$

Using Young's inequality for the last five terms in the right-hand side gives (3.42) under the condition $k=k_0$.

Lemma 3.11. *Let $(\varphi,\psi,\omega,\theta,q,z_1,z_2)$ be a solution of (2.11)-(2.12). Then the functional*

$$\mathcal{I}_7(t) = -\rho_1\int_0^L (\varphi\varphi_t + \omega\omega_t)\,dx - \frac{\mu_1}{2}\int_0^L \varphi^2 dx - \frac{\lambda_1}{2}\int_0^L \omega^2 dx$$

satisfies, for $c > 0$, *the estimate*

$$\mathcal{I}_7'(t) \leq -\rho_1 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \omega_t^2 dx + c \int_0^L (\varphi_x + \psi + l\omega)^2 dx \tag{3.43}$$

$$+c \int_0^L (\omega_x - l\varphi)^2 dx + c \int_0^L \psi_x^2 dx + \frac{\mu_2^2}{2} \int_0^L z_1^2(x,1,t) dx$$

$$+\frac{\lambda_2^2}{2} \int_0^L z_2^2(x,1,t) dx. \tag{3.44}$$

Proof. Taking the derivative of $\mathcal{I}_7$, by using equations in (2.11), we get

$$\mathcal{I}_7'(t) = -\rho_1 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \omega_t^2 dx + k \int_0^L (\varphi_x + \psi + l\omega)^2 dx \tag{3.45}$$

$$+k_0 \int_0^L (\omega_x - l\varphi)^2 dx - k \int_0^L (\varphi_x + \psi + lw) \psi dx$$

$$+\mu_2 \int_0^L \varphi z_1(x,1,t) dx + \lambda_2 \int_0^L \omega z_2(x,1,t) dx, \tag{3.46}$$

according to (3.17), we have the following relation where c is a positive constant

$$\int_0^L \left[\varphi_x^2 + \psi_x^2 + \omega_x^2\right] dx \leq c \int_0^L \left[(\varphi_x + \psi + l\omega)^2 + (\omega_x - l\varphi)^2 + \psi_x^2\right] dx. \tag{3.47}$$

We obtain the result by using (3.47) and Young's inequality.

Lemma 3.12. *Let* $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ *be the solution of (2.11)-(2.12). Then the functional* $\mathcal{I}_8$ *defined by*

$$\mathcal{I}_8(t) = \tau_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1^2(x,\rho,t) d\rho dx \tag{3.48}$$

satisfies

$$\mathcal{I}_8'(t) \leq -2I_8(t) - C_1 \int_0^L z_1^2(x,1,t) dx + \int_0^L \varphi_t^2 dx. \tag{3.49}$$

Proof. *By differentiating* $\mathcal{I}_8$, *then by using* $(2.11)_2$ *and* $(2.11)_5$, *and integrating by parts, we get*

$$\begin{aligned}
\mathcal{I}_8'(t) &= -2 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1 z_{1_\rho}(x,\rho,t) d\rho dx \\
&= -2\tau_1 \int_0^L \int_0^1 e^{-2\tau_1 \rho} z_1^2(x,\rho,t) d\rho dx - \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-2\tau_1 \rho} z_1^2(x,\rho,t)\right) d\rho dx \\
&= -2I_8(t) - \int_0^L e^{-2\tau_1} z_1^2(x,1,t) dx + \int_0^L \varphi_t^2 dx. \\
&= -2I_8(t) - C_1 \int_0^L z_1^2(x,1,t) dx + \int_0^L \varphi_t^2 dx
\end{aligned}$$

for $C_1 > 0$.

Lemma 3.13. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12). Then the functional $\mathcal{I}_8$ defined by*

$$\mathcal{I}_9(t) = \tau_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2^2(x, \rho, t) d\rho dx \tag{3.50}$$

satisfies

$$\mathcal{I}_9'(t) \leq -2I_9(t) - C_2 \int_0^L z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx. \tag{3.51}$$

Proof. *By differentiating $\mathcal{I}_8$, then by using $(2.11)_2$ and $(2.11)_5$, and integrating by parts, we get*

$$\begin{aligned}
\mathcal{I}_9'(t) &= -2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2 z_{2_\rho}(x, \rho, t) d\rho dx \\
&= -2\tau_2 \int_0^L \int_0^1 e^{-2\tau_2 \rho} z_2^2(x, \rho, t) d\rho dx - \int_0^L \int_0^1 \frac{d}{d\rho} \left(e^{-2\tau_2 \rho} z_2^2(x, \rho, t) \right) d\rho dx \\
&= -2I_9(t) - \int_0^L e^{-2\tau_2} z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx. \\
&= -2I_9(t) - C_2 \int_0^L z_2^2(x, 1, t) dx + \int_0^L \omega_t^2 dx
\end{aligned}$$

for $C_2 > 0$.

