



International Journal of Maps in Mathematics

Volume (3), Issue (1), (2020), Pages:(28–38)

ISSN: 2636-7467 (Online)

www.journalmim.com

f – BIHARMONIC NORMAL SECTION CURVES

FEYZA ESRA ERDOĞAN*, SELCEN YÜKSEL PERKTAŞ, AND GÜLISTAN POLAT

ABSTRACT. In this paper, We study f – biharmonic and bi– f harmonic normal section curves. We have obtain sufficient and necessary conditions to be f – biharmonic and bi– f – harmonic of a 3 - planar normal section curve.

1. INTRODUCTION

During the examination of the geometry of submanifolds, the classification of submanifolds has a great importance in applications. While classifying the submanifolds, many authors take advantage of distributions on the submanifolds.

Then, they try to simplify the carrying out operations by imposing totally geodesic, totally umbilical and totally integrability conditions and search for the characteristics of submanifolds. This is a functional method though, it is time consuming in practice. The most basic and simplest method for studying submanifold geometry is to study on the curve. In this respect, Chen ([1],[3]) described the normal section curves and used such curves to examine the geometry of submanifolds. Chen ([1],[2]), Kim ([4],[5],[6]) and Li and Chen [1] etc.

Received:2019-08-22

Accepted:2019-11-21

2010 Mathematics Subject Classification. 53A04, 53C40, 58E20.

Key words: Normal section curve, Planar normal section, bi– f harmonic curve, f – biharmonic curve,

* Corresponding author

* Dedicated to Professor Sadık Keleş on the occasion of his retirement from Inonu University.

created classification of submanifolds with the concept of a normal section curve. On the other hand the first comprehensive study of harmonic convergence between Riemannian manifolds was conducted by Eells and Sampson [7]. Then harmonic convergence shed light on many geometers ([14],[9],[11]), a great number studies have been carried out in this area and have been one of the areas of great interest until today. Harmonic transformations are related to variational calculation.

This calculation requires studying with the most appropriate selected objects. This situation is that the appropriate function selected for roughly harmonic convergence equals zero at the appropriate point. From this point of view, it is known that the critical points of the variation functionalities of harmonic convergence identify geodesic curves and minimal surfaces. Therefore, variations of the harmonic convergence of Riemannian manifolds are investigated by identifying connections on the energy and tension of convergence, cotangent space and pull- back tangent bundle in order to study on harmonic convergence of Riemannian manifolds. Eells and Lemaire [14] proposed a k - harmonic convergence in 1993. For $k = 2$, Jiang[8] obtained the variation formulas of such convergence. Today, these convergence are called biharmonic convergence.

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced in [15]. Chen [16] defined biharmonic submanifolds of the Euclidean space and stated a well-known conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, thus minimal.

Harmonic maps between Riemannian manifolds were first introduced and studied by Lichnerowicz in 1970 (see also [23]). He has also some physical meanings by considering them as solutions of continuous spin systems and inhomogenous Heisenberg spin systems [22]. Moreover, there is a strong relationship between f -harmonic maps and gradient Ricci solitons [19].

Biharmonic maps and f -harmonic maps can be associated in two different ways. The first way put forward by Zhao and Lu [20] by following the concept of biharmonic maps. The authors extended bienergy functional to bi- f -energy functional and obtained a new type of harmonic maps called bi- f -harmonic maps. This idea was already considered by Ouakkas, Nasri and Djaa [18]. The second way is that to extend the f -energy functional to the f -bienergy functional by following the definition of f -harmonic map, and obtain another type of harmonic maps which are called f -biharmonic maps as critical points of f -bienergy functional. As a generalization of biharmonic maps, f -biharmonic of maps was

introduced by Lu [17]. A differentiable map between Riemannian manifolds is said to be f -biharmonic if it is a critical point of the f -bienergy functional defined by integral of f times the square-norm of the tension field, where f is a smooth positive function on the domain. If $f = 1$, then f -biharmonic maps are biharmonic. To avoid the confusion with the types of maps called by the same name in [18] and defined as critical points of the square-norm of the f -tension field, some authors (see [17], [21]) called the map defined in [12] as *bi- f -harmonic map*.

