



ON QUASI BI-SLANT SUBMANIFOLDS OF A KÄHLER MANIFOLD IMMERSED AS SEQUENTIAL WARPED PRODUCTS

SOFIA SINGLA , MOHAMMAD SHUAIB , AND SANGEET KUMAR *

Abstract. The present paper focuses on the study of a new class of sequential warped product submanifolds considered by harmonizing one invariant submanifold and two proper slant submanifolds of Kähler manifolds with the concept of sequential warped products. In particular, we examine the existence (or non-existence) of proper quasi bi-slant warped product submanifolds of Kähler manifolds, conceptualized as sequential warped products. Our claim demonstrates that a Kähler manifold does not admit any proper quasi bi-slant warped product submanifolds (in the sequential warped product sense). The study advances the understanding of submanifold geometry in complex manifolds, offering significant insights into the non-existence of such warped product submanifolds and their geometric implications in Kähler geometry.

Keywords: Warped product submanifolds, Kähler manifolds, slant submanifold, quasi bi-slant submanifolds.

MSC (2020): 53B25, 53B30, 53B35.

1. INTRODUCTION

The geometry of submanifolds, studying subsets of higher-dimensional spaces with geometric structures, emerged in the 19th century through the work of Gauss and Riemann. Their foundational contributions in differential geometry enabled the analysis of submanifolds' intrinsic and extrinsic properties. The study of invariant, anti-invariant, and CR -submanifolds of almost Hermitian manifolds began in the mid-20th century, building on Hermitian geometry foundations from Riemann's theory. Invariant and anti-invariant submanifolds, defined by their behavior under the almost complex structure, along with CR -submanifolds combining both properties, were formalized and advanced by Bejancu in the 1970s through studies on submanifolds in Kähler and complex manifolds [2, 3].

The study of slant submanifolds in almost Hermitian manifolds originated with B.Y. Chen's [8] introduction of the concept, defining them as submanifolds where the angle between the tangent space and its image under the almost complex structure is constant. This

Received: 2026.04.23

Revised: 2026.07.14

Accepted: 2026.07.25

* Corresponding author

Sofia Singla (sofia.e17646@cumail.in) $\diamond$ ORCID:0000-0001-8917-299X

Mohammad Shuaib (shuaibyouusuf6@gmail.com) $\diamond$ ORCID:0000-0003-0022-893X

Sangeet Kumar (sp7maths@gmail.com) $\diamond$ ORCID:0000-0002-3736-4751

notion was further extended by N. Papaghiuc [14] to semi-slant submanifolds, which decompose the tangent space into invariant and slant distributions. In 2000, A. Carriazo [6] introduced bi-slant submanifolds, featuring two orthogonal slant distributions with potentially different angles, encompassing semi-slant and hemi-slant types as special cases, where hemi-slant involves one totally real and one slant distribution. These generalizations have since been explored in various geometric settings [23], providing a unified framework for intermediate submanifold classes between invariant and anti-invariant ones.

On the other hand, warped product manifolds were first introduced by R. L. Bishop and B. O'Neill [5], generalizing Riemannian products to create manifolds with negative sectional curvature, facilitating the study of geometric properties such as completeness and embeddings. In Kähler manifolds, Chen [9] work on CR -warped product submanifolds established key inequalities and classifications. B. Şahin [17] furthered investigating slant warped product submanifolds in Kähler manifolds, proving non-existence for non-trivial cases through inequalities tied to the warping function and slant angle. Further, Şahin [17, 18] studied semi-slant and pointwise semi-slant submanifolds of Kähler manifolds immersed as warped products. Later, several authors have investigated such warped products from a geometric perspective and obtained a variety of interesting results [13, 12, 21, 1].

