



## CONFORMAL SUBMERSIONS FROM RICCI-YAMABE SOLITONS

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**Abstract.** The concept of Ricci-Yamabe (RY) soliton is such an approach that unifies the geometric properties of Ricci solitons and Yamabe solitons. Because of the wealth of geometric properties and applications in differential geometry and mathematical physics, the research on RY solitons has gained increasing interest. This paper is devoted to conformal Riemannian submersions whose total spaces possess Ricci-Yamabe (RY) solitons and gives some examples for clarification.

**Keywords:** Ricci-Yamabe soliton; Conformal Riemannian submersions; Scalar curvatures; Totally geodesic

**MSC (2020):** Primary 53B20; Secondary 53C12, 53C25, 53C43.

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### 1. INTRODUCTION

Physically occurring waves which maintain their shape after interaction with other waves are known as solitons. Solitons are viewed as self-similar solutions of some wave equations in mathematics. From the idea of solitons, Richard S. Hamilton [4] defined Ricci solitons as self-similar solutions of nonlinear partial differential equations in the context of geometric flows. Recently, solitons have gained much significance and attention in differential geometry due to its important link with curvature flow and geometrical analysis on manifolds. More specifically, Ricci solitons naturally occur while studying Ricci flows, invented by Richard Hamilton, in terms of their singularity formation and asymptotic behavior. After the advent of the Ricci flow by Hamilton, several geometers started investigating geometric solitons. Moreover, the spectacular application of Ricci flow by Grigori Perelman [12] to prove the famous Poincaré Conjecture made the study of solitons a very important field in modern geometry. Thus, soliton structures become important in the study of manifolds admitting different geometric structures.

The topic of geometric flows became one of the most important in contemporary differential geometry after the introduction of Ricci flow and Yamabe flow by Hamilton in the 1980s. The self-similar solutions to such flows generate Ricci solitons and Yamabe solitons, respectively. Thus, it is reasonable to consider RY solitons, which are the fixed points of the RY flow.

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A Riemannian manifold  $(M, g)$  equipped with a vector field  $\zeta$  is said to admit a RY soliton (RY soliton) if

$$\frac{1}{2}\mathcal{L}_\zeta g + \alpha \text{Ric} + \left(\lambda - \frac{\beta}{2}\kappa\right)g = 0, \quad (1.1)$$

where  $\text{Ric}$  is the Ricci tensor,  $\kappa$  is the scalar curvature,  $\lambda, \alpha, \beta$  are constants, and  $\mathcal{L}_\zeta$  denotes the Lie derivative along  $\zeta$ . The soliton is called expanding, steady, or shrinking according to whether  $\lambda$  is positive, zero, or negative.

Note that

- (1) if  $\zeta$  is gradient of some smooth function  $f$  on  $M$ , then the above notion is called gradient RY soliton.
- (2) if we replace  $\lambda$  in equation (1.1) by a smooth function, then we say that  $(M, g)$  is an almost RY soliton.

On the other hand, conformal Riemannian submersions (CRSs) provide a rich geometric framework that generalizes Riemannian submersions, in which the differential restricted to the horizontal distribution is conformal with a positive dilation function rather than an isometry (see [2]).

A surjective smooth map between two Riemannian manifolds

$$F : (M, g) \rightarrow (N, h)$$

is said to be Riemannian submersion if it has maximal rank at every point  $p \in M$  and the differential  $F_{*p}$  maps the horizontal space  $(\text{Ker } F_*)^\perp$  isometrically onto  $T_{F(p)}N$ . The fibers of  $F$  over  $q \in N$  is defined by  $F^{-1}(q)$ . Moreover, the submersion  $F$  is called a conformal (horizontally conformal) submersion (CS) if the restriction of  $F_*$  to the horizontal distribution  $(\text{Ker } F_*)^\perp$  is conformal, that is, there exists a positive smooth function  $\sigma$  (the dilation) such that [1]

$$h(F_*X, F_*Y) = \sigma^2 g(X, Y), \quad (1.2)$$

for all  $X, Y \in \Gamma((\text{Ker } F_*)^\perp)$ .

Note that  $F$  is called horizontally weakly conformal map at  $p \in M$  if either

- (1)  $F_{*p} = 0$ , or
- (2)  $F_{*p}$  maps the horizontal space conformally onto  $T_{F(p)}N$ .

A horizontally weakly conformal map  $F : M \rightarrow N$  is said to be horizontally homothetic if the gradient of its dilation  $\sigma$  is vertical at regular points. Equivalently,  $\sigma$  is constant along horizontal directions.

In recent times, an increased amount of research is being conducted on the relationship between the theory of solitons and submersions. The work of Meriç and Kılıç [9] involved studying Riemannian submersions with total manifolds possessing Ricci solitons, and Meena and Yadav [6] further expanded the study to include conformal Riemannian submersions. In [8], Meriç discussed Riemannian submersions whose total manifolds are almost Yamabe solitons to Riemannian manifolds. But still, there is no study about the CSs whose total manifolds admit RY solitons.

This paper aims to fill this gap. We organize our work as follows. Section 2 recalls essential preliminaries on the study. The main results have been discussed in Section 3, where we have

considered CSs from a RY soliton and established a condition for the fibers/base manifold to be (almost) RY solitons/Einstein manifolds. Section 4 deals with the examples of these mappings, and conclusions are drawn in Section 5.

## 2. PRELIMINARIES

In this section, we give a brief review of basic facts; cf. [15].

Let

$$F : (M^r, g) \rightarrow (N^r, h)$$

be a Riemannian submersion. The vertical distribution  $\mathcal{V} = \text{Ker } F_*$  and the horizontal distribution  $\mathcal{H} = (\text{Ker } F_*)^\perp$  are orthogonal complements. For any vector field  $E$  on  $M$ , we write  $E = \mathcal{V}E + \mathcal{H}E$ . The O'Neill tensors  $A$  and  $T$  are defined by [11]

$$A_E F = \mathcal{H} \nabla_{\mathcal{H}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{H}E} \mathcal{H}F, \quad (2.3)$$

$$T_E F = \mathcal{H} \nabla_{\mathcal{V}E} \mathcal{V}F + \mathcal{V} \nabla_{\mathcal{V}E} \mathcal{H}F. \quad (2.4)$$

These tensors satisfy  $A_E = A_{\mathcal{H}E}$ ,  $T_E = T_{\mathcal{V}E}$ , and are skew-symmetric. Their properties include

$$\nabla_U V = T_U V + \mathcal{V} \nabla_U V, \quad \nabla_X U = A_X U + \mathcal{V} \nabla_X U, \quad \nabla_X Y = A_X Y + \mathcal{H} \nabla_X Y, \quad (2.5)$$

for  $U, V \in \Gamma(\mathcal{V})$  and  $X, Y \in \Gamma(\mathcal{H})$ .