Now, we are ready to state and prove the main result of this section. First, we define a Lyapunov functional $\mathcal{L}$ as follows

$$\mathcal{L}(t) = N\mathcal{E}(t) + \sum_{i=1}^{9} N_i \mathcal{I}_i(t) \tag{3.52}$$

satisfies, for N_i, $i = 1, 2, ..., 9$ are positive constants to be proprerly chosen later, with sufficiently large N, one can easily prove that

$$\alpha_1 \mathcal{E}(t) \leq \mathcal{L}(t) \leq \alpha_2 \mathcal{E}(t), \qquad \forall t \geq 0 \tag{3.53}$$

where α_1 and α_2 are positive constants.

Theorem 3.1. *Let $(\varphi, \psi, \omega, \theta, q, z_1, z_2)$ be the solution of (2.11)-(2.12) and assume that $(A_1), (A_2)$, $k = k_0$, $\mu_1 > |\mu_2|$ and $\lambda_1 > |\lambda_2|$ hold. Then, the energy functional (2.13) satisfies,*

$$\mathcal{E}(t) \leq c_1 e^{-c_2 \int_{t_0}^t \eta(s) ds}, \qquad \forall t \geq 0$$

where c_1 and c_2 are positive constants.

Proof. *From the estimates of the previous lemmas we have*

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & \{-n_0N + cN_4 + l\rho_1 N_6 - \rho_1 N_7 + N_8\} \int_0^L \varphi_t^2 dx + \left\{-\rho_2 g_0 N_1 - \frac{\rho_2}{\gamma} N_2 + \rho_2 N_5\right\} \int_0^L \psi_t^2 dx \\
& + \left\{-n_0' N - l\rho_1 N_6 - \rho_1 N_7 + N_9\right\} \int_0^L \omega_t^2 dx + \{-\beta N + cN_2 + cN_3\} \int_0^L q^2 dx \\
& + \left\{-N\frac{\delta}{2} g(t) + (\epsilon_2 + 2g_1\epsilon_3) N_2 + \left(\frac{-b}{2} + \frac{\gamma^2}{\epsilon_4}\right) N_5 + cN_7\right\} \int_0^L \psi_x^2 dx \\
& + \left\{\frac{N_1}{2} + cN_2 \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} + \frac{1}{\epsilon_3} + 1\right) - \frac{\rho_3}{2} N_3 + \epsilon_4 N_5\right\} \int_0^L \theta^2 dx \\
& + \left\{-\frac{lk_0}{2} N_4 - lk_0 N_6 + cN_7\right\} \int_0^L (\omega_x - l\varphi)^2 dx \\
& + \left\{\epsilon_1 N_2 - \frac{k}{2} N_4 + \frac{k^2}{b} N_5 + lkN_6 + cN_7\right\} \int_0^L (\varphi_x + \psi + l\omega)^2 dx \\
& + \left\{-n_0 N + cN_4 + \frac{\mu_2^2}{2} N_7 - C_1 N_8\right\} \int_0^L z_1^2(x,1,t) dx \\
& + \left\{-n_0' N + \frac{\lambda_2^2}{2} N_7 - C_1' N_9\right\} \int_0^L z_2^2(x,1,t) dx \\
& + \{-mN_8\} \int_0^L z_1^2(x,\rho,t) dx + \{-mN_9\} \int_0^L z_2^2(x,\rho,t) dx \\
& + \left\{c(\delta' N_1 + 2g_1\epsilon_3 N_2)\right\} \int_0^L (g \circ \psi_x)\, dx + N\frac{\delta}{2} \int_0^L \left(g' \circ \psi_x\right) dx \\
& + \delta' \int_0^L \Big[(N_3 + \rho_2 N_1 + cN_4 + N_6)\, \psi_t^2 + (2N_6 + kN_1)(\varphi_x + \psi + l\omega)^2 + 2N_6 (\omega_x - l\varphi)^2 \\
& + \left[\left(b^2 + \delta^2 - 2\delta b g_0\right) N_1 + 2g_1 N_5\right] \psi_x^2 + 2g_1 N_5 (g \circ \psi_x)\Big] dx \\
& + \frac{1}{\delta'} \int_0^L \left[\frac{\rho_2 g(0)}{4} N_1 \left(g' \circ \psi_x\right) + \frac{N_3}{4} c q^2 + \frac{\delta^2}{4} N_5 \psi_x^2 + \left(\frac{\rho_1^2}{4} + \frac{\lambda_1^2}{4}\right) N_6 \omega_t^2\right] dx \\
& + \frac{1}{\delta'} \int_0^L \left[\left(\frac{\mu_1^2}{4} N_6 + \frac{N_4}{4}\right) \varphi_t^2 + \frac{\mu_2^2}{4} N_6 z_1^2(x,1,t) + \frac{\lambda_2^2}{4} N_6 z_2^2(x,1,t)\right] dx.
\end{aligned}$$