Moreover, sequential warped product manifolds, initially introduced by Shenawy [22], represent a generalization of traditional warped product manifolds. These structures were thoroughly investigated by De, Shenawy, and Ünal [10, 24], focusing on their properties as submanifolds in various geometric settings. Additionally, sequential warped products have been recognized as effective frameworks for modeling generalized Robertson–Walker space-time and standard static space-time, as demonstrated in further studies. Building on this, Şahin [19], advanced their application in almost Hermitian manifolds, particularly Kähler manifolds.

Inspired by this concept, this paper proposes a sequential warped product characterized by two proper slant distributions and one invariant distribution, which we designate as a quasi bi-slant sequential warped product. Additionally, we explore the existence of such proper sequential warped product manifolds of Kähler manifolds that feature two proper slant distributions and one invariant distribution. The structure of this paper is organized as follows: Section 2 presents the foundational concepts and preliminaries concerning almost Hermitian manifolds, with a particular emphasis on Kähler manifolds. In Section 3, we provide an overview of essential properties and characteristics of sequential warped product submanifolds of almost Hermitian manifolds. Additionally, this section investigates the existence of such sequential warped product submanifolds specifically in the context of Kähler manifolds.

2. PRELIMINARIES

An almost complex manifold is defined as a pair (M, J) , where M is a real differentiable manifold and J is a smooth field of endomorphisms on the tangent bundle (i.e., $J_p : T_p M \rightarrow T_p M$, for each $p \in M$) satisfying $J^2 = -I$. When M is paracompact, there always exists a Riemannian metric g that respects this structure. Such a metric, characterized by the invariance condition $g(JU_1, JU_2) = g(U_1, U_2)$, is referred to as a Hermitian metric on the almost complex manifold. A differentiable manifold is called almost Hermitian if it is almost

complex and possesses a Hermitian metric. In the integrable case, such a manifold is simply called a Hermitian manifold. By definition, a Hermitian manifold (M, J, g) therefore carries a canonical geometric structure, which includes the complex structure J , the Hermitian metric g and its unique torsion-free metric connection, the Levi-Civita connection $\bar{\nabla}$.

The fundamental 2-form Ω associated with an almost complex structure J and Hermitian metric g of an almost Hermitian manifold $\bar{M}$ is defined by

$$\Omega(U_1, U_2) = g(U_1, JU_2), \quad (2.1)$$

for all vector fields U_1, U_2 on $\bar{M}$. Since g is invariant under J so is Ω , i.e.

$$\Omega(JU_1, JU_2) = \Omega(U_1, U_2). \quad (2.2)$$

Definition 2.1. [25] *An almost complex manifold endowed with a Hermitian metric is called a Kähler manifold if its fundamental 2-form Ω is closed. Equivalently, a Hermitian manifold $\bar{M}$ is Kähler if and only if its almost complex structure J is parallel with respect to the Hermitian connection $\bar{\nabla}$, i.e.,*

$$(\bar{\nabla}_{U_1} J)U_2 = 0 \quad \text{for all } U_1, U_2 \in T\bar{M}. \quad (2.3)$$

When this condition holds, the connection $\bar{\nabla}$ is referred to as the Kähler connection.

Let M and $\bar{M}$ be differentiable manifolds of dimensions n and m respectively. Hence, for every point $p \in M$, the tangent space $T_p\bar{M}$ of $\bar{M}$ admits the following decomposition

$$T_p\bar{M} = T_pM \oplus T_p^\perp M.$$

The Riemannian connection $\bar{\nabla}$ of $\bar{M}$ induces canonically the connection ∇ and $\nabla^\perp$ on TM and on the normal bundle $T^\perp M$ respectively governed by the Gauss and Weingarten formulae, viz.

