International Journal of Maps in Mathematics
Volume (2), Issue (2), (2019), Pages:(207-218)
www.journalmim.com

DECOMPOSITION ON $QTAG$-MODULES WITH CERTAIN SUBMODULES

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ABSTRACT. A module M over an associative ring R with unity is a $QTAG$-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. In this paper, we study some decomposition results on $QTAG$-modules with certain submodules in terms of the cardinality $g(M)$.

1. Introduction and Terminology

Let R be any ring with unity. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

Received:2019-05-03 *Revised:2019-07-27* *Accepted:2019-08-22*

*2018 Mathematics Subject Classification.*16K20.

Key words: QTAG-modules, upper and lower basic submodules, cardinality.

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A module M_R is called a TAG-module if it satisfies the following two conditions:

(I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f: W \to V$ can be extended to a homomorphism $g: U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M_R satisfying only condition (I) is called a $QTAG$-module. The study of $QTAG$-modules was initiated by Singh [11]. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of $QTAG$-modules have been studied, and a theory was developed, introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been obtained, but there is still a lot to explore.

Everywhere in the text of the present article; let it be agreed that all the rings are associative with unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. For a uniform element $x \in M,\ e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. As defined in [5], the module M is h-divisible if $M = M^1 = \bigcap\limits_{k=0}^{\infty} H_k(M)$. The module M is h-reduced if it does not contain any h-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded, if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$.

A submodule N of M is h-pure [3] in M if $N \cap H_n(M) = H_n(N)$, for every integer $n \geq 0$. A submodule $B \subseteq M$ is a basic submodule [5] of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible. A submodule $N \subseteq M$ is said to be high [4], if it is a complement of M^1 i.e. $M = N \oplus M^1$. The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M (see [6]) and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

Imitating [8], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$.

It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [9]. Many results of this paper are the generalization of [10]. Our notations and terminology generally agree with those in [1] and [2].

2. Elementary results

We start here with a recollection of the following notions from [7].

Definition 2.1. *A basic submodule B^u of a $QTAG$-module M is said to be an upper basic submodule if $g(M/B^u) = min\{g(M/B) \mid B$ is a basic submodule of $M\}$.*

Definition 2.2. *A basic submodule B^l is said to be a lower basic submodule of M if $g(M/B^l) =$ fin $g(M) = min\{H_k(M)\}$.*

To develop the study, we need to prove some elementary but helpful lemmas.

Lemma 2.1. *If M is a $QTAG$-module without elements of infinite height such that every basic submodule of M is both an upper and lower basic submodule of M, and such that fin $g(M) = g(M)$. Then M cannot be decomposed as $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules, and $g(M_1) < g(M)$.*

Proof. Suppose such a decomposition of M does exist, and let B be a basic submodule of M_1. Now $B \oplus M_2$ is a basic submodule of M and

$$g(M/(B \oplus M_2)) = g(M_1/B) + g(M_2/M_2) = g(M_1/B) \leq g(M_1) < g(M).$$

Since fin $g(M) = g(M)$ there exists a basic submodule B' of M such that $g(M/B') = g(M)$. But these two facts contradict the hypothesis that every basic submodule of M is both an upper and lower basic submodule of M.

Lemma 2.2. *Let M be a $QTAG$-module without elements of infinite height. Suppose $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules, and suppose that every basic submodule of M_1 is both an upper and lower basic submodule of M_1. If fin $g(M_1) = g(M_1)$,*

$g(M_2) < g(M_1)$, and B is a basic submodule of M_1, then $B \oplus M_2$ is an upper basic submodule of M.