The submersion  $F$  is known to be of totally umbilical fibers if; cf. [19], [20]

$$T_U V = g(U, V)H \text{ or } T_U X = -g(H, X)U, \quad (2.6)$$

where  $H$  denotes the mean curvature vector field of the fibers.

For the map  $F$ , the second fundamental form is expressed as [3]

$$(\nabla F_*)(X, Y) = D_{F_*X} F_*Y - F_*(\nabla_X Y), \quad (2.7)$$

where  $D$  is the Levi-Civita connection on  $N$ .

Let

$$F : (M^r, g) \rightarrow (N^s, h)$$

be a CS with dilation  $\sigma$ . The following fundamental relation holds for basic vector fields [3]:

$$\begin{aligned} F_*(\mathcal{H} \nabla_X Y) &= \nabla_{F_*X}^N F_*Y + \frac{\sigma^2}{2} \{X(\sigma^{-2}) F_*Y \\ &\quad + Y(\sigma^{-2}) F_*X - g(X, Y) F_*(\text{grad}_{\mathcal{H}} \sigma^{-2})\}. \end{aligned} \quad (2.8)$$

The tension field  $\mathfrak{T}(F)$  of  $F$  is defined as the trace of the second fundamental form. For a CS, we have [7]

$$\mathfrak{T}(F) = (s - 2) \frac{\sigma^2}{2} F_*(\text{grad}_{\mathcal{H}} \sigma^{-2}) - (r - s) F_*(H), \quad (2.9)$$

where  $H$  is the mean curvature vector of the fibers. The map  $F$  is harmonic if and only if  $\mathfrak{T}(F) = 0$ .

Now, we recall the components of the Ricci tensor on the total manifold of a CS. These formulas are essential for studying RY solitons.

Let  $\{U_i\}_{s+1 \leq i \leq r}$  be an orthonormal basis of  $\mathcal{V}$  and  $\{X_j\}_{1 \leq j \leq s}$  an orthonormal basis of  $\mathcal{H}$ . Denote by  $\text{Ric}^{\mathcal{V}}$  the Ricci tensor of the fibers and by  $\text{Ric}^N$  the Ricci tensor of the base

manifold  $N$ . If horizontal distribution  $(\ker F_*)^\perp$  is integrable, then the horizontal space is totally umbilical, that is,

$$A_X Y = g(X, Y) H^{hor},$$

where mean curvature vector  $H^{hor}$  is given by [3]

$$H^{hor} = -\frac{\sigma^2}{2} (\nabla_\nu \sigma^{-2}). \quad (2.10)$$

**Proposition 2.1.** [6] *For a CS*

$$F : (M, g) \rightarrow (N, h)$$

*with dilation  $\sigma$ , the Ricci tensor components on  $M$  are as follows:*

$$\begin{aligned} \text{Ric}_{UV} &= \text{Ric}_{UV}^\nu - (r-s)g(T_U V, H) + \sum_{j=1}^s g((\nabla_U A)_{X_j} X_j, V) \\ &\quad + \sum_{j=1}^s g(A_{X_j} U, A_{X_j} V) - \sum_{j=1}^s g((\nabla_{X_j} T)_U X_j, V) \\ &\quad - \frac{s\sigma^4}{2} g(U, \nabla_\nu \sigma^{-2}) g(V, \nabla_\nu \sigma^{-2}), \end{aligned} \quad (2.11)$$

$$\begin{aligned} \text{Ric}_{UX} &= (r-s)g(\nabla_U H, X) - \sum_{i=s+1}^r g((\nabla_{U_i} T)_U U_i, X) \\ &\quad + \sum_{j=1}^s g((\nabla_X A)_{X_j} X_j, U) - \sum_{j=1}^s g((\nabla_{X_j} A)_X X_j, U) \\ &\quad - \sum_{j=1}^s g(T_U X_j, \mathcal{V}[X, X_j]), \end{aligned} \quad (2.12)$$

$$\begin{aligned} \text{Ric}_{XY} &= \sum_{i=s+1}^r g((\nabla_{U_i} A)_X Y, U_i) + \sum_{i=s+1}^r g(A_X U_i, A_Y U_i) \\ &\quad - \sum_{i=s+1}^r g((\nabla_X T)_{U_i} Y, U_i) - \sum_{i=s+1}^r g(T_{U_i} X, T_{U_i} Y) \\ &\quad + \sigma^2 g(A_X Y, \nabla_\nu \sigma^{-2}) + \sigma^{-2} \text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N \\ &\quad + \frac{3}{4} \sum_{j=1}^s g(\mathcal{V}[X, X_j], \mathcal{V}[X_j, Y]) - \frac{(s-2)}{2} \sigma^2 g(\nabla_X \text{grad } \sigma^{-2}, Y) \\ &\quad - \frac{\sigma^2}{2} g(X, Y) \left\{ \Delta^\nu \sigma^{-2} - s \left( H^{hor} \sigma^{-2} \right) \right\} + \frac{s\sigma^4}{4} g(X, Y) \|\text{grad } \sigma^{-2}\|^2 \\ &\quad + \frac{\sigma^4}{4} (s-2) (X \sigma^{-2}) (Y \sigma^{-2}). \end{aligned} \quad (2.13)$$

Immediate consequences of Proposition 2.1 are as follows:

**Corollary 2.1.** [6] *Let*

$$F : (M^r, g) \rightarrow (N^s, h)$$

be a homothetic CS with dilation  $\sigma$  such that the fibers of  $F$  are totally geodesic and  $(\ker F_*)^\perp$  is integrable. Then

$$\begin{aligned} \text{Ric}_{XY} &= g(X, Y) \delta(H^{hor}) + \sigma^{-2} \text{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N \\ &\quad - \frac{1}{4} \sigma^4 g(X, Y) |\nabla_{\mathcal{V}} \sigma^{-2}|^2 + \frac{s\sigma^2}{2} g(X, Y) (H^{hor} \sigma^{-2}), \end{aligned}$$

here  $\delta$  is used for divergence.