$$\bar{\nabla}_{U_1} U_2 = \nabla_{U_1} U_2 + h(U_1, U_2), \quad (2.4)$$

$$\bar{\nabla}_{U_1} N = -A_N U_1 + \nabla_{U_1}^\perp N, \quad (2.5)$$

where U_1, U_2 are tangent vector fields on M and $N \in T^\perp M$. h and A are the second fundamental form and the shape operator respectively and are related as

$$g(h(U_1, U_2), N) = g(A_N U_1, U_2). \quad (2.6)$$

A submanifold M isometrically immersed in a Kähler manifold $\bar{M}$ with an almost complex structure J is said to be CR -submanifold, if there exists invariant $\mathcal{H}^T$ and anti-invariant distribution $\mathcal{H}^\perp$ such that $J\mathcal{H}^T = \mathcal{H}^T$ and $J\mathcal{H}^\perp \subset \mathcal{H}^\perp$ [4]. A submanifold M of a Kähler manifold $\bar{M}$ is called a slant submanifold [7] if, for every point $p \in M$ and every nonzero tangent vector $X \in T_pM$, the angle $\theta(X)$ (known as the Wirtinger angle or slant angle) between JX and T_pM is constant, independent of the choice of p and $X \in T_pM$. This constant angle $\theta \in [0, \pi/2]$ characterizes the submanifold globally. Special cases include, $\theta = 0$: M is invariant (holomorphic), $\theta = \pi/2$: M is anti-invariant (totally real) and $0 < \theta < \pi/2$: Proper slant. A submanifold M of a Kähler manifold $\bar{M}$ with almost complex structure J is called a semi-slant submanifold [14] if the tangent bundle TM admits an orthogonal decomposition $TM = \mathcal{H}^T \oplus \mathcal{H}^\theta$, where $\mathcal{H}^T$ is an invariant distribution, i.e., $J(\mathcal{H}^T) = \mathcal{H}^T$ and $\mathcal{H}^\theta$ is a slant distribution. A submanifold M of a Kähler manifold $\bar{M}$ said to be hemi-slant

submanifold if $\mathcal{H}^\perp$ is an anti-invariant distribution, i.e., $J(\mathcal{H}^\perp) \subseteq T^\perp M$ and $\mathcal{H}^\theta$ is slant distribution [20]. A submanifold M of a Kähler manifold $\bar{M}$ is called a bi-slant submanifold if there exists two distributions $\mathcal{H}^{\theta_1}, \mathcal{H}^{\theta_2}$ on M such that the distribution $\mathcal{H}^{\theta_1}$ (resp. $\mathcal{H}^{\theta_2}$) is slant, i.e., for every point $p \in M$ (resp. $q \in M$) and every nonzero vector $X \in (\mathcal{H}^{\theta_1})_p$, (resp. $Y \in (\mathcal{H}^{\theta_2})_q$) the Wirtinger angle $\theta_1(X)$ (resp. $\theta_2(Y)$) between JX and $T_p M$ (resp. $T_q M$) is constant, independent of p (resp. q) in M and $X \in T_p M$ (resp. $T_q M$) [6].

In this context, we revisit the definition of quasi bi-slant submanifolds of almost Hermitian manifolds as established by R. Prasad et al., [15].

Definition 2.2. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be a quasi bi-slant submanifold if there exists distribution $\mathcal{H}^T, \mathcal{H}^{\theta_1}$ and $\mathcal{H}^{\theta_2}$ such that

$$(i) \quad TM = \mathcal{H}^T \oplus \mathcal{H}^{\theta_1} \oplus \mathcal{H}^{\theta_2},$$

$$(ii) \quad J\mathcal{H}^T = \mathcal{H}^T,$$

$$(iii) \quad J\mathcal{H}^{\theta_1} \perp \mathcal{H}^{\theta_2}, \quad J\mathcal{H}^{\theta_2} \perp \mathcal{H}^{\theta_1},$$

$$(iv) \quad \text{for every point } p \in M \text{ and every nonzero vector field } Y_1 \in (\mathcal{H}^{\theta_1})_p, \text{ the angle } \theta_1 \text{ between } JY_1 \text{ and } (\mathcal{H}^{\theta_1})_p \text{ is constant, independent of the choice of } p \in M \text{ and } Y_1 \in (\mathcal{H}^{\theta_1})_p,$$

$$(v) \quad \text{for every point } q \in M \text{ and every nonzero vector field } Y_2 \in (\mathcal{H}^{\theta_2})_q, \text{ the angle } \theta_2 \text{ between } JY_2 \text{ and } (\mathcal{H}^{\theta_2})_q \text{ is constant, independent of the choice of } q \in M \text{ and } Y_2 \in (\mathcal{H}^{\theta_2})_q.$$

The angles θ_1 and θ_2 are called the slant angles.