Proof. Suppose that $B \oplus M_2$ is not an upper basic submodule of M, and let B^u be an upper basic submodule of M. Let $M = S \oplus P_1$ and $B^u = P_1 \oplus P_2$ where $g(S) = \max\{\aleph_0,\ g(M/B^u)\}$. If $g(S) \leq \aleph_0$ then we get that M is a direct sum of uniserial modules, and therefore M_1 is a direct sum of uniserial modules. Thus M_1 must be bounded since each of its basic submodules is both an upper and lower basic submodule. Therefore $B = M_1$ and so $B \oplus M_2 = M_1 \oplus M_2$ is an upper basic submodule of M. Now we assume that $\aleph_0 < g(S) = g(M/B^u) < g(M/(B \oplus M_2)) \leq g(M_1)$. Now we write $M = S \oplus Q_1 \oplus Q_2$ where $P_1 = Q_1 \oplus Q_2$, and $S \oplus Q_1$ contains M_2 and $g(S \oplus Q_1) < g(M_1)$. But we know that $M_1 \cong M/M_2 \cong [(S \oplus Q_1)/M_2] \oplus Q_2$ which contradicts Lemma 2.1 when applied to M_1, therefore $B \oplus M_2$ must be an upper basic submodule of M.

Lemma 2.3. *Let M be a QTAG-module without elements of infinite height such that $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules, fin $g(M_1) = g(M_2)$, and every basic submodule of M_1 is both an upper and lower basic submodule of M_1. If B is a basic submodule of M_1, then $B \oplus M_2$ is an upper basic submodule of M.*

Proof. Suppose that $B \oplus M_2$ is not an upper basic submodule of M, and let B^u be an upper basic submodule of M. As in Lemma 2.2 we can assume that $g(M/B^u) > \aleph_0$. Write $M = S \oplus P_1$ and $B^u = P_1 \oplus P_2$ where $g(S) = g(M/B^u) < g(M/(B \oplus M_2)) \leq g(M_1)$. Now we can write $M = M_1 \oplus R_1 \oplus R_2$ where $M_2 = R_1 \oplus R_2$ and $M_1 \oplus R_1$ contains S and $g(R_1 + S) < g(M_2)$. Consider the module $M_1 \oplus R_1$. By Lemma 2.2 we know that $B \oplus R_1$ is an upper basic submodule of $M_1 \oplus R_1$. But $M_1 \oplus R_1$ contains S which is a summand of M so that we can write $M_1 \oplus R_1 = S \oplus [(M_1 \oplus R_1) \cap P_1]$. Let $T = (M_1 \oplus R_1) \cap P_1$. Now observe that $g((M_1 \oplus R_1)/(P_2 \oplus T)) = g(S/P_2) \leq g(S) < g(M_1)$. Since fin $g(M_1) = g(M_1)$, and every basic submodule of M_1 is both an upper and lower basic submodule of M_1 with $g(M_1/B) = g(M_1)$ which contradicts $B \oplus R_1$ being an upper basic submodule of $M_1 \oplus R_1$. Thus $B \oplus M_2$ must be an upper basic submodule of M.

Lemma 2.4. *Let M be an h-reduced QTAG-module such that every basic submodule of M is an upper and lower basic submodule of M. Suppose $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules. Then M_1 has the property that each of its basic submodules is both an upper and lower basic submodule of M_1.*

Proof. Suppose there exists two basic submodules B_1 and B_2 of M_1 such that $g(M_1/B_1) \neq g(M_1/B_2)$. Then $B_1 \oplus M_2$ and $B_2 \oplus M_2$ are basic submodules of M such that $g(M/(B_1 \oplus M_2)) = g(M_1/B_1) \neq g(M_1/B_2) = g(M/(B_2 \oplus M_2))$, and this contradicts the hypothesis on M.

Lemma 2.5. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where fin $g(M) = g(M)$, fin $g(M_1) = g(M_1)$, fin $g(M_2) = g(M_2)$, and M_1 and M_2 have the property that every basic submodule is an upper and lower basic submodule. If $M = M_3 \oplus M_4$ where $g(M_3) < g(M)$, and M_4 is a direct sum of uniserial modules, then $g(M) = g(M_1) = g(M_2)$.*

Proof. Suppose that $g(M_1) < g(M)$, then $g(M_2) = g(M)$. Now we write $M = M_3 \oplus T_1 \oplus T_2$ where $M_3 \oplus T_1$ contains M_1, and $g(M_3 \oplus T_1) < g(M)$. This is possible since $g(M_3) < g(M)$, and $g(M_1) < g(M)$. Now notice that $M_2 \cong M/M_1 \cong [(M_3 \oplus T_1)/M_1] \oplus T_2$, and $g((M_3 \oplus T_1)/M_1) \leq g(M_3 \oplus T_1) < g(M) = g(M_2)$. But this contradicts Lemma 2.1 when applied to M_2, therefore we have $g(M) = g(M_1) = g(M_2)$.