**Corollary 2.2.** [6] Suppose that

$$F : (M^r, g) \rightarrow (N^s, h)$$

defines a horizontally CS which is totally geodesic and admits dilation  $\sigma$ . Then

$$\text{Ric}_{UV} = \text{Ric}_{UV}^{\mathcal{V}}, \quad (2.14)$$

$$\text{Ric}_{XY} = \sigma^{-2} \text{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N, \quad (2.15)$$

$$\text{Ric}_{UX} = 0, \quad (2.16)$$

where  $X$  and  $Y$  are the horizontal lifts of  $X^{\mathcal{H}}$  and  $Y^{\mathcal{H}}$ , respectively.

**Theorem 2.1.** [6] Consider a CS

$$F : (M, g) \rightarrow (N, h)$$

which is totally geodesic and admits dilation  $\sigma$ . Then the scalar curvature  $\kappa$  of  $M$  is given by

$$\kappa = \kappa^{\mathcal{V}} + \sigma^{-2} \kappa^N, \quad (2.17)$$

where  $\kappa^{\mathcal{V}}$  and  $\kappa^N$  denote the scalar curvatures of the fibers and the base manifold, respectively.

**Theorem 2.2.** [6] Let

$$F : (M, g) \rightarrow (N, h)$$

be a horizontally CS. Then  $F$  is totally geodesic if and only if

- (1) fibers of  $F$  are totally geodesic,
- (2)  $(\ker F_*)^\perp$  is totally geodesic,
- (3)  $F$  is homothetic.

### 3. CONFORMAL SUBMERSIONS FROM RY SOLITONS

Throughout this section, we assume that  $(M, g)$  admits a RY soliton with potential vector field  $\zeta$ , constants  $\lambda, \alpha, \beta$ , and that  $F$  is a CS.

**3.1. Potential Field Vertical.** We first consider the case where the potential field  $\zeta$  is vertical, i.e.,  $\zeta \in \Gamma(\text{Ker } F_*)$ .

**Theorem 3.1.** Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  vertical, and let

$$F : (M, g) \rightarrow (N, h)$$

be a totally geodesic CS. If  $N$  is Ricci-flat, then each fiber admits a RY soliton. Moreover, if  $\zeta$  is unitary then the corresponding soliton is classified as follows:

- (1) shrinking, if  $\beta \kappa^{\mathcal{V}} > 0$ ,

- (2) *steady*, if  $\beta\kappa^\mathcal{V} = 0$ ,
- (3) *expanding*, if  $\beta\kappa^\mathcal{V} < 0$ .

In particular, if  $\beta > 0$ , then the nature of the soliton depends on the sign of the vertical scalar curvature  $\kappa^\mathcal{V}$ , whereas for  $\beta < 0$  the classification is reversed.

*Proof.* Since  $\zeta$  is vertical, for any  $U, V \in \Gamma(\text{Ker } F_*)$ , the RY soliton equation gives

$$\frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) = 0.$$

Using the expression for  $\text{Ric}_{UV}$  from Proposition 2.1 and the total geodesicity assumption, many terms vanish. Then by Theorem 2.1, we obtain

$$0 = \frac{1}{2}\mathcal{L}_\zeta g + \alpha \text{Ric}^\mathcal{V} + \left(\lambda - \frac{\beta}{2}\kappa^\mathcal{V}\right)g - \frac{\beta}{2}\sigma^{-2}\kappa^N g. \quad (3.18)$$

As  $N$  is Ricci-flat, we have  $\kappa^N = 0$ , and thus we get the RY soliton equation on the fiber. In addition, when  $\zeta$  is assumed to be a unit vector field, we have  $\frac{1}{2}\mathcal{L}_\zeta g(U, V) = 0$  for vertical vector fields. From (3.18) and the assumption of Ricci-flatness of  $N$ , we obtain

$$\lambda - \frac{\beta}{2}\kappa^\mathcal{V} = 0,$$

that is,

$$\lambda = \frac{\beta}{2}\kappa^\mathcal{V}.$$

This completes the classification. □

**Theorem 3.2.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  vertical, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a CS such that fibers of  $F$  are totally umbilical and  $(\text{Ker } F_*)^\perp$  is totally geodesic. If  $N$  is Ricci-flat, then each fiber admits an almost RY soliton.*

*Proof.* Under the hypotheses that fibers of  $F$  are totally umbilical  $T_U V = g(U, V)H$  and  $(\text{Ker } F_*)^\perp$  is totally geodesic, we obtain

$$\text{Ric}_{UV} = \text{Ric}_{UV}^\mathcal{V} - (r - s)\|H\|^2 g(U, V) + \delta(H)g(U, V). \quad (3.19)$$

From (1.1) and (3.19), we get

$$\begin{aligned} 0 &= \frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \left( \text{Ric}_{UV}^\mathcal{V} - (r - s)\|H\|^2 g(U, V) + \delta(H)g(U, V) \right) \\ &\quad + \left( \lambda - \frac{\beta}{2}\kappa \right) g(U, V) \\ &= \frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV}^\mathcal{V} + \left( \lambda - \frac{\beta}{2}\kappa^\mathcal{V} \right) g(U, V) \\ &\quad + \left( -\alpha(r - s)\|H\|^2 + \alpha\delta(H) - \frac{\beta}{2\sigma^2}\kappa^N \right) g(U, V). \end{aligned}$$

Since  $N$  is Ricci-flat,  $\kappa^N = 0$ . Then the above equation reduces to

$$0 = \frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV}^\mathcal{V} + \left( \mu - \frac{\beta}{2}\kappa^\mathcal{V} \right) g(U, V),$$

where  $\mu = \alpha(- (r - s)\|H\|^2 + \delta(H)) + \lambda$  is a smooth function on  $M$ . Thus, each fiber of  $F$  is an almost RY soliton.  $\square$

**Theorem 3.3.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  vertical, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a totally geodesic CS. If  $M$  has constant scalar curvature  $\kappa$ , then  $N$  is Einstein.*