As the generalization of warped product and doubly warped product, here is the definition of sequential warped product manifolds.

Definition 2.3. [11] For pseudo-Riemannian manifolds (M_i, g_i) ($i = 1, 2, 3$), let $f : M_1 \rightarrow (0, \infty)$ and $h : M_1 \times M_2 \rightarrow (0, \infty)$ be smooth positive functions. The sequential warped product, denoted $(M_1 \times_f M_2) \times_h M_3$, is the product manifold $M_1 \times M_2 \times M_3$ endowed with the metric $\bar{g} = g_1 \oplus f^2 g_2 \oplus h^2 g_3$. Here, f and h are the warping functions. Notably, if all (M_i, g_i) are Riemannian, then this construction yields a Riemannian manifold.

Proposition 2.1. [11] Consider a sequential warped product manifold $(M_1 \times_f M_2) \times_h M_3$ equipped with the metric $\bar{g} = (g_1 \oplus f^2 g_2) \oplus h^2 g_3$. Let $X_i, Y_i, Z_i \in \mathfrak{X}(M_i)$ denote vector fields on the respective factor manifolds for $i = 1, 2, 3$. Then

$$(i) \quad \bar{\nabla}_{W_1} W_2 = \bar{\nabla}_{W_2} W_1 = W_1(\ln f)W_2,$$

$$(ii) \quad \bar{\nabla}_{W_3} W_1 = \bar{\nabla}_{W_1} W_3 = W_1(\ln h)W_3,$$

$$(iii) \quad \bar{\nabla}_{W_2} W_3 = \bar{\nabla}_{W_3} W_2 = W_2(\ln h)W_3,$$

$$(iv) \quad \bar{R}(W_i, W_3)W_j = \frac{-1}{h}\mathcal{H}^h(W_i, W_j)W_3, \text{ for } i, j = 1, 2.$$

Remark 2.1. Consider smooth positive functions $f : M_1 \rightarrow (0, \infty)$ and $h : M_1 \times M_2 \rightarrow (0, \infty)$ defined on the Riemannian manifolds M_1 and $M_1 \times M_2$, respectively. Then the sequential warped product $M = (M_1 \times_f M_2) \times_h M_3$ is said to be proper if the following three conditions hold for all non-zero vector fields $X_1 \in \Gamma(TM_1)$ and $X_2 \in \Gamma(TM_2)$:

$$X_1(\ln f) \neq 0, \quad X_1(\ln h) \neq 0, \quad X_2(\ln h) \neq 0.$$

3. QUASI BI-SLANT WARPED PRODUCT SUBAMNIFOLDS AS SEQUENTIAL WARPED PRODUCT SUBAMANIFOLDS

B. Sahin [19] introduced the concept of sequential warped products in Kähler manifolds, considering factors that are holomorphic, totally real, and pointwise slant. He further established fundamental inequalities and a characterization for the associated warping functions.

As a natural consequence, it is obvious to seek an answer to the following question.

Question: Does a Kähler manifold admit any proper sequential warped product submanifold with one holomorphic factor and two proper slant factors?

To answer this affirmative, we investigate the geometry of a quasi bi-slant submanifold of a Kähler manifold immersed as a sequential warped product.