2. PRELIMINARIES

2.1. Normal Section Curves and Curvatures on Riemannian Manifolds

Let $\gamma : I \subset \mathbb{R} \rightarrow E^m$ be a unit speed curve in E^m . The curve is called Frenet curve of osculating order r if its higher order derivatives $\gamma'(s), \gamma''(s), \dots, \gamma^{(r)}(s)$ are linearly independent and $\gamma'(s), \gamma''(s), \dots, \gamma^{(r+1)}(s)$ linearly dependent, for all $s \in I$. For each Frenet curve of osculating order r , one can associate an orthonormal r -frame v_1, v_2, \dots, v_r along γ such that the Frenet formulas defined by in the usual way

$$\begin{aligned}
 T'(s) &= v_1(s) = k_1(s)v_2(s), \\
 v_2'(s) &= -k_1(s)T(s) + k_2(s)v_3(s), \\
 &\vdots \\
 v_i'(s) &= -k_{i-1}(s)v_{i-1}(s) + k_i(s)v_{i+1}(s), \\
 v_{i+1}'(s) &= -k_i(s)v_i(s),
 \end{aligned} \tag{2.1}$$

where k_1, k_2, \dots, k_{r-1} are called the Frenet curvatures. Let M be a differentiable n -dimensional submanifold in $(n+r)$ -dimensional Euclidean space E^{n+r} . If each normal sections γ of M is a W -curve of rank r in M then M is called weak helical submanifold of order r . If each r -planar normal section is a geodesic then the submanifold M is said to have geodesic normal sections. For every geodesic normal sections of M if it is a W -curve of rank r in M is called weak geodesic helical submanifold of order r [13]. Assume that γ is a normal section curve of a differentiable n -dimensional submanifold M in E^{n+2} and M has 3-planar normal

sections. Then by using Frenet formulas given by (2.1) we write

$$\left\{ \begin{array}{l} \gamma'(s) = T(s) = v_1(s), \\ \gamma''(s) = k_1(s)v_2(s), \\ \gamma'''(s) = -k_1(s)T(s) + k_1'(s)v_2(s) + k_1(s)k_2(s)v_3(s), \\ \gamma^{(w)}(s) = (-3k_1(s)k_1'(s))T(s) \\ \quad + (k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s))v_2(s) \\ \quad + (2k_1'(s)k_2(s) + k_1(s)k_2'(s))v_3(s). \end{array} \right. \quad (2.2)$$

Hence, from (2.2)if we suppose M has 3-planar normal sections, we find

$$\left\{ \begin{array}{l} k_1^3(s)(2k_1'(s)k_2(s) + k_1(s)k_2'(s)) = 0, \\ k_1(s)k_1'(s)((2k_1'(s)k_2(s) + k_1(s)k_2'(s)) = 0, \\ k_1(s)k_1'(s)((k_1''(s) - k_1^3(s) - k_1(s)k_2^2(s)) + 3k_1^4(s)k_1'(s) = 0, \end{array} \right. \quad (2.3)$$

for all $s \in I$.

2.2. f - Biharmonic and Bi- f -harmonic maps between Riemannian manifolds

Let (M, g) and (N, h) be Riemannian manifolds and $\Psi : (M, g) \longrightarrow (N, h)$ be a smooth map.

The tension field of Ψ is given by $\tau(\Psi) = \text{trace } \nabla d\Psi$, where $\nabla d\Psi$ is the second fundamental form of Ψ defined by

$$\nabla d\Psi(X, Y) = \nabla_X^\Psi d\Psi(Y) - d\Psi(\nabla_X^M Y) \quad (2.4)$$

$$\Delta^\Psi V = - \sum_{i=1}^m \left\{ \nabla_{e_i}^\Psi \nabla_{e_i}^\Psi V - \nabla_{\nabla_{e_i}^M e_i}^\Psi V \right\}, \quad V \in \Gamma(\Psi^{-1}TN) \quad (2.5)$$