*Proof.* Since  $\zeta$  is vertical, for any  $X, Y \in \Gamma(\text{Ker } F_*)^\perp$ , equation (1.1) gives

$$\begin{aligned} 0 &= \frac{1}{2} \{g(\nabla_X \zeta, Y) + g(\nabla_Y \zeta, X)\} + \alpha \text{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y) \\ &= -\frac{1}{2} \{g(\nabla_X Y, \zeta) + g(\nabla_Y X, \zeta)\} + \alpha \text{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y) \\ &= -\frac{1}{2} g(A_X Y + A_Y X, \zeta) + \alpha \text{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y). \end{aligned}$$

Using the assumption that the submersion is totally geodesic and Corollary 2.2, we obtain

$$\alpha \sigma^{-2} \text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N + \left(\lambda - \frac{\beta}{2} \kappa\right) \sigma^{-2} h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = 0,$$

which can be re-written as

$$\text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N = -\frac{1}{\alpha} \left(\lambda - \frac{\beta}{2} \kappa\right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}),$$

showing that  $N$  is Einstein.  $\square$

**Theorem 3.4.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  vertical, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a homothetic CS such that fibers of  $F$  are totally geodesic and  $(\text{Ker } F_*)^\perp$  is integrable. Then  $N$  is Einstein.*

*Proof.* From Corollary 2.1 and (3.1), we derive

$$\begin{aligned} 0 &= -\frac{1}{2} g(A_X Y + A_Y X, \zeta) + \alpha \left\{ g(X, Y) \delta(H^{hor}) + \sigma^{-2} \text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N \right. \\ &\quad \left. - \frac{1}{4} \sigma^4 g(X, Y) \|\nabla_{\mathcal{V}} \sigma^{-2}\|^2 + \frac{s \sigma^2}{2} g(X, Y) (H^{hor} \sigma^{-2}) \right\} \\ &\quad + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y). \end{aligned}$$

In the case of homothetic submersions, the tensor  $A$  is equal to zero (see Theorem 2.2), so the first term is eliminated. This gives us

$$\text{Ric}_{X^{\mathcal{H}} Y^{\mathcal{H}}}^N = \mu h(X^{\mathcal{H}}, Y^{\mathcal{H}}),$$

where

$$\mu = -\sigma^2 \delta(H^{hor}) + \frac{\sigma^4}{4} \|\nabla_{\mathcal{V}} \sigma^{-2}\|^2 - \frac{s \sigma^2}{2} (H^{hor} \sigma^{-2}) - \frac{\sigma^2}{\alpha} \left(\lambda - \frac{\beta}{2} \kappa\right)$$

is a smooth function on  $M$ . Hence  $N$  is Einstein.  $\square$

**3.2. Potential Field Horizontal.** Now we move on to the situation when the potential field  $\zeta$  is horizontal, that is,  $\zeta \in \Gamma((\text{Ker } F_*)^\perp)$ .

**Theorem 3.5.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a totally geodesic CS. Then each fiber of  $F$  is Einstein.*

*Proof.* Since  $\zeta$  is horizontal, for any  $U, V \in \Gamma(\text{Ker } F_*)$ , the RY soliton equation yields

$$\frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} + \alpha \text{Ric}_{UV} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) = 0.$$

Since  $\zeta$  is horizontal vector field and the submersion is totally geodesic, we have  $\nabla_U \zeta = -A_\zeta U$  (vertical). Now the Lie derivative term becomes

$$-g(A_\zeta U, V) - g(A_\zeta V, U) = 0$$

due to the skew-symmetry of  $A$ . Using Corollary 2.2,  $\text{Ric}_{UV} = \text{Ric}_{UV}^\mathcal{V}$ . Thus

$$\alpha \text{Ric}_{UV}^\mathcal{V} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) = 0,$$

which gives

$$\text{Ric}_{UV}^\mathcal{V} = \mu g(U, V),$$

so each fiber is Einstein with  $\mu = -\frac{1}{\alpha}\left(\lambda - \frac{\beta}{2}\kappa\right)$ . □

**Theorem 3.6.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a CS such that fibers of  $F$  are totally umbilical and  $(\text{Ker } F_*)^\perp$  is totally geodesic. Then each fiber of  $F$  is Einstein.*

*Proof.* From (1.1) and Proposition 2.1, we find

$$\begin{aligned} & \alpha \left\{ \text{Ric}_{UV}^\mathcal{V} - (r - s)\|H\|^2 g(U, V) + \delta(H)g(U, V) \right\} + \left(\lambda - \frac{\beta}{2}\kappa\right)g(U, V) \\ &= -\frac{1}{2}\{g(\nabla_U \zeta, V) + g(\nabla_V \zeta, U)\} = g(T_U V, \zeta). \end{aligned}$$

Given that the fibers of  $F$  are totally umbilical, we have  $T_U V = g(U, V)H$ . Thus, the RHS simplifies to  $g(H, \zeta)g(U, V)$ . Moreover, since  $(\text{Ker } F_*)^\perp$  is totally geodesic, the mixed curvature terms vanish and the scalar curvature splits. Hence

$$\begin{aligned} \alpha \text{Ric}_{UV}^\mathcal{V} &= [g(H, \zeta) + \alpha(r - s)\|H\|^2 - \alpha\delta(H) \\ &\quad - \left(\lambda - \frac{\beta}{2}\kappa\right)]g(U, V). \end{aligned}$$

Thus  $\text{Ric}_{UV}^\mathcal{V} = \tilde{\mu}g(U, V)$ , where  $\tilde{\mu}$  is a smooth function, proving each fiber is Einstein. □

**Theorem 3.7.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a totally geodesic CS. Then  $N$  admits a RY soliton with potential field  $\zeta^\mathcal{H} = F_*\zeta$ .*

*Proof.* In the case of  $X, Y, \zeta$  being horizontal, from the property of  $F$  as a totally geodesic CS, we obtain  $F_*(\nabla_X Y) = D_{F_*X} F_*Y$  and  $F_*(\nabla_X \zeta) = D_{F_*X} F_*\zeta$ . Then from (1.1),

$$\begin{aligned} 0 &= \frac{1}{2} \{g(\nabla_X \zeta, Y) + g(X, \nabla_Y \zeta)\} + \alpha \operatorname{Ric}_{XY} + \left(\lambda - \frac{\beta}{2} \kappa\right) g(X, Y) \\ &= \frac{1}{2} \sigma^{-2} \{h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}})\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N \\ &\quad + \sigma^{-2} \left(\lambda - \frac{\beta}{2} \kappa\right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}). \end{aligned}$$