By taking $\epsilon_2 = \epsilon_3 = \epsilon_4 = N_5 = N_6 = N_7 = 1$, $N_1 = N_2$ *and* $N_8 = N_9$, *we arrive at*

$$\begin{aligned}
\mathcal{L}'(t) \leq\ & \{-n_0N + cN_4 + l\rho_1 - \rho_1 + N_8\} \int_0^L \varphi_t^2 dx + \left\{\left(-\rho_2 g_0 - \frac{\rho_2}{\gamma}\right) N_1 + \rho_2\right\} \int_0^L \psi_t^2 dx \\
& + \left\{-n_0' N - l\rho_1 - \rho_1 + N_8\right\} \int_0^L \omega_t^2 dx + \{-\beta N + cN_2 + cN_3\} \int_0^L q^2 dx \\
& + \left\{-N\frac{\delta}{2} g(t) + (1 + 2g_1) N_1 + \frac{-b}{2} + \gamma^2 + c\right\} \int_0^L \psi_x^2 dx \\
& + \left\{\left(\frac{1}{2} + c\left(\frac{1}{\epsilon_1} + 3\right)\right) N_1 - \frac{\rho_3}{2} N_3 + 1\right\} \int_0^L \theta^2 dx \\
& + \left\{-\frac{lk_0}{2} N_4 - lk_0 + c\right\} \int_0^L (\omega_x - l\varphi)^2 dx
\end{aligned}$$

$$+\left\{\epsilon_1 N_1 - \frac{k}{2}N_4 + \frac{k^2}{b} + lk + c\right\}\int_0^L (\varphi_x + \psi + l\omega)^2\,dx$$
$$+\left\{-n_0 N + cN_4 + \frac{\mu_2^2}{2} - C_1 N_8\right\}\int_0^L z_1^2(x,1,t)dx$$
$$+\left\{-n_0^{'} N + \frac{\lambda_2^2}{2} - C_1^{'} N_8\right\}\int_0^L z_2^2(x,1,t)dx$$
$$+\{-mN_8\}\left(\int_0^L z_1^2(x,\rho,t)dx + \int_0^L z_2^2(x,\rho,t)dx\right)$$
$$+\left\{\left(c(\delta^{'}) + 2g_1\right)N_1\right\}\int_0^L (g\circ\psi_x)\,dx + N\frac{\delta}{2}\int_0^L \left(g^{'}\circ\psi_x\right)dx$$
$$+\delta^{'} C_1(N_1,N_3,N_4)E(t) - \frac{1}{\delta^{'}}C_2(N_1,N_3,N_4)E^{'}(t).$$

Let us choose N_4 *large enough such that*

$$-\frac{lk_0}{2}N_4 - lk_0 + c < 0.$$

Picking N_4 *and choose* N_1 *large enough so that*

$$\left(-\rho_2 g_0 - \frac{\rho_2}{\gamma}\right)N_1 + \rho_2 < 0,$$

choose ϵ_1 *small enough so that*

$$\epsilon_1 N_1 - \frac{k}{2}N_4 + \frac{k^2}{b} + lk + c < 0.$$

Next, we select N_3 *large enough such that*

$$\left(\frac{1}{2} + c\left(\frac{1}{\epsilon_1} + 3\right)\right)N_1 - \frac{\rho_3}{2}N_3 + 1 < 0.$$