where ∇^Ψ is the pull-back connection on the pull-back bundle $\Psi^{-1}TN$ and $\{e_i\}_{i=1}^m$ is an orthonormal frame on M . Let M be a Riemannian manifold and $\gamma : I \rightarrow M$ be a differentiable curve parameterized by arc length. By using the definition of the tension field, we have

$$\tau(\gamma) \equiv \nabla_{\frac{\partial}{\partial s}}^\gamma d_\gamma \left(\frac{\partial}{\partial s} \right) = \nabla_T T, \quad (2.6)$$

where $T = \gamma'$. In this case biharmonic equation [7] for the curve γ reduces to

$$\nabla_T^3 T - R(T, \nabla_T T)T = 0, \quad (2.7)$$

that is, γ is called a biharmonic curve if it is a solution of the equation (2.7).

The map Ψ is a f -harmonic map with a differentiable function $f : M \rightarrow R$, if it is a critical point of f -energy

$$E_f(\Psi) = \frac{1}{2} \int_{\Omega} f |d\Psi|^2 dv_g, \quad (2.8)$$

where Ω is a compact domain of M . The Euler-Lagrange equation of $E_f(\Psi)$ is,

$$\tau_f(\Psi) = f\tau(\Psi) + d\Psi(\text{grad}f) = 0, \quad (2.9)$$

where $\tau_f(\Psi)$ is the f -tension field of Ψ . The map Ψ is said to be f -biharmonic, if it is a critical point of the f -bienergy functional

$$E_{2,f}(\Psi) = \frac{1}{2} \int_{\Omega} f |\tau(\Psi)|^2 v_g \quad (2.10)$$

where Ω is a compact domain of M . The Euler-Lagrange equation for the f -bienergy functional is given by

$$\tau_{2,f}(\Psi) = f\tau_2(\Psi) + \Delta f\tau(\Psi) + 2\nabla_{(\text{grad}f)}^{\Psi}\tau(\Psi) = 0, \quad (2.11)$$

where $\tau_{2,f}(\Psi)$ is the f -bitension field of Ψ . If an f -biharmonic map is neither harmonic nor biharmonic then we call it by proper f -biharmonic and if f is a constant, then an f -biharmonic map turns into a biharmonic map.

Bi- f -harmonic maps $\Psi : (M, g) \rightarrow (N, h)$ between two Riemannian manifolds are critical points of the bi- f -energy functional:

$$E_{f,2}(\Psi) = \frac{1}{2} \int_{\Omega} |\tau_f(\Psi)|^2 v_g, \quad (2.12)$$

where Ω is a compact domain of M . The corresponding Euler-Lagrange equation is

$$\tau_{f,2}(\Psi) = -\text{trace}(\nabla^{\Psi} f(\nabla^{\Psi} \tau_f(\Psi))) - f\nabla_{\nabla^{\Psi} M}^{\Psi} \tau_f(\Psi) + fR^N(\tau_f(\Psi), d\Psi)d\Psi = 0, \quad (2.13)$$

where $\tau_f(\Psi)$ is the f -tension field of Ψ . $\tau_{f,2}(\Psi)$ is called the bi- f -tension field of the map Ψ .

3. f -BIHARMONIC AND BI- f -HARMONIC NORMAL SECTION CURVES

An important special case of f -biharmonic maps is an f -biharmonic curve. Let $\gamma = \gamma(s)$ be a differentiable curve on N parameterized by arclength $s \in (a, b)$, where $a, b \in \mathbb{R}$. Then, putting $e_1 = \frac{\partial}{\partial s}$ as an orthonormal frame on $((a, b), ds^2)$, we write $d\gamma(e_1) = d\gamma(\frac{\partial}{\partial s}) = \gamma'$. Thus, the tension field of the curve is given by

$$\tau(\gamma) = \nabla_{e_1}^{\gamma} d\gamma(e_1) = \nabla_{\gamma'}^N \gamma'. \quad (3.1)$$

It is also easy to see that for a function $f : (a, b) \rightarrow (0, \infty)$, $\Delta f = f''$ and $\nabla_{\text{grad}f}^{\gamma} \tau(\gamma) = f' \nabla_{\gamma'}^N \nabla_{\gamma'}^N \gamma'$. If we put them in the f -biharmonic map equation,

$$f(\nabla_{\gamma'}^N \nabla_{\gamma'}^N \nabla_{\gamma'}^N \gamma' - R^N(\gamma', \nabla_{\gamma'}^N \gamma')\gamma') + 2f' \nabla_{\gamma'}^N \nabla_{\gamma'}^N \gamma' + f'' \nabla_{\gamma'}^N \gamma' = 0 \quad (3.2)$$

the biharmonicity equation of γ is obtained (see [9]).