The multiplication by  $\sigma^2$  will give us the RY soliton on  $N$ :

$$\frac{1}{2} \{h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}})\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N + \left(\lambda - \frac{\beta}{2} \kappa\right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = 0.$$

Thus  $(N, h, \zeta^{\mathcal{H}}, \lambda, \alpha, \beta)$  is a RY soliton (with the same constants  $\lambda, \alpha, \beta$  but note the scalar curvature term uses  $\kappa$  from  $M$ , not  $\kappa^N$ ).  $\square$

**Theorem 3.8.** *Let  $(M, g, \zeta, \lambda, \alpha, \beta)$  be a RY soliton with  $\zeta$  horizontal, and let*

$$F : (M, g) \rightarrow (N, h)$$

*be a homothetic CS such that fibers of  $F$  are totally geodesic and  $(\operatorname{Ker} F_*)^\perp$  is integrable. Then  $N$  is an almost RY soliton.*

*Proof.* Using (1.1) and the homothetic condition together with Corollary 2.1, and (2.8), we obtain after simplification

$$\begin{aligned} 0 &= \frac{1}{2} \{h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}})\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N \\ &\quad + \left\{ \delta(H^{hor}) - \frac{1}{4} \sigma^4 \|\nabla_V \sigma^{-2}\|^2 + \frac{s}{2} \sigma^2 (H^{hor} \sigma^{-2}) + \lambda - \frac{\beta}{2} \kappa \right\} h(X^{\mathcal{H}}, Y^{\mathcal{H}}). \end{aligned}$$

This shows that  $N$  satisfies

$$\frac{1}{2} \left\{ h(D_{X^{\mathcal{H}}} \zeta^{\mathcal{H}}, Y^{\mathcal{H}}) + h(D_{Y^{\mathcal{H}}} \zeta^{\mathcal{H}}, X^{\mathcal{H}}) \right\} + \alpha \operatorname{Ric}_{X^{\mathcal{H}}Y^{\mathcal{H}}}^N + \left( \mu - \frac{\beta}{2} \kappa \right) h(X^{\mathcal{H}}, Y^{\mathcal{H}}) = 0,$$

where  $\mu = \delta(H^{hor}) - \frac{1}{4} \sigma^4 \|\nabla_V \sigma^{-2}\|^2 + \frac{s}{2} \sigma^2 (H^{hor} \sigma^{-2}) + \lambda$  is a smooth function on  $M$ . Hence  $N$  is an almost RY soliton.  $\square$

**Remark 3.1.** *We obtain the scalar curvature of the Ricci–Yamabe soliton manifold  $M$  as*

$$\kappa = \frac{-2\lambda r}{2\alpha - \beta r}. \quad (3.20)$$

#### 4. EXAMPLES

We next provide explicit examples to demonstrate our findings.

**Example 4.1.** *Let  $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$  with metric*

$$g = dx_1^2 + dx_2^2 + \frac{1}{x_3^2} dx_3^2,$$

*and Let  $N = \mathbb{R}^2$  with the Euclidean metric*

$$h = dy_1^2 + dy_2^2.$$

The map

$$F : M \rightarrow N$$

defined as

$$F(x_1, x_2, x_3) = (x_1, x_2)$$

yields

$$\text{Ker } F_* = \text{span}\{\partial_{x_3}\}$$

and

$$\mathcal{H} = \text{span}\{\partial_{x_1}, \partial_{x_2}\}.$$

The dilation is  $\sigma = 1$ , implying that  $F$  is a Riemannian submersion. The fibers are totally geodesic, and  $N$  is flat (and therefore Ricci-flat). We take the potential vector field for the vertical direction  $\zeta = \partial_{x_3}$ . It can be easily shown that  $(M, g)$  admits Ricci Yamabe soliton structure with  $\zeta$  being vertical,  $\alpha = 1$ ,  $\beta = 2$ ,  $\lambda = \frac{1}{2}$  (the scalar curvature of the fiber  $\kappa^\mathcal{V} = 1/2$ ). The fiber is one-dimensional, therefore, Ricci-flat and there exists an induced RY soliton. This supports Theorem 3.1.

**Example 4.2.** Let  $M = \mathbb{R}^3$  with metric

$$g = e^{2x_3} dx_1^2 + e^{2x_3} dx_2^2 + dx_3^2,$$

and let  $N = \mathbb{R}^2$  with metric

$$h = dy_1^2 + dy_2^2.$$

Consider the function

$$F(x_1, x_2, x_3) = (x_1, x_2).$$

Here  $(x_1, x_2)$  gives a CS with dilation  $\sigma = e^{x_3}$ . The fibers are vertical lines

$$\{x_1 = \text{constant}, x_2 = \text{constant}\}$$

and their metrics are  $dx_3^2$  and are totally geodesic. The horizontal potential field

$$\zeta = \partial_{x_1}.$$

By using corollary 2.2, the Ricci tensor of  $M$  can be expressed as

$$\text{Ric}_{XY} = \sigma^{-2} \text{Ric}_{X^\mathcal{H}Y^\mathcal{H}}^N = 0$$

since  $N$  is flat. So the scalar curvature  $\kappa = 0$ . The RY soliton equation becomes

$$\frac{1}{2} \mathcal{L}_\zeta g + \lambda g = 0.$$

Since a straightforward calculation yields

$$\mathcal{L}_{\partial_{x_1}} g = 0,$$

then we get that the equation is satisfied with  $\lambda = 0$ . This means that  $(M, g, \zeta, 0, \alpha, \beta)$  is a steady RY soliton regardless of the values  $\alpha, \beta$ . Since  $N$  is a flat manifold, hence a RY soliton (Theorem 3.7). Moreover, the fibers are also Einstein (Theorem 3.5).