Finally, we choose N *sufficiently large to satisfy*

$$-n_0 N + cN_4 + N_8 + \rho_1(l-1) < 0, \quad -n_0^{'}N - C_1^{'}N_8 + \tfrac{\lambda_2^2}{2} < 0.$$

$$-n_0^{'}N + N_8 - \rho_1(l+1) < 0, \quad -n_0 N + cN_4 - C_1 N_8 + \tfrac{\mu_2^2}{2} < 0,$$

$$-\beta N + cN_1 + cN_3 < 0, \quad -N\tfrac{\delta}{2}g(t) + (1+2g_1)N_1 + \tfrac{-b}{2} + \gamma^2 + c < 0.$$

Therefore, (3.54) *takes the form*

$$\mathcal{L}^{'}(t) \le -\left[C_0 - C_1(N_1,N_3,N_4)\delta^{'}\right]\mathcal{E}(t) - \frac{C_2(N_1,N_3,N_4)}{\delta^{'}}\mathcal{E}^{'}(t) + C_3\int_0^L (g\circ\psi_x)dx$$

for some positive constants C_0, C_1, C_2, C_3*. At this point, we take* $\delta^{'} < \frac{C_0}{C_1}$*, then for some* $m_0 > 0$*, we obtain*

$$\mathcal{L}^{'}(t) \le -m_0\mathcal{E}(t) + C_3\int_0^L (g\circ\psi_x)dx - \frac{C_2}{\delta^{'}}\mathcal{E}^{'}(t). \tag{3.54}$$

Multiplying (3.54) *by* $\eta(t)$ *gives*

$$\eta(t)\mathcal{L}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) + C_3\eta(t)\int_0^L (g\circ\psi_x)dx - \frac{C_2}{\delta'}\eta(t)\mathcal{E}^{'}(t). \tag{3.55}$$

The second term can be estimated, using (A_2), *as follows*

$$\begin{aligned} C_3\eta(t)\int_0^L (g\circ\psi_x)dx &= C_3\eta(t)\int_0^L\int_0^t g(t-s)\left(\psi_x(t)-\psi_x(s)\right)^2 dsdx \\ &\leq -\frac{2C_3}{\beta}\mathcal{E}^{'}(t), \end{aligned}$$

so for some $C_4 > 0$, (3.55) *becomes as follows*

$$\eta(t)\mathcal{L}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) - C_4^{'}\mathcal{E}(t) - \frac{C_2}{\delta'}\eta(t)\mathcal{E}^{'}(t). \tag{3.56}$$

We have

$$\mathcal{F}(t) = \eta(t)\left(\mathcal{L}(t) + \frac{C_2}{\delta'}\mathcal{E}(t)\right) \sim \mathcal{E}(t)$$

Therefore, using (3.56) *and the fact that* $\eta^{'}(t) \leq 0$, *we arrive at,*

$$\mathcal{F}^{'}(t) = \eta^{'}(t)\left(\mathcal{L}(t) + \frac{C_2}{\delta'}\mathcal{E}(t)\right) + \eta(t)\left(\mathcal{L}^{'}(t) + \frac{C_2}{\delta'}\mathcal{E}^{'}(t)\right) \leq \eta(t)\left(\mathcal{L}^{'}(t) + \frac{C_2}{\delta'}\mathcal{E}^{'}(t)\right).$$

So

$$\mathcal{F}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t) - C_4^{'}\mathcal{E}(t).$$

Now, we set

$$\mathcal{G}(t) = \mathcal{F}(t) + C_4\mathcal{E}(t) \sim \mathcal{E}(t),$$

gives

$$\mathcal{G}^{'}(t) = \mathcal{F}^{'}(t) + C_4\mathcal{E}^{'}(t) \leq -m_0\eta(t)\mathcal{E}(t). \tag{3.57}$$

A simple integration of (3.57) *over* (t_0, t) *leads to*

$$\mathcal{G}(t) \leq \mathcal{G}(t_0)e^{-m_0\int_{t_0}^t \eta(s)ds}. \tag{3.58}$$

Recalling (3.53) *and estimate* (3.58) *completes the proof.*

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