Let $N^n(c)$ be a Riemannian space form and $\gamma : (a, b) \rightarrow N^n(c)$ be a curve with arclength parametrization. Let $\{F_i, i = 1, 2, \dots, n\}$ be the Frenet frame along the curve $\gamma(s)$, which is obtained as the orthonormalisation of the n -tuple $\{\nabla_{\frac{\partial}{\partial s}}^{(k)} d\gamma(\frac{\partial}{\partial s}) \mid k = 1, 2, \dots, n\}$. Then we have the following Frenet formula along the curve γ given by

$$\begin{aligned} \nabla_{\frac{\partial}{\partial s}}^\gamma F_1 &= k_1 F_2, \\ \nabla_{\frac{\partial}{\partial s}}^\gamma F_i &= -k_{i-1} F_{i-1} + k_i F_{i+1}, \quad \forall i = 2, 3, \dots, n-1, \\ \nabla_{\frac{\partial}{\partial s}}^\gamma F_n &= -k_{n-1} F_{n-1}, \end{aligned} \tag{3.3}$$

where $\{k_1, k_2, \dots, k_{n-1}\}$ are the curvatures of γ . Using the Frenet formulas one finds the tension and the bitension fields of γ , respectively, as follows:

$$\tau(\gamma) = \nabla_{\gamma'}^N \gamma' = k_1 F_2, \tag{3.4}$$

$$\tau_2(\gamma) = -3k_1 k_1' F_1 + (k_1'' - k_1 k_2^2 - k_1^3 + k_1 c) F_2 \tag{3.5}$$

$$+ (2k_1' k_2 + k_1 k_2') F_3 + k_1 k_2 k_3 F_4. \tag{3.6}$$

Substituting these into the f -biharmonic curve equation (3.2) and comparing the coefficients of both sides we say that γ is an f -biharmonic curve if and only if

$$\left\{ \begin{array}{l} 3k_1 k_1' f + 2f' k_1^2 = 0, \\ f k_1'' - f k_1^3 - f k_1 k_2^2 + f c k_1 + 2f' k_1' + f'' k_1 = 0, \\ f k_1' k_2 + f (k_1 k_2)' + f' k_1 k_2 = 0, \\ k_1 k_2 k_3 = 0, \end{array} \right. \tag{3.7}$$

(for details, we refer [9]).

Case 3.1. *If $k_1 = \text{constant} \neq 0$, then the first equation of (3.7) implies that f is constant and the curve γ is biharmonic. Also, if $k_2 = \text{constant} \neq 0$, then the first and third equations (3.7) imply that f is constant and thus the curve γ is biharmonic again.*

Case 3.2. *If $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then the f -biharmonic curve equation (3.7) is equivalent to*

$$\left\{ \begin{array}{l} f' k_1^2 = 0, \\ -f k_1^3 + f c k_1 + f'' k_1 = 0. \end{array} \right. \tag{3.8}$$

Here we conclude

Theorem 3.1. *If $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then γ is an f -biharmonic curve if and only if f is a non-zero constant function and γ is a curve with $k_1 = \sqrt{c}$.*

Case 3.3. If $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$. In this case the f -biharmonic curve equation (3.7) is equivalent to

$$\begin{cases} f' k_1^2 = 0, \\ f k_1^3 + f k_1 k_2^2 - f c k_1 = 0, \\ k_3 = 0. \end{cases} \quad (3.9)$$

Then we give the following conclusion

Theorem 3.2. If $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$, then γ is an f -biharmonic curve if and only if f is a non-zero constant function, γ is a helix and $k_3 = 0$.

