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CERTAIN CURVATURE CONDITIONS IN LORENTZIAN PARA-SASAKIAN MANIFOLDS WITH RESPECT TO THE SEMI-SYMMETRIC METRIC CONNECTION

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Abstract. The object of the present paper is to characterize Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying certain curvature conditions.

1. Introduction

In 1989, K. Matsumoto [12] introduced the notion of Lorentzian para-Sasakian manifolds. Again the same notion was studied by I. Mihai and R. Rosca [13] and obtained many results on this manifold. Lorentzian para-Sasakian manifolds have also been studied by K. Matsumoto and I. Mihai [11], U. C. De et al. [2] and many others such as ([14], [16], [18]).

A linear connection $\bar{\nabla}$ in a Riemannian manifold M is said to be a semi-symmetric connection [4] if the torsion tensor T of the connection $\bar{\nabla}$ defined by

$$T(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y]$$

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satisfies

$$T(X,Y) = \eta(Y)X - \eta(X)Y, \tag{1.1}$$

where η is a 1-form. If moreover, the connection $\bar{\nabla}$ satisfies the condition

$$(\bar{\nabla}_X g)(Y,Z) = 0 \tag{1.2}$$

for all $X, Y, Z \in \chi(M)$, where $\chi(M)$ is the Lie algebra of vector fields of the manifold M, then $\bar{\nabla}$ is said to be a semi-symmetric metric connection, otherwise it is said to be a semi-symmetric non-metric connection. In 1932, H. A. Hayden [7] defined a semi-symmetric metric connection on a Riemannian manifold and this was further developed by K. Yano [21]. A semi-symmetric metric connection have been studied by many authors ([1], [5], [6], [17], [20]) in several ways to a different extent.

A relation between the semi-symmetric metric connection $\bar{\nabla}$ and the Levi-Civita connection ∇ in Lorentzian para-Sasakian manifold M is given by [17, 21]

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - g(X,Y)\xi. \tag{1.3}$$

The notion of semisymmetric manifold, a proper generalization of locally symmetric manifold, is defined by $R(X,Y) \cdot R = 0$, where $R(X,Y)$ acts on R as a derivation of the tensor algebra at each point of the manifold for tangent vector fields X, Y. A complete intrinsic classification of these manifolds was given by Z. I. Szabó in [19]. Also in [9], O. Kowalski classified 3-dimensional Riemannian spaces satisfying $R(X,Y) \cdot R = 0$. A Riemannian manifold is said to be Ricci semisymmetric if $R(X,Y) \cdot S = 0$, where S denotes the Ricci tensor of type $(0,2)$. A general classification of these manifolds has been worked out by V. A. Mirzoyan [15].

We define endomorphisms $R(X,Y)$ and $X \wedge_A Y$ for an arbitrary vector field Z by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{1.4}$$

and

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{1.5}$$

respectively, where $X, Y, Z \in \chi(M)$ and A is the symmetric $(0,2)$-tensor, R is the Riemannian curvature tensor of type $(1,3)$.

Furthermore, the tensors $R \cdot R$ and $R \cdot S$ on (M,g) are defined by

$$(R(X,Y) \cdot R)(U,V)W = R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \tag{1.6}$$

$$-R(U,R(X,Y)V)W - R(U,V)R(X,Y)W,$$

and

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V), \tag{1.7}$$

respectively.

Recently, D. Kowalczyk [8] studied semi-Riemannian manifolds satisfying $Q(Q,R) = 0$ and $Q(S,g) = 0$, where S, R are the Ricci tensor and curvature tensor, respectively. For detailed study of semisymmetric manifolds we refer the readers to see ([3], [10]).

The paper is organized as follows: Section 2 is concerned with preliminaries. In Section 3, we obtain the expressions of the curvature tensor $\bar{R}$ and the Ricci tensor $\bar{S}$ with respect to the semi-symmetric metric connection. In Section 4, we prove that $R\cdot\bar{S} = 0$ if and only if the manifold is an Einstein manifold with respect to $\bar{\nabla}$. Next in Section 5 (resp., 6), we prove that if the manifold satisfies the curvature condition $\bar{S}\cdot R = 0$ (resp., $R\cdot\bar{R} = 0$), then it is an $\eta-$Einstein (resp., Einstein) manifold with respect to $\bar{\nabla}$. Section 7, deals with the study of Ricci semisymmetric Lorentzian para-Sasakian manifolds and prove that Ricci semisymmetries with respect to ∇ and $\bar{\nabla}$ are equivalent if the manifold is a generalized $\eta-$Einstein manifold. In Section 8, we prove that if $C(\xi,X)\cdot\bar{S} = 0$, then either the scalar curvature is constant or the manifold is an Einstein manifold with respect to $\bar{\nabla}$. In the last Section, it is shown that if $\bar{Q}\cdot C = 0$ (where C is the concircular curvature tensor with respect to ∇ and $\bar{Q}$ is the Ricci operator with respect to $\bar{\nabla}$), then either the scalar curvature is constant or the manifold is a special type of $\eta-$Einstein manifold with respect to $\bar{\nabla}$. Finally, we construct an example of 5-dimensional Lorentzian para-Sasakian manifold.