Lemma 2.6. *Let M be a QTAG-module without elements of infinite height and suppose that $M = M_1 \oplus M_2$, where fin $g(M_1) = g(M_1)$, and fin $g(M_2) = g(M_2)$, fin $g(M) = g(M)$, and every basic submodule of M_1 or M_2 is both an upper and lower basic submodule. Let B_1 and B_2 be basic submodules of M_1 and M_2 respectively. If either $g(M_1) < g(M)$ or $g(M_2) < g(M)$, then $B_1 \oplus B_2$ is an upper basic submodule of M.*

Proof. Assume that $B_1 \oplus B_2$ is not an upper basic submodule of M. Let B^u be an upper basic submodule of M. Write $M = M_3 \oplus M_4$ where M_4 is a direct sum of uniserial modules, and $g(M_3) = \max(\aleph_0, g(M/B^u))$. If $g(M_3) \leq \aleph_0$, then M is a direct sum of uniserial modules. But this means that M_1 and M_2 are bounded since the bounded direct sum of uniserial modules are the only direct sums of uniserial modules which have the property that every basic submodule is both an upper and lower basic submodule. Thus M is a bounded module and hence only basic submodule which contradicts the assumption that $B_1 \oplus B_2$ is not an upper basic submodule of M. Therefore $\aleph_0 < g(M/B^u)$, and $g(M_3) = g(M/B^u)$.

Since $B_1 \oplus B_2$ is not an upper basic submodule of M, we get that $g(M/B^u) < g(M)$, but this contradicts Lemma 2.5. Thus $B_1 \oplus B_2$ must be an upper basic submodule of M.

3. Main results

Through the proceeding series of lemmas, we are in a position to proceed by proving the following.

Theorem 3.1. *Let M be a $QTAG$-module without elements of infinite height such that $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules. If B^u is an upper basic submodule of M_1, then $B^u \oplus M_2$ is an upper basic submodule of M.*

Proof. If M_1 is a finitely generated, module then $B^u \oplus M_2 = M$ since B^u is basic submodule in M_1, thus $B^u \oplus M_2$ is upper basic submodule. Since B^u is an upper basic submodule of M_1, we can write, $M_1 = S_1 \oplus T_1$ and $B^u = T_1 \oplus T_2$ where $g(S_1) = \max(\aleph_0, g(M_1/B^u))$, and every basic submodule of S_1 is both an upper and lower basic submodule of S_1. As in Lemma 2.2, we assume that $g(M_1/B^u) > \aleph_0$. Now if fin $g(S_1) < g(S_1)$ then we write $S_1 = S_2 \oplus S_3$ and $T_1 = K \oplus S_3$ where fin $g(S_2) < g(S_2)$. By Lemma 2.4 every basic submodule of S_2 is both an upper and lower basic submodule of S_2. Thus $M = S_2 \oplus S_3 \oplus T_2 \oplus M_2$ where $S_3 \oplus T_2 \oplus M_2$ is a direct sum of uniserial modules, and hence, by Lemma 2.3 we have that $K \oplus S_3 \oplus T_2 \oplus M_2$ is an upper basic submodule of M, but $K \oplus S_3 \oplus T_2 \oplus M_2 = B^u \oplus M_2$.

With the last statement in hand, we establish the following corollaries about decomposition of $QTAG$-modules.

Corollary 3.1. *Let M be a $QTAG$-module without elements of infinite height such that $M = M_1 \oplus M_2 = M_3 \oplus M_4$, where fin $g(M_1) = g(M_1)$ and fin $g(M_3) = g(M_3)$. Suppose that M_2 and M_4 are direct sums of uniserial modules, and every basic submodule of M_1 and M_3 is an upper and lower basic submodule. Then fin $g(M_1) =$ fin $g(M_3)$.*

Proof. Let B_1 and B_2 be basic submodule of M_1 and M_3 respectively. Now we have fin $g(M_1) = g(M_1/B_1) = g(M/(B_1 \oplus M_2)) = g(M/(B_2 \oplus M_4)) =$ fin $g(M_3)$, since $B_1 \oplus M_2$ and $B_2 \oplus M_4$ are upper basic submodules of M by Theorem 3.1.

