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SYMPLECTOSUBMERSIONS

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Abstract. In this paper, we introduce a new submersion between almost symplectic manifolds, give examples and investigate the geometry of the base manifold when the total manifold has some special cases.

1. Introduction

In Riemannian geometry, there are two basic maps; isometric immersions and Riemannian submersions. Isometric immersions (Riemannian submanifolds) are basic such maps between Riemannian manifolds and they are characterized by their Riemannian metrics and Jacobian matrices. More precisely, a smooth map $F:(M,g_{_M})\longrightarrow(N,g_{_N})$ between Riemannian manifolds $(M,g_{_M})$ and $(N,g_{_N})$ is called an isometric immersion (submanifold) if F_* is injective and

$$g_{_N}(F_*X,F_*Y)=g_{_M}(X,Y)$$

for vector fields X,Y tangent to M; here F_* denotes the derivative map. A smooth map $F:(M_1,g_1)\longrightarrow(M_2,g_2)$ is called a Riemannian submersion if F_* is onto and it

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satisfies the above equation for vector fields tangent to the horizontal space $(kerF_*)^\perp$. Riemannian submersions between Riemannian manifolds were first studied by O'Neill [8] and Gray [6]; see also [4].

Submanifolds of complex manifolds (holomorphic, totally real, CR-submanifold, etc..) and Riemannian submersions (holomorphic, anti-invariant, semi-invariant etc...) between complex manifolds have been studied widely, see for instance [11], [4] and [9]. On the other hand, submanifolds of symplectic manifolds have been also studied by many authors and this research area is an active research area. But as far as we know, a submersion analog with Riemannian submersion (or holomorphic submersion) has been not studied. By considering applications of symplectic manifolds and Riemannian submersions [7] in mathematical physics, it would be interesting to consider as analog of holomorphic submersion for symplectic manifolds.

In this paper, we introduce a new submersion, namely symplectosubmersion, between almost symplectic manifolds. We provide examples and check the existence of symplectic connection on the base manifold. We note that, in [3], the authors have considered a submersion f from an open manifold with a symplectic form Ω to a manifold N with $dimN < dimM$, and they proved that such submersion with symplectic fibres satisfy the $h-$ principle.

2. Preliminaries

A differentiable manifold M is said to be an *almost complex manifold* if there exists a linear map $J : TM \longrightarrow TM$ satisfying $J^2 = -id$ and J is said to be an *almost complex structure* of M. The tensor field N of type (1, 2) defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J([X, JY] + [JX, Y]), \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$, is called *Nijenhuis tensor field* of J. Then, J defines a complex structure [11] on M if and only if N vanishes on M. Now consider a Riemannian metric g on an almost complex manifold (M, J). We say that the pair (J, g) is an *almost Hermitian structure* on M, and M is an *almost Hermitian manifold* if

$$g(JX, JY) = g(X, Y), \qquad \forall X, Y \in \Gamma(M). \tag{2.2}$$

Moreover, if J defines a complex structure on M, then (J, g) and M are called *Hermitian structure* and *Hermitian manifold*, respectively. The *fundamental* 2*-form* Ω of an almost

Hermitian manifold is defined by

$$\Omega(X,\,Y) = g(X,\,JY), \quad \forall X, Y \in \Gamma(M). \tag{2.3}$$

A Hermitian metric on an almost complex M is called a *Kähler metric* and then M is called a *Kähler manifold* if Ω is closed, i.e.,

$$d\,\Omega(X,\,Y,\,Z) = 0, \quad \forall X, Y \in \Gamma(M). \tag{2.4}$$

It is known (see [11]) that the Kählerian condition (2.4) is equivalent to

$$(\nabla_X J)Y = 0, \forall X, Y \in \Gamma(M), \tag{2.5}$$

where ∇ is the Riemannian connection of g. We note that submanifolds of an almost Hermitian manifolds are defined with respect to behaviour of the almost complex structure J. We will not give details of these submanifolds here, we refer the book [2] for various submanifolds in complex geometry.

Riemannian submersions as a dual notion of isometric immersions have been studied in complex settings in the early 1970s. As an analogue of holomorphic submanifolds, Watson [10] defined almost Hermitian submersions between almost Hermitian manifolds and he showed that the base manifold and each fiber have the same kind of structure as the total space, in most cases.