By definition of a quasi bi-slant submanifold M of a Kähler manifold $\bar{M}$, considering the decomposition of the tangent bundle of M , we have $TM = \mathcal{H}^T \oplus \mathcal{H}^{\theta_1} \oplus \mathcal{H}^{\theta_2}$, which indicates that TM is decomposed into three distributions of which one is holomorphic (invariant) while the other two are proper slant distributions. This yields that a quasi bi-slant submanifold of a Kähler manifold can be immersed either as a doubly warped product submanifold or as a sequential warped product submanifold. From the stated two type of warped products, the most simple possible warped products for quasi bi-slant submanifolds are sequential warped products and in this context, the possible sequential warped products for this class of submanifolds are as follows:

$$(i) \quad (M_T \times_{f_1} M_{\theta_i}) \times_{f_2} M_{\theta_j},$$

$$(ii) \quad (M_{\theta_i} \times_{f_1} M_T) \times_{f_2} M_{\theta_j},$$

$$(iii) \quad (M_{\theta_i} \times_{f_1} M_{\theta_j}) \times_{f_2} M_T,$$

where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are proper slant submanifolds of $\bar{M}$ for $i, j = 1, 2$.

For any vector fields $U_1 \in \Gamma(TM)$ and $U_2 \in \Gamma(TM^\perp)$, we have

$$JU_1 = tU_1 + nU_1 \tag{3.7}$$

and

$$JU_2 = bU_2 + cU_2, \tag{3.8}$$

where tU_1 and bU_2 are the tangential components of JU_1 , whereas nU_1 and cU_2 are the normal components of JU_2 .

For any vector fields $U_1 \in TM_T$ and $X_1 \in TM_{\theta_i}$, by using Proposition 2.1, we have

$$g(\nabla_{tX_1} U_1, X_1) = (U_1 \ln f_1)g(tX_1, X_1) = 0.$$

From Eqs. (2.3), (2.4) and (3.7), we have

$$\begin{aligned} 0 &= g(\nabla_{tX_1} U_1, X_1) \\ &= g(\bar{\nabla}_{tX_1} U_1 - h(tX_1, U_1), X_1) \\ &= -g(JU_1, \bar{\nabla}_{tX_1} tX_1) - g(JU_1, \bar{\nabla}_{tX_1} nX_1). \end{aligned}$$

In the light of Eqs. (2.5), (2.6) and Proposition 2.1, we get

$$(JU_1 \ln f_1) \cos^2 \theta_i g(X_1, X_1) = -g(h(JU_1, tX_1), nX_1). \quad (3.9)$$

On replacing U_1 by JU_1 in Eq. (3.9), we have

$$(U_1 \ln f_1) \cos^2 \theta_i g(X_1, X_1) = -g(h(U_1, tX_1), nX_1). \quad (3.10)$$

By replacing X_1 by tX_1 in Eq. (3.10), we acquire

$$g(h(U_1, X_1), ntX_1) = \cos^2 \theta_i (U_1 \ln f_1) g(X_1, X_1). \quad (3.11)$$

On the other hand, for any $U_1 \in TM_T$ and $X_1, X_2 \in TM_{\theta_i}$, by using Eqs. (2.3), (2.4) and (3.7), we have

$$\begin{aligned} g(h(tX_1, U_1), nX_2) &= g(\bar{\nabla}_{U_1} tX_1 - \nabla_{U_1} tX_1, nX_2) \\ &= -g(\bar{\nabla}_{U_1} nX_2, tX_1) \\ &= -g(\bar{\nabla}_{U_1} JX_2, tX_1) + g(\bar{\nabla}_{U_1} tX_2, tX_1) \\ &= g(t^2 X_1 + ntX_1, \bar{\nabla}_{U_1} X_2) + g(tX_1, \nabla_{U_1} tX_2) \\ &= g(t^2 X_1, \bar{\nabla}_{U_1} X_2) + g(ntX_1, \bar{\nabla}_{U_1} X_2) + g(tX_1, \nabla_{U_1} tX_2). \end{aligned}$$