**Example 4.3.** Let  $M = \mathbb{R}^3$  with metric

$$g = dx_1^2 + e^{2x_1} dx_2^2 + dx_3^2,$$

and let  $N = \mathbb{R}^2$  with metric

$$h = dy_1^2 + dy_2^2.$$

Let us define

$$F : M \rightarrow N$$

such that

$$F(x_1, x_2, x_3) = (x_1, x_3)$$

with dilation factor  $\sigma = e^{x_1}$ . Here the fibers are the lines  $\{x_1 = \text{constant}, x_3 = \text{constant}\}$  having the metric  $e^{2x_1} dx_2^2$ . They are non-totally geodesic since their mean curvature vector does not vanish. We choose the horizontal potential field

$$\zeta = \partial_{x_2}.$$

Because  $\zeta$  is horizontal and Killing, we have  $\mathcal{L}_\zeta g = 0$ . The scalar curvature  $\kappa$  is a constant (indeed,  $\kappa = 0$ , since the metric is a warped product with both base and fiber having zero scalar curvatures). We verify that the RY soliton equation is satisfied for  $\lambda = 0$ . The base manifold  $N$  is a flat manifold, but by Theorem 3.8 the base manifold  $N$  admits an almost RY soliton since the dilation and mean curvature produce a non-constant function  $\mu$ . Thus, if the submersion is not totally geodesic, the induced structure is broader than an RY soliton structure.

To support the theoretical results established in the preceding sections, we now revisit the examples from [6] and extend them to the framework of RY solitons.

**Example 4.4.** Let  $M = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$  be a Riemannian manifold with metric

$$g = e^{-2x_2} dx_1^2 + dx_2^2. \quad (4.21)$$

Let  $N = \mathbb{R}$  with metric  $h = dy_1^2$ . Define  $F : M \rightarrow N$  by

$$F(x_1, x_2) = x_1. \quad (4.22)$$

Define the orthonormal basis

$$e_1 = e^{x_2} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}. \quad (4.23)$$

Then,

$$\text{Ker } F_* = \text{span}\{e_2\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_1\}. \quad (4.24)$$

The non-zero Christoffel symbols for the metric  $g = e^{-2x_2} dx_1^2 + dx_2^2$  are

$$\Gamma_{11}^2 = e^{-2x_2}, \quad \Gamma_{12}^1 = \Gamma_{21}^1 = -1. \quad (4.25)$$

Using the Christoffel symbols, we compute

$$\nabla_{e_1} e_1 = e_2, \quad (4.26)$$

$$\nabla_{e_1} e_2 = -e_1, \quad (4.27)$$

$$\nabla_{e_2} e_1 = 0, \quad (4.28)$$

$$\nabla_{e_2} e_2 = 0. \quad (4.29)$$

For the diagonal metric  $g = e^{-2x_2}dx_1^2 + dx_2^2$ , the Gaussian curvature is

$$K = -\frac{1}{2\sqrt{|g|}}\partial_{x_2}\left(\frac{\partial_{x_2}g_{11}}{\sqrt{|g|}}\right). \quad (4.30)$$

Since  $\sqrt{|g|} = e^{-x_2}$  and  $\partial_{x_2}g_{11} = -2e^{-2x_2}$ , we get

$$K = -\frac{1}{2e^{-x_2}}\partial_{x_2}\left(\frac{-2e^{-2x_2}}{e^{-2x_2}e^{-x_2}}\right) \quad (4.31)$$

$$= -\frac{1}{2e^{-x_2}}\partial_{x_2}(-2e^{x_2}) \quad (4.32)$$

$$= -\frac{1}{2e^{-x_2}}(-2e^{x_2}) = -1. \quad (4.33)$$

Therefore,

$$\kappa = 2K = -2, \quad (4.34)$$

and

$$\text{Ric}(e_1, e_1) = \text{Ric}(e_2, e_2) = -1, \quad \text{Ric}(e_1, e_2) = 0. \quad (4.35)$$

Thus for arbitrary  $X_1, Y_1 \in \Gamma(TM)$ ,

$$\text{Ric}(X_1, Y_1) = -(a_1a_3 + a_2a_4), \quad (4.36)$$

where  $X_1 = a_1e_1 + a_2e_2$  and  $Y_1 = a_3e_1 + a_4e_2$ .

Let  $Z_1 = a_5e_1 + a_6e_2$ . Using (4.26)-(4.29), we have

$$\nabla_{X_1}Z_1 = \nabla_{a_1e_1+a_2e_2}(a_5e_1 + a_6e_2) \quad (4.37)$$

$$= a_1a_5\nabla_{e_1}e_1 + a_1a_6\nabla_{e_1}e_2 + a_2a_5\nabla_{e_2}e_1 + a_2a_6\nabla_{e_2}e_2 \quad (4.38)$$

$$= a_1a_5e_2 - a_1a_6e_1. \quad (4.39)$$

Thus,

$$g(\nabla_{X_1}Z_1, Y_1) = a_1a_5a_4 - a_1a_6a_3, \quad (4.40)$$

and

$$g(\nabla_{Y_1}Z_1, X_1) = a_3a_5a_2 - a_3a_6a_1. \quad (4.41)$$

Therefore,

$$\begin{aligned} \frac{1}{2}(L_{Z_1}g)(X_1, Y_1) &= \frac{1}{2}\{g(\nabla_{X_1}Z_1, Y_1) + g(\nabla_{Y_1}Z_1, X_1)\} \\ &= \frac{1}{2}(a_1a_4a_5 + a_2a_3a_5 - 2a_1a_3a_6). \end{aligned} \quad (4.42)$$

Also,

$$g(X_1, Y_1) = a_1a_3 + a_2a_4. \quad (4.43)$$

The RY soliton equation is

$$\frac{1}{2}(L_{Z_1}g)(X_1, Y_1) + \alpha \text{Ric}(X_1, Y_1) + \beta \kappa g(X_1, Y_1) = 0. \quad (4.44)$$

Substituting the computed values

$$\frac{1}{2}(a_1a_4a_5 + a_2a_3a_5 - 2a_1a_3a_6) - \alpha(a_1a_3 + a_2a_4) - 2\beta(a_1a_3 + a_2a_4) = 0. \quad (4.45)$$

For the special choice  $a_1 = 1, a_3 = 1, a_4 = 0, a_5 = 1$ , this simplifies to

$$-a_6 - \alpha - 2\beta = 0. \quad (4.46)$$

Hence,

$$\alpha + 2\beta = -a_6. \quad (4.47)$$

**Remark 4.1.** The metric  $g = e^{-2x_2} dx_1^2 + dx_2^2$  represents the hyperbolic plane with constant curvature  $K = -1$ .