A *symplectic manifold* is an even dimensional differentiable manifold M with a global $2-$form Ω which is closed $d\Omega = 0$ and of maximal rank $\Omega^n \neq 0$. A Kähler manifold M with its fundamental $2-$form is a symplectic manifold. However, there are symplectic manifolds that do not admit any complex structures. A pair of a manifold M and non-degenerate form Ω, not necessarily closed is called an almost symplectic manifold. Given a linear subspace W of a symplectic vector space (V, Ω), its symplectic orthogonal W^Ω is the linear subspace defined by $W^\Omega = \{v \in V \mid \Omega(u, v) = 0, \forall u \in W\}$. Now, Let (N, Ω) be a $2n$-dimensional symplectic manifold and $I : M \to N$ an immersed submanifold of N. Then M is called a symplectic submanifold if $I^*\Omega$ is symplectic, i.e. the induced bilinear form Ω is nondegenerate and closed on the tangent bundle of the submanifold. M is called an isotropic submanifold if $I^*\Omega = 0$. M is a Lagrangian submanifold if $I^*\Omega = 0$ and $dimM = \frac{1}{2}N$. We note that since $I^*\Omega = 0$, there is no induced structure. Finally M is called a coisotropic submanifold if $(T_pM)^\Omega \subseteq T_pM$ for every $p \in M$, for more information see:[1]

3. A submersion between almost symplectic manifolds

By inspiring Riemannian submersions, we present the following notion.

Definition 3.1. *Let (M, ω_M) and (N, ω_N) be almost symplectic manifolds and F a submersion. If the following two conditions are satisfied, then F is called symplectosubmersion between symplectic manifolds;*
(S1). *The fibers $F^{-1}(q), q \in N$, are symplectic submanifolds of M.*
(S2). $\omega_N(F_*X, F_*Y) = \omega_M(X, Y)$ *for* $X, Y \in \Gamma\left((KerF_*)^{\perp}\right)$.

We first note that, since the fibers are symplectic submanifolds it follows that $(KerF_*)^{\perp}$ is a symplectic distribution on M, i.e. $(KerF_*)^{\perp} \cap KerF_* = \{0\}$.

We now give two examples of symplectic submersions. But we first recall the notion of holomorphic submersions [4]. Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be almost Hermition manifolds. A surjective map $\Pi : M_1 \to M_2$ is called almost Hermitian (holomorphic) submersion and an almost complex map; i.e.

$$\Pi_* J_1 = J_2 \Pi_*. \tag{3.6}$$

Example 3.1. *Let (M_1, J_1, g_1) and (M_2, J_2, g_2) be Kähler manifolds and $\Pi : M_1 \to M_2$ an almost Hermitian submersion. Then (M_1, J_1, g_1) and (M_2, J_2, g_2) are symplectic manifolds with symplectic forms $\Omega_1 = g_1(X, J_1Y)$ and $\Omega_2 = g_2(U, J_2V)$ for $X, Y \in T(M_1)$ and $U, V \in T(M_2)$. Since Π is an almost complex map, we get*

$$\Omega_2(F_*X, F_*Y) = g_2(F_*X, J_2F_*Y)$$

and

$$\Omega_2(F_*X, F_*Y) = g_2(F_*X, F_*J_1Y).$$

Then Riemannian submersion Π implies that

$$\Omega_2(F_*X, F_*Y) = g_1(X, J_1Y).$$

Hence, we get

$$\Omega_2(F_*X, F_*Y) = \Omega_1(X, Y).$$

On the other hand, since g_1 is a Riemannian metric, $(KerF_*)$ is a symplectic distribution.

Example 3.2. *Consider the following submersion defined by*

$$\begin{array}{rccc} F: & (\mathbb{R}^4, \Omega_4) & \to & (\mathbb{R}^2, \Omega_2) \\ & (x_1, x_2, x_3, x_4) & \to & \left(\dfrac{x_1 + x_2}{\sqrt{2}}, \dfrac{x_3 + x_4}{\sqrt{2}}\right) \end{array}$$

where Ω_4 and Ω_2 are canonical symplectic structure of $\mathbb{R}^4$ and $\mathbb{R}^2$. By direct computation we have

$$KerF_* = Sp\left\{X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, X_2 = \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4}\right\}$$

and

$$(KerF_*)^{(\Omega_4)_\perp} = Sp\left\{X_3 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}, X_4 = \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4}\right\},$$

where $(\Omega_4)_\perp$ denotes the orthogonality with respect to the symplectic form of Euclidean $4-$ space. It is easy to see that $(KerF_*)$ and $(KerF_*)^\perp$ are symplectic subspace of $\left(\mathbb{R}^4, \Omega_4\right)$. On the other hand, we have

$$F_*X_3 = \sqrt{2}\frac{\partial}{\partial y_1}, \qquad F_*X_4 = \sqrt{2}\frac{\partial}{\partial y_2}.$$

Then we get

$$\Omega_4\left(X_3, X_4\right) = \Omega_2\left(F_*X_3, F_*X_4\right) = 2.$$

This shows that F is a symplectosubmersion.

It is known that symplectic connection of a symplectic manifold is not unique. In the sequel we show that if the total manifold of a symplectosubmersion has a unique symplectic connection, then, the base manifold has also a unique symplectic connection. A symplectic connection ∇ is a connection that is both torsion free and $\nabla\omega = 0$. We recall that a symplectic manifold with a fixed symplectic connection is called a Fedosov manifold [5].