Case 3.4. If $k_1 \neq \text{constant}$ and $k_2 = \text{constant} \neq 0$. In this case from the f -biharmonic curve equation (3.7) we obtain that $f = 0$, which is impossible.

So we have

Theorem 3.3. If $k_1 \neq \text{constant}$ and $k_2 = \text{constant} \neq 0$, there does not exist an f -biharmonic curve.

Case 3.5. If $k_1 \neq \text{constant}$ and $k_2 \neq \text{constant}$, then the system (3.7) is equivalent to

$$\begin{cases} f^2 k_1^3 = c_1^2, \\ (f k_1)'' = f k_1 (k_1^2 + k_2^2 - c), \\ f^2 k_1^2 k_2 = c_2, \\ k_3 = 0. \end{cases} \quad (3.10)$$

Solving the first equation of (3.10), we find $f = c_1 k_1^{-3/2}$. Substituting the first equation into the third one in (3.10) we have $k_2/k_1 = c_3$. Therefore, we conclude that

Theorem 3.4. If $k_1 \neq \text{constant}$ and $k_2 \neq \text{constant}$, then γ is a f -biharmonic curve if and only if $f = c_1 k_1^{-3/2}$, $k_2/k_1 = c_3$, $k_3 = 0$.

Now we shall examine necessary and sufficient conditions for a normal section curve γ to be f -biharmonic in the Riemannian space form $N(c)$. Note that we concentrate on non-geodesic cases:

Theorem 3.5. Let M be an n -dimensional submanifold of Riemannian space form $N(c)$, ($\dim N = (n + 3)$) and γ be the normal section curve of M with $k_1 = \text{constant} \neq 0$, $k_2 = 0$. Then M has planar normal sections if and only if the normal section curve γ of M is an f -biharmonic curve satisfying f is a non-zero constant function and $k_1 = \sqrt{c}$.

Theorem 3.6. *Let M be an n -dimensional submanifold of Riemannian space form $N(c)$, ($\dim N = (n + 3)$) and γ be the normal section curve of M with $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$. Then M has planar normal sections if and only if the normal section curve γ of M is an f -biharmonic curve satisfying f is a non-zero constant function and $k_1^2 + k_2^2 = c$, $k_3 = 0$.*

Theorem 3.7. *Let M be an n -dimensional submanifold of Riemannian space form $N(c)$, ($\dim N = (n + 3)$) and γ be the normal section curve of M with $k_1 \neq \text{constant}$ and $k_2 \neq \text{constant}$. Then M has planar normal sections if and only if the normal section curve γ of M is an f -biharmonic curve satisfying $f = c_1 k_1^{-3/2}$ and $k_2/k_1 = c_3$, $k_3 = 0$.*

Next we shall investigate bi- f -harmonicity of planar normal sections curves.

Let $\gamma : I \rightarrow M(c)$ be a differentiable curve in a Riemannian manifold $M(c)$, parameterized by its arclength. Then γ is a bi- f -harmonic curve if and only if (for details, see [12])

$$\left\{ \begin{array}{l} -3k_1 k_1' f^2 - 4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^3 f^2 - k_1 k_2^2 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 f f'' + 2k_1 (f')^2 + c k_1 f^2 = 0, \\ 2k_1' k_2 f + k_1 k_2' f + 4k_1 k_2 f' = 0, \\ k_1 k_2 k_3 = 0. \end{array} \right. \quad (3.11)$$

We assume that $\gamma : I \rightarrow E^n$ is a differentiable curve in the n -dimensional Euclidean space, defined on an open real interval I and parameterized by its arclength. Since E^n is a Riemannian space form with $c = 0$, from the bi- f -harmonic curve equation given by (3.11) we have [12]

Theorem 3.8. *Let $\gamma : I \rightarrow E^n$ be a curve in the n -dimensional Euclidean space parameterized by its arclength. Then γ is a bi- f -harmonic curve if and only if*

$$\left\{ \begin{array}{l} -3k_1 k_1' f^2 - 4k_1^2 f f' + f f''' + f' f'' = 0, \\ -k_1^3 f^2 - k_1 k_2^2 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 f f'' + 2k_1 (f')^2 = 0, \\ 2k_1' k_2 f + k_1 k_2' f + 4k_1 k_2 f' = 0, \\ k_1 k_2 k_3 = 0. \end{array} \right. \quad (3.12)$$