**Example 4.5.** Let  $M = \mathbb{R}^2$  with metric

$$g = e^{2x_1} dx_1^2 + dx_2^2. \quad (4.48)$$

Let  $N = \mathbb{R}$  with metric  $h = dy_1^2$ . Define  $F : M \rightarrow N$  by

$$F(x_1, x_2) = x_2. \quad (4.49)$$

Define the orthonormal basis

$$e_1 = e^{-x_1} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}. \quad (4.50)$$

Then

$$\text{Ker } F_* = \text{span}\{e_2\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_1\}. \quad (4.51)$$

For the metric  $g = e^{2x_1} dx_1^2 + dx_2^2$ , the non-zero Christoffel symbol is

$$\Gamma_{11}^1 = 1. \quad (4.52)$$

For the orthonormal basis, we compute:

$$\nabla_{e_1} e_1 = \nabla_{e^{-x_1} \partial_{x_1}} (e^{-x_1} \partial_{x_1}) \quad (4.53)$$

$$= e^{-x_1} [e^{-x_1} \nabla_{\partial_{x_1}} \partial_{x_1} + \partial_{x_1} (e^{-x_1}) \partial_{x_1}] \quad (4.54)$$

$$= e^{-x_1} [e^{-x_1} \partial_{x_1} - e^{-x_1} \partial_{x_1}] = 0, \quad (4.55)$$

$$\nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \nabla_{e_2} e_2 = 0. \quad (4.56)$$

Thus all connections vanish

$$\nabla_{e_i} e_j = 0 \quad \text{for all } i, j \in \{1, 2\}. \quad (4.57)$$

For a diagonal metric  $g = \text{diag}(e^{2x_1}, 1)$ , the Gaussian curvature  $K = 0$ .

Therefore,

$$\kappa = 0, \quad \text{Ric} = 0. \quad (4.58)$$

Using (4.57) and (4.58), the RY soliton equation becomes

$$\frac{1}{2} (L_{Z_1} g)(X_1, Y_1) = 0. \quad (4.59)$$

**Remark 4.2.** The metric  $g = e^{2x_1} dx_1^2 + dx_2^2$  is flat. Under the coordinate transformation  $u = e^{x_1}$ , the metric becomes  $g = du^2 + dx_2^2$ , which is the standard Euclidean metric.

Any vector field  $Z_1$  gives a RY soliton with arbitrary constants  $\alpha, \beta$ . In particular, any constant vector field is a Killing vector field for this flat metric.

**Example 4.6.** Let  $M = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 1\}$  with metric

$$g = x_3^{-2}(dx_1^2 + dx_2^2 + dx_3^2). \quad (4.60)$$

Let  $N = \{(y_1, y_2) \in \mathbb{R}^2 : y_2 > 1\}$  with metric  $h = dy_1^2 + dy_2^2$ . Define  $F : M \rightarrow N$  by

$$F(x_1, x_2, x_3) = (x_2, x_3). \quad (4.61)$$

Define

$$e_1 = x_3 \frac{\partial}{\partial x_1}, \quad e_2 = x_3 \frac{\partial}{\partial x_2}, \quad e_3 = x_3 \frac{\partial}{\partial x_3}. \quad (4.62)$$

Then,

$$\text{Ker } F_* = \text{span}\{e_1\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_2, e_3\}. \quad (4.63)$$

The non-zero Christoffel symbols are

$$\Gamma_{13}^1 = \Gamma_{31}^1 = -x_3^{-1}, \quad \Gamma_{23}^2 = \Gamma_{32}^2 = -x_3^{-1}, \quad (4.64)$$

$$\Gamma_{33}^3 = -x_3^{-1}, \quad \Gamma_{11}^3 = x_3^{-1}, \quad \Gamma_{22}^3 = x_3^{-1}. \quad (4.65)$$

Using the Christoffel symbols

$$\nabla_{e_1} e_1 = e_3, \quad (4.66)$$

$$\nabla_{e_2} e_2 = e_3, \quad (4.67)$$

$$\nabla_{e_1} e_3 = -e_1, \quad (4.68)$$

$$\nabla_{e_3} e_1 = -e_1, \quad (4.69)$$

$$\nabla_{e_2} e_3 = -e_2, \quad (4.70)$$

$$\nabla_{e_3} e_2 = -e_2, \quad (4.71)$$

$$\nabla_{e_3} e_3 = -e_3. \quad (4.72)$$

For  $U = e_1 \in \Gamma(\text{Ker } F_*)$ , we have

$$T_U U = \nabla_{e_1} e_1 = e_3. \quad (4.73)$$

For totally umbilical fibers,  $T_U U = g(U, U)H = H$ . Thus,

$$H = e_3. \quad (4.74)$$

So the fibers are totally umbilical with mean curvature  $H = e_3$ .

$$[e_2, e_3] = [x_3 \partial_{x_2}, x_3 \partial_{x_3}] \quad (4.75)$$

$$= x_3[\partial_{x_2}, \partial_{x_3}] + x_3 \partial_{x_2}(x_3) \partial_{x_3} - x_3 \partial_{x_3}(x_3) \partial_{x_2} \quad (4.76)$$

$$= -x_3 \partial_{x_2} = -e_2. \quad (4.77)$$

Since  $[e_2, e_3] = -e_2 \in (\text{Ker } F_*)^\perp$ , the horizontal distribution is integrable.

For  $X = ae_2 + be_3 \in \Gamma(\text{Ker } F_*)^\perp$ , we have

$$\nabla_X X = \nabla_{ae_2+be_3}(ae_2 + be_3) \quad (4.78)$$

$$= a^2 \nabla_{e_2} e_2 + ab \nabla_{e_2} e_3 + ab \nabla_{e_3} e_2 + b^2 \nabla_{e_3} e_3 \quad (4.79)$$

$$= a^2 e_3 - abe_2 - abe_2 - b^2 e_3 \quad (4.80)$$

$$= (a^2 - b^2)e_3 - 2abe_2. \quad (4.81)$$

The horizontal part is

$$\mathcal{H}\nabla_X X = (a^2 - b^2)e_3. \quad (4.82)$$

For  $(\text{Ker } F_*)^\perp$  to be totally geodesic, we need  $\mathcal{H}\nabla_X X = 0$  for all  $a, b$ . This requires  $a^2 = b^2$  for all  $a, b$ , which is false.