In the light of Eq. (2.4) with Proposition 2.1, we get

$$\begin{aligned} g(h(tX_1, U_1), nX_2) &= -\cos^2 \theta_i g(X_1, \bar{\nabla}_{U_1} X_2) + g(ntX_1, \bar{\nabla}_{U_1} X_2) + g(tX_1, \nabla_{U_1} tX_2) \\ &= -\cos^2 \theta_i g(X_1, \nabla_{U_1} X_2) + g(ntX_1, h(U_1, X_2)) + g(tX_1, \nabla_{U_1} tX_2) \\ &= -\cos^2 \theta_i (U_1 \ln f_1) g(X_1, X_2) + g(ntX_1, h(U_1, X_2)) + (U_1 \ln f_1) g(tX_1, tX_2) \\ &= g(ntX_1, h(U_1, X_2)) - \cos^2 \theta_i (U_1 \ln f_1) g(X_1, X_2) + (U_1 \ln f_1) g(tX_1, tX_2). \end{aligned}$$

From the above equation, it is easy to obtain that

$$g(h(tX_1, U_1), nX_2) = g(ntX_1, h(U_1, X_2)). \quad (3.12)$$

Taking $X_1 = X_2$ in Eq. (3.12), we get

$$g(h(tX_1, U_1), nX_1) = g(ntX_1, h(U_1, X_1)). \quad (3.13)$$

By using Eqs. (3.10), (3.11) and (3.13), we have

$$(U_1 \ln f_1) \cos^2 \theta_i g(X_1, X_1) = 0.$$

As M_{θ_i} is proper slant submanifold and X_1 is non-null vector, therefore we get $U_1 \ln f_1 = 0$, which implies that f_1 is constant on M_T .

Based on the above, we can assert the following result:

Theorem 3.1. *There do not exist any proper sequential warped product submanifold of a Kähler manifold $\bar{M}$ of the form $(M_T \times_{f_1} M_{\theta_i}) \times_{f_2} M_{\theta_j}$ where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are the slant submanifolds of a Kähler manifold $\bar{M}$ for $i, j = 1, 2$.*

Moreover, from the computation in the prior section, with $U_1 \in TM_T$ and $X_1, X_2 \in TM_{\theta_i}$, we infer that

Theorem 3.2. *There do not exist any proper sequential warped product submanifold of a Kähler manifold $\bar{M}$ of the form $(M_{\theta_i} \times_{f_1} M_T) \times_{f_2} M_{\theta_j}$, where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are the slant submanifolds of a Kähler manifold $\bar{M}$ for $i, j = 1, 2$.*

The subsequent theorem demonstrates that a Kähler manifold $\bar{M}$ does not admit a quasi bi-slant sequential warped product submanifold of the form $(M_{\theta_i} \times_{f_1} M_{\theta_j}) \times_{f_2} M_T$.

Theorem 3.3. *There do not exist any proper sequential warped product submanifold of a Kähler manifold $\bar{M}$ of the form $(M_{\theta_i} \times_{f_1} M_{\theta_j}) \times_{f_2} M_T$ where M_T is invariant submanifold and $M_{\theta_i}, M_{\theta_j}$ are the slant submanifolds of a Kähler manifold $\bar{M}$ for $i, j = 1, 2$.*

Proof. Let us suppose for any $U_1 \in TM_T$ and $X_1 \in TM_{\theta_j}$, by using Proposition 2.1, we get

$$g(\nabla_{JU_1} X_1, U_1) = (X_1 \ln f_2)g(JU_1, U_1) = 0.$$

From Gauss formula and considering the above fact, we have

$$g(\nabla_{JU_1} X_1, U_1) = g(\bar{\nabla}_{JU_1} X_1, U_1).$$

By using Eqs. (2.3) and (3.7), we may write

$$g(\nabla_{JU_1} X_1, U_1) = -g(tX_1 + nX_1, \bar{\nabla}_{JU_1} JU_1).$$

Considering Eq. (2.4) and Proposition 2.1, we get

$$(tX_1 \ln f_2)g(U_1, U_1) = g(h(JU_1, JU_1), nX_1). \quad (3.14)$$

By using the polarization identity for any $U_1, V_1 \in TM_T$ and $X_1 \in TM_{\theta_j}$, we have