Case 3.6. *If $k_1 = \text{constant} \neq 0$ and $k_2 = 0$, then (3.12) reduces to*

$$\begin{aligned} -4k_1^2 f f' + f f''' + f' f'' &= 0, \\ -k_1^2 f^2 + 3f f'' + 2(f')^2 &= 0. \end{aligned} \quad (3.13)$$

From the second equation above we obtain

$$f'(5k_1^2 f^2 + 2f'') = 0, \quad (3.14)$$

via the first equation of system (3.14) and we get

Theorem 3.9. *Let $\gamma : I \rightarrow E^n$ be a curve in the n -dimensional Euclidean space, parameterized by its arclength, with $k_1 = \text{constant} \neq 0$ and $k_2 = 0$. Then γ is a bi- f -harmonic curve if and only if either f is a constant function or f is given by*

$$f(s) = c_1 \cos\left(\sqrt{\frac{5}{2}}k_1 s\right) + c_2 \sin\left(\sqrt{\frac{5}{2}}k_1 s\right), \quad (3.15)$$

for $s \in I$ and $c_1, c_2 \in R$.

Case 3.7. *If $k_1 = \text{constant} \neq 0$ and $k_2 = \text{constant} \neq 0$, then (3.12) reduces to*

$$\begin{aligned} -4k_1^2 f f' + f f''' + f' f'' &= 0, \\ -k_1^2 f^2 - k_2^2 f^2 + 3k_1 f f'' + 2k_1 (f')^2 &= 0, \\ f' &= 0, \\ k_3 &= 0, \end{aligned} \quad (3.16)$$

and we conclude

Theorem 3.10. *Let $\gamma : I \rightarrow E^n$ be a curve in the n -dimensional Euclidean space, parameterized by its arclength, with $k_1 = \text{constant} \neq 0$ and $k_2 = 0$. Then γ is a bi- f -harmonic curve if and only if the curvatures k_1 and k_2 satisfy:*

$$-3k_1 k_1' f^2 - 4k_1^2 f f' + f f''' + f' f'' = 0, \quad (3.17)$$

$$-k_1^3 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 (f')^2 = 0. \quad (3.18)$$

Let us examine necessary and sufficient conditions for which normal section curve γ be bi- f -harmonic in the Riemannian space form. If we search non-geodesic solution.

Theorem 3.11. *Let N be a submanifold of $M(c)$. Then N has 3-planar normal sections bi- f -harmonic for $f(s) = c_1 \cos\left(\sqrt{\frac{5}{2}}k_1 s\right) + c_2 \sin\left(\sqrt{\frac{5}{2}}k_1 s\right)$ if and only if curvatures of planar normal section curves are $k_1 = \text{constant} \neq 0$ and $k_2 = 0$ being solution of system (3.14).*

Theorem 3.12. *Let N be a submanifold of $M(c)$. Then N has 3-planar normal sections bi- f -harmonic if and only if curvatures of planar normal section curves are $k_1 = \text{constant} \neq 0$ and $k_2 = 0$ satisfy;*