Corollary 3.2. *Let M be a $QTAG$-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where fin $g(M_1) = g(M_1)$ and $g(M_2) < g(M_1)$, and every basic submodule of M_1 is both an upper and lower basic submodule of M_1. If B^u is an upper basic submodule of M_2, and B is a basic submodule of M_1, then $B \oplus B^u$ is an upper basic submodule of M.*

Proof. If M is finitely generated, the proof is trivial. Now we write $M_2 = S \oplus T_1$ and $B^u = T_1 \oplus T_2$ where every basic submodule of S is both an upper and lower basic submodule of S. We can also assume that fin $g(S) = g(S)$. Consider the module $M_1 \oplus S$ satisfies the hypothesis of Lemma 2.6, and so $B \oplus T_2$ is an upper basic submodule of $M_1 \oplus S$. Now by Theorem 3.1, we have $B \oplus T_1 \oplus T_2 = B \oplus B^u$ is an upper basic submodule of M.

And so, we are ready to prove the following.

Proposition 3.1. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where fin $g(M_1) = g(M_1)$ and fin $g(M_2) = g(M_2)$, fin $g(M) = g(M)$, and every basic submodule of M_1 and M_2 is both an upper and lower basic submodule. If B_1 and B_2 are basic submodules of M_1 and M_2 respectively, then $B_1 \oplus B_2$ is an upper basic submodule of M.*

Proof. By Lemma 2.6 we assume that $g(M_1) = g(M_2) = g(M)$. Suppose that $B_1 \oplus B_2$ is not upper basic submodule of M, and let B^u be an upper basic submodule of M. Now we write $M = S \oplus Q_1$ and $B^u = Q_1 \oplus Q_2$ where S has the property that every basic submodule of S is both an upper and a lower basic submodule of S, and $g(S) = \max(\aleph_0, g(M/B^u))$. As in the proof of Lemma 2.6 we can assume that $\aleph_0 < g(S) = g(M/B^u) < g(M)$. We may also assume that fin $g(S) = g(S)$. Consider the module $M_1 + S$. Since $M_1 + S$ contains the modules M_1 and S, both of which are summands of M, we have $M_1 + S = M_1 \oplus [(M_1 + S) \cap T]$, and $M_1 + S = S \oplus [(M_1 + S) \cap Q_1]$. Let $U = (M_1 + S) \cap T$, and let B_1^u be an upper basic submodule of U. Observing that

$$\begin{aligned} g(S) &= g(S/Q_2), \\ &= g((S + M_1)/(Q_2 \oplus [(M_1 + S) \cap Q_1])), \\ &= g((M_1 + S)/(B_1 \oplus B_1^u)), \\ &= g(M_1/B_1) + g(U/B_1^u) \end{aligned}$$

Now $g(S) = g(M_1/B_1) + g(U/B_1^u) = g(M_1) = g(U/B_1^u)$, since fin $g(M_1) = g(M_1)$, and every basic submodule of M_1 is both an upper and a lower basic submodule of M_1. Thus we have that $g(M_1) \leq g(S)$, but this a contradiction since $g(S) < g(M) = g(M_1)$. Therefore $B_1 \oplus B_2$ must be an upper basic submodule of M.

Theorem 3.2. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, and let B_1^u and B_2^u be upper basic submodules of M_1 and M_2 respectively. Then $B_1^u \oplus B_2^u$ is an upper basic submodule of M.*

Proof. If either M_1 or M_2 is finitely generated, then by Theorem 3.1, $B_1^u \oplus B_2^u$ is an upper basic submodule in M. Now we write $M_1 = S_1 \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where every basic submodule of S_1 is both an upper and a lower basic submodule of S_1. Similarly we can write $U = V_1 \oplus T_1$ and $B_2^u = T_1 \oplus T_2$ where every basic submodule of V_1 is both an upper and a lower submodule of V_1. Now we have $S_1 = S_2 \oplus P_3$ and $P_2 = P_3 \oplus W_1$ where fin $g(S_2) = g(S_2)$. Similarly we have $V_1 = V_2 \oplus T_3$ and $T_2 = T_3 \oplus W_2$ where fin $g(V_2) = g(V_2)$. Therefore by Lemma 2.4 we get that S_2 and V_2 have the property that every basic submodule is both an upper and a lower basic submodule. Thus $M = (S_2 \oplus V_2) \oplus (P_1 \oplus P_3 \oplus T_1 \oplus T_3)$ and by applying Theorem 3.1 and Proposition 3.1 the proof is completed.