$$(tX_1 \ln f_2)g(U_1, V_1) = g(h(JU_1, JV_1), nX_1). \quad (3.15)$$

On the other hand, considering

$$g(A_{nX_1} JU_1, JV_1) = g(-\bar{\nabla}_{JU_1} nX_1 + \nabla_{JU_1}^\perp nX_1, JV_1),$$

by using Eq. (2.5). Further, in the light of fact from Eqs. (2.3) and (3.7), we can write

$$\begin{aligned} g(A_{nX_1} JU_1, JV_1) &= g(JX_1, J\bar{\nabla}_{JU_1} V_1) - g(tX_1, \bar{\nabla}_{JU_1} JV_1) \\ &= g(X_1, \nabla_{JU_1} V_1 + h(JU_1, V_1)) + g(\bar{\nabla}_{JU_1} tX_1, JV_1) \\ &= g(X_1, \nabla_{JU_1} V_1) + g(\nabla_{JU_1} tX_1, JV_1). \end{aligned}$$

In the light of Proposition 2.1 and Eq. (2.4), we get

$$\begin{aligned} g(A_{nX_1}JU_1, JV_1) &= g(X_1, \bar{\nabla}_{JU_1}V_1) + (tX_1 \ln f_2)g(JU_1, JV_1) \\ &= -g(\nabla_{JU_1}X_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1) \\ &= -g(\nabla_{JU_1}X_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1) \\ &= -(X_1 \ln f_2)g(JU_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1), \end{aligned}$$

which implies that

$$g(h(JU_1, JV_1), nX_1) = -(X_1 \ln f_2)g(JU_1, V_1) + (tX_1 \ln f_2)g(U_1, V_1). \quad (3.16)$$

From Eqs. (3.15) and (3.16), we have

$$(X_1 \ln f_2)g(JU_1, V_1) = 0.$$

Then taking $U_1 = JU_1$ in the last equation, we get $(X_1 \ln f_2)g(U_1, V_1) = 0$. As M_T is non-degenerate, thus we have $\ln f_2 = 0$, which implies that f_2 is constant on M_{θ_j} . $\square$

4. CONCLUSION

Inspired by B. Şahin's [19] investigation into the existence of sequential warped product submanifolds in almost Hermitian manifolds, where such warped products were found to exist, we explored the existence of proper sequential warped product submanifolds in Kähler manifolds, which we name this type of warped products as proper quasi bi-slant warped product submanifolds. Our findings reveal that no such proper warped product submanifolds exist in Kähler manifolds. This opens a scope for future research to investigate whether warped product submanifolds of this type exist in nearly Kähler manifolds or almost contact metric manifolds. On a similar note, one may look to characterize the notion of sequential warped products for submanifolds involving one or two pointwise slant factors in different ambient space settings.

Acknowledgments. The authors would like to express sincere gratitude to the anonymous referees and the editor for their valuable suggestions that helped us to improve the manuscript.

Conflict of interest. The authors declare no potential conflict of interests.

REFERENCES

- [1] Al-Solamy, F. R. (2014). Hemi-slant warped product submanifolds of nearly Kähler manifolds. *Abstract and Applied Analysis*, 2014, Article 404851.
- [2] Bejancu, A. (1978). CR submanifolds of a Kähler manifold I. *Proceedings of the American Mathematical Society*, 69, 135–142.
- [3] Bejancu, A. (1979). CR submanifolds of a Kähler manifold II. *Transactions of the American Mathematical Society*, 250, 333–345.
- [4] Bejancu, A. (1986). *Geometry of CR-submanifolds*. Kluwer Academic Publishers.
- [5] Bishop, R. L., & O'Neil, B. (1969). Manifolds of negative curvature. *Transactions of the American Mathematical Society*, 145, 1–49.
- [6] Carriazo, A. (2000). Bi-slant immersions. In J. C. Misra & S. B. Sinha (Eds.), *Proceedings ICRAMS* (pp. 88–97). Narosa Publishing House.