Theorem 3.1. *Let M_1 be a Fedosov manifold and M_2 a symplectic manifold. If $F : M_1 \to M_2$ is a symplectosubmersion then M_2 is also a Fedosov manifold.*

Proof. Since M_1 is a Fedosov manifold then it has a unique symplectic connection. Thus we have

$$\left(\overset{1}{\nabla}_X w_1\right)(Y, Z) = Xw_1(Y, Z) - w_1\left(H\overset{1}{\nabla}_X Y, Z\right) - w_1\left(Y, H\overset{1}{\nabla}_X Z\right) = 0$$

for $X, Y, Z \in \Gamma((KerF_*)^\perp)$, where H is the projection morphism from TM_1 to $(KerF_*)^\perp$. Since F is a symplectosubmersion, we obtain

$$Xw_2(F_*Y, F_*Z) - w_2\left(F_*H\overset{1}{\nabla}_X Y, F_*Z\right) - w_2\left(F_*Y, F_*H\overset{1}{\nabla}_X Z\right) = \left(\overset{2}{\nabla}_X w_2\right)(F_*Y, F_*Z).$$

Thus, since $\overset{1}{\nabla}$ is unique symplectic connection, it follows that $\overset{2}{\nabla}$ is also a unique symplectic connection on M_2.

We also have the following theorem.

Theorem 3.2. *Let F be a symplectosubmersion from symplectic manifold M_1 to an almost symplectic manifold M_2. Then M_2 is a symplectic manifold.*

Proof. Let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be vector fields on an open subset of M_2, and X, Y and Z be their horizontal lifts to M_1. Since M_1 is a symplectic manifold then there is a closed nondegenerate $2-$form w_1 on M_1. Thus we get

$$\begin{aligned} 3dw_1(X,Y,Z) &= Xw_1(Y,Z) + Yw_1(Z,X) + Zw_2(X,Y) - w_1([X,Y],Z) \\ &\quad -w_1([Y,Z],X) - w_1([Z,X],Y). \end{aligned}$$

Then symplectosubmersion F implies that

$$\begin{aligned} 3dw_1(X,Y,Z) &= \tilde{X}w_2\left(\tilde{Y},\tilde{Z}\right) + \tilde{Y}w_2\left(\tilde{Z},\tilde{X}\right) + F_*Zw_2\left(\tilde{X},\tilde{Y}\right) \\ &\quad -w_2\left(\left[\tilde{X},\tilde{Y}\right],\tilde{Z}\right) - w_2\left(\left[\tilde{Y},\tilde{Z}\right],\tilde{X}\right) - w_2\left(\left[\tilde{Z},\tilde{X}\right],\tilde{Y}\right) \\ &= 3dw_2\left(\tilde{X},\tilde{Y},\tilde{Z}\right). \end{aligned}$$

which proves the theorem.

It is known that, if M_1 is a Kähler manifold with the Riemannian metric g_{M_1} and complex structure J. Then (M_1, Ω_1) is a symplectic manifold with $\Omega_1 = (X,Y) = g_1(X, JY)$. Since g_1 is a Riemannian metric it follows that the Levi-Civita connection ∇ is a unique symplectic connection. As a result, (M_1, Ω) is a Fedosov manifold.

Theorem 3.3. *Let (M_1, g_1) be a Kähler manifold and (M_2, Ω_2) a symplectic manifold. If F is a symplectosubmersion from (M_1, Ω_1) to (M_2, Ω_2), then (M_2, Ω_2) is a Fedosov manifold, where $\Omega_1(X,Y) = g_1(X, J_1Y)$ for almost complex structure J_1 and vector fields $X, Y \in \Gamma(TM_1)$.*

References

[1] Cannas da Silva, A., Lectures on Symplectic Geometry, Lecture Notes in Mathematics (Book 1764), 2008.

[2] Chen, B. Y., Differential geometry of warped product manifolds and submanifolds, World Scientific, 2017.

[3] Datta, M., Islam, M. R. Submersions on open symplectic manifolds, Topology and its Applications 156 (2009) 1801-1806

[4] Falcitelli, M., Pastore, A. M., Ianus, S., Riemannian Submersions and Related Topics, World Scientific, 2004.

[5] Gelfand, I., Retakh, V., Shubin, M.: Fedosov manifolds. Adv. Math. 136(1), 104-140 (1998)

[6] Gray, A. Pseudo-Riemannian almost product manifolds and submersions. J Math Mech 1967; 16: 715-737.

[7] Hogan, P. A., Kaluza-Klein theory derived from a Riemannian submersion. J. Math. Phys. 25(7), (1984), 2301-2305.

[8] O'Neill, B. The fundamental equations of a submersion. Mich Math J 1966; 13: 459-469.

[9] Şahin, B., Riemannian Submersions, Riemannian Maps in Hermitian Geometry and Their Applications, Elsevier, 2017.

[10] B. Watson, Almost Hermitian submersions. J. Differential Geometry, 11(1)(1976), 147-165.

[11] Yano, K., Kon, M., Structures on Manifolds, World Scientific, 1984.

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