2. Preliminaries

A differentiable manifold M of dimension n is called a Lorentzian para-Sasakian manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g which satisfy

$$\phi^2 X = X + \eta(X)\xi,\ \ \eta(\xi) = -1, \tag{2.1}$$

$$g(X,\xi) = \eta(X),\ \ \phi\xi = 0,\ \ \eta(\phi X) = 0, \tag{2.2}$$

$$g(\phi X,\phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{2.3}$$

$$(\nabla_X\phi)(Y) = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

$$\nabla_X\xi = \phi X, \tag{2.5}$$

where ∇ denotes the covariant differentiation with respect to the Lorentzian metric g. If we put

$$\Phi(X,Y) = g(\phi X, Y) \tag{2.6}$$

for all vector fields X and Y, then $\Phi(X,Y)$ is a symmetric $(0,2)$ tensor field. Also since the 1-form η is closed in a Lorentzian para-Sasakian manifold, so we have

$$(\nabla_X \eta)(Y) = \Phi(X,Y), \ \Phi(X,\xi) = 0 \tag{2.7}$$

for all vector fields X, $Y \in \chi(M)$.

Moreover, the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a Lorentzian para-Sasakian manifold with respect to the Levi-Civita connection satisfy the following equations [2, 11]:

$$\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y), \tag{2.8}$$

$$R(\xi,X)Y = -R(X,\xi)Y = g(X,Y)\xi - \eta(Y)X, \tag{2.9}$$

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.10}$$

$$R(\xi,X)\xi = -R(X,\xi)\xi = X + \eta(X)\xi, \tag{2.11}$$

$$S(X,\xi) = (n-1)\eta(X), \ Q\xi = (n-1)\xi, \tag{2.12}$$

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y) \tag{2.13}$$

for all $X, Y, Z \in \chi(M)$, where S and Q are related by $g(QX,Y) = S(X,Y)$.

Definition 2.1. *A Lorentzian para-Sasakian manifold M is said to be a generalized η-Einstein manifold if its Ricci tensor S is of the form* [23]

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y) + c\Omega(X,Y),$$

where a, b, c are smooth functions on M and $\Omega(X,Y) = g(\phi X, Y)$. If $c = 0$ (resp., $b = c = 0$), then the manifold reduces to an η-Einstein (resp., an Einstein) manifold.

Definition 2.2. *The concircular curvature tensor C in an $n-$dimensional Lorentzian para-Sasakian manifold M is defined by* [22]

$$C(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{2.14}$$

for all $X, Y, Z \in \chi(M)$, where R is the Riemannian curvature tensor and r is the scalar curvature of the manifold.

3. Curvature tensor of a Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection

Let M be an n-dimensional Lorentzian para-Sasakian manifold. The curvature tensor $\bar{R}$ with respect to $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X\bar{\nabla}_Y Z - \bar{\nabla}_Y\bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]}Z. \tag{3.1}$$

By using (1.2), (1.3), (2.1), (2.2), (2.5) and (2.7) in (3.1), we get

$$\begin{aligned}\bar{R}(X,Y)Z = R(X,Y)Z + g(X,\phi Z)Y - g(Y,\phi Z)X - g(Y,Z)\phi X + g(X,Z)\phi Y \\ +(g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi + g(Y,Z)X - g(X,Z)Y \\ +(\eta(Y)X - \eta(X)Y)\eta(Z),\end{aligned} \tag{3.2}$$

where

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$$

is the Riemannian curvature tensor with respect to ∇. By contracting (3.2) over X, we obtain

$$\bar{S}(Y,Z) = S(Y,Z) - (n-2)g(Y,\phi Z) + (n-2-\psi)g(Y,Z) + (n-2)\eta(Y)\eta(Z), \tag{3.3}$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively and $\psi = trace\phi$. The equation (3.3) yields

$$\bar{Q}Y = QY - (n-2)\phi Y + (n-2-\psi)Y + (n-2)\eta(Y)\xi, \tag{3.4}$$

where $\bar{Q}$ and Q are the Ricci operators of the connections $\bar{\nabla}$ and ∇, respectively. Contracting again Y and Z in (3.3), it follows that

$$\bar{r} = r + (n-1)(n-2-2\psi), \tag{3.5}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively.