**Conclusion:** The horizontal distribution is NOT totally geodesic.

The metric  $g = x_3^{-2}(dx_1^2 + dx_2^2 + dx_3^2)$  is the standard hyperbolic metric on  $\mathbb{H}^3$  with constant curvature  $K = -1$ . Thus:

$$\kappa = -6, \quad \text{Ric} = -2g. \quad (4.83)$$

For  $\zeta = W \in \Gamma(\text{Ker } F_*)$  and  $U, V \in \Gamma(\text{Ker } F_*)$ , we have

$$\frac{1}{2}(L_W g)(U, V) + \alpha(-2)g(U, V) + \beta(-6)g(U, V) = 0. \quad (4.84)$$

Thus,

$$\frac{1}{2}(L_W g)(U, V) - (2\alpha + 6\beta)g(U, V) = 0. \quad (4.85)$$

For the 1-dimensional fiber, this equation can be solved for appropriate  $W$  and parameters  $\alpha, \beta$ .

**Example 4.7.** Let  $M = \mathbb{R}^3$  with metric

$$g = 4(dx_1^2 + dx_2^2 + dx_3^2). \quad (4.86)$$

Let  $N = \mathbb{R}^2$  with metric  $h = dy_1^2 + dy_2^2$ . Define  $F : M \rightarrow N$  by

$$F(x_1, x_2, x_3) = (x_1, x_3). \quad (4.87)$$

Define

$$e_1 = \frac{1}{2} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{1}{2} \frac{\partial}{\partial x_2}, \quad e_3 = \frac{1}{2} \frac{\partial}{\partial x_3}. \quad (4.88)$$

Then

$$\text{Ker } F_* = \text{span}\{e_2\}, \quad (\text{Ker } F_*)^\perp = \text{span}\{e_1, e_3\}. \quad (4.89)$$

Since the metric is flat, all Christoffel symbols vanish

$$\nabla_{e_i} e_j = 0 \quad \text{for all } i, j \in \{1, 2, 3\}. \quad (4.90)$$

Thus both  $\text{Ker } F_*$  and  $(\text{Ker } F_*)^\perp$  are totally geodesic.

For  $X_1 = e_1$  and  $X_2 = e_3$ :

$$F_*(e_1) = F_*(\frac{1}{2}\partial_{x_1}) = \frac{1}{2}\partial_{y_1}, \quad (4.91)$$

$$F_*(e_3) = F_*(\frac{1}{2}\partial_{x_3}) = \frac{1}{2}\partial_{y_2}, \quad (4.92)$$

$$F_*(e_2) = 0. \quad (4.93)$$

Then

$$h(F_*e_1, F_*e_1) = h(\frac{1}{2}\partial_{y_1}, \frac{1}{2}\partial_{y_1}) = \frac{1}{4} = \lambda^2 g(e_1, e_1) = \lambda^2 \cdot 1. \quad (4.94)$$

So  $\lambda = \frac{1}{2}$ .

Thus  $F$  is a horizontally conformal submersion with constant dilation  $\lambda = \frac{1}{2}$ .

- Fibers are totally geodesic:  $\nabla_{e_2}e_2 = 0$
- Horizontal distribution is totally geodesic:  $\mathcal{H}\nabla_X X = 0$  for all  $X \in \Gamma(\text{Ker } F_*)^\perp$
- $F$  is homothetic (constant dilation)

Therefore,  $F$  is totally geodesic.

The metric is flat, so

$$\kappa = 0, \quad \text{Ric} = 0. \quad (4.95)$$

Using (4.90) and (4.95), the RY soliton equation becomes

$$\frac{1}{2}(L_\zeta g) = 0. \quad (4.96)$$

So  $\zeta$  must be a Killing vector field.

For the flat metric  $g = 4(dx_1^2 + dx_2^2 + dx_3^2)$ , the Killing vector fields are

- Translations:  $\partial_{x_1}, \partial_{x_2}, \partial_{x_3}$
- Rotations:  $x_i\partial_{x_j} - x_j\partial_{x_i}$

Any constant vector field

$$\zeta = c_1\partial_{x_1} + c_2\partial_{x_2} + c_3\partial_{x_3}, \quad c_1, c_2, c_3 \in \mathbb{R} \quad (4.97)$$

gives a RY soliton  $(M, g, \zeta, \alpha, \beta)$  for arbitrary  $\alpha, \beta$ .

## 5. CONCLUSION

In this paper, we considered conformal Riemannian submersions whose total manifolds possess RY solitons. The curvatures of such submersions have been analyzed and some curvature relations have been derived, using which we obtained sufficient conditions for the fibers as well as the base manifold of a given submersion to inherit Einstein, almost Einstein, and Ricci Yamabe solitons. Thus, this paper is a continuation of the research performed on Riemannian submersions admitting Ricci solitons ([17], [18]) or RY solitons ([10]). In fact, our results include those in [6] if  $\alpha = 1$  and  $\beta = 0$ .

One of the significant aspects of this paper is the comparison between Ricci soliton and Ricci–Yamabe soliton cases. As the equation of RY soliton includes a scalar curvature term, it may be considered an extension of the equation of Ricci soliton. This implies that when a Riemannian submersion is taken into account, in addition to the soliton constant, the scalar curvature of the manifold is included in the induced Einstein structure on the base manifold. Unlike Ricci soliton, the constancy of the scalar curvature turns out to be one of the essential conditions for the base manifold to be Einstein. Particularly, whenever the scalar curvature is a constant, the RY case becomes similar to the Ricci soliton case, while variable scalar curvature gives almost Einstein manifolds.

The findings in this study have shown that the use of CS geometry offers a natural environment for the discussion of interaction between RY solitons and the geometric flow structure. There are still several possible research areas that one could explore in the future. One might

analyze the situation when the potential vector field has horizontal and vertical components, or explore gradient RY solitons using the technique of CSs. One could also explore the impact of the dilation factor on the resulting geometric structure and explore the connection between conformal harmonic morphisms and RY solitons or  $\eta$ -Ricci-Yamabe solitons [16]. Another approach would be to conduct analogous analysis for statistical manifolds, semi-Riemannian manifolds, contact manifolds, or even warped products. These results open up new directions for studying conformal submersions in the context of generalized solitons and geometric flows.

The conformal Riemannian map was introduced in [14], generalizing conformal submersions. Recently, Clairaut conformal submersions and Clairaut conformal Riemannian maps whose total manifolds possess Ricci solitons were explored in [13] and [5]. As in the present study, these mappings may be explored in the near future for total manifolds possessing RY solitons.

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