For freely use in the sequel, we state the following.

Corollary 3.3. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where every basic submodule of M_1 or M_2 is both an upper and a lower basic submodule, and suppose that fin $g(M) = g(M)$. Then every basic submodule of M is an upper and lower basic submodule of M.*

Proof. Let B_1^u and B_2^u be upper basic submodules of M_1 and M_2 respectively. By Theorem 3.2, we have $B_1^u \oplus B_2^u$ is an upper basic submodule of M. Notice that fin $g(M) =$ fin $g(M_1) +$ fin $g(M_2) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M/(B_1^u \oplus B_2^u))$, and we are done.

Now we are able to prove the following.

Theorem 3.3. *Let M be an h-reduced QTAG-module, and let N be a high submodule of M. If B^u is an upper basic submodule of N, then B^u is an upper basic submodule of M.*

Proof. Suppose that B^u is not upper basic submodule in M, and let B_1^u be an upper basic submodule of M and K a high submodule of M containing B_1^u with $g(M/B_1^u) < g(M)$. We have two cases to consider.

Case (i). Suppose that $g(K/B_1^u) \leq \aleph_0$, then M is a Σ-module and $g(M/N) = g(M/K)$. Since B^u is an upper basic submodule of N and M is a Σ-module, then $B^u = N$, and thus B^u is an upper basic submodule of M.

Case (ii). Suppose that $g(K/B_1^u) > \aleph_0$, then $g(M/B^u) > \aleph_0$ and we have $M = S \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where $g(S) = g(M/B_1^u)$. Now K contains P_1 and hence $K = P_1 \oplus T$ where $T = S \cap K$. Let $\phi : M \to M/M^1$ be the natural quotient map such that $N \cong \phi(N)$, $K \cong \phi(K)$, $B_1^u \cong \phi(B_1^u)$, and $Soc(\phi(N)) = Soc(\phi(K))$. Since $K = P_1 \oplus T$ we have $\phi(K) = \phi(B_1^u) \oplus \phi(T)$. Now $Soc(\phi(B_1^u)) = \cup_{i=1}^{\infty} Q_i$ where Q_i is a submodule of elements of bounded height in $\phi(M)$

and consequently in $\phi(N)$ and $\phi(K)$ since both are h-pure in $\phi(M)$. Therefore there exists a basic submodule B_1 of $\phi(N)$ such that $B_1 \supset Soc(\phi(B_1^u))$. Let $B_2 = \phi^{-1}(B_1) \cap N$, since ϕ is an isomorphism between N and $\phi(N)$ and B_2 is a basic submodule of N. If $g(N/B_2) \leq \aleph_0$ an argument as in Case (i) would complete the proof. Thus assume that $g(N/B_2) > \aleph_0$ and consider

$$\begin{aligned} g(N/B_2) &= g(\phi(N)/B_1) \\ &= g(Soc(\phi(N)/B_1)) \\ &= g(Soc(\phi(N))/Soc(B_1)) \\ &= g(Soc(\phi(K))/Soc(B_1)) \\ &= g(Soc(\phi(P_1)) \oplus Soc(\phi(T))/Soc(B_1)), \end{aligned}$$

but $Soc(B_1)$ contains $Soc(\phi(P_1))$. Hence $g(N/B_2) \leq g(Soc(T)) \leq g(Soc(S)) = g(M/B_1^u)$. Notice that $M/B_2 \cong N/B_2 \oplus M/N$, and hence

$$g(M/B_2) = g(N/B_2) + g(M/N) \leq g(M/B_1^u) + g(M/N) = g(M/B_1^u) + g(M/K),$$

and since $g(M/K) \leq g(M/B_1^u)$, we have $g(M/B_2) \leq g(M/B_1^u) + g(M/B_1^u) = g(M/B_1^u)$. Therefore B_2 is an upper basic submodule of M. We assume that B^u is not upper basic submodule of M, and so $g(M/B^u) > g(M/B_2)$. Notice that $g(M/B^u) = g(M/N) + g(N/B^u)$, and $g(M/B_2) = g(M/N) + g(N/B_2)$, so that $g(N/B^u) > g(N/B_2)$ which contradicts B^u being an upper basic submodule of N. Therefore B^u is an upper basic submodule of M.