- [7] Chen, B. Y. (1990a). *Geometry of slant submanifolds*. Katholieke Universiteit Leuven.
- [8] Chen, B. Y. (1990b). Slant immersions. *Bulletin of the Australian Mathematical Society*, 41, 135–147.
- [9] Chen, B. Y. (2002). Geometry of warped product CR-submanifolds in Kähler manifolds. *Monatshefte für Mathematik*, 133, 177–195.
- [10] De, A., Shenawy, S., & Ünal, B. (2017). Sequential warped product submanifolds in Kähler manifolds. *Filomat*, 31, 6051–6062.
- [11] De, U. C., Shenawy, S., & Ünal, B. (2019). Sequential warped products: Curvature and conformal vector fields. *Filomat*, 33, 4071–4083.
- [12] Khan, V. A. (2012). Pseudo-slant submanifolds of nearly Kähler manifolds. *Journal of Geometry and Symmetry in Physics*, 28, 1–12.
- [13] Khan, V. A., & Khan, M. A. (2010). Slant submersions from almost Hermitian manifolds. *Kyungpook Mathematical Journal*, 50, 549–566.
- [14] Papaghiuc, N. (1994). Semi-slant submanifolds of a Kählerian manifold. *Analele Științifice ale Universității “Al. I. Cuza” din Iași*, 40, 55–61.
- [15] Prasad, R., Akyol, M. A., Verma, S. K., & Kumar, S. (2022). Quasi bi-slant submanifolds of Kähler manifolds. *International Electronic Journal of Geometry*, 15, 57–68.
- [16] Şahin, B. (2006). Non-existence of warped product semi-slant submanifolds of Kähler manifolds. *Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie*, 49, 205–214.
- [17] Şahin, B. (2009). Warped product submanifolds of Kähler manifolds with a slant factor. *Annales Polonici Mathematici*, 95, 207–226.
- [18] Şahin, B. (2013). Warped product pointwise semi-slant submanifolds of Kähler manifolds. *Portugaliae Mathematica*, 70, 279–298.
- [19] Şahin, B. (2022). Sequential warped product submanifolds having holomorphic, totally real and pointwise slant factors. *Periodica Mathematica Hungarica*, 85, 128–139.
- [20] Şahin, F. (2014). Cohomology of hemi slant submanifolds of a Kähler manifold. *Journal of Advanced Studies in Topology*, 5, 27–31.
- [21] Sharma, S. K. (2017). Pointwise pseudo-slant warped product submanifolds in a Kähler manifold. *Mediterranean Journal of Mathematics*, 14, Article 25.
- [22] Shenawy, S. (n.d.). *A note on sequential warped product manifolds* [Preprint]. arXiv, arXiv:1506.06056v1.
- [23] Taştan, H. M., Şahin, B., & Yanan, S. (2016). Hemi-slant submersions. *Mediterranean Journal of Mathematics*, 13, 2171–2184.
- [24] Ünal, B. (2016). Sequential warped products and their applications in general relativity. *Journal of Geometry and Physics*, 106, 279–288.
- [25] Yano, K., & Kon, M. (1984). *Structures on manifolds*. World Scientific Press.

(S. Singla) DEPARTMENT OF MATHEMATICS, CHANDIGARH UNIVERSITY, GHARUAN, MOHALI 140413, INDIA

(M. Shuaib) DEPARTMENT OF MATHEMATICS, LOVELY PROFESSIONAL UNIVERSITY, PHAGWARA 144402, INDIA

(S. Kumar) DEPARTMENT OF MATHEMATICS, SRI GURU TEG BAHADUR KHALSA COLLEGE, SRI ANANDPUR SAHIB 140118, INDIA