$$-3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \quad (3.19)$$

$$-k_1^3f^2 + k_1''f^2 + 4k_1'ff' + 3k_1(f')^2 = 0. \quad (3.20)$$

REFERENCES

- [1] B.Y. Chen ve S. J. Li, "Classification of surfaces with pointwise planar normal sections and its application to Fomenko's conjecture", J.Geom., vol. 26, pp. 21-34, 1986.
- [2] B.Y. Chen ve P. Verheyen, "Submanifolds with geodesic normal sections", Math. Ann., vol. 269, pp. 417-429, 1984.
- [3] B.Y. Chen, "Classification of surfaces with planar normal sections", J. Of Geometry, vol. 20, pp. 122-127, 1983.
- [4] Y.H. Kim, "Minimal surfaces of pseudo-euclidean spaces with geodesic normal sections", Differential geometry and its applications, vol.5, pp. 321-329, 1995.
- [5] Y.H. Kim, "Pseudo-Riemannian submanifolds with pointwise planar normal Sections", Math J. Okayama Univ., vol:34, pp. 249-257, 1992.
- [6] Y.H. Kim, "Surfaces in a pseudo-euclidean space with planar normal sections", J. Geom., vol.35, pp. 120-131, 1989.
- [7] J. Eells ve J.H. Sampson, "Harmonic mappings of Riemannian manifolds", Amer. J. Math., vol. 86, No. 1, pp. 109-160, 1964.
- [8] G. Y. Jiang , 2- harmonic maps and their first and second variational formulas ,Chinese Ann. Math. Ser. A, vol.7, pp. 109-117, 1989.
- [9] Y. L. Ou, On f-biharmonic maps ve f-biharmonic submanifolds , Pacific J. Math. ,vol. 271, no. 2, pp. 461-477, 2014.
- [10] F. E. Erdoğan, S.Y. Perktaş, Normal section curves and curvatures , Honam Mathematical Journal, vol. 39, no. 4, pp. 515-525, 2017.
- [11] B.Şahin,Harmonic Riemannian maps on locally conformal Kaehler manifolds, Proc. Indian Acad.Sci. (Math. Sci.) vol. 118, no. 4, pp. 573-581, November 2008.
- [12] S. Y. Perktaş, A. M. Blaga, F. E. Erdoğan, B. E. Acet, "Bi-f-harmonic curves and hypersurfaces", Filomat, vol.33, no.16, pp.5167-5180, (2019).
- [13] B. Kılıç, K. Arslan, "On curve and surfaces of AW(k) type, BAÜ Fen. Bil. Enst. Dergisi, vol.6, no.1, pp.52-61, 2004.
- [14] J. Eells and L. Lemaire, A report on harmonic maps, Bull. London Math. Soc. vol.10 , pp.1-68, 1978.
- [15] J. Eells and J.H. Sampson, Harmonic mappings of Riemannian manifolds, Amer. J. Math. vol.86, pp.109-160, 1964.

- [16] B.Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.* vol.17, pp.169-188, 1991.
- [17] W.J. Lu, On f -biharmonic maps and bi- f -harmonic maps between Riemannian manifolds, *Sci. China Math.* vol.58(B) , pp.1483-1498, 2015.
- [18] S. Ouakkas, R. Nasri, and M. Djaa, On the f -harmonic and f -biharmonic maps, *JP J. Geom. Topol.* vol.10(1), pp.11-27, 2010.
- [19] M. Rimoldi and G. Veronelli, f -harmonic maps and applications to gradient Ricci solitons, arXiv:1112.3637, 2011.
- [20] C.-L. Zhao and W.-J. Lu, Bi- f -harmonic map equations on singly warped product manifold, *Appl. Math. J. Chinese Univ.* vol.30(1) , pp.111-126, (2010).
- [21] Y.-L. Ou, On f -biharmonic maps and f -biharmonic submanifolds, *Pacific J. Math.* vol.271(2), pp.461-477, 2014.
- [22] P. Baird and J.C. Wood, Harmonic morphisms between Riemannian manifolds, London Math. Soc. Monogr. (N.S.) No. 29, Oxford Univ. Press, 2003
- [23] Lichnerowicz A. Applications harmoniques et varietes kahleriennes. In: *Symposia Mathematica*, vol. III. London: Academic Press, pp. 341–402, 1970.

FACULTY OF SCIENCE, DEPARTMENT OF MATHEMATICS, EGE UNIVERSITY, TURKEY

Email address: feyza.esra.erdogan@ege.edu.tr

ART AND SCIENCE FACULTY, DEPARTMENT OF MATHEMATICS, ADIYAMAN UNIVERSITY TURKEY

Email address: sperktas@adiyaman.edu.tr

ART AND SCIENCE FACULTY, DEPARTMENT OF MATHEMATICS, ADIYAMAN UNIVERSITY TURKEY

Email address: glstnyldrmm@gmail.com