Corollary 3.4. *Let M be a $QTAG$-module, and let N_1 and N_2 be high submodules of M, and let B_1^u and B_2^u be upper basic submodules of N_1 and N_2 respectively. Then $g(N_1/B_1^u) = g(N_2/B_2^u)$.*

Proof. Follows easily from the proof of the last theorem.

4. Some extended results

The purpose of the present section is to extending the results of Theorems 3.1 and 3.2. Several such structural consequences are now presented. In this view we first prove the following.

Theorem 4.1. *Let M be an h-reduced QTAG-module such that $M = M_1 \oplus M_2$. Let B_1^u and B_2^u be upper basic submodules of M_1 and M_2 respectively. Then $B_1^u \oplus B_2^u$ is an upper basic submodule of M.*

Proof. Let N_1 and N_2 be high submodules of M_1 and M_2 respectively, which contain B_1^u and B_2^u respectively. Now suppose that B_3^u and B_4^u are upper basic submodules of N_1 and N_2 respectively. By Theorem 3.2 we know that $B_3^u \oplus B_4^u$ is an upper basic submodules of $N_1 \oplus N_2$, and hence by Theorem 3.3, $B_3^u \oplus B_4^u$ is an upper basic submodule of M. Now $g(M/(B_3^u \oplus B_4^u)) = g(M_1/B_3^u) + g(M_2/B_4^u)$, and since B_3^u and B_4^u are upper basic submodules of N_1 and N_2 respectively, we have by Theorem 3.3, they are basic submodules of M_1 and M_2 respectively. Thus we know that $g(M_1/B_3^u) = g(M_1/B_1^u)$, and $g(M_2/B_4^u) = g(M_2/B_2^u)$. Therefore $g(M/(B_3^u \oplus B_4^u)) = g(M_1/B_3^u) + g(M_2/B_4^u) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M/(B_1^u \oplus B_2^u))$, and hence $B_1^u \oplus B_2^u$ is an upper basic submodule of M.

Theorem 4.2. *Let M be an h-reduced QTAG-module such that $M = M_1 \oplus M_2$ where M_2 is a direct sum of uniserial modules. Let B^u be an upper basic submodules of M_1. Then $B^u \oplus M_2$ is an upper basic submodule of M.*

Proof. The proof follows easily from Theorem 4.1.

Lemma 4.1. *Let M be a QTAG-module such that $M = M_1 \oplus M_2$ be a direct sum of uniserial modules, and suppose that $g(M_2) < g(M)$ and that $\aleph_0 < g(M_2)$ is not a limit cardinal. Let N be an h-pure submodule of M_2, and let $B_1 \oplus N$ be a basic submodule of M such that $g(M/(B_1 \oplus N)) > g(M_2)$. Then there exists a basic submodule $B_2 \oplus N$ such that $B_2 \oplus N$ contains $B_1 \oplus N$, and $g(M/(B_2 \oplus N)) \leq g(M_2)$.*

Proof. Consider $M/B_1 = M_3/B_1 \oplus M_4/B_1$ where M_4/B_1 is an h-reduced and contain $(B_1 \oplus N)/B_1$, and where M_3/B_1 is h-divisible. Notice that $M_3 \cap N = 0$ since $M_3 \cap B_1 = B_1$ and $B_1 \cap N = 0$. Thus $M_3 + N = M_3 \oplus N$, and as a submodule of a direct sum of uniserial modules is itself a direct sum of uniserial modules. To show that $M_3 \oplus N$ is a basic submodule, we need only prove that $M_3 \oplus N$ is h-pure, but $(M_3 \oplus N)/(B_1 \oplus N) \cong M_3/B_1$ which is h-divisible and hence $M_3 \oplus N$ is h-pure. Therefore $M_3 \oplus N$ is a basic submodule of M which contains $B_1 \oplus N$, and notice that $g(M/(M_3 \oplus N)) \leq g(M/M_3) = g(M_4/B_1)$, and $g(M_4/B_1) \leq g(M_2)$ since $g(M_2)$ is not a limit ordinal. This completes the proof.