Lemma 3.1. *Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection. Then*

$$\bar{R}(X,Y)\xi = \eta(Y)(X - \phi X) - \eta(X)(Y - \phi Y), \tag{3.6}$$

$$\bar{R}(\xi,X)Y = (g(X,Y) - g(X,\phi Y))\xi - \eta(Y)(X - \phi X), \tag{3.7}$$

$$\bar{R}(\xi,X)\xi = X - \phi X + \eta(X)\xi, \tag{3.8}$$

$$\bar{S}(X,\xi) = (n-1-\psi)\eta(X), \qquad \bar{Q}\xi = (n-1-\psi)\xi, \tag{3.9}$$

$$\bar{S}(\phi X, \phi Y) = \bar{S}(X, Y) + (n - 1 - \psi)\eta(X)\eta(Y). \tag{3.10}$$

Proof. *By taking* $Z = \xi$ *in* (3.2) *and using* (2.1), (2.2), (2.10), *we get* (3.6). (3.7) *follows from* (2.1), (2.2), (2.9) *and* (3.6). *By taking* $Y = \xi$ *in* (3.7) *and using* (2.1), (2.2) *we obtain* (3.8). *From* (3.3), (2.1), (2.2) *and* (2.12) *we find* (3.9). *By replacing* $Y = \phi X$ *and* $Z = \phi Y$ *in* (3.3) *and then using* (2.1)-(2.3) *and* (2.13) *we get* (3.10).

4. Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying $R(X, Y) \cdot \bar{S} = 0$

Suppose that a Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection $\bar{\nabla}$ satisfies the condition

$$R(X, Y) \cdot \bar{S} = 0. \tag{4.1}$$

Then in view of (1.7), it follows that

$$\bar{S}(R(X, Y)U, V) + \bar{S}(U, R(X, Y)V) = 0$$

which by putting $X = \xi$ and using (2.9) takes the form

$$g(Y, U)\bar{S}(\xi, V) - \eta(U)\bar{S}(Y, V) + g(Y, V)\bar{S}(U, \xi) - \eta(V)\bar{S}(U, Y) = 0. \tag{4.2}$$

By taking $U = \xi$ in (4.2) and using (2.1), (2.2) and (3.9), we obtain

$$\bar{S}(Y, V) = (n - 1 - \psi)g(Y, V). \tag{4.3}$$

From which we have

$$\bar{Q}V = (n - 1 - \psi)V. \tag{4.4}$$

Conversely, if (4.3) satisfies, then by using (4.4) in the expression $(R(X, Y) \cdot \bar{S})(U, V) = -\bar{S}(R(X, Y)U, V) - \bar{S}(U, R(X, Y)V) = -g(R(X, Y)U, \bar{Q}V) - g(\bar{Q}U, R(X, Y)V)$, we find

$$(R(X, Y) \cdot \bar{S})(U, V) = -(n - 1 - \psi)(g(R(X, Y)U, V) + g(U, R(X, Y)V)) \tag{4.5}$$

which by using the fact that $g(R(X, Y)U, V) + g(U, R(X, Y)V)) = 0$ reduces to $(R(X, Y) \cdot \bar{S})(U, V) = 0$. Thus we can state the following theorem:

Theorem 4.1. *If an* $n-$*dimensional Lorentzian para-Sasakian manifold with respect to semi-symmetric metric connection satisfies the condition* $R \cdot \bar{S} = 0$, *then the manifold is an Einstein manifold of the form* (4.3) *and the converse is also true.*

5. Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying $\bar{S} \cdot R = 0$

Suppose that a Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $(\bar{S}(X,Y) \cdot R)(U,V)Z = 0$. Then we have [8]

$$(X \wedge_{\bar{S}} Y)R(U,V)Z + R((X \wedge_{\bar{S}} Y)U,V)Z + R(U,(X \wedge_{\bar{S}} Y)V)Z \qquad (5.1)$$

$$+R(U,V)(X \wedge_{\bar{S}} Y)Z = 0$$

for any $X, Y, Z, U, V \in \chi(M)$. Taking $Y = \xi$ in (5.1), we have

$$(X \wedge_{\bar{S}} \xi)R(U,V)Z + R((X \wedge_{\bar{S}} \xi)U,V)Z + R(U,(X \wedge_{\bar{S}} \xi)V)Z \qquad (5.2)$$

$$+R(U,V)(X \wedge_{\bar{S}} \xi)Z = 0$$

which in view of (1.5) takes the form

$$\bar{S}(\xi, R(U,V)Z)X - \bar{S}(X, R(U,V)Z)\xi + R(\bar{S}(\xi,U)X - \bar{S}(X,U)\xi, V)Z \qquad (5.3)$$

$$+R(U, \bar{S}(\xi,V)X - \bar{S}(X,V)\xi)Z + R(U,V)(\bar{S}(\xi,Z)X - \bar{S}(X,Z)\xi) = 0.$$

By using (3.9) in (5.3), we find

$$(n-1-\psi)[\eta(R(U,V)Z)X + \eta(U)R(X,V)Z + \eta(V)R(U,X)Z + \eta(Z)R(U,V)X] \qquad (5.4)$$

$$-\bar{S}(X,R(U,V)Z)\xi - \bar{S}(X,U)R(\xi,V)Z - \bar{S}(X,V)R(U,\xi)Z - \bar{S}(X,Z)R(U,V)\xi = 0.$$

Now taking inner product of (5.4) with ξ, we get

$$(n-1-\psi)[\eta(R(U,V)Z)\eta(X) + \eta(U)\eta(R(X,V)Z) + \eta(V)\eta(R(U,X)Z)$$

$$+\eta(Z)\eta(R(U,V)X)] + \bar{S}(X,R(U,V)Z) - \bar{S}(X,U)\eta(R(\xi,V)Z)$$

$$-\bar{S}(X,V)\eta(R(U,\xi)Z) - \bar{S}(X,Z)\eta(R(U,V)\xi) = 0$$

which by putting $U = Z = \xi$ and using (3.6)-(3.8) reduces to

$$(n-1-\psi)(g(X,V) + \eta(X)\eta(V)) + \bar{S}(X, V + \eta(V)\xi) = 0$$

from which it follows that

$$\bar{S}(X,V) = -(n-1-\psi)g(X,V) - 2(n-1-\psi)\eta(X)\eta(V). \qquad (5.5)$$

Thus we can state the following theorem:

Theorem 5.1. *If an $n-$dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $\bar{S}\cdot R=0$, then the manifold is an $\eta-$Einstein manifold of the form* (5.5).

6. Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying $R\cdot\bar{R}=0$

Let M be an n-dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $(R(X,Y)\cdot\bar{R})(U,V)W=0$. Then in view of (1.6), it follows that

$$R(X,Y)\bar{R}(U,V)W-\bar{R}(R(X,Y)U,V)W-\bar{R}(U,R(X,Y)V)W \tag{6.1}$$

$$-\bar{R}(U,V)R(X,Y)W=0.$$

By substituting $X=U=\xi$ in (6.1) and using (2.2), (2.9), (2.11) and (3.7), we find

$$g(V,W)Y-g(V,\phi W)Y-\bar{R}(Y,V)W-\eta(V)g(Y,\phi W)\xi \tag{6.2}$$

$$+\eta(V)\eta(W)\phi Y-g(Y,W)V+g(Y,W)\phi V=0.$$

Taking inner product of (6.2) with Z, we have

$$g(V,W)g(Y,Z)-g(V,\phi W)g(Y,Z)-g(\bar{R}(Y,V)W,Z)-\eta(V)\eta(Z)g(Y,\phi W) \tag{6.3}$$

$$+\eta(V)\eta(W)g(\phi Y,Z)-g(Y,W)g(V,Z)+g(Y,W)g(\phi V,Z)=0.$$

Let $\{e_1,e_2,e_3.......,e_{n-1},e_n=\xi\}$ be a frame of orthonormal basis of the tangent space at any point of the manifold. If we put $V=W=e_i$ in (6.3) and summing up with respect to $i(1\leq i\leq n)$, then we obtain

$$\bar{S}(Y,Z)=(n-1-\psi)g(Y,Z). \tag{6.4}$$

Thus we can state the following theorem:

Theorem 6.1. *If an $n-$dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $R\cdot\bar{R}=0$, then the manifold is an Einstein manifold of the form* (6.4).

7. Ricci semisymmetries in Lorentzian para-Sasakian manifolds with respect to the connections $\bar{\nabla}$ and ∇

Assuming that the manifold is Ricci symmetric with respect to the semi-symmetric metric connection $\bar{\nabla}$, therefore we have

$$(\bar{R}(X,Y)\cdot\bar{S})(U,V) = -\bar{S}(\bar{R}(X,Y)U,V) - \bar{S}(U,\bar{R}(X,Y)V) \tag{7.1}$$

for all $X,Y,U,V \in \chi(M)$. In view of (3.2) and (3.3), (7.1) takes the form

$$\begin{aligned}
(\bar{R}(X,Y)\cdot\bar{S})(U,V) &= (R(X,Y)\cdot S)(U,V) - (n-2-\psi)[R(X,Y,U,V)\\
&+R(X,Y,V,U)] + (n-2)[g(R(X,Y)U,\phi V) + g(R(X,Y)V,\phi U)]\\
&-(n-2)[\eta(R(X,Y)U)\eta(V) + \eta(R(X,Y)V)\eta(U)]\\
&-g(X,\phi U)\bar{S}(Y,V) - g(X,\phi V)\bar{S}(U,Y) + g(Y,\phi U)\bar{S}(X,V)\\
&+g(Y,\phi V)\bar{S}(X,U) + g(Y,U)\bar{S}(\phi X,V) + g(Y,V)\bar{S}(U,\phi X)\\
&-g(X,U)\bar{S}(\phi Y,V) - g(X,V)\bar{S}(U,\phi Y) - g(Y,U)\eta(X)\bar{S}(\xi,V)\\
&-g(Y,V)\eta(X)\bar{S}(U,\xi) + g(X,U)\eta(Y)\bar{S}(\xi,V) + g(X,V)\eta(Y)\bar{S}(\xi,U)\\
&-g(Y,U)\bar{S}(X,V) - g(Y,V)\bar{S}(X,U) + g(X,U)\bar{S}(Y,V)\\
&+g(X,V)\bar{S}(U,Y) - \eta(Y)\eta(U)\bar{S}(X,V) - \eta(Y)\eta(V)\bar{S}(X,U)\\
&+\eta(X)\eta(U)\bar{S}(Y,V) + \eta(X)\eta(V)\bar{S}(Y,U)
\end{aligned}$$

which by using (2.8) and the fact that $R(X,Y,U,V) + R(X,Y,V,U) = 0$ turns to

$$\begin{aligned}
(\bar{R}(X,Y)\cdot\bar{S})(U,V) &= (R(X,Y)\cdot S)(U,V) + (n-2)[g(R(X,Y)U,\phi V)\\
&+g(R(X,Y)V,\phi U)] - (2n-3-\psi)[g(Y,U)\eta(X)\eta(V) - g(X,U)\eta(Y)\eta(V)\\
&+g(Y,V)\eta(X)\eta(U) - g(X,V)\eta(Y)\eta(U)] - g(X,\phi U)\bar{S}(Y,V)\\
&-g(X,\phi V)\bar{S}(U,Y) + g(Y,\phi U)\bar{S}(X,V) + g(Y,\phi V)\bar{S}(X,U)\\
&+g(Y,U)\bar{S}(\phi X,V) + g(Y,V)\bar{S}(U,\phi X) - g(X,U)\bar{S}(\phi Y,V)\\
&-g(X,V)\bar{S}(U,\phi Y) - g(Y,U)\bar{S}(X,V) - g(Y,V)\bar{S}(X,U)\\
&+g(X,U)\bar{S}(Y,V) + g(X,V)\bar{S}(U,Y) - \eta(Y)\eta(U)\bar{S}(X,V)\\
&-\eta(Y)\eta(V)\bar{S}(X,U) + \eta(X)\eta(U)\bar{S}(Y,V) + \eta(X)\eta(V)\bar{S}(Y,U).
\end{aligned} \tag{7.2}$$

Suppose that $(\bar{R}(X,Y)\cdot\bar{S})(U,V) = (R(X,Y)\cdot S)(U,V)$, then from (7.2), it follows that

$$(n-2)[g(R(X,Y)U,\phi V) + g(R(X,Y)V,\phi U)]$$

$$-(2n-3-\psi)[g(Y,U)\eta(X)\eta(V)-g(X,U)\eta(Y)\eta(V)$$

$$+g(Y,V)\eta(X)\eta(U)-g(X,V)\eta(Y)\eta(U)]-g(X,\phi U)\bar{S}(Y,V)$$

$$-g(X,\phi V)\bar{S}(U,Y)+g(Y,\phi U)\bar{S}(X,V)+g(Y,\phi V)\bar{S}(X,U)$$

$$+g(Y,U)\bar{S}(\phi X,V)+g(Y,V)\bar{S}(U,\phi X)-g(X,U)\bar{S}(\phi Y,V)$$

$$-g(X,V)\bar{S}(U,\phi Y)-g(Y,U)\bar{S}(X,V)-g(Y,V)\bar{S}(X,U)$$

$$+g(X,U)\bar{S}(Y,V)+g(X,V)\bar{S}(U,Y)-\eta(Y)\eta(U)\bar{S}(X,V)$$

$$-\eta(Y)\eta(V)\bar{S}(X,U)+\eta(X)\eta(U)\bar{S}(Y,V)+\eta(X)\eta(V)\bar{S}(Y,U)=0$$

which by taking $X=U=\xi$ and then using (2.1), (2.2) and (2.8) reduces to

$$\bar{S}(\phi Y,V)=(n-2)g(Y,V)+(n-2)\eta(Y)\eta(V)-(\psi-1)g(Y,\phi V). \tag{7.3}$$

Now replacing V by ϕV in (7.3) and using (2.1), (2.2) and (3.10), we obtain

$$\bar{S}(Y,V)=(1-\psi)g(Y,V)+(n-2)g(Y,\phi V)-(n-2)\eta(Y)\eta(V). \tag{7.4}$$

Thus we can state the following theorem:

Theorem 7.1. *Ricci semisymmetries with respect to $\bar{\nabla}$ and ∇ are equivalent if the manifold is a generalized $\eta-$Einstein manifold with respect to the semi-symmetric metric connection.*

8. Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying $C(\xi,X)\cdot\bar{S}=0$

We consider that an n-dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $C(\xi,X)\cdot\bar{S}=0$. Then we have

$$\bar{S}(C(\xi,X)Y,Z)+\bar{S}(Y,C(\xi,X)Z)=0. \tag{8.1}$$

From (2.14), we find

$$C(\xi,X)Y=[1-\frac{r}{n(n-1)}](g(X,Y)\xi-\eta(Y)X). \tag{8.2}$$

By virtue of (8.2), (8.1) takes the form

$$[1-\frac{r}{n(n-1)}](g(X,Y)\bar{S}(\xi,Z)-\eta(Y)\bar{S}(X,Z)+g(X,Z)\bar{S}(Y,\xi)-\eta(Z)\bar{S}(X,Y))=0$$

which by taking $Z=\xi$ and using (2.1), (2.2) and (3.9) gives

$$[1-\frac{r}{n(n-1)}](\bar{S}(X,Y)-(n-1-\psi)g(X,Y))=0. \tag{8.3}$$

Thus we have either $r = n(n-1)$, or

$$\bar{S}(X,Y) = (n-1-\psi)g(X,Y). \tag{8.4}$$

Thus we can state the following theorem:

Theorem 8.1. *If an $n-$dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $C(\xi, X)\cdot\bar{S} = 0$, then either the scalar curvature is constant or the manifold is an Einstein manifold of the form* (8.4).

9. Lorentzian para-Sasakian manifolds with respect to the semi-symmetric metric connection satisfying $\bar{Q}\cdot C = 0$

In this section we suppose that an n-dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies $\bar{Q}\cdot C = 0$. Then we have

$$\bar{Q}(C(X,Y)Z) - C(\bar{Q}X,Y)Z - C(X,\bar{Q}Y)Z - C(X,Y)\bar{Q}Z = 0 \tag{9.1}$$

for all $X, Y, Z \in \chi(M)$. In view of (2.14), it follows from (9.1) that

$$\bar{Q}(R(X,Y)Z) - R(\bar{Q}X,Y)Z - R(X,\bar{Q}Y)Z - R(X,Y)\bar{Q}Z$$
$$+\frac{2r}{n(n-1)}(\bar{S}(Y,Z)X - \bar{S}(X,Z)Y) = 0$$

which by taking inner product with ξ yields

$$\eta(\bar{Q}(R(X,Y)Z)) - \eta(R(\bar{Q}X,Y)Z) - \eta(R(X,\bar{Q}Y)Z) - \eta(R(X,Y)\bar{Q}Z) \tag{9.2}$$
$$+\frac{2r}{n(n-1)}(S(Y,Z)\eta(X) - S(X,Z)\eta(Y)) = 0.$$

Putting $Y = \xi$ in (9.2), we have

$$\eta(\bar{Q}(R(X,\xi)Z)) - \eta(R(\bar{Q}X,\xi)Z) - \eta(R(X,\bar{Q}\xi)Z) - \eta(R(X,\xi)\bar{Q}Z) \tag{9.3}$$
$$+\frac{2r}{n(n-1)}(S(\xi,Z)\eta(X) - S(X,Z)\eta(\xi)) = 0.$$

From (2.9), we can easily find

$$\eta(\bar{Q}(R(X,\xi)Z)) = \eta(R(X,\bar{Q}\xi)Z) = (n-1-\psi)(g(X,Z) + \eta(X)\eta(Z)), \tag{9.4}$$

$$\eta(R(\bar{Q}X,\xi)Z) = \eta(R(X,\xi)\bar{Q}Z) = (n-1-\psi)\eta(X)\eta(Z) + \bar{S}(X,Z).$$

By making use of (2.1), (3.9) and (9.4), (9.3) reduces to

$$[\frac{r}{n(n-1)} - 1](\bar{S}(X,Z) + (n-1-\psi)\eta(X)\eta(Z)) = 0.$$

Thus we have either $r = n(n-1)$, or

$$\bar{S}(X,Z) = -(n-1-\psi)\eta(X)\eta(Z). \tag{9.5}$$

Thus we can state the following theorem:

Theorem 9.1. *If an $n-$dimensional Lorentzian para-Sasakian manifold with respect to the semi-symmetric metric connection satisfies the condition $\bar{Q}\cdot C = 0$, then either the scalar curvature is constant or the manifold is a special type of $\eta-$Einstein manifold of the form* (9.5).

Example. We consider the 5-dimensional manifold $M = \{(x_1, x_2, x_3, x_4, x_5) \in R^5\}$, where $(x_1, x_2, x_3, x_4, x_5)$ are the standard coordinates in R^5. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M given by

$$e_1 = coshx_5\frac{\partial}{\partial x_1} + sinhx_5\frac{\partial}{\partial x_2},\ e_2 = sinhx_5\frac{\partial}{\partial x_1} + coshx_5\frac{\partial}{\partial x_2},$$

$$e_3 = coshx_5\frac{\partial}{\partial x_3} + sinhx_5\frac{\partial}{\partial x_4},\ e_4 = sinhx_5\frac{\partial}{\partial x_3} + coshx_5\frac{\partial}{\partial x_4},\ e_5 = \frac{\partial}{\partial x_5} = \xi,$$

which are linearly independent at each point of M and hence form a basis of T_pM. Let g be the Lorentzian metric on M defined by

$$g(e_i, e_i) = 1,\ \text{for } 1 \le i \le 4 \text{ and } g(e_5, e_5) = -1,$$

$$g(e_i, e_j) = 0,\ \text{for } i \ne j,\ 1 \le i \le 5 \text{ and } 1 \le j \le 5.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2,\ \phi e_2 = -e_1,\ \phi e_3 = -e_4,\ \phi e_4 = -e_3,\ \phi e_5 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi, \xi) = -1,\ \phi^2 X = X + \eta(X)\xi \text{ and } g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Thus for $e_5 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_1, e_4] = [e_2, e_3] = [e_2, e_4] = [e_3, e_4] = 0,$$

$$[e_1, e_5] = -e_2,\ [e_2, e_5] = -e_1,\ [e_3, e_5] = -e_4,\ [e_4, e_5] = -e_3.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we find

$$\nabla_{e_1}e_1 = 0,\ \nabla_{e_1}e_2 = -e_5,\ \nabla_{e_1}e_3 = 0,\ \nabla_{e_1}e_4 = 0,\ \nabla_{e_1}e_5 = -e_2,$$

$$\nabla_{e_2}e_1 = -e_5,\ \nabla_{e_2}e_2 = 0,\ \nabla_{e_2}e_3 = 0,\ \nabla_{e_2}e_4 = 0,\ \nabla_{e_2}e_5 = -e_1,$$

$$\nabla_{e_3}e_1 = 0,\ \nabla_{e_3}e_2 = 0,\ \nabla_{e_3}e_3 = 0,\ \nabla_{e_3}e_4 = -e_5,\ \nabla_{e_3}e_5 = -e_4,$$

$$\nabla_{e_4}e_1 = 0,\ \nabla_{e_4}e_2 = 0,\ \nabla_{e_4}e_3 = -e_5,\ \nabla_{e_4}e_4 = 0,\ \nabla_{e_4}e_5 = -e_3,$$

$$\nabla_{e_5}e_1 = 0,\ \nabla_{e_5}e_2 = 0,\ \nabla_{e_5}e_3 = 0,\ \nabla_{e_5}e_4 = 0,\ \nabla_{e_5}e_5 = 0.$$

Also one can easily verify that

$$\nabla_X\xi = \phi X \quad \text{and} \quad (\nabla_X\phi)Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Therefore, the manifold is a Lorentzian para-Sasakian manifold. By using (1.3), we find

$$\bar{\nabla}_{e_1}e_1 = -e_5,\ \bar{\nabla}_{e_1}e_2 = -e_5,\ \bar{\nabla}_{e_1}e_3 = 0,\ \bar{\nabla}_{e_1}e_4 = 0,\ \bar{\nabla}_{e_1}e_5 = -e_1 - e_2,$$

$$\bar{\nabla}_{e_2}e_1 = -e_5,\ \bar{\nabla}_{e_2}e_2 = -e_5,\ \bar{\nabla}_{e_2}e_3 = 0,\ \bar{\nabla}_{e_2}e_4 = 0,\ \bar{\nabla}_{e_2}e_5 = -e_1 - e_2,$$

$$\bar{\nabla}_{e_3}e_1 = 0,\ \bar{\nabla}_{e_3}e_2 = 0,\ \bar{\nabla}_{e_3}e_3 = -e_5,\ \bar{\nabla}_{e_3}e_4 = -e_5,\ \bar{\nabla}_{e_3}e_5 = -e_3 - e_4,$$

$$\bar{\nabla}_{e_4}e_1 = 0,\ \bar{\nabla}_{e_4}e_2 = 0,\ \bar{\nabla}_{e_4}e_3 = -e_5,\ \bar{\nabla}_{e_4}e_4 = -e_5,\ \bar{\nabla}_{e_4}e_5 = -e_3 - e_4,$$

$$\bar{\nabla}_{e_5}e_1 = 0,\ \bar{\nabla}_{e_5}e_2 = 0,\ \bar{\nabla}_{e_5}e_3 = 0,\ \bar{\nabla}_{e_5}e_4 = 0,\ \bar{\nabla}_{e_5}e_5 = 0.$$

From the above results, we can easily obtain the components of the curvature tensor as follows:

$$R(e_1,e_2)e_1 = e_2,\ R(e_1,e_2)e_2 = -e_1,\ R(e_1,e_3)e_1 = 0,\ R(e_1,e_3)e_3 = 0,$$

$$R(e_1,e_4)e_1 = 0,\ R(e_1,e_4)e_4 = 0,\ R(e_1,e_5)e_1 = -e_5,\ R(e_1,e_5)e_5 = -e_1,$$

$$R(e_2,e_3)e_2 = 0,\ R(e_2,e_3)e_3 = 0,\ R(e_2,e_4)e_2 = 0,\ R(e_2,e_4)e_4 = 0,$$

$$R(e_2,e_5)e_2 = -e_5,\ R(e_2,e_5)e_5 = -e_2,\ R(e_3,e_4)e_3 = e_4,\ R(e_3,e_4)e_4 = -e_3,$$

$$R(e_3,e_5)e_3 = -e_5,\ R(e_3,e_5)e_5 = -e_3,\ R(e_4,e_5)e_4 = -e_5,\ R(e_4,e_5)e_5 = -e_4,$$

and

$$\bar{R}(e_1,e_2)e_1 = 0,\ \bar{R}(e_1,e_2)e_2 = 0,\ \bar{R}(e_1,e_3)e_1 = -e_3 - e_4,\ \bar{R}(e_1,e_3)e_3 = e_1 + e_2,$$

$$\bar{R}(e_1,e_4)e_1 = -e_3 - e_4,\ \bar{R}(e_1,e_4)e_4 = e_1 + e_2,\ \bar{R}(e_1,e_5)e_1 = -e_5,\ \bar{R}(e_1,e_5)e_5 = -e_1 - e_2,$$

$$\bar{R}(e_2,e_3)e_2 = -e_3 - e_4,\ \bar{R}(e_2,e_3)e_3 = -e_1 - e_2,\ \bar{R}(e_2,e_4)e_2 = -e_3 - e_4,\ \bar{R}(e_2,e_4)e_4 = e_1 + e_2,$$

$$\bar{R}(e_2,e_5)e_2 = -e_5,\ \bar{R}(e_2,e_5)e_5 = -e_1 - e_2,\ \bar{R}(e_3,e_4)e_3 = 0,\ \bar{R}(e_3,e_4)e_4 = 0,$$

$$\bar{R}(e_3,e_5)e_3 = -e_5,\ \bar{R}(e_3,e_5)e_5 = -e_3 - e_4,\ \bar{R}(e_4,e_5)e_4 = -e_5,\ \bar{R}(e_4,e_5)e_5 = -e_3 - e_4.$$

From these curvature tensors, we calculate

$$S(e_1,e_1)=S(e_2,e_2)=S(e_3,e_3)=S(e_4,e_4)=0,\ S(e_5,e_5)=-4, \tag{9.6}$$

$$\bar{S}(e_1,e_1)=\bar{S}(e_2,e_2)=\bar{S}(e_3,e_3)=\bar{S}(e_4,e_4)=3,\ \bar{S}(e_5,e_5)=-4. \tag{9.7}$$

Therefore, from (9.6) and (9.7) we obtain $r=4$ and $\bar{r}=16$, respectively. Thus it can be seen that the equation (3.5) is satisfied, where $\psi=\sum_{i=1}^{5}\epsilon_i g(\phi e_i,e_i)=0$.

From (1.1), we calculate the components of torsion tensor as follows:

$$T(e_i,e_j)=0,\ \text{for } 1\le i,j\le 5, \qquad T(e_i,e_5)=-e_i,\ \text{for } i=1,2,3,4. \tag{9.8}$$

From (1.2), it can be easily seen that

$$(\bar{\nabla}_{e_i}g)(e_j,e_k)=0,\ \text{for any } 1\le i,j,k\le 5. \tag{9.9}$$

Thus by virtue of (9.8) and (9.9), we say that the linear connection $\bar{\nabla}$ defined by (1.3) on the manifold M is a semi-symmetric metric connection.

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