Theorem 4.3. *Let M be a QTAG-module without elements of infinite height, and let B be a basic submodule of M. Let B^u be an upper basic submodules of M, and suppose that*

$g(M/B^u)$ is not a limit cardinal larger than $\aleph_0$. Then B is contained in an upper basic submodule of M.

Proof. If $g(M/B^u) \leq \aleph_0$, then M is a direct sum of uniserial modules and hence B is contained in an upper basic submodule of M, namely, M itself. Thus we may assume that $\aleph_0 < g(S) = g(M/B^u)$, $M = S \oplus P_1$ where $B^u = P_1 \oplus P_2$ and $g(S) = g(M/B^u)$. Let S be the homomorphic image of the free module T with h-pure kernel K and we can assume $g(T) = g(S)$. Now $(P_1 \oplus T)/K \cong S \oplus P_1$, and suppose $(Q_1 \oplus K)/K \cong B$. If $g(M/B) = g(M/B^u)$ we know that B is already an upper basic submodule of M and we are done, so that we can assume that $g(M/B) > g(M/B^u) = g(S)$. Thus $g[(T \oplus P_1)/(Q_1 \oplus K)] > g(S) = g(T)$, and by Lemma 4.1 there exists a basic submodule $Q_2 \oplus K$ containing $Q_1 \oplus K$ and such that $g[(T \oplus P_1)/(Q_2 \oplus K)] = g(T)$. Let $R \cong (Q_2 \oplus K)/K$. We know that R is a basic submodule of M which contains B and R is upper basic submodule of M since $g(M/R) = g[(T \oplus P_1)/(Q_2 \oplus K)] = g(S) = g(M/B^u)$.

Theorem 4.4. *Let M be an h-reduced $QTAG$-module and let B be a basic submodule of M. If there exists a high submodule N of M whcih contains B, and an upper basic submodule B^u of N containing B, then B is contained in an upper basic submodule of M.*

Proof. If B^u is an upper basic submodule of N, then B^u is an upper basic submodule of M by Theorem 3.3.

Let $\mathfrak{B}$ be the class of $QTAG$-modules which have the property that every basic submodule is contained in an upper basic submodule.

Theorem 4.5. *The class $\mathfrak{B}$ contains all $QTAG$-modules which are direct sum of an h-divisible and a bounded module.*

Proof. This follows immediately from the fact that such modules have only basic submodule, and it is by necessity an upper basic submodule.

Recall that a module M is a Σ-module (see [4]) if some its high submodule is a direct sum of uniserial modules.

Theorem 4.6. *The class $\mathfrak{B}$ contains all Σ-modules.*

Proof. Let M be Σ-module, and B be a basic submodule of M. Now B can be embedded in a high submodule of N of M, and since M is a Σ-module we get that N is an upper basic submodule of M.

Theorem 4.7. *Let M be a QTAG-module without elements of infinite height. Suppose that $fin\ g(M)$ is equal to its cardinality, and that B is a basic submodule of M. If $g(M/B)$ is equal to the cardinality of $\overline{M}$, the closure of M, then $M \in \mathfrak{B}$.*

Proof. This follows from Theorem 4.3.

Corollary 4.1. *The class $\mathfrak{B}$ contains all closed modules.*

5. Open problems

In closing, we pose the following questions of interest:

Problem 5.1. *If M is an h-reduced QTAG-module such that $M = \sum_{\alpha \in I} M_\alpha$, and B_α is an upper basic submodule of M_α, then is it true that $\sum_{\alpha \in I} B_\alpha$ is upeer basic submodule of M?*

Problem 5.2. *Does the class $\mathfrak{B}$ defined above indeed contains all h-reduced QTAG-